Pricing formulae for derivatives in insurance

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Ruin Theory framework

Cumulative Loss process :

$$L_t := \sum_{i=1}^{N_t} X_i, \quad t \in [0, T],$$

- Frequency: Claims arrival modeled by a jump process $N := (N_t)_{t\in[0,T]}$, jumping at time $(\tau_i)_{i\in\mathbb{N}^*}$,
- Severity: claims sizes $(X_i)_{i\in\mathbb{N}^*}$

Classical **Cramer-Lundberg** model

- N is a Poisson process independent of the claims sizes (X_i),
- (X_i) iid random variables.
- but the independence assumptions are in practice often too restrictive

Different models of dependencies

Explicit dependency between claim size (X_i) and interarrival time $(\tau_i - \tau_{i-1})$

- the distribution of $(\tau_i - \tau_{i-1})$ depends on previous claim size X_{i-1}. Albrecher and Boxma (2004): $(\tau_i - \tau_{i-1})$ follows a mixing of two exponential distribution (extended to an Erlang distribution in Sajithamony and K.K. Thampi (2015)), whose mixing probability is the probability that X_{i-1} is larger than a given threshold.
- the distribution of the next claim X_i depends on the last interarrival time. Boudreault et al. (2006): X_i follows a mixing of two distributions, whose mixing parameter is $e^{-\beta(\tau_i - \tau_{i-1})}$. similar model proposed by Kwan and Yang (2007) and Zhang, Meng and Guo (2008), with mixing parameter is the probability that the $(\tau_i - \tau_{i-1})$ is larger than a threshold.

Different models of dependencies (continued)

Dependency via mixing through a frailty parameter
(Albrecher, Constantinescu and Loisel (2011))

- parameter pertaining to the distribution of the interarrival times , and/or of the claim sizes, is itself considered to be a random variable.
- mixing over the distribution of this parameter
- implies an exchangeable family for (X_i)

Different quantities of interest

- Ruin probability $\psi(u) := \mathbb{P}(\exists t \in [0, T], u + ct - L_t < 0)$.
- Expected discounted penalty function at ruin : Gerber-Shiu function (1998).
- Valuation of (re)-insurance contracts.

Our framework

"Pricing formulae for derivatives in insurance using Malliavin calculus", Hillairet, Jiao, Réveillac. Probability, Uncertainty and Quantitative Risk, volume 3 (2018)

- General framework of dependencies between claims arrival N and claims sizes (X_i).
  - general setting of dependencies
  - we do not assume a Markovian framework
  - extend the mixing approach by allowing of non-exchangeable family of random variables for the claim size.

- Provide pricing formulae for insurance contracts
  - decomposition formula into "building blocks" (in analogy with the Black-Scholes formula)
  - using Malliavin calculus

Loss processes in insurance

Cumulative Loss process :

$$L_t := \sum_{i=1}^{N_t} f(\tau_i, \Lambda_{\tau_i}, \varepsilon_i)\, e^{-\kappa(t-\tau_i)}, \quad t \in [0, T],$$

- $N := (N_t)_{t\in[0,T]}$ is a Cox process (doubly stochastic Poisson process) with intensity $\lambda := (\lambda_t)_{t\in[0,T]}$, ($\Lambda_t := \int_0^t \lambda_s ds$),
- $(\varepsilon_i)_{i\geq 1}$ is a sequence of iid random variables,
- $\kappa \geq 0$ is a discount factor,
- $\tau_i := \inf\{t > 0,\ N_t = i\}$,
- $f : [0, T] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is a bounded deterministic function.

Loss processes in insurance

Cumulative Loss process :

$$L_t := \sum_{i=1}^{N_t} f(\tau_i, \Lambda_{\tau_i}, \varepsilon_i)\, e^{-\kappa(t-\tau_i)}, \quad t \in [0, T],$$

Modified cumulative Loss process :

$$\hat{L}_t := \sum_{i=1}^{N_t} g(\tau_i, \Lambda_{\tau_i}, \varepsilon_i, \vartheta_i) e^{-\kappa(t-\tau_i)}, \quad t \in [0, T],$$

- $(\varepsilon_i, \vartheta_i)_{i\geq 1}$ is a sequence of iid rv with distribution μ,
- $g : [0, T] \times \mathbb{R}_+^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ is a bounded deterministic function,
- $\Lambda_t = \int_0^t \lambda_s ds, \quad t \in [0, T]$.

