Income Taxation with Frictional Labor Supply

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Abstract

I study nonlinear income taxation in a dynamic environment where individual labor supply is subject to frictions, e.g., hours constraints within firms. Specifically, consistent with the empirical evidence, agents incur a fixed cost of adjusting their hours of work in response to wage or tax changes. This generates an endogenous extensive margin of labor supply, conditional on participation. I derive a formula that characterizes the optimal long-run progressive tax schedule in this economy, and show in particular that the standard frictionless models, which ignore the lumpiness of labor supply, underestimate the long-run costs of raising tax progressivity.

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Theoretical models of optimal labor income taxation typically assume that labor supply can be adjusted costlessly and optimally on the intensive margin in response to wage or tax changes: a 1 percent increase in the marginal tax rate induces an ε percent decrease in hours of work or gross income. Given this simple modeling of labor supply, optimal tax rates are then determined by the trade-off between their effect on government revenue (including these behavioral losses driven by the elasticity ε) and their effect on welfare, measured by the marginal utility of consumption (normalized by the shadow value of public funds, also called marginal social welfare weights). Even models of taxation that incorporate explicitly an extensive margin of labor supply typically consider the binary decision of whether to participate in the labor force, keeping the assumption that conditional on participation, hours are either exogenously fixed or fully flexible on the intensive margin.

One of the most common criticisms of this framework is that this modeling of labor supply is unrealistic. A large (and growing) body of empirical evidence shows that the adjustment of labor supply in response to productivity, wage or tax changes, is subject to substantial frictions. The empirical literature (e.g., among others, Altonji and Paxson (1992), Chetty, Friedman, Olsen, and Pistaferri (2011)) finds that workers face hour constraints set by firms and must change jobs in order to adjust their hours of work. This entails large fixed (search) costs. The presence of such fixed adjustment costs generates endogenously an extensive margin of labor supply conditional on participation, where the thresholds of adjustments (i.e., their timing and size) are chosen optimally by the agent.

Despite this empirical evidence, there is little theoretical work that explicitly incorporates such labor supply adjustment frictions into models of income taxation. The reason (besides tractability concerns) is that it is commonly believed, perhaps loosely, that in the *long-run*, when individuals have had the time to fully adjust their labor supply, the effects of a tax change should be accurately captured by the standard frictionless optimal tax formulas, where ε is interpreted as a long-run labor supply elasticity; in other words, there may exist short-run adjustment frictions and sluggish responses to taxes at the individual level, but they become irrelevant in the long-run.

The question I address in this paper is whether long-run optimal taxes, in a dynamic model with lumpy individual labor supply, differ from those derived in the standard frictionless model with intensive margin behavior, and if so, what are the theoretical forces that determine optimal policy in the frictional economy.[1] The main result is that the long-run effects of non-linear taxation are not correctly captured by the frictionless tax formulas in the presence of fixed costs at the micro level. I derive novel formulas for long-run optimal taxes with lumpy labor supply and show that the frictionless tax formulas implicitly make an implausibly strong assumption about the effects of wage changes on optimal individual behavior.

I set up a dynamic model in which individuals choose their labor supply as a function of their stochastic idiosyncratic wage shocks and the non-linear tax schedule. In order to adjust their hours of work in response to wage or tax changes, they must pay a fixed cost. This fixed adjustment cost can be thought of as the cost of searching for a new job in an economy where hours are constrained within the firm, and is assumed proportional to the worker's foregone utility of consumption from the search activity. Once they decide

[1] In this paper I call "long-run" the time until the new steady state of the economy is reached after a tax change, which is driven by the time it takes for workers to react and adjust their labor supply. This should be thought of as a horizon of, say, five to ten years. I am not concerned with "very" responses to taxes which would take into account, for instance, distorted human capital accumulation of future generations – a problem analyzed by Stantcheva (2015), Findeisen and Sachs (2015), Badel and Huggett (2015).

to pay the fixed cost, i.e., to start searching for a new job, they receive a job offer (a costless adjustment opportunity) at an exogenous Poisson rate, which captures in a reduced-form way the frictions on the demand side of the labor market. Several assumptions allow me to keep the model analytically tractable: the utility function has no income effects; the wage follows a random growth process; the nonlinear tax schedule has a constant rate of progressivity; individuals are born (or enter the labor force) and die (or retire) at an exogenous Poisson rate; and they cannot save or borrow. As a result, hours of work evolve in a lumpy manner at the individual level: workers remain inactive, that is they keep the same job, until their productivity (or wage) is such that their optimal desired labor supply is far enough from their current, actual labor supply; at this point, which I characterize analytically, they decide to pay the fixed cost and start searching for a new job.[2]

The government chooses the tax schedule given this optimal individual behavior. Its objective is to maximize a long-run utilitarian social objective (the sum of individual value functions, weighted by the stationary distribution of agents), subject to a budget constraint. In the frictionless model, the optimum tax schedule is characterized by sufficient statistic formulas standard in the public finance literature. In the frictional model, I show two results. First, the long-run effect on social welfare of a uniform increase in marginal tax rates is given by the same formula as in the frictionless environment; this neutrality result formalizes the intuition described in the third paragraph and justifies the use of the frictionless intensive margin model for the analysis of long-run (linear) taxation. Second, the long-run effect on social welfare of an

[2]For salaried workers, an alternative setup would place the fixed cost on income rather than hours of work. All the results of this paper would continue to hold in this model, see Werquin (2014) for details.

increase in the progressivity of the tax schedule is *not* accurately captured by the frictionless formula: it depends on several new (extensive margin) labor supply elasticities and marginal social welfare weights. The main force is the following. An increase in progressivity reduces the volatility of the income process, as higher incomes are taxed at a higher rate. This indirectly reduces the option value of waiting to adjust labor supply in response to wage changes. The lumpiness of labor supply generates extensive margin effects on welfare unless these two effects exactly cancel out. In other words, the frictionless tax formula implicitly makes a strong assumption about the effect of taxes on the option value, namely, that it is exactly as large as their effect on the volatility of income shocks. However, the option value effect is typically negligible relative to the direct volatility effect of raising progressivity: individuals adjust their labor supply whenever their current hours of work are 10 percent away (say) from their optimum, and this value is hardly affected by taxes. Relative to the benchmark (frictionless equivalent) case where the two effects exactly cancel out, an increase in progressivity leads to a wider dispersion of individual incomes around their desired frictionless values, which adversely affects welfare. I show numerically in a calibrated version of the model that the magnitude of these effects can be large (the true welfare effects of raising taxes differ by up to 7 percent from those computed using the frictionless formulas), and is larger for smaller values of the labor supply elasticity ε.[3]

[3]There is another, "composition" effect of raising progressivity that is not captured by the frictionless formulas: the presence of adjustment frictions implies that individuals who earn the same income differ in their utility, as the least productive of them (i.e., those with a lower wage) are working more hours to earn the same income; this non-degenerate distribution is itself endogenous to the tax schedule. By treating the population earning a given income as a representative agent, the frictionless model thus ignores this endogenous heterogeneity and miscalculates the welfare effects of perturbing taxes.

Related literature. This paper is related to three distinct strands in the literature. First, it is motivated by the large empirical literature that points to the presence of frictions in the adjustment of labor supply. Altonji and Paxson (1992) show that changes in labor supply preferences have a much larger effect on hours of work when individual change jobs, suggesting that hours are constrained within firms and adjusting behavior entails substantial fixed costs. Other papers have similarly argued that labor supply is subject to fixed adjustment costs,[4] e.g., Cogan (1981), Altonji and Paxson (1988), Dickens and Lundberg (1993), Holmlund and Söderström (2008), Chetty, Guren, Manoli, and Weber (2012), Gelber, Jones, and Sacks (2013). Two closely related papers to mine are Chetty, Friedman, Olsen, and Pistaferri (2011) and Chetty (2012), who argue that adjustment frictions (in particular, fixed costs) can reconcile several puzzles observed in the large literature that estimates labor supply elasticities (see, e.g., Saez, Slemrod, and Giertz (2012) and Keane and Rogerson (2012) for surveys), e.g., the wide range of empirical elasticity estimates (from 0.1 to 1 for the micro Hicksian elasticity), or the fact that larger tax changes generate larger responses. My contribution is to incorporate explicitly these fixed costs into a theoretical dynamic model and derive the consequences for long-run optimal income taxes.

Second, this paper relates to the optimal taxation literature. The literature on the sufficient statistic approach to optimal taxation (e.g., Saez (2001), Golosov, Tsyvinski, and Werquin (2013), Hendren (2014, 2015), Jacquet and Lehmann (2015), derive optimal tax formulas for a large class of underlying functional forms for the utility function, the sources of heterogeneity, etc, but generally assumes that labor supply is set optimally. Recently, Chetty,

[4]Other types of fixed costs than hour requirements within firms can also be present, e.g., the cognitive cost of paying attention to a tax change.

Looney, and Kroft (2009) and Farhi and Gabaix (2015) have extended these results to cases with boundedly rational agents, but in static settings. My focus is instead on the *long-run* effects of taxes in a dynamic environment, when individuals *have* had the time to adjust their behavior in response to the tax change.

This paper also relates to the literature on extensive margin labor supply responses to taxes: Saez (2002), Rogerson and Wallenius (2009), Shourideh and Troshkin (2009), Chone and Laroque (2010), Ljungqvist and Sargent (2011), Jacquet, Lehmann, and Van der Linden (2013), study optimal taxation models where individuals face a fixed cost of working, leading to binary participation decisions and an intensive margin of hours. Alvarez, Borovicková, and Shimer (2015) model labor supply in a similar way as I do in this paper, although they consider only one adjustment (participation) threshold and do not focus on optimal taxation. My paper can be seen as a generalization of their insights, by deriving tax formulas that account for the many more (endogenous) extensive margins of labor supply observed in practice, conditional on participation. It would be straightforward to include an explicit participation margin in my setting (similar to Alvarez, Borovicková, and Shimer (2015)), although empirically the long-run responses to taxation seems to be driven mostly by hours worked conditional on employment (see Davis and Henrekson (2004), and Chetty, Guren, Manoli, and Weber (2012) for analyses of labor supply across countries with different tax regimes).

Finally, the technical tools that I use to model lumpy individual behavior are those developed in the impulse control literature originally developed to analyze operations research questions. Dixit and Pindyck (1994) and Stokey (2008) summarize many references and applications of these models to economics, primarily for monetary and investment topics. To cite only a few recent papers, Caplin and Leahy (1991), Bertola and Caballero (1990), Grossman and Laroque (1990), Caballero and Engel (1999), Alvarez and Lippi (2014), and Alvarez, Bihan, and Lippi (2014), have made important economic contributions to this literature, on which this paper builds. I contribute to this literature by studying long-run optimal policy in this class of models.[5] Finally, in public finance, there is a rich literature on investment in the presence of adjustment costs: Hall and Jorgenson (1967), Summers (1981), Abel (1983), Auerbach and Hines Jr (1987), Auerbach (1989), Auerbach and Hassett (1992). I bring this literature to the study of labor supply, since labor supply (and not just capital) adjustment costs are important.

The structure of the paper is as follows. Section 2 sets up the environment and describes the maximization problems of the individual and the government. Section 3 analyzes the optimal individual behavior. Section 4 characterizes the aggregate steady-state of the economy. Section 5 derives the main formulas for optimal taxes. Section 6 calibrates the model and runs numerical exercises. Section 7 concludes. The proofs of all the results and additional details are gathered in the Appendix.

[5]This paper finds that non-linear policy tools interact with fixed adjustment costs to yield long-run aggregate real effects on welfare. This is in contrast with many papers with indivisibilities or fixed costs at the micro level, where the aggregate economy behaves as a frictionless (representative agent) model (e.g., Caplin and Spulber (1987), Rogerson (1988)). I show in this paper that this insight holds only if the available policy instruments are linear. This insight could be applied more generally to models with fixed costs and non-linear policy instruments, beyond the taxation framework.

1 Environment

There is a continuum of mass one of individuals in the economy. Time is continuous.

Preferences. Individuals have the following Greenwood, Hercowitz, and Huffman (1988) utility of consumption c and hours of work (labor supply) h, with isoelastic disutility of labor:

$$U(c,h) = \frac{1}{1-\gamma}\left(c - \frac{1}{1+1/\varepsilon}h^{1+1/\varepsilon}\right)^{1-\gamma}, \tag{1}$$

with $\gamma \in [0,1)$. They discount the future at rate ρ_1. They are born (or enter the labor force) and die (or retire) at an exogenous and constant Poisson rate ρ_2. I denote $g(x) = (1-\gamma)^{-1} x^{1-\gamma}$.

Technology. Individual productivity θ is exogenous. The production function is linear in the labor input, so that workers' wages w_t are equal to θ_t for all $t \geq 0$. An individual's log-wage (i.e., log-productivity) at birth, w_0, is drawn from a normal distribution $f_{\ln w_0}(\cdot)$ with mean m_w and variance s_w^2. The idiosyncratic wage w_t then evolves stochastically over time according to a geometric Brownian motion with drift μ_w and volatility σ_w:

$$d \ln w_t = \mu_w dt + \sigma_w d\mathcal{W}_t, \tag{2}$$

The reduced-form equation (2) for the exogenous wage process can be micro-founded.[6] A large empirical literature[7] estimates wage specifications of this form and its findings are consistent with the presence of a unit root in the

[6] See, e.g. Gabaix, Lasry, Lions, and Moll (2015) and the references therein.
[7] See Meghir and Pistaferri (2011) for a survey.

wage process w_t.[8]

Budget constraint and taxes. An individual with wage w who works h hours earns taxable labor income $y = w \times h$ and pays taxes $T(y)$ to the government. I assume that she cannot save or borrow, so that her consumption equals her net income, $c = y - T(y)$. The tax-and-transfer system is restricted within a class of two-parameter tax schedules, defined as

$$T(y) = y - \frac{1-\tau}{1-p} y^{1-p}, \tag{3}$$

with $\tau \in \mathbb{R}$ and $p \in (-\infty, 1)$. If $p = 0$, the income tax schedule is linear with constant marginal tax rate τ. If $p \in (0,1)$, the ratio of the marginal tax rate to the average tax rate is $T'(y) / [T(y)/y] > 1$, so that the tax schedule is progressive; if $p < 0$, the tax schedule is regressive. The parameter p is the coefficient of marginal rate progression.[9] It is equal to the elasticity of the net-of-tax rate with respect to taxable income,

$$p = -\frac{d \ln (1 - T'(y))}{d \ln y}.$$

This functional form for the tax schedule approximates extremely accurately the U.S. tax system.[10] Figure 2 in the Appendix plots this tax function calibrated to the U.S. tax code ($p = 0.151$).

[8]The results of this paper would not be affected if wages were allowed to jump. This extension (letting jumps be drawn from a double-Pareto distribution) would also make the wage process consistent with the evidence presented in Guvenen, Song, Ozkan, and Karahan (2015), who find that the distribution of income growth rates $d \ln w_t$ has Pareto tails.

[9]See Musgrave and Thin (1948). Recent papers by Benabou (2002) and Heathcote, Storesletten, and Violante (2014) study optimal taxes using this functional form assumption.

[10]See Figure 1.(a) in Heathcote, Storesletten, and Violante (2014). They fit this tax function to the PSID data and find an R^2 equal to 0.96.

Individual problem. In a frictionless environment, individuals choose their labor supply h_t optimally and costlessly at every instant t. A worker with current wage w_0 solves

$$\mathcal{V}^*(w_0) = \max_{\{h_t\}_{t\geq 0}} \mathbb{E}_0 \left[\int_0^\infty e^{-(\rho_1+\rho_2)t} U(w_t h_t - T(w_t h_t), h_t) \, dt \right], \tag{4}$$

subject to (2). The solution to this problem gives the agent's frictionless, or desired, labor supply $\{h_t^*\}_{t\geq 0}$, consumption $\{c_t^*\}_{t\geq 0}$, and value function $\mathcal{V}^*(w_0)$.

I now suppose that in order to adjust their labor supply from h to h', individuals must pay a fixed utility cost $\boldsymbol{\kappa} \geq 0$, which can be interpreted as the search cost of finding a new job.[11] After paying this fixed adjustment cost, a job offer arrives at rate $q > 0$, at which time the agent can costlessly and optimally adjust her hours; intuitively, q captures in a reduced-form way the frictions on the demand side of the labor market.[12] I assume that $\boldsymbol{\kappa}$ is proportional to the utility from the foregone (frictionless) disposable income due to the search activity, which is itself proportional to the instantaneous utility $g(c_0^*)$:[13]

$$\boldsymbol{\kappa} = \tilde{\kappa} \times \mathbb{E}_0 \left[\int_0^{\hat{\tau}} e^{-(\rho_1+\rho_2)t} U(c_t^*, h_t^*) \, dt \right] = \kappa \times g(c_0^*), \tag{5}$$

with $\kappa \geq 0$ and $\hat{\tau}$ is the Poisson random time at which a job offer is received after paying the fixed cost at time 0 (the second equality is proved in the

[11]See the empirical evidence presented in, e.g., Altonji and Paxson (1992) and Chetty, Friedman, Olsen, and Pistaferri (2011).

[12]In this paper I focus on the labor supply effects of taxation, and thus assume that q is exogenous to taxes.

[13]Assuming $\gamma < 1$ in (1) ensures that the fixed cost $\boldsymbol{\kappa}$ is strictly positive and increasing in income.

Appendix). When $\kappa > 0$ and $q = \infty$, the environment reduces to an (S, s)[14] model. When $\kappa = 0$ and $q < \infty$, it becomes a model à la Calvo (1983).[15]

In this frictional environment, individuals decide when and by how much to adjust their labor supply as their wage evolves. They can choose their hours optimally and costlessly at birth. I define an impulse control policy $\mathfrak{p}$ as a sequence of stopping times (at which the fixed adjustment cost is paid) $0 < \tau_1 < \ldots < \tau_i < \ldots$ adapted to the filtration $\{\mathcal{F}_t\}$ generated by $\mathcal{W}_t$, and a sequence of random variables (the jumps in log-hours upon receiving an adjustment offer) $\Delta_0, \Delta_1, \ldots, \Delta_i, \ldots$, constructed inductively as follows. Let $\tau_0 = \hat{\tau}_0 = 0$ and $\{\hat{\tau}_i\}_{i \geq 1}$ be a sequence of i.i.d. random variables with exponential distribution $\mathcal{E}(q)$, i.e., $\mathbb{P}(\hat{\tau}_i \geq t) = e^{-qt}$ for all $t \geq 0$. For all $i \geq 0$, $\Delta_i \in \mathbb{R}$ is measurable with respect to the minimum σ-algebra $\mathcal{F}_{\tau_i + \hat{\tau}_i}$ of events up to time $\tau_i + \hat{\tau}_i$, and $\tau_{i+1} > \tau_i + \hat{\tau}_i$. I denote by $\mathcal{P}$ the set of such impulse control policies $\mathfrak{p}$.

An individual with current wage w_0 who just received an adjustment offer chooses hours h_0 and a stopping time $\tau_1 > 0$ (the next time at which he will

[14] See Dixit and Pindyck (1994) and Stokey (2008), and the references therein.

[15] In a previous version of this paper (Werquin (2014)), I considered salaried workers instead, and assumed that the total taxable income y, rather than hours h, is subject to the fixed adjustment cost. In this case, w and h are interpreted respectively as productivity and effort, and their product y is the agent's effective labor supply, or income. As the individual becomes more productive and stays in her current job (w increases and y remains constant), she needs to provide less effort (h decreases) to produce the required amount y. She adjusts her income upwards (resp., downwards) when she becomes so productive (resp., unproductive) that she spends most of her time idle (resp., when she must provide too much effort) to produce y. The results of the paper are unaffected by this alternative specification.

pay the fixed cost) to solve:

$$\tilde{\mathcal{V}}\left(w_0\right) = \max_{\{h_0,\mathrm{\tau}_1\}} \mathbb{E}_0\left[\int_0^{\infty} q e^{-q\hat{\mathrm{\tau}}}\left\{\int_0^{\mathrm{\tau}_1+\hat{\mathrm{\tau}}} e^{-(\rho_1+\rho_2)t} U\left(w_t h_0 - T\left(w_t h_0\right), h_0\right) dt \right.\right.$$
$$\left.\left. - e^{-(\rho_1+\rho_2)\mathrm{\tau}_1}\boldsymbol{\kappa}_{\mathrm{\tau}_1} + e^{-(\rho_1+\rho_2)\left(\mathrm{\tau}_1+\hat{\mathrm{\tau}}_1\right)}\tilde{\mathcal{V}}\left(w_{\mathrm{\tau}_1+\hat{\mathrm{\tau}}_1}\right)\right\} d\hat{\mathrm{\tau}}\right], \tag{6}$$

subject to (2). An individual is "inactive" if she has not yet paid the fixed adjustment cost since she started working at her current job, and is "searching" otherwise. Let $\mathcal{V}_i\left(w,h\right)$ and $\mathcal{V}_s\left(w,h\right)$ denote the corresponding value functions.

Government's problem. The government chooses the tax schedule (τ, p) to maximize long-run utilitarian social welfare[16] over all living individuals, subject to a budget balance constraint with an exogenous revenue requirement $\bar{R}$. Assuming their existence, let $f^i_{w,h}\left(\cdot,\cdot\right)$ and $f^s_{w,h}\left(\cdot,\cdot\right)$ denote the stationary joint densities of wages w and hours h for inactive and searching individuals, respectively. The government solves:

$$\max_{\{\tau,p\}} \quad \int_0^{\infty}\int_0^{\infty} \sum_{x\in\{i,s\}} \mathcal{V}_x\left(w,h\right) f^x_{w,h}\left(w,h\right) dw dh \tag{7}$$
$$\text{s.t.} \quad \int_0^{\infty}\int_0^{\infty} \sum_{x\in\{i,s\}} T\left(wh\right) f^x_{w,h}\left(w,h\right) dw dh \geq \bar{R}. \tag{8}$$

Let λ denote the marginal value of public funds, i.e. the Lagrange multiplier associated with the budget constraint (8), $\mathscr{R}\left(T\right)$ denote tax revenue, i.e. the left-hand side of (8), and $\mathscr{W}\left(T\right)$ denote social welfare expressed in monetary units, i.e. the maximand in (7) normalized by λ.

[16] Due to the assumption of exponential deaths, individuals receive equal weights independently of their age.

2 Individual behavior

Frictionless model. The first-order conditions of the frictionless problem (4) imply that the individual's optimal labor supply h_t^* at each instant t is an increasing function of her wage w_t and her net-of-tax rate $(1 - T'(w_t h_t^*))$. The frictionless taxable income $y_t^* = w_t h_t^*$ and disposable income c_t^* are then given by the following expressions as functions of the labor supply elasticity and the parameters of the tax schedule:

$$\begin{aligned} y_t^* &= (1 - T'(y_t^*))^{\varepsilon} w_t^{1+\varepsilon} = (1-\tau)^{\frac{\varepsilon}{1+p\varepsilon}} w_t^{\frac{1+\varepsilon}{1+p\varepsilon}}, \\ c_t^* &= y_t^* - T(y_t^*) = \frac{1}{1-p}(1-\tau)^{\frac{1+\varepsilon}{1+p\varepsilon}} w_t^{\frac{(1-p)(1+\varepsilon)}{1+p\varepsilon}}. \end{aligned} \tag{9}$$

Equations (9) imply that the effects of taxes on frictionless incomes at each time t are given by

$$\frac{d\ln y^*}{d\ln(1-\tau)} = \frac{\varepsilon}{1+p\varepsilon}, \quad \text{and} \quad \frac{d\ln y^*}{dp} = -\frac{\varepsilon}{1+p\varepsilon}\ln y^*. \tag{10}$$

The interpretation is as follows. The behavioral change in income y^* following a tax increase (both in τ and p) is determined by the elasticity ε. If the baseline tax system is linear, i.e. $p = 0$, (10) implies immediately that the elasticity of labor income y^* with respect to the net-of tax rate $1 - \tau$ is equal to ε. If the baseline tax system is non-linear, i.e. $p \neq 0$, a change in the marginal tax rate $T'(y^*)$ induces a direct reduction of labor income y^* by ε, which triggers in turn an indirect change $d(T'(y)) = T''(y)\,dy$ in the marginal tax rate faced by the individual, due to the non-linearity of the tax schedule, and hence a further income adjustment. Therefore

$$\frac{d\ln y^*}{d\ln\left(1-T'\left(y^*\right)\right)}=\frac{\varepsilon}{1+T''\left(y^*\right)\frac{y^*\varepsilon}{1-T'(y^*)}}=\frac{\varepsilon}{1+p\varepsilon}. \tag{11}$$

Thus the effect on income of a perturbation of the parameter $(1-\tau)$ is equivalent to uniformly perturbing by the same amount the net-of tax rates $(1-T'(y))$, while a perturbation of p is equivalent to perturbing the marginal tax rates at each income level by an amount proportional to log-income.

Equations (9) imply moreover that taxable and disposable incomes evolve according to random growth processes with endogenous drifts and volatilities:

$$d\ln y_t^* = \mu_y dt+\sigma_y d\mathcal{W}_t, \text{ with } \{\mu_y,\sigma_y\}=\frac{1+\varepsilon}{1+p\varepsilon}\{\mu_w,\sigma_w\}, \tag{12}$$

$$d\ln c_t^* = \mu_c dt+\sigma_c d\mathcal{W}_t, \text{ with } \{\mu_c,\sigma_c\}=(1-p)\frac{1+\varepsilon}{1+p\varepsilon}\{\mu_w,\sigma_w\}. \tag{13}$$

These expressions show that a higher rate of progressivity of the tax schedule lowers the drift and the volatility of both the taxable and the disposable income processes,

$$\frac{d\ln\{|\mu_y|,|\sigma_y|\}}{dp}=-\frac{\varepsilon}{1+p\varepsilon}, \text{ and } \frac{d\ln\{|\mu_c|,|\sigma_c|\}}{dp}=-\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}. \tag{14}$$

Intuitively, individual income responses to increases in wages are attenuated by the fact that higher incomes pay higher marginal tax rates if the tax schedule is progressive, so that tax progressivity dampens the volatility of income fluctuations. A uniform change in the marginal tax rates (i.e., a change in τ), on the other hand, does not affect the volatility of the income processes since all incomes are shifted by a proportional amount.

