Necessary Conditions and Sufficient Conditions for Global Existence in the Nonlinear Schrödinger Equation

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Abstract

In this paper, we consider the nonlinear Schrödinger equation with the super critical power of nonlinearity in the attractive case. We give a sufficient condition and a necessary condition to obtain global or blowing up solutions. These conditions coincide in the critical case, thereby extending the results of Weinstein [26, 27]. Furthermore, we improve a blow-up condition.

1 Introduction and notations

We consider the following nonlinear Schrödinger equation,

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u + \lambda |u|^\alpha u = 0, \ (t,x) \in (-T_*, T^*) \times \mathbb{R}^N, \\ u(0) = \varphi, \ \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $0 \leqslant \alpha < \dfrac{4}{N-2}$ ($0 \leqslant \alpha < \infty$ if $N=1$) and φ a given initial data.

It is well-known that for every $\varphi \in H^1(\mathbb{R}^N)$, (1.1) has a unique solution $u \in C((-T_*, T^*); H^1(\mathbb{R}^N))$ which satisfies the blow-up alternative and the conservation of charge and energy. In other words, if $T^* < \infty$ then $\lim\limits_{t \nearrow T^*} \|u(t)\|_{H^1} = \infty$. In the same way, if $T_* < \infty$ then $\lim\limits_{t \searrow -T_*} \|u(t)\|_{H^1} = \infty$. And for all $t \in (-T_*, T^*)$, $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ and $E(u(t)) = E(\varphi)$, where $E(\varphi) \overset{\text{def}}{=} \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 - \frac{\lambda}{\alpha+2}\|\varphi\|_{L^{\alpha+2}}^{\alpha+2}$. If $\varphi \in X \overset{\text{def}}{=} H^1(\mathbb{R}^N) \cap L^2(|x|^2; dx)$ then $u \in C((-T_*, T^*); X)$. Moreover, if $\lambda \leqslant 0$, if $\alpha < \frac{4}{N}$ or if

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$\|\varphi\|_{H^1}$ is small enough then $T^*=T_*=\infty$ and $\|u\|_{L^\infty(\mathbb{R};H^1)}<\infty$. Finally, is $\alpha\geqslant\frac{4}{N}$ then there exist initial values $\varphi\in H^1(\mathbb{R}^N)$ such that the corresponding solution of (1.1) blows up in finite time. See Cazenave [9], Ginibre and Velo [11, 12, 13, 14], Glassey [15], Kato [17].

In the attractive and critical case ($\lambda>0$ and $\alpha=\frac{4}{N}$), there is a sharp condition to obtain global solutions (see Weinstein [26, 27]). It is given in terms of the solution of a related elliptic problem. But in the super critical case ($\alpha>\frac{4}{N}$), we only know that there exists $\varepsilon>0$ sufficiently small such that if $\|\varphi\|_{H^1}\leqslant\varepsilon$, then the corresponding solution is global in time.

In this paper, we try to extend the results of Weinstein [26, 27] to the super critical case $\alpha>\frac{4}{N}$. As we will see, we are not able to establish such a result, but we can give two explicit real values functions γ_* and r_* with $0<\gamma_*<r_*$ such that if $\|\varphi\|_{L^2}\leqslant\gamma_*(\|\nabla\varphi\|_{L^2})$, then the corresponding solution is global in time. Furthermore, for every $(a,b)\in(0,\infty)\times(0,\infty)$ such that $a>r_*(b)$, there exists $\varphi_{a,b}\in H^1(\mathbb{R}^N)$ with $\|\varphi_{a,b}\|_{L^2}=a$ and $\|\nabla\varphi_{a,b}\|_{L^2}=b$ such that the associated solution blows up in finite time for both $t<0$ et $t>0$ (see Theorem 4.1 below). Despite of the fact we do not obtain a sharp condition (since $\gamma_*<r_*$), we recover the results of Weinstein [26, 27] as $\alpha\searrow\frac{4}{N}$. Setting $\mathcal{A}=\{\varphi\in H^1(\mathbb{R}^N);\ \|\varphi\|_{L^2}\leqslant\gamma_*(\|\nabla\varphi\|_{L^2})\}$, it follows that for every $\varphi\in\mathcal{A}$, the corresponding solution of (1.1) is global in time and uniformly bounded in $H^1(\mathbb{R}^N)$. It is interesting to note that $\mathcal{A}$ is an unbounded subset of $H^1(\mathbb{R}^N)$ as for the case $\alpha=\frac{4}{N}$. We also improve some results about blow-up (Theorems 2.1 and 2.2).

This paper is organized as follows. In Section 2, we give a sufficient blow-up condition. In Section 3, we recall the best constant in a Gagliardo-Nirenberg's inequality. In Section 4, we give the main result of this paper, that is necessary conditions and sufficient conditions to obtain global solutions. In Section 5, we prove the result given in Section 4.