Some contracts in (Re-)insurance

$$L_T := \sum_{i=1}^{N_T} f(\tau_i, \Lambda_{\tau_i}, \varepsilon_i)\, e^{-\kappa(T-\tau_i)}$$

Stop-loss Contrats : provides to its buyer (another insurance company), the protection against losses which are larger than a given level K and its payoff function is given by a "call" function.

$$\Phi(L_T) = \begin{cases} 0, & \text{if } L_T < K \\ L_T - K, & \text{if } K \leq L_T < M \\ M-K, & \text{if } L_T \geq M \end{cases} .$$

Evaluating stop-loss contracts relies in computing

$$\mathbb{E}[\Phi(L_T)] = \mathbb{E}\left[L_T \mathbf{1}_{\{L_T \in [K,M]\}}\right] - K\mathbb{P}\left[L_T \in [K,M]\right] + (M-K)\mathbb{P}\left[L_T \geq M\right].$$

Our aim : Compute $\mathbb{E}\left[L_T \mathbf{1}_{\{L_T \in [K,M]\}}\right]$ in terms of the building block $x \mapsto \mathbb{P}\left[L_T \in [K-x, M-x]\right]$.

Some contracts in (Re-)insurance (continued)

$$L_T := \sum_{i=1}^{N_T} f(\tau_i, \Lambda_{\tau_i}, \varepsilon_i)\, e^{-\kappa(T-\tau_i)}, \quad \hat{L}_T := \sum_{i=1}^{N_T} g(\tau_i, \Lambda_{\tau_i}, \varepsilon_i, \vartheta_i) e^{-\kappa(T-\tau_i)}$$

Generalized Stop-loss Contrats : Our approach allows us to go beyond the case of stop loss contracts. Consider now a contract where the reinsurance company pays

$$\widetilde{\Phi}(L_T, \hat{L}_T) = \begin{cases} 0, & \text{if } L_T \leq K \\ \hat{L}_T - K, & \text{if } K \leq L_T \leq M \\ M - K, & \text{if } L_T \geq M \end{cases},$$

More precisely, when the insurance contract is triggered by the loss process L, the compensation amount can depend on some other exogenous factors $(\vartheta_i)_{i\in\mathbb{N}}$.

Some contracts in (Re-)insurance (continued)

$$L_T := \sum_{i=1}^{N_T} f(\tau_i, \Lambda_{\tau_i}, \varepsilon_i)\, e^{-\kappa(T-\tau_i)}, \quad \hat{L}_T := \sum_{i=1}^{N_T} g(\tau_i, \Lambda_{\tau_i}, \varepsilon_i, \vartheta_i) e^{-\kappa(T-\tau_i)}$$

Generalized Stop-loss Contrats :

$$\widetilde{\Phi}(L_T, \hat{L}_T) = \begin{cases} 0, & \text{if } L_T \leq K \\ \hat{L}_T - K, & \text{if } K \leq L_T \leq M \\ M - K, & \text{if } L_T \geq M \end{cases}.$$

Then the price of such a contract would be :

$$\mathbb{E}\left[\hat{L}_T \mathbf{1}_{\{K \leq L_T \leq K\}}\right] - K\mathbb{P}\left[L_T \in [K, M]\right] + (M - K)\mathbb{P}\left[L_T \geq M\right],$$

Our aim : Compute $\mathbb{E}\left[\hat{L}_T \mathbf{1}_{\{L_T > K\}}\right]$ in terms of the building block $x \mapsto \mathbb{P}\left[L_T \in [K - x, M - x]\right]$ (or an equivalent quantity).

A related quantity :

$$L_T := \sum_{i=1}^{N_T} f(\tau_i, \Lambda_{\tau_i}, \varepsilon_i)\, e^{-\kappa(T-\tau_i)}$$

Expected Shortfall (risk measure) : The expected shortfall is a useful risk measure, that takes into account the size of the expected loss above the value at risk.

$$ES_\alpha(-L_T) = \mathbb{E}[-L_T | -L_T > V@R_\alpha(-L_T)], \quad \alpha \in (0,1).$$

$$ES_\alpha(-L_T) = AV@R(-L_T) := \frac{1}{1-\alpha}\int_\alpha^1 V@R_s(-L_T)ds,$$

if the law of L_T is continuous, which is NOT the case here. The latter property fails already in the case where the size claims X_i are constant. So one needs an explicit computation of

$$ES_\alpha(-L_T) = \frac{-\mathbb{E}[L_T \mathbf{1}_{\{L_T<\beta\}}]}{\mathbb{P}(L_T < \beta)}, \quad \beta := -V@R_\alpha(-L_T)$$

