From Hotelling to Nakamoto: The Economic Meaning of Bitcoin Mining

Wei JIANG
Hong Kong University of Science & Technology

jointly with
Min DAI (NUS), Steven KOU (BU), and Cong QIN (SoochowU)

The 2nd Tokenomics Conference
Toulouse, France, 2020

Agenda

Introduction

The Model

Calibration

Quantitative Analysis

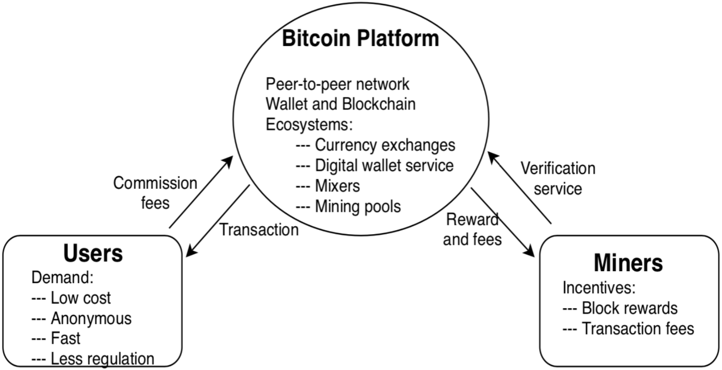
Introduction

The Model

Calibration

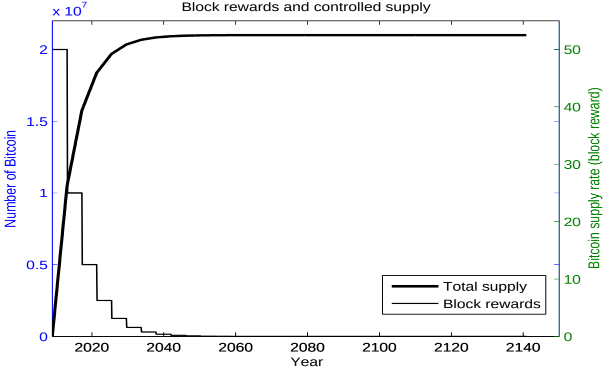
Quantitative Analysis

Bitcoin System



Participants: (a) Users, and (b) Miners

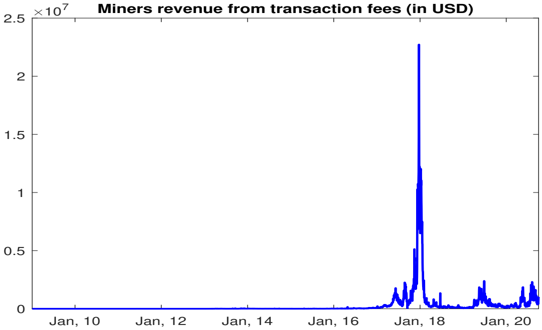
Block Rewards



Block rewards: Deterministic, Exogenous, and Scarce (terminate in 2140)

Scarcity $\Longrightarrow$ Bitcoin is an exhaustible resource!

Transaction fees in USD versus Bitcoin Price



Note. Daily total transaction fees in USD.

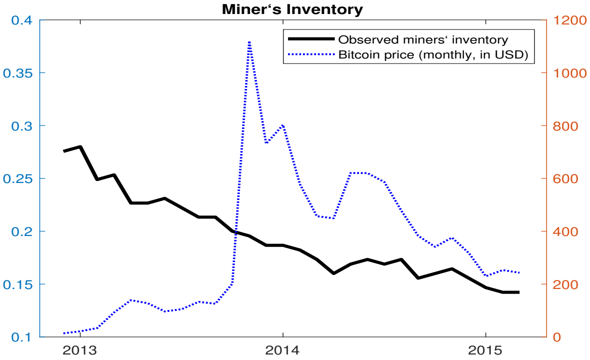
Transaction fees: Stochastic, Endogenous, and Unlimited