The following notations will be used throughout this paper. $\Delta=\sum\limits_{j=1}^N\frac{\partial^2}{\partial x_j^2}$ and we denote by $B(0,R)$, for $R>0$, the ball of $\mathbb{R}^N$ of center 0 with radius R. For $1\leqslant p\leqslant\infty$, we design by $L^p(\mathbb{R}^N)=L^p(\mathbb{R}^N;\mathbb{C})$, with norm $\|\,.\,\|_{L^p}$, the usual Lebesgue spaces and by $H^1(\mathbb{R}^N)=H^1(\mathbb{R}^N;\mathbb{C})$, with norm $\|\,.\,\|_{H^1}$, the Sobolev space. For $k\in\mathbb{N}\cup\{0\}$ and $0<\gamma<1$, we denote by $C^{k,\gamma}(\mathbb{R}^N)=C^{k,\gamma}(\mathbb{R}^N;\mathbb{C})$ the Hölder spaces and we introduce the Hilbert space $X=\left\{\psi\in H^1(\mathbb{R}^N;\mathbb{C});\ \|\psi\|_X<\infty\right\}$ with norm $\|\psi\|_X^2=\|\psi\|_{H^1(\mathbb{R}^N)}^2+\int\limits_{\mathbb{R}^N}|x|^2|\psi(x)|^2dx$. For a normed functional space $E\subset L^1_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{C})$, we denote by E_{rad} the space of functions $f\in E$ such that f is spherically symmetric. E_{rad} is endowed with the norm of E. Finally, C are auxiliary positive constants.

2 Blow-up

The first two results are an improvement of a blow-up condition (see Glassey [15], Ogawa and Tsutsumi [23]). We know that if a solution has a negative energy, then it blows up in finite time. We extend this result for any nontrivial solution with nonpositive energy.

Theorem 2.1. *Let $\lambda > 0$, $\frac{4}{N} < \alpha < \frac{4}{N-2}$ ($4 < \alpha < \infty$ if $N = 1$) and $\varphi \in X$, $\varphi \not\equiv 0$. If $E(\varphi) \leqslant 0$ then the corresponding solution $u \in C((-T_*, T^*); X)$ of (1.1) blows up in finite time for both $t > 0$ and $t < 0$. In other words, $T^* < \infty$ and $T_* < \infty$.*

Theorem 2.2. *Let $\lambda > 0$, $N \geqslant 2$, $\frac{4}{N} < \alpha < \frac{4}{N-2}$ ($2 < \alpha \leqslant 4$ if $N = 2$) and $\varphi \in H^1_{\rm rad}(\mathbb{R}^N)$, $\varphi \not\equiv 0$. If $E(\varphi) \leqslant 0$ then the corresponding solution $u \in C((-T_*, T^*); H^1(\mathbb{R}^N))$ of (1.1) blows up in finite time for both $t > 0$ and $t < 0$. In other words, $T^* < \infty$ and $T_* < \infty$.*

Remark 2.3. When $E(\varphi) = 0$, the conclusion of Theorems 2.1 and 2.2 is false for $\alpha = \frac{4}{N}$. Indeed, let $\varphi \in X_{\rm rad}$, $\varphi \not\equiv 0$, be a solution of $-\Delta\varphi + \varphi = \lambda|\varphi|^{\frac{4}{N}}\varphi$, in $\mathbb{R}^N$. Then $E(\varphi) = 0$ from (3.5) but $u(t,x) = \varphi(x)e^{it}$ is the solution of (1.1) and so $T^* = T_* = \infty$.

Similar results exist for the critical case. See Nawa [19, 21]. It is shown that if $\varphi \in H^1(\mathbb{R}^N)$ satisfies $E(\varphi) < \frac{(\varphi', i\varphi)^2}{\|\varphi\|^2_{L^2}}$, when $N = 1$, or if $E(\varphi) < 0$, when $N \geqslant 2$, then the corresponding solution of (1.1) blows up in finite time or grows up at infinity, the first case always occurring when $N = 1$. Here, $(\, , \,)$ denotes the scalar product in $L^2(\mathbb{R}^N)$. See also Nawa [20, 22]. Note that in the case $N = 1$, the result of Nawa [21] slightly improves that of Ogawa and Tsutsumi [24], since it allows to make blow-up some solution with nonnegative energy.

Proof of Theorem 2.1. We argue by contradiction. Set for every $t \in (-T_*, T^*)$, $h(t) = \|xu(t)\|^2_{L^2}$. Then $h \in C^2((-T_*, T^*); \mathbb{R})$ and

$$\forall t \in (-T_*, T^*),\ h''(t) = 4N\alpha E(\varphi) - 2(N\alpha - 4)\|\nabla u(t)\|^2_{L^2} \tag{2.1}$$

(Glassey [15]). Since $E(\varphi) \leqslant 0$, we have by Gagliardo-Nirenberg's inequality (Proposition 3.1) and conservation of energy and charge, $\|\nabla u(t)\|^2_{L^2} \leqslant \frac{2\lambda}{\alpha+2}\|u(t)\|^{\alpha+2}_{L^{\alpha+2}} \leqslant C\|\nabla u(t)\|^{\frac{N\alpha}{2}}_{L^2}$, for every $t \in (-T_*, T^*)$. Since $\alpha > \frac{4}{N}$ and $\varphi \not\equiv 0$, we deduce that $\inf\limits_{t\in(-T_*,T^*)} \|\nabla u(t)\|_{L^2} > 0$ and with (2.1), we obtain

$$\forall t \in (-T_*, T^*),\ h''(t) \leqslant -C.$$

So, if $T^* = \infty$ or if $T_* = \infty$ then there exists $S \in (-T_*, T^*)$ with $|S|$ large enough such that $h(S) < 0$ which is absurd since $h > 0$. Hence the result. $\square$

Proof of Theorem 2.2. For $\Psi \in W^{4,\infty}(\mathbb{R}^N;\mathbb{R})$, $\Psi \geqslant 0$, we set

$$\forall t \in (-T_*, T^*),\ V(t) = \int\limits_{\mathbb{R}^N} \Psi(x)|u(t,x)|^2 dx.$$

We know that there exists $\Psi \in W^{4,\infty}(\mathbb{R}^N;\mathbb{R})$, $\Psi \geqslant 0$, such that $V \in C^2((-T_*, T^*);\mathbb{R})$ and

$$\forall t \in (-T_*, T^*),\ V''(t) \leqslant 2N\alpha E(\varphi) - 2(N\alpha - 4)\|\nabla u(t)\|_{L^2}^2,$$

(see the proof of Theorem 2.7 of Cazenave [8] and Remark 2.13 of this reference). We conclude in the same way that for Theorem 2.1. $\square$

3 Sharp estimate

In this section, we recall the sharp estimate in a Gagliardo-Nirenberg's inequality (Proposition 3.1) and a result concerning the *ground states*.