General Payoffs :

$$L_T := \sum_{i=1}^{N_T} f(\tau_i, \Lambda_{\tau_i}, \varepsilon_i)\, e^{-\kappa(T-\tau_i)}, \quad \hat{L}_T := \sum_{i=1}^{N_T} g(\tau_i, \Lambda_{\tau_i}, \varepsilon_i, \vartheta_i) e^{-\kappa(T-\tau_i)}$$

Goal : compute quantities of the form

$$\mathbb{E}\left[\hat{L}_T h(L_T)\right],$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borelian map with $\mathbb{E}[h(L_T)] < \infty$ in terms of the building block

$$\varphi_\lambda^h(x) := \mathbb{E}\left[h(L_T + x) | \mathcal{F}_T^\lambda\right], \quad x \in \mathbb{R}_+.$$

- In the classical Stop Loss contract $h := \mathbf{1}_{[K,M]}$ and so $\varphi_\lambda^h(x) = \mathbb{P}\left[L_T \in [K - x, M - x] | \mathcal{F}_T^\lambda\right]$.

Analysis

$$L_T := \sum_{i=1}^{N_T} f(\tau_i, \Lambda_{\tau_i}, \varepsilon_i)\, e^{-\kappa(T-\tau_i)}, \quad \hat{L}_T := \sum_{i=1}^{N_T} g(\tau_i, \Lambda_{\tau_i}, \varepsilon_i, \vartheta_i) e^{-\kappa(T-\tau_i)}$$

We want to compute : $\mathbb{E}\left[\hat{L}_T h(L_T)\right]$.

$$\text{Note that} \quad \hat{L}_T = \int_0^T \hat{Z}_s dN_s,$$

$$\hat{Z}_s := \sum_{i=1}^{+\infty} g(s, \Lambda_s, \varepsilon_i, \vartheta_i) e^{-\kappa(T-s)} \mathbf{1}_{(\tau_{i-1}, \tau_i]}(s), \quad s \in [0, T].$$

So

$$\mathbb{E}\left[\hat{L}_T h(L_T)\right] = \mathbb{E}\left[\int_0^T \hat{Z}_t dN_t h(L_T)\right].$$

A quantum of Malliavin calculus

A Malliavin integration by parts formula on the Poisson space: For u a predictable process and F an integrable random variable, it holds that

$$\mathbb{E}\left[F\int_0^T u_t dN_t | \mathcal{F}_T^\lambda \vee \mathcal{F}^{\varepsilon,\vartheta}\right] = \mathbb{E}\left[\int_0^T u_t\ F(\cdot \cup \{t\})\lambda_t dt | \mathcal{F}_T^\lambda \vee \mathcal{F}^{\varepsilon,\vartheta}\right],$$

where $\cdot \cup \{t\}$ denotes the creation operator which consists in adding one jump at time t to the Poisson process.

- Coming back to our problem we thus have :

$$\begin{aligned}
&\mathbb{E}\left[\hat{L}_T h\left(L_T\right)\right] \\
&= \mathbb{E}\left[\int_0^T \hat{Z}_t dN_t h\left(L_T\right)\right] \\
&= \mathbb{E}\left[\int_0^T \hat{Z}_t h\left(L_T(\cdot \cup \{t\})\right)\lambda_t dt\right].
\end{aligned}$$

Main result

We proved that

Theorem

Assume that $(\varepsilon_i, \vartheta_i)$ and $(\bar{\varepsilon}, \bar{\vartheta})$ are iid with common law μ, and independent of λ. It holds that

$$\mathbb{E}\left[\hat{L}_T h(L_T)\right]$$
$$= \int_0^T e^{-\kappa(T-t)} \mathbb{E}\left[g(t, \Lambda_t, \bar{\varepsilon}, \bar{\vartheta})\, \lambda_t\, \varphi_\lambda^h\left(f(t, \Lambda_t, \bar{\varepsilon}) e^{-\kappa(T-t)}\right)\right] dt,$$

(recall that $\varphi_\lambda^h(x) := \mathbb{E}\left[h(L_T + x) | (\lambda_t)_{t\in[0,T]}\right]$).