I assume that $\boldsymbol{\rho}\equiv\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2>0$, which ensures that lifetime utility is finite. The value function $\mathcal{V}^*(y)$ of an individual with income y is given by (see Appendix):

$$\mathcal{V}^*(y) = \frac{1}{\boldsymbol{\rho}} g\left(\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p} y^{1-p}\right). \tag{15}$$

Note that $\boldsymbol{\rho}$ depends on the growth rate of (the utility from) consumption, so that the relevant discount rate to compute the present value of utility is endogenous to taxes.

Frictional model. To analyze the frictional problem (6), introduce the labor supply deviation δ, defined as the log-difference between the actual and the frictionless hours of work h and h^*,

$$\delta_t \equiv \ln(h_t) - \ln(h_t^*) = \ln(y_t) - \ln(y_t^*). \tag{16}$$

While the individual remains inactive, δ_t evolves according to

$$d\delta_t = -\, d\ln h_t^* = \mu_\delta dt + \sigma_\delta d\mathcal{W}_t, \text{ with } \{\mu_\delta, \sigma_\delta\} = -\frac{(1-p)\,\varepsilon}{1+p\varepsilon}\{\mu_w, \sigma_w\}, \tag{17}$$

and when she adjusts her labor supply from h to h' at time $\hat{\tau}$, it jumps from $\delta_{\hat{\tau}^-}$ to $\delta_{\hat{\tau}^-} + (\ln h' - \ln h)$.

We can replace the state variables (w, h) by either (y^*, δ) or (y, δ) in the individual's problem. I denote accordingly the utility and value functions by $U(y^*, \delta)$, $\mathcal{V}_x(y^*, \delta)$ for $x \in \{i, s\}$, and the value function of an individual with actual, rather than frictionless, income y as $\hat{\mathcal{V}}_x(y, \delta) \equiv \mathcal{V}_x\left(ye^{-\delta}, \delta\right)$. I finally denote by $\bar{\mathcal{V}}(y, \delta)$ the average (over employment states x) welfare of individuals with states (y, δ) (see Appendix for a formal definition).

In the Appendix I show that the utility function can be written as $U(y^*, \delta) =$

$u(\delta) \times g(c^*)$, where $u(\delta)$ is (approximately) quadratic around the frictionless optimum $\delta = 0$.[17] The homogeneity of the utility function and of the fixed adjustment cost (5), along with the random growth processes for the frictionless disposable income (13), allow us to crucially reduce the dimensionality of the state space.[18]

Proposition 1. *The policy functions and the value functions of inactive and searching individuals with frictionless taxable income y^* and labor supply deviation δ are homogeneous of degree one in the utility of frictionless consumption $g(c^*)$:*

$$\mathcal{V}_x(y^*, \delta) = v_x(\delta) \times g\left(\frac{1-\tau}{1-p} y^{*(1-p)}\right), \quad \forall x \in \{i, s\}, \tag{18}$$

for some functions $v_i, v_s : \mathbb{R} \to \mathbb{R}$.

Proof. See Appendix. The value functions $v_i(\delta), v_s(\delta)$ are defined formally in (42) and plotted in Figure 3. □

An implication of Proposition 1 and equation (15) is that the value functions $\hat{\mathcal{V}}_x(y, \delta)$ of individuals with income y and deviation δ can be expressed as the value $\mathcal{V}^*(y)$ that the planner would compute for an individual with income y wrongly assuming that the world is frictionless, times a correction factor $\hat{v}_x(\delta)$ which depends only on her (unobservable) labor supply deviation δ. Unlike in the frictionless environment, where there is a representative agent at each income level y, there is now a heterogeneous population of individuals who earn the same income y but reach different levels of utility, because their

[17]Because the exact expression (derived in the Appendix) for the function $u(\delta)$ is not well defined for δ far away from 0, I assume for simplicity that the utility loss from failing to optimize is given by its quadratic approximation for any $\delta \in \mathbb{R}$. Alternatively we can keep the exact expression if $\gamma = 0$, and add curvature to the social welfare function to give the government a redistributive motive; none of the qualitative results would be affected.

[18]See Alvarez and Stokey (1998) for a general analysis of this property.

wage-hour bundles (i.e., their deviation δ) and their employment state x (inactive or searching) differ.[19] This heterogeneity is summarized by the functions $\hat{v}_x(\delta)$ which are strictly decreasing in δ (individuals who work less but earn the same income are better off) and are endogenous to tax policy (see Figure 4 and Appendix for details).

I now analyze the solution to the impulse control problem (6). Define a "$\{\underline{\delta}, \delta^*, \bar{\delta}\}$-policy", where $\underline{\delta} < \delta^* < \bar{\delta}$, as follows. For any labor supply h, the individual remains inactive as long as the state process δ_t is in $(\underline{\delta}, \bar{\delta})$. When δ_t hits or is below $\underline{\delta}$ or above $\bar{\delta}$, she pays the fixed cost $\boldsymbol{\kappa}$ and waits until she receives an adjustment opportunity, which occurs at the random time $\hat{\tau} \sim \mathcal{E}(q)$. She then adjusts the state to δ^*, so that hours jump from h to $h' = h \exp(\delta^* - \delta_{\hat{\tau}^-})$.

To find the solution to the individual's problem and characterize her optimal labor supply behavior, guess that the optimal control is a $\{\underline{\delta}, \delta^*, \bar{\delta}\}$-policy. The value of inaction $v_i(\delta)$ must satisfy the following Hamilton-Jacobi-Bellman equation within the inaction region: $\forall \delta \in (\underline{\delta}, \bar{\delta})$,

$$\mathcal{L}v_i(\delta) - \boldsymbol{\rho} v_i(\delta) = -u(\delta), \quad \text{where} \quad \mathcal{L}v \equiv \frac{1}{2}\sigma_\delta^2 v'' + [\mu_\delta + (1-\gamma)\sigma_c\sigma_\delta] v', \tag{19}$$

which has a standard asset pricing interpretation (see Appendix), subject to the following boundary conditions: (i) value-matching,

$$v_i(\bar{\delta}) = v_s(\bar{\delta}) - \kappa, \quad \text{and} \quad v_i(\underline{\delta}) \quad = v_s(\underline{\delta}) - \kappa, \tag{20}$$

which impose that at the time the agent decides to pay the fixed cost and search

[19] A key friction in this paper is that the government is restricted to using a tax $T(\cdot)$ on observed incomes y, and hence cannot differentiate between various individuals who earn the same amount but have different wage-hour pairs and employment states.

for a new job she is indifferent between doing so and remaining inactive; (*ii*) smooth-pasting,

$$v_i'\left(\bar{\delta}^+\right) = v_i'\left(\bar{\delta}^-\right) = v_s'\left(\bar{\delta}\right), \quad \text{and} \quad v_i'\left(\underline{\delta}^+\right) = v_i'\left(\underline{\delta}^-\right) = v_s'\left(\underline{\delta}\right), \tag{21}$$

which impose that at the time the agent decides to pay the fixed cost the marginal value and the marginal cost of starting to search are equal; and (*iii*) optimality,

$$v_i'\left(\delta^*\right) = 0, \tag{22}$$

which imposes that the value function at the optimal adjustment target δ^* is maximized. The value of searching $v_s(\delta)$ is equal to the sum of: the flow utility from the time at which the fixed cost is paid until the adjustment occurs; and the expected value of returning to δ^*. That is, for all $\delta \in \mathbb{R}$, letting $\hat{\tau} \sim \mathcal{E}(q)$,

$$v_s(\delta) = \mathbb{E}_0\left[\int_0^{\hat{\tau}} e^{-(\rho_1+\rho_2)t} U\left(y_t^*, \delta_t\right) dt \,\middle|\, \delta_0 = \delta\right] + \frac{q}{\boldsymbol{\rho} + q} v_i\left(\delta^*\right). \tag{23}$$

Equation (44) in the Appendix gives a closed-form expression for $v_s(\delta)$. I also show a heuristic derivation of this system of equations (19)-(23), and provide detailed interpretations.

The following proposition shows under which conditions the $\left\{\underline{\delta}, \delta^*, \bar{\delta}\right\}$-policy described above is the solution to the individual's problem:[20]

Proposition 2. *Suppose that there exist* $\underline{\delta} < \delta^* < \bar{\delta}$ *and* $v_i \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2\left(\mathbb{R} \setminus \left\{\underline{\delta}, \bar{\delta}\right\}\right)$ *that solve the system of equations (19) to (23). Suppose more-*

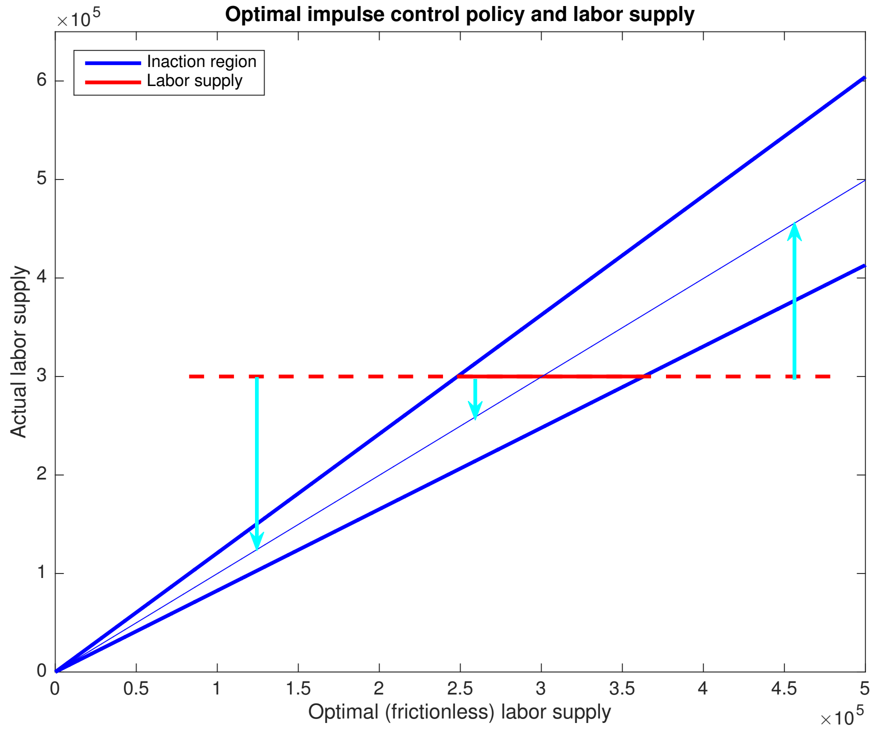
[20]Note that equations (19) to (23) define a *free-boundary* problem, as the boundaries of the inaction region are part of the solution $\left(v_i, v_s, \underline{\delta}, \delta^*, \bar{\delta}\right)$. For any set of three fixed (potentially suboptimal) thresholds $\underline{\delta} < \delta^* < \bar{\delta}$, equations (19), (20) and (23) define a *fixed-boundary* problem which can be solved in closed-form by integrating the second-order differential equation and inverting the linear system of two equations and two unknowns formed by the boundary conditions.

over that the following quasi-variational inequalities hold: $\boldsymbol{\mathcal{L}}v_s - \boldsymbol{\rho}v_s \leq -u$ *for all* $\delta \in \mathbb{R} \setminus \left(\underline{\delta}, \bar{\delta}\right)$ *and* $v_i \geq v_s - \kappa$ *for all* $\delta \in \left(\underline{\delta}, \bar{\delta}\right)$*, and that the technical condition (48) in the Appendix is satisfied. Then* v_i *and* v_s *are the value functions of inactives and searchers, respectively, and the optimal impulse control policy is the* $\left\{\underline{\delta}, \delta^*, \bar{\delta}\right\}$*-policy.*

Proof. See Appendix. □

Individual labor supply behavior is depicted graphically in Figure 1. The variable on the x-axis is the individual's frictionless labor supply h^*, and that on the y-axis is her actual labor supply h. In the frictionless environment, labor supply would evolve along the 45°-line. In the frictional environment, labor supply is lumpy: an inactive individual moves along a horizontal (red) line, that is, her actual labor supply h stays constant while her desired labor supply h^* tracks the evolution of her productivity. When she reaches the (optimally chosen) boundaries of her inaction region (the thick blue lines $h^* = e^{-\bar{\delta}}h$ or $h^* = e^{-\underline{\delta}}h$), she starts searching and, as soon as she receives an offer, she adjusts (up or down) to the new labor supply level on the central blue line $h' = e^{\delta^*}h^*$.

Figure 1: Optimal labor supply behavior



The optimal labor supply adjustment behavior $\left\{\underline{\delta}, \delta^{*}, \bar{\delta}\right\}$ is not affected by the parameter τ of the tax schedule, while a lower progressivity p increases the size of the inaction region:

$$\frac{d \ln \left\{\left|\underline{\delta}\right|, \bar{\delta}\right\}}{d \ln (1-\tau)}=0, \quad \text{and} \quad \frac{d \ln \left\{\left|\underline{\delta}\right|, \bar{\delta}\right\}}{d p}<0.$$

This is because a decrease in p acts as a positive volatility shock (see equation (14)), i.e., it increases the volatilities σ_y, σ_δ of the income and the deviation processes. This in turn raises the option value of waiting to adjust labor supply, and therefore widens the optimal inaction region. Intuitively, a less progressive tax schedule magnifies the unexpected shocks to the wage, which raises the incentives for the individual to keep her current job and wait to observe the evolution of her productivity before carrying out the costly adjustment, in order to save on new search costs.

It follows that a lower rate of progressivity has an ambiguous effect on the frequency of adjustment of labor supply: on the one hand, the higher volatility of the deviation process makes current labor supply diverge from its frictionless optimum at a faster rate, so that individuals reach more quickly the boundaries of their inaction region; on the other hand, the inaction region is wider, which tends to make them adjust less often. Importantly, there is no reason to expect a priori that the (direct) volatility-dampening effect of progressivity has exactly the same magnitude as the (indirect) option value effect on the boundaries of the inaction region; in fact, the reduction in the volatility of income typically vastly dominates the option value effect.[21] Another way to express this heuristically is to say that an individual adjusts her labor supply in response to productivity changes whenever her current hours of work are 10 percent (say) away from their optimum amount, and this maximum tolerated deviation is hardly affected by taxes. On the other hand, progressivity has a large dampening effect on the volatility of income fluctuations. The discrepancy between these two opposing forces plays a crucial role in the optimal taxation analysis of Section 4.

Figures 3 and 4 in the Appendix show the optimal thresholds $\underline{\delta}, \delta^{*}, \bar{\delta}$, the value functions $v_i(\delta), v_s(\delta)$ and $\hat{v}_i(\delta), \hat{v}_s(\delta)$, and their responses to a change

[21] Hence a less progressive tax schedule increases the frequency of adjustment. This result, which I show numerically in Section 5, is common to all the papers in this literature, see e.g. Vavra (2014) (with the exception of Bloom (2009), because he focuses on the impact effect of shocks, rather than on the long-run effects). Theoretically, Dixit (1993) shows in a simple version of the model that the option value effect is several orders of magnitude smaller than the direct volatility effect. The optimal taxation formulas I derive in Section 4 hold of course for any relative value of these two opposing forces.

in progressivity.

3 Aggregation

In this section I characterize the long-run wage and income distributions obtained by aggregating the optimal individual policies described in Section 2.

3.1 Stationary wage distribution

A variable x has a double-Pareto-lognormal distribution (DPLN) (or $\ln x$ has a Normal-Laplace distribution) with parameters (r_1, r_2, m, s^2) (where $r_1 < 0 < r_2$) if its density is given by

$$f_x(x) = \frac{|r_1| r_2}{|r_1| + r_2} \left\{ e^{\frac{1}{2}r_1^2 s^2 - r_1 m} x^{r_1 - 1} \Phi\left(\frac{\ln x - m}{s} + r_1 s\right) \right. \\ \left. + e^{\frac{1}{2}r_2^2 s^2 - r_2 m} x^{r_2 - 1} \Phi^c\left(\frac{\ln x - m}{s} + r_2 s\right) \right\}. \tag{24}$$

The double-Pareto-lognormal distribution closely approximates the actual wage and income distributions observed empirically.[22] In particular, the DPLN distribution exhibits power-law behavior in both tails, with Pareto coefficients on the right and left tail respectively given by (r_1, r_2), that is,

$$f_x(x) \underset{x \to 0}{\sim} w^{r_2 - 1}, \text{ and } f_x(x) \underset{x \to \infty}{\sim} w^{r_1 - 1}. \tag{25}$$

Define, for any $x \in \{w, y, c, \delta\}$ and $\rho > 0$,

$$r_{1,x}^{\rho} \equiv \frac{\mu_x}{\sigma_x^2} - \sqrt{\frac{\mu_x^2}{\sigma_x^4} + \frac{2\rho}{\sigma_x^2}}, \text{ and } r_{2,x}^{\rho} \equiv \frac{\mu_x}{\sigma_x^2} + \sqrt{\frac{\mu_x^2}{\sigma_x^4} + \frac{2\rho}{\sigma_x^2}}, \tag{26}$$

[22]See, e.g., Reed (2003), Reed and Jorgensen (2004), Toda (2012).

The following proposition shows that the wage distribution converges to a DPLN stationary distribution:

Proposition 3. *The distribution of wages w converges towards a unique stationary distribution $f_w(\cdot)$ which is double-Pareto-lognormal with parameters $\left(r_{1,w}^{\rho_2}, r_{2,w}^{\rho_2}, m_w, s_w^2\right)$, where $r_{1,w}^{\rho_2}, r_{2,w}^{\rho_2}$ are defined in (26) and m_w, s_w^2 are the mean and variance of the log-wage distribution at birth.*[23]

Proof. See Toda (2012). □

The aggregation of the random growth individual wage processes naturally generates the wage distribution's Pareto tails,[24] which is one of the most robust empirical stylized facts (as well as an important determinant of optimal taxes, see Saez (2001)). The smaller the Pareto coefficient (in absolute value) $\left|r_{1,w}^{\rho_2}\right|$, the thicker the right tail, the more unequal the distribution. A higher drift μ_w and volatility σ_w, and a lower death rate ρ_2, lead to a more unequal distribution.

3.2 Stationary income distributions

I now characterize the stationary joint distributions $f_{\ln y^*,\delta}^i$ and $f_{\ln y^*,\delta}^s$ of frictionless taxable incomes $\ln y^*$ and labor supply deviations δ for inactive and searching individuals, respectively. Denote by f_1, f_2 their partial derivatives with respect to the first and second variables, and by f_{11}, f_{12}, f_{22} their second partial derivatives. We have $f^i = 0$ for all $\delta < \underline{\delta}$ and $\delta > \bar{\delta}$. Moreover, for all $\ln y^* \in \mathbb{R}$, all $\delta \in (\underline{\delta}, \delta^*) \cup \left(\delta^*, \bar{\delta}\right)$ if $f = f^i$, and all $\delta \in \mathbb{R} \setminus \left\{\underline{\delta}, \bar{\delta}\right\}$ if $f = f^s$, these

[23]The frictionless taxable and disposable incomes y^*, c^* are also log-normally distributed at birth with respective mean and variance (m_y, s_y) and (m_c, s_c) and follow random growth processes (12,13) from then on. Hence their stationary distributions f_{y^*}, f_{c^*} are also double-Pareto lognormal with respective parameters $\left(r_{1,y}^{\rho_2}, r_{2,y}^{\rho_2}, m_y, s_y^2\right)$ and $\left(r_{1,c}^{\rho_2}, r_{2,c}^{\rho_2}, m_c, s_c^2\right)$.

[24]See, e.g., Nirei and Souma (2007), Gabaix (2009).

distributions are the solutions to the following Kolmogorov-forward equations (KFE):

$$0 = -\left(\rho_2 + q\mathbb{I}_{f^s}\right) f - \mu_y f_1 + \mu_\delta f_2 + \frac{1}{2}\sigma_y^2 f_{11} + \frac{1}{2}\sigma_\delta^2 f_{22} - \sigma_y \sigma_\delta f_{12}, \tag{27}$$

where $\mathbb{I}_{f^s}$ is equal to one if $f = f^s_{\ln y^*,\delta}$ and zero if $f = f^i_{\ln y^*,\delta}$. The boundary conditions and the derivation and interpretation of these equations, which equate the inflows and outflows at any $(\ln y^*, \delta) \in \mathbb{R}^2$, are laid out in the Appendix.

Proposition 4. *Assuming their existence, the stationary distributions $f^i_{\ln y^*,\delta}$ and $f^s_{\ln y^*,\delta}$ of inactive and searching individuals are the solution to the Kolmogorov-forward partial differential equations (27) subject to the boundary conditions (49), (50), (51), (52), (53), (54), and (55) in the Appendix. If $q = \infty$ the stationary distributions of taxable and disposable incomes f_y, f_c have Pareto right and left tails with respective coefficients $\left(r^{\rho_2}_{1,y}, r^{\rho_2}_{2,y}\right)$ and $\left(r^{\rho_2}_{1,c}, r^{\rho_2}_{2,c}\right)$.*

Proof. See Appendix. $\square$

Ignoring the search period for simplicity ($q = \infty$), Proposition 4 shows that the Pareto coefficients of the tails of the taxable and disposable income distributions are given in closed form by

$$\left\{r^{\rho_2}_{1,y}, r^{\rho_2}_{2,y}\right\} = \frac{1+p\varepsilon}{1+\varepsilon}\left\{r^{\rho_2}_{1,w}, r^{\rho_2}_{2,w}\right\}, \text{ and } \left\{r^{\rho_2}_{1,c}, r^{\rho_2}_{2,c}\right\} = \frac{1}{1-p}\frac{1+p\varepsilon}{1+\varepsilon}\left\{r^{\rho_2}_{1,w}, r^{\rho_2}_{2,w}\right\}. \tag{28}$$

These expressions show that the labor supply elasticity ε and the progressivity p determine the amount by which inequality in exogenous productivities (measured by the Pareto coefficient $r^{\rho_2}_{1,w}$) translates into inequality in taxable and disposable incomes. The distribution of taxable income y is more unequal (thicker tail) than the wage distribution, because wage differences are amplified by the labor supply responses due to the positive elasticity ε.[25] Moreover, both the before-tax and (more strongly so) the after-tax income distributions become more equal when the tax schedule is more progressive: $\frac{d}{dp}\ln\left|r_{1,c}^{\rho_2}\right| > \frac{d}{dp}\ln\left|r_{1,y}^{\rho_2}\right| > 0$.[26] Figure 5 in the Appendix summarizes these results graphically and plots the stationary distributions of wages w, incomes y, c, and labor supply deviations δ, as well as the effects of taxes on these distributions.

4 Optimal taxation

In this section, I analyze the effects of taxes on long-run social welfare (comparative statics across steady-states) in order to characterize the optimal tax schedule, that is, the solution to the government's problem (7,8), given that individual labor supply is characterized by Proposition 2. Before deriving these formulas, I formally define and characterize two sets of key variables for the analysis of tax policy: the labor income elasticities and the marginal social welfare weights.

4.1 Labor income elasticities

I first define the (Hicksian) frictionless intensive margin labor income elasticity, $\boldsymbol{\varepsilon}^*(y^*)$, as the elasticity of an individual's frictionless taxable income y^* with respect to the net of tax rate $(1-T'(y^*))$. We saw in Section 2 that

[25] This is consistent with the findings of Krueger, Perri, Pistaferri, and Violante (2010).

[26] Note also that $r_{1,y}^{\rho_2} \leq r_{1,c}^{\rho_2} < r_{1,w}^{\rho_2}$ if $p \geq 0$ and $r_{1,c}^{\rho_2} \leq r_{1,y}^{\rho_2} < r_{1,w}^{\rho_2}$ if $p \leq 0$, with strict inequalities if $p \neq 0$, so that the distribution of frictionless disposable income is less unequal than the distribution of desired taxable income if and only if the tax schedule is progressive.

this elasticity is constant accross individuals and is given by the (normalized) structural elasticity parameter ε.

Definition 1. The *frictionless intensive margin labor income elasticity* is defined as

$$\boldsymbol{\varepsilon}^*(y^*) \equiv \frac{d\ln y^*}{d\ln(1-T'(y^*))} = \frac{\varepsilon}{1+p\varepsilon}, \quad \forall y^* \in \mathbb{R}_+, \tag{29}$$

where the second equality follows from equations (10) and (11).

In a frictionless world ($\kappa = 0$), this variable $\boldsymbol{\varepsilon}^*(y)$ would be equal to the response of the individual's true income to a change in marginal tax rates. It could be estimated empirically by observing the magnitude of the change in income following a change in statutory net-of-tax rates (see, e.g., Gruber and Saez (2002)). However, when the adjustment of labor supply in response to tax changes is frictional ($\kappa > 0$), the individual elasticity of *actual* (observed) income is not well defined: it is in general equal to zero, if the agent has not yet adjusted her income in response to the tax change, and is infinite at the time of adjustment since a small tax increase then induces a discrete jump in income. In this environment we can nevertheless define and observe empirically the *long-run* elasticity of *aggregate* labor income with respect to a *uniform* change in net-of-tax rates (that is, a change in $(1-\tau)$). Formally,

Definition 2. The *macro labor income elasticity* $\boldsymbol{E}$ is defined as

$$\boldsymbol{E} \equiv \frac{d\ln\int_0^\infty y f_y(y)\,dy}{d\ln(1-\tau)}, \tag{30}$$

where $f_y(\cdot)$ is the stationary density of incomes.