Let $\lambda > 0$, $\omega > 0$ and $0 < \alpha < \dfrac{4}{N-2}$ ($0 < \alpha < \infty$ if $N = 1$). We consider the following elliptic equations.

$$\begin{cases} -\Delta R + R = |R|^\alpha R, \text{ in } \mathbb{R}^N, \\ R \in H^1(\mathbb{R}^N;\mathbb{R}),\ R \not\equiv 0, \end{cases} \tag{3.1}$$

$$\begin{cases} -\Delta \Phi + \omega\Phi = \lambda|\Phi|^\alpha \Phi, \text{ in } \mathbb{R}^N, \\ \Phi \in H^1(\mathbb{R}^N;\mathbb{R}),\ \Phi \not\equiv 0. \end{cases} \tag{3.2}$$

It is well-known that the equation (3.2) possesses at less one solution ψ. Furthermore, each solution ψ of (3.2) satisfies $\psi \in C^{2,\gamma}(\mathbb{R}^N) \cap W^{3,p}(\mathbb{R}^N)$, $\forall \gamma \in (0,1)$, $\forall p \in [2,\infty)$, $|\psi(x)| \leqslant Ce^{-\delta|x|}$, for all $x \in \mathbb{R}^N$, where C and δ are two positive constants which do not depend on x, $\lim\limits_{|x|\to\infty} |D^\beta \psi(x)| = 0$, $\forall |\beta| \leqslant 2$ multi-index. Finally, ψ satisfies the following identities.

$$\|\nabla\psi\|_{L^2}^2 = \frac{\omega N\alpha}{4 - \alpha(N-2)}\|\psi\|_{L^2}^2, \tag{3.3}$$

$$\|\psi\|_{L^{\alpha+2}}^{\alpha+2} = \frac{2\omega(\alpha+2)}{\lambda(4 - \alpha(N-2))}\|\psi\|_{L^2}^2, \tag{3.4}$$

$$\|\psi\|_{L^{\alpha+2}}^{\alpha+2} = \frac{2(\alpha+2)}{\lambda N\alpha}\|\nabla\psi\|_{L^2}^2. \tag{3.5}$$

Such solutions are called *bound states* solutions. Furthermore, (3.2) has a unique solution Φ satisfying the following additional properties. $\Phi \in \mathcal{S}_{\text{rad}}(\mathbb{R}^N;\mathbb{R})$; $\Phi > 0$ over $\mathbb{R}^N$; Φ is decreasing with respect to $r = |x|$; for every multi-index $\beta \in \mathbb{N}^N$, there exist two constants $C > 0$ and $\delta > 0$ such that for every $x \in \mathbb{R}^N$, $|\Phi(x)| + |D^\beta \Phi(x)| \leqslant Ce^{-\delta|x|}$. Finally, for every solution ψ of (3.2), we have

$$\|\Phi\|_{L^2} \leqslant \|\psi\|_{L^2}. \tag{3.6}$$

Such a solution is called a *ground state* of the equation (3.2).

Equation (3.2) is studied in the following references. Berestycki, Gallouët and Kavian [3]; Berestycki and Lions [4, 5] ; Berestycki, Lions and Peletier [6]; Gidas, Ni and Nirenberg [10]; Jones and Küpper [16]; Kwong [18]; Strauss [25]. See also Cazenave [9], Section 8.

Proposition 3.1. *Let* $0 < \alpha < \dfrac{4}{N-2}$ $(0 < \alpha < \infty$ *if* $N = 1)$ *and* R *be the ground state solution of* (3.1)*. Then the best constant* $C_* > 0$ *in the Gagliardo-Nirenberg's inequality,*

$$\forall f \in H^1(\mathbb{R}^N),\ \|f\|_{L^{\alpha+2}}^{\alpha+2} \leqslant C_* \|f\|_{L^2}^{\frac{4-\alpha(N-2)}{2}} \|\nabla f\|_{L^2}^{\frac{N\alpha}{2}}, \tag{3.7}$$

is given by

$$C_* = \frac{2(\alpha+2)}{N\alpha}\left(\frac{4-\alpha(N-2)}{N\alpha}\right)^{\frac{N\alpha-4}{4}} \|R\|_{L^2}^{-\alpha}. \tag{3.8}$$

See Weinstein [26] for the proof in the case $N \geqslant 2$. See also Lemma 3.4 of Cazenave [8] in the case $\alpha = \dfrac{4}{N}$. But for convenience, we give the proof. It makes use of a compactness result which is an adaptation of the compactness lemma due to Strauss (Strauss [25]).

