Horizon-Dependent Risk Aversion and the Timing and Pricing of Uncertainty

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Abstract

We propose a model that addresses two fundamental challenges concerning the timing and pricing of uncertainty: established equilibrium asset pricing models require a controversial degree of preference for early resolution of uncertainty; and do not generate the downward-sloping term structure of risk premia suggested by the data. Inspired by experimental evidence, we construct dynamically inconsistent preferences in which risk aversion decreases with the temporal horizon. The resulting pricing model can generate a term structure of risk premia consistent with empirical evidence, without forcing a particular preference for resolution of uncertainty or compromising the ability to match standard moments.

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1 Introduction

The finance literature has been successful in explaining many features of observed equilibrium asset prices as well as their dynamics (Cochrane, 2016). However, recent work has posed two puzzles concerning the timing and the pricing of uncertainty. First, empirical evidence shows unexpected patterns in the pricing of risk in the term structure, suggesting that average risk premia are higher for short-term risks than for long-term risks (e.g. van Binsbergen, Brandt, and Koijen, 2012; Giglio, Maggiori, and Stroebel, 2014).[1] These findings pose a fundamental challenge because they are inconsistent with established asset pricing models: the term structure of risk premia is upward-sloping in the widely used long-run risk model of Bansal and Yaron (2004) as well as in the habit-formation model of Campbell and Cochrane (1999), whereas the term structure is flat in the rare disaster models of Gabaix (2012) and Wachter (2013). Second, the long-run risk model has recently come under attack also on conceptual grounds: Epstein, Farhi, and Strzalecki (2014) show that calibrating the model to match asset pricing moments requires a surprisingly strong preference for early resolution of uncertainty, which the authors argue is difficult to reconcile with the limited micro evidence and introspection.

A conceptually sound and empirically consistent framework for understanding the pricing of risk at different horizons is important for various fields in economics beyond asset pricing and has immediate policy implications, e.g. for climate change policy (Gollier, 2013; Giglio, Maggiori, Stroebel, and Weber, 2015). To address these challenges, we propose a model that relaxes the assumption, standard in the economics literature, that risk aversion is constant across temporal horizons. Inspired by experimental evidence,[2] we generalize Epstein and Zin (1989) preferences to accommodate the case of agents that are more averse to immediate than to delayed risks. Doing so renders preferences dynamically inconsistent with respect to risk-taking, which makes the existing toolbox of asset pricing inapplicable.[3] We therefore investigate how existing tools can be generalized, and if such a generalization is useful for understanding the empirical patterns concerning the timing and pricing of risk. We find that combining the standard long-run risk endowment economy and a representative agent with horizon-dependent risk aversion can address both challenges: the model can speak to the recent empirical evidence on the term structure of

[1] For a review of the literature, see van Binsbergen and Koijen (2016).

[2] Jones and Johnson (1973); Onculer (2000); Sagristano et al. (2002); Noussair and Wu (2006); Coble and Lusk (2010); Baucells and Heukamp (2010); Abdellaoui et al. (2011).

[3] Eisenbach and Schmalz (2016) show in a static model with time-separable utility that horizon-dependent risk aversion is conceptually orthogonal to other non-standard preferences such as time-varying risk aversion (Constantinides, 1990; Campbell and Cochrane, 1999) or non-exponential time discounting (Phelps and Pollak, 1968; Laibson, 1997).

risk premia—while still matching the usual asset pricing moments—and it can also mitigate (or even reverse) the implied preference for early resolution of uncertainty. Hence, the long-run risk model can be adapted to address both challenges raised by the recent literature.

The paper makes three contributions. The first contribution is methodological. We show that commonly used recursive techniques can be adapted to a setting of pseudo-recursive preferences with horizon-dependent risk aversion while still allowing for closed-form solutions. Our framework generalizes the standard recursive utility model of Epstein and Zin (1989) and thus builds on the success of the separation of risk and time preferences when combined with long-run risk to explain asset pricing moments. We can accommodate numerous extensions, be it on the valuation of risk (habit, disappointment aversion, loss aversion, etc.), or on the quantity of risk (rare disasters, production-based models, etc.). Further, our model implies dynamically inconsistent *risk* preferences while maintaining dynamically consistent *time* preferences: intra-temporal allocations across risky assets depend on horizon-dependent risk aversion; but intertemporal decisions for deterministic payoffs are unchanged from the standard, time consistent model. We can therefore study the pricing impact of horizon dependent risk aversion in isolation from quasi-hyperbolic discounting, which has only limited implications for asset pricing (Harris and Laibson, 2001; Luttmer and Mariotti, 2003).

The second contribution concerns the preferences for early or late resolution of uncertainty. Specifically, we formally derive how two consumption streams with ex-ante identical risk but different timing for the resolution of uncertainty are valued. As in the model of Epstein and Zin (1989), our agents value these consumption streams differently. Whether and how the relative valuations differ depends on the degree of horizon dependent risk aversion. In a standard long-run risk framework that uses Epstein and Zin (1989) preferences, the level of risk aversion *and* elasticity of intertemporal substitution that are necessary to match observed asset pricing moments imply that agents have a seemingly excessive preference for early resolutions of uncertainty (Epstein et al., 2014). In contrast, our model not only mitigates but can even reverse this result: we are able to calibrate both asset pricing moments and reasonable preferences for either early or late resolution of uncertainty.

As a third contribution, we apply our utility model and methodology to equilibrium asset pricing, with a particular focus on the term structure of risk premia. In the spirit of Strotz (1955), we assume that our agents are perfectly rational and aware of their horizon-dependent risk aversion preferences. We consider a representative agent who trades and clears the market every period, and, as such, cannot pre-commit to any specific strategy:

unable to commit to future behavior but aware of her dynamic inconsistency, the agent optimizes in the current period, fully anticipating re-optimization in future periods. Solving our model this way yields a one-period pricing problem in which the Euler equation is satisfied.

Obtaining a decreasing term structure of risk premia from a model with a decreasing term structure of risk aversion may seem trivial. However, solving the problem is far from tautological. The agent's choices, and thus equilibrium prices, are determined dynamically from one period to the next. At time t, the agent chooses how to allocate her wealth between t and $t+1$—a time frame over which only her immediate risk aversion matters: in this context, why and how horizon-dependent risk aversion should affect pricing is a complex question, with non-obvious answers. We formally derive the stochastic discount factor of our pseudo-recursive model, and show that it nests the standard Epstein and Zin (1989) case, with a new multiplicative term arising from the preferences' dynamic inconsistency. The new term reflects the wedge between the continuation value used for optimization at any period t and the actual valuation at $t+1$. Its impact on risk prices is rather subtle.

We investigate the implications of our model both on the level and on the slope of the term structure of risk premia in a Lucas-tree endowment economy. Horizon-dependent risk aversion does not concern inter-temporal decisions. As such, we formally show that both the risk-free rate and the pricing of shocks that impact consumption *levels* are unchanged from the standard model. Further, if risk is constant in the economy, equilibrium asset prices are unaffected by our model of dynamically inconsistent risk preferences. By contrast, the pricing of shocks that impact consumption *risk*, or volatility, are modified by horizon-dependent risk aversion. In a standard log-normal consumption growth setting with stochastic volatility, our model can simultaneously match the average level of risk prices and generate a downward-sloping term structure of risk premia (van Binsbergen et al., 2012, 2013; van Binsbergen and Koijen, 2016).

In sum, we develop a new model that can address both the "early versus late resolution of uncertainty" puzzle of Epstein et al. (2014) as well as the observed term structure of risk premia, a puzzle first emphasized by van Binsbergen et al. (2012). The success at solving these hotly debated problems regarding the timing and pricing of uncertainty is achieved without compromising the model's ability to match the usual asset pricing moments as in Bansal and Yaron (2004), and without departing significantly from the widely-used preference structure of Epstein and Zin (1989).

After a short overview of the literature, we present our model of preferences in Section 3. We analyze the preference for early or late resolution of uncertainty in Section 4. In

Section 5, we derive the stochastic discount factor and the formal risk pricing formulas of our model. Section 6 presents and discusses the models' quantitative predictions. Section 7 concludes.

2 Related literature

This paper is the first to solve for equilibrium asset prices in an economy populated by agents with dynamically inconsistent *risk* preferences. It complements Luttmer and Mariotti (2003), who show that dynamically inconsistent *time* preferences of the kind examined by Harris and Laibson (2001) have little power to explain cross-sectional variation in asset returns. Given that cross-sectional asset pricing involves intra-period risk-return trade-offs, it is indeed quite intuitive that horizon-dependent *time* preferences are not suitable to address puzzles related to risk premia.

Our model generalizes Epstein and Zin (1989) preferences by relaxing the dynamic consistency axiom of Kreps and Porteus (1978) to analyze the subtle relationship between the timing and pricing of uncertainty. By contrast, Routledge and Zin (2010), Bonomo et al. (2011) and Schreindorfer (2014) follow Gul (1991) and relax the independence axiom of Kreps and Porteus (1978) to analyze the asset pricing impact of generalized disappointment aversion within a recursive framework. They find their model generates endogenous predictability (Routledge and Zin, 2010); matches various asset pricing moments (Bonomo et al., 2011); prices the cross-section of options better than the standard model (Schreindorfer, 2014). Their models, however, do not address the "excessive preference for early resolutions of uncertainty puzzle", pointed out by Epstein et al. (2014) or quantitatively match the term structure of risk prices.[4]

Our formal results on the term structure of risk pricing are consistent with patterns uncovered by the recent empirical literature. Van Binsbergen et al. (2012) show that the expected excess returns for short-term dividend strips are higher than for long-term dividend strips (see also Boguth et al., 2012; van Binsbergen and Koijen, 2011; van Binsbergen et al., 2013). Van Binsbergen and Koijen (2016) review the recent literature documenting downward sloping Sharpe ratios of risky assets' excess returns, across a variety of markets. Giglio et al. (2014) show a similar pattern exists for discount rates over much longer horizons using real estate markets. Lustig et al. (2016) document a downward-sloping term structure of currency carry trade risk premia. Weber (2016) sorts stocks by the duration

[4]Just like the standard Epstein and Zin (1989) model, our model can accommodate generalized disappointment aversion for the valuation of risk. Such a framework might be of interest for future research.

of their cash flows and finds significantly higher returns for short-duration stocks. Dew-Becker et al. (2016) use data on variance swaps to show, first, that volatility risk is priced (crucial to our model), and second, that investors mostly price it at the 1-month horizon and are essentially indifferent to news about future volatility at horizons ranging from 1 month to 14 years. Using different methodologies and standard index option data, Andries et al. (2016) find a negative price of variance risk for maturities up to 4 months, and a strongly nonlinear downward sloping term structure (in absolute value). The importance of a volatility risk channel, central to our qualitative and quantitative asset pricing results, for other asset pricing implications is supported by Campbell et al. (2016), who show that it is an important driver of asset returns in a CAPM framework, and relates to numerous other works on the relation between volatility risk and returns (Ang et al., 2006; Adrian and Rosenberg, 2008; Bollerslev and Todorov, 2011; Menkhoff et al., 2012; Boguth and Kuehn, 2013).

While the empirical findings concerning the term structure of risk premia are not uncontroversial—it is as of yet uncertain how robust through time some of the evidence will be—they are provocative enough to have triggered a significant literature that aims to explain these patterns. Our model of preferences implies a downward sloping pricing of risk in a simple endowment economy. By contrast, other approaches typically generate the desired implications by making structural assumptions about the economy or about the priced shocks driving the stochastic discount factor directly. For example, in a model with financial intermediaries, Muir (2016) uses time-variation in institutional frictions to explain why the term structure of risky asset returns changes over time. Ai et al. (2015) derive similar results in a production-based real business cycle model in which capital vintages face heterogeneous shocks to aggregate productivity; Zhang (2005) explains the value premium with costly reversibility and a countercyclical price of risk. Other production-based models with implications for the term structure of equity risk are, e.g. Kogan and Papanikolaou (2010, 2014), and Gârleanu et al. (2012). Favilukis and Lin (2015), Belo et al. (2015), and Marfe (2015) offer wage rigidities as an explanation why risk levels and thus risk premia could be higher at short horizons. Croce et al. (2015) use informational frictions to generate a downward-sloping equity term structure. Backus et al. (2016) propose the inclusion of jumps to account for the discrepancy between short-horizon and long-horizon returns. By contrast, our contribution is about risk prices, and, though we derive predictions under standard log-normal consumption growth with time-varying volatility, our framework can accommodate other risk evolutions, such as those employed in the above-cited work. Our methodology is thus broadly applicable.

Other models focusing on the price, rather than the quantity, of risk are Andries (2015) and Curatola (2015) who propose preferences with first order-risk aversion to explain the observed term structure patterns; or Khapko (2015) and Guo (2015), who both study other dynamic extensions to Eisenbach and Schmalz (2016). However, they do so in a time-separable model, which confounds dynamically inconsistent risk preferences with dynamically inconsistent time preferences (hyperbolic discounting). That approach makes the two ingredients' relative contributions opaque. Further, the approach does not accommodate formal solutions, and thus formal interpretations. Chabi-Yo (2016) uses a two-period model to derive higher order conditions on utility over final wealth such that the term structure of volatility risk premia is downward-sloping (in absolute value), and upward-sloping in the bond market.

All of the above mentioned work focuses on matching the recently found evidence on the term structure of risk prices. None of the cited papers addresses the challenge raised by Epstein et al. (2014) regarding the seemingly excessive preference for early resolution implied by the standard models. Our paper addresses both puzzles.

3 Preferences with horizon-dependent risk aversion

We generalize the model of Epstein and Zin (1989) by relaxing the dynamic consistency axiom of Kreps and Porteus (1978). To simplify exposition, we present the model with two levels of risk aversion $\gamma, \tilde{\gamma}$ with $\gamma > \tilde{\gamma}$. Appendix A has the model for general sequences $\{\gamma_h\}_{h\geq 1}$ of risk aversion at horizon h. Here, we assume that the agent treats immediate uncertainty with risk aversion γ, and all delayed uncertainty with risk aversion $\tilde{\gamma}$, where $\gamma > \tilde{\gamma} \geq 1$. Our approach with only two levels of risk aversion is analogous to the β-δ framework (Phelps and Pollak, 1968; Laibson, 1997) as a special case of the general non-exponential discounting framework of Strotz (1955).

The benefit of using the non-separable utility specification of Epstein and Zin (1989) is to disentangle the risk aversion from the elasticity of intertemporal substitution, two features of preferences that are conceptually distinct but artificially linked in the standard model with time-separable utility. However, standard Epstein and Zin (1989) are dynamically consistent (by definition). We modify the model to introduce horizon-dependent risk aversion, and assume that the agent's utility in period t is given by

$$V_t = \left((1-\beta)\, C_t^{1-\rho} + \beta E_t \left[\tilde{V}_{t+1}^{1-\gamma} \right]^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}}, \tag{1}$$

where the continuation value $\tilde{V}_{t+1}$ satisfies the recursion

$$\tilde{V}_{t+1} = \left((1-\beta)\, C_{t+1}^{1-\rho} + \beta E_{t+1} \left[\tilde{V}_{t+2}^{1-\tilde{\gamma}} \right]^{\frac{1-\rho}{1-\tilde{\gamma}}} \right)^{\frac{1}{1-\rho}} . \tag{2}$$

As in the Epstein-Zin model, utility V_t depends on the deterministic current consumption C_t and on the certainty equivalent $E_t \left[\tilde{V}_{t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}$ of the uncertain continuation value $\tilde{V}_{t+1}$, where the aggregation of the two periods occurs with constant elasticity of intertemporal substitution given by $1/\rho$. However, in contrast to the Epstein-Zin model, the certainty equivalent of consumption starting at $t+1$ is calculated with relative risk aversion γ, wherein the certainty equivalents of consumption starting at $t+2$ and beyond are calculated with relative risk aversion $\tilde{\gamma}$.

This is the concept of horizon-dependent risk aversion applied to the recursive valuation of certainty equivalents, as in the Epstein-Zin model, but with risk aversion γ for imminent uncertainty and risk aversion $\tilde{\gamma}$ for delayed uncertainty. Our model nests the Epstein-Zin model if we set $\gamma = \tilde{\gamma}$, and, in turn, nests the standard CRRA time-separable model if $\gamma = \tilde{\gamma} = \rho$.

The horizon-dependent valuation of risk implies a dynamic inconsistency, as the uncertain consumption stream starting at $t+1$ is evaluated as $\tilde{V}_{t+1}$ by the agent's self at t and as V_{t+1} by the agent's self at $t+1$:

$$\begin{aligned} \tilde{V}_{t+1} &= \left((1-\beta)\, C_{t+1}^{1-\rho} + \beta E_{t+1} \left[\tilde{V}_{t+2}^{1-\tilde{\gamma}} \right]^{\frac{1-\rho}{1-\tilde{\gamma}}} \right)^{\frac{1}{1-\rho}} \\ \neq V_{t+1} &= \left((1-\beta)\, C_{t+1}^{1-\rho} + \beta E_{t+1} \left[\tilde{V}_{t+2}^{1-\gamma} \right]^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}} \end{aligned} \tag{3}$$

Crucially, this disagreement between the agent's continuation value $\tilde{V}_{t+1}$ at t and the agent's utility V_{t+1} at $t+1$ arises only for *uncertain* consumption streams. For any *deterministic* consumption stream the horizon-dependence in (1) becomes irrelevant and we have

$$\tilde{V}_{t+1} = V_{t+1} = \left((1-\beta) \textstyle\sum_{h \geq 0} \beta^h C_{t+1+h}^{1-\rho} \right)^{\frac{1}{1-\rho}} .$$

Our model therefore implies dynamically inconsistent *risk* preferences while maintaining dynamically consistent *time* preferences. The results we obtain in the analysis that follows can be attributed to horizon dependent risk aversion, orthogonal to extant models of time inconsistency, such as hyperbolic discounting.