Key incentive to miners after the end of block rewards

Bitcoin Mining v.s. Gold Mining

- **Frictions:** no storage costs+liquidation costs v.s. storage costs + low liquidation costs.
- **Output:** preditermined block rewards + transaction fees v.s. no transaction fees
- **Uncertainty:** mining lottery v.s. no lottery
- **Policy:** adjusting inventory v.s ajusting production

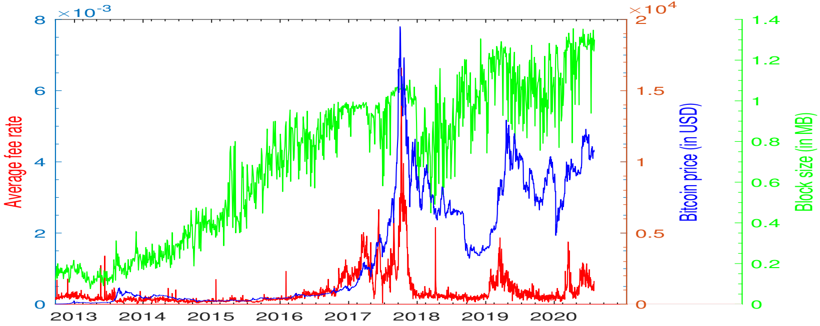
Stylized Facts: Miner's Inventory



Why the miners kept on reducing inventory?

Propotional inventory $= \frac{\text{Miners' aggregate inventory at time t}}{\text{Cumulative Bitcoin supply at time t}}$ (Athey et al. 2016).

Stylized Facts: Exchange Rate & Average Transaction Fee Rate



Why the fee rate is flat first and increases later?

Average fee rate at t $= \frac{\text{Total transactin fees at t}}{\text{Processed transaction volume at t}}$.

Our Main Results

- We develop a continuous-time dynamic model for Bitcoin mining by borrowing idea of the classic Hotelling model for exhaustible resources.
- Our model can calibrate to empirical data and explain the aforementioned two stylized facts.
- Our model has many interesting implications including
    - We find that high jump risk is one of major forces driving miners to sell their Bitcoin holdings at an early stage even when Bitcoin prices are quite low or very volatile.
    - Our model suggests that a high (low) Bitcoin demand leads to a high (low) transaction fee rate.

Literature Review

- Model on transaction fees:

Easley, O'Hara, and Basu(2019, JFE)	One period	Nash equilibrium of users' fee paying strategy
Our model	Continuous time dynamic model	Transaction fees from miner's perspective incorporating declined block rewards and miners' inventory

- Resource models: Hotelling (1931, JPE); Levhari and Pindyck (1981, QJE); Pindyck (2001);
  - Hotelling: storage cost $\Longrightarrow$ optimal production without inventory;
  - Our model: predetermined rewards + no storage cost $\Longrightarrow$ optimal inventory strategy; transaction fees as feedback supply.
- Bitcoin as currency: Athey et al. (2016); Gandal and Halaburda (2015); Halaburda and Sarvary (2016); Bolt et al. (2016); Jermann (2018).
- Others: Cong, He, and Li (2018); Dixon (1980); Bass (2004).

Introduction

The Model

Calibration

Quantitative Analysis

A Resource Production Model

Originating from Hotelling (1931, JPE), resource mining problem can be written in general as:

$$\sup_{Q_u \geq 0} \mathbb{E}_t \Big[\int_t^\infty e^{-\beta\,(u-t)} \Big(\mathrm{Rev}(Q_u) - \mathrm{Cost}(Q_u) \Big) du \Big]$$

where $\beta > 0$ is a discount factor, and

- $\mathrm{Rev}(Q_u) = P_u Q_u$
- $\mathrm{Cost}(Q_u) = \psi P_u Q_u^2 / H_u$ [mainly liquidation cost] $+ C_m$ [running]
- Q : Miner's selling rate
- P : Bitcoin Price
- H : Holding Inventory

Note. For simplicity, we can assume $C_m = 0$ since the constant mining cost does not affect the miner's decision.