The following proposition proves a neutrality result that characterizes the relationship between the frictionless individual elasticity and the macro elasticity:

Proposition 5. *The frictionless intensive margin elasticity and the macro elasticity are equal, that is,*

$$\boldsymbol{E} = \boldsymbol{\varepsilon}^* (y) = \frac{\varepsilon}{1 + p\varepsilon}, \quad \forall y \in \mathbb{R}_+. \tag{31}$$

Proof. See Appendix. □

Proposition 5 shows that in the frictional model, the long-run aggregate income elasticity $\boldsymbol{E}$ is equal to the elasticity of frictionless individual income $\boldsymbol{\varepsilon}^*$, even though there is always (even in the steady-state) a non-degenerate distribution of individuals with actual incomes y at each frictionless level y^*. Intuitively, in the long-run, individuals have had the time to fully adjust their behavior to the new tax schedule and even though they almost never actually earn their desired income y^*, the individual errors wash out in the aggregate: after the tax change, individuals revolve around their new optimum which is ε percent away from the previous one, so that the aggregate (log-)income distribution is uniformly shifted by ε. In other words, in the case of a uniform increase in the net-of-tax rates (i.e., a change in the parameter τ), the economy behaves in the long-run as if there were a representative frictionless agent at each income level.[27,28] Note in particular that this result allows us to recover empirically the structural parameter ε when individual labor supply is lumpy.

Next, I define three extensive margin labor income elasticities as the ef-

[27]This neutrality result is related to those of Caplin and Spulber (1987) and Rogerson (1988) who build models where frictions at the micro (individual) level are irrelevant at the macro (aggregate) level.

[28]This neutrality result formalizes the intuition (explained in the Introduction of this paper) that the standard frictionless model should be interpreted as the long-run outcome of an environment with sluggish adjustments in the short-run. Of course this result depends on the assumptions I have made in the previous sections about individual behavior. It should be interpreted as follows: the assumptions of my model are giving the strongest possible chances to the frictionless model's tax formulas (39), (40) to capture the long-run effects of taxation in the frictional environment.

fects of changes in the adjustment thresholds $\underline{\delta}, \delta^*, \bar{\delta}$ on the stationary income distribution.

Definition 3. The *extensive margin labor income elasticities* $\underline{\Xi}(y)$, $\Xi^*(y)$, and $\bar{\Xi}(y)$ are defined as

$$\underline{\Xi}(y) \equiv \frac{\partial \ln f_y(y)}{\partial \ln |\underline{\delta}|}, \quad \Xi^*(y) \equiv \frac{\partial \ln f_y(y)}{\partial \ln |\delta^*|}, \text{ and } \bar{\Xi}(y) \equiv \frac{\partial \ln f_y(y)}{\partial \ln \bar{\delta}}, \tag{32}$$

where $f_y(\cdot)$ is the stationary density of incomes.

These elasticities capture the effects of percentage variations in the inaction thresholds on the number of employed workers at income y. The definition of these elasticities is similar to the participation elasticities in, e.g., Saez (2002); the key difference is that in Definition 3 the income thresholds are not exogenously given as in the case of a binary participation decision, but instead are endogenously and optimally chosen by the individual.

4.2 Marginal social welfare weights

The social welfare effects of taxation can be characterized using the notion of marginal social welfare weights.[29] In the standard frictionless static model with a utilitarian social objective, the social weight at income y is defined as the individual's marginal utility of consumption normalized by the shadow value of public funds, $\lambda^{-1}c_0^{-\gamma}$,[30] and represents intuitively the increase in social welfare, expressed in terms of public revenue, of distributing an additional unit of consumption uniformly among individuals who earn income y. In this section I define formally and generalize to the dynamic and frictional environment the relevant notions of marginal social welfare weights.

[29]See, e.g., Saez and Stantcheva (2016) for a recent general exposition.
[30]Note that these marginal social welfare weights are endogenous.

First consider, in the frictionless model, the effect of giving an additional marginal consumption stream $\{\hat{c}_t\}_{t\geq 0}$ to an individual with current income y, where $\hat{c}_t$ evolves stochastically according to the same process (13) as the frictionless disposable income c_t^*. The frictionless social weight $\boldsymbol{\omega}^*(y)$ is defined as the change in the individual's value function (and hence, in utilitarian social welfare) due to this additional consumption stream:

$$\boldsymbol{\omega}^*(y) \equiv \frac{1}{\lambda^*}\frac{d}{d\hat{c}_0}\,\mathbb{E}_0\left[\int_0^\infty e^{-(\rho_1+\rho_2)t}u\left(c_t^*+\hat{c}_t-\frac{(h_t^*)^{1+1/\varepsilon}}{1+1/\varepsilon}\right)dt\Big|y_0=y\right]\Bigg|_{\hat{c}_0=0}, \tag{33}$$

where λ^* is the marginal value of public funds in the frictionless model. A closed-form expression for $\boldsymbol{\omega}^*(y)$ is derived in the Appendix.

Now, in the frictional model, giving individuals with income y the additional marginal consumption stream $\{\hat{c}_t\}_{t\geq 0}$ defined above has different welfare effects depending on their deviations δ (i.e., their wage-hours bundles) and employment states $x\in\{i,s\}$. Since the income tax system treats all individuals with the same income identically, we define the average social weight at income y, $\boldsymbol{\omega}(y)$, as follows:

Definition 4. The *static intensive margin social weight* $\boldsymbol{\omega}(y)$ is defined as

$$\boldsymbol{\omega}(y) = \frac{\lambda^*}{\lambda}\boldsymbol{\omega}^*(y)\times\int_{-\infty}^{\infty}\bar{v}(y,\delta)\,f_{\delta|y}(\delta\,|y)\,d\delta, \tag{34}$$

where $\boldsymbol{\omega}^*(y)$ is the corresponding frictionless welfare weight defined in equation (33), $\bar{v}(y,\delta)$ is the average (over employment states x) welfare of individuals with income and deviation (y,δ), defined formally in equation (43), $f_{\delta|y} = f^i_{\delta|y}+f^s_{\delta|y}$ is the total density of deviations conditional on an actual income y, and λ is the marginal value of public funds in the frictional model.

Second, we saw that a permanent change in progressivity affects the drift and volatility of the consumption process, and hence the discount rate $\boldsymbol{\rho}$ used to compute welfare. I thus define the dynamic intensive margin social weight $\hat{\boldsymbol{\omega}}(y)$ as the welfare effect of a percentage decrease in $\boldsymbol{\rho}$. In the frictionless model, this is given by

$$\hat{\boldsymbol{\omega}}^*(y) = -\frac{1}{\lambda^*}\frac{\partial \mathcal{V}^*(y)}{\partial \ln \boldsymbol{\rho}}, \tag{35}$$

for which a closed-form expression is given in the Appendix. Similarly, in the frictional model I define:

Definition 5. The *dynamic intensive margin social weight* $\hat{\boldsymbol{\omega}}(y)$ is defined as

$$\hat{\boldsymbol{\omega}}(y) = \frac{\lambda^*}{\lambda}\hat{\boldsymbol{\omega}}^*(y) \times \int_{-\infty}^{\infty} \bar{v}(y,\delta)\, f_{\delta|y}(\delta\,|y)\, d\delta. \tag{36}$$

Third, I define the extensive margin social weights $\underline{\boldsymbol{\Omega}}(y)$, $\boldsymbol{\Omega}^*(y)$, $\bar{\boldsymbol{\Omega}}(y)$ as the effects of changes in the thresholds $\underline{\delta}, \delta^*, \bar{\delta}$ on total welfare at the income level y:

Definition 6. Let $\{\delta_i\}_{1\le i\le 3} \equiv \{\underline{\delta}, \delta^*, \bar{\delta}\}$. The *extensive margin social weights* $\{\boldsymbol{\Omega}_i(y)\}_{1\le i\le 3} = \{\underline{\boldsymbol{\Omega}}(y), \boldsymbol{\Omega}^*(y), \bar{\boldsymbol{\Omega}}(y)\}$ are defined as

$$\boldsymbol{\Omega}_i(y) \equiv \frac{1}{\lambda}\int_{-\infty}^{\infty} \frac{\partial \ln f_{y,\delta}(y,\delta)}{\partial \ln |\delta_i|}\bar{\mathcal{V}}(y,\delta)\, f_{\delta|y}(\delta\,|y)\, d\delta, \;\; \forall i \in \{1,2,3\}, \tag{37}$$

where $\bar{\mathcal{V}}(y,\delta)$ is defined in equation (43).

Finally, the equilibrium composition of each income group y, summarized by the value functions $\bar{v}(y,\delta)$, is endogenously affected by progressivity, which in turn has an effect on social welfare. I thus define the composition margin social weight $\tilde{\boldsymbol{\Omega}}(y)$ as:

Definition 7. The *composition margin social weight* $\tilde{\boldsymbol{\Omega}}(y)$ is defined as

$$\tilde{\boldsymbol{\Omega}}(y) \equiv \frac{1}{\lambda}\int_{-\infty}^{\infty}\left(\frac{\partial \ln \bar{v}(y,\delta)}{\partial p}+\frac{\partial \ln \bar{v}(y,\delta)}{\partial \ln y}\frac{\partial \ln y}{\partial p}\right)\bar{\mathcal{V}}(y,\delta)\, f_{\delta|y}(\delta\,|y)\, d\delta, \tag{38}$$

where $\frac{\partial \ln y}{\partial p}$ is given by (10). In particular, when $q=\infty$, $\tilde{\boldsymbol{\Omega}}$ is simply equal to $\frac{1}{\lambda}\frac{\partial \mathbb{E}[\bar{v}(\delta)|y]}{\partial p}\mathcal{V}^*(y)$, that is, the effect of progressivity on the average welfare at income y.

4.3 Tax reforms and optimal tax schedule

In this section I derive the first-order social welfare effects of two "local tax reforms" of a given baseline (potentially suboptimal) tax system T: an (infinitesimal) perturbation of the parameter τ by $d\tau$, and a perturbation of the progressivity p by dp. In the former case, this implies that the tax liability at income y, $T(y)=\frac{1-\tau}{1-p}y^{1-p}$, is replaced by the perturbed tax liability $T(y)+\Psi_\tau(y)\,d\tau$, and in the latter case by $T(y)+\Psi_p(y)\,dp$, where for all $y\in\mathbb{R}_+$,

$$\Psi_\tau(y)=\frac{y^{1-p}}{\int_0^\infty x^{1-p}f_y(x)\,dx}, \quad \text{and} \quad \Psi_p(y)=\frac{\left(\ln y-\frac{1}{1-p}\right)y^{1-p}}{\int_0^\infty\left(\ln x-\frac{1}{1-p}\right)x^{1-p}f_y(x)\,dx},$$

where $f_y(x)$ denotes the stationary density of incomes given the baseline tax schedule (τ,p). The denominators of these expressions are simply a normalization ensuring that in the absence of any behavioral responses (i.e., if all individuals earned the same taxable income before and after the tax reform), the total ("statutory") additional tax revenue collected would be 1 (in monetary units, say dollars).

I denote the corresponding first-order social welfare changes by Γ_τ and Γ_p, defined as

$$\Gamma_\tau = \lim_{d\tau \to 0} \frac{\mathscr{W}\left(T + \Psi_\tau d\tau\right) - \mathscr{W}\left(T\right)}{d\tau}, \quad \text{and} \quad \Gamma_p = \lim_{dp \to 0} \frac{d\mathscr{W}\left(T + \Psi_p dp\right) - \mathscr{W}\left(T\right)}{dp},$$

where $\mathscr{W}\left(\mathcal{T}\right)$ denotes the social welfare when the tax schedule is $\mathcal{T}$ (see (7)). Formally, this is the Gateaux derivative of social welfare with respect to the tax reforms Ψ_τ and Ψ_p, or simply the derivatives $d\mathscr{W}/d\tau$ and $d\mathscr{W}/dp$, normalized as explained above.[31] The optimal tax schedule (τ^*, p^*) is then characterized by imposing that the first-order welfare effects Γ_τ and Γ_p are equal to zero.

The following proposition characterizes the welfare effects of the perturbation Ψ_τ.

Proposition 6. *The first-order social welfare effects of uniformly perturbing the marginal tax rates of any baseline tax schedule T are given by:*[32]

$$\Gamma_\tau = 1 - \int_0^\infty T'\left(y\right) \frac{y\boldsymbol{\varepsilon}^*\left(y\right)}{1 - T'\left(y\right)} \Psi'_\tau\left(y\right) f_y\left(y\right) dy - \int_0^\infty \boldsymbol{\omega}\left(y\right) \Psi_\tau\left(y\right) f_y\left(y\right) dy. \tag{39}$$

In particular, the optimal tax schedule (τ^*, p^*) satisfies $\Gamma_\tau = 0$. In the frictionless model, the same equation holds, except that the marginal social welfare weights $\boldsymbol{\omega}\left(y\right)$ are replaced by their frictionless counterparts $\boldsymbol{\omega}^*\left(y\right)$.

Proof. See Appendix. □

The interpretation of equation (39) is as follows. The first and second terms on the right hand side measure the actual change in government tax revenue

[31] See details in Golosov, Tsyvinski, and Werquin (2013).

[32] This equation can be thought of as pinning down the marginal value of public funds λ given a tax schedule (τ, p). Intuitively, λ (the Lagrange multiplier associated with the constraint (8)) is equal to the social value of redistributing a dollar of tax revenue through an decrease in τ by $d\tau$, i.e., through a uniform increase in the net of tax rates (taking into account the behavioral responses that this perturbation induces).

of a one-dollar statutory increase in taxes through a uniform perturbation of the marginal tax rates, taking into account the induced change in individual behavior. The additional tax liability levied at the income level y after the tax reform is implemented is given, to a first order in $d\tau \to 0$, by $\Psi_\tau(y)\,d\tau$, and the marginal tax rate changes by $\Psi'_\tau(y)\,d\tau$. The first term in the right hand side of (39) is the *mechanical effect* of the perturbation, i.e., the statutory increase in government revenue absent behavioral responses. It is equal to one (dollar) by construction of the normalization of the magnitude of the perturbation Ψ_τ. The second term in the right hand side of equation (39) is the *behavioral effect* of the perturbation. The increase $dT' \equiv \Psi'_\tau(y)\,d\tau$ in the marginal tax rate of an individual with income y induces her to decrease her taxable income by $\frac{y}{1-T'(y)}\boldsymbol{\varepsilon}^*(y)\,dT'$. This behavioral income response generates a loss in government revenue proportional to the marginal tax rate $T'(y)$. Summing over individuals using the density of incomes $f_y(\cdot)$ yields the total revenue loss, i.e. the second term in (39). Finally, the third term in (39) is the *welfare effect* of the perturbation, expressed in monetary units. An increase in the tax liability of individual y by $dT \equiv \Psi_\tau(y)\,d\tau$ directly reduces her utility and hence social welfare by $\boldsymbol{\omega}(y) \times dT$, by construction of the marginal social welfare weights (33).

This equation (39) has a structure that is identical to the "sufficient statistic" formulas derived by, e.g., Saez (2001), Golosov, Tsyvinski, and Werquin (2013), Hendren (2014, 2015), Jacquet and Lehmann (2015), to characterize the optimal tax systems in frictionless models. Note in particular that all the variables other than the marginal social welfare weights in equation (39) (elasticities, tax schedule, income distribution) are empirically observable.[33]

[33] Using the formula $\Gamma_\tau = 0$ to characterize the optimum tax schedule requires evaluating these endogenous variables at the optimum. Strictly speaking, the values estimated given

Proposition 6 extends the neutrality result of Proposition 5. Its main insight is that the long-run effects on social welfare of a uniform change in marginal tax rates are the same as we would calculate by assuming that the economy is frictionless and has a representative agent at each income level y. In particular, as in Proposition 5, the relevant elasticity is the individual frictionless elasticity $\boldsymbol{\varepsilon}^*(y)$ defined in (29). In short, the frictions at the individual level wash out in the long-run of the aggregate economy.[34]

The next proposition characterizes the welfare effects of perturbing the progressivity p of the tax schedule. It is the main theoretical result of the paper.

Proposition 7. *In the frictionless model, the first-order social welfare effects of perturbing the progressivity of any baseline tax schedule T are given by:*

$$\Gamma_p^* = 1 - \int_0^\infty T' \frac{y}{1-T'} \boldsymbol{\varepsilon}^* \Psi_p' dF_y - \int_0^\infty \left[\boldsymbol{\omega}^* \Psi_p + \frac{d \ln \boldsymbol{\rho}}{dp} \hat{\boldsymbol{\omega}}^* \right] dF_y. \tag{40}$$

In the frictional model, these effects are given by:

$$\begin{aligned} \Gamma_p = 1 &- \int_0^\infty T' \frac{y}{1-T'} \boldsymbol{\varepsilon}^* \Psi_p' dF_y - \int_0^\infty \left[\boldsymbol{\omega} \Psi_p + \frac{d \ln \boldsymbol{\rho}}{dp} \hat{\boldsymbol{\omega}} \right] dF_y + \boldsymbol{A} \\ &+ \int_0^\infty T \sum_{i=1}^{3} \frac{d \ln \frac{|\delta_i|}{|\sigma_\delta|}}{dp} \boldsymbol{\Xi}_i dF_y + \int_0^\infty \left[\sum_{i=1}^{3} \frac{d \ln \frac{|\delta_i|}{|\sigma_\delta|}}{dp} \boldsymbol{\Omega}_i + \tilde{\boldsymbol{\Omega}} \right] dF_y, \end{aligned} \tag{41}$$

where $\{\delta_i\}_{1 \le i \le 3} = \{\underline{\delta}, \delta^*, \bar{\delta}\}$, $\{\boldsymbol{\Xi}_i\}_{1 \le i \le 3} = \{\underline{\boldsymbol{\Xi}}, \boldsymbol{\Xi}^*, \bar{\boldsymbol{\Xi}}\}$, $\{\boldsymbol{\Omega}_i\}_{1 \le i \le 3} = \{\underline{\boldsymbol{\Omega}}, \boldsymbol{\Omega}^*, \bar{\boldsymbol{\Omega}}\}$,

the current tax code (in particular, the current U.S. income distribution) can only be used to quantify the welfare effects of *local tax reforms*, as explained above.

[34] There is one difference, however, between the frictionless and the frictional versions of the optimal tax formula (39): the frictional marginal social welfare weights $\boldsymbol{\omega}(y)$ must be computed by taking into account the non-degenerate distribution of utilities $\mathbb{E}[\bar{v}(y,\delta)|y]$ within income groups. In general this correction term varies with income y, so that the schedule of frictional social weights is not perfectly homothetic to the schedule of frictionless weights, and the effective redistributive tastes of the government have to be adjusted relative to a model with a representative agent at each income level.

and $\boldsymbol{A} = \frac{d \ln \frac{\sigma_y}{|\sigma_\delta|}}{dp} \int \left(T' y - \frac{1}{\lambda} \frac{\partial \bar{\mathcal{V}}}{\partial \ln y} \right) \delta dF_{y,\delta}$. *In particular, the optimal tax schedule* (τ^*, p^*) *is fully characterized by (8), (39), and* $\Gamma_p = 0$.

Proof. See Appendix. □

The first result of Proposition 7, equation (40), characterizes the effects of an increase in progressivity (and in particular, the optimal tax schedule) in the frictionless model. Its interpretation is identical to that of equation (39), except that the welfare effect depends in addition on the social weight $\hat{\boldsymbol{\omega}}^*(y)$.[35] It has the same structure as the standard sufficient statistic formulas in static models.

The second, and most important, result of Proposition 7 is the derivation of equation (41) which characterizes the long-run welfare effects of raising progressivity in the frictional model. The first line of this expression is the same as the frictionless formula (40), replacing the social weights with their frictional counterparts $\boldsymbol{\omega}(y), \hat{\boldsymbol{\omega}}(y)$ (and with the exception of the unimportant term $\boldsymbol{A}$).[36] The presence of the second line implies that the frictionless formula does *not* correctly account for all of the long-run effects of nonlinear taxes. The novel effects are: (i) behavioral (revenue) effects driven by the extensive margin labor income elasticities $\underline{\boldsymbol{\Xi}}(y)$, $\boldsymbol{\Xi}^*(y)$, $\bar{\boldsymbol{\Xi}}(y)$; (ii) welfare effects driven by the extensive margin social weights $\underline{\boldsymbol{\Omega}}(y)$, $\boldsymbol{\Omega}^*(y)$, $\bar{\boldsymbol{\Omega}}(y)$; and (iii) a welfare

[35]These social weights are absent from expression (39) and of the optimal tax formulas derived in standard static models. This highlights the need to properly microfound the "long-run" as the steady-state of a dynamic economy (rather than simply applying the static formulas) to capture the full welfare effects of permanent changes in taxes.

[36]This term appears because because a change in progressivity affects the *relative* processes driving the frictionless income variables $\{\mu_y, \sigma_y, m_y, s_y\}$ (with elasticity $\frac{d \ln \sigma_y}{dp}$) and the deviation (or hour) variables $\{\mu_\delta, \sigma_\delta, \underline{\delta}, \delta^*, \bar{\delta}\}$ (with elasticity $\frac{d \ln \sigma_\delta}{dp}$); this decoupling affects the long-run density of incomes, see Appendix for details. If the fixed adjustment cost were on income y rather than hours h (see footnote 15), the volatility of deviations σ_δ would always be equal to (minus) that of frictionless incomes σ_y, and this term would disappear from the optimal tax formula (41); see the earlier version of this paper (Werquin (2014)). Moreover, this term has negligible magnitude in practice; see Section 5.

effect driven by the composition margin social weight $\tilde{\mathbf{\Omega}}(y)$.[37] The extensive margin terms arise because of the endogenous lumpiness of labor supply, and the composition term arises because of the endogenous heterogeneity within income groups.[38]

Importantly, the extensive margin elasticities $\mathbf{\Xi}_i(y)$ and social weights $\mathbf{\Omega}_i(y)$ in (41) are multiplied by the terms $\left(\frac{d\ln|\delta_i|}{dp} - \frac{d\ln|\sigma_\delta|}{dp}\right)$. This means that the standard frictionless optimal tax formula (40) amounts to implicitly making an (implausibly strong) assumption on behavior: namely, that raising the progressivity of tax schedule shrinks the inaction region, i.e. lowers the option value of waiting to adjust labor supply, by the same amount as it reduces the volatility of income shocks. If this does not hold, then the frictionless optimal tax formula miscalculates the welfare effects of the extensive margin nature of labor supply adjustments.[39] This condition *is* satisfied in the case of a uniform change in marginal tax rates, because τ affects neither the volatility nor the optimal inaction region; this explains the neutrality results of Propositions 5 and 6. In general, however, this condition is violated: the non-linearity of the policy instrument (here, the progressivity of the tax schedule) interacts with

[37]Note that while the intensive margin elasticities are multiplied by the marginal tax rate $T'(y)$ to obtain the behavioral (revenue) effect of the tax change, because an infinitesimal change in income dy reduces tax revenue by $T'(y)\,dy$, the three extensive margin terms (corresponding to the three variables $\underline{\delta}, \delta^*, \bar{\delta}$ of discrete adjustment), on the other hand, are multiplied by the average tax rate $T(y)$.

[38]This is a stronger result than those in the literature that analyzes optimal policy in static models with an extensive margin or fixed costs (e.g., Saez (2002), Chetty, Looney, and Kroft (2009), Chetty, Friedman, Olsen, and Pistaferri (2011)). In these models a fraction of the population finds it optimal to adjust their labor supply in response to the tax change, while the rest of the population doesn't. This is occuring in my dynamic model only in the *short-run*: eventually individuals will all optimally adjust their behavior to the new tax code as their characteristics evolve over time.

[39]The frictionless tax formula would also be correct if the effects of locally varying the thresholds of the inaction region had a second-order effect on the density of incomes and deviations at each income level, so that the extensive margin elasticities $\mathbf{\Xi}_i$ and social weights $\mathbf{\Omega}_i$ would be equal to zero. I do not have a sharp theoretical result proving that this is not the case, but it does not seem to be happening numerically.

the lumpiness of individual behavior to generate long-term real effects of tax changes.

We can already anticipate the direction of these effects. In general, the narrowing of the inaction region due to a higher progressivity is vastly dominated by the decrease in the volatility of incomes. Intuitively, taxes hardly affect the tolerance of individuals for deviations of their labor supply away from their optimum (they will adjust when their hours are 10 percent away, say, from their optimum, regardless of the progressivity), while progressivity does dampen the volatility of income fluctuations by a first-order amount. Thus, relative to the frictionless benchmark where the two effects exactly compensate each other, raising p is equivalent to a wider dispersion of individual incomes around their desired values. This in turn adversely affects welfare, so that the extensive margin effects on welfare $\underline{\boldsymbol{\Omega}}, \boldsymbol{\Omega}^*, \bar{\boldsymbol{\Omega}}$ will tend to reduce the gains (or increase the losses) of raising progressivity. I analyze the quantitative magnitude of the novel effects of equation (41) in Section 5, and show that this negative effects on welfare is the dominant force. In particular, the composition effects (coming from the endogeneity to taxes of the average utility within the population earning income y, which is no longer a representative agent when labor supply is frictional) play a less important role in practice.