In order to analyze the dynamic choices of our time-inconsistent agent, we follow the tradition of Strotz (1955), and assume that she is fully rational and sophisticated about her preferences when making choices in period t to maximize V_t. Self t realizes that its valuation of future consumption, given by $\tilde{V}_{t+1}$, differs from the objective function V_{t+1} which self $t+1$ will maximize. The solution then corresponds to the subgame-perfect equilibrium in the sequential game played among the agent's different selves (see Appendix B).

An alternative approach to solving the model would be to assume that the agent is naive about the disagreement between her temporal selves, and, at t, thinks of $\tilde{V}_{t+1}$ as the objective function she will optimize at $t+1$. The valuation of early versus late resolutions of uncertainty, a static problem, is, naturally, the same for naive and sophisticated investors. Moreover, analyzing the differences in the pricing of risk for naive versus sophisticated representative investors does not present any conceptual challenge, and we find that, in many cases, including the one we consider in quantitative Section 6, the asset pricing implications are the same.

Yet another approach would be to let the sophisticated agent commit to certain strategies. Studying the differences between optimization with or without commitment is interesting when dealing with individual decision making under horizon-dependent risk aversion (see Eisenbach and Schmalz, 2016). However, such an approach is not appropriate for the analysis of a representative agent who trades and clears the market at all times, and cannot pre-commit to a strategy. In our analysis of equilibrium asset prices, we therefore focus on the fully sophisticated case with no commitment, similar to the approach of Luttmer and Mariotti (2003) for non-geometric discounting.

4 Preference for early or late resolution of uncertainty

Because risk aversion is disentangled from the elasticity of intertemporal substitution, in the preferences of Epstein and Zin (1989), as well as in the pseudo-recursive model of equations (1) and (2), two consumption streams with ex-ante identical risks, but different timing for the resolution of uncertainty, can have different values. Calibrations tailored to match various asset pricing moments, e.g. Bansal and Yaron (2004) for the model of Epstein and Zin (1989) or Section 5 for ours, have strict implications for the relative values of early versus late resolutions of uncertainty.

An agent with Epstein-Zin preferences strictly prefers an early resolution of uncertainty if and only if $\gamma > \rho$. Epstein et al. (2014) point out that the parameters commonly used in the long-run risk literature imply too strong a preference for early resolutions of uncertainty. For example, in the calibration of Bansal and Yaron (2004), the representative agent would be willing to forgo up to 35% of her consumption stream in exchange for all uncertainty to be resolved the next month instead of gradually over time. Epstein et al. (2014) argue that this "timing premium" seems excessive, especially since the ex-ante amount of risk is unchanged by an early rather than late resolution of uncertainty: the agent cannot act on early information to change the consumption stream she will receive. Besides, in numerous cases, in both the empirical and the theoretical literatures, agents *prefer* not to observe early information, even when they can act on it, suggesting a preference for late rather than early resolution of uncertainty (see Golman et al., 2016; Andries and Haddad, 2015). This makes the magnitude of the timing premium under the standard long-run risk model all the more puzzling.

As in Epstein et al. (2014), we assume a unit elasticity of intertemporal substitution, $\rho = 1$, and log-normal consumption growth with time varying drift, i.e. long-run risk, to replicate their formal analysis under our assumption of horizon-dependent risk aversion. Using lower-case letters to denote logs, i.e. $c_t = \log C_t$, $v_t = \log V_t$, $\tilde{v}_t = \log \tilde{V}_t$, we have:

$$\begin{aligned} c_{t+1} - c_t &= \mu_c + \phi_c x_t + \alpha_c \sigma W_{c,t+1} \\ x_{t+1} &= \nu_x x_t + \alpha_x \sigma W_{x,t+1} \end{aligned} \tag{4}$$

The drift is stationary, i.e. ν_x is contracting. For simplicity, we assume x_t is one-dimensional and the shocks α_c and α_x are orthogonal.

Lemma 1. *An agent with horizon-dependent risk aversion $\gamma > \tilde{\gamma} \geq 1$ and $\rho = 1$ values the consumption process (4) as*

$$v_t = c_t + \frac{\beta}{1-\beta}\mu_c + \frac{\beta\phi_c}{1-\beta\nu_x}x_t + \frac{1}{2}\left(1 - \gamma + \beta\left(\gamma - \tilde{\gamma}\right)\right)\frac{\beta}{1-\beta}\alpha_v^2\sigma^2,$$

where $\alpha_v^2 = \alpha_c^2 + \left(\frac{\beta\phi_c}{1-\beta\nu_x}\right)^2 \alpha_x^2$.

Consumption risk is treated with an "effective risk aversion" given by

$$\gamma - \beta\left(\gamma - \tilde{\gamma}\right) < \gamma.$$

This effective risk aversion is the initial risk aversion γ net of the discounted decline in risk aversion between imminent and delayed risk, and is therefore lower than γ itself. The risk in the consumption stream, as represented by the volatility σ, is less penalized than in the standard model with $\gamma = \tilde{\gamma}$, as only part of it is immediate and subject to a high risk aversion.

Denoting by V_t^* the agent's utility at t if all uncertainty (i.e. the entire sequence of shocks $\{W_\tau\}_{\tau \geq t+1}$ in the consumption process (4)) is resolved at $t+1$, Epstein et al. (2014) define the timing premium as the fraction of utility the agent is willing to pay for the early resolution of uncertainty:

$$\mathrm{TP_t} = \frac{V_t^* - V_t}{V_t^*}$$

Proposition 1. *The timing premium for an agent with horizon-dependent risk $\gamma > \tilde{\gamma} \geq 1$ aversion and $\rho = 1$, facing the consumption process* (4) *is*

$$\mathrm{TP} = 1 - \exp\left(\frac{1}{2}\left(1 - \gamma + (1+\beta)(\gamma - \tilde{\gamma})\right)\frac{\beta^2}{1-\beta^2}\alpha_v^2\sigma^2\right).$$

Compared to the timing premium for an Epstein-Zin agent with risk aversion γ, $\mathrm{TP}|_{\gamma=\tilde{\gamma}} = 1 - \exp\left(\frac{1}{2}(1-\gamma)\frac{\beta^2}{1-\beta^2}\alpha_v^2\sigma^2\right)$, the timing premium for an agent with horizon dependent risk aversion is *lower* since

$$\gamma - (1+\beta)(\gamma - \tilde{\gamma}) < \gamma.$$

Our model unambiguously reduces the timing premium.

The lower timing premium is partially due to the lower "effective risk aversion" of an agent with horizon dependent risk aversion (Lemma 1). In addition, a consumption stream with an early resolution of uncertainty concentrates all the risk on the first period, over which the agent is the most risk averse, with immediate risk aversion γ. In contrast, a consumption stream with late resolution of uncertainty has risk spread over multiple horizons, over some of which the agent is moderately risk averse, with risk aversion $\tilde{\gamma} < \gamma$. This has important implications for whether or not the agent prefers early or late resolution.

Consider cases when the timing premium turns negative, indicating a preference for *late* resolution. For an Epstein-Zin agent, this happens when $\gamma < \rho$. In our model, however, the timing premium turns negative when

$$\gamma < 1 + (1+\beta)(\gamma - \tilde{\gamma}). \tag{5}$$

Corollary 1. *An agent with horizon-dependent risk aversion can prefer late resolution of the consumption process* (4) *even when all risk aversions exceed the inverse elasticity of intertemporal substitution, i.e. when $\gamma > \tilde{\gamma} > \rho$.*

Since, for $\gamma > \tilde{\gamma}$, the right-hand side of (5) is greater than $\rho = 1$, the agent with horizon-dependent risk aversion can have a preference for late resolution, even when *both* risk aversions γ and $\tilde{\gamma}$ are greater, even considerably so, than the inverse elasticity of intertemporal substitution, as long as the decline in risk aversion across horizons is sufficiently large. For example, suppose we set immediate risk aversion $\gamma = 10$ and β close to 1. Then the agent will prefer uncertainty to be resolved late rather than early according to condition (5) as long as $\tilde{\gamma} < 5.5$ which is substantially larger than $\rho = 1$.

The result of Corollary 1 is of particular interest because extant calibrations of the long-run risk model with Epstein and Zin (1989) preferences require γ greater than ρ by an order of magnitude, to match equilibrium asset pricing moments. Under horizon-dependent risk aversion, such a calibration for γ and ρ, combined with long-run risk, no longer automatically implies an excessive preference for early resolutions of uncertainty, as we show in Section 6, when we consider the joint quantitative implications for asset pricing moments, term structures, and preferences for early or late resolution of uncertainty.

5 Pricing of risk and the term structure

We now turn to the marginal pricing of risk in our model, in a standard consumption-based asset pricing framework. We assume a fully sophisticated representative agent, who re-optimizes every period, and thus cannot commit, similar to the approach of Luttmer and Mariotti (2003). All decisions are made in sequential one-period problems, raising the question whether the term structure of risk aversions beyond the first period is relevant at all for pricing. We formally analyze whether it is the case, and if yes, how horizon dependent risk aversion impacts equilibrium prices.

5.1 Stochastic discount factor

For asset pricing purposes, the object of interest is the stochastic discount factor (SDF) resulting from the preferences in equations (1) and (2). To satisfy the one-period Euler equation, the SDF's derivation is based on the intertemporal marginal rate of substitution:

$$\Pi_{t,t+1} = \frac{dV_t/dW_{t+1}}{dV_t/dC_t}.$$

We decompose the marginal utility of next-period wealth as $\frac{dV_t}{dW_{t+1}} = \frac{dV_t}{d\tilde{V}_{t+1}} \times \frac{d\tilde{V}_{t+1}}{dW_{t+1}}$, and appeal to the envelope condition at $t+1$: $dV_{t+1}/dW_{t+1} = dV_{t+1}/dC_{t+1}$. Note the envelope condition does not apply to $\tilde{V}_{t+1}$, the value self t attaches to future consumption, but to V_{t+1}, the objective function of self $t+1$. However, due to the homotheticity of our preferences, we can rely on the fact that both $\tilde{V}_{t+1}$ and V_{t+1} are homogeneous of degree one in wealth:

$$\frac{d\tilde{V}_{t+1}/dW_{t+1}}{dV_{t+1}/dW_{t+1}} = \frac{\tilde{V}_{t+1}}{V_{t+1}}.$$

This allows us to formally derive the SDF as:

$$\Pi_{t,t+1} = \frac{dV_{t+1}/dC_{t+1}}{dV_t/dC_t} \times \frac{dV_t}{d\tilde{V}_{t+1}} \times \frac{\tilde{V}_{t+1}}{V_{t+1}}.$$

Proposition 2. *An agent with horizon-dependent risk aversion preferences* (1) *and* (2) *has a one-period stochastic discount factor given by*

$$\Pi_{t,t+1} = \underbrace{\beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho}}_{\text{(I)}} \times \underbrace{\left(\frac{\tilde{V}_{t+1}}{E_t\left[\tilde{V}_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\rho-\gamma}}_{\text{(II)}} \times \underbrace{\left(\frac{\tilde{V}_{t+1}}{V_{t+1}}\right)^{1-\rho}}_{\text{(III)}}. \tag{6}$$

The SDF consists of three multiplicative parts. The first term (I) is standard, capturing intertemporal substitution between t and $t+1$, and is governed by the time discount factor β and the EIS $1/\rho$.

The second term (II) captures uncertainty realized in $t+1$, comparing the ex-post realized $t+1$ utility $\tilde{V}_{t+1}$ to its ex-ante certainty equivalent $E_t\left[\tilde{V}_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$; both the comparison as well as the certainty equivalent are evaluated with immediate risk aversion γ. This term is similar to the corresponding one in Epstein-Zin except that the $t+1$ utility is that of self t ($\tilde{V}_{t+1}$) and not that of self $t+1$ (V_{t+1}).

Finally, the third term (III) directly captures the disagreement between self t and self $t+1$ by comparing their $t+1$ utility. Since self t values future consumption uncertainty with lower risk aversion than self $t+1$, the ratio $\tilde{V}_{t+1}/V_{t+1}$ is greater than 1 and increasing in the disagreement between the two selves. Assets that pay off in states where the disagreement is large are valued highly by self t since they partially compensate for the decisions self $t+1$ takes based on V_{t+1} compared to the ones self t would prefer based on $\tilde{V}_{t+1}$.

Horizon-dependent risk aversion affects the pricing of shocks through terms (II) and (III) in the stochastic discount factor of equation (6). Comparing the expressions for V_{t+1} and $\tilde{V}_{t+1}$ in equation (3), we see that the two selves do *not* disagree about the effect of C_{t+1} on $t+1$ utility so shocks to C_{t+1} will *not* be priced differently than in standard Epstein-Zin. The two selves do, however, disagree about the effect of uncertainty realized at $t+2$ on $t+1$ utility, and horizon-dependent risk aversion *can* therefore affect the pricing of shocks to the value function: shocks to $t+1$ utility will be priced differently than in Epstein-Zin, to the extent that self t and self $t+1$ disagree about their impact, i.e. to the extent that

$\tilde{V}_{t+1}$'s impulse responses differ from those of V_{t+1}.

Asset liquidity Proposition 2 assumes retrading in every period, i.e. fully liquid assets, as appropriate for the asset pricing moments we consider in Section 6. In dynamically consistent models, this is an innocuous assumption: the SDF for pricing an asset at t that can be retraded at $t+2$ is the same as the product of the SDF between t and $t+1$ with the SDF between $t+1$ and $t+2$. For dynamically inconsistent preferences however, illiquidity is similar to a form of forced commitment. In Appendix B, we derive the two-period stochastic discount factor $\Pi_{t,t+2}$, and show how it differs from the product of the two one-period stochastic discount factors $\Pi_{t,t+1}\Pi_{t+1,t+2}$. The pricing impact of our form of dynamic inconsistency therefore differs for liquid and illiquid assets.

Agent sophistication If the agent is naive about her time-inconsistency, she wrongly assumes that the the envelope condition at $t+1$ applies to $\tilde{V}_{t+1}$ and her SDF becomes:

$$\Pi^{\text{naive}}_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{\tilde{V}_{t+1}}{E_t\left[\tilde{V}_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\rho-\gamma}$$

This naive SDF is equal to the sophisticated one in equation (6) when $\rho = 1$: for unit elasticity of intertemporal substitution, naive and sophisticated agents have same risk prices.

Closed form solutions To derive closed-form solutions for the pricing of risk under horizon-dependent risk aversion, we again focus on the case of unit elasticity of intertemporal substitution.[5] Further, we maintain the standard Lucas-tree endowment economy but generalize the consumption process (4) by adding stochastic volatility, in line with the long-run risk literature (e.g. Bansal and Yaron, 2004; Bansal et al., 2009):

$$\begin{aligned} c_{t+1} - c_t &= \mu_c + \phi_c x_t + \alpha_c \sigma_t W_{c,t+1} \\ x_{t+1} &= \nu_x x_t + \alpha_x \sigma_t W_{x,t+1} \\ \sigma^2_{t+1} &= \sigma^2 + \nu_\sigma \left(\sigma_t^2 - \sigma^2\right) + \alpha_\sigma \sigma_t W_{\sigma,t+1} \end{aligned} \tag{7}$$

Both state variables are stationary, i.e. ν_x and ν_σ are contracting with $\nu_\sigma < 1 - \frac{1}{2}\alpha_\sigma^2/\sigma^2$. For simplicity, we assume x_t is one-dimensional and the three shocks α_c, α_x, and α_σ are

[5]In Appendix D, we consider $\rho \neq 1$ and the approximation of a rate of time discount close to zero, $\beta \approx 1$, and show our main results remain valid as long as the elasticity of intertemporal substitution is greater or equal to one ($1/\rho \geq 1$).

orthogonal.[6]

With $\rho = 1$, the SDF in equation (6) becomes

$$\pi_{t,t+1} = \log\beta - (c_{t+1} - c_t) + \underbrace{(1-\gamma)\left(\tilde{v}_{t+1} - E_t\tilde{v}_{t+1} - \frac{1}{2}(1-\gamma)\operatorname{var}_t \tilde{v}_{t+1}\right)}_{\text{shock to utility } \tilde{v}_{t+1} \text{ evaluated with risk aversion } \gamma}. \tag{8}$$

The shocks to the continuation value are priced with immediate risk aversion γ, as in Epstein-Zin. The sole difference is that the SDF involves shocks to $\tilde{v}_{t+1}$ (which evaluates future uncertainty with risk aversion $\tilde{\gamma}$) rather than v_{t+1} (which evaluates future uncertainty with risk aversion γ). To understand the pricing implications of horizon-dependent risk aversion, we first consider how the $t+1$ utilities $\tilde{v}_{t+1}$ and v_{t+1} differ.

Lemma 2. *Under the Lucas-tree endowment process (7) and $\rho = 1$,*

$$\tilde{v}_{t+1} - v_{t+1} = \frac{1}{2}\beta(\gamma - \tilde{\gamma})\left(\alpha_c^2 + \phi_v^2\alpha_x^2 + \psi_v(\tilde{\gamma})^2\alpha_\sigma^2\right)\sigma_{t+1}^2, \tag{9}$$

where $\phi_v = \frac{\beta\phi_c}{1-\beta\nu_x}$, independent of both γ and $\tilde{\gamma}$.