A Resource Production Model

Originating from Hotelling (1931, JPE), resource mining problem can be written in general as:

$$\sup_{Q_u \geq 0} \mathbb{E}_t \left[\int_t^{\infty} e^{-\beta\,(u-t)} \Big(\text{Rev}(Q_u) - \text{Cost}(Q_u) \Big) du \right]$$

where $\beta > 0$ is a discount factor, and

- $\text{Rev}(Q_u) = P_u Q_u$
- $\text{Cost}(Q_u) = \psi P_u Q_u^2 / H_u$ [liquidation]
- Q : Miner's selling rate
- P : Bitcoin Price $P_t = \theta_p X_t$, where

$$dX_t = \mu(\xi_t, X_t)dt + \sigma(\xi_t, X_t)dW_t - X_{t-} d\left(\sum_{i=1}^{N_t} (1 - Z_i) \right)$$

- H : Holding Inventory

$$dH_u = \{(b_u[\text{block}] + I_u[\text{transaction}])\pi[\text{probability}] - Q_u\}du$$

Modeling Bitcoin Price

- Bitcoin price satisfies an inverse demand function.
- Bitcoin price is determined by quantity equation of money (Bolt et al. 2016, WP; Fisher 1911; Friedman 1973):

$$P_t = \theta_p X_t.$$

where the constant θ_p is determined by Bitcoin supply and velocity.

Modeling Demand Shock

Demand shock (jump diffusion with a mean reverting drift):

$$dX_t = \mu(X_t, i_t)dt + \sigma(X_t, i_t)dW_t - X_{t-}d\left(\sum_{i=1}^{N_t}(1 - Z_i)\right)$$

- $i_t \in \{\mathbb{H}, \mathbb{L}\}$ represent two transaction states: High-active/Low-active markets, with transition intensities $\zeta = (\zeta_{\mathbb{H}}, \zeta_{\mathbb{L}})$.
- $\mu(x, i) = \kappa_i(\nu_i - \ln x)x$, and $\sigma(x, i) = \sigma_i x$ denote the adoption term and volatility term respectively in state i (Gompertz Model, Gronwald, 2015).
- N_t is a jump process with intensity λ_J, and $1 - Z$ is the proportional jump size (Weil, 1987).

Miner's Inventory

Miner's inventory H_t satisfies

$$dH_t = [(b_t + I_t)\pi - Q_t]dt,$$

- $\pi = \frac{\omega}{D \times 2^{32}/600}$ is the probability of successful validations and D is the difficulty level (Hayes 2017).
- b_t is the block reward at t with total supply $\bar{S} = \int_0^{\infty} b_t dt = \int_0^{T} b_t dt < \infty$
- I_t is the transaction fees in candidate blocks at t.

Note. $D \times 2^{32}/600$ is also called network hash rate.

Modeling Transaction Fees

- Total volume of submitted orders by others:

$$L(H, X_t; i, S) = \theta_{p,i}(S - H)\log(1 + X_t) \text{ with } i \in \{\mathbb{H}, \mathbb{L}\}.$$

  Here S is the total supply, H is the inventory holding of miners, X_t is the demand shock. The particular functional form is not crucial. For example, one can use $\log(X)$ instead of $\log(1 + X)$
- The distribution of orders with different fee rate:

$$f(\phi), \quad \phi \in (0, \bar{\phi}) \quad \text{with C.D.F.} \quad F(\phi).$$

- Each time, a fixed number of orders G can be processed by miners.

Transaction Fees

- The miner selects fee threshold Φ_t to solve

$$I = \max_{\Phi \in [0,\bar{\phi}]} K(\Phi)L$$
$$s.t. \quad k(\Phi)L \leq G,$$

where $k(\Phi) = \int_{\Phi}^{\bar{\phi}} f(\phi)d\phi$, and $K(\Phi) = \int_{\Phi}^{\bar{\phi}} f(\phi)\phi d\phi$.

- Optimal fee threshold satisfies:

$$\Phi^* = \begin{cases} F^{-1}(1 - \frac{G}{L}), & \text{if } L > G, \\ 0 & \text{if } L \leq G, \end{cases}$$

- $\Phi^* = 0$ means that the miner takes all the orders.
- First-price auction problem and symmetric Bayes-Nash equilibrium (Basu et al., 2018).