Proof of Proposition 3.1. We define for every $f \in H^1(\mathbb{R}^N)$, $f \not\equiv 0$, the functional

$$J(f) = \frac{\|f\|_{L^2}^{\frac{4-\alpha(N-2)}{2}} \|\nabla f\|_{L^2}^{\frac{N\alpha}{2}}}{\|f\|_{L^{\alpha+2}}^{\alpha+2}},$$

and we set $\sigma = \inf\limits_{f \in H^1 \setminus \{0\}} J(f)$. Then $\sigma \in (0,\infty)$ by (3.7). We have to show that $\sigma = C_*^{-1}$ where C_* is defined by (3.8). Let $(f_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^N)$ be a minimizing sequence. Let

$$\mu_n = \frac{\|f_n\|_{L^2}^{\frac{N-2}{2}}}{\|\nabla f_n\|_{L^2}^{\frac{N}{2}}},\ \lambda_n = \frac{\|f_n\|_{L^2}}{\|\nabla f_n\|_{L^2}} \ \text{ and } \ \forall x \in \mathbb{R}^N,\ v_n(x) = \mu_n f_n(\lambda_n x).$$

Then $\|v_n\|_{L^2} = \|\nabla v_n\|_{L^2} = 1$ and $J(f_n) = J(v_n) = \|v_n\|_{L^{\alpha+2}}^{-(\alpha+2)} \xrightarrow{n\to\infty} \sigma$. Let v_n^* be the symmetrization of Schwarz of $|v_n|$ (see Bandle [1]; Berestycki and Lions [4], Appendix A.III). Then $J(v_n^*) \xrightarrow{n\to\infty} \sigma$ and by compactness, $v_{n_\ell}^* \rightharpoonup v$ as $\ell \longrightarrow \infty$ in $H^1_{\mathrm{w}}(\mathbb{R}^N)$ (and in particular, in $L^{\alpha+2}_{\mathrm{w}}(\Omega)$ for every subset $\Omega \subset \mathbb{R}^N$) and $v^*_{n_\ell} \xrightarrow[\ell\to\infty]{L^{\alpha+2}} v$, for a subsequence $(v^*_{n_\ell})_\ell \subset (v^*_n)_n$ and for some $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Indeed, since $(v^*_n)_{n\in\mathbb{N}}$ is bounded in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and nonincreasing with respect to $|x|$, then $\forall \ell \in \mathbb{N}$ and $\forall x \in \mathbb{R}^N$, $|v^*_{n_\ell}(x)| \leqslant C|x|^{-\frac{N}{2}}$, where $C>0$ does not depend on ℓ and x (Berestycki and Lions [4], Appendix A.II, Radial Lemma A.IV). From this and Hölder's inequality, we deduce that $\forall \ell \in \mathbb{N}$ and $\forall R>0$, $\|v^*_{n_\ell}\|_{L^{\alpha+2}(\mathbb{R}^N\setminus B(0,R))} \leqslant CR^{-\frac{N\alpha}{2(\alpha+2)}}$, for a constant $C>0$ which does not depend on ℓ. Then $\forall R>0$, $\|v\|_{L^{\alpha+2}(\mathbb{R}^N\setminus B(0,R))} \leqslant \liminf\limits_{\ell\to\infty} \|v^*_{n_\ell}\|_{L^{\alpha+2}(\mathbb{R}^N\setminus B(0,R))} \leqslant CR^{-\frac{N\alpha}{2(\alpha+2)}}$. The strong convergence in $L^{\alpha+2}(\mathbb{R}^N)$ follows easily from the two above estimates and from the compact embedding $H^1(B(0,R)) \hookrightarrow L^{\alpha+2}(B(0,R))$, which holds for every $R>0$. Since $\|v_n\|_{L^{\alpha+2}} = \|v^*_n\|_{L^{\alpha+2}}$, it follows that $\|v\|^{\alpha+2}_{L^{\alpha+2}} = \sigma^{-1}$ and then $v \not\equiv 0$. Thus, $J(v)=\sigma$ and $\|v\|_{L^2} = \|\nabla v\|_{L^2} = 1$. It follows that $\forall w \in H^1(\mathbb{R}^N)$, $\frac{d}{dt}J(v+tw)_{|t=0} = 0$. So v satisfies $-\Delta v + \frac{4-\alpha(N-2)}{N\alpha}v = \sigma\frac{2(\alpha+2)}{N\alpha}|v|^\alpha v$, in $\mathbb{R}^N$. Set $a = \left(\frac{N\alpha}{4-\alpha(N-2)}\right)^{\frac12}$, $b = \left(\frac{2\sigma(\alpha+2)}{4-\alpha(N-2)}\right)^{\frac1\alpha}$ and $\forall x \in \mathbb{R}^N$, $u(x) = bv(ax)$. Then $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ is a solution of (3.1) and $J(u)=\sigma$. By (3.3)–(3.4), we obtain $J(u) = C_*^{-1}\frac{\|u\|^\alpha_{L^2}}{\|R\|^\alpha_{L^2}} = \sigma$ and $J(R) = C_*^{-1} \geqslant \sigma$ (since R also satisfies (3.1)). Then $\|u\|_{L^2} \leqslant \|R\|_{L^2}$ and so with (3.6), $\|u\|_{L^2} = \|R\|_{L^2}$. Hence the result. $\square$