$\psi_v(\tilde{\gamma}) < 0$ is implicitly defined by

$$\psi_v(\tilde{\gamma}) = \frac{1}{2}\frac{\beta(1-\tilde{\gamma})}{1-\beta\nu_\sigma}\left(\alpha_c^2 + \phi_v^2\alpha_x^2 + \psi_v(\tilde{\gamma})^2\alpha_\sigma^2\right),$$

and is independent of γ.[7]

Equation (9) reflects that the $t+1$ value of self t ($\tilde{V}_{t+1}$) and that of self $t+1$ (V_{t+1}) only differ in their $t+1$ valuation of uncertain consumption starting in $t+2$, which is governed by volatility σ_{t+1}^2. Self t evaluates this uncertainty with low risk aversion $\tilde{\gamma}$ while self $t+1$ evaluates it with high risk aversion γ; $\tilde{v}_{t+1} - v_{t+1}$ is therefore positive and increasing in $\gamma - \tilde{\gamma}$, and in the amount of uncertainty driven by current volatility σ_{t+1}^2. We obtain the following central result:

Proposition 3. *If volatility is constant, $\sigma_t = \sigma\ \forall t$ in the consumption process (7), horizon dependent risk aversion does not affect equilibrium risk prices.*

Under constant volatility, the agent can fully anticipate how her future self will re-optimize, and her time inconsistency does not cause any additional uncertainty in her

[6] These assumptions are not crucial to our results and can be generalized. We employ them here to make our results comparable to those of Bansal and Yaron (2004) and Bansal et al. (2009).

[7] As in Hansen and Scheinkman (2009), only one solution of the second order equation, which determines our choice for ψ_v, results in a stationary distribution for $\tilde{v}$.

one-period ahead decision making. Only unanticipated changes in her intra-temporal decisions, when the quantity of risk varies through time, get priced in the risky assets' excess returns. This result crucially hinges on the fact that, in our preference framework, only intra-temporal decisions are time inconsistent: intertemporal decisions are unchanged from the standard model.

If $\sigma_t = \sigma$, i.e., volatility is constant at all times, $\tilde{V}_{t+1}$ and V_{t+1} only differ by a constant wedge (equation (9)), and any shock impacts $\tilde{V}_{t+1}$ and V_{t+1} one-for-one. The difference between the two turns inconsequential for the stochastic discount factor of equation (10) which becomes

$$\Pi_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{\tilde{V}_{t+1}}{E_t \left[\tilde{V}_{t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{E_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma},$$

unaffected by the dynamic time inconsistency of horizon-dependent risk aversion. The result of Proposition 3 can be extended to any endowment process, e.g. jumps or regime switches, where uncertainty is constant through time such that unexpected shocks affect V and $\tilde{V}$ identically. Self t and self $t+1$ disagree *only* about the risk aversion applied to future uncertainty and *not* about consumption in $t+1$ or any deterministic part of future consumption. The result of Proposition 3 is also not specific to the knife-edge case of a unit elasticity of intertemporal substitution, $\rho = 1$, as we show in Appendix D.

Quantitatively, Proposition 3 implies that, when risk in the economy is constant, risk prices are entirely determined by the calibration of the immediate risk aversion, γ, and the elasticity of intertemporal substitution, $1/\rho$, whereas the timing premium is greatly dependent on the wedge between the immediate and "long-term" risk aversions (see equation (5)). This striking result formally proves horizon-dependent risk aversion can solve the "excessive preference for early resolutions of uncertainty puzzle" of Epstein et al. (2014), without compromising on the model's ability to match the usual asset pricing moments.

We decompose the shocks to $\tilde{v}_{t+1}$ into the components due to the three processes in (7).

Lemma 3. *Under the Lucas-tree endowment process (7) and $\rho = 1$,*

$$\tilde{v}_{t+1} - E_t \tilde{v}_{t+1} = c_{t+1} - E_t c_{t+1} + \phi_v \left(x_{t+1} - E_t x_{t+1} \right) + \psi_v(\tilde{\gamma}) \left(\sigma_{t+1}^2 - E_t \sigma_{t+1}^2 \right).$$

Positive shocks to immediate consumption, $c_{t+1} - E_t c_{t+1}$, or to expected consumption growth, $x_{t+1} - E_t x_{t+1}$, naturally increase the value of the consumption stream starting at $t+1$, $\tilde{v}_{t+1}$ ($\phi_v > 0$). Increases in uncertainty, $\sigma_{t+1}^2 - E_t \sigma_{t+1}^2$, on the other hand, reduce

$\tilde{v}_{t+1}$ ($\psi_v(\tilde{\gamma}) < 0$). Both immediate consumption shocks and shocks to the drift, captured by the coefficient ϕ_v, are unaffected by horizon-dependent risk aversion. These shocks affect intertemporal consumption smoothing decisions only, and, as such, their valuation is governed by the elasticity of intertemporal substitution and not by risk aversion, nor the dynamic risk inconsistency of our model. Long-run risk aversion $\tilde{\gamma}$ only matters for shocks to volatility, as indicated by Proposition 3.

From Lemma 3 we obtain the SDF as follows.

Proposition 4. *Under the Lucas-tree endowment process* (7), *and* $\rho = 1$, *the stochastic discount factor satisfies*

$$\pi_{t,t+1} = \bar{\pi}_t - \gamma\alpha_c\sigma_t W_{c,t+1} + (1-\gamma)\,\phi_v\alpha_x\sigma_t W_{x,t+1} \\ + (1-\gamma)\,\psi_v(\tilde{\gamma})\,\alpha_\sigma\sigma_t W_{\sigma,t+1}, \tag{10}$$

where $\bar{\pi}_t = E_t[\pi_{t,t+1}]$.

The risk free rate is independent of $\tilde{\gamma}$*:*

$$r_{f,t} = -\log\beta + \mu_c + \phi_c x_t + \left(\frac{1}{2} - \gamma\right)\alpha_c^2\sigma_t^2 \tag{11}$$

The pricing of the immediate consumption shocks, given by the term $\gamma\alpha_c\sigma_t W_{c,t+1}$ in equation (10), depends only on short-term risk aversion γ. It is unchanged from the standard Epstein-Zin model (and from the expected utility model with CRRA preferences).

The pricing of drift shocks, the term $(1-\gamma)\,\phi_v\alpha_x\sigma_t W_{x,t+1}$ in equation (10), as well as the risk-free rate (equation (11)) also depends only on immediate risk aversion γ. In line with the results of Lemma 3, these assets hinge on one-period intertemporal decisions only, and are therefore unaffected by the intra-temporal dynamic inconsistency of horizon-dependent risk aversion.

Our model yields a negative price for volatility shocks, the term $(1-\gamma)\,\psi_v(\tilde{\gamma})\,\alpha_\sigma\sigma_t W_{\sigma,t+1}$ in equation (10), consistent with the existing long-run risk literature and the observed data for one-period returns (see Dew-Becker et al., 2016, and Andries et al., 2016 for recent examples). Volatility shocks pricing depends on both the immediate risk aversion γ, and on the "long-term" one through $\psi_v(\tilde{\gamma})$. Any novel pricing effects we obtain—both in levels and in the term structure—derive only from the time varying volatility risk channel, and the pricing of its shocks. The combined results of Propositions 3 and 4 make clear the subtlety in the link between risk-aversion and risk premia in the term structure. It is neither one-for-one, as seems most intuitive at first, nor non-existent, as the one-period-ahead pricing framework may suggest—unless risk is constant in the economy.

We analyze next if the model can, as successfully as for the valuations of early versus late resolutions of risk (Corollary 1), match the observed evidence on the term structure of risk premia, when risk varies through time.

5.2 Term structure of returns

We turn to the variations in risk prices across the term structure. We focus our analysis on three specific assets: risk-free bonds, dividend strip futures and variance swaps. This allows us to compare our calibrated term structure results to the empirical evidence from van Binsbergen and Koijen (2016) for the bond and dividend strips futures markets; and Dew-Becker et al. (2016) and Andries et al. (2016) for the variance swaps and options markets, respectively.

Risk-free bonds We write $B_{t,h}$ the price of a risk-free zero-coupon bond with maturity h at time t.

Lemma 4. *The price of a risk-free zero-coupon bond with maturity h at time t is*

$$B_{t,h} = \exp\left(\mu_{b,h} + \phi_{b,h} x_t + \psi_{b,h} \sigma_t^2\right),$$

where $\phi_{b,h} = -\phi_c \frac{1-\nu_x^h}{1-\nu_x}$ *does not depend on* γ *or* $\tilde{\gamma}$*, but both* $\psi_{b,h}$ *and* $\mu_{b,h}$ *do. For all* $h \geq 0$, $\phi_{b,h} < 0$ *and strictly decreasing with the horizon h. For* $\tilde{\gamma} > 1$*, and for all* $h \geq 0$, $\psi_{b,h} > 0$ *and strictly increasing with the horizon h.* $\mu_{b,h}$ *is not monotone in h.*[8]

This result is reminiscent of and consistent with that of Lemma 2. Higher expected consumption growth reduces the incentive to save, and thus bond prices ($\phi_{b,h} < 0$), the more so the longer the savings horizon. Higher volatility increases precautionary savings motives, and thus bond prices ($\psi_{b,h} > 0$). As the quantity of risk increases with the horizon in the long-run risk setting of process (7), so does the precautionary savings motive and thus $\psi_{b,h}$. Horizon-dependent risk aversion, and its lower risk aversion in the long-run, mitigates this effect (as we show below in the calibration of our model), but only partially, as per Lemma 4.

Dividend strip futures In line with the long-run risk literature (Bansal and Yaron, 2004; Bansal et al., 2009), and consistent with the consumption growth process (7), we assume

[8] If $\log \beta - \mu_c > 0$, $\mu_{b,h}$ is increasing in h. This condition is not satisfied under the calibration of Section 6.

dividends follow a lognormal growth given by:

$$d_{t+1} - d_t = \mu_d + \phi_d x_t + \chi \alpha_c \sigma_t W_{c,t+1} + \alpha_d \sigma_t W_{d,t+1}, \tag{12}$$

where $W_{d,t+1}$ is orthogonal to the consumption shocks $W_{c,t+1}$, $W_{x,t+1}$ and $W_{\sigma,t+1}$. ϕ_d captures the link between mean consumption growth and mean dividend growth; χ the correlation between immediate consumption and dividend shocks in the business cycle.

We denote the value at time t for a dividend strip with horizon h, i.e. the claim to the aggregate dividend at horizon $t+h$, as $D_{t,h}$. In the spirit of van Binsbergen and Koijen (2016) and in order to compare our calibrated results with their empirical ones, we study one-period holding returns on dividend strip futures, $\frac{D_{t+1,h-1}/D_{t,h}}{B_{t+1,h-1}/B_{t,h}} - 1$.

Lemma 5. *The price of a dividend strip with maturity h at time t is*

$$\frac{D_{t,h}}{D_t} = \exp\Big(\mu_{d,h} + \phi_{d,h} x_t + \psi_{d,h} \sigma_t^2\Big) ,$$

where $\phi_{d,h} = (-\phi_c + \phi_d) \frac{1-\nu_x^h}{1-\nu_x}$ *does not depend on* γ *or* $\tilde{\gamma}$*, but both* $\psi_{d,h}$ *and* $\mu_{d,h}$ *do. If* $\phi_d > \phi_c$ *(*$\phi_d < \phi_c$*), then* $\phi_{b,h} > 0$ *(*$\phi_{b,h} < 0$*) and strictly increasing (decreasing) with the horizon h.* $\mu_{d,h}$ *and* $\psi_{d,h}$ *are not monotone in h.*

This result is, once again, consistent with that of Lemma 2: horizon-dependent risk aversion affects the pricing of volatility (through $\psi_{d,h}$), and the average discount for risk (through $\psi_{d,h}$ and $\mu_{d,h}$).

Lemma 6. *Conditional Sharpe ratios of one-period holding returns on futures are approximately proportional to the time-varying volatility* σ_t^2*: the slope of the term structure at any horizon increases in absolute value with volatility.*[9]

Van Binsbergen et al. (2013) find, for the dividend strips futures market, a steeper downward sloping term structure in times of high volatility. As per Lemma 6, such dynamics naturally obtain when the term structure of unconditional Sharpe ratios of dividend strip futures is downward sloping, as in the calibration of Section 6.

Variance swaps At any time t, the payoff of a variance swap with horizon h is approximately proportional to the future variance σ_{t+h}^2 (see Appendix C for details), with the following pricing result:

[9] First-order approximation in $\alpha_c^2 \sigma_t^2 \sim \alpha_x^2 \sigma_t^2 \sim \alpha_\sigma^2 \sigma_t^2 \ll 1$.

Table 1: Calibration.

(a) Parameters.

Process	Parameters
c_t	$\mu_c = 0.15\%$
	$\phi_c = 1$
	$\alpha_c = 1$
x_t	$\nu_x = 0.92$
	$\alpha_x = 0.11$
σ_t	$\nu_\sigma = 0.992$
	$\sigma = 0.72\%$
	$\alpha_\sigma = 0.04\%$
d_t	$\mu_d = 0.15\%$
	$\phi_d = 4$
	$\alpha_d = 5.96$
	$\chi = 2.6$

(b) Results.

	Data	Calibration
$E\left[d_{cons}\right]$	0.02	0.02
$\sigma\left[d_{cons}\right]$	0.03	0.03
$AC_1\left[d_{cons}\right]$	0.29	0.23
$AC_2\left[d_{cons}\right]$	0.03	0.07
$AC_3\left[d_{cons}\right]$	−0.17	0.02
$AC_4\left[d_{cons}\right]$	−0.22	−0.02
$AC_5\left[d_{cons}\right]$	0.03	0.01
$E\left[d_{div}\right]$	0.01	0.02
$\sigma\left[d_{div}\right]$	0.11	0.18
$AC_1\left[d_{div}\right]$	0.18	0.13
$\rho\left(d_{div}, d_{cons}\right)$	0.52	0.57

Lemma 7. *The price at time t of an asset with payoff σ^2_{t+h} at horizon $h \geq 0$ is given by*

$$S_{t,h} = \left(\mu_{\sigma^2,h}\sigma^2 + \psi_{\sigma^2,h}\sigma_t^2\right) B_{t,h},$$

where both $\mu_{\sigma^2,h}$ and $\psi_{\sigma^2,h}$ depend on immediate risk aversion γ and "long-run" risk aversion $\tilde{\gamma}$, for $h \geq 1$. $\mu_{\sigma^2,h} \geq 0$ and strictly increasing with the horizon h. $\psi_{\sigma^2,h} \geq 0$ and not monotone in h.[10] *$B_{t,h}$ is the price of a bond with maturity h as above.*[11]

The pricing results of Lemma 4, 5 and 7 show horizon-dependent risk aversion affects the pricing of all assets of interest, in levels and term structures, when volatility is time varying. We assess the magnitude of this effect in the calibration that follows.

6 Quantitative results

We calibrate the consumption and dividend growth processes (7) and (12) to fit moments in the data as closely as possible, within the constraints of our framework (Tables 1a and 1b, data source from Shiller's website, annual data 1926–2009). Note that fitting both the

[10] If $|\psi_v(\tilde{\gamma})| > \frac{1-\nu_\sigma}{\alpha_\sigma^2(\gamma-1)}$, $\psi_{\sigma^2,h}$ is increasing in h. This condition is not satisfied under the calibration of Section 6.

[11] Initial conditions are $\mu_{\sigma^2,0} = 0$, $\psi_{\sigma^2,0} = 1$.

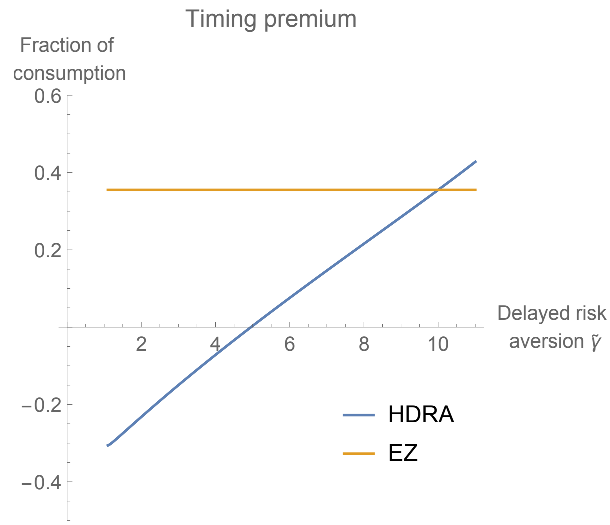

Figure 1: Effect of horizon dependent risk aversion (HDRA) on willingness to pay for early resolution of uncertainty (timing premium), compared to Epstein-Zin preferences (EZ) with $\gamma = 10$.

strongly positive autocorrelation for consumption growth at the one-year frequency and the strongly negative one at the four-year frequency is difficult when the time varying drift follows an AR(1) process (see Bryzgalova and Julliard, 2015, for a recent analysis of consumption growth in the data). Our calibration for the dividend and volatility processes is very similar to Bansal et al. (2014).[12] This choice, instead of a GMM approach incorporating term structure moments which could improve the fit of Figures 2, 3 and 4, allows us to highlight how our preference model—rather than changes in the calibration for the endowment process—affect prices. In line with the literature, we use $\beta = 0.999$ for the monthly rate of time discount, and the elasticity of intertemporal substitution is one throughout (see Appendix D for $\rho \neq 1$ approximations results).

6.1 Timing premium

We first study the quantitative implications of horizon-dependent risk aversion on the timing premium—the agent's willingness to pay for early resolution of all consumption uncertainty. Figure 1 plots the timing premium for both horizon-dependent risk aversion and for standard Epstein-Zin preferences when $\gamma = 10$, using the calibration of Table 1a.[13]

As pointed out by Epstein et al. (2014), a standard Epstein-Zin agent has a high willing-

[12] We assume a higher volatility and lower persistence for the consumption drift process, closer to direct evidence in the data (Hansen et al., 2008).

[13] In Section 4, we analyze the term-premium under the constant volatility process (4), as in Epstein et al. (2014). We formally derive the term-premium under the stochastic volatility process (7) in Appendix C.

Table 2: Equity premium versus timing premium.