5 Quantitative analysis

In this section I calibrate the model analyzed in the previous sections, and evaluate quantitatively the welfare effects of raising the progressivity of the U.S. tax schedule (equation (41)) and the errors made by wrongly assuming a frictionless economy.

5.1 Calibration

I calibrate the marginal tax rates and the rate of progressivity (τ, p) of the tax schedule in the U.S. using the empirical estimates from PSID data of Heathcote, Storesletten, and Violante (2014): $\tau = -3$ and $p = 0.151$.[40,41] I take a coefficient of risk aversion[42] $\gamma = 0.9$ and a discount rate ρ_1 such that $(1 + \rho_1 + \rho_2)^{-1} = 0.95$ (see below for the calibration of ρ_2). In an environment closely related to this paper's model, Chetty (2012) finds a value for the structural parameter $\varepsilon = 0.33$ from a meta analysis of the elasticities estimated in the literature.[43] I use this value for the baseline calibration and show how the results are affected when $\varepsilon = 1$.

The Pareto coefficients and the mean and variance of the U.S. income distribution are observable and lead to $\left(r_{y,1}^{\rho_2}, r_{y,2}^{\rho_2}, m_y, s_y\right) = (-1.9, 1.4, 10.46, 0.43)$.[44] The volatility of idiosyncratic wage risk σ_y^2 in my model corresponds to the variance of the permanent component of the individual log-income process in the

[40]This value of p implies that earning twice as high a gross income leads to a 15.1 percent decrease in the net-of-tax rate.

[41]These parameters yield a value for total U.S. government revenue $\bar{R} = \$2.33$tn for a population of 320m. I keep this value constant throughout the numerical analysis, so that whenever I vary the progressivity p I adjust τ in order to keep the revenue $\bar{R}$ unchanged.

[42]Recall that the theoretical analysis requires γ to be strictly below 1. Even though allowing for $\gamma > 1$ would be more realistic, this assumption is not crucial, since the goal of this section is to compute optimal tax rates *relative* to those in the frictionless model. The optimal progressivity would be higher with a higher risk aversion in the frictionless model, but here I focus on the difference between the frictionless and the frictional formulas.

[43]There is substantial controversy in the literature about the value of the *observed* taxable income elasticity ε. The micro literature typically finds values lower than 0.3, while the macro literature and some structural estimates find it to be closer to 1. See Saez, Slemrod, and Giertz (2012) and Keane and Rogerson (2012) for an overview of the two strands.

[44]For the Pareto coefficients, see, e.g., Reed (2003), Reed and Jorgensen (2004), Toda (2012). For the mean m_y and variance s_y^2 of the density of incomes at birth, I use $\mathbb{E}[\ln y] = 10.3$ and $\mathbb{V}[\ln y] = 1$. In the frictionless model, we have $\mathbb{E}[\ln y] = m_y - r_{y,1}^{-1} - r_{y,2}^{-1}$ and $\mathbb{V}[\ln y] = s_y^2 + r_{y,1}^{-2} + r_{y,2}^{-2}$.

literature (e.g., Meghir and Pistaferri (2004, 2011)); I take $\sigma_y^2 = 0.01$.[45,46] The parameters $\left(\mu_w, \sigma_w, m_w, s_w, r_{1,w}^{\rho_2}, r_{2,w}^{\rho_2}\right)$ and $\left(\mu_c, \sigma_c, m_c, s_c, r_{1,c}^{\rho_2}, r_{2,c}^{\rho_2}\right)$ are then obtained from the relationships $r_{y,1}^{\rho_2} + r_{y,2}^{\rho_2} = \frac{2\mu_y}{\sigma_y^2}$ and $r_{y,1}^{\rho_2} r_{y,2}^{\rho_2} = -\frac{2\rho_2}{\sigma_y^2}$, which pin down the drift μ_y and the death rate ρ_2,[47] and equations (13) and (28).

The fixed adjustment cost κ and the arrival rate of costless adjustment opportunities q are calibrated using the average duration of searching for a new job (I take $t_s = q^{-1} = 1$ month) and the average duration of a job, which I take equal to $t_i + t_s = 5$ years. For $\varepsilon = 0.33$ (resp., $\varepsilon = 1$), I obtain $\kappa = 0.0038$ (resp., $\kappa = 0.015$), which implies that the cost of searching for a new job, $\boldsymbol{\kappa}$, is equal to 1.2 percent (resp., 5.1 percent) of the average total (frictionless) utility received during the duration of the search, and an individual starts searching when her hours are approximately $|\underline{\delta}| = \bar{\delta} \approx 9$ percent (resp., 20 percent) away from their optimal value and she then adjusts to $\delta^* = -0.1$ percent below her current optimal value.

5.2 Numerical results

In this section, I compute Γ_p in formula (41), i.e., the revenue and social welfare effects, expressed in dollars, of a \$1 statutory increase in tax revenue through an increase in the rate of progressivity in the U.S. economy calibrated in Section 5.1. The advantage of using a tax reform (around the U.S. tax code) rather than an optimal tax approach for quantitative purposes is that the endogenous variables that appear in the formula can all be easily evaluated

[45] See also Jones and Kim (2014), for an estimate in a frictionless model similar to this paper and further references to the empirical literature.

[46] If the analysis were extended to allow for jumps in the wage process, the corresponding parameters could be calibrated from Guvenen, Song, Ozkan, and Karahan (2015) who find that the distribution of earnings growth rates is double-Pareto.

[47] Note that this leads to a negative drift of income μ_y, but the growth rate $\mu_y + \frac{1}{2}\sigma_y^2$ is positive.

empirically given the actual U.S. data, without the need to extrapolate their values at the optimum tax schedule.[48]

The extensive margin elasticities $\left(\frac{d\ln|\delta|}{dp} - \frac{d\ln|\sigma_\delta|}{dp}\right)\left\{\underline{\boldsymbol{\Xi}}\left(y\right), \bar{\boldsymbol{\Xi}}\left(y\right)\right\}$ are plotted in Figure 6 in the Appendix for $\varepsilon = 0.33$ (left panel) and $\varepsilon = 1$ (right panel). Figure 7 plots the revenue effects of the tax reform disaggregated by income for $\varepsilon = 0.33$ and $\varepsilon = 1$ in the frictionless and the frictional models, that is, $-T' \frac{y\boldsymbol{\varepsilon}^*}{1-T'}\Psi'_p + T\sum_{i=1}^{3}\frac{d\ln|\delta_i|/|\sigma_\delta|}{dp}\boldsymbol{\Xi}_i$. The elasticities $\boldsymbol{\Xi}_i$ are non-negligible (of the order of 0.1 to 0.3 in absolute value). However the revenue effects are nearly identical in the frictionless and the frictional models at every income level. This is because the extensive margin elasticities are bounded, while the increase in progressivity induces an unbounded increase in marginal tax rates Ψ'_p and hence an unbounded intensive margin effect.

Figure 8 plots the welfare effects of increasing progressivity disaggregated by income for $\varepsilon = 0.33$ and $\varepsilon = 1$, that is, $-\left[\boldsymbol{\omega}\Psi_p + \frac{d\ln\boldsymbol{\rho}}{dp}\hat{\boldsymbol{\omega}}\right] + \sum_{i=1}^{3}\frac{d\ln|\delta_i|/|\sigma_\delta|}{dp}\boldsymbol{\Omega}_i + \tilde{\boldsymbol{\Omega}}$. These effects are large for the smaller value of the labor income elasticity $\varepsilon = 0.33$, and almost zero for the larger value $\varepsilon = 1$. The composition effects (captured by the welfare weight $\tilde{\boldsymbol{\Omega}}$) play little role in the discrepancy between the two curves in the left panel. Instead, the extensive margin effects on welfare tend to reduce the gains of raising progressivity, as explained in Section 4. Thus, ignoring the stickiness in the labor supply decisions of individuals leads to substantially mis-estimating the welfare costs of raising the progressivity of the tax schedule. These effects disappear as the labor income elasticity gets higher, in which case the standard intensive margin welfare effects dominate the extensive margin effects.

I finally sum these welfare effects over the whole population to obtain the

[48] This is strictly speaking what the "sufficient statistic" approach of Chetty (2009) allows us to do.

full right hand side of (41). For $\varepsilon = 0.33$, in the frictionless model the total behavioral (revenue) loss of a \$1 statutory increase in p is c11.10, and the total welfare loss is c83.12.[49] The frictional effect of perturbing progressivity on revenue is 2.25 percent away from this frictionless effect (the behavioral loss is c0.25 higher), and the frictional effect on welfare is 7.3 percent away from the frictionless effect (the welfare loss is \$6.05 higher), which is a sizeable error. In contrast, when $\varepsilon = 1$, the total behavioral revenue response of an increase in p is c33.44, and the total welfare loss is c66.80,[50] and the frictional effects of perturbing progressivity are 0.46 percent and 0.75 percent away from the frictionless effects on revenue and welfare, respectively; the static model's calculations are very accurate in this case.

6 Conclusion

This paper analyzes the effects on nonlinear income taxation in a model where individual labor supply is subject to fixed adjustment costs. Several long-run effects arise, captured by novel elasticities and marginal social welfare weights. This paper can be seen as a generalization of the optimal taxation models studied by, e.g., Saez (2002) and Jacquet, Lehmann, and Van der Linden (2013), who derive optimal tax formulas in static models with a 0-1 decision whether to participate in the labor force. Here, I derive a tax formula where the extensive margins of adjustment arise endogenously and optimally conditional on participation because of the fixed cost of adjusting labor supply,

[49] Note that the sum of the two is lower than the mechanical effect \$1, implying that the U.S. tax code is not progressive enough for the parameters of the calibration. The optimum rate of progressivity is increasing the risk aversion γ and decreasing in the elasticity ε.

[50] Note that the sum of the two is slightly larger than the mechanical effect \$1, implying that the U.S. tax code is slightly too progressive for this higher value of the labor income elasticity.

consistent with a large body of empirical evidence. I show that the nonlinearity of the policy instruments interacts with the lumpiness of individual behavior to yield long-run real effects on social welfare, an insight that can be applied more generally to other settings than income taxation.

The results of this paper rely on several simplifying assumptions that allow the model to be analytically tractable (e.g., specific utility function, wage process, tax schedule, fixed cost proportional to utility, no savings and borrowings). Most of these assumptions were made to allow for a reduction in the dimensionality of the state space, leading to a sharp characterization of individual behavior (Proposition 2 and Figure 1).

On the theoretical side, it should be possible to substantially generalize the results of this paper by directly postulating a given shape (not necessarily a cone) for the individual inaction region. This would allow in particular an analysis of the effects of general non-linear income tax schedules. Note that a mechanism design approach would in addition allow taxes to depend on the time at which the individual adjusted her income, and the size of the corresponding jump (for instance, if productivity has a drift, this would give information about how far a given individual is from its optimum labor supply).

On the quantative side, it would be valuable to estimate numerically the novel effects highlighted theoretically in this paper in a more sophisticated structural model, allowing for, e.g., savings and borrowings, life-cycle labor supply decisions, non-proportional fixed adjustment costs, transitory as well as permanent wage shocks, more general non-linear taxes and transfer programs, and other dimensions of labor supply adjustment choices (e.g., job satisfaction). Such a model would also provide realistic estimates of the speed of adjustment of the economy in response to tax changes, and hence allow us to infer the long-run elasticities from the empirically observed short-run elasticities in the presence of sluggish individual adjustments to tax changes.

I leave these investigations for future research.

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Income Taxation with Frictional Labor Supply

Online Appendix

Nicolas Werquin

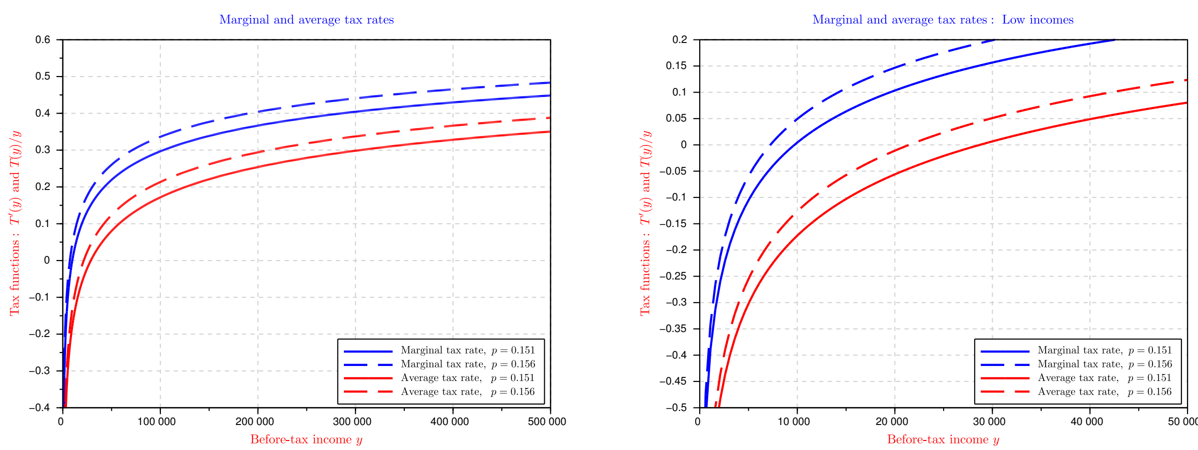
A Appendix

A.1 Proofs and additional results for Sections 1 to 3

Tax system

The tax schedule (3), calibrated to the U.S. tax code, has a rate a progressivity equal to $p = 0.151$. This means that the net-of-tax rate decreases by 15% when gross income doubles. Figure 1.(a) in Heathcote, Storesletten, and Violante (2014) fits this tax function to the U.S. tax-and-transfer system using PSID data and finds an extremely good fit ($R^2 = 0.96$). Figure 2 shows this tax schedule for $p = 0.151$ and $p = 0.156$. The right panel zooms in at the bottom of the income distribution, where marginal and average tax rates are negative.

Figure 2: Tax and transfer schedule



Individual behavior and welfare: frictionless model

I first derive the properties of individual welfare in the frictionless model. I show the second equality in equation (5), i.e., that the average expected frictionless utility during the search period (and hence the fixed cost $\boldsymbol{\kappa}$) is proportional to the instantaneous utility of frictionless consumption. I also compute the value function in the frictionless model, i.e., equation (15), which is also proportional to the current utility of consumption. I finally derive the marginal social welfare weights (33).

Proof of equations (5), (15), and (33). The expected frictionless utility from time 0 to the random time $\hat{\tau} \sim \mathcal{E}(q)$ is then equal to

$$\begin{aligned}
\boldsymbol{\kappa} =& \tilde{\kappa} \times \mathbb{E}_0 \left[\int_0^{\hat{\tau}} e^{-(\rho_1+\rho_2)t} \frac{1}{1-\gamma} \left(c_t^* - \frac{1}{1+1/\varepsilon} \left(\frac{y_t^*}{w_t} \right)^{1+1/\varepsilon} \right)^{1-\gamma} dt \right] \\
=& \tilde{\kappa} \left(\frac{1+p\varepsilon}{1+\varepsilon} \right)^{1-\gamma} \mathbb{E}_0 \left[\int_0^{\hat{\tau}} e^{-(\rho_1+\rho_2)t} g\left(c_t^*\right) dt \right],
\end{aligned}$$

where the equality follows from the first-order conditions (9). Moreover, equation (13) shows that c^* follows a geometric Brownian motion, which implies

$$dg\left(c_t^*\right) = \left[(1-\gamma)\mu_c + \frac{1}{2}(1-\gamma)^2 \sigma_c^2 \right] g\left(c_t^*\right) dt + \left[(1-\gamma)\sigma_c \right] g\left(c_t^*\right) d\mathcal{W}_t,$$

and hence $g\left(c_t^*\right) = g\left(c_0^*\right) e^{(1-\gamma)\mu_c t + (1-\gamma)\sigma_c \mathcal{W}_t}$. Using the facts that $\mathcal{W}_t \sim \mathcal{N}(0,t)$ and $\mathbb{E}\left[e^X\right] = e^{\mu+\sigma^2/2}$ if $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$, we thus obtain

$$\begin{aligned}
\boldsymbol{\kappa} =& \tilde{\kappa} \left(\frac{1+p\varepsilon}{1+\varepsilon} \right)^{1-\gamma} g\left(c_0^*\right) \mathbb{E}_0 \left[\int_0^{\hat{\tau}} e^{-\left(\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2\right)t} \right] \\
=& \tilde{\kappa} \left(\frac{1+p\varepsilon}{1+\varepsilon} \right)^{1-\gamma} g\left(c_0^*\right) \int_0^{\infty} q e^{-q\hat{\tau}} \int_0^{\hat{\tau}} e^{-\left(\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2\right)t} d\hat{\tau} \\
=& \frac{\tilde{\kappa} \left(\frac{1+p\varepsilon}{1+\varepsilon} \right)^{1-\gamma}}{\rho_1+\rho_2+q-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2} g\left(c_0^*\right),
\end{aligned}$$

which proves equation (5).

Similarly, the individual frictionless value function is given by

$$\begin{aligned}
\mathcal{V}^*(y) =& \mathbb{E}\left[\int_0^{\infty} e^{-(\rho_1+\rho_2)t} \frac{1}{1-\gamma} \left(c_t^* - \frac{1}{1+1/\varepsilon} \left(\frac{y_t^*}{w_t} \right)^{1+1/\varepsilon} \right)^{1-\gamma} dt \,\middle|\, y = y^*(w_0) \right] \\
=& \frac{\left(\frac{1+p\varepsilon}{1+\varepsilon}\right)^{1-\gamma}}{1-\gamma} \mathbb{E}_0 \left[\int_0^{\infty} e^{-(\rho_1+\rho_2)t} \left(c_t^*\right)^{1-\gamma} dt \right] = \frac{\left(\frac{1+p\varepsilon}{1+\varepsilon} c_0^*\right)^{1-\gamma}}{1-\gamma} \int_0^{\infty} e^{-\left(\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2\right)t},
\end{aligned}$$

from which equation (15) follows.

Finally, let $\hat{c}_t$ follow the same geometric Brownian motion process as c_t^* (equation (13)), which implies that $\frac{1+p\varepsilon}{1+\varepsilon}c_t^* + \hat{c}_t$ also follows the same process. The same steps as above imply

$$\begin{aligned}
\boldsymbol{\omega}^*(y) =& \frac{1}{\lambda^*}\frac{d}{d\hat{c}_0}\left(\mathbb{E}_0\left[\int_0^\infty e^{-(\rho_1+\rho_2)t}u\left(c_t^*+\hat{c}_t-\frac{1}{1+1/\varepsilon}\left(h_t^*\right)^{1+1/\varepsilon}\right)dt\middle| y_0=y\right]\right)_{\hat{c}_0=0} \\
=& \frac{1}{\lambda^*}\frac{d}{d\hat{c}_0}\left(\frac{\frac{1}{1-\gamma}\left(\frac{1+p\varepsilon}{1+\varepsilon}c_0^*+\hat{c}_0\right)^{1-\gamma}}{\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2}\right)_{\hat{c}_0=0} \\
=& \frac{1}{\lambda^*}\frac{\left(\frac{1+p\varepsilon}{1+\varepsilon}\right)^{-\gamma}\left(\frac{1-\tau}{1-p}\right)^{-\gamma}y^{-\gamma(1-p)}}{\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2},
\end{aligned}$$

which gives a closed-form expression for the static marginal social welfare weights in the frictionless setting. Moreover,

$$\hat{\boldsymbol{\omega}}^*(y) = -\frac{1}{\lambda^*}\frac{\partial\mathcal{V}^*(y)}{\partial\ln\boldsymbol{\rho}} = \frac{1}{\lambda^*}\frac{\frac{1}{1-\gamma}\left(\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p}y^{1-p}\right)^{1-\gamma}}{\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2},$$

which gives a closed-form expression for the dynamic marginal social welfare weights in the frictionless setting. □

Individual behavior and welfare: frictional model

I now derive the properties of individual welfare in the frictional model. I show the homogeneity of the utility function and the value functions $\mathcal{V}_x(y^*,\delta)$ (Proposition 1), and define the value functions $\hat{\mathcal{V}}_x(y,\delta)$ (i.e., the welfare of individuals with actual, rather than frictionless, income y, deviation δ, and employment state x) and $\bar{\mathcal{V}}(y,\delta)$ (i.e., the average welfare of individuals with actual income y and deviation δ).

Proof of Proposition 1. Substituting y^* for w using the optimality condition (9) into the flow utility $U\left(y-T(y),\frac{y}{w}\right)$ yields

$$\begin{aligned}
U =& \frac{1}{1-\gamma}\left[\frac{1-\tau}{1-p}\left(y^*\right)^{1-p}\right]^{1-\gamma}\times\left[\left(\frac{y}{y^*}\right)^{1-p}-\frac{1-p}{1+1/\varepsilon}\left(\frac{y}{y^*}\right)^{1+1/\varepsilon}\right]^{1-\gamma} \\
=& \frac{1}{1-\gamma}\left[\frac{1-\tau}{1-p}\left(y^*\right)^{1-p}\right]^{1-\gamma}\left[e^{(1-p)\delta}-\frac{1-p}{1+1/\varepsilon}e^{\left(1+\frac{1}{\varepsilon}\right)\delta}\right]^{1-\gamma} \equiv g\left(c^*\right)\times u(\delta),
\end{aligned}$$

which implies that the flow utility $U(y^*,\delta)$ is homogeneous in the utility of frictionless disposable income c^*. A second-order Taylor approximation of the function $u(\delta)$ around the frictionless optimum $\delta=0$ shows that the utility loss from failing to optimize is locally quadratic,

$$u(\delta) \underset{\delta\to 0}{\sim} \left(\frac{1+p\varepsilon}{1+\varepsilon}\right)^{1-\gamma}\left[1-\frac{1}{2}(1-\gamma)(1-p)\left(1+\frac{1}{\varepsilon}\right)\delta^2\right].$$

The value function $\mathcal{V}_i(y^*, \delta)$ of inactive individuals is equal to

$$\mathcal{V}_i(y_0^*, \delta_0) = \max_{\tau_1} \mathbb{E}_0 \left[\int_0^{\tau_1} e^{-(\rho_1+\rho_2)t} g\left(\frac{1-\tau}{1-p} y_t^{*1-p} \right) u(\delta_t)\, dt + e^{-(\rho_1+\rho_2)\tau_1} \left\{ \mathcal{V}_s\left(y_{\tau_1}^*, \delta_{\tau_1}\right) - \boldsymbol{\kappa}_{\tau_1} \right\} \right],$$

subject to the laws of motion (12) and (17), where τ_1 is the optimal stopping time at which the individual starts searching by paying the fixed cost. The value function $\mathcal{V}_s(y^*, \delta)$ of searchers is equal to

$$\mathcal{V}_s(y_0^*, \delta_0) = \max_{\delta_{\hat{\tau}}^*} \mathbb{E}_0 \left[\int_0^{\hat{\tau}} e^{-(\rho_1+\rho_2)t} U(y_t^*, \delta_t)\, dt + e^{-(\rho_1+\rho_2)\hat{\tau}} \mathcal{V}_i\left(y_{\hat{\tau}}^*, \delta_{\hat{\tau}}^*\right) \right],$$

subject to the laws of motion (12) and (17), where $\hat{\tau}$ is a stopping time with an exponential distribution with parameter q, and $\delta_{\hat{\tau}}^*$ is the optimal deviation that the individual chooses upon reception of an adjustment opportunity at time $\hat{\tau}$. From these equations we can write a sequential formulation of the value functions $\mathcal{V}_i$ and $\mathcal{V}_s$ and obtain that $\frac{\mathcal{V}_x(y^*,\delta)}{g\left(\frac{1-\tau}{1-p} y^{*1-p}\right)} = v_x(\delta)$ for $x \in \{i, s\}$, where

$$\begin{aligned} v_i(\delta_0) &= \max_{\tau_1} \mathbb{E}_0 \left[\int_0^{\tau_1} e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)t+(1-\gamma)\sigma_c \mathcal{W}_t} u(\delta_t)\, dt \right. \\ &\qquad \left. + e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)\tau_1+(1-\gamma)\sigma_c \mathcal{W}_{\tau_1}} \left\{ v_s(\delta_{\tau_1}) - \kappa \right\} \right] \\ v_s(\delta_0) &= \max_{\delta_{\hat{\tau}}^*} \mathbb{E}_0 \left[\int_0^{\hat{\tau}} e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)t+(1-\gamma)\sigma_c \mathcal{W}_t} u(\delta_t)\, dt \right. \\ &\qquad \left. + e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)\hat{\tau}+(1-\gamma)\sigma_c \mathcal{W}_{\hat{\tau}}} v_i\left(\delta_{\hat{\tau}}^*\right) \right]. \end{aligned} \tag{42}$$

Proposition 1 follows.

These results imply that the value functions $\hat{\mathcal{V}}_x(y, \delta)$ (for an actual income y) are equal to $\hat{\mathcal{V}}_x(y, \delta) = \mathcal{V}^*(y) \times \hat{v}_x(\delta)$, where the value functions $\hat{v}_i(\delta), \hat{v}_s(\delta)$ are given by

$$\hat{v}_x(\delta) = \boldsymbol{\rho} \left(\frac{1+p\varepsilon}{1+\varepsilon} \right)^{\gamma-1} e^{-(1-p)(1-\gamma)\delta} v_x(\delta),$$

for $x \in \{i, s\}$. I finally denote by $\bar{\mathcal{V}}(y, \delta)$ and $\bar{v}(y, \delta)$ the the value functions averaged over the employment state (inactive and searching), that is,

$$\bar{v}(y, \delta) = \bar{v}_i(\delta) \frac{f_{y,\delta}^i(y, \delta)}{f_{y,\delta}^i(y, \delta) + f_{y,\delta}^s(y, \delta)} + \bar{v}_s(\delta) \frac{f_{y,\delta}^s(y, \delta)}{f_{y,\delta}^i(y, \delta) + f_{y,\delta}^s(y, \delta)}, \quad \text{and} \quad \bar{\mathcal{V}}(y, \delta) = \mathcal{V}^*(y)\, \bar{v}(y, \delta), \tag{43}$$

where $f_{y,\delta}^i, f_{y,\delta}^s$ denote the stationary joint densities of inactive and searching individuals at income and deviation (y, δ). □

Next, I characterize the solution to the individual's impulse control problem (Proposition 2). As a first step, I calculate explicitly the value of searching v_s.