4 Necessary condition and sufficient condition for global existence

Theorem 4.1. *Let $\lambda>0$, $\dfrac{4}{N} < \alpha < \dfrac{4}{N-2}$ ($4<\alpha<\infty$ if $N=1$) and R be the ground state solution of (3.1). We define for every $a>0$,*

$$r_*(a) = \left(\frac{N\alpha}{4-\alpha(N-2)}\right)^{\frac{N\alpha-4}{2(4-\alpha(N-2))}} \left(\lambda^{-\frac1\alpha}\|R\|_{L^2}\right)^{\frac{2\alpha}{4-\alpha(N-2)}} a^{-\frac{N\alpha-4}{4-\alpha(N-2)}}, \tag{4.1}$$

$$\gamma_*(a) = \left(\frac{N\alpha-4}{N\alpha}\right)^{\frac{N\alpha-4}{2(4-\alpha(N-2))}} r_*(a). \tag{4.2}$$

1. *If $\varphi \in H^1(\mathbb{R}^N)$ satisfies*

$$\|\varphi\|_{L^2} \leqslant \gamma_*(\|\nabla\varphi\|_{L^2}), \tag{4.3}$$

then the corresponding solution $u \in C((-T_*,T^*);H^1(\mathbb{R}^N))$ of (1.1) is global in time, that is $T^*=T_*=\infty$, and the following estimates hold.

$$\forall t \in \mathbb{R}, \quad \begin{cases} \|\nabla u(t)\|^2_{L^2} < \dfrac{2N\alpha}{N\alpha-4}E(\varphi), \\ \|\nabla u(t)\|_{L^2} < r_*^{-1}(\|\varphi\|_{L^2}), \end{cases}$$

where r_*^{-1} is the function defined by (4.5). In particular, $E(\varphi) > \dfrac{N\alpha-4}{2N\alpha}\|\nabla\varphi\|^2_{L^2}$.

2. *For every $a > 0$ and for every $b > 0$ satisfying $a > r_*(b)$, there exists $\varphi_{a,b} \in H^1(\mathbb{R}^N)$ with $\|\varphi_{a,b}\|_{L^2} = a$ and $\|\nabla\varphi_{a,b}\|_{L^2} = b$ such that the associated solution $u_{a,b} \in C((-T_*, T^*); H^1(\mathbb{R}^N))$ of (1.1) blows up in finite time for both $t > 0$ et $t < 0$. In other words, $T^* < \infty$ and $T_* < \infty$. Furthermore, $E(\varphi_{a,b}) > 0 \iff r_*(b) < a < \rho_*(b)$ and $E(\varphi_{a,b}) = 0 \iff a = \rho_*(b)$, where for every $a > 0$,*

$$\rho_*(a) = \left(\frac{N\alpha}{4}\right)^{\frac{2}{4-\alpha(N-2)}} r_*(a). \tag{4.4}$$

Finally, $E(\varphi_{a,b}) < \dfrac{N\alpha - 4}{2N\alpha}\|\nabla\varphi_{a,b}\|_{L^2}^2$.

Remark 4.2. Let γ_* be the function defined by (4.2). Set

$$\mathcal{A} = \{\varphi \in H^1(\mathbb{R}^N);\ \|\varphi\|_{L^2} \leqslant \gamma_*(\|\nabla\varphi\|_{L^2})\}.$$

By Theorem 4.1, for every $\varphi \in \mathcal{A}$, the corresponding solution of (1.1) is global in time and uniformly bounded in $H^1(\mathbb{R}^N)$. It is interesting to note that $\mathcal{A}$ is an unbounded subset of $H^1(\mathbb{R}^N)$. So Theorem 4.1 gives a general result for global existence for which we can take initial values with the $H^1(\mathbb{R}^N)$ norm large as we want.

Remark 4.3. Let γ_*, r_*, and ρ_* be the functions defined respectively by (4.2), (4.1) and (4.4). It is clear that since $\alpha > \frac{4}{N}$, γ_*, γ_*^{-1}, r_*, r_*^{-1}, ρ_* and ρ_*^{-1} are decreasing and bijective functions from $(0,\infty)$ to $(0,\infty)$ and for every $a > 0$,

$$\begin{aligned}
\gamma_*^{-1}(a) &= \left(\frac{N\alpha-4}{N\alpha}\right)^{\frac{1}{2}} r_*^{-1}(a),\\
r_*^{-1}(a) &= \left(\frac{N\alpha}{4-\alpha(N-2)}\right)^{\frac{1}{2}} \left(\lambda^{-\frac{1}{\alpha}}\|R\|_{L^2}\right)^{\frac{2\alpha}{N\alpha-4}} a^{-\frac{4-\alpha(N-2)}{N\alpha-4}},\\
\rho_*^{-1}(a) &= \left(\frac{N\alpha}{4}\right)^{\frac{2}{N\alpha-4}} r_*^{-1}(a).
\end{aligned} \tag{4.5}$$

So the condition condition (4.3) is equivalent to the condition $\|\nabla\varphi\|_{L^2} \leqslant \gamma_*^{-1}(\|\varphi\|_{L^2})$. Furthermore, $\gamma_* < r_* < \rho_*$ and $\gamma_*^{-1} < r_*^{-1} < \rho_*^{-1}$.

Remark 4.4. Let γ_*, r_*, and ρ_* be the functions defined respectively by (4.2), (4.1) and (4.4). Then $\gamma_* \xrightarrow{\alpha\searrow\frac{4}{N}} \lambda^{-\frac{1}{\alpha}}\|R\|_{L^2}$ and $r_* \xrightarrow{\alpha\searrow\frac{4}{N}} \lambda^{-\frac{1}{\alpha}}\|R\|_{L^2}$ (and even, $\rho_* \xrightarrow{\alpha\searrow\frac{4}{N}} \lambda^{-\frac{1}{\alpha}}\|R\|_{L^2}$). So we obtain the sharp condition for global existence, $\|\varphi\|_{L^2} < \lambda^{-\frac{1}{\alpha}}\|R\|_{L^2}$ which coincide with the results obtained by Weinstein [26, 27]. However, we do not know if γ_* or r_* are optimum.