		Equity premium	Timing premium
Data		6.64%	–
Calibration	$\tilde{\gamma} \approx 1$	6.20%	-31%
	$\tilde{\gamma} = 2$	6.23%	-24%
	$\tilde{\gamma} = 3$	6.27%	-15%
	$\tilde{\gamma} = 5$	6.35%	0%
	$\tilde{\gamma} = 7$	6.45%	14%
	$\tilde{\gamma} = \gamma = 10$	6.65%	35%

Annualized returns under $\gamma = 10$, $\rho = 1$; calibration of Table 1a; data is from Bob Shiller's website.

ness to pay for early resolution—about 35% of expected consumption. In contrast, under our calibration, an agent with horizon-dependent risk aversion can have a significantly lower willingness to pay for early resolution. In fact, for delayed risk aversion $\tilde{\gamma} < 5$, the agent with horizon-dependent risk aversion has a preference for *late* resolution of risk.

This result is of particular interest for two reasons. First, as briefly discussed in Section 4, apart from the fact that a 35% premium seems unrealistically large, there is no clear consensus concerning the "right" value for the timing premium: how large it should be, or whether it should even be positive. With horizon-dependent risk aversion, and the calibration of Table 1a, the possible values for the timing premia range from -30% to $+35\%$: our framework can accommodate any "reasonable" value for the valuation of early versus late resolutions of uncertainty. Second, and crucially, the average risk free rate and equity premium are mostly determined by the calibration for immediate risk aversion γ (and $\rho = 1$), with $\tilde{\gamma}$ playing a limited role (or no role whatsoever for the risk-free rate). These results are presented in Table 2: calibrating the usual asset pricing moments no longer precludes a reasonable timing premium.

6.2 Term structures

We now turn to the pricing of risk in the term structure. We present results for $\tilde{\gamma} \approx 1$, under which horizon-dependent risk aversion is the most impactful. Results with higher $\tilde{\gamma}$ are provided in Appendix E, and show that, as long as $\tilde{\gamma}$ remains relatively low (less than $\tilde{\gamma} = 5$) our model remains quantitatively distinguishable from the standard Epstein-Zin model.

Figure 2 shows the term structure of one-month holding returns on risk-free bonds.

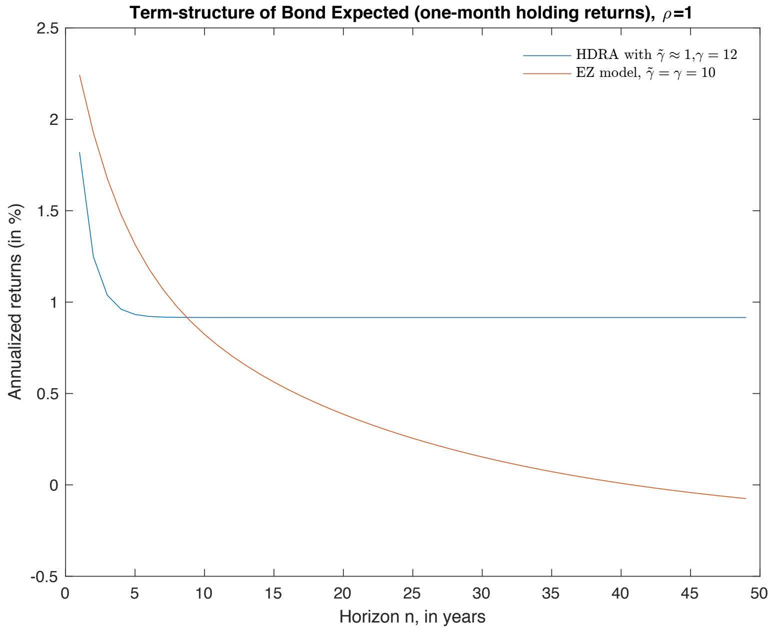

Figure 2: Calibrated term structure of risk-free bond returns under horizon-dependent risk aversion (HDRA). The Epstein-Zin curve (EZ) is done with the calibration of Bansal et al. (2014).

Because our model assumes a low risk aversion for delayed risk, the motive for long-term precautionary savings is lower than in the standard model, and we obtain a higher equilibrium risk-free rate at the back-end of the curve. Van Binsbergen and Koijen (2016) find, using CRSP Treasury bond portfolios (from 1952 to 2013), real returns of 0.6% at 1 year, 1.3% at 3 years and 1.8% at 10 years. Even though horizon-dependent risk aversion does not deliver an upward sloping yield curve (see the discussion of Lemma 4), our model modifies the standard Epstein-Zin model in an empirically useful direction.

Figure 3 plots the term structure for the Sharpe ratios of one-month holding returns on dividend strip futures. Our model, in which the term structure is upward sloping for the first 5 years, and downward sloping thereafter, does well with regards to the empirical evidence of van Binsbergen and Koijen (2016). They find, on the US SPX (from 2002 to 2014), Sharpe ratios of 0.12 at 1 year, 0.14 at 3 years and 0.16 at 5 years, but 0.04 for the index, indicating a sharply decreasing term structure for the medium to long-term, a puzzle for the standard Epstein-Zin model.

Finally, Figure 4 plots the term structure for Sharpe ratios of one-month holding returns on variance swaps. Our calibration with horizon-dependent risk aversion results in an upward sloping term structure, equivalent to a downward sloping price of volatility risk, in absolute value. This result is in line with the recent evidence from Dew-Becker et al.

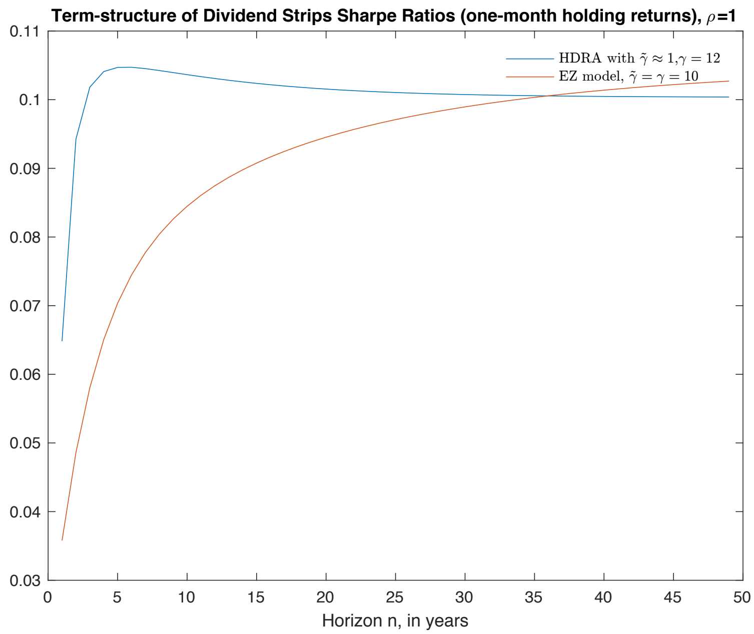

Figure 3: Calibrated term structure of Sharpe ratios under horizon-dependent risk aversion (HDRA). The Epstein-Zin curve (EZ) is done with the calibration of Bansal et al. (2014).

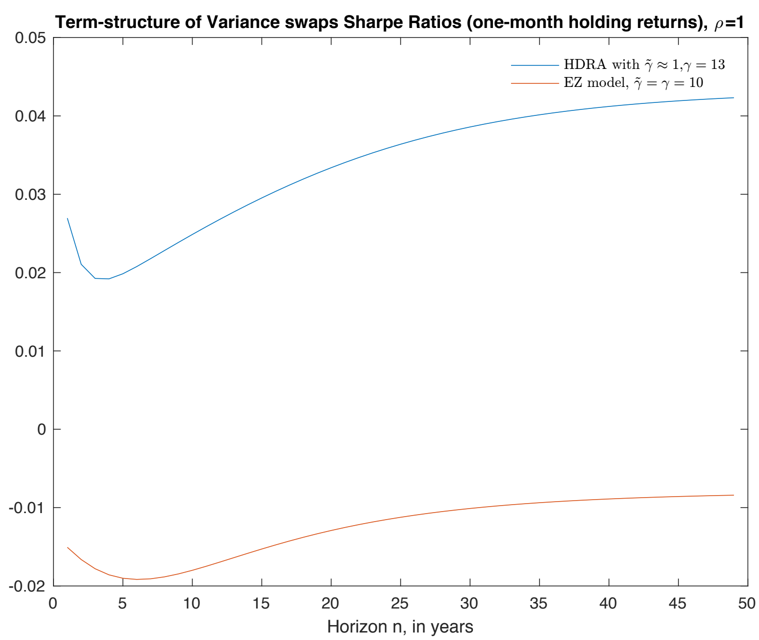

Figure 4: Calibrated term structure of variance swaps returns Sharpe ratios under horizon-dependent risk aversion (HDRA). The Epstein-Zin curve (EZ) is done with the calibration of Bansal et al. (2014).

(2016), using US variance swaps data, and Andries et al. (2016), on US option straddles data. Expected returns are positive in our calibrated model, in line with Dew-Becker et al. (2016), who find positive values beyond the 3-months horizon. However, Dew-Becker et al. (2016) find strong negative values for the first month horizon, followed by a sharp increase. Andries et al. (2016) show a similar, though attenuated, pattern. These results indicate the presence, and pricing, of immediate, transitory, shocks, which the consumption and dividend growth processes (7) and (12) do not account for. Introducing such shocks into our framework is left for future research.

7 Conclusion

The "long-run risk" model of Bansal and Yaron (2004) has recently been criticized because it makes qualitatively counterfactual predictions about the term structure of risk prices (e.g., van Binsbergen et al., 2012, 2013; van Binsbergen and Koijen, 2016) and because its calibrations are difficult to reconcile with the microeconomic foundations of the preferences it employs (Epstein et al., 2014). We show that these criticisms do not imply that the model needs to be discarded. Instead, relaxing the restriction of Epstein and Zin (1989) that risk preferences be constant across horizons allows researchers to retain the desirable pricing properties of the long-run risk model, and simultaneously match the sign of the term structure of risk prices and obtain reasonable implications for the timing of the resolution of uncertainty.

Our analysis is accomplished with considerable technical difficulty and is not due to a tautological relationship between risk aversion and risk pricing at different maturities. In particular, we show how to solve for general equilibrium asset prices in an economy populated by agents with dynamically inconsistent risk preferences. In such a model, the price of risk depends on the horizon, but only if volatility is stochastic. This insight leads to several testable predictions. Some of them, such as a declining term structure of the price of risk, have found support in the recent empirical literature. Others constitute opportunities for future research. We conclude that relaxing the common assumption that risk preferences are constant across maturities—and specifically, replacing it with the assumption that short-horizon risk aversion is higher than long-horizon risk aversion—is a useful new tool for future research in asset pricing and macro-finance.

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Appendix (for online publication)

A General sequence of risk aversions

Let $\{\gamma_h\}_{h\geq 1}$ be a decreasing sequence representing risk aversion at horizon h. In period t, the agent evaluates a consumption stream starting in period $t+h$ by

$$V_{t,t+h} = \left((1-\beta)\, C_{t+h}^{1-\rho} + \beta E_{t+h}\left[V_{t,t+h+1}^{1-\gamma_{h+1}} \right]^{\frac{1-\rho}{1-\gamma_{h+1}}} \right)^{\frac{1}{1-\rho}} \quad \text{for all} \quad h \geq 0. \tag{13}$$

The agent's utility in period t is given by setting $h=0$ in (13) which we denote by $V_t \equiv V_{t,t}$ for all t:

$$V_t = \left((1-\beta)\, C_t^{1-\rho} + \beta E_t\left[V_{t,t+1}^{1-\gamma_1} \right]^{\frac{1-\rho}{1-\gamma_1}} \right)^{\frac{1}{1-\rho}}$$

As in the Epstein-Zin model, utility V_t depends on deterministic current consumption C_t and a certainty equivalent $E_t\left[V_{t,t+1}^{1-\gamma_1} \right]^{\frac{1}{1-\gamma_1}}$ of uncertain continuation values $V_{t,t+1}$, where the aggregation of the two periods occurs with constant elasticity of intertemporal substitution given by $1/\rho$, regardless of the horizon h. However, in contrast to the Epstein-Zin model, the certainty equivalent of consumption starting at $t+1$ is calculated with relative risk aversion γ_1, wherein the certainty equivalent of consumption starting at $t+2$ is calculated with relative risk aversion γ_2, and so on. This is the concept of horizon-dependent risk aversion applied to the nested valuation of certainty equivalents, as in the Epstein-Zin model, but with relative risk aversion γ_h for the certainty equivalent formed at horizon h. Our model therefore nests the Epstein-Zin model if we set $\gamma_h = \gamma$ for all h, which, in turn, nests the standard time-separable model for $\gamma = \rho$.

An interesting question is the possibility to axiomatize the horizon-dependent risk aversion preferences we propose. Our dynamic model builds on the functional form of Epstein and Zin (1989) which captures non-time-separable preferences of the form axiomatized by Kreps and Porteus (1978). However, our generalization of Epstein and Zin (1989) explicitly violates Axiom 3.1 (temporal consistency) of Kreps and Porteus (1978) which is necessary for the recursive structure. In contrast to Epstein-Zin, the preference of our model captured by $V_t \equiv V_{t,t}$ is *not* recursive since $V_{t+1} \equiv V_{t+1,t+1}$ does not recur in the definition of V_t.

In order to derive the closed-form solution for $V_t \equiv V_{t,t}$, we assume that risk aversion is decreasing until some horizon H and constant thereafter, $\gamma_h > \gamma_{h+1}$ for $h < H$ and $\gamma_h = \tilde{\gamma}$

for $h \geq H$. Starting with $V_{t,t+H}$, our model then corresponds to the standard Epstein-Zin recursion with risk aversion $\tilde{\gamma}$ for which we can use the standard solution. Determining V_t then is just a matter of solving backwards.

B Stochastic discount factor

We present the derivation of the stochastic discount factor with a general sequence of risk aversions $\{\gamma_h\}_{h\geq 1}$. The equations simplify to the ones in the main text by setting $\gamma_1 = \gamma$ and $\gamma_h = \tilde{\gamma}$ for $h \geq 2$.

Proof of Proposition 2. This appendix derives the stochastic discount factor of our dynamic model using an approach similar to the one used by Luttmer and Mariotti (2003) for dynamic inconsistency due to non-geometric discounting. In every period t the agent chooses consumption C_t for the current period and state-contingent levels of wealth $\{W_{t+1,s}\}$ for the next period to maximize current utility V_t subject to a budget constraint *and* anticipating optimal choice C^*_{t+h} in all following periods ($h \geq 1$):

$$
\begin{aligned}
&\max_{C_t,\{W_{t+1}\}} \left((1-\beta)\, C_t^{1-\rho} + \beta E_t \left[\left(V^*_{t,t+1}\right)^{1-\gamma_1} \right]^{\frac{1-\rho}{1-\gamma_1}} \right)^{\frac{1}{1-\rho}} \\
&\quad \text{s.t.} \quad \Pi_t C_t + E_t[\Pi_{t+1} W_{t+1}] \leq \Pi_t W_t \\
&\qquad\qquad V^*_{t,t+h} = \left((1-\beta)\left(C^*_{t+h}\right)^{1-\rho} + \beta E_{t+h}\left[\left(V^*_{t,t+h+1}\right)^{1-\gamma_{h+1}} \right]^{\frac{1-\rho}{1-\gamma_{h+1}}} \right)^{\frac{1}{1-\rho}} \qquad \text{for all} \quad h \geq 1.
\end{aligned}
$$

Denoting by λ_t the Lagrange multiplier on the budget constraint for the period-t problem, the first order conditions are:[14]

- For C_t:

$$
\left((1-\beta)\, C_t^{1-\rho} + \beta E_t \left[V_{t,t+1}^{1-\gamma_1} \right]^{\frac{1-\rho}{1-\gamma_1}} \right)^{\frac{1}{1-\rho}-1} (1-\beta)\, C_t^{-\rho} = \lambda_t.
$$

[14] For notational ease we drop the star from all Cs and Vs in the following optimality conditions but it should be kept in mind that all consumption values are the ones optimally chosen by the corresponding self.