- Requires only the law of L_T and not the joint law $(L_T, \hat{L}_T)$.
- If h is convex (resp. concave) one can give a lower (resp. upper) bound on $\mathbb{E}\left[\hat{L}_T h(L_T)\right]$.

A Black-Scholes type formula for generalized Stop Loss contracts :

For $h := \mathbf{1}_{[K,M]}$, with $K < M$,

$$\varphi_\lambda(x) := \varphi_\lambda^h(x) = \mathbb{P}\left[L_T \in [K-x, M-x] | \mathcal{F}_T^\lambda\right], \quad x \in \mathbb{R}_+.$$

The theorem above becomes

Corollary

$$\mathbb{E}\left[\hat{L}_T \mathbf{1}_{L_T \in [K,M]}\right]$$
$$= \int_0^T e^{-\kappa(T-t)} \mathbb{E}\left[g(t, \Lambda_t, \bar{\varepsilon}, \bar{\vartheta})\, \lambda_t\, \varphi_\lambda\left(f(t, \Lambda_t, \bar{\varepsilon}) e^{-\kappa(T-t)}\right)\right] dt.$$

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Corollary

If $\lambda_t = \lambda > 0$, then

$$\mathbb{E}\left[\hat{L}_T \mathbf{1}_{L_T \in [K,M]}\right]$$
$$= \lambda \int_0^T \int_{\mathbb{R}_+^2} e^{-\kappa(T-t)} g(t,x,y)\, \varphi_\lambda\left(f(t,x)e^{-\kappa(T-t)}\right) \mu(dx,dy)dt,$$

(recall that $\mu := \mathcal{L}_{(\bar{\varepsilon},\bar{\vartheta})}$).

Examples

Explicit computations for some cases, for example:

- **Model on** $(\varepsilon_i, \vartheta_i)$ **:** $(\varepsilon_i, \vartheta_i)_{i\in\mathbb{N}^*}$ i.i.d. random vectors, with marginal distributions following Pareto distributions $\mathcal{P}(\alpha_\varepsilon, \beta_\varepsilon)$ and $\mathcal{P}(\alpha_\vartheta, \beta_\vartheta)$ and dependence structure modeled through a Clayton copula with parameter $\theta > 0$
  $C(u, v) := \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}$
- **Joint law of** (λ_t, Λ_t)**:** the intensity process $(\lambda_t)_{t\in[0,T]}$ given by $\lambda_t = \lambda_0 \exp(2\beta W_t)$ where W is a Brownian motion.

$\rightarrow$ Analytical formula for the pricing (stop-loss contract).

Illustration in the classic Cramer-Lundberg model

In the literature, for the classic Cramer-Lundberg model (N homogeneous Poisson process with constant intensity $\lambda_0 > 0$, $h := \mathbf{1}_{[K,M]}$)

- the pricing of Stop-Loss contracts relies on the computation of a term of the form $\int_K^M y dF(y)$ with F being the cumulative distribution function of the loss process L_T,
- the discussion mainly focuses on the derivation of the compound distribution function F (usually calculated recursively, using the Panjer recursion formula and numerical methods/approximations, cf Panjer (1981), Gerber (1982))

Our Malliavin approach provides another formula which reads as

$$\mathbb{E}\left[\hat{L}_T \mathbf{1}_{L_T \in [K,M]}\right] = \lambda_0 T \int_{\mathbb{R}_+} x\,(F(M-x) - F(K-x))\mu(dx).$$

if one translates results of Gerber (1982) in a general setting

$$ydF(y) = \lambda_0 T \int_{\mathbb{R}_+} xdF(y-x)\mu(dx),$$

from which one deduces that

$$\begin{aligned}\int_K^M ydF(y) &= \lambda_0 T \int_K^M \int_{\mathbb{R}_+} x\mu(dx)dF(y-x) \\ &= \lambda_0 T \int_{\mathbb{R}_+} x \int_K^M dF(y-x)\mu(dx) \\ &= \lambda_0 T \int_{\mathbb{R}_+} x(F(K-x) - F(M-x))\mu(dx).\end{aligned}$$

$\rightarrow$**For the Cramer-Lundberg model, our formula coincides with Gerber's formula**

Summary

- Efficient formula for the pricing of Stop-Loss contracts numerics
- It allows to handle general dependencies framework
- Once the building block is calculated (via analytical formula in some cases, or Monte-Carlo simulations), the computation (for pricing and sensitivity analysis) is easy
- Outgoing work: extension with N a Hawkes process (self exciting process)

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