5 Proof of Theorem 4.1

In order to prove the blowing up result (2 of Theorem 4.1), we need of several lemmas. We follow the method of Berestycki and Cazenave [2] (see also Cazenave [7] and Cazenave [9], Section 8.2). *A priori*, we would expect to use Theorem 2.1, that is to construct initial values in X with nonpositive energy, which is the case for $\alpha = \frac{4}{N}$. But it will not be enough because we have to make blow-up some solutions whose the initial values have a positive energy.

We define the following functionals and sets. Let $\lambda > 0$, $\omega > 0$, $\beta > 0$, $0 < \alpha < \frac{4}{N-2}$ $(0 < \alpha < \infty$ if $N = 1)$ and $\psi \in H^1(\mathbb{R}^N)$.

$$\begin{cases} \beta^*(\psi)^{\frac{N\alpha-4}{2}} = \dfrac{2(\alpha+2)}{\lambda N\alpha} \dfrac{\|\nabla\psi\|_{L^2}^2}{\|\psi\|_{L^{\alpha+2}}^{\alpha+2}}, \text{ if } \psi \not\equiv 0, \\ Q(\psi) = \|\nabla\psi\|_{L^2}^2 - \dfrac{\lambda N\alpha}{2(\alpha+2)} \|\psi\|_{L^{\alpha+2}}^{\alpha+2}, \\ S(\psi) = \dfrac{1}{2}\|\nabla\psi\|_{L^2}^2 - \dfrac{\lambda}{\alpha+2}\|\psi\|_{L^{\alpha+2}}^{\alpha+2} + \dfrac{\omega}{2}\|\psi\|_{L^2}^2, \\ \mathcal{P}(\beta,\psi)(x) = \beta^{\frac{N}{2}}\psi(\beta x), \text{ for almost every } x \in \mathbb{R}^N, \\ M = \left\{\psi \in H^1(\mathbb{R}^N);\ \psi \not\equiv 0 \text{ and } Q(\psi) = 0\right\}, \\ A = \left\{\psi \in H^1(\mathbb{R}^N);\ \psi \not\equiv 0 \text{ and } -\Delta\psi + \omega\psi = \lambda|\psi|^\alpha\psi, \text{ in } \mathbb{R}^N\right\}, \\ G = \{\psi \in A;\ \forall\phi \in A,\ S(\psi) \leqslant S(\phi)\}. \end{cases}$$

Note that by the discussion at the beginning of Section 3 and (3.3)–(3.6), $M \neq \emptyset$, $A \neq \emptyset$ and $G \neq \emptyset$.

Lemma 5.1. *We have the following results.*

1. $\forall\beta > 0,\ \beta \neq \beta^*(\psi),\ S(\mathcal{P}(\beta,\psi)) < S(\mathcal{P}(\beta^*(\psi),\psi))$.

2. *The following equivalence holds.*

$$\psi \in G \quad \Longleftrightarrow \quad \begin{cases} \psi \in M, \\ S(\psi) = \min\limits_{\phi\in M} S(\phi), \end{cases}$$

3. *Let* $m \overset{def}{=} \min\limits_{\phi\in M} S(\phi)$. *Then* $\forall\phi \in H^1(\mathbb{R}^N)$ *with* $Q(\phi) < 0,\ Q(\phi) \leqslant S(\phi) - m$.

See Cazenave [9], Lemma 8.2.5 for the proof of 1; Proposition 8.2.4 for the proof of 2; Corollary 8.2.6 for the proof of 3. There is a mistake in the formula (8.2.4) of this reference. Replace the expression $\lambda^*(u)^{\frac{n\alpha-4}{2}} = \frac{\alpha+2}{2}\left(\int_{\mathbb{R}^n}|\nabla u|^2\right)\left(\int_{\mathbb{R}^n}|u|^{\alpha+2}\right)^{-1}$ with $\lambda^*(u)^{\frac{n\alpha-4}{2}} = \frac{2(\alpha+2)}{n\alpha}\left(\int_{\mathbb{R}^n}|\nabla u|^2\right)\left(\int_{\mathbb{R}^n}|u|^{\alpha+2}\right)^{-1}$.

The proof of 1 of Theorem 4.1 relies on the following lemma.

Lemma 5.2. *Let $I \subseteq \mathbb{R}$, be an open interval, $t_0 \in I$, $p > 1$, $a > 0$, $b > 0$ and $\Phi \in C(I; \mathbb{R}^+)$. We set, $\forall x \geqslant 0$, $f(x) = a - x + bx^p$, $\underline{x} = (bp)^{-\frac{1}{p-1}}$ and $b_* = \dfrac{p-1}{p}\underline{x}$. Assume that $\Phi(t_0) < \underline{x}$, $a \leqslant b_*$ and that $f \circ \Phi > 0$. Then, $\forall t \in I$, $\Phi(t) < \underline{x}$.*

Proof. Since $\Phi(t_0) < \underline{x}$ and Φ is a continuous function, there exists $\eta > 0$ with $(t_0 - \eta, t_0 + \eta) \subseteq I$ such that, $\forall t \in (t_0 - \eta, t_0 + \eta)$, $\Phi(t) < \underline{x}$. If $\Phi(t_*) = \underline{x}$ for some $t_* \in I$, then $f \circ \Phi(t_*) = f(\underline{x}) = a - b_* \leqslant 0$. But $f \circ \Phi > 0$. Then, $\forall t \in I$, $\Phi(t) < \underline{x}$. $\square$

The proof of 2 of Theorem 4.1 makes use the following lemma.