HJB Equation

- Short-run case: $t < T$, there are block rewards.
- $(t, i_t, X_t, H_t) = (t, i, x, h) \in (0, \infty) \times \{\mathbb{H}, \mathbb{L}\} \times (0, \infty) \times [0, S(t)]$,

$$\frac{\partial V_i}{\partial t} + \mathcal{L}V_i + \max_{\{q \geq 0\}} \left\{ \left[\Big(b_t + K(\Phi^*(L))L \Big) \pi - q) \right] \frac{\partial V_i}{\partial h} + Pq - C(q,h) \right\} + \mathcal{J}V_i = \beta V_i$$

where

$$\mathcal{L}V_i = \frac{1}{2}\sigma^2(x,i)\frac{\partial^2 V_i}{\partial x^2} + \mu(x,i)\frac{\partial V_i}{\partial x},$$

$$\mathcal{J}V_i = \lambda_J \Big[V_i(t, Zx, h) - V_i(t,x,h) \Big] + \zeta_i \Big[V_\iota(t,x,h) - V_i(t,x,h) \Big]$$

- Long-run case: $b_t = 0$ for $t \geq T$
- $V_i(t, X, H) = V_i(T, X, H) := V_i^T(X, H)$ for any $t \geq T$.

Optimal Selling Strategies

- In state i, optimal inventory strategy q_i^* satisfies:

$$q_i^* = \frac{h}{2\psi P}\left(P - \frac{\partial V_i}{\partial h}\right)^+.$$

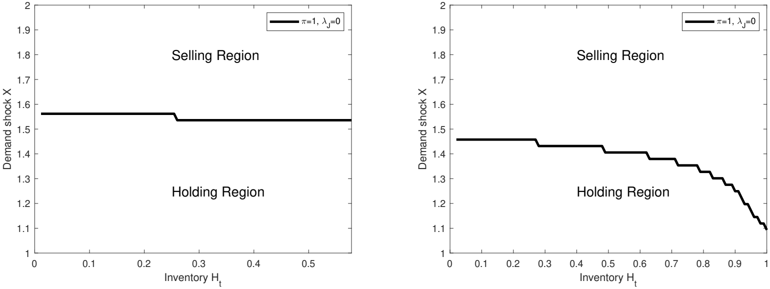
- **Holding / Selling regions:**
  - i-Selling Region:

$$\left\{(t,x,h)\,\middle|\, P > \frac{\partial V_i(t,x,h)}{\partial h}\right\}$$

  - i-Holding Region:

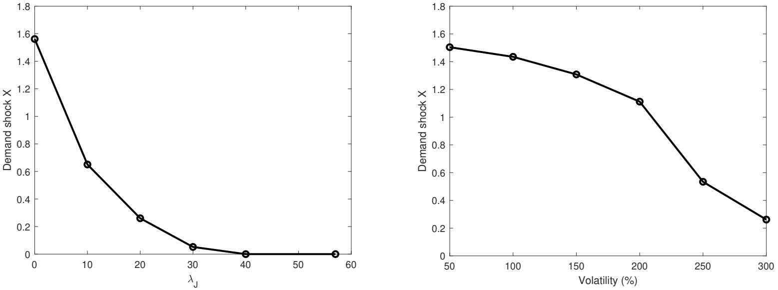
$$\left\{(t,x,h)\,\middle|\, P \leq \frac{\partial V_i(t,x,h)}{\partial h}\right\}.$$

Selling barriers, no jumps



Note. We assume no jump risk in this case, i.e. $\lambda_J = 0$. Other parameter values are based on calibration results. The left panel shows the short-run case, i.e. $t = 2014$ and $H_t/S(t) = 0.1$ with $S(t) = 0.5871$, while the right panel shows the long-run case.

Jump risk lowers selling barrier



Note. Parameter values are based on calibration results. We consider short-run case in a High-active market and choose $t = 2014$, $H_t/S(t) = 0.1$ with $S(t) = 0.5871$.