- For each $W_{t+1,s}$:

$$\frac{1}{1-\rho}\left(\left(1-\beta\right)C_t^{1-\rho}+\beta E_t\left[V_{t,t+1}^{1-\gamma_1}\right]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1}{1-\rho}-1}\beta\frac{d}{dW_{t+1,s}}\beta E_t\left[V_{t,t+1}^{1-\gamma_1}\right]^{\frac{1-\rho}{1-\gamma_1}}$$
$$=\Pr[t+1,s]\,\frac{\Pi_{t+1,s}}{\Pi_t}\lambda_t.$$

Combining the two, we get an initial equation for the SDF:

$$\frac{\Pi_{t+1,s}}{\Pi_t}=\beta\,\frac{\frac{1}{1-\rho}\frac{1}{\Pr[t+1,s]}\frac{d}{dW_{t+1,s}}E_t\left[V_{t,t+1}^{1-\gamma_1}\right]^{\frac{1-\rho}{1-\gamma_1}}}{1}\frac{1}{\left(1-\beta\right)C_t^{-\rho}}. \tag{14}$$

The agent in state s at $t+1$ maximizes

$$\left(\left(1-\beta\right)C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}\left[\left(V_{t+1,s,t+2}^{*}\right)^{1-\gamma_1}\right]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1}{1-\rho}}$$

and has the analogous first order condition for $C_{t+1,s}$:

$$\left(\left(1-\beta\right)C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}\left[V_{t+1,s,t+2}^{1-\gamma_1}\right]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1}{1-\rho}-1}\left(1-\beta\right)C_{t+1,s}^{-\rho}=\lambda_{t+1,s}.$$

The Lagrange multiplier $\lambda_{t+1,s}$ is equal to the marginal utility of an extra unit of wealth in state $t+1,s$:

$$\lambda_{t+1,s}=\frac{1}{1-\rho}\left(\left(1-\beta\right)C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}\left[V_{t+1,s,t+2}^{1-\gamma_1}\right]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1}{1-\rho}-1}$$
$$\times\frac{d}{dW_{t+1,s}}\left(\left(1-\beta\right)C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}\left[V_{t+1,s,t+2}^{1-\gamma_1}\right]^{\frac{1-\rho}{1-\gamma_1}}\right).$$

Eliminating the Lagrange multiplier $\lambda_{t+1,s}$ and combining with the initial equation (14) for the SDF, we get:

$$\frac{\Pi_{t+1,s}}{\Pi_t}=\beta\,\frac{\frac{1}{\Pr[t+1,s]}\frac{d}{dW_{t+1,s}}E_t\left[V_{t,t+1}^{1-\gamma_1}\right]^{\frac{1-\rho}{1-\gamma_1}}}{\frac{d}{dW_{t+1,s}}\left(\left(1-\beta\right)C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}\left[V_{t+1,s,t+2}^{1-\gamma_1}\right]^{\frac{1-\rho}{1-\gamma_1}}\right)}\left(\frac{C_{t+1,s}}{C_t}\right)^{-\rho}.$$

Expanding the V expressions, we can proceed with the differentiation in the numerator:

$$\begin{aligned}
\frac{\Pi_{t+1,s}}{\Pi_t} = E_t & \left[\left((1-\beta)\, C_{t+1}^{1-\rho} + \beta E_{t+1}[\dots]^{\frac{1-\rho}{1-\gamma_2}} \right)^{\frac{1-\gamma_1}{1-\rho}} \right]^{\frac{1-\rho}{1-\gamma_1}-1} \\
& \times \left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}[\dots]^{\frac{1-\rho}{1-\gamma_2}} \right)^{\frac{1-\gamma_1}{1-\rho}-1} \\
& \qquad \times \beta \, \frac{\frac{d}{dW_{t+1,s}} \left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}[\dots]^{\frac{1-\rho}{1-\gamma_2}} \right)}{\frac{d}{dW_{t+1,s}} \left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}[\dots]^{\frac{1-\rho}{1-\gamma_1}} \right)} \left(\frac{C_{t+1,s}}{C_t} \right)^{-\rho}.
\end{aligned} \tag{15}$$

For Markov consumption $C = \phi W$, we can divide by $C_{t+1,s}$ and solve both differentiations:

- For the numerator:

$$\begin{aligned}
& \frac{d}{dW_{t+1,s}} \left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s} \left[\left((1-\beta)\, C_{t+2}^{1-\rho} + \beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_3}} \right)^{\frac{1-\gamma_2}{1-\rho}} \right]^{\frac{1-\rho}{1-\gamma_2}} \right) \\
& = \left((1-\beta)\, 1 + \beta E_{t+1,s} \left[\left((1-\beta) \left(\frac{C_{t+2}}{C_{t+1,s}} \right)^{1-\rho} + \beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_3}} \right)^{\frac{1-\gamma_2}{1-\rho}} \right]^{\frac{1-\rho}{1-\gamma_2}} \right) \\
& \qquad \times \phi_{t+1,s}^{1-\rho} W_{t+1,s}^{-\rho}.
\end{aligned}$$

- For the denominator:

$$\begin{aligned}
& \frac{d}{dW_{t+1,s}} \left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s} \left[\left((1-\beta)\, C_{t+2}^{1-\rho} + \beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_2}} \right)^{\frac{1-\gamma_1}{1-\rho}} \right]^{\frac{1-\rho}{1-\gamma_1}} \right) \\
& = \left((1-\beta)\, 1 + \beta E_{t+1,s} \left[\left((1-\beta) \left(\frac{C_{t+2}}{C_{t+1,s}} \right)^{1-\rho} + \beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_2}} \right)^{\frac{1-\gamma_1}{1-\rho}} \right]^{\frac{1-\rho}{1-\gamma_1}} \right) \\
& \qquad \times \phi_{t+1,s}^{1-\rho} W_{t+1,s}^{-\rho}.
\end{aligned}$$

Substituting these into equation (15) and canceling we get:

$$\frac{\Pi_{t+1,s}}{\Pi_t} = \frac{(1-\beta)\,C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}\left[\left((1-\beta)\,C_{t+2}^{1-\rho} + \beta E_{t+2}[\ldots]^{\frac{1-\rho}{1-\gamma_3}}\right)^{\frac{1-\gamma_2}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_2}}}{(1-\beta)\,C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}\left[\left((1-\beta)\,C_{t+2}^{1-\rho} + \beta E_{t+2}[\ldots]^{\frac{1-\rho}{1-\gamma_2}}\right)^{\frac{1-\gamma_1}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_1}}}$$

$$\times \beta \left(\frac{C_{t+1,s}}{C_t}\right)^{-\rho} \left(\frac{(1-\beta)\,C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}[\ldots]^{\frac{1-\rho}{1-\gamma_2}}}{E_t\left[\left((1-\beta)\,C_{t+1,s}^{1-\rho} + \beta E_{t+1}[\ldots]^{\frac{1-\rho}{1-\gamma_2}}\right)^{\frac{1-\gamma_1}{1-\rho}}\right]}\right)^{\rho-\gamma_1}.$$

Simplifying and cleaning up notation, we arrive at

$$\Pi_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{V_{t,t+1}}{E_t\left[V_{t,t+1}^{1-\gamma_1}\right]^{\frac{1}{1-\gamma_1}}}\right)^{\rho-\gamma_1} \left(\frac{V_{t,t+1}}{V_{t+1}}\right)^{1-\rho},$$

as stated in the text. □

To derive the two-period ahead stochastic discount factor, we use the intertemporal marginal rate of substitution:

$$\Pi_{t,t+2} = \frac{dV_t/dW_{t+2}}{dV_t/dC_t}.$$

While the marginal utility of current consumption, dV_t/dC_t, is unchanged from the standard model,

$$\frac{dV_t}{dC_t} = V_t^{\rho}\,(1-\beta)\,C_t^{-\rho},$$

the marginal utility of wealth two periods away is different:

$$\frac{dV_t}{dW_{t+2}} = \frac{dV_t}{dV_{t,t+2}} \times \frac{dV_{t,t+2}}{dW_{t+2}}.$$

In particular, we cannot appeal to the envelope condition at $t+1$ to replace the term $dV_{t,t+2}/dW_{t+2}$ by $dV_{t,t+2}/dC_{t+1}$ because $V_{t,t+2}$ is the value self t attaches to future consumption while the envelope condition at $t+2$ is in terms of V_{t+2}, the objective function of self $t+2$:

$$\frac{dV_{t+2}}{dW_{t+2}} = \frac{dV_{t+2}}{dC_{t+2}} = V_{t+2}^{\rho}\,(1-\beta)\,C_{t+2}^{-\rho}.$$

However, due to the homotheticity of our preferences, we can rely on the fact that both $V_{t,t+2}$ and V_{t+2} are homogeneous of degree one which implies that

$$\frac{dV_{t,t+2}/dW_{t+2}}{dV_{t+2}/dW_{t+2}} = \frac{V_{t,t+2}}{V_{t+2}}.$$

This allows us to derive the two-period SDF $\Pi_{t,t+2}$ and compare it two the sequence of one-period SDFs $\Pi_{t,t+1}\Pi_{t+1,t+2}$:

$$\begin{aligned}
\Pi_{t,t+2} = \beta^2 \left(\frac{C_{t+2}}{C_t}\right)^{-\rho} & \left(\frac{V_{t,t+1}}{E_t\left[V_{t,t+1}^{1-\gamma_1}\right]^{\frac{1}{1-\gamma_1}}}\right)^{\rho-\gamma_1} \\
& \times \left(\frac{V_{t,t+2}}{E_{t+1}\left[V_{t,t+2}^{1-\gamma_2}\right]^{\frac{1}{1-\gamma_2}}}\right)^{\rho-\gamma_2} \left(\frac{V_{t,t+2}}{V_{t+2}}\right)^{1-\rho} \\
\Pi_{t,t+1}\Pi_{t+1,t+2} = \beta^2 \left(\frac{C_{t+2}}{C_t}\right)^{-\rho} & \left(\frac{V_{t,t+1}}{E_t\left[V_{t,t+1}^{1-\gamma_1}\right]^{\frac{1}{1-\gamma_1}}}\right)^{\rho-\gamma_1} \\
& \times \left(\frac{V_{t+1,t+2}}{E_{t+1}\left[V_{t+1,t+2}^{1-\gamma_1}\right]^{\frac{1}{1-\gamma_1}}}\right)^{\rho-\gamma_1} \left(\frac{V_{t,t+1}}{V_{t+1}}\right)^{1-\rho} \left(\frac{V_{t+1,t+2}}{V_{t+2}}\right)^{1-\rho}.
\end{aligned}$$

C Exact solutions for $\rho = 1$

This appendix presents the exact solutions derived for unit elasticity of intertemporal substitution, $1/\rho = 1$, and log-normal uncertainty. Denoting logs by lowercase letters, our general model (13) becomes

$$v_t = (1-\beta)\,c_t + \beta\left(E_t[v_{t,t+1}] + \frac{1}{2}\,(1-\gamma_1)\,\mathrm{var}_t(v_{t,t+1})\right), \tag{16}$$

with the continuation value $v_{t,t+1}$ satisfying the recursion

$$v_{t,t+h} = (1-\beta)\, c_{t+h} + \beta \left(E_{t+1}[v_{t,t+h+1}] + \frac{1}{2}\,(1-\gamma_{h+1})\,\mathrm{var}_{t+1}(v_{t,t+h+1}) \right).$$

C.1 Valuation of risk and temporal resolution

Proof of Lemma 1. Starting at horizon $t+1$, equation (16) corresponds to the standard recursion

$$\tilde{v}_{t+1} = (1-\beta)\, c_{t+1} + \frac{\beta}{1-\tilde{\gamma}} \log(E_{t+1}[\exp\left((1-\tilde{\gamma})\,\tilde{v}_{t+2}\right)])\,.$$

If consumption follows process (4), guess and verify that the solution to the recursion satisfies

$$\tilde{v}_t - c_t = \tilde{\mu}_v + \tilde{\phi}_v x_t.$$

Substituting in and matching coefficients yields

$$\tilde{v}_t - c_t = \frac{\beta}{1-\beta}\mu_c + \frac{\beta\phi_c}{1-\beta\nu_x}x_t + \frac{1}{2}\frac{\beta\,(1-\tilde{\gamma})}{1-\beta}\left(\alpha_c^2 + \left(\frac{\beta\phi_c}{1-\beta\nu_x}\right)^2 \alpha_x^2\right)\sigma^2.$$

From the perspective of period t,

$$v_t = (1-\beta)\, c_t + \frac{\beta}{1-\gamma}\log(E_t[\exp\left((1-\gamma)\,\tilde{v}_{t+1}\right)])$$

and

$$v_t - c_t = \frac{\beta}{1-\beta}\mu_c + \frac{\beta\phi_c}{1-\beta\nu_x}x_t + \frac{1}{2}\frac{\beta}{1-\beta}\left(\alpha_c^2 + \left(\frac{\beta\phi_c}{1-\beta\nu_x}\right)^2 \alpha_x^2\right)\sigma^2\left((1-\gamma) + \beta\,(\gamma-\tilde{\gamma})\right),$$

as stated in the text. □

Proof of Proposition 1. If all risk is resolved at $t+1$, log continuation utility $v^*_{t,t+1}$ is given by

$$\begin{aligned} v^*_{t+1} &= (1-\beta)\, c_{t+1} + \beta\left((1-\beta)\, c_{t+2} + \beta\left((1-\beta)\, c_{t+3} + \cdots\right)\right) \\ &= c_{t+1} + \sum_{h=1}^{\infty} \beta^h \left(c_{t+h+1} - c_{t+h}\right). \end{aligned}$$

From the perspective of period t, this continuation utility is normally distributed with mean and variance given by

$$
\begin{aligned}
E[v^*_{t+1}] &= c_t + \frac{1}{1-\beta}\mu + \frac{\phi_c}{1-\beta\nu_x}x_t, \\
\mathrm{var}(v^*_{t+1}) &= \frac{1}{1-\beta^2}\sigma^2\left(\alpha_c^2 + \left(\frac{\beta\phi_c}{1-\beta\nu_x}\right)^2\alpha_x^2\right).
\end{aligned}
$$

Using these expressions, we can derive the early resolution utility at t as

$$
v^*_t - c_t = \frac{\beta}{1-\beta}\mu_c + \frac{\beta\phi_c}{1-\beta\nu_x}x_t + \frac{1}{2}\frac{\beta(1-\gamma)}{1-\beta^2}\left(\alpha_c^2 + \left(\frac{\beta\phi_c}{1-\beta\nu_x}\right)^2\alpha_x^2\right)\sigma^2.
$$

Subtracting this from the utility v_t under gradual resolution, we arrive at a timing premium given by

$$
\mathrm{TP} = 1 - \exp\left(\frac{1}{2}\frac{\beta^2(1-\gamma)}{1-\beta}\left(\alpha_c^2 + \left(\frac{\beta\phi_c}{1-\beta\nu_x}\right)^2\alpha_x^2\right)\sigma^2\left(\frac{\gamma-\tilde{\gamma}}{1-\gamma} + \frac{1}{1+\beta}\right)\right),
$$

as stated in the text. □

Case with stochastic volatility: If consumption follows process (7) with stochastic volatility, guess and verify that the solution to the recursion for $\tilde{v}_t$ satisfies

$$
\tilde{v}_t - c_t = \tilde{\mu}_v + \phi_v x_t + \tilde{\psi}_v\sigma_t^2
$$

where

$$
\begin{aligned}
\tilde{\mu}_v &= \frac{\beta}{1-\beta}\left(\mu_c + \tilde{\psi}_v\sigma^2(1-\nu_\sigma)\right) \\
\phi_v &= \frac{\beta\phi_c}{1-\beta\nu_x} \\
\tilde{\psi}_v &= \frac{1}{2}\frac{\beta(1-\tilde{\gamma})}{1-\beta\nu_\sigma}\left(\alpha_c^2 + \phi_v^2\alpha_x^2 + (\tilde{\psi}_v)^2\alpha_\sigma^2\right).
\end{aligned}
$$

We then obtain:

$$
v_t - c_t = \tilde{\mu}_v + \phi_v x_t + \tilde{\psi}_v\sigma_t^2\left(1 - (1-\beta\nu_\sigma)\frac{\gamma-\tilde{\gamma}}{1-\tilde{\gamma}}\right)
$$

If all risk is resolved at $t+1$, log continuation utility $v^*_{t,t+1}$ is given by

$$
\begin{aligned}
v^*_{t+1} &= (1-\beta)\, c_{t+1} + \beta \Big((1-\beta)\, c_{t+2} + \beta \left((1-\beta)\, c_{t+3} + \cdots \right) \Big) \\
&= c_{t+1} + \sum_{h=1}^{\infty} \beta^h \left(c_{t+h+1} - c_{t+h} \right).
\end{aligned}
$$

From the perspective of period t, this continuation utility is normally distributed with mean and variance given by

$$
\begin{aligned}
E_t[v^*_{t+1}] &= c_t + \frac{1}{1-\beta}\,\mu + \frac{\phi_c}{1-\beta\nu_x} x_t, \\
\mathrm{var}_t(v^*_{t+1}) &= \frac{1}{1-\beta^2\nu_\sigma} \left(\sigma_t^2 + \frac{\beta^2}{1-\beta^2}\sigma^2 \left(1-\nu_\sigma\right) \right) \left(\alpha_c^2 + \left(\frac{\beta\phi_c}{1-\beta\nu_x} \right)^2 \alpha_x^2 \right).
\end{aligned}
$$

Using these expressions, we can derive the early resolution utility at t as

$$
v^*_t - c_t = \frac{\beta}{1-\beta}\mu_c + \frac{\beta\phi_c}{1-\beta\nu_x} x_t + \frac{1}{2}\frac{\beta\left(1-\gamma\right)}{1-\beta^2\nu_\sigma} \left(\alpha_c^2 + \left(\frac{\beta\phi_c}{1-\beta\nu_x} \right)^2 \alpha_x^2 \right) \left(\sigma_t^2 + \frac{\beta^2}{1-\beta^2}\sigma^2 \left(1-\nu_\sigma\right) \right)
$$

and

$$
\begin{aligned}
v_t - v^*_t = {} & \frac{\beta}{1-\beta}\tilde{\psi}_v \sigma^2 \left(1-\nu_\sigma\right) \left(1 - \frac{1-\gamma}{1-\tilde{\gamma}} \frac{\beta}{1+\beta} \frac{1-\beta\nu_\sigma}{1-\beta^2\nu_\sigma} + \frac{1}{2}\frac{1-\gamma}{1-\beta^2\nu_\sigma}\frac{\beta^2}{1+\beta}\tilde{\psi}_v \alpha_\sigma^2 \right) \\
& + \tilde{\psi}_v \sigma_t^2 \left(1 - \frac{\left(1-\beta\nu_\sigma\right)}{\left(1-\tilde{\gamma}\right)} \left(\gamma - \tilde{\gamma} + \frac{\beta\left(1-\gamma\right)}{1-\beta^2\nu_\sigma} \right) + \frac{1}{2}\frac{\beta\left(1-\gamma\right)}{1-\beta^2\nu_\sigma}\tilde{\psi}_v \alpha_\sigma^2 \right).
\end{aligned}
$$

C.2 Stochastic discount factor

We now specialize to the case of two levels of risk aversion, setting $\gamma_1 = \gamma$ and $\gamma_h = \tilde{\gamma}$ for $h \geq 2$.