Lemma 5.3. *Let $\lambda > 0$, $\omega > 0$ and $\dfrac{4}{N} < \alpha < \dfrac{4}{N-2}$ ($4 < \alpha < \infty$ if $N = 1$). We set for every $\beta > 0$ and for every $\psi \in H^1(\mathbb{R}^N)$, $\varphi_\beta = \mathcal{P}(\beta, \psi)$. Let $u_\beta \in C((-T_*, T^*); H^1(\mathbb{R}^N))$ be the solution of (1.1) with initial value φ_β. Then we have, $\forall \psi \in G$, $\forall \beta > 1$, $T_* < \infty$ and $T^* < \infty$.*

Proof. Let $\psi \in G$. By (3.5), we have

$$\beta^*(\varphi_\beta)^{\frac{N\alpha-4}{2}} = \frac{2(\alpha+2)}{\lambda N\alpha}\frac{\beta^2\|\nabla\psi\|_{L^2}^2}{\beta^{\frac{N\alpha}{2}}\|\psi\|_{L^{\alpha+2}}^{\alpha+2}},$$
$$Q(\varphi_\beta) = \beta^2\left(\|\nabla\psi\|_{L^2}^2 - \frac{\lambda N\alpha}{2(\alpha+2)}\beta^{\frac{N\alpha-4}{2}}\|\psi\|_{L^{\alpha+2}}^{\alpha+2}\right),$$
$$\frac{\lambda N\alpha}{2(\alpha+2)}\|\psi\|_{L^{\alpha+2}}^{\alpha+2} = \|\nabla\psi\|_{L^2}^2.$$

So, $\beta^*(\varphi_\beta)^{\frac{N\alpha-4}{2}} = \beta^{-\frac{N\alpha-4}{2}}$, $Q(\varphi_\beta) = -\beta^2\|\nabla\psi\|_{L^2}^2\left(\beta^{\frac{N\alpha-4}{2}} - 1\right)$ and $\beta^*(\psi) = 1$. From these three last equalities, from 1 and 2 of Lemmas 5.1 and by conservation of charge and energy, we have

$$\forall \beta > 1,\ Q(\varphi_\beta) < 0, \tag{5.1}$$

$$\forall \beta \neq 1,\ S(\varphi_\beta) < S(\psi) \equiv m, \tag{5.2}$$

$$\forall \beta > 0,\ \forall t \in (-T_*, T^*),\ S(u_\beta(t)) = S(\varphi_\beta). \tag{5.3}$$

By continuity of u_β, by (5.1)–(5.3) and from 3 of Lemma 5.1, we have for every $\beta > 1$,

$$\forall t \in (-T_*, T^*),\ Q(u_\beta(t)) \leqslant S(\varphi_\beta) - m < 0. \tag{5.4}$$

Set $\forall t \in (-T_*, T^*)$, $h(t) = \|xu_\beta(t)\|_{L^2}^2$. Then we have by Glassey [15], $h \in C^2((-T_*, T^*); \mathbb{R})$ and $\forall t \in (-T_*, T^*)$, $h''(t) = 8\|\nabla u_\beta(t)\|_{L^2}^2 - \frac{4\lambda N\alpha}{\alpha+2}\|u_\beta(t)\|_{L^{\alpha+2}}^{\alpha+2} \equiv 8Q(u_\beta(t))$. So with (5.4),

$$\forall t \in (-T_*, T^*),\ h''(t) \leqslant 8(S(\varphi_\beta) - m) < 0,$$

for every $\beta > 1$. It follows that $T_* < \infty$ and $T^* < \infty$. Hence the result. $\square$

Proof of Theorem 4.1. We proceed in two steps.

Step 1. We have 1.

Let C_* be the constant defined by (3.8). We set $I = (-T_*, T^*)$, $t_0 = 0$, $p = \frac{N\alpha}{4}$, $a = \|\nabla\varphi\|_{L^2}^2$, $b = \frac{2\lambda}{\alpha+2} C_* \|\varphi\|_{L^2}^{\frac{4-\alpha(N-2)}{2}}$, $\underline{x} = (bp)^{-\frac{1}{p-1}}$, $b_* = \frac{p-1}{p}\underline{x}$, $\forall t \in I$, $\Phi(t) = \|\nabla u(t)\|_{L^2}^2$ and for any $x \geqslant 0$, $f(x) = a - x + bx^p$. Then by conservation of energy, by Proposition 3.1 and by conservation of charge, we have

$$\begin{aligned}
\forall t \in I,\ \|\nabla u(t)\|_{L^2}^2 &= 2E(\varphi) + \frac{2\lambda}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \\
&< \|\nabla\varphi\|_{L^2}^2 + \frac{2\lambda}{\alpha+2} C_* \|\varphi\|_{L^2}^{\frac{4-\alpha(N-2)}{2}} (\|\nabla u(t)\|_{L^2}^2)^{\frac{N\alpha}{4}}.
\end{aligned}$$