Introduction

The Model

Calibration

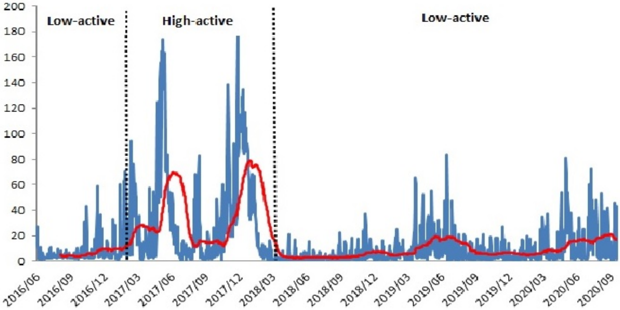
Quantitative Analysis

Calibration: Data

- Observed data (source: https://www.blockchain.com):
  - Monthly Bitcoin price from 2013 to 2020.
  - Monthly Miners' aggregate inventory from 2013 to 2015,
  - Monthly Market average fee rate from 2013 to 2020.
  - Monthly Aggregate transaction fees from 2013 to 2020.
  - Yearly Difficulty level from 2013 to 2020.
  - Daily Mempool transaction count in Bitcoin from Apr, 2016 to Sep, 2020
- Bitcoin prices are informative to parameters $\Theta_1 = \{\kappa, \nu, \sigma_{\mathbb{H}}, \sigma_{\mathbb{L}}\}$.
- Miners' aggregate inventory, average fee rate, and aggregate fee income are informative to parameters $\Theta_2 = \{\psi, \theta_{\mathbb{H}}, \theta_{\mathbb{L}}\}$.

High-active/low-active Market

- Detect the high-active and low-active market by Mempool transaction count (High-active: count > 20).
- Low-active: 2013Q1-2016Q3; High-active: 2016Q4-2017Q4; Low-active: 2018Q1-2020Q3



Note. In thousand. The red line is the 60-day moving average.

Calibration Method

- **Step 1:** Set $\beta = 0.06; \bar{S} = 1; G = 10; \theta_p = 100;$ $\lambda_J = 57; Z = 0.9$. The $f(\cdot)$ satisfies Beta distribution with parameters $(a, b) = (0.1, 99.9)$.
- **Step 2:** Estimate $\Theta_1 = (\kappa, \nu, \sigma_{\mathbb{H}}, \sigma_{\mathbb{L}})$ with Bitcoin price data.
- **Step 3:** Given $\Theta_2 = (\psi, \theta_{\mathbb{H}}, \theta_{\mathbb{L}})$ and observed Bitcoin price, we can compute the path of demand shock $\{\widetilde{X}_t; t = 1, \cdots, T_1\}$.
  - implied transaction fees $\{\widetilde{I}_t(\Theta_2); t = 1, \cdots, T_1\}$,
  - implied average fee rate $\{\widetilde{r}_t(\Theta_2); t = 1, \cdots, T_1\}$.
  - implied inventory $\{\widetilde{H}_t(\Theta_2); t = 1, \cdots, T_2\}$,

Calibration Method

- **Step 3 (continue):**

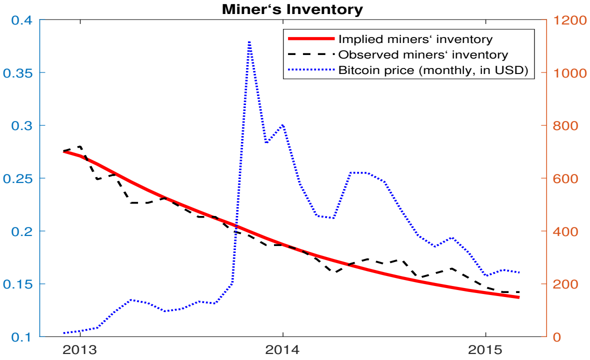
  We estimate $\widehat{\Theta}_2$ by minimizing the weighted least square relative error between model output and observed data:

$$\min_{\Theta_2} \frac{1}{T_1} \sum_{t=1}^{T_1} \left\{ \frac{1}{(r_t^A)^2} (r_t^A - \widetilde{r}_t(\Theta_2))^2 + \frac{1}{(I_t^A)^2} (I_t^A - \widetilde{I}_t(\Theta_2))^2 \right\} + \frac{1}{T_2} \sum_{t=1}^{T_2} \left\{ \frac{1}{(H_t^A)^2} (H_t^A - \widetilde{H_t})^2 \right\}.$$

