Multi-Product Supply function equilibria

Work in Progress

Pär Holmberg[1,2] Keith Ruddell[1] Bert Willems[3]

[1]Research Institute of Industrial Economics (IFN)

[2]University of Cambridge (EPRG)

[3]Tilburg University, Toulouse School of Economics

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Multi-Product Divisible Good Auctions

- Many auctions deal with heterogenous but closely related goods
  Production cost of good 1 depends on quantity supplied of good 2:
  (Dis)economies of scope
- We **propose an auction with complex bids** in which the producer offer of good 1 to depend on price of ***all*** products

$$\{\text{Supply good}_1(p_1, p_2), \text{Supply good}_2(p_1, p_2)\}$$

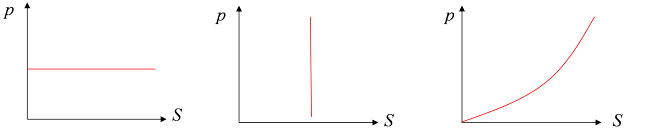
- We study strategic behavior in such an auction in a setting with **divisible** goods

Multi-Product Divisible Good Auctions: Applications

1. **Electricity markets:** different delivery periods (ramping constraints, start-up costs, storage facilities)
   [Also delivery locations; day-ahead, reserve, and ancillary services]
2. **Commodities:** milk in New Zealand, oil distillates
3. **Securities:** government bonds with different duration, risk profile, bank assets
4. **Derivatives market**: illiquid option contracts with risk for leg/execution risks
5. **Transport services:** back-haul capacity might matter

Look for Supply Function Equilibrium (Klemperer & Meyer, 1989)

- Price-contingent supply $s(p)$
  - Generalization of Cournot and Betrand models
  - Flexible strategy to deal with demand uncertainty



Bertrand NE: Best price given competitors' bids. Certain demand

Cournot NE: Best supply given competitors' bids. Certain demand

SFE: Best supply curve given competitors' bid curves. Uncertain demand

- Our contribution: We look at a multiple product market

Methodological contribution: Product bundles

- A **"bundling"** is set of independent bundles that span the product space
  - a bundle consist of a fixed proportion of goods 1 and 2
  - change of basis vectors $\rightarrow$ formulate game in bundle coordinates
- The **equilibrium is invariant** to bundling (isomorphic)
- There exist a bundling that separates markets : **separating bundles**
  - No economics of scope in production of bundles
  - Demand for bundles: neither substitutes nor complements
- In separating bundle coordinates, firms **do not make conditional bids**
- Solve for equilibrium in each single bundle market
- Map bundle equilibrium into product space

Example Methodology: Power Market - 2 time periods

- Storage operator (storage → dis-economies of scope)
  - Buys in period 1 and resells in period 2
  - Its willingness to supply electricity in period 2 depends on price in period 1, $\partial s_2 / \partial p_1 < 0$
  - Introduce the **storage bundle** $\mathbf{q}_{storage} = (-1, 1)$
  - Its bid now only depends on the price of this storage bundle
- Inflexible generator (ramping constraints → economies of scope)
  - Has to sell the same amount in period 1 and period 2
  - Competitive firm: Willingness to supply electricity in period 2 depends on average price over both periods: $\partial s_2 / \partial p_1 > 0$
  - Introduce the **block bundle** $\mathbf{q}_{block} = (1, 1)$
  - Its bid now only depends on the price of this block bundle
- The **storage bundle** and the **block bundle** span the full produce space

Example Methodology: Power Market - 2 time periods

- General power plant with production cost

$$C = c \cdot \frac{q_1^2}{2} + c \cdot \frac{q_2^2}{2} + d \cdot q_1 q_2$$

  - marginal production cost of q_1 depends on q_2
  - Costs can be separated:

$$C = \underbrace{\frac{c+d}{4}(q_1 + q_2)^2}_{\text{Cost Block bundle}} + \underbrace{\frac{c-d}{4}(q_1 - q_2)^2}_{\text{Cost Storage Bundle}}$$

  - Firm does not need to make bundle bids conditional on price of other bundle

Alternative Implementation: Firms report costs

- Firms report a cost function $k(\mathbf{q})$ for producing quantities $\mathbf{q} = (q_1, q_2)$
- Market clearing
  - auctioneer collects cost reports of suppliers, and consumers' utility function
  - maximizes total market surplus (assuming reports are truthful)
  - market prices = shadow price of goods balance (Uniform Price Auction)
- Equivalence between two approaches: Inverse of reported marginal cost function corresponds to conditional supply function:

$$\frac{\partial k(\mathbf{s}(\mathbf{p}))}{\partial \mathbf{q}} = \mathbf{p}$$

Alternative to auction with Complex Bids

- Run two standard auctions with simple bids $\{s_1(p_1), s_2(p_2,)\}$

- In competitive market less efficient, as cost interactions cannot be represented
- In oligopoly: result will depend on strategic effects (follow-up paper)