Proof of Lemma 2. Under the stochastic process (7), we can guess and verify that

$$
\tilde{v}_{t+1} = c_{t+1} + \frac{\beta\phi_c}{1-\beta\nu_x} x_{t+1} + \frac{\beta}{1-\beta}\mu_c + \psi_v(\tilde{\gamma}) \left(\frac{\beta}{1-\beta}\sigma^2 \left(1-\nu_\sigma\right) + \sigma_{t+1}^2 \right)
$$

where $\psi_v(\tilde{\gamma}) < 0$ is implicitly defined by

$$
\psi_v(\tilde{\gamma}) = \frac{1-\tilde{\gamma}}{2}\frac{\beta}{1-\beta\nu_\sigma} \left(\alpha_c^2 + \frac{\beta^2\phi_c^2\alpha_x^2}{\left(1-\beta\nu_x\right)^2} + \psi_v(\tilde{\gamma})^2\, \alpha_\sigma^2 \right).
$$

We therefore have

$$
\begin{aligned}
E_t[\tilde{v}_{t+1}] &= c_t + \frac{1}{1-\beta\nu_x}\phi_c x_t + \frac{1}{1-\beta}\mu_c + \psi_v(\tilde{\gamma})\left(\frac{1}{1-\beta}\sigma^2\left(1-\nu_\sigma\right) + \nu_\sigma\sigma_t^2\right), \\
\mathrm{var}_t(\tilde{v}_{t+1}) &= \left(\alpha_c^2 + \frac{\beta^2\phi_c^2\alpha_x^2}{\left(1-\beta\nu_x\right)^2} + \psi_v(\tilde{\gamma})^2\,\alpha_\sigma^2\right)\sigma_t^2.
\end{aligned}
$$

Substituting these into (16), we arrive at the solution for v_t:

$$
\begin{aligned}
v_t = c_t &+ \frac{\beta\phi_c}{1-\beta\nu_x}x_t + \frac{\beta}{1-\beta}\mu_c \\
&+ \frac{\beta}{1-\beta}\left(\alpha_c^2 + \frac{\beta^2\phi_c^2\alpha_x^2}{\left(1-\beta\nu_x\right)^2} + \psi_v(\tilde{\gamma})^2\,\alpha_\sigma^2\right) \\
&\qquad \times\left(\beta\frac{1-\tilde{\gamma}}{2}\frac{\sigma^2\left(1-\nu_\sigma\right) + \left(1-\beta\right)\nu_\sigma\sigma_t^2}{1-\beta\nu_\sigma} + \left(1-\beta\right)\frac{1-\gamma}{2}\sigma_t^2\right).
\end{aligned}
$$

Taking the difference $v_t - \tilde{v}_t$ yields the result in the text. □

Proof of Lemma 3. The result follows directly from the expression for $\tilde{v}_{t+1}$ in the proof of Lemma 2. □

Proof of Proposition 4. Using the results of Lemmas 2 and (16), the expression for the SDF follows from equation (8):

$$
\begin{aligned}
\pi_{t,t+1} = &\overbrace{\log\beta - \mu_c - \phi_c x_t - \left(1-\gamma\right)^2\frac{1-\beta\nu_\sigma}{\beta\left(1-\tilde{\gamma}\right)}\psi_v(\tilde{\gamma})\,\sigma_t^2}^{\bar{\pi}_t} \\
&- \gamma\alpha_c\sigma_t W_{c,t+1} + \left(1-\gamma\right)\phi_v\alpha_x\sigma_t W_{x,t+1} \\
&+ \left(1-\gamma\right)\psi_v(\tilde{\gamma})\,\alpha_\sigma\sigma_t W_{\sigma,t+1},
\end{aligned}
$$

The risk-free rate is defined as $r_{f,t} = -\log E_t\left(\Pi_{t,t+1}\right)$ and simplifies to

$$
r_{f,t} = -\log\beta + \mu_c + \phi_c x_t + \left(\frac{1}{2} - \gamma\right)\alpha_c^2\sigma_t^2
$$

as stated in the text. □

C.3 Term structure of returns

C.3.1 General claims

To make the problem as general as possible, we analyze horizon-dependent claims that are priced recursively as

$$Y_{t,h} = E_t\left[\Pi_{t,t+1} G_{y,t+1} Y_{t+1,h-1}\right],$$

that is

$$y_{t,h} = E_t\left[\pi_{t,t+1} + g_{y,t+1} + y_{t+1,h-1}\right] + \frac{1}{2}\mathrm{var}_t\left(\pi_{t,t+1} + g_{y,t+1} + y_{t+1,h-1}\right),$$

where

$$\begin{aligned} g_{y,t+1} &= \mu_y + \phi_y x_t + \psi_y \sigma_t^2 \\ &\qquad + \alpha_{y,c}\alpha_c \sigma_t W_{c,t+1} + \alpha_{y,x}\alpha_x \sigma_t W_{x,t+1} + \alpha_{y,\sigma}\alpha_\sigma \sigma_t W_{\sigma,t+1} + \alpha_{y,d}\alpha_d \sigma_t W_{d,t+1}, \end{aligned}$$

and $Y_{t,0} = 1$.

Guess that

$$Y_{t,h} = \exp\left(\mu_{y,h} + \phi_{y,h} x_t + \psi_{y,h} \sigma_t^2\right).$$

Suppose $h \geq 1$, then:

$$\begin{aligned} &\pi_{t,t+1} + g_{y,t+1} + y_{t+1,h-1} \\ &= \log\beta - \mu_c + \mu_y + \mu_{y,h-1} + \psi_{y,h-1}\left(1-\nu_\sigma\right)\sigma^2 \\ &\qquad + \left(-\phi_c + \phi_y + \phi_{y,h-1}\nu_x\right) x_t + \left(-\left(1-\gamma\right)^2 \frac{1-\beta\nu_\sigma}{\beta\left(1-\tilde{\gamma}\right)}\psi_v(\tilde{\gamma}) + \psi_y + \psi_{y,h-1}\nu_\sigma\right)\sigma_t^2 \\ &\qquad + \left(-\gamma + \alpha_{y,c}\right)\alpha_c\sigma_t W_{c,t+1} + \left(\left(1-\gamma\right)\phi_v + \alpha_{y,x} + \phi_{y,h-1}\right)\alpha_x\sigma_t W_{x,t+1} \\ &\qquad + \left(\left(1-\gamma\right)\psi_v(\tilde{\gamma}) + \alpha_{y,\sigma} + \psi_{y,h-1}\right)\alpha_\sigma\sigma_t W_{\sigma,t+1} + \alpha_{y,d}\alpha_d\sigma_t W_{d,t+1}, \end{aligned}$$

and therefore

$$
\begin{aligned}
y_{t,h} = \log\beta - \mu_c + \mu_y + \mu_{y,h-1} + \psi_{y,h-1}\left(1-\nu_\sigma\right)\sigma^2 + \left(-\phi_c + \phi_y + \phi_{y,h-1}\nu_x\right)x_t \\
+ \Bigg[- \left(1-\gamma\right)^2 \frac{1-\beta\nu_\sigma}{\beta\left(1-\tilde{\gamma}\right)}\psi_v(\tilde{\gamma}) + \psi_y + \psi_{y,h-1}\nu_\sigma \\
+ \frac{1}{2}\Big(\left(-\gamma + \alpha_{y,c}\right)^2 \alpha_c^2 + \left(\left(1-\gamma\right)\phi_v + \alpha_{y,x} + \phi_{y,h-1}\right)^2 \alpha_x^2 \\
+ \left(\left(1-\gamma\right)\psi_v(\tilde{\gamma}) + \alpha_{y,\sigma} + \psi_{y,h-1}\right)^2 \alpha_\sigma^2 + \alpha_{y,d}^2\alpha_d^2 \Big) \Bigg] \sigma_t^2.
\end{aligned}
$$

Matching coefficients, we find the recursions, for $h \geq 1$:

- Constant:

$$
\begin{aligned}
& \mu_{y,h} = \log\beta - \mu_c + \mu_y + \mu_{y,h-1} + \psi_{y,h-1}\left(1-\nu_\sigma\right)\sigma^2 \\
\Rightarrow \quad & \mu_{y,h} = \left(\log\beta - \mu_c + \mu_y\right)h + \left(1-\nu_\sigma\right)\sigma^2 \sum_{i=0}^{h-1} \psi_{y,i}
\end{aligned}
$$

- Terms in x_t:

$$
\begin{aligned}
& \phi_{y,h} = -\phi_c + \phi_y + \phi_{y,h-1}\nu_x \\
\Rightarrow \quad & \phi_{y,h} = \left(-\phi_c + \phi_y\right)\frac{1-\nu_x^h}{1-\nu_x}
\end{aligned}
$$

- Terms in σ_t^2:

$$
\begin{aligned}
\psi_{y,h} = & - \left(1-\gamma\right)^2 \frac{1-\beta\nu_\sigma}{\beta\left(1-\tilde{\gamma}\right)}\psi_v(\tilde{\gamma}) + \psi_y + \psi_{y,h-1}\nu_\sigma \\
& + \frac{1}{2}\Big(\left(-\gamma + \alpha_{y,c}\right)^2 \alpha_c^2 + \left(\left(1-\gamma\right)\phi_v + \alpha_{y,x} + \phi_{y,h-1}\right)^2 \alpha_x^2 \\
& + \left(\left(1-\gamma\right)\psi_v(\tilde{\gamma}) + \alpha_{y,\sigma} + \psi_{y,h-1}\right)^2 \alpha_\sigma^2 + \alpha_{y,d}^2\alpha_d^2 \Big),
\end{aligned}
$$

  which has no simple solution.

Further, we consider one-period holding returns for these claims of the form

$$
\begin{aligned}
1 + R_{t+1,h}^{Y} &= \frac{G_{y,t+1}Y_{t+1,h-1}}{Y_{t,h}} = \frac{G_{y,t+1}Y_{t+1,h-1}}{E_t\left[\Pi_{t,t+1}G_{y,t+1}Y_{t+1,h-1}\right]} \\
&= R_{f,t}\frac{E_t[\Pi_{t,t+1}]\,G_{y,t+1}Y_{t+1,h-1}}{E_t\left[\Pi_{t,t+1}G_{y,t+1}Y_{t+1,h-1}\right]},
\end{aligned}
$$

with the risk-free rate

$$R_{f,t} = \frac{1}{E_t[\Pi_{t,t+1}]}.$$

The conditional Sharpe Ratio is

$$\begin{aligned}
\mathrm{SR}^{Y}_{t,h} &= \frac{E_t\left[1+R^{Y}_{t+1,h}\right]-1}{\sqrt{\mathrm{var}_t\left(1+R^{Y}_{t+1,h}\right)}} \\
&= \frac{E_t\left(1+R^{Y}_{t+1,h}\right)-1}{\sqrt{E_t\left(\left(1+R^{Y}_{t+1,h}\right)^2\right)-\left(E_t\left(1+R^{Y}_{t+1,h}\right)\right)^2}} \\
&\approx \frac{r_{f,t}+\sigma_t^2\begin{cases}\gamma\alpha_{y,c}\alpha_c^2-(1-\gamma)\,\phi_v\left(\alpha_{y,x}+\phi_{y,h-1}\right)\alpha_x^2 \\ -(1-\gamma)\,\tilde{\psi}_v\left(\alpha_{y,\sigma}+\psi_{y,h-1}\right)\alpha_\sigma^2\end{cases}}{\sigma_t\sqrt{\alpha_{y,c}^2\alpha_c^2+\left(\alpha_{y,x}+\phi_{y,h-1}\right)^2\alpha_x^2+\left(\alpha_{y,\sigma}+\psi_{y,h-1}\right)^2\alpha_\sigma^2+\alpha_{y,d}^2\alpha_d^2}}.
\end{aligned}$$

Futures returns $R^{F,Y}_{t+1,h}$ for asset Y at time t and horizon h are of the form

$$\begin{aligned}
R^{F,Y}_{t+1,h}+1 = \frac{1+R^{Y}_{t+1,h}}{1+R^{B}_{t+1,h}} &= \frac{G_{y,t+1}Y_{t+1,h-1}}{Y_{t,h}}\frac{B_{t,h}}{B_{t+1,h-1}} \\
&= \frac{G_{y,t+1}Y_{t+1,h-1}}{E_t\left(\Pi_{t,t+1}G_{y,t+1}Y_{t+1,h-1}\right)}\frac{E_t\left(\Pi_{t,t+1}B_{t+1,h-1}\right)}{B_{t+1,h-1}},
\end{aligned}$$

where $B_{t,h}$ is the price of \$1. Their conditional Sharpe Ratio is

$$\begin{aligned}
\mathrm{SR}^{F,Y}_{t,h} &= \frac{E_t\left(1+R^{F,Y}_{t+1,h}\right)-1}{\sqrt{\mathrm{var}_t\left(1+R^{F,Y}_{t+1,h}\right)}} \\
&= \frac{E_t\left(1+R^{F,Y}_{t+1,h}\right)-1}{\sqrt{E_t\left(\left(1+R^{F,Y}_{t+1,h}\right)^2\right)-\left(E_t\left(1+R^{F,Y}_{t+1,h}\right)\right)^2}} \\
&\approx \sigma_t\frac{\begin{cases}\gamma\alpha_{y,c}\alpha_c^2-\left(\alpha_{y,x}+\phi_{y,h-1}-\phi_{b,h-1}\right)\left((1-\gamma)\,\phi_v+\phi_{b,h-1}\right)\alpha_x^2 \\ -\left(\alpha_{y,\sigma}+\psi_{y,h-1}-\psi_{b,h-1}\right)\left((1-\gamma)\,\tilde{\psi}_v+\psi_{b,h-1}\right)\alpha_\sigma^2\end{cases}}{\sqrt{\alpha_{y,c}^2\alpha_c^2+\left(\alpha_{y,x}+\phi_{y,h-1}-\phi_{b,h-1}\right)^2\alpha_x^2+\left(\alpha_{y,\sigma}+\psi_{y,h-1}-\psi_{b,h-1}\right)^2\alpha_\sigma^2+\alpha_{y,d}^2\alpha_d^2}}.
\end{aligned}$$

For the *unconditional* Sharpe ratio observe that the volatility process

$$\sigma_{t+1}^2 - \sigma^2 = \nu_\sigma \left(\sigma_t^2 - \sigma^2\right) + \alpha_\sigma \sigma_t W_{t+1}$$

is a square-root Feller process, which is stationary under the constraint $\nu_\sigma < 1 - \frac{\alpha_\sigma^2}{2\sigma^2}$ with a stationary distribution

$$\sigma_t^2 = \frac{\alpha_\sigma^2}{4\left(1-\nu_\sigma\right)} x$$

where

$$x \sim \chi^2 \left(\frac{4\left(1-\nu_\sigma\right)\sigma^2}{\alpha_\sigma^2}\right)$$

The moment generating function is

$$\begin{aligned}
E\left(\exp\left(a\sigma_t^2\right)\right) &= E\left(\exp\left(a\frac{\alpha_\sigma^2}{4\left(1-\nu_\sigma\right)}x\right)\right) \\
&= \exp\left(\sum_{h=1}^{\infty} k_h \frac{\left(a\frac{\alpha_\sigma^2}{4(1-\nu_\sigma)}\right)^h}{h!}\right),
\end{aligned}$$

where $k_h = 2^{h-1}\left(h-1\right)!\left(\frac{4(1-\nu_\sigma)\sigma^2}{\alpha_\sigma^2}\right)$. For $a\frac{\alpha_\sigma^2}{2(1-\nu_\sigma)} < 1$, which is the case in the calibration, we therefore have

$$\begin{aligned}
E\left(\exp\left(a\sigma_t^2\right)\right) &= \exp\left(\left(\frac{2\left(1-\nu_\sigma\right)\sigma^2}{\alpha_\sigma^2}\right)\sum_{h=1}^{\infty}\frac{\left(a\frac{\alpha_\sigma^2}{2(1-\nu_\sigma)}\right)^h}{h}\right) \\
&= \exp\left(-\left(\frac{2\left(1-\nu_\sigma\right)\sigma^2}{\alpha_\sigma^2}\right)\log\left(1-a\frac{\alpha_\sigma^2}{2\left(1-\nu_\sigma\right)}\right)\right).
\end{aligned}$$

Note that $a\frac{\alpha_\sigma^2}{2(1-\nu_\sigma)} < a\sigma^2 \lll 1$ and therefore $E\left(\exp\left(a\sigma_t^2\right)\right) \approx \exp\left(a\sigma^2\right)$. The unconditional Sharpe Ratio is therefore

$$
\begin{aligned}
\mathrm{SR}_{t,h}^{F,Y} &= \frac{E\left(1+R_{t+1,h}^{F,Y}\right)-1}{\operatorname{var}\left(1+R_{t+1,h}^{F,Y}\right)} \\
&= \frac{EE_t\left(1+R_{t+1,h}^{F,Y}\right)-1}{\sqrt{EE_t\left(\left(1+R_{t+1,h}^{F,Y}\right)^2\right)-\left(EE_t\left(1+R_{t+1,h}^{F,Y}\right)\right)^2}} \\
&\approx -\sigma\frac{\begin{cases}\gamma\alpha_{y,c}\alpha_c^2-\left(\alpha_{y,x}+\phi_{y,h-1}-\phi_{b,h-1}\right)\left(\left(1-\gamma\right)\phi_v+\phi_{b,h-1}\right)\alpha_x^2 \\ -\left(\alpha_{y,\sigma}+\psi_{y,h-1}-\psi_{b,h-1}\right)\left(\left(1-\gamma\right)\tilde{\psi}_v+\psi_{b,h-1}\right)\alpha_\sigma^2\end{cases}}{\sqrt{\alpha_{y,c}^2\alpha_c^2+\left(\alpha_{y,x}+\phi_{y,h-1}-\phi_{b,h-1}\right)^2\alpha_x^2+\left(\alpha_{y,\sigma}+\psi_{y,h-1}-\psi_{b,h-1}\right)^2\alpha_\sigma^2+\alpha_{y,d}^2\alpha_d^2}}.
\end{aligned}
$$