And so, $\forall t \in I$, $a - \|\nabla u(t)\|_{L^2}^2 + b(\|\nabla u(t)\|_{L^2}^2)^p > 0$, that is $f \circ \Phi > 0$. Furthermore, $\Phi(t_0) \equiv a \leqslant b_* < \underline{x}$. Indeed, by Remark 4.3, we have

$$\Phi(t_0) \leqslant b_* \iff \|\nabla\varphi\|_{L^2} \leqslant \gamma_*^{-1}(\|\varphi\|_{L^2}) \iff \|\varphi\|_{L^2} \leqslant \gamma_*(\|\nabla\varphi\|_{L^2}).$$

So by Lemma 5.2, $\Phi(t) < \underline{x} \equiv [r_*^{-1}(\|\varphi\|_{L^2})]^2$, $\forall t \in I$. Thus, $I = \mathbb{R}$ and for every $t \in \mathbb{R}$,

$$\|\nabla u(t)\|_{L^2} < r_*^{-1}(\|\varphi\|_{L^2}).$$

It follows from conservation of charge and energy, (3.7), (3.8), and the above inequality, that

$$\begin{aligned}
\forall t \in \mathbb{R},\ E(\varphi) &\geqslant \frac{1}{2}\left(\|\nabla u(t)\|_{L^2}^2 - \frac{2\lambda}{\alpha+2} C_* \|\varphi\|_{L^2}^{\frac{4-\alpha(N-2)}{2}} \|\nabla u(t)\|_{L^2}^{\frac{N\alpha}{2}}\right) \\
&= \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 \left(1 - \frac{4}{N\alpha}\left[r_*^{-1}(\|\varphi\|_{L^2})\|\nabla u(t)\|_{L^2}^{-1}\right]^{-\frac{N\alpha-4}{2}}\right) \\
&> \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 \left(1 - \frac{4}{N\alpha}\right) \\
&= \frac{N\alpha - 4}{2N\alpha}\|\nabla u(t)\|_{L^2}^2.
\end{aligned}$$

Hence 1.

Step 2. We have 2.

Let R be the ground state solution of (3.1). Let first remark from the assumptions and from Remark 4.3, we have $b > r_*^{-1}(a)$. We set

$$\nu = [r_*^{-1}(a)]^{\frac{N}{2}} \left(\tfrac{4-\alpha(N-2)}{N\alpha}\right)^{\frac{N}{4}} a^{-\frac{N-2}{2}} \|R\|_{L^2}^{-1}, \quad \omega = [r_*^{-1}(a)]^2 \tfrac{4-\alpha(N-2)}{N\alpha} a^{-2} = \left(\lambda^{-\frac{1}{\alpha}} \|R\|_{L^2} a^{-1}\right)^{\frac{4\alpha}{N\alpha-4}},$$

and for every $x \in \mathbb{R}^N$, $\psi(x) = \nu R(\sqrt{\omega}x)$. Then $\psi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^N) \cap A$. Since R satisfies (3.1)–(3.6), it follows that $\psi \in G$. Furthermore, $\|\psi\|_{L^2} = a$ and $\|\nabla\psi\|_{L^2} = r_*^{-1}(a)$. Let $\beta = \frac{b}{r_*^{-1}(a)} > 1$. Set for every

$x \in \mathbb{R}^N$, $\varphi_{a,b}(x) = \varphi_\beta(x) = \mathcal{P}(\beta, \psi)(x)$. In particular, $\varphi_{a,b} \in \mathcal{S}_{\text{rad}}(\mathbb{R}^N)$ and $\varphi_{a,b}$ satisfies

$$-\Delta\varphi_{a,b} + \omega\beta^2\varphi_{a,b} = \lambda\beta^{-\frac{N\alpha-4}{2}}|\varphi_{a,b}|^\alpha\varphi_{a,b}, \text{ in } \mathbb{R}^N.$$

Denote $u_{a,b} \in C((-T_*, T^*); H^2(\mathbb{R}^N) \cap X_{\text{rad}})$ the solution of (1.1) with initial value $\varphi_{a,b}$. Then by Lemma 5.3, $T_* < \infty$ and $T^* < \infty$. Moreover, $\|\varphi_{a,b}\|_{L^2} = a$, $\|\nabla\varphi_{a,b}\|_{L^2} = b$ and by (3.5),

$$\begin{aligned}
E(\varphi_{a,b}) &= \frac{1}{2}\|\nabla\varphi_{a,b}\|_{L^2}^2 - \frac{\lambda}{\alpha+2}\|\varphi_{a,b}\|_{L^{\alpha+2}}^{\alpha+2} \\
&= \frac{1}{2}\|\nabla\varphi_{a,b}\|_{L^2}^2 - \frac{\lambda}{\alpha+2}\beta^{\frac{N\alpha}{2}}\|\psi\|_{L^{\alpha+2}}^{\alpha+2} \\
&= \frac{\|\nabla\varphi_{a,b}\|_{L^2}^2}{2N\alpha}\left(N\alpha - 4\beta^{\frac{N\alpha-4}{2}}\right) \\
&= \frac{\|\nabla\varphi_{a,b}\|_{L^2}^2}{2N\alpha}\left(N\alpha - 4\beta^{\frac{N\alpha-4}{2}}\right).
\end{aligned}$$

By Remark 4.3, it follows that

$$E(\varphi_{a,b}) \leqslant 0 \iff \beta \geqslant \left(\frac{N\alpha}{4}\right)^{\frac{2}{N\alpha-4}} \iff b \geqslant \left(\frac{N\alpha}{4}\right)^{\frac{2}{N\alpha-4}} r_*^{-1}(a) \equiv \rho_*^{-1}(a) \iff a \geqslant \rho_*(b).$$

Hence the result. ◻

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