- However **if separating bundles** can be found: then simple and complex auctions are equivalent
- Suggestion on how to bundle goods (e.g literature on financial innovation in repackaging derivatives to lower transaction costs)

Set-up

Demand and Supply

- Two goods: prices $\mathbf{p} = [p_1, p_2]^\top$ and quantities $\mathbf{q} = [q_1, q_2]^\top$
- Stochastic demand function $\mathbf{q} = \mathbf{d}(\mathbf{p}) + \boldsymbol{\varepsilon}$
  - additive demand shock $\boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2]^\top$
    joint cumulative distribution function $\Phi(\boldsymbol{\varepsilon})$ on $\mathcal{E} \subset \mathbb{R}^2$.
  - linear demand $\mathbf{d} = -\boldsymbol{D}\mathbf{p}$ with $\boldsymbol{D}>0$ (i.e. positive definite)
- Quadratic cost: $\partial c / \partial \mathbf{q} = \boldsymbol{C}\mathbf{q}$ with $\boldsymbol{C} > 0$
- Profit of supplier $n \in \mathcal{N}$ producing $\mathbf{q}$ at price $\mathbf{p}$:

$$\pi(\mathbf{q}, \mathbf{p}) = \mathbf{p}^\top \mathbf{q} - c(\mathbf{q})$$

Set-up

Bidding and equilibrium

- Firm n bids supply function $\mathbf{q} = \mathbf{s}_n(\mathbf{p})$, upward sloping: $(\partial \mathbf{s}_n / \partial \mathbf{p} > 0)$
- Market equilibruim price $\mathbf{p}^{eq}(\varepsilon)$ is determined by market clearing

$$\mathbf{d}(\mathbf{p}) + \varepsilon = \sum_n \mathbf{s}_n(\mathbf{p}) \tag{1}$$

- Firm n maximizes expected profit

$$\Pi_n = \int_{\mathcal{E}} \pi(\mathbf{p}^{eq}(\varepsilon), \mathbf{s}_n(\mathbf{p}^{eq}(\varepsilon)))\, d\Phi(\varepsilon) \tag{2}$$

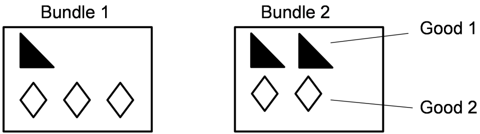
Definition: Bundle

Consider the procurement of two bundles $i' = 1', 2'$

- Each bundle is divisible and consists of fixed proportions of goods 1 and 2.
- Bundle $1'$ consists of $A_{1'1}$ units of good 1 and $A_{1'2}$ units of good 2, etc.
- So: $\tilde{\mathbf{q}}$ bundles contain $\mathbf{q}$ goods:

$$\mathbf{q} = A\tilde{\mathbf{q}}$$

Example: matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$



Game in Bundle Coordinates

- If the goods price is $\mathbf{p}$, then corresponding bundle price $\tilde{\mathbf{p}}$ should satisfy

$$\tilde{\mathbf{p}} = A^\top \mathbf{p}$$

- Marginal production cost of bundles:

$$\frac{\partial \tilde{c}_k(\tilde{\mathbf{q}})}{\partial \tilde{\mathbf{q}}} = \underbrace{A^\top C A}_{=\tilde{C}} \tilde{\mathbf{q}}$$

- Demand for bundles:

$$\tilde{\mathbf{d}}(\tilde{\mathbf{p}}) + \tilde{\varepsilon} = \underbrace{A^{-1} D A^{-\top}}_{=\tilde{D}} \tilde{\mathbf{p}} + \underbrace{A^{-1} \varepsilon}_{=\tilde{\varepsilon}}$$

- If $\tilde{\mathbf{s}}(\tilde{\mathbf{p}})$ is a Nash Equilibrium for bundles then $\mathbf{s}(\mathbf{p}) = A\tilde{\mathbf{s}}(A^\top \tilde{\mathbf{p}})$ is a Nash Equilibrium for goods

Separating Bundles

Lemma

Let bundling matrix $\boldsymbol{A}$ *be the eigenvectors of* $\mathbf{DC}$*, then demand and cost for* ***bundles*** *are separated. Matrices* $\tilde{\mathbf{D}}$ *and* $\tilde{\mathbf{C}}$ *are diagonal.*

Intuition: One separating bundle solves:

$$\max_{\mathbf{q}} \text{Net Consumer Utility}$$
$$\text{s.t.} \quad \text{Competitive Firm's Profit} = 1$$

Other separating bundle is the worst bundle for consumers

Example

Linear demand $\mathbf{d} = \varepsilon - \boldsymbol{D}\mathbf{p}$ and quadratic costs $c = \frac{1}{2}\mathbf{q}^\top \boldsymbol{C}\mathbf{q}$ with

$$\boldsymbol{D} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

Then, bundling matrix $\boldsymbol{A}$

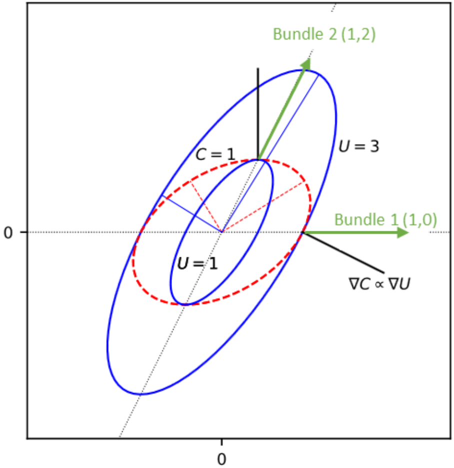
$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

diagonalizes $\boldsymbol{C}$ and $\boldsymbol{D}$ to

$$\tilde{\boldsymbol{C}} = \boldsymbol{A}^\top \boldsymbol{C}\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix} \qquad \text{and} \qquad \tilde{\boldsymbol{D}} = \boldsymbol{A}^{-1}\boldsymbol{D}(\boldsymbol{A}^{-1})^\top = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example: Separating Bundles

Net utility $\frac{1}{2}\mathbf{q}^\top \boldsymbol{D}^{-1}\mathbf{q}$ (Blue)
is tangent to
Competitive firm's profit $\frac{1}{2}\mathbf{q}^\top \boldsymbol{C}\mathbf{q}$ (Red)



Best response by firm n

- Response is ex-post optimal (once ε is observed, no reason to change strategy)
- Bid surfaces $\mathbf{s}(\mathbf{p})$ satisfy FOC of optimization problem :

$$\mathbf{s}_n + \frac{\partial \mathbf{d}_n}{\partial \mathbf{p}} \left(\mathbf{p} - \frac{\partial c}{\partial \mathbf{q}} \right) = \mathbf{0} \qquad \text{(Best Response)}$$

  and market clearing condition: $\mathbf{s}^n(\mathbf{p}) = \mathbf{d}^n(\mathbf{p}, \varepsilon) \equiv \mathbf{d}(\mathbf{p}, \varepsilon) - \sum_{m \neq n} \mathbf{s}_m(\mathbf{p})$
- This corresponds to multi-good monopoly pricing

Symmetric Nash Equilibrium

- We assume a **symmetric** equilibrium: Residual demand $\mathbf{d}^n = \mathbf{d} - (N-1)\mathbf{s}$
- 2-dimensional Klemperer and Meyer FOCs

$$\mathbf{s} + \left((n-1)\frac{\partial \mathbf{s}}{\partial \mathbf{p}} - \frac{\partial \mathbf{d}}{\partial \mathbf{p}} \right) \left(\mathbf{p} - \frac{\partial c}{\partial \mathbf{q}} \right) = \mathbf{0} \tag{2-D K\&M.}$$

Full Separation

- Assume $\mathbf{s}^*(\mathbf{p})$ is a solution of 2-D K&M

Theorem (Full Separation)

With the separating bundles $\mathbf{A}$, bid function $\tilde{\mathbf{s}}^*$ is separated. That is

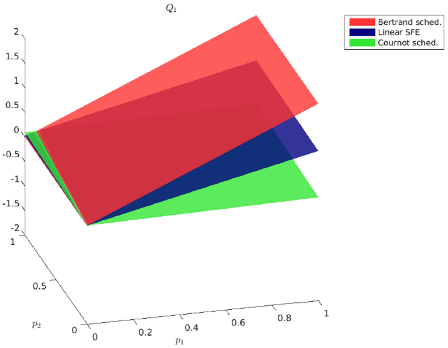
$$\frac{\partial \tilde{\mathbf{s}}^*(\tilde{\mathbf{p}}_0)}{\partial \tilde{\mathbf{p}}} \textit{ is a diagonal matrix}$$

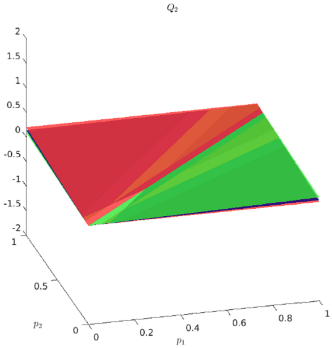
- From single 2-D problem $\rightarrow$ two 1-D Klemperer and Meyer problems:

$$\mathbf{F}\left(\mathbf{s}, \mathbf{p}, \frac{\partial \mathbf{s}}{\partial \mathbf{p}}\right) = \mathbf{0} \Rightarrow \tilde{F}_1\left(\tilde{s}_1, \tilde{p}_1, \frac{\partial \tilde{s}_1}{\partial \tilde{p}_1}\right) = 0 \text{ and } \tilde{F}_2\left(\tilde{s}_2, \tilde{p}_2, \frac{\partial \tilde{s}_2}{\partial \tilde{p}_2}\right) = 0$$

Unbounded shocks: Equilibrium

The linear supply function $\mathbf{s} = \boldsymbol{S}\mathbf{p}$ is the unique SFE





$n = 3$, $\boldsymbol{C} = \begin{pmatrix} .5 & 0 \\ 0 & 5 \end{pmatrix}$, $\boldsymbol{D} = \begin{pmatrix} 0.2 & 0.1999 \\ 0.199 & .2 \end{pmatrix}$ and unbounded shocks

Welfare analysis

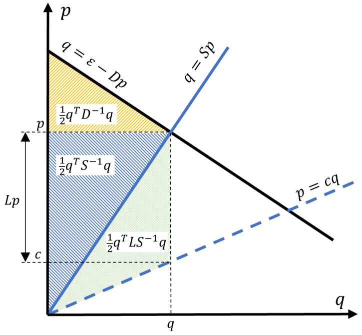
- Define ***Lerner tensor*** $\boldsymbol{L} = \boldsymbol{I} - \boldsymbol{CS}$.
  - It maps any price vector $\mathbf{p}$ on the price mark-up:

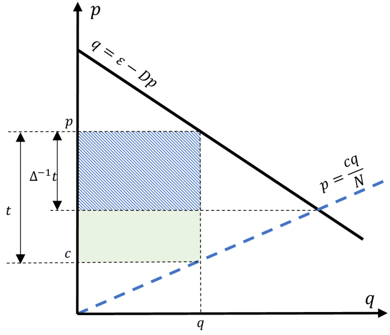
$$\boldsymbol{L} \cdot \mathbf{p} = \mathbf{p} - \frac{\partial c}{\partial \mathbf{q}}$$

  - Measures competitiveness of market
- Define ***Pass-through tensor*** $\boldsymbol{\rho} = (\boldsymbol{I} + \frac{1}{N}\boldsymbol{CD})^{-1}$.
  - It measures marginal effect of tax increase $\mathbf{t}$ on the consumer price $\mathbf{p}$

$$d\mathbf{p} = \boldsymbol{\rho} \cdot d\mathbf{t}$$

Lerner and Pass-through tensors





Lerner and Pass-through tensors: Properties

- $\boldsymbol{L}$ and $\boldsymbol{\rho}$ have identical eigenvectors (corresponding to the separating bundles)
- The eigenvalues of $\boldsymbol{L}$ and $\boldsymbol{\rho}$ are invariant to bundling.
- The eigenvalues of the Lerner tensor are between 0 and 1.
- Lerner Tensor $\boldsymbol{L}$ depends only on pass-through tensor $\boldsymbol{\rho}$ and number of firms N
  - With more firms N, eigenvalues of $\boldsymbol{L}$ decrease.
  - Higher eigenvalue pass-through tensor $\boldsymbol{\rho}$, increases corresponding eigenvalue of $\boldsymbol{L}$

Conclusion

- We consider auction that trades related divisible goods of different varieties, such as commodities/securities of different qualities.
- We solve for multi-product Supply Function Equilibria (SFE).
- We use separating bundles to find the equilibrium.
- Same bundling technique could potentially be used in practice.
- Extensions:
  - Non-quadratic costs: local unbundling and a numerical approach
  - Cost type is private information