Summary of Parameters

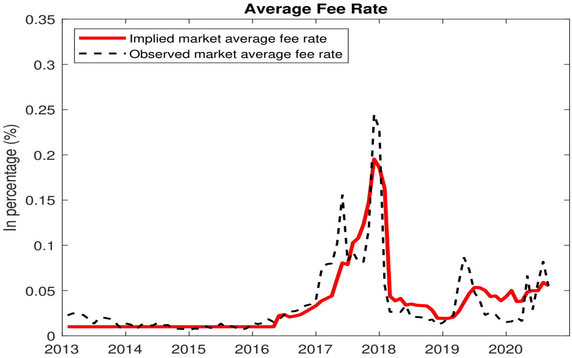
Parameters	Symbol	Value
Risk-free rate	β	0.06
Total supply of Bitcoin	$\bar{S}$	1
Capacity of blocks per unit of time	G	10
Hash rate per miner (TH/s)		5.2
Coefficient in quantity equation (Billion USD per unit)	θ_p	100
Upper bound of fee rate	$\bar{\phi}$	10%
Beta distribution parameters	(a, b)	(0.1, 99.9)
Adoption speed of Bitcoin	κ	1.1742
Log carrying capacity	ν	0.7793
Volatility of demand shock in high-active market	$\sigma_{\mathbb{H}}$	0.7910
Volatility of demand shock in low-active market	$\sigma_{\mathbb{L}}$	0.6225
State transition intensity	$(\zeta_{\mathbb{H}}, \zeta_{\mathbb{L}})$	(0.8, 0.3)
Jump parameters	(λ_J, Z)	(57, 0.9)
parameter in utility cost in liquidation	ψ	0.51
Sensitivity of volume to demand in high-active market	$\theta_{\mathbb{H}}$	260.5
Sensitivity of volume to demand in low-active market	$\theta_{\mathbb{L}}$	32.3

Implied Inventory



Note. Proportional inventory.

Implied Average Fee Rate



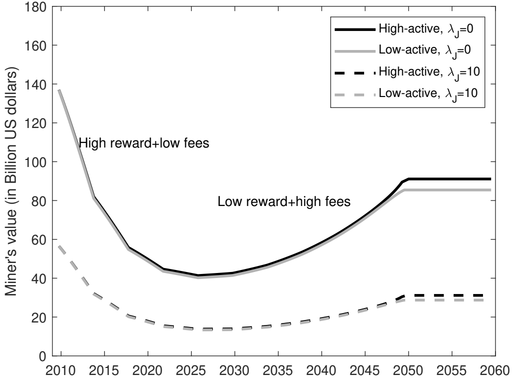
Introduction

The Model

Calibration

Quantitative Analysis

Miners' value in "U" shape



Note. Here we assume that the miner's inventory is zero, i.e. $H = 0$. The constant demand shock is $X = 1$.

Conclusion

- The Bitcoin mining model extends the classical Hotelling (1931) model with inventory and a feedback supply. A quantitative justification of miner's value based only on transaction fees is provided.
- The model is calibrated to data and can explain the dynamics of average transaction fee rate and miners' inventory holdings in observed data.
- Jump risk is a key factor to understand miners' inventory holdings.

Q&A

Thanks!

Proposition on transaction fees

Proposition

Assume that the miner's average fee rate r is given by $r = \frac{K(\Phi^*)}{k(\Phi^*)}$.

(i) *If $L \leq G$, then $\Phi^* = 0$ and $r = K(0)/k(0)$.*

(ii) *If $L > G$, then Φ^* and r are strictly increasing with X (or L). In particular, we have $\lim_{X \to \infty} \Phi^* = \lim_{X \to \infty} r = \bar{\phi}$.*

Optimal selling rate: q^*/H_t