C.3.2 Bonds

Bond prices: Let the price at time t for \$1 in h periods be $B_{t,h}$ with $B_{t,0}=1$. For $h \geq 1$, we have

$$B_{t,h} = E_t[\Pi_{t,t+1}B_{t+1,h-1}]$$

This is the general problem from above with $g_{y,t+1}=0$ for all t and therefore

$$b_{t,h} = \mu_{b,h} + \phi_{b,h}x_t + \psi_{b,h}\sigma_t^2,$$

with

$$
\begin{aligned}
\phi_{b,h} &= -\phi_c\frac{1-\nu_x^h}{1-\nu_x} \\
\psi_{b,h} &= -\left(1-\gamma\right)^2\frac{1-\beta\nu_\sigma}{\beta\left(1-\tilde{\gamma}\right)}\tilde{\psi}_v+\psi_{b,h-1}\nu_\sigma \\
&\quad+\frac{1}{2}\begin{cases}\gamma^2\alpha_c^2+\left(\left(1-\gamma\right)\phi_v+\phi_{b,h-1}\right)^2\alpha_x^2 \\ +\left(\left(1-\gamma\right)\tilde{\psi}_v+\psi_{b,h-1}\right)^2\alpha_\sigma^2\end{cases} \\
\psi_{b,1} &= \left(\gamma-\frac{1}{2}\right)\alpha_c^2 > 0
\end{aligned}
$$

and $\psi_{b,h} > 0$ for all h, and $\psi_{b,h}$ increasing in h. Further,

$$\mu_{b,h}-\mu_{b,h-1} = \log\beta-\mu_c+\sigma^2\left(1-\nu_\sigma\right)\psi_{b,h-1},$$

and thus the solution, for $h \geq 1$ is:

$$\mu_{b,h} = h\left(\log\beta - \mu_c\right) + \sigma^2\left(1 - \nu_\sigma\right)\sum_{i=0}^{h-1}\psi_{b,i}.$$

Bond returns and Sharpe ratios: The one-period returns are given by:

$$R^B_{t+1,h} = \frac{B_{t+1,h-1}}{B_{t,h}} - 1$$

and therefore

$$\begin{aligned}\log\left(R^B_{t+1,h} + 1\right) = & -\log\beta + \mu_c + \phi_c x_t + \left(\psi_{b,h-1}\nu_\sigma - \psi_{b,h}\right)\sigma_t^2 \\ & + \psi_{b,h-1}\alpha_\sigma\sigma_t W_{t+1} + \phi_{b,h-1}\alpha_x\sigma_t W_{t+1}.\end{aligned}$$

The term structure of conditional Sharpe ratios

$$\mathrm{SR}_t\left(R^B_{t+1,h}\right) \approx \frac{-\log\beta + \mu_c + \phi_c x_t + \left(\psi_{b,h-1}\nu_\sigma - \psi_{b,h} + \frac{1}{2}\left(\psi_{b,h-1}\right)^2\alpha_\sigma^2 + \frac{1}{2}\left(\phi_{b,h-1}\right)^2\alpha_x^2\right)\sigma_t^2}{\sqrt{\left(\left(\psi_{b,h-1}\right)^2\alpha_\sigma^2 + \left(\phi_{b,h-1}\right)^2\alpha_x^2\right)\sigma_t^2}}.$$

For the unconditional Sharpe ratio, recall that

$$E\left(\exp\left(a\sigma_t^2\right)\right) \approx \exp\left(a\sigma^2\right).$$

Further, we have

$$x_{t+1} = \nu_x x_t + \alpha_x\sigma_t W_{t+1}$$

an AR1 process, stationary under the constraint $\nu_x < 1$, with distribution

$$x_t \sim \mathcal{N}\left(0, \Sigma_x\right),$$

where

$$\Sigma_x = \frac{\alpha_x^2}{1 - \nu_x^2}\sigma^2,$$

and

$$E\left(\exp\left(a x_t\right)\right) \approx \exp\left(\frac{1}{2}a^2\Sigma_x\right).$$

Since x_t and σ_t have independent stationary distributions, we obtain:

$$\mathrm{SR}\left(R^B_{t+1,h}\right) \approx \frac{-\log\beta+\mu_c+\frac{1}{2}\phi_c^2\Sigma_x+\left(\psi_{b,h-1}\nu_\sigma-\psi_{b,h}+\frac{1}{2}\left(\psi_{b,h-1}\right)^2\alpha_\sigma^2+\frac{1}{2}\left(\phi_{b,h-1}\right)^2\alpha_x^2\right)\sigma^2}{\sqrt{\phi_c^2\Sigma_x+\left(\left(\psi_{b,h-1}\right)^2\alpha_\sigma^2+\left(\phi_{b,h-1}\right)^2\alpha_x^2\right)\sigma^2}}.$$

The effect of horizon h enters through the term in the numerator

$$\begin{aligned}
&\psi_{b,h-1}\nu_\sigma-\psi_{b,h}+\frac{1}{2}\left(\psi_{b,h-1}\right)^2\alpha_\sigma^2+\frac{1}{2}\left(\phi_{b,h-1}\right)^2\alpha_x^2\\
&\quad=(1-\gamma)^2\frac{1-\beta\nu_\sigma}{\beta\left(1-\tilde{\gamma}\right)}\tilde{\psi}_v-\frac{1}{2}\left(\gamma^2\alpha_c^2+\left((1-\gamma)\phi_v\right)^2\alpha_x^2+\left((1-\gamma)\tilde{\psi}_v\right)^2\alpha_\sigma^2\right)\\
&\qquad-(1-\gamma)\phi_v\phi_{b,h-1}\alpha_x^2-(1-\gamma)\tilde{\psi}_v\psi_{b,h-1}\alpha_\sigma^2,
\end{aligned}$$

with

$$\begin{aligned}
&\frac{\partial}{\partial h}\left(\psi_{b,h-1}\nu_\sigma-\psi_{b,h}+\frac{1}{2}\left(\psi_{b,h-1}\right)^2\alpha_\sigma^2+\frac{1}{2}\left(\phi_{b,h-1}\right)^2\alpha_x^2\right)\\
&\quad=-(1-\gamma)\phi_v\underbrace{\frac{\partial\phi_{b,h-1}}{\partial h}}_{>0}\alpha_x^2-(1-\gamma)\tilde{\psi}_v\underbrace{\frac{\partial\psi_{b,h-1}}{\partial h}}_{>0}\alpha_\sigma^2.
\end{aligned}$$

Risk-free rate: The risk-free rate is given by

$$\begin{aligned}
r_{f,t}&=-\log B_{t,1}\\
&=-\log\beta+\mu_c+\phi_c x_t+(1-\gamma)^2\frac{1-\beta\nu_\sigma}{\beta\left(1-\tilde{\gamma}\right)}\tilde{\psi}_v^1\sigma_t^2\\
&\quad-\frac{1}{2}\sigma_t^2\begin{cases}\gamma^2\alpha_c^2+\left((1-\gamma)\frac{\beta\phi_c}{1-\beta\nu_x}\right)^2\alpha_x^2\\+\left((1-\gamma)\tilde{\psi}_v^1\right)^2\alpha_\sigma^2\end{cases},
\end{aligned}$$

where

$$\tilde{\psi}_v=\frac{\frac{1-\beta\nu_\sigma}{\beta(1-\tilde{\gamma})}+\sqrt{\left(\frac{1-\beta\nu_\sigma}{\beta(1-\tilde{\gamma})}\right)^2-\alpha_\sigma^2\left(\alpha_c^2+\left(\frac{\beta\phi_c}{1-\beta\nu_x}\right)^2\alpha_x^2\right)}}{\alpha_\sigma^2}.$$

We thus have

$$E\left(R_{f,t}\right) = \exp\left(-\mu_{b,1} + \frac{1}{2}\left(\phi_c\right)^2 \Sigma_x - \psi_{b,1}\sigma^2\right)$$
$$\text{var}\left(R_{f,t}\right) = \exp\left(-\mu_{b,1} + \frac{1}{2}\left(\phi_c\right)^2 \Sigma_x - \psi_{b,1}\sigma^2\right)\phi_c\sqrt{\Sigma_x}.$$

C.3.3 Dividend strips

Let the price at time t for the full dividend D_{t+h} in h periods be $P_{t,h}$ with $P_{t,0} = D_t$. Then for $h \geq 1$:

$$\frac{P_{t,h}}{D_t} = E_t\left(\Pi_{t,t+1}\frac{D_{t+1}}{D_t}\frac{P_{t+1,h-1}}{D_{t+1}}\right),$$

which is the general problem from above with

$$g_{p,t+1} = d_{t+1} - d_t = \mu_d + \phi_d x_t + \chi\alpha_c\sigma_t W_{t+1} + \alpha_d\sigma_t W_{t+1},$$

for all t and therefore

$$p_{t,h} - d_t = \mu_{p,h} + \phi_{p,h}x_t + \psi_{p,h}\sigma_t^2,$$

with

$$\begin{aligned}
\phi_{p,h} &= \left(-\phi_c + \phi_d\right)\frac{1-\nu_x^h}{1-\nu_x} \\
\psi_{p,h} &= -\left(1-\gamma\right)^2\frac{1-\beta\nu_\sigma}{\beta\left(1-\tilde{\gamma}\right)}\tilde{\psi}_v + \psi_{p,h-1}\nu_\sigma \\
&\quad + \frac{1}{2}\begin{cases}\left(-\gamma+\chi\right)^2\alpha_c^2 + \left(\left(1-\gamma\right)\phi_v + \phi_{p,h-1}\right)^2\alpha_x^2 \\ + \left(\left(1-\gamma\right)\tilde{\psi}_v + \psi_{p,h-1}\right)^2\alpha_\sigma^2 + \alpha_d^2\end{cases} \\
\frac{1-\beta\nu_\sigma}{\beta\left(1-\tilde{\gamma}\right)}\tilde{\psi}_v &= \frac{1}{2}\left(\alpha_c^2 + \left(\phi_v\right)^2\alpha_x^2 + \left(\tilde{\psi}_v\right)^2\alpha_\sigma^2\right) < 0 \\
\psi_{p,1} &= \frac{1}{2}\alpha_d^2 + \left(\chi+1-2\gamma\right)\left(\chi-1\right)\frac{1}{2}\alpha_c^2,
\end{aligned}$$

where the sign depends on the parameters of the model. Further,

$$\mu_{p,h} - \mu_{p,h-1} = \log\beta - \mu_c + \mu_d + \sigma^2\left(1-\nu_\sigma\right)\psi_{p,h-1}.$$

For the dividend strips, the spot one-period returns are given by

$$R_{t+1,h}^P + 1 = \frac{P_{t+1,h-1}/D_{t+1}}{P_{t,h}/D_t}\frac{D_{t+1}}{D_t},$$

and the future one-period returns are given by

$$R_{t+1,h}^{F,P} + 1 = \frac{1 + R_{t+1,h}^{P}}{1 + R_{t+1,h}^{B}}.$$

The conditional expected future one-period returns are

$$E_t\left(R_{t+1,h}^{F,P}\right) \approx \sigma_t^2 \begin{cases} \gamma\chi\alpha_c^2 - \left(\phi_{p,h-1} - \phi_{b,h-1}\right)\left(\left(1-\gamma\right)\phi_v + \phi_{b,h-1}\right)\alpha_x^2 \\ - \left(\psi_{p,h-1} - \psi_{b,h-1}\right)\left(\left(1-\gamma\right)\tilde{\psi}_v + \psi_{b,h-1}\right)\alpha_\sigma^2 \end{cases},$$

and the conditional Sharpe ratio term structure is given by

$$\mathrm{SR}_{t,h}^{F,P} \approx \sigma_t \frac{\begin{cases} \gamma\chi\alpha_c^2 - \left(\phi_{p,h-1} - \phi_{b,h-1}\right)\left(\left(1-\gamma\right)\phi_v + \phi_{b,h-1}\right)\alpha_x^2 \\ - \left(\psi_{p,h-1} - \psi_{b,h-1}\right)\left(\left(1-\gamma\right)\tilde{\psi}_v + \psi_{b,h-1}\right)\alpha_\sigma^2 \end{cases}}{\sqrt{\chi^2\alpha_c^2 + \left(\phi_{p,h-1} - \phi_{b,h-1}\right)^2\alpha_x^2 + \left(\psi_{p,h-1} - \psi_{b,h-1}\right)^2\alpha_\sigma^2 + \alpha_d^2}},$$

where we can replace σ_t by σ for the unconditional one.

C.3.4 Variance swaps

Under log-linearization, the log-linear normal returns assumption is consistent with the lognormal consumption and dividend growth model. Indeed, under log-linearization of the index returns, we have

$$r_{t+1} = \Delta d_{t+1} + k_1^1 z_{t+1} + k_0^1 - z_t,$$

where $z_t = p_t - d_t$, the price dividend ratio, and $k_1^1 = \frac{e^{\bar{z}}}{1+e^{\bar{z}}}$, $k_0^1 = \log\left(1 + e^{\bar{z}}\right) - k_1^1\bar{z}$.

From $E_t\left(\Pi_{t,t+1}R_{t+1}\right) = 1$, we get

$$z_t = \mu_{pd} + \phi_{pd}x_t + \psi_{pd}\sigma_t^2$$

with

$$
\begin{aligned}
\phi_{pd} &= \frac{-\phi_c + \phi_d}{1 - k_1 \nu_x} \\
\psi_{pd}\left(1 - k_1 \nu_\sigma\right) &= -\left(1-\gamma\right)^2 \frac{1 - \beta\nu_\sigma}{\beta\left(1-\tilde{\gamma}\right)} \tilde{\psi}_v \\
&\quad + \frac{1}{2} \begin{cases} \left(-\gamma + \chi\right)^2 \alpha_c^2 + \left(\left(1-\gamma\right)\phi_v + k_1 \phi_{pd}\right)^2 \alpha_x^2 \\ + \left(\left(1-\gamma\right)\tilde{\psi}_v + k_1 \psi_{pd}\right)^2 \alpha_\sigma^2 + \alpha_d^2 \end{cases} \\
\mu_{pd}\left(1 - k_1\right) &= k_0 + \log\beta - \mu_c + \mu_d + \sigma^2\left(1 - \nu_\sigma\right) k_1 \psi_{pd}
\end{aligned}
$$

and k_0 and k_1 are determined by the fixed-point problem $\bar{z} = \mu_{pd} + \psi_{pd}\sigma^2$.

Under this approximation, returns on the index follow the evolution

$$
r_{t,t+1} = \mu_{r,t} + p_c \alpha_c \sigma_t W_{t+1} + p_x \alpha_x \sigma_t W_{t+1} + p_\sigma \alpha_\sigma \sigma_t W_{t+1} + p_d \alpha_d \sigma_t W_{t+1},
$$

where p_c, p_x, p_σ, p_d are the prices of risk and the h-month zero-coupon variance claim is a claim to the variance[15]

$$
\left| p_c \alpha_c + p_x \alpha_x + p_\sigma \alpha_\sigma + p_d \alpha_d \right|^2 \sigma_{t+h}^2 = p_r^2 \sigma_{t+h}^2.
$$

For the spot one-period returns on variance swaps, p_r is irrelevant. Let's price the returns of payoffs σ_{t+h}^2 with price $P\sigma_{t,h}$. We show the solution is of the form

$$
P\sigma_{t,h} = \left(\mu_{\sigma,h}\sigma^2 + \psi_{\sigma,h}\sigma_t^2\right) B_{t,h},
$$

with $\mu_{\sigma,0} = 0$, $\psi_{\sigma,0} = 1$ and

$$
\begin{aligned}
\psi_{\sigma,h} &= \psi_{\sigma,h-1}\left(\nu_\sigma + \left(\left(1-\gamma\right)\tilde{\psi}_v + \psi_{b,h-1}\right)\alpha_\sigma^2\right) \\
\mu_{\sigma,h} &= \left(\mu_{\sigma,h-1} + \psi_{\sigma,h-1}\left(1 - \nu_\sigma\right)\right)
\end{aligned}
$$

The spot one-period returns, on which we have empirical evidence, are given by

$$
R\sigma_{t+1,h}^S + 1 = \frac{P\sigma_{t+1,h-1}}{P\sigma_{t,h}}.
$$

[15]To validate the lognormal assumption, we simulate the returns $R_{t+1} = \frac{P_{t+1}+D_{t+1}}{P_t}$, where $\frac{P_t}{D_t} = \sum_1^{+\infty} \frac{P_{t,h}}{D_t}$, for various levels of σ_t, and verify the correlation between $\sigma^2\left(R_{t+1}\right)$ and σ_t^2 is close to 1.

We therefore have

$$R\sigma^S_{t+1,h} + 1 = \frac{\left(\mu_{\sigma,h}\sigma^2 + \psi_{\sigma,h-1}\nu_\sigma\sigma_t^2 + \psi_{\sigma,h-1}\alpha_\sigma\sigma_t W_{t+1}\right) B_{t+1,h-1}}{\left(\mu_{\sigma,h}\sigma^2 + \psi_{\sigma,h}\sigma_t^2\right) B_{t,h}},$$

and so

$$E_t\left(R\sigma^S_{t+1,h} + 1\right) = \frac{\left(\mu_{\sigma,h}\sigma^2 + \psi_{\sigma,h-1}\left(\nu_\sigma + \psi_{b,h-1}\alpha_\sigma^2\right)\sigma_t^2\right)}{\left(\mu_{\sigma,h}\sigma^2 + \psi_{\sigma,h}\sigma_t^2\right)} E_t\left(R^B_{t+1,h} + 1\right).$$

We simulate to obtain the unconditional moments and the sharpe ratios.