Closed-form expression for (23). I show that

$$
\begin{aligned}
\hat{v}_s(\delta) \equiv & \mathbb{E}_0\left[\int_0^{\hat{\tau}} e^{-(\rho_1+\rho_2)t} U\left(y_t^*, \delta_t\right) dt \,|\delta_0 = \delta\right] \\
= & \frac{\int_{-\infty}^{\infty}\left[e^{r_{2,\delta}^{\rho_1+\rho_2+q}x}\mathbb{I}_{\{x\leq 0\}} + e^{r_{1,\delta}^{\rho_1+\rho_2+q}x}\mathbb{I}_{\{x>0\}}\right] u(x+\delta)\, dx}{\frac{\sigma_\delta^2}{2} e^{(1-\gamma)\left(1+\frac{1}{\varepsilon}\right)}\left(r_{2,\delta}^{\rho_1+\rho_2+q} - r_{1,\delta}^{\rho_1+\rho_2+q}\right)},
\end{aligned}
\tag{44}
$$

where $r_{1,x}^{\rho}, r_{2,x}^{\rho}$ are defined in (26). We have, letting $\hat{\tau} \sim \mathcal{E}(q)$,

$$
\begin{aligned}
& \mathbb{E}_0\left[\int_0^{\hat{\tau}} e^{-(\rho_1+\rho_2)t} g\left(c_t^*\right) u\left(\delta_t\right) dt + e^{-(\rho_1+\rho_2)\hat{\tau}} g\left(c_{\hat{\tau}}^*\right) v_i\left(\delta^*\right)\right] \\
= & g\left(c_0^*\right)\int_0^{\infty} q e^{-qt} \mathbb{E}_0\Bigg[\int_0^t e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)s+(1-\gamma)\sigma_c \mathcal{W}_s} u\left(\delta_s\right) ds \\
& \qquad + e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)t+(1-\gamma)\sigma_c\mathcal{W}_t} v_i\left(\delta^*\right)\Bigg] dt.
\end{aligned}
$$

Using Fubini's theorem, the fact that $\mathcal{W}_t \sim \mathcal{N}(0,t)$, and the quadratic approximation $u(\delta) = \alpha_0 + \alpha_1\delta + \alpha_2\delta^2$, we obtain that the stochastic integral (for fixed t) is equal to, with $\delta_s = \delta_0 + \mu_\delta s + \sigma_\delta \mathcal{W}_s$:

$$
\begin{aligned}
& \mathbb{E}_0\left[\int_0^t e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)s} e^{(1-\gamma)\sigma_c\mathcal{W}_s} u\left(\delta_s\right) ds\right] \\
= & \frac{1}{\boldsymbol{\rho}^3}\alpha_2\left((1-\gamma)\sigma_c\sigma_\delta + \mu_\delta\right)^2\left(2 - e^{-\boldsymbol{\rho}t}\left(\boldsymbol{\rho}^2 t^2 + 2\boldsymbol{\rho}t + 2\right)\right) + \frac{1}{\boldsymbol{\rho}}\left(\alpha_0 + \alpha_1\delta_0 + \alpha_2\delta_0^2\right)\left(1 - e^{-\boldsymbol{\rho}t}\right) \\
& + \frac{1}{\boldsymbol{\rho}^2}\left(\alpha_1\left(\mu_\delta + (1-\gamma)\sigma_c\sigma_\delta\right) + \alpha_2\sigma_\delta^2 + 2\alpha_2\left(\mu_\delta + (1-\gamma)\sigma_c\sigma_\delta\right)\delta_0\right)\left(1 - e^{-\boldsymbol{\rho}t}\left(\boldsymbol{\rho}t + 1\right)\right).
\end{aligned}
$$

Integrating over the stopping time t then yields:

$$
\begin{aligned}
& \int_0^{\infty} q e^{-qt}\mathbb{E}_0\left[\int_0^t e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)s} e^{(1-\gamma)\sigma_c\mathcal{W}_s} u\left(\delta_s\right) ds\right] dt \\
= & \left(\frac{\alpha_2}{\boldsymbol{\rho}+q}\right)\delta_0^2 + \left(\frac{\alpha_1}{\boldsymbol{\rho}+q} + 2\alpha_2\frac{\mu_\delta + (1-\gamma)\sigma_c\sigma_\delta}{(\boldsymbol{\rho}+q)^2}\right)\delta_0 \\
& + \left(\frac{\alpha_0}{\boldsymbol{\rho}+q} + \alpha_1\frac{\left(\mu_\delta + (1-\gamma)\sigma_c\sigma_\delta\right)}{(\boldsymbol{\rho}+q)^2} + \alpha_2\frac{2\left(\mu_\delta + (1-\gamma)\sigma_c\sigma_\delta\right)^2 + (\boldsymbol{\rho}+q)\sigma_\delta^2}{(\boldsymbol{\rho}+q)^3}\right).
\end{aligned}
$$

Straightforward algebra finally shows that the right hand side of (44) is equal to this expression. (Formula 44 is obtained by solving directly the HJB equation for searchers with the relevant boundary conditions at $\pm\infty$.) Finally the value of returning to δ^* (assuming that the optimal impulse control policy is the $\{\underline{\delta}, \delta^*, \bar{\delta}\}$-policy, see below) is given by

$$\begin{aligned}
&\mathbb{E}\left[e^{-(\rho_1+\rho_2)\hat{\tau}}\mathcal{V}_i\left(y^*_{\hat{\tau}}, \delta^*_{\hat{\tau}}\right)\right] \\
&= \int_0^\infty qe^{-qt}\mathbb{E}_0\left[e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)t+(1-\gamma)\sigma_c\mathcal{W}_t}v_i\left(\delta^*\right)\right]dt = \frac{qv_i\left(\delta^*\right)}{\rho_1+\rho_2+q-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2},
\end{aligned}$$

which is the second term of equation (23). □

I now define the operators $\mathcal{L}, \mathcal{M} : \mathcal{C}^2(\mathbb{R}) \to \mathcal{C}^0(\mathbb{R})$ by

$$\begin{aligned}
\mathcal{L}v(\delta) &= \left[\mu_\delta + (1-\gamma)\sigma_c\sigma_\delta\right]v'(\delta) + \frac{1}{2}\sigma_\delta^2 v''(\delta), \\
\mathcal{M}v(\delta) &= \frac{q}{\boldsymbol{\rho}+q}\sup_{\delta'\in\mathbb{R}} v\left(\delta'\right) + \hat{v}_s(\delta) - \kappa.
\end{aligned}$$

The following *Verification Lemma* provides sufficient conditions for optimality and characterizes the optimal policy.

Lemma 1. *Suppose we can find a function $v : \mathbb{R} \to \mathbb{R}$ that satisfies:*

$$v \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}\setminus\mathcal{D}), \text{ where } \mathcal{D} = \{\delta\in\mathbb{R} : v(\delta) > \mathcal{M}v(\delta)\}, \tag{45}$$

$$\mathcal{L}v(\delta) + u(\delta) \le \boldsymbol{\rho}v(\delta), \quad \forall\delta\in\mathbb{R}\setminus\partial\mathcal{D}, \text{ with equality in } \mathcal{D}, \tag{46}$$

$$\mathcal{M}v(\delta) \le v(\delta), \qquad \forall\delta\in\mathbb{R}. \tag{47}$$

Suppose moreover that v has locally bounded derivatives near $\partial\mathcal{D}$ and for all $\boldsymbol{\tau}\in\mathcal{T}$, $\mathfrak{p}\in\mathcal{P}$, $\delta_0\in\mathbb{R}$,

$$\mathbb{E}\left[e^{-(\rho_1+\rho_2)\boldsymbol{\tau}}\left|v\left(\delta^{\mathfrak{p}}_{\boldsymbol{\tau}}\right)\right| + \int_0^\infty e^{-(\rho_1+\rho_2)t}\left|\mathcal{L}v\left(\delta^{\mathfrak{p}}_t\right)\right|dt\right] < \infty. \tag{48}$$

Then $v(\delta) = v_i(\delta)$ and $\mathcal{M}v(\delta) = v_s(\delta)$ for all $\delta\in\mathbb{R}$, where v_i and v_s are the value functions (42). Moreover, if $\{v(\delta^{\mathfrak{p}^*}_{\boldsymbol{\tau}}) : \boldsymbol{\tau}\in\mathcal{T}\}$ is uniformly integrable the optimal impulse control policy $\mathfrak{p}^*\in\mathcal{P}$ is given by: for all $j\ge 1$,

$$\boldsymbol{\tau}^*_j = \inf\left\{t > \boldsymbol{\tau}^*_{j-1} + \hat{\tau}_{j-1} : \delta^{\mathfrak{p}^*_{j-1}}_t \notin \mathcal{D}\right\}, \text{ and } \Delta^*_j = \sup_{\delta'\in\mathbb{R}} v\left(\delta'\right) - v\left(\delta^{\mathfrak{p}^*_{j-1}}_{\boldsymbol{\tau}^*_j+\hat{\tau}^-_j}\right),$$

where $\delta^{\mathfrak{p}^*_{j-1}}_t$ is the process resulting from applying $\mathfrak{p}^*_j = \left(\boldsymbol{\tau}^*_1, \ldots, \boldsymbol{\tau}^*_{j-1}, \Delta^*_1(\hat{\tau}_1), \ldots, \Delta^*_{j-1}(\hat{\tau}_{j-1})\right)$ to δ.

Proof. Fix an impulse control policy $\mathfrak{p} = \left(\{\tau_j, \Delta_j(\hat{\tau}_j)\}_{j\ge1}\right) \in \mathcal{P}$. By condition (i) and Theorem 2.1. in Øksendal and Sulem (2005), we can assume that $v\in\mathcal{C}^2(\mathbb{R})$. Hence using condition (iv) we can apply the localized version of Dynkin's formula (Theorem 1.24. in Øksendal and Sulem (2005) modified to take into account the discounting) to get, for $j \geq 0$,

$$
\begin{aligned}
&\mathbb{E}\left[e^{-(\rho_1+\rho_2)\left(\tau_j+\hat{\tau}_j\right)} g\left(c^*_{\tau_j+\hat{\tau}_j}\right) v\left(\delta_{\tau_j+\hat{\tau}_j}\right)\right]-\mathbb{E}\left[e^{-(\rho_1+\rho_2)\tau_{j+1}} g\left(c^*_{\tau_{j+1}}\right) v\left(\delta_{\tau_{j+1}}\right)\right] \\
=&-g\left(c_0^*\right) \mathbb{E}\left[\int_{\tau_j+\hat{\tau}_j}^{\tau_{j+1}} e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)t+(1-\gamma)\sigma_c \mathcal{W}_t}\left(-\boldsymbol{\rho} v\left(\delta_t\right)+\mathcal{L} v\left(\delta_t\right)\right) dt\right] \\
\geq\; & g\left(c_0^*\right) \mathbb{E}\left[\int_{\tau_j+\hat{\tau}_j}^{\tau_{j+1}} e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)t+(1-\gamma)\sigma_c \mathcal{W}_t} u\left(\delta_t\right) dt\right],
\end{aligned}
$$

where the inequality follows from condition (47), and it becomes an equality if $\mathfrak{p}=\mathfrak{p}^*$. Moreover, we have

$$
\begin{aligned}
&\mathbb{E}\left[e^{-(\rho_1+\rho_2)\tau_{j+1}} g\left(c^*_{\tau_{j+1}}\right) v\left(\delta_{\tau_{j+1}}\right)\right] \\
&-\mathbb{E}\left[e^{-(\rho_1+\rho_2)\left(\tau_{j+1}+\hat{\tau}_{j+1}\right)} g\left(c^*_{\tau_{j+1}+\hat{\tau}_{j+1}}\right) v\left(\delta_{\tau_{j+1}+\hat{\tau}_{j+1}^-}+\Delta_{j+1}\left(\hat{\tau}_{j+1}\right)\right)\right] \\
\geq & g\left(c_0^*\right)\left\{\quad \mathbb{E}\left[e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)\tau_{j+1}+(1-\gamma)\sigma_c \mathcal{W}_{\tau_{j+1}}} v\left(\delta_{\tau_{j+1}}\right)\right]\right. \\
&\left.-\mathbb{E}\left[e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)\left(\tau_{j+1}+\hat{\tau}_{j+1}\right)+(1-\gamma)\sigma_c \mathcal{W}_{\tau_{j+1}+\hat{\tau}_{j+1}}}\right] \sup_{\delta' \in \mathbb{R}} v\left(\delta'\right)\right\} \\
= & g\left(c_0^*\right) \mathbb{E}\left[e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)\tau_{j+1}+(1-\gamma)\sigma_c \mathcal{W}_{\tau_{j+1}}}\left(v\left(\delta_{\tau_{j+1}}\right)-\mathcal{M} v\left(\delta_{\tau_{j+1}}\right)+\hat{v}_s\left(\delta_{\tau_{j+1}}\right)-\kappa\right)\right] \\
\geq & g\left(c_0^*\right) \mathbb{E}\left[e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)\tau_{j+1}+(1-\gamma)\sigma_c \mathcal{W}_{\tau_{j+1}}}\left(\hat{v}_s\left(\delta_{\tau_{j+1}}\right)-\kappa\right)\right],
\end{aligned}
$$

where the last equality follows from condition (ii). Both inequalities become equalities if $\mathfrak{p}=\mathfrak{p}^*$. Thus, we obtain, summing the previous equations from $j=0$ to $j=N \geq 1$,

$$
\begin{aligned}
& g\left(c_0^*\right) v\left(\delta_0\right)-\mathbb{E}\left[e^{-(\rho_1+\rho_2)\left(\tau_{N+1}+\hat{\tau}_{N+1}\right)} g\left(c^*_{\tau_{N+1}+\hat{\tau}_{N+1}}\right) v\left(\delta_{\hat{\tau}_{N+1}}\right)\right] \\
\geq & g\left(c_0^*\right) \mathbb{E}\left[\quad \sum_{j=0}^{N} \int_{\tau_j+\hat{\tau}_j}^{\tau_{j+1}} e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)t+(1-\gamma)\sigma_c \mathcal{W}_t} u\left(\delta_t\right) dt\right. \\
&\left.\qquad+\sum_{j=0}^{N} e^{-(\rho_1+\rho_2-(1-\gamma)\mu_c)\tau_{j+1}+(1-\gamma)\sigma_c \mathcal{W}_{\tau_{j+1}}}\left(\hat{v}_s\left(\delta_{\tau_{j+1}}\right)-\kappa\right)\right] \\
= & \mathbb{E}\left[\int_0^{\tau_{N+1}+\hat{\tau}_{N+1}} e^{-(\rho_1+\rho_2)t} g\left(c_t^*\right) u\left(\delta_t^{\mathfrak{p}}\right) dt-\sum_{j=0}^{N} e^{-(\rho_1+\rho_2)\tau_{j+1}} \kappa g\left(c^*_{\tau_{j+1}}\right)\right],
\end{aligned}
$$

(with equality if $\mathfrak{p}=\mathfrak{p}^*$), where the equality follows from (44). Now, as $N \to \infty$, we have $\tau_N \to \infty$ so that the second term on the l.h.s. of the previous equation converges to zero. Therefore we obtain

$$
\mathcal{V}\left(y_0^*, \delta_0\right) \geq \mathbb{E}\left[\int_0^{\infty} e^{-(\rho_1+\rho_2)t} U\left(y_t^*, \delta_t^{\mathfrak{p}}\right) dt-\sum_{j=0}^{\infty} e^{-(\rho_1+\rho_2)\tau_{j+1}} \boldsymbol{\kappa}_{\tau_{j+1}}\right], \ \forall \mathfrak{p} \in \mathcal{P} \quad ,
$$

with equality if $\mathfrak{p} = \mathfrak{p}^*$. (We restrict the set $\mathcal{P}$ of admissible controls to those $\mathfrak{p}$ that satisfy $\mathbb{E}\left[\int_0^\infty e^{-(\rho_1+\rho_2)t}\left|u\left(\delta_t^{\mathfrak{p}}\right)\right| dt\right] < \infty$.) But the right hand side is the lifetime utility of an individual at birth, after δ_0 has been chosen, under the control policy $\mathfrak{p}$ (see (6)). This concludes the proof. $\square$

Condition (47) is the Hamilton-Jacobi-Bellman Quasi-Variational Inequality (QVI), (46) is the value-matching QVI, and (45) is the smooth-pasting property.[51] The optimal policy is such that that the agent remains inactive when the state process δ stays within the continuation region $\mathcal{D}$, and starts searching (by paying the fixed cost) when it attempts to leave $\mathcal{D}$. By Proposition 1, this solution coincides with the optimal value functions and control policy if v_s satisfies (46) on $\left(\underline{\delta}, \bar{\delta}\right)$, (47) on $\mathbb{R} \setminus \left(\underline{\delta}, \bar{\delta}\right)$, and (48). In the case where paying the fixed cost leads to immediate adjustment (i.e., $q = \infty$), we can verify these conditions directly:

Corollary 1. *Let $q = \infty$, so that $\hat{v}_s(\delta) = 0$. Suppose that there exist $\left(\underline{\delta}, \delta^*, \bar{\delta}\right)$ such that the function $v_i : \mathbb{R} \to \mathbb{R}$ solves the differential equation problem (19), (23), (20), and the conditions (21) and (22) hold. Suppose finally that $v_i'(\delta) > 0$ (resp., $v_i'(\delta) < 0$) on an interval $\delta \in (\underline{\delta}, \underline{\delta} + \varepsilon)$ (resp., $\delta \in \left(\bar{\delta} - \varepsilon, \bar{\delta}\right)$). Then $v_i(\delta)$ is the value function of the individual's problem and the optimal individual policy is characterized by the control band $\left\{\underline{\delta}, \delta^*, \bar{\delta}\right\}$.*

Proof. To prove that the conjectured policy is indeed optimal, we need to show that if the value function satisfies (19,20,21,22,23), then it satisfies the assumptions of the Verification Proposition 1 and the quasi-variational inequalities (46,47). If $q = \infty$, then the technical conditions required for the Verification theorem 1 to hold are simpler (see Richard (1977)): v' must be absolutely continuous and bounded and v'' must be in $L^2(\mathbb{R})$; these are easily verified. It remains to check that the QVI are satisfied.

First, I show that the lower bound of the conjectured inaction region is non-positive, $\underline{\delta} \leq 0$, and the upper bound is non-negative, $\bar{\delta} \geq 0$, so that the argmax of the flow utility $u(\delta)$ lies within the inaction region. Note first that $v''\left(\underline{\delta}^+\right) > 0$, $v''\left(\bar{\delta}^-\right) < 0$ and $v''(\delta^*) \leq 0$, where the first two inequalities follow from the continuity of v' and the assumption that $v' > 0$ (resp., $v' < 0$) in a neighborhood to the right of $\underline{\delta}$ (resp. to the left of $\bar{\delta}$). Define the first and the last inflection points of v on $[\underline{\delta}, \delta^*]$ by We know that such values exist in $[\underline{\delta}, \delta^*]$, because $v''\left(\underline{\delta}^+\right) > 0$ and $v'(\delta^*) = 0$. Since $u, v, v' \in \mathcal{C}^1((\underline{\delta}, \delta^*))$, the HJB equation (19) implies that v'' is continuously differentiable. Taking left derivatives of the HJB and evaluating at δ_M^* yields

$$\frac{\sigma_\delta^2}{2} v'''\left(\delta_M^{*-}\right) = \boldsymbol{\rho} v'\left(\delta_M^{*-}\right) - u'\left(\delta_M^{*-}\right) \geq \boldsymbol{\rho} v'(\delta^*) - u'(\delta_M^*) = -u'(\delta_M^*),$$

where the inequality follows from the definition of δ_M^* and $v''(\delta^*) \leq 0$. Since $v'''\left(\delta_M^{*-}\right) \leq 0$, we obtain $-u'(\delta_M^*) \leq 0$. But u is concave with a unique global maximum at $\delta = 0$, hence $\delta_M^* \leq 0$ and $u'(\delta) > u'(\delta_M^*)$ for all $\delta < \delta_M^*$.

Second, I show that the conjectured value function v is unimodal, that is, $v(\delta)$ is strictly increasing on $(\underline{\delta}, \delta^*)$ and strictly decreasing on $\left(\delta^*, \bar{\delta}\right)$. If $\delta_m^* = \delta_M^*$, then $v'(\delta) > 0$ for all $\delta \in (\underline{\delta}, \delta^*)$.

[51] See Bensoussan et al. (1982).

Suppose that $\delta_m^* < \delta_M^*$ and there exists $\delta \in (\underline{\delta}, \delta^*)$ such that $v'(\delta) \leq 0$. Then there exists $\tilde{\delta} \in (\delta_m^*, \delta_M^*)$ which is a local minimizer of v', with $v'\left(\tilde{\delta}\right) < 0$ and $v'''\left(\tilde{\delta}^-\right) \geq 0$. Taking left derivatives in the HJB equation (19) and evaluating at $\tilde{\delta}$ yields:

$$u'\left(\tilde{\delta}^-\right) = -\frac{1}{2}\frac{\sigma_c^2}{(1-p)^2}v'''\left(\tilde{\delta}^-\right) + \boldsymbol{\rho} v'\left(\tilde{\delta}^-\right) < 0.$$

But we saw above that $u'(\delta_M^*) \geq 0$, which implies $u'\left(\tilde{\delta}\right) > 0$ since $\tilde{\delta} < \delta_M^*$ and u is strictly concave and unimodal, a contradiction.

I now show that the conjectured value function satisfies the QVI (47), i.e., $\mathcal{L}v(\delta) - \boldsymbol{\rho} v(\delta) + u(\delta) \leq 0$ for all $\delta \in \mathbb{R}$. The HJB equation (19) and a symmetry argument imply that it is sufficient to check this inequality on $(-\infty, \underline{\delta})$. Fix $\delta < \underline{\delta}$. We have

$$\begin{aligned}
&[\mathcal{L}v(\delta) - \boldsymbol{\rho} v(\delta) + u(\delta)] \\
=& \left[\mathcal{L}v\left(\underline{\delta}^+\right) - \boldsymbol{\rho} v(\underline{\delta}) + u(\underline{\delta})\right] - \boldsymbol{\rho}\left(v(\delta) - v(\underline{\delta})\right) - [\mu_\delta + (1-\gamma)\sigma_c\sigma_\delta]\left(v'(\delta) - v'(\underline{\delta})\right) \\
&+ \frac{1}{2}\sigma_\delta^2\left(v''(\delta) - v''\left(\underline{\delta}^+\right)\right) + \left(u(\delta) - u(\underline{\delta})\right).
\end{aligned}$$

The HJB equation (19), the value-matching condition (20) and the smooth-pasting conditions (21) imply that the first line of the right hand side is equal to zero. Moreover, we saw above that $v''(\delta) = 0 < v''\left(\underline{\delta}^+\right)$ and $u(\delta) \leq u(\underline{\delta})$, which concludes the proof.

Finally, I show that the conjectured value function satisfies the QVI (46), i.e., $\mathcal{M}v(\delta) - v(\delta) \leq 0$ for all $\delta \in \mathbb{R}$, where $\mathcal{M}v(\delta) = v(\delta^*) - \kappa$. For $\delta \in \mathbb{R} \setminus \left(\underline{\delta}, \bar{\delta}\right)$, this QVI is satisfied with equality by construction. For $\delta \in (\underline{\delta}, \delta^*)$, we saw that $v'(\delta) > 0$ which implies $\mathcal{M}v(\delta) < v(\delta)$, and a similar argument applies for $\delta \in \left(\delta^*, \bar{\delta}\right)$. □

I now provide an intuitive derivation of the QVI of Proposition 1, i.e., of the HJB equation and the value-matching and smooth-pasting conditions.

Heuristic derivation of equations (46) and (47). Suppose that an optimal policy $\mathfrak{p}^*$ exists. If the individual adopts an arbitrary control for an infinitesimal amount of time and then switches back to the optimal control $\mathfrak{p}^*$, then the resulting value function cannot be better than the optimal one. If the individual is currently inactive, there are only two possible choices of control during that infinitesimal period: not impose any control (QVI (**47**)), and pay the fixed cost to begin a search period (QVI (**46**)). Finally, one of the two quasi-variational inequalities must hold with equality, since one of these two choices of control must be optimal.

Consider an inactive individual with frictionless income and deviation (y_t^*, δ_t), who remains inactive during the time interval $[t, t+\Delta t)$ for some small $\Delta t > 0$, then reverts back to the optimal policy $\mathfrak{p}^*$. Her value function $\mathcal{V}(y_t^*, \delta_t)$ satisfies

$$\mathcal{V}(y_t^*, \delta_t) \geq U(y_t^*, \delta_t)\Delta t + \frac{1-\rho_2\Delta t}{1+\rho_1\Delta t}\mathbb{E}_t\left[\mathcal{V}(y_t^* + \Delta y^*, \delta_t + \Delta\delta)\right],$$

where $\Delta\delta = -\frac{(1-p)\varepsilon}{1+\varepsilon}\Delta \ln y^*$. Multiplying by $(1+\rho_1\Delta t)$, substracting $(1-\rho_2\Delta t)\,\mathcal{V}(y_t^*,\delta_t)$ and dividing by Δt on both sides, we obtain, letting $\Delta t \to 0$,

$$(\rho_1+\rho_2)\,\mathcal{V}(y^*,\delta) \geq U(y^*,\delta) + \frac{\mathbb{E}_t\left[d\mathcal{V}(y^*,\delta)\right]}{dt}.$$

Using Itô's formula and the laws of motion of y_t^* and δ_t, we find

$$\frac{\mathbb{E}_t\left[d\mathcal{V}(y_t^*,\delta_t)\right]}{dt} = \left(\mu_y + \frac{1}{2}\sigma_y^2\right) y_t^* \frac{\partial \mathcal{V}}{\partial y^*} + \mu_\delta \frac{\partial \mathcal{V}}{\partial \delta} + \frac{1}{2}\sigma_y^2 (y_t^*)^2 \frac{\partial^2 \mathcal{V}}{\partial (y^*)^2} + \frac{1}{2}\sigma_\delta^2 \frac{\partial^2 \mathcal{V}}{\partial \delta^2} + \sigma_y\sigma_\delta y_t^* \frac{\partial^2 \mathcal{V}}{\partial y^* \partial \delta}.$$

Since the value function is homogeneous in $g(c^*)$, we can replace $\mathcal{V}(y_t^*,\delta_t)$ with $g(c_t^*)\,v(\delta_y)$ in the resulting equation and divide through by $g(c_t^*)$ to obtain $\boldsymbol{\rho} v(\delta_t) \geq \mathcal{L}v(\delta_t) + u(\delta_t)$.

Next suppose that the individual pays the fixed adjustment cost at time t, and hence becomes a searcher. We have

$$\mathcal{V}(y_t^*,\delta_t) \geq \mathcal{V}_s(y_t^*,\delta_t) - \kappa g(c_t^*).$$

Dividing both sides by $g(c_t^*)$ and using the expression derived above for $v_s(\delta)$, we thus obtain $v(\delta_t) \geq \mathcal{M}v(\delta_t)$.

We can similarly derive heuristically the smooth-pasting conditions. $\square$

The interpretation of the HJB equation (19) is as follows. Interpreting the entitlement to the flow of disposable incomes and deviations as an asset, and $\mathcal{V}_s(y^*,\delta)$ as its value, we can write:

$$(\rho_1+\rho_2)\,\mathcal{V}_i(y^*,\delta) = U(y^*,\delta) + \frac{\mathbb{E}_t\left[d\mathcal{V}_i(y^*,\delta)\right]}{dt}.$$

The left hand side gives the normal return per unit time that an individual, using $(\rho_1+\rho_2)$ as the discount rate, would require for holding this asset. The right hand side is the expected total return per unit time from holding the asset. The first term is the immediate payout or dividend from the asset. The second term is its expected rate of capital gain or loss. The equality is a no-arbitrage condition, expressing the investor's willingness to hold the asset. Using Itô's formula, we can express the second term in the right hand side as a function of the first and second partial derivatives of the value function $\mathcal{V}_i$ and the drifts and volatilities of the income and deviation processes. We then obtain the HJB equation (19) for $v_i(\delta)$ using the homogeneity of the value function shown in Proposition 1.

Feng and Muthuraman (2010) provide an algorithm to compute numerically the optimal individual policy solution to (19,20,20,22), which is easily extended to this paper's environment.

Figure 3 shows the value functions $v_i(\delta)$ and $v_s(\delta)$ of inactives (in blue) and searchers (in red) along with the optimal thresholds $\underline{\delta}, \delta^*, \bar{\delta}$, as well as the function $v_s(\delta) - \kappa$ (dashed red), which illustrates the value-matching and smooth-pasting conditions.

Figure 3: Value functions $v_i(\delta), v_s(\delta)$

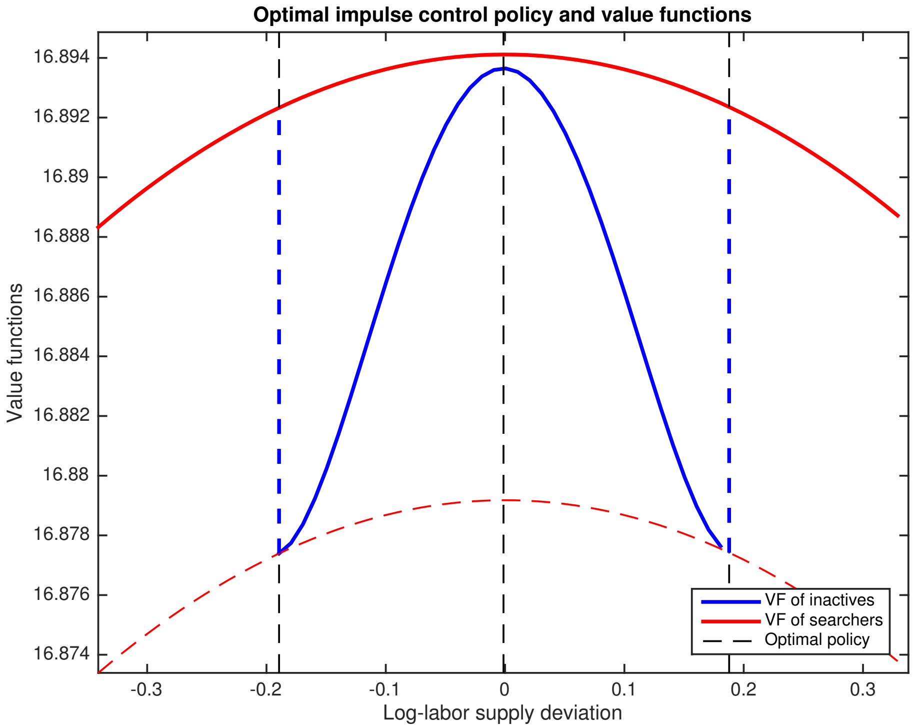
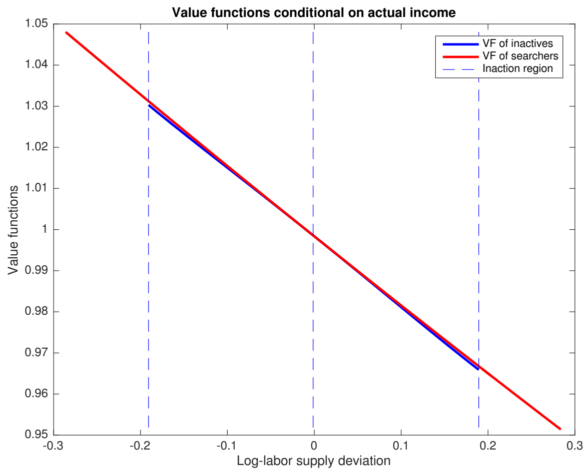
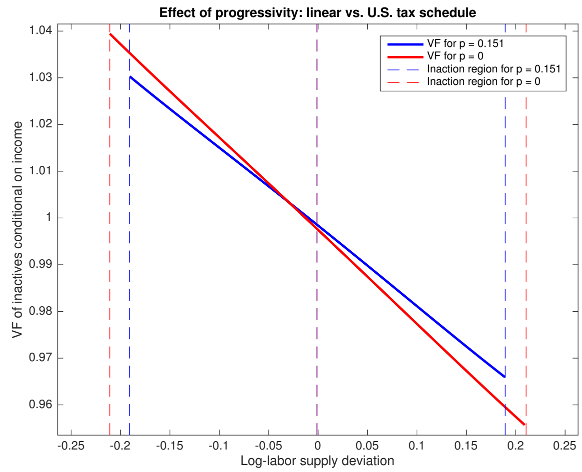


Figure 4 shows graphically the value functions $\hat{v}_i(\delta), \hat{v}_s(\delta)$ conditional on (actual) income y (left panel) and the effects of perturbing the progressivity of the tax schedule on the optimal inaction region and on the value of inactives $\hat{v}_i$ (right panel). These figures show that (both inactive and searching) individuals with higher deviation but the same income are worse off (i.e., individuals who work more hours but earn the same income, and hence have a lower wage, reach a lower level of utility), and that within the inaction region the value of searching is always strictly higher than the value of inactivity. The right panel shows that as the progressivity of the tax schedule decreases, the inaction region widens, and the distribution of individual utilities $\hat{v}_i(\cdot)$ adjusts endogenously within the new bands.

Figure 4: Value functions conditional on observed income $\bar{v}(\delta)$ and effects of progressivity p





I now characterize the stationary income and deviation distributions, assuming their existence. I derive the KFE equations (27) and show that their boundary conditions are the following. First, the density functions $f^i_{\ln y^*,\delta}$ and $f^s_{\ln y^*,\delta}$ are continuous in δ^* and $\{\underline{\delta},\bar{\delta}\}$ respectively: for all $u\in\mathbb{R}$,

$$f^{i,s}_{\ln y^*,\delta}\left(u,\delta^-\right)=f^{i,s}_{\ln y^*,\delta}\left(u,\delta^+\right),\ \text{for } \delta\in\left\{\underline{\delta},\delta^*,\bar{\delta}\right\}. \tag{49}$$

Second, the boundaries $\underline{\delta}$ and $\bar{\delta}$ are absorbing for $f^i_{\ln y^*,\delta}$, so that there is no mass of inactive individuals at the edges of the inaction region: for all $u\in\mathbb{R}$,

$$f^i_{\ln y^*,\delta}\left(u,\underline{\delta}\right)=f^i_{\ln y^*,\delta}\left(u,\bar{\delta}\right)=0. \tag{50}$$

Intuitively, this is because individuals who reach a boundary of their inaction region immediately start searching and leave the inaction state. Third, the density of searchers in a given job converges to zero as $\delta\to\pm\infty$: for all $u\in\mathbb{R}$,

$$\lim_{\delta\to\pm\infty} f^s_{\ln y^*,\delta}\left(u-\frac{\sigma_y}{\sigma_\delta}\delta,\delta\right)=0. \tag{51}$$

Fourth, total flows in and out of $\underline{\delta},\delta^*,\bar{\delta}$ must balance, which yields three functional equations linking the density functions $f^i_{\ln y^*,\delta}$ and $f^s_{\ln y^*,\delta}$. Letting $\hat{f}^x$ denote the function $\frac{\sigma_y}{\sigma_\delta}f^x_1+f^x_2$ for $x\in\{i,s\}$, these conditions write: for all $u\in\mathbb{R}$,

$$\begin{aligned}\hat{f}^i\left(u,\delta^{*+}\right)-\hat{f}^i\left(u,\delta^{*-}\right) &= \frac{2}{\sigma^2_\delta}\left(\rho_2 f_{\ln y^*_0}(u)+q f^s_{\ln y^*}(u)\right), &(52)\\ \hat{f}^i\left(u,\underline{\delta}^+\right)+\hat{f}^s\left(u,\underline{\delta}^+\right)-\hat{f}^s\left(u,\underline{\delta}^-\right) &= 0, &(53)\\ \hat{f}^i\left(u,\bar{\delta}^-\right)+\hat{f}^s\left(u,\bar{\delta}^-\right)-\hat{f}^s\left(u,\bar{\delta}^+\right) &= 0. &(54)\end{aligned}$$

These equations equate the inflows and outflows of individuals going from one state (inaction, search, non-participation) into another, following a change in their wage and hence desired hours, the reception of a new job opportunity, or a "birth". Finally, a normalizing condition imposing that the total mass of individuals in the population is equal to 1 completes the full characterization of the economy's steady-state:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[f^i_{\ln y^*,\delta}\left(u,\delta\right)+f^s_{\ln y^*,\delta}\left(u,\delta\right)\right]dud\delta=1. \tag{55}$$

Proof of Proposition 4. To derive the KFE equations (27) that must be satisfied by the stationary joint distributions $f^i_{\ln y^*,\delta}$ and $f^s_{\ln y^*,\delta}$, discretize the processes $(\ln y^*,\delta)$ on a two-dimensional grid with size $\left(\Delta h,\frac{1-p}{1+1/\varepsilon}\Delta h\right)$. In the time unit Δt, $\ln y^*$ moves up by $\Delta h=\sigma_y\sqrt{\Delta t}$ and δ moves down by $\frac{1-p}{1+1/\varepsilon}\Delta h$ with probability $\frac{1}{2}\left(1+\frac{\mu_y}{\sigma_y^2}\Delta h\right)$. The balanced flow equations for $f^i_{\ln y^*,\delta}$ at point $(u,\delta)\in\mathbb{R}\times\left\{(\underline{\delta},\delta^*)\cup\left(\delta^*,\bar{\delta}\right)\right\}$ write:

$$\begin{aligned}f^i_{\ln y^*,\delta}\left(u,\delta\right)=(1-\rho_2\Delta t)\Bigg\{\quad&\frac{1}{2}\left(1+\frac{\mu_y}{\sigma_y^2}\Delta h\right)f^i_{\ln y^*,\delta}\left(u-\Delta h,\delta+\frac{1-p}{1+1/\varepsilon}\Delta h\right)\\&+\frac{1}{2}\left(1-\frac{\mu_y}{\sigma_y^2}\Delta h\right)f^i_{\ln y^*,\delta}\left(u+\Delta h,\delta-\frac{1-p}{1+1/\varepsilon}\Delta h\right)\Bigg\}.\end{aligned}$$

Taking a second-order Taylor expansion in Δh of this equation around 0 and rearranging terms easily yields (27). We also have $f^i_{\ln y^*,\delta}=0$ if $\delta\notin\left(\underline{\delta},\bar{\delta}\right)$. The balanced flow equations for $f^s_{\ln y^*,\delta}$ at point $(u,\delta)\in\mathbb{R}\times\left\{(-\infty,\underline{\delta})\cup\left(\underline{\delta},\bar{\delta}\right)\cup\left(\bar{\delta},\infty\right)\right\}$ write identically, except that the right hand side is multiplied by the probability $(1-q\Delta t)$ of exiting the search region in $[t,t+\Delta t)$ due to the arrival of an adjustment opportunity.

The boundary conditions (50) to (54) can be derived as follows. The balanced-flow equation for $f^i_{\ln y^*,\delta}$ at the point $\left(u,\bar{\delta}\right)$ writes:

$$f^i_{\ln y^*,\delta}\left(u,\bar{\delta}\right)=(1-\rho_2\Delta t)\left\{\frac{1}{2}\left(1-\frac{\mu_y}{\sigma_y^2}\Delta h\right)f^i_{\ln y^*,\delta}\left(u+\Delta h,\bar{\delta}-\frac{1-p}{1+1/\varepsilon}\Delta h\right)\right\}.$$

A first-order Taylor expansion in Δh around 0 yields $f^i_{\ln y^*,\delta}\left(u,\bar{\delta}\right)=0$, and similarly $f^i_{\ln y^*,\delta}\left(u,\underline{\delta}\right)=0$. Similarly, the balanced flow condition at the boundaries $\pm\infty$ for $f^s_{\ln y^*,\delta}$ writes

$$\lim_{\delta\to\pm\infty}f^s\left(u,\delta\right)|_h\text{ constant}=0.$$

Noting that the condition "h constant" is equivalent to "$\frac{(1-p)\varepsilon}{1+\varepsilon}\ln y^*+\delta$ constant", we obtain (51).

The balanced-flow equation for $f^i_{\ln y^*,\delta}$ at the point (u,δ^*) writes:

$$\begin{aligned} f^i(u,\delta^*) = (1-\rho_2\Delta t)\Bigg\{ & \frac{1}{2}\left(1+\frac{\mu_y}{\sigma_y^2}\Delta h\right) f^i\left(u-\Delta h, \delta^* + \frac{1-p}{1+1/\varepsilon}\Delta h\right) \\ & + \frac{1}{2}\left(1-\frac{\mu_y}{\sigma_y^2}\Delta h\right) f^i\left(u+\Delta h, \delta^* - \frac{1-p}{1+1/\varepsilon}\Delta h\right)\Bigg\} \\ + (1-\rho_2\Delta t)(q\Delta t)\Bigg\{ & \frac{1}{2}\left(1+\frac{\mu_y}{\sigma_y^2}\Delta h\right)\sum_{\delta\in\mathcal{G}} f^s(u-\Delta h,\delta) \\ & + \frac{1}{2}\left(1-\frac{\mu_y}{\sigma_y^2}\Delta h\right)\sum_{\delta\in\mathcal{G}} f^s(u+\Delta h,\delta)\Bigg\} \\ + (\rho_2\Delta t)&\left(\frac{1-p}{1+1/\varepsilon}\Delta h\right)^{-1} f_{\ln y_0^*}(u), \end{aligned}$$

where $\mathcal{G}$ denotes the grid of δ and $f_{\ln y_0^*}$ denotes the density of frictionless log-incomes at birth. Taking a first-order Taylor expansion in Δh around 0 using

$$\sum_{\delta\in\mathcal{G}} f^s_{\ln y^*,\delta}(\ln y^*,\delta) \xrightarrow[\Delta h\to 0]{} \left(\frac{1-p}{1+1/\varepsilon}\Delta h\right)^{-1} f^s_{\ln y^*}(\ln y^*)$$

yields (52). Finally, the balanced flow condition at the boundary $\bar{\delta}$ for $f^s_{\ln y^*,\delta}$ writes:

$$\begin{aligned} f^s(u,\bar{\delta}) = (1-\rho_2\Delta t)(1-q\Delta t)\Bigg\{ & \frac{1}{2}\left(1+\frac{\mu_y}{\sigma_y^2}\Delta h\right) f^s\left(u-\Delta h, \bar{\delta} + \frac{1-p}{1+1/\varepsilon}\Delta h\right) \\ & + \frac{1}{2}\left(1-\frac{\mu_y}{\sigma_y^2}\Delta h\right) f^s\left(u+\Delta h, \bar{\delta} - \frac{1-p}{1+1/\varepsilon}\Delta h\right)\Bigg\} \\ + (1-\rho_2\Delta t)\Bigg\{ & \frac{1}{2}\left(1-\frac{\mu_y}{\sigma_y^2}\Delta h\right) f^i\left(u+\Delta h, \bar{\delta} - \frac{1-p}{1+1/\varepsilon}\Delta h\right)\Bigg\}. \end{aligned}$$

A first-order Taylor expansion in Δh yields (54). Equation (53) is obtained similarly.

Note that changing variables and defining the density

$$g^i(u,\delta) = \frac{1-p}{1+1/\varepsilon} f^i_{\ln y^*,\delta}\left(u - \frac{1-p}{1+1/\varepsilon}\delta, \delta\right)$$

implies that g^i satisfies the PDE

$$\frac{1}{2}\sigma_\delta^2 g^i_{22} + \mu_\delta g^i_2 - \rho_2 g^i = 0 \quad \text{on } \mathbb{R}\times\left\{(\underline{\delta},\delta^*)\cup\left(\delta^*,\bar{\delta}\right)\right\}.$$

Imposing the boundary conditions above on the explicit solution to this differential equation yields, after tedious algebra and letting $\underline{\Delta} \equiv \delta^* - \underline{\delta}$ and $\bar{\Delta} \equiv \bar{\delta} - \delta^*$,

$$f^i_{\ln y^*,\delta}(u,\delta) = \frac{\tilde{f}\left(u + \frac{1+1/\varepsilon}{1-p}\delta\right)}{r^{\rho_2}_{2,\delta} - r^{\rho_2}_{1,\delta}} \left\{ \frac{e^{-r^{\rho_2}_{1,\delta}(\delta-\underline{\delta})} - e^{-r^{\rho_2}_{2,\delta}(\delta-\underline{\delta})}}{e^{-r_{1,\delta}(\rho_2)\underline{\Delta}} - e^{-r_{2,\delta}(\rho_2)\underline{\Delta}}} \mathbb{I}_{[\underline{\delta},\delta^*]} + \frac{e^{r^{\rho_2}_{1,\delta}(\bar{\delta}-\delta)} - e^{r^{\rho_2}_{2,\delta}(\bar{\delta}-\delta)}}{e^{r^{\rho_2}_{1,\delta}\bar{\Delta}} - e^{r^{\rho_2}_{2,\delta}\bar{\Delta}}} \mathbb{I}_{(\delta^*,\bar{\delta}]} \right\}$$

$$f^s_{\ln y^*,\delta}(u,\delta) = \frac{\tilde{f}\left(u + \frac{1+1/\varepsilon}{1-p}\delta\right)}{r^{\rho_2+q}_{2,\delta} - r^{\rho_2+q}_{1,\delta}} \left\{ \frac{e^{\left(r^{\rho_2+q}_{1,\delta}\mathbb{I}_{(-\infty,\underline{\delta}]} + r^{\rho_2+q}_{2,\delta}\mathbb{I}_{(\underline{\delta},\infty)}\right)(\delta-\underline{\delta})}}{e^{-r^{\rho_2+q}_{1,\delta}\underline{\Delta}} - e^{-r^{\rho_2+q}_{2,\delta}\underline{\Delta}}} - \frac{e^{\left(r^{\rho_2+q}_{1,\delta}\mathbb{I}_{(-\infty,\bar{\delta}]} + r^{\rho_2+q}_{2,\delta}\mathbb{I}_{(\bar{\delta},\infty)}\right)(\bar{\delta}-\delta)}}{e^{r^{\rho_2+q}_{1,\delta}\bar{\Delta}} - e^{r^{\rho_2+q}_{2,\delta}\bar{\Delta}}} \right\}$$

where the function $\tilde{f}(\cdot)$ satisfies the normalization (55) and the integral equation

$$\tilde{f}(u) = \frac{\left(e^{-r^{\rho_2}_{1,\delta}\underline{\Delta}} - e^{-r^{\rho_2}_{2,\delta}\underline{\Delta}}\right)\left(e^{r^{\rho_2}_{1,\delta}\bar{\Delta}} - e^{r^{\rho_2}_{2,\delta}\bar{\Delta}}\right)}{e^{r^{\rho_2}_{2,\delta}\bar{\Delta}}e^{-r^{\rho_2}_{1,\delta}\underline{\Delta}} - e^{-r^{\rho_2}_{2,\delta}\underline{\Delta}}e^{r^{\rho_2}_{1,\delta}\bar{\Delta}}} \left\{ \frac{\frac{2\rho_2}{\sigma^2_\delta} f_{\ln y^*_0}\left(u - \frac{1+1/\varepsilon}{1-p}\delta^*\right)}{r^{\rho_2}_{1,\delta} - r^{\rho_2}_{2,\delta}} + \dots \right.$$

$$\left. \dots + \frac{\frac{2q}{\sigma^2_\delta}}{r^{\rho_2+q}_{1,\delta} - r^{\rho_2+q}_{2,\delta}} \int_{-\infty}^{\infty} f^s_{\ln y^*,\delta}\left(u - \frac{1+1/\varepsilon}{1-p}\delta^*, \delta\right) d\delta \right\}$$

for all $u \in \mathbb{R}$. Now suppose $q = \infty$. The functional equation satisfied by $\tilde{f}$ is simpler:

$$\frac{\frac{2\rho_2}{\sigma_y\sigma_\delta}\frac{1+1/\varepsilon}{1-p}}{r^{\rho_2}_{2,\delta} - r^{\rho_2}_{1,\delta}} f_{\ln y^*_0}(u) = \frac{\tilde{f}\left(u + \frac{1+1/\varepsilon}{1-p}\underline{\delta}\right)}{e^{-r^{\rho_2}_{2,\delta}\underline{\Delta}} - e^{-r^{\rho_2}_{1,\delta}\underline{\Delta}}} - \frac{\tilde{f}\left(u + \frac{1+1/\varepsilon}{1-p}\bar{\delta}\right)}{e^{r^{\rho_2}_{2,\delta}\bar{\Delta}} - e^{r^{\rho_2}_{1,\delta}\bar{\Delta}}}$$

$$- \left(\frac{r^{\rho_2}_{2,\delta}e^{-r^{\rho_2}_{2,\delta}\underline{\Delta}} - r^{\rho_2}_{1,\delta}e^{-r^{\rho_2}_{1,\delta}\underline{\Delta}}}{e^{-r^{\rho_2}_{2,\delta}\underline{\Delta}} - e^{-r^{\rho_2}_{1,\delta}\underline{\Delta}}} - \frac{r^{\rho_2}_{2,\delta}e^{r^{\rho_2}_{2,\delta}\bar{\Delta}} - r^{\rho_2}_{1,\delta}e^{r^{\rho_2}_{1,\delta}\bar{\Delta}}}{e^{r^{\rho_2}_{2,\delta}\bar{\Delta}} - e^{r^{\rho_2}_{1,\delta}\bar{\Delta}}} \right) \tilde{f}\left(u + \frac{1+1/\varepsilon}{1-p}\delta^*\right).$$

Letting $\tilde{f}(u) = e^{\frac{1-p}{1+1/\varepsilon}r^{\rho_2}_{1,\delta}u} g(u)$, it is easy to check that the solution $g(u)$ to the equation converges to a constant as $u \to \infty$. This implies that

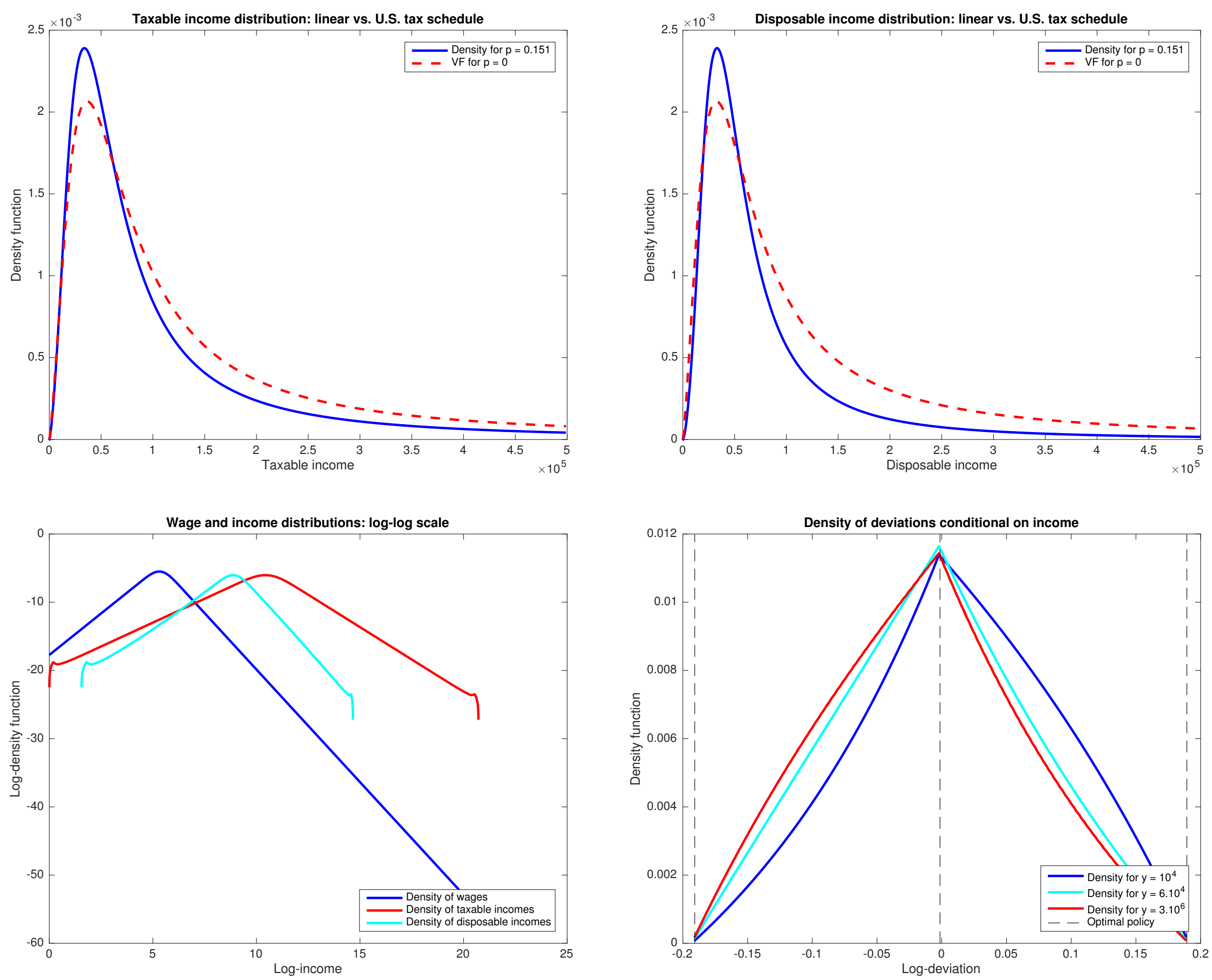
$$f^i_{\ln y^*}(u) = \int_{\underline{\delta}}^{\bar{\delta}} f^i_{\ln y^*,\delta}(u,\delta)\, d\delta \underset{u\to\infty}{\sim} e^{r^{\rho_2}_{1,y}u},$$

i.e., that the stationary income distribution has a Pareto right tail with coefficient $r^{\rho_2}_{1,y}$. □

Equation (27) has the following interpretation. At a given point $(\ln y^*, \delta)$, the density is reduced by the agents who move away from there, and is increased by those who move there from a former deviation δ', following an increase (resp., decrease) in their wage if $\delta' > \delta$ (resp., $\delta' < \delta$). These flows occur both because of the drift μ_w (second and third terms of (27)) and the volatility σ_w (fourth to sixth terms of (27)) of individual productivities. Moreover, the distribution loses mass at the death rate ρ_2, plus the hours adjustment rate q for the searchers. In the steady-state, these flows in and out of $(\ln y^*, \delta)$ must balance on net and are thus equal to zero. Note that these equations do not hold at δ^* for $f^i_{\ln y^*,\delta}$, and at $\{\underline{\delta}, \bar{\delta}\}$ for $f^s_{\ln y^*,\delta}$, where the inflows from births and from endogenous adjustments produce kinks in the densities.

The top two graphs of Figure 5 show the distribution of taxable incomes (left panel) and disposable incomes (right panel), and how they change when the tax schedule goes from the U.S. rate of progressivity to a linear tax rate. The mean and variance of both distributions are lower when p is higher; the tails are thinner, to a much larger extent in the case of the disposable income distribution than of the pre-tax income distribution. The bottom left panel of Figure 5 shows the wage, taxable income and disposable income distributions in log-log scale. This representation illustrates clearly the fact that those distributions all have left and right Pareto tails, corresponding to the asymptotic straight lines whose slopes are equal to the Pareto coefficients. The smaller the slopes in absolute value, the more unequal the distribution: the wage (or productivity) distribution is the most equal, the taxable income distribution is the most unequal (due to the positive labor supply elasticity); the inequality of disposable incomes is smaller than that of taxable incomes and closer to that of wages due to the positive rate of progressivity. Finally, the bottom right panel of Figure 5 shows the inactive individuals' stationary distributions of deviations δ conditional on income y for several values of y, along with the boundaries of the optimal inaction region $\left(\underline{\delta}, \delta^{*}, \bar{\delta}\right)$.

Figure 5: Stationary income distributions



A.2 Proofs and additional results for Sections 4 and 5

I first derive the formulas (39) and (40) characterizing the optimal taxes in the frictionless model.

Proof of equations (39) and (40). Consider a perturbation $(d\tau, dp)$ of the baseline tax system. The first-order change in the tax liability at income y is, to a first-order in $(d\tau, dp)$ (ignoring the term $o\left(d\tau, dp\right)$),

$$\begin{aligned}\tilde{T}\left(y\right)-T\left(y\right)=&\left(y-\frac{1-\tau-d\tau}{1-p-dp}y^{1-p-dp}\right)-\left(y-\frac{1-\tau}{1-p}y^{1-p}\right)\\=&\frac{1}{1-p}y^{1-p}d\tau+\left(\ln y-\frac{1}{1-p}\right)\frac{1-\tau}{1-p}y^{1-p}dp=\partial_{\tau}\left(y\right)d\tau+\partial_{p}\left(y\right)dp,\end{aligned}$$

and the first-order change in the marginal tax rate at income y is

$$\tilde{T}'\left(y\right)-T'\left(y\right)=y^{-p}d\tau+\left(1-\tau\right)y^{-p}\ln y dp=\partial_{\tau}'\left(y\right)d\tau+\partial_{p}'\left(y\right)dp.$$

Thus, letting $f_y\left(y\right)$ denote the stationary density of incomes, we have

$$\begin{aligned}\int_0^\infty\left[\partial_{\tau}\left(y\right)-\frac{T'\left(y\right)}{1-T'\left(y\right)}\frac{\varepsilon}{1+p\varepsilon}y\partial_{\tau}'\left(y\right)\right]f_y\left(y\right)dy=&\frac{1}{1-\tau}\left\{\frac{1+\varepsilon}{1+p\varepsilon}\frac{1-\tau}{1-p}\mathbb{E}\left[y^{1-p}\right]-\frac{\varepsilon}{1+p\varepsilon}\mathbb{E}\left[y\right]\right\},\\\int_0^\infty\left[\partial_{p}\left(y\right)-\frac{T'\left(y\right)}{1-T'\left(y\right)}\frac{\varepsilon}{1+p\varepsilon}y\partial_{p}'\left(y\right)\right]f_y\left(y\right)dy=&-\frac{1}{1-p}\frac{1-\tau}{1-p}\mathbb{E}\left[y^{1-p}\right]-\frac{\varepsilon}{1+p\varepsilon}\mathbb{E}\left[y\ln y\right]\\&+\frac{1+\varepsilon}{1+p\varepsilon}\frac{1-\tau}{1-p}\mathbb{E}\left[y^{1-p}\ln y\right].\end{aligned}$$

In the frictionless model, the income distribution $f_y\left(y\right)$ is double-Pareto-lognormal with parameters $\left(r^{\rho_2}_{1,y}, r^{\rho_2}_{2,y}, m_y, s_y^2\right)$, where $m_y=\frac{1+\varepsilon}{1+p\varepsilon}m_w+\frac{\varepsilon\ln(1-\tau)}{1+p\varepsilon}$ and $s_y=\frac{1+\varepsilon}{1+p\varepsilon}s_w$. We can thus derive directly formulas (39) and (40). The parameters of the economy are affected by the perturbation in the following way:

$$\begin{aligned}&\frac{dm_y}{d\tau}=-\frac{1}{1-\tau}\frac{\varepsilon}{1+p\varepsilon},\quad\frac{d}{d\tau}\left\{s_y,\mu_y,\sigma_y,r^{\rho_2}_{1,y},r^{\rho_2}_{2,y}\right\}=0,\\&\frac{d}{dp}\left\{m_y,s_y,\mu_y,\sigma_y,r^{\rho_2}_{1,y},r^{\rho_2}_{2,y}\right\}=\frac{\varepsilon}{1+p\varepsilon}\left\{-m_y,-s_y,-\mu_y,-\sigma_y,r^{\rho_2}_{1,y},r^{\rho_2}_{2,y}\right\}.\end{aligned}$$

Thus the density of incomes satisfies, letting $\Phi_i\equiv\Phi\left(\frac{\ln y-m_y}{s_y}+r^{\rho_2}_{i,y}s_y\right)$ for $i\in\{1,2\}$ with similar definitions of Φ_i^c and φ_i, and $e_i \equiv \frac{\left|r_{1,y}^{\rho 2}\right| r_{2,y}^{\rho 2}}{\left|r_{1,y}^{\rho 2}\right| + r_{2,y}^{\rho 2}} e^{\frac{1}{2}\left(r_{i,y}^{\rho 2}\right)^2 s_y^2 - r_{i,y}^{\rho 2} m_y}$,

$$
\begin{aligned}
f_y(y) = & \frac{\left|r_{1,y}^{\rho 2}\right| r_{2,y}^{\rho 2}}{\left|r_{1,y}^{\rho 2}\right| + r_{2,y}^{\rho 2}} \left\{ e^{\frac{1}{2}\left(r_{1,y}^{\rho 2}\right)^2 s_y^2 - r_{1,y}^{\rho 2} m_y} y^{r_{1,y}^{\rho 2}-1} \Phi_1 + e^{\frac{1}{2}\left(r_{2,y}^{\rho 2}\right)^2 s_y^2 - r_{2,y}^{\rho 2} m_y} y^{r_{2,y}^{\rho 2}-1} \Phi_2^c \right\}, \\
\frac{df_y(y)}{d\tau} = & \frac{1}{1-\tau} \frac{\varepsilon}{1+p\varepsilon} \frac{\left|r_{1,y}^{\rho 2}\right| r_{2,y}^{\rho 2}}{\left|r_{1,y}^{\rho 2}\right| + r_{2,y}^{\rho 2}} \left\{ e^{\frac{1}{2}\left(r_{1,y}^{\rho 2}\right)^2 s_y^2 - r_{1,y}^{\rho 2} m_y} y^{r_{1,y}^{\rho 2}-1} \left[r_{1,y}^{\rho 2} \Phi_1 + \frac{1}{s_y} \varphi_1 \right] \right. \\
& \left. + e^{\frac{1}{2}\left(r_{2,y}^{\rho 2}\right)^2 s_y^2 - r_{2,y}^{\rho 2} m_y} y^{r_{2,y}^{\rho 2}-1} \left[r_{2,y}^{\rho 2} \Phi_2^c - \frac{1}{s_y} \varphi_2 \right] \right\}, \\
\frac{df_y(y)}{dp} = & \frac{\varepsilon}{1+p\varepsilon} \frac{\left|r_{1,y}^{\rho 2}\right| r_{2,y}^{\rho 2}}{\left|r_{1,y}^{\rho 2}\right| + r_{2,y}^{\rho 2}} \left\{ e^{\frac{1}{2}\left(r_{1,y}^{\rho 2}\right)^2 s_y^2 - r_{1,y}^{\rho 2} m_y} y^{r_{1,y}^{\rho 2}-1} \left[\left(1 + r_{1,y}^{\rho 2} \ln y\right) \Phi_1 + \ln y \frac{\varphi_1}{s_y} \right] \right. \\
& \left. + e^{\frac{1}{2}\left(r_{2,y}^{\rho 2}\right)^2 s_y^2 - r_{2,y}^{\rho 2} m_y} y^{r_{2,y}^{\rho 2}-1} \left[\left(1 + r_{2,y}^{\rho 2} \ln y\right) \Phi_2^c - \ln y \frac{\varphi_2^c}{s_y} \right] \right\}.
\end{aligned}
$$

The first-order change in tax revenue due to a perturbation $d\tau$ in the frictionless model, $\frac{d\mathscr{R}^*(T)}{d\tau}$, is given by

$$
\begin{aligned}
\frac{d\mathscr{R}^*}{d\tau} = & \frac{d}{d\tau} \left\{ \int_0^\infty \left(y - \frac{1-\tau}{1-p} y^{1-p} \right) f_y(y)\, dy \right\} \\
= & \frac{1}{1-p} \int_0^\infty y^{1-p} f_y(y)\, dy + \int_0^\infty \left(y - \frac{1-\tau}{1-p} y^{1-p} \right) \frac{df_y(y)}{d\tau} dy \\
= & - \frac{1}{1-\tau} \frac{\varepsilon}{1+p\varepsilon} \mathbb{E}[y] + \frac{1}{1-\tau} \frac{1+\varepsilon}{1+p\varepsilon} \frac{1-\tau}{1-p} \mathbb{E}\left[y^{1-p}\right],
\end{aligned}
$$

where the last line is obtained by integrating by parts to compute the integrals of the form $\int_0^\infty e^{\left(r_{i,y}^{\rho 2}-1+\alpha\right)\ln y} \frac{1}{s_y} \varphi_i dy$ with $r_{i,y}^{\rho 2} + \alpha < 0$. Similarly, the effect of a perturbation dp is given by

$$
\begin{aligned}
\frac{d\mathscr{R}^*}{dp} = & \int_0^\infty \left(-\frac{1}{1-p} \frac{1-\tau}{1-p} y^{1-p} + \frac{1-\tau}{1-p} y^{1-p} \ln y \right) f_y(y)\, dy + \int_0^\infty \left(y - \frac{1-\tau}{1-p} y^{1-p} \right) \frac{df_y(y)}{dp} dy \\
= & - \frac{1}{1-p} \frac{1-\tau}{1-p} \mathbb{E}\left[y^{1-p}\right] - \frac{\varepsilon}{1+p\varepsilon} \mathbb{E}[y \ln y] + \frac{1+\varepsilon}{1+p\varepsilon} \frac{1-\tau}{1-p} \mathbb{E}\left[y^{1-p} \ln y\right],
\end{aligned}
$$

again integrating by parts to compute the integrals of the form $\int_0^\infty e^{\left(r_{i,y}^{\rho 2}-1+\alpha\right)\ln y} \ln y \frac{1}{s_y} \varphi_i dy$. Note that a different way of showing these results is to use the KFE characterization of the income distribution to deduce that

$$
\begin{aligned}
f_y^{\tau+d\tau}(y) & \underset{d\tau \to 0}{=} \left(1 + \frac{\varepsilon}{1+p\varepsilon} \frac{d\tau}{1-\tau}\right) f_y^\tau \left(\left(1 + \frac{\varepsilon}{1+p\varepsilon} \frac{d\tau}{1-\tau}\right) y \right) + o(d\tau), \\
f_y^{p+dp}(y) & \underset{dp \to 0}{=} \left(1 + \frac{\varepsilon}{1+p\varepsilon} dp\right) y^{\frac{\varepsilon}{1+p\varepsilon} dp} f_y^p \left(y^{1+\frac{\varepsilon}{1+p\varepsilon} dp} \right) + o(dp),
\end{aligned}
$$

which mean that the individuals with income y before the perturbation $d\tau$ (resp., dp) end up earning $y' = \left(1 - \frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau}\right) y$ (resp., $y' = y^{1-\frac{\varepsilon}{1+p\varepsilon}dp}$) after the perturbation, so that $f_y^{\tau+d\tau}(y')\, dy' = f_y^{\tau}(y)\, dy$ (resp., $f_y^{p+dp}(y')\, dy' = f_y^{p}(y)\, dy$). It is then straightforward (with a change of variables in the integral) to obtain the formulas above:

$$
\begin{aligned}
\frac{d\mathscr{R}^*}{d\tau} =& \frac{1}{d\tau}\left\{\left(1+\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau}\right)\int_0^\infty \left(y - \frac{1-\tau-d\tau}{1-p}y^{1-p}\right) f_y^{\tau+d\tau}\left(\left(1+\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau}\right)y\right) dy - \mathscr{R}\right\} \\
=& \frac{1}{d\tau}\left\{\left(-\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau}\right)\int_0^\infty y f_y^{\tau}(y)\, dy + \left(\frac{1+\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau}\right)\int_0^\infty \frac{1-\tau}{1-p}y^{1-p} f_y^{\tau}(y)\, dy\right\},
\end{aligned}
$$

and similarly for a perturbation dp.

The first-order change in the government objective due to a perturbation $d\tau$ in the frictionless model is given by:

$$
\begin{aligned}
\frac{d\mathscr{W}^*(T)}{d\tau} =& \frac{d}{d\tau}\left\{\int_0^\infty \mathcal{V}(y) f_y(y)\, dy\right\} \\
=& \frac{1}{d\tau}\left\{\int_0^\infty \frac{\frac{1}{1-\gamma}\left(\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau-d\tau}{1-p}y^{1-p}\right)^{1-\gamma}}{\rho+\beta-(1-\gamma)\mu_c - \frac{1}{2}(1-\gamma)^2\sigma_c^2} f_y^{\tau+d\tau}(y)\, dy - \int_0^\infty \mathcal{V}(y) f_y(y)\, dy\right\} \\
=& \frac{1}{d\tau}\int_0^\infty \left\{\frac{\frac{1}{1-\gamma}\left(\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p}\left\{\left(1-\frac{d\tau}{1-\tau}\right)\left(1-\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau}\right)^{1-p}\right\}y^{1-p}\right)^{1-\gamma}}{\rho+\beta-(1-\gamma)\mu_c - \frac{1}{2}(1-\gamma)^2\sigma_c^2}\right. \\
& \left. - \frac{\frac{1}{1-\gamma}\left(\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p}y^{1-p}\right)^{1-\gamma}}{\rho+\beta-(1-\gamma)\mu_c - \frac{1}{2}(1-\gamma)^2\sigma_c^2}\right\} f_y(y)\, dy \\
=& -\int_0^\infty \frac{\left(\frac{1+p\varepsilon}{1+\varepsilon}\right)^{-\gamma}\left(\frac{1-\tau}{1-p}y^{1-p}\right)^{-\gamma}}{\rho+\beta-(1-\gamma)\mu_c - \frac{1}{2}(1-\gamma)^2\sigma_c^2}\frac{y^{1-p}}{1-p} f_y(y)\, dy \\
=& -\lambda^*\int_0^\infty \boldsymbol{\omega}^*(y)\, \partial_\tau(y)\, f_y(y)\, dy,
\end{aligned}
$$

and similarly for a perturbation dp:

$$\begin{aligned}\frac{d\mathscr{W}^*(T)}{dp} =& \frac{1}{dp}\left\{\int_0^\infty \frac{\frac{1}{1-\gamma}\left(\frac{1+(p+dp)\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-(p+dp)}y^{1-(p+dp)}\right)^{1-\gamma}}{\rho+\beta-(1-\gamma)\left(1-\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\mu_c-\frac{1}{2}(1-\gamma)^2\left(1-\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)^2\sigma_c^2}\times\right. \\ &\left. \ldots\times f_y^{p+dp}(y)\,dy-\mathscr{W}^*(T)\right\} \\ =& \frac{1}{dp}\left\{\left(1+\frac{1-\gamma}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp-\frac{\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}\left((1-\gamma)\mu_c+(1-\gamma)^2\sigma_c^2\right)}{\rho+\beta-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2}dp\right)\times\ldots\right. \\ &\left.\ldots\times\int_0^\infty \frac{\frac{1}{1-\gamma}\left(\left(1-\frac{1+\varepsilon}{1+p\varepsilon}\ln y dp\right)\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p}y^{1-p}\right)^{1-\gamma}}{\rho+\beta-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2}f_y(y)\,dy-\mathscr{W}^*(T)\right\} \\ =& -\lambda^*\int_0^\infty \boldsymbol{\omega}^*(y)\,\partial_p(y)\,f_y(y)\,dy-\lambda^*\int_0^\infty \frac{d\boldsymbol{\rho}}{dp}\hat{\boldsymbol{\omega}}^*(y)\,f_y(y)\,dy.\end{aligned}$$

We finally obtain the optimal tax schedule by imposing that

$$0=\frac{d\mathscr{W}^*}{d\tau}+\lambda^*\frac{d\mathscr{R}^*}{d\tau}=\frac{d\mathscr{W}^*}{dp}+\lambda^*\frac{d\mathscr{R}^*}{dp},$$

which proves formulas (39) and (40) in the frictionless model. □

Next I first formulas (31) and (39) in the frictional model.

Proof of Propositions 5 and 6. I show that a perturbation $d\tau$ of the tax schedule has the following first-order effects on the density functions in the frictional model: for all u,δ,

$$\begin{aligned} f_{\ln y^*,\delta}^{x,\tau+d\tau}(u,\delta) &= f_{\ln y^*,\delta}^{x,\tau}\left(u+\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau},\delta\right) \\ \text{i.e., } f_{y^*,\delta}^{x,\tau+d\tau}(y,\delta) &\underset{d\tau\to 0}{=} \left(1+\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau}\right)f_{y^*,\delta}^{x,\tau}\left(\left(1+\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau}\right)y,\delta\right),\end{aligned} \tag{56}$$

for $x\in\{i,s\}$. To see this, consider the functions

$$g^x(u,\delta)\equiv f_{\ln y^*,\delta}^{x,\tau}\left(u+\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau},\delta\right).$$

I show that g^i, g^s satisfy the KFE and boundary conditions that define the functions $f_{\ln y^*,\delta}^{i,\tau+d\tau}$ and $f_{\ln y^*,\delta}^{s,\tau+d\tau}$, respectively, which will imply the result. First, note that $\mu_y,\sigma_y,\mu_\delta,\sigma_\delta,\underline{\delta},\delta^*,\bar{\delta}$ do not depend on τ. We have, for all $u\in\mathbb{R}$, all $\delta\in(\underline{\delta},\delta^*)\cup(\delta^*,\bar{\delta})$ if $x=i$, and all $\delta\in\mathbb{R}\setminus\{\underline{\delta},\bar{\delta}\}$ if $x=s$, letting

$\hat{u} \equiv u + \frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau}$,

$$\begin{aligned}
&-\left(\beta + q\mathbb{I}_{f^s}\right) g\left(u,\delta\right) - \mu_y g_1\left(u,\delta\right) + \mu_\delta g_2\left(u,\delta\right) + \frac{1}{2}\sigma_y^2 g_{11}\left(u,\delta\right) + \frac{1}{2}\sigma_\delta^2 g_{22}\left(u,\delta\right) - \sigma_y\sigma_\delta g_{12}\left(u,\delta\right) \\
=& -\left(\beta + q\mathbb{I}_{f^s}\right) f^{x,\tau}_{\ln y^*,\delta}\left(\hat{u},\delta\right) - \mu_y \partial_1 f^{x,\tau}_{\ln y^*,\delta}\left(\hat{u},\delta\right) + \mu_\delta \partial_2 f^{x,\tau}_{\ln y^*,\delta}\left(\hat{u},\delta\right) \\
&+ \frac{1}{2}\sigma_y^2 \partial_{11} f^{x,\tau}_{\ln y^*,\delta}\left(\hat{u},\delta\right) + \frac{1}{2}\sigma_\delta^2 \partial_{22} f^{x,\tau}_{\ln y^*,\delta}\left(\hat{u},\delta\right) - \sigma_y\sigma_\delta \partial_{12} f^{x,\tau}_{\ln y^*,\delta}\left(\hat{u},\delta\right) = 0,
\end{aligned}$$

where the last equality follows from the KFE solved by $f^{x,\tau}_{\ln y^*,\delta}$, evaluated at $(\hat{u},\delta)$. Next, note that the density of incomes at birth satisfies

$$f^{\tau+d\tau}_{\ln y_0^*}\left(u\right) = \frac{1}{s_y\sqrt{2\pi}} e^{-\frac{1}{2s_y^2}\left(u - \frac{\varepsilon}{1+p\varepsilon}\ln(1-\tau-d\tau) - \frac{1+\varepsilon}{1+p\varepsilon}m_w\right)^2} = f^{\tau}_{\ln y_0^*}\left(u + \frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau}\right).$$

Thus we have

$$\begin{aligned}
&g_1^s\left(u,\bar{\delta}^+\right) - g_1^s\left(u,\bar{\delta}^-\right) - \frac{1-p}{1+1/\varepsilon}\left(g_2^s\left(u,\bar{\delta}^+\right) - g_2^s\left(u,\bar{\delta}^-\right)\right) - \left(g_1^i\left(u,\bar{\delta}^-\right) - \frac{1-p}{1+1/\varepsilon} g_2^i\left(u,\bar{\delta}^-\right)\right) \\
=& \partial_1 f^{s,\tau}_{\ln y^*,\delta}\left(\hat{u},\bar{\delta}^+\right) - \partial_1 f^{s,\tau}_{\ln y^*,\delta}\left(\hat{u},\bar{\delta}^-\right) - \frac{1-p}{1+1/\varepsilon}\left(\partial_2 f^{s,\tau}_{\ln y^*,\delta}\left(\hat{u},\bar{\delta}^+\right) - \partial_2 f^{s,\tau}_{\ln y^*,\delta}\left(\hat{u},\bar{\delta}^-\right)\right) \\
&- \left(\partial_1 f^{i,\tau}_{\ln y^*,\delta}\left(\hat{u},\bar{\delta}^-\right) - \frac{1-p}{1+1/\varepsilon}\partial_2 f^{i,\tau}_{\ln y^*,\delta}\left(\hat{u},\bar{\delta}^-\right)\right) = 0,
\end{aligned}$$

where the last equality follows from equation (54) satisfied by $f^{\tau}_{\ln y^*,\delta}$, evaluated at $(\hat{u},\delta)$. The corresponding equation (53) for g^i, g^s is shown in the same way. Similarly,

$$\begin{aligned}
&g_1^i\left(u,\delta^{*+}\right) - g_1^i\left(u,\delta^{*-}\right) - \frac{1-p}{1+1/\varepsilon}\left(g_2^i\left(u,\delta^{*+}\right) - g_2^i\left(u,\delta^{*-}\right)\right) - \frac{2}{\sigma_y\sigma_\delta}\left(\beta f^{\tau}_{\ln y_0^*}\left(\hat{u}\right) + q f^{s,\tau}\left(\hat{u}\right)\right) \\
=& \partial_1 f^{i,\tau}_{\ln y^*,\delta}\left(\hat{u},\delta^{*+}\right) - \partial_1 f^{i,\tau}_{\ln y^*,\delta}\left(\hat{u},\delta^{*-}\right) - \frac{1-p}{1+1/\varepsilon}\left(\partial_2 f^{i,\tau}_{\ln y^*,\delta}\left(\hat{u},\delta^{*+}\right) - \partial_2 f^{i,\tau}_{\ln y^*,\delta}\left(\hat{u},\delta^{*-}\right)\right) \\
&- \frac{2}{\sigma_y\sigma_\delta}\left(\beta f^{\tau}_{\ln y_0^*}\left(\hat{u}\right) + q f^{s,\tau}\left(\hat{u}\right)\right) = 0,
\end{aligned}$$

where the last equality follows from the third conservation law satisfied by $f^{\tau}_{\ln y^*,\delta}$, evaluated at $(\hat{u},\delta)$. Finally check the other boundary conditions: we have $g^i\left(u,\bar{\delta}\right) = f^{i,\tau}_{\ln y^*,\delta}\left(\hat{u},\bar{\delta}\right) = 0$ and similarly $g^i\left(u,\underline{\delta}\right) = 0$, where the last equalities follow from the corresponding boundary conditions of f^i. Similarly, we have, for all $h \in \mathbb{R}$,

$$\lim_{\delta\to\pm\infty} g^s\left(h - \frac{1+1/\varepsilon}{1-p}\delta, \delta\right) = f^s_{\ln y^*,\delta}\left(h + \frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau} - \frac{1+1/\varepsilon}{1-p}\delta, \delta\right) = 0.$$

Finally, we have $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{g^i + g^s\right\}\left(u,\delta\right) du d\delta = 1$, which completes the proof that $g^i = f^{i,\tau+d\tau}_{\ln y^*,\delta}$ and $g^s = f^{s,\tau+d\tau}_{\ln y^*,\delta}$.

Equation (56) implies that, for $x \in \{i,s\}$, for all v,δ,

$$f^{x,\tau+d\tau}_{\ln y,\delta}(v,\delta) = f^{x,\tau+d\tau}_{\ln y^*,\delta}(v-\delta,\delta) = f^{x,\tau}_{\ln y^*,\delta}\left(v+\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau}-\delta,\delta\right) = f^{x,\tau}_{\ln y,\delta}\left(v+\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau},\delta\right),$$

and thus the following relationship between the marginal densities of income given taxes τ and $\tau+d\tau$ holds:

$$f^{\tau+d\tau}_{\ln y}(v) = \int_{-\infty}^{\infty}\left\{f^{i,\tau+d\tau}_{\ln y,\delta}(v,\delta)+f^{s,\tau+d\tau}_{\ln y,\delta}(v,\delta)\right\}d\delta$$
$$= \int_{-\infty}^{\infty}\left\{f^{i,\tau}_{\ln y,\delta}\left(v+\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau},\delta\right)+f^{s,\tau}_{\ln y,\delta}\left(v+\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau},\delta\right)\right\}d\delta = f^{\tau}_{\ln y}\left(v+\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau}\right).$$

We therefore find, with the same change of variables as in the previous proof:

$$\begin{aligned}
\frac{d\mathscr{R}}{d\tau} &= \frac{d}{d\tau}\left\{\int_{-\infty}^{\infty}\left(e^u-\frac{1-\tau}{1-p}e^{(1-p)u}\right)f_{\ln y}(u)\,du\right\}\\
&= \frac{1}{d\tau}\left\{\int_{-\infty}^{\infty}\left(e^u-\frac{1-\tau-d\tau}{1-p}e^{(1-p)u}\right)f_{\ln y}\left(u+\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau}\right)du-\mathscr{R}\right\}\\
&= \int_{-\infty}^{\infty}\left(-\frac{\varepsilon}{1+p\varepsilon}\frac{1}{1-\tau}e^u+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}e^{(1-p)u}\right)f_{\ln y}(u)\,du\\
&= -\frac{\varepsilon}{1+p\varepsilon}\frac{1}{1-\tau}\mathbb{E}[y]+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}\mathbb{E}\left[y^{1-p}\right].
\end{aligned}$$

Note that the same computations (keeping only the term e^u in the integral) imply that $\frac{d}{d\tau}\mathbb{E}[y] = -\frac{\varepsilon}{1+p\varepsilon}\frac{1}{1-\tau}\mathbb{E}[y]$, proving equation (31). Finally,

$$\begin{aligned}
\frac{d\mathscr{W}}{d\tau} &= \frac{d}{d\tau}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{V}_i(e^u,\delta)\,f^i_{\ln y^*,\delta}(u,\delta)\,dud\delta+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{V}_s(e^u,\delta)\,f^s_{\ln y^*,\delta}(u,\delta)\,dud\delta\right\}\\
&= \sum_{x\in\{i,s\}}\frac{1}{d\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\mathcal{V}^{*,\tau+d\tau}\left(e^{v-\frac{\varepsilon}{1+p\varepsilon}\frac{d\tau}{1-\tau}}\right)-\mathcal{V}^{*,\tau}(e^v)\right\}\tilde{v}_x(\delta)\,f^{x,(\tau)}_{\ln y,\delta}(v,\delta)\,dvd\delta\\
&= -\lambda\int_0^{\infty}\boldsymbol{\omega}(y)\,\partial_\tau(y)\,f_y(y)\,dy.
\end{aligned}$$

This concludes the proof of Proposition 6. □

I finally prove equation (41).

Proof of Proposition 7. Suppose first that the following assumption (∗) holds:

$$\left\{\underline{\delta},\delta^*,\bar{\delta}\right\}(p+dp) = \left(1-\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\left\{\underline{\delta},\delta^*,\bar{\delta}\right\} = \frac{\sigma_\delta(p+dp)}{\sigma_\delta}\left\{\underline{\delta},\delta^*,\bar{\delta}\right\}.$$

Then the perturbation dp affects the densities of income as follows:

$$
\begin{aligned}
f^{x,p+dp}_{\ln y^*,\delta}(u,\delta) = \left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\times\dots \\
f^{x,p}_{\ln y^*,\delta}\left(\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)u,\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\delta\right) \equiv g^x(u,\delta),
\end{aligned}
\tag{57}
$$

for $x\in\{i,s\}$. To see this, I show that g^i, g^s satisfy the KFE and boundary conditions that define the functions $f^{i,p+dp}_{\ln y^*,\delta}$ and $f^{s,p+dp}_{\ln y^*,\delta}$, respectively.

For any δ,u, let $\check{\delta}\equiv\left(1-\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\delta$, $\hat{\delta}\equiv\left(1-\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\delta$, and $\hat{u}=\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)u$. First, we have, for all $u\in\mathbb{R}$, all $\delta\in\left(\check{\underline{\delta}},\check{\delta}^*\right)\cup\left(\check{\delta}^*,\check{\bar{\delta}}\right)$ if $x=i$, and all $\delta\in\mathbb{R}\setminus\left\{\check{\underline{\delta}},\check{\bar{\delta}}\right\}$ if $x=s$,

$$
\begin{aligned}
& -(\beta+q\mathbb{I}_{f^s})g^x-\left(1-\frac{\varepsilon}{1+p\varepsilon}dp\right)\mu_y g^x_1+\frac{1-p}{1+\frac{1}{\varepsilon}}\left(1-\frac{1}{1-p}dp\right)\left(1-\frac{\varepsilon}{1+p\varepsilon}dp\right)\mu_y g^x_2 \\
& +\frac{1}{2}\left(1-\frac{\varepsilon}{1+p\varepsilon}dp\right)^2\sigma_y^2 g^x_{11}+\frac{1}{2}\left(\frac{1-p}{1+\frac{1}{\varepsilon}}\right)^2\left(1-\frac{1}{1-p}dp\right)^2\left(1-\frac{\varepsilon}{1+p\varepsilon}dp\right)^2\sigma_y^2 g^x_{22} \\
& -\frac{1-p}{1+\frac{1}{\varepsilon}}\left(1-\frac{1}{1-p}dp\right)\left(1-\frac{\varepsilon}{1+p\varepsilon}dp\right)^2\sigma_y^2 g^x_{12} \\
= & \left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\left\{-(\beta+q\mathbb{I}_{f^s})f^{i,(p)}_{\ln y^*,\delta}-\mu_y\frac{\partial f^p_{\ln y^*,\delta}}{\partial u}+\frac{1-p}{1+\frac{1}{\varepsilon}}\mu_y\frac{\partial f^p_{\ln y^*,\delta}}{\partial\delta}\right. \\
& \left.\left.\dots+\frac{1}{2}\sigma_y^2\frac{\partial^2 f^p_{\ln y^*,\delta}}{\partial u^2}+\frac{1}{2}\left(\frac{1-p}{1+\frac{1}{\varepsilon}}\right)^2\sigma_y^2\frac{\partial^2 f^p_{\ln y^*,\delta}}{\partial^2\delta}-\frac{1-p}{1+\frac{1}{\varepsilon}}\sigma_y^2\frac{\partial^2 f^p_{\ln y^*,\delta}}{\partial\delta\partial u}\right\}\right|_{(\hat{u},\hat{\delta})}=0,
\end{aligned}
$$

where the last equality follows from the KFE solved by $f^p_{\ln y^*,\delta}$, evaluated at $\left(\hat{u},\hat{\delta}\right)$. Thus g^x satisfies the KFE of $f^{p+dp}_{\ln y^*,\delta}$ for all $\hat{\delta}\in(\underline{\delta},\delta^*)\cup(\delta^*,\bar{\delta})$ if $x=i$ and $\hat{\delta}\in\mathbb{R}\setminus\{\underline{\delta},\bar{\delta}\}$ if $x=s$, i.e. the same domains as those of $f^{p+dp}_{\ln y^*,\delta}$. Next, we have, for $\delta\in\{\bar{\delta},\underline{\delta}\}$,

$$
\begin{aligned}
& g^s_1(u,\delta^+)-g^s_1(u,\delta^-)-\frac{1-p}{1+1/\varepsilon}\left(1-\frac{dp}{1-p}\right)\left(g^s_2(u,\delta^+)-g^s_2(u,\delta^-)\right) \\
& -\left(g^i_1(u,\delta^-)-\frac{1-p}{1+1/\varepsilon}\left(1-\frac{dp}{1-p}\right)g^i_2(u,\delta^-)\right) \\
= & \left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)^2\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\left\{\frac{\partial f^{s,p}_{\ln y^*,\delta}}{\partial u}\left(\hat{u},\hat{\delta}^+\right)-\frac{\partial f^{s,p}_{\ln y^*,\delta}}{\partial u}\left(\hat{u},\hat{\delta}^-\right)\right\} \\
& -\frac{1-p}{1+1/\varepsilon}\left(1-\frac{dp}{1-p}\right)\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)^2\left\{\frac{\partial f^{s,p}_{\ln y^*,\delta}}{\partial\delta}\left(\hat{u},\hat{\delta}^+\right)-\frac{\partial f^{s,p}_{\ln y^*,\delta}}{\partial\delta}\left(\hat{u},\hat{\delta}^-\right)\right\} \\
& -\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)^2\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\left\{\frac{\partial f^{i,p}_{\ln y^*,\delta}}{\partial u}\left(\hat{u},\hat{\delta}^-\right)-\frac{1-p}{1+1/\varepsilon}\frac{\partial f^{i,p}_{\ln y^*,\delta}}{\partial\delta}\left(\hat{u},\hat{\delta}^-\right)\right\}=0,
\end{aligned}
$$

where the last equality follows from the corresponding conservation law satisfied by $f^p_{\ln y^*,\delta}$, evaluated at $\left(\hat{u},\hat{\delta}\right)$, and, noting that

$$\beta f_{\ln y_0^*}^{p+dp}(u)+qf_{\ln y^*}^{s,p+dp}(u)=\frac{\beta}{\left(1-\frac{\varepsilon}{1+p\varepsilon}dp\right)s_y\sqrt{2\pi}}e^{-\frac{1}{2\left(1-\frac{\varepsilon}{1+p\varepsilon}dp\right)^2 s_y^2}\left(u-\left(1-\frac{\varepsilon}{1+p\varepsilon}dp\right)m_y\right)^2}$$
$$+q\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\int_{\underline{\delta}(p+dp)}^{\bar{\delta}(p+dp)}f_{\ln y^*,\delta}^{s,p}\left(\hat{u},\hat{\delta}\right)d\delta,$$

we obtain

$$g_1^i\left(u,\delta^{*+}\right)-g_1^i\left(u,\delta^{*-}\right)-\frac{1-p}{1+1/\varepsilon}\left(1-\frac{dp}{1-p}\right)\left(g_2^i\left(u,\delta^{*+}\right)-g_2^i\left(u,\delta^{*-}\right)\right)$$
$$--\frac{\frac{2}{\sigma_y\sigma_\delta}}{\left(1-\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\left(1-\frac{\varepsilon}{1+p\varepsilon}dp\right)^2}\left\{\frac{\beta}{s_y\sqrt{2\pi}}e^{-\frac{1}{2s_y^2}(\hat{u}-m_y)^2}+q\int_{\underline{\delta}}^{\bar{\delta}}f_{\ln y^*,\delta}^{s,p}(\hat{u},\delta)\,d\delta\right\}$$
$$=\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)^2\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\left\{\frac{\partial f_{\ln y^*,\delta}^{i,p}}{\partial u}\left(\hat{u},\hat{\delta}^{*+}\right)-\frac{\partial f_{\ln y^*,\delta}^{i,p}}{\partial u}\left(\hat{u},\hat{\delta}^{*-}\right)\right\}$$
$$-\frac{1-p}{1+1/\varepsilon}\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)^2\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\left\{\frac{\partial f_{\ln y^*,\delta}^{i,p}}{\partial \delta}\left(\hat{u},\hat{\delta}^{*+}\right)-\frac{\partial f_{\ln y^*,\delta}^{i,p}}{\partial \delta}\left(\hat{u},\hat{\delta}^{*-}\right)\right\}$$
$$-\frac{1}{\left(1-\frac{dp}{1-p}\right)\left(1-\frac{\varepsilon}{1+p\varepsilon}dp\right)\left(1-\frac{\varepsilon}{1+p\varepsilon}dp\right)^2}\frac{2}{\sigma_y\sigma_\delta}\left\{\frac{\beta}{s_y\sqrt{2\pi}}e^{-\frac{1}{2s_y^2}(\hat{u}-m_y)^2}+qf_{\ln y^*}^{s,p}(\hat{u})\right\}=0,$$

where the last equality follows from the corresponding conservation law satisfied by $f_{\ln y^*,\delta}^{p}$, evaluated at $\left(\hat{u},\hat{\delta}^*\right)$. The remaining boundary conditions are proved similarly, which concludes the proof. Note that equation (57) implies

$$f_{y,\delta}^{(p+dp)}(y,\delta)=e^{-\delta}f_{y^*,\delta}^{(p+dp)}\left(ye^{-\delta},\delta\right)=\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)e^{\frac{\varepsilon}{1+p\varepsilon}\ln y dp}\times\dots$$
$$\times\dots e^{\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right]\delta dp}f_{y,\delta}^{(p)}\left(e^{\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\ln y+\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right]\delta dp},\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\delta\right).$$

Second, I compute the effect of a perturbation dp on government revenue:

$$\frac{d\mathscr{R}}{dp}=\frac{d}{dp}\left\{\int_0^\infty\left(y-\frac{1-\tau}{1-p}y^{1-p}\right)f_y^p(y)\,dy\right\}$$
$$=\int_0^\infty\left(\ln y-\frac{1}{1-p}\right)\frac{1-\tau}{1-p}y^{1-p}f_y^p(y)\,dy+\frac{1}{dp}\int_0^\infty\left(y-\frac{1-\tau}{1-p}y^{1-p}\right)\left\{f_y^{p+dp}(y)-f_y^p(y)\right\}dy.$$

The first term in the right hand side is the standard mechanical effect M of the perturbation, as in the frictionless model. I now decompose the second integral into three parts: $\frac{d\mathscr{R}}{dp}=M+$

$\frac{1}{dp}\left(B_1+B_2+B_3\right)$, where

$$B_1=\int_0^\infty\int_{-\infty}^\infty T\left(y\right)\left\{\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)^2 y^{\frac{\varepsilon}{1+p\varepsilon}dp}f_{y,\delta}^p\left(y^{1+\frac{\varepsilon}{1+p\varepsilon}dp},\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\delta\right)-\ldots\right.$$

$$\left.\ldots-f_{y,\delta}^p\left(y,\delta\right)\right\}dyd\delta$$

$$B_2=\int_0^\infty\int_{-\infty}^\infty T\left(y\right)\left\{\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)e^{\frac{\varepsilon}{1+p\varepsilon}\ln ydp}e^{\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right]\delta dp}\times\ldots\right.$$

$$\ldots\times f_{y,\delta}^p\left(e^{\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\ln y+\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right]\delta dp},\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\delta\right)$$

$$\left.-\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)^2 y^{\frac{\varepsilon}{1+p\varepsilon}dp}f_{y,\delta}^p\left(y^{1+\frac{\varepsilon}{1+p\varepsilon}dp},\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\delta\right)\right\}dyd\delta$$

$$B_3=\int_0^\infty\int_{-\infty}^\infty T\left(y\right)\left\{f_{y,\delta}^{p+dp}\left(y,\delta\right)-\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)e^{\frac{\varepsilon}{1+p\varepsilon}\ln ydp}\times\ldots\right.$$

$$\left.\times e^{\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right]\delta dp}f_{y,\delta}^p\left(e^{\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\ln y+\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right]\delta dp},\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\delta\right)\right\}dyd\delta,$$

and I compute each term in turn.

$[B_1]$: We have

$$\begin{aligned}B_1=&\int_0^\infty\left(\left(1-\frac{\varepsilon}{1+p\varepsilon}\ln ydp\right)y-\frac{1-\tau}{1-p}\left(1-\frac{\varepsilon\left(1-p\right)}{1+p\varepsilon}\ln ydp\right)y^{1-p}\right)f_y^p\left(y\right)dy\\&-\int_0^\infty\left(y-\frac{1-\tau}{1-p}y^{1-p}\right)f_y^p\left(y\right)dy\\&=-\frac{\varepsilon}{1+p\varepsilon}dp\int_0^\infty\left(y-\left(1-\tau\right)y^{1-p}\right)\ln yf_y^p\left(y\right)dy\\&=-\frac{\varepsilon}{1+p\varepsilon}dp\int_0^\infty\frac{T'\left(y\right)}{1-T'\left(y\right)}y\partial'_p\left(y\right)f_y^p\left(y\right)dy,\end{aligned}$$

which is the standard behavioral effect found in frictionless models.

$[B_2]$: We have

$$\begin{aligned}B_2&=\int_0^\infty\int_{-\infty}^\infty\left\{T\left(y^{1-\frac{\varepsilon}{1+p\varepsilon}dp}e^{-\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right]\delta dp}\right)-T\left(y^{1-\frac{\varepsilon}{1+p\varepsilon}dp}\right)\right\}f_{y,\delta}^p\left(y,\delta\right)dyd\delta\\&=-\left(\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right)dp\int_0^\infty T'\left(y\right)\left[\int_{-\infty}^\infty\delta f_{\delta|y}\left(\delta\left|y\right.\right)d\delta\right]yf_y\left(y\right)dy.\end{aligned}$$

We can show that this term is related to the elasticity of income with respect to a proportional change in the parameters $\mathbb{Y}=\{\mu_y,\sigma_y,m_y,s_y\}$ (which all have elasticity $-\frac{\varepsilon}{1+p\varepsilon}$ with respect to p), keeping the parameters $\mathbb{D}=\left\{\mu_\delta,\sigma_\delta,\underline{\delta},\delta^*,\bar{\delta}\right\}$ constant, i.e., $\frac{1-F_y(y)}{yf_y(y)}\left.\frac{\partial\ln(1-F_y(y))}{\partial\ln\mathbb{Y}}\right|_{\mathbb{D}}=\frac{\partial\ln y}{\partial\ln\mathbb{Y}}$.

$[B_3]$: We have

$$
\begin{aligned}
B_3 =& \int_0^\infty \int_{-\infty}^\infty T(y) \left\{ f_{y,\delta}^{\left\{\tilde{\underline{\delta}},\tilde{\delta}^*,\tilde{\bar{\delta}}\right\}}(y,\delta) - f_{y,\delta}^p(y,\delta) \right\} dyd\delta \\
=& \int_0^\infty \int_{-\infty}^\infty T(y) \left\{ \left(\tilde{\underline{\delta}} - \underline{\delta}\right) \frac{\partial f_{y,\delta}(y,\delta)}{\partial \underline{\delta}} + \left(\tilde{\delta}^* - \delta^*\right) \frac{\partial f_{y,\delta}(y,\delta)}{\partial \delta^*} + \left(\tilde{\bar{\delta}} - \bar{\delta}\right) \frac{\partial f_{y,\delta}(y,\delta)}{\partial \bar{\delta}} \right\} dyd\delta \\
=& dp \int_0^\infty \int_{-\infty}^\infty T(y) \left\{ \sum_{\delta_i \in \left\{\underline{\delta},\delta^*,\bar{\delta}\right\}} \delta_i \left(\frac{\partial \ln|\delta_i|}{\partial p} - \left(-\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp \right) \right) \frac{\partial f_{y,\delta}(y,\delta)}{\partial \delta_i} \right\} dyd\delta,
\end{aligned}
$$

where the first equality follows from the observation that the KFE and boundary conditions defining the density $f_{\ln y^*,\delta}^{x,p+dp}(u,\delta)$ is solved by the function

$$
f_{\ln y^*,\delta}^{x,p+dp}(u,\delta) = \left(1 + \frac{\varepsilon}{1+p\varepsilon}dp\right)\left(1 + \frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right) f_{\ln y^*,\delta}^{x,\left\{\tilde{\underline{\delta}},\tilde{\delta}^*,\tilde{\bar{\delta}}\right\}}\left(\hat{u},\hat{\delta}\right),
$$

where the r.h.s. is the density given the progressivity p and the parameters

$$
\left\{\tilde{\underline{\delta}},\tilde{\delta}^*,\tilde{\bar{\delta}}\right\} \equiv \left(1 + \frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\left\{\underline{\delta}(p+dp), \delta^*(p+dp), \bar{\delta}(p+dp)\right\},
$$

which implies that

$$
\begin{aligned}
f_{y,\delta}^{x,p+dp}(y,\delta) =& \left(1 + \frac{\varepsilon}{1+p\varepsilon}dp\right)\left(1 + \frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right) e^{\frac{\varepsilon}{1+p\varepsilon}\ln y dp} \times \dots \\
& \dots \times e^{\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon} - \frac{\varepsilon}{1+p\varepsilon}\right]\delta dp} f_{y,\delta}^{x,\left\{\tilde{\underline{\delta}},\tilde{\delta}^*,\tilde{\bar{\delta}}\right\}}\left(e^{\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\ln y} e^{\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon} - \frac{\varepsilon}{1+p\varepsilon}\right]\delta dp}, \left(1 + \frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\delta\right)
\end{aligned}
$$

and leads to the expression above after a change of variables. Gathering all the terms, we obtain

$$
\begin{aligned}
\frac{d\mathscr{R}}{dp} =& \int_0^\infty \left(\ln y - \frac{1}{1-p}\right)\frac{1-\tau}{1-p} y^{1-p} f_y(y)\, dy \\
& - \left(\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon} - \frac{\varepsilon}{1+p\varepsilon}\right)\int_0^\infty T'(y)\, y\mathbb{E}[\delta\,|y]\, f_y(y)\, dy \\
& - \frac{\varepsilon}{1+p\varepsilon}\int_0^\infty \frac{T'(y)}{1-T'(y)} y\partial'_p(y)\, f_y(y)\, dy \\
& + \int_0^\infty T(y)