D Approximation for $\beta \approx 1$

As in Appendix C, consider the simplified model with only two levels of risk aversion:

$$V_t = \left[(1-\beta)C_t^{1-\rho} + \beta\left(\mathcal{R}_{t,\gamma}\left(\widetilde{V}_{t+1}\right)\right)^{1-\rho}\right]^{\frac{1}{1-\rho}},$$
$$\tilde{V}_t = \left[(1-\beta)C_t^{1-\rho} + \beta\left(\mathcal{R}_{t,\tilde{\gamma}}\left(\widetilde{V}_{t+1}\right)\right)^{1-\rho}\right]^{\frac{1}{1-\rho}},$$

where

$$\mathcal{R}_{t,\lambda}(X) = \left(E_t\left(X^{1-\lambda}\right)\right)^{\frac{1}{1-\lambda}}.$$

Also, as in Appendix C, take the evolutions:

$$c_{t+1} - c_t = \mu + \phi_c x_t + \alpha_c\sigma_t W_{t+1},$$
$$x_{t+1} = \nu_x x_t + \alpha_x \sigma_t W_{t+1},$$
$$\sigma^2_{t+1} - \sigma^2 = \nu_\sigma\left(\sigma_t^2 - \sigma^2\right) + \alpha_\sigma\sigma_t W_{t+1},$$

and suppose the three shocks are independent. (We can relax this assumption.)

For β close to 1, we have:

$$\left(\frac{\tilde{V}_t}{C_t}\right)^{1-\tilde{\gamma}} \approx \beta^{\frac{1-\tilde{\gamma}}{1-\rho}} E_t\left[\left(\frac{\tilde{V}_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{1-\tilde{\gamma}}\right].$$

This is an eigenfunction problem with eigenvalue $\beta^{-\frac{1-\tilde{\gamma}}{1-\rho}}$ and eigenfunction $\left(\tilde{V}/C\right)^{1-\tilde{\gamma}}$

known up to a multiplier. Let's assume:

$$\tilde{v}_t - c_t = \mu_v + \phi_v x_t + \psi_v \sigma_t^2.$$

Then we have:

- Terms in x_t (standard formula with $\beta = 1$):

$$\phi_v = \phi_c \left(I - \nu_x\right)^{-1}$$

- Terms in σ_t^2:

$$\psi_v = \frac{1}{2}\frac{1-\tilde{\gamma}}{1-\nu_\sigma}\left(\alpha_c^2 + \phi_v^2\alpha_x^2 + \psi_v^2\alpha_\sigma^2\right) < 0$$

- Constant terms:

$$\log\beta = -\left(1-\rho\right)\left(\mu + \psi_v\sigma^2\left(1-\nu_\sigma\right)\right)$$

For β close to 1, we have:

$$\frac{V_t}{\tilde{V}_t} \approx \frac{\mathcal{R}_{t,\gamma}\left(\tilde{V}_{t+1}\right)}{\mathcal{R}_{t,\tilde{\gamma}}\left(\tilde{V}_{t+1}\right)} = \frac{\left(E_t\left[\left(\frac{\tilde{V}_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}}{\left(E_t\left[\left(\frac{\tilde{V}_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{1-\tilde{\gamma}}\right]\right)^{\frac{1}{1-\tilde{\gamma}}}},$$

and therefore:

$$\begin{aligned}
& v_t - \tilde{v}_t = -\frac{1}{2}\left(\gamma - \tilde{\gamma}\right)\left(\alpha_c^2 + \phi_v^2\alpha_x^2 + \psi_v^2\alpha_\sigma^2\right)\sigma_t^2, \\
\Rightarrow \quad & v_t - \tilde{v}_t = -\left(\gamma - \tilde{\gamma}\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma_t^2 < 0.
\end{aligned}$$

The stochastic discount factor becomes:

$$\begin{aligned}
\pi_{t,t+1} = \bar{\pi}_t - \gamma\alpha_c\sigma_t W_{t+1} + \left(\rho - \gamma\right)\phi_v\alpha_x\sigma_t W_{t+1} \\
+ \left(\left(\rho - \gamma\right) + \left(1-\rho\right)\left(\gamma - \tilde{\gamma}\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v\alpha_\sigma\sigma_t W_{t+1},
\end{aligned}$$

where

$$\bar{\pi}_t = -\mu - \rho\phi_c x_t - (1-\rho)\,\psi_v\sigma^2\,(1-\nu_\sigma)\left(1 - (\gamma-\tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)$$
$$-\left((\rho-\gamma)\,(1-\gamma) - (1-\rho)\,(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma_t^2.$$

Observe that in all the analysis the impact and the pricing of the state variable x_t is unaffected by the horizon-dependent model. We can therefore simplify the analysis by setting $x_t = 0$ for all t. Going forward, take the evolutions:

$$c_{t+1} - c_t = \mu + \alpha_c\sigma_t W_{t+1},$$
$$\sigma_{t+1}^2 - \sigma^2 = \nu_\sigma\left(\sigma_t^2 - \sigma^2\right) + \alpha_\sigma\sigma_t W_{t+1},$$

and suppose the two shocks are independent.

We have

$$\tilde{v}_t - c_t = \mu_v + \psi_v\sigma_t^2,$$

where

$$\psi_v = \frac{1}{2}\frac{(1-\tilde{\gamma})}{1-\nu_\sigma}\left(\alpha_c^2 + \psi_v^2\alpha_\sigma^2\right) < 0,$$

and

$$\log\beta = -(1-\rho)\left(\mu + \psi_v\sigma^2\,(1-\nu_\sigma)\right).$$

As before, we have

$$v_t - \tilde{v}_t = -(\gamma-\tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma_t^2 < 0,$$

as well as

$$v_t - c_t = \mu_v + \psi_v\sigma_t^2\left(1 - (\gamma-\tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right).$$

The stochastic discount factor becomes:

$$\pi_{t,t+1} = \bar{\pi}_t - \gamma\alpha_c\sigma_t W_{t+1} + \left((\rho-\gamma) + (1-\rho)\,(\gamma-\tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v\alpha_\sigma\sigma_t W_{t+1},$$

where

$$\bar{\pi}_t = -\mu - (1-\gamma)^2 \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \psi_v \sigma^2$$
$$- \left((1-\gamma)^2 - (1-\rho)\left(1-\gamma + (\gamma - \tilde{\gamma})\,\nu_\sigma\right) \right) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \psi_v \left(\sigma_t^2 - \sigma^2\right).$$

Let the period-t price for the endowment consumption in h periods be $P_{t,h}$. For $h = 0$, we have $P_{t,0} = C_t$. For $h \geq 1$ we have:

$$\frac{P_{t,h}}{C_t} = E_t \left(\Pi_{t,t+1} \frac{C_{t+1}}{C_t} \frac{P_{t+1,h-1}}{C_{t+1}} \right).$$

We can guess that

$$\frac{P_{t,h}}{C_t} = \exp\left(a_h + A_h \sigma_t^2\right),$$

with $a_0 = 0$ and $A_0 = 0$. Suppose $h \geq 1$, then:

$$\begin{aligned}
\log \Pi_{t,t+1} \frac{C_{t+1}}{C_t} \frac{P_{t+1,h-1}}{C_{t+1}} = & -(1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \psi_v \sigma^2 \\
& - \left((1-\gamma)^2 - (1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right) \right) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \psi_v \sigma_t^2 \\
& + a_{h-1} + A_{h-1}\sigma^2 (1-\nu_\sigma) + A_{h-1}\nu_\sigma \sigma_t^2 \\
& + (1-\gamma)\,\alpha_c \sigma_t W_{t+1} \\
& + \left(\left((\rho-\gamma) + (1-\rho)(\gamma-\tilde{\gamma}) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \right) \psi_v + A_{h-1} \right) \alpha_\sigma \sigma_t W_{t+1}.
\end{aligned}$$

We find the recursion

$$\begin{aligned}
A_h = & -\left((1-\gamma)^2 - (1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right) \right) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \psi_v + A_{h-1}\nu_\sigma \\
& + \frac{1}{2} \left(\left(\rho - \gamma + (1-\rho)(\gamma-\tilde{\gamma}) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \right) \psi_v + A_{h-1} \right)^2 \alpha_\sigma^2 + \frac{1}{2}(1-\gamma)^2 \alpha_c^2,
\end{aligned}$$

and

$$a_h = -(1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \psi_v \sigma^2 + a_{h-1} + A_{h-1}\sigma^2 (1-\nu_\sigma).$$

The one-period excess returns on the dividend strips are given by:

$$R^h_{t+1} = \frac{P_{t+1,h-1} - P_{t,h}}{P_{t,h}} = \frac{\frac{P_{t+1,h-1}}{C_{t+1}}\frac{C_{t+1}}{C_t}}{\frac{P_{t,h}}{C_t}} - 1.$$

We have:

$$\begin{aligned}
\log\left(R^h_{t+1}+1\right) = {} & \overbrace{\mu + (1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma^2}^{\equiv\bar{r}} \\
& + \left(A_{h-1}\nu_\sigma - A_h\right)\sigma_t^2 + \left(\alpha_c + A_{h-1}\alpha_\sigma\right)\sigma_t W_{t+1}.
\end{aligned}$$

So the conditional Sharpe ratio term structure is given by:

$$\begin{aligned}
\mathrm{SR}_t\left(R^h_{t+1}\right) &= \frac{\exp\left(\bar{r} + \left(A_{h-1}\nu_\sigma - A_h + \frac{1}{2}\left(\alpha_c^2 + A_{h-1}^2\alpha_\sigma^2\right)\right)\sigma_t^2\right) - 1}{\sqrt{\begin{cases}\exp\left(2\bar{r} + 2\left(A_{h-1}\nu_\sigma - A_h + \left(\alpha_c^2 + A_{h-1}^2\alpha_\sigma^2\right)\right)\sigma_t^2\right) \\ -\left[\exp\left(\bar{r} + \left(A_{h-1}\nu_\sigma - A_h + \frac{1}{2}\left(\alpha_c^2 + A_{h-1}^2\alpha_\sigma^2\right)\right)\sigma_t^2\right)\right]^2\end{cases}}} \\
&= \frac{1 - \exp\left[-\left(\bar{r} + \left(A_{h-1}\nu_\sigma - A_h + \frac{1}{2}\left(\alpha_c^2 + A_{h-1}^2\alpha_\sigma^2\right)\right)\sigma_t^2\right)\right]}{\sqrt{\exp\left(\left(\alpha_c^2 + A_{h-1}^2\alpha_\sigma^2\right)\sigma_t^2\right) - 1}}.
\end{aligned}$$

Observe that:

$$\begin{aligned}
& A_{h-1}\nu_\sigma - A_h + \frac{1}{2}\left(\alpha_c^2 + A_{h-1}\alpha_\sigma^2\right) \\
&\quad = \frac{(1-\rho)}{1-\tilde{\gamma}}\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\left(\rho-\gamma+(\gamma-\tilde{\gamma})\left((1-\rho)\frac{1-\nu_\sigma}{1-\tilde{\gamma}} - 1\right)\right)\frac{1}{2}\psi_v^2\alpha_\sigma^2 \\
&\qquad + \left(1-(1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\right)\frac{1}{2}\alpha_c^2 \\
&\qquad - \left(\rho-\gamma+(1-\rho)(\gamma-\tilde{\gamma})\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v A_{h-1}\alpha_\sigma^2,
\end{aligned}$$

which we can re-write as:

$$A_{h-1}\nu_\sigma - A_h + \frac{1}{2}\left(\alpha_c^2 + A_{h-1}^2\alpha_\sigma^2\right) = A - \left(\rho-\gamma+(1-\rho)(\gamma-\tilde{\gamma})\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v A_{h-1}\alpha_\sigma^2,$$

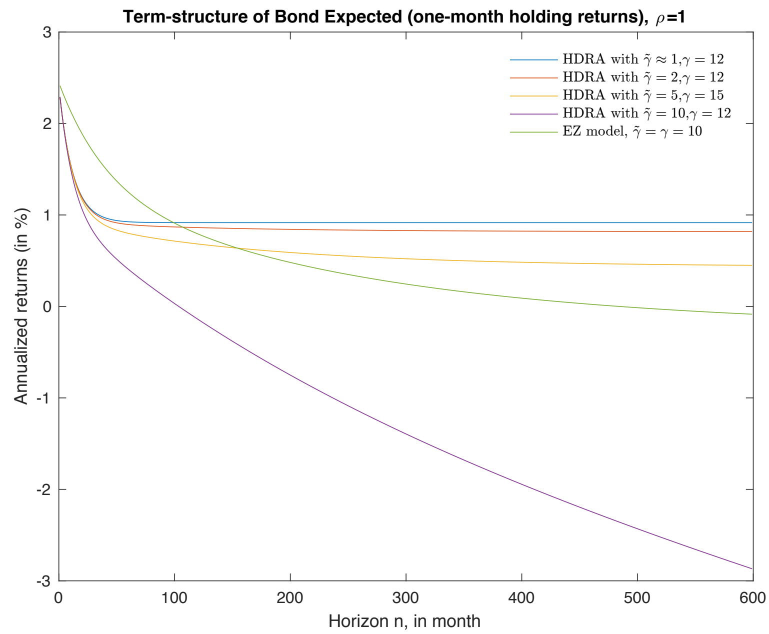

Figure 5: Calibrated term structure of risk-free bond returns under horizon-dependent risk aversion (HDRA). The Epstein-Zin curve (EZ) is done with the calibration of Bansal et al. (2014).

where

$$A = \frac{(1-\rho)}{1-\tilde{\gamma}}\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\left(\rho-\gamma+(\gamma-\tilde{\gamma})\left((1-\rho)\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}-1\right)\right)\frac{1}{2}\psi_v^2\alpha_\sigma^2 + \left(1-(1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\right)\frac{1}{2}\alpha_c^2.$$

We therefore have:

$$\mathrm{SR}_t\left(R_{t+1}^h\right) = \frac{1-\exp\left[-\left(\bar{r}+A\sigma_t^2-\left(\rho-\gamma+(1-\rho)\,(\gamma-\tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v A_{h-1}\alpha_\sigma^2\sigma_t^2\right)\right]}{\sqrt{\exp\left(\left(\alpha_c^2+A_{h-1}^2\alpha_\sigma^2\right)\sigma_t^2\right)-1}}.$$

E Additional figures

Figures 5–7 present the term structures of Figures 2–4 for various other combinations of immediate risk aversion γ and delayed risk aversion $\tilde{\gamma}$.

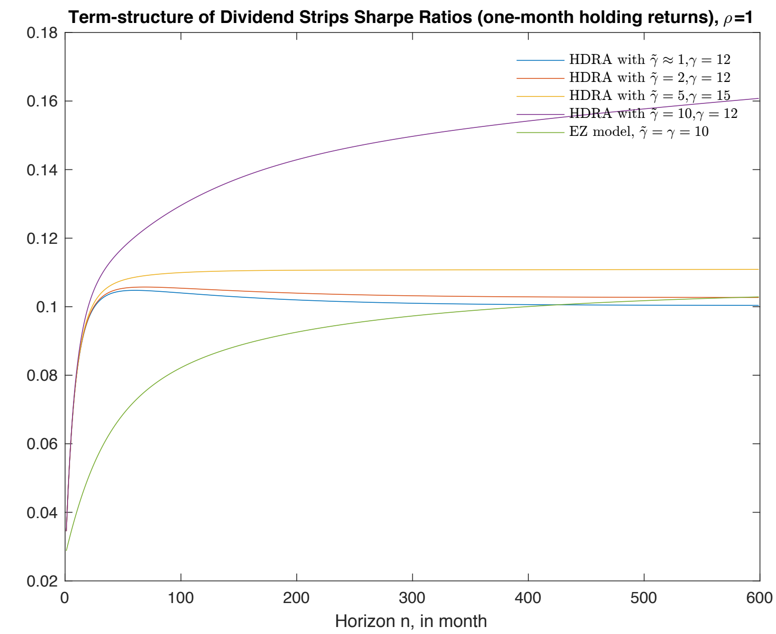

Figure 6: Calibrated term structure of Sharpe ratios under horizon-dependent risk aversion (HDRA). The Epstein-Zin curve (EZ) is done with the calibration ofBansal et al. (2014).

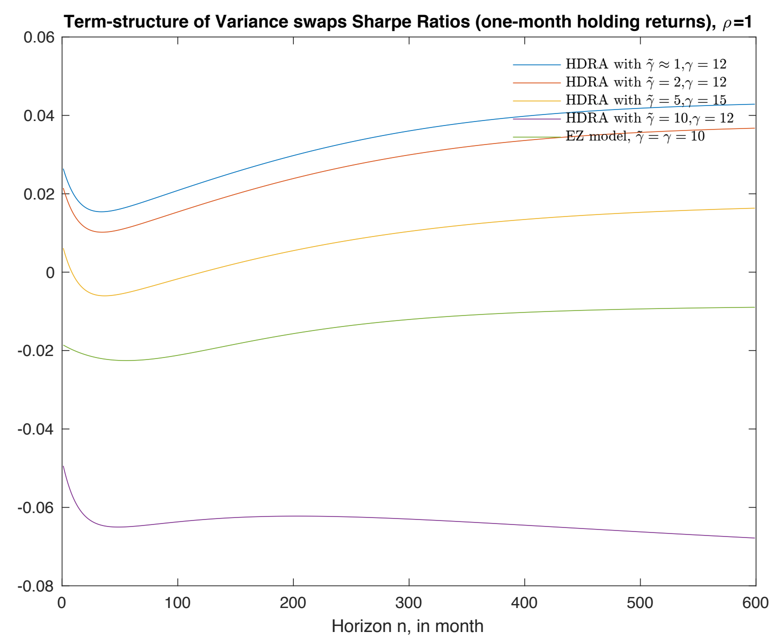

Figure 7: Calibrated term structure of variance swaps returns Sharpe ratios under horizon-dependent risk aversion (HDRA). The Epstein-Zin curve (EZ) is done with the calibration of Bansal et al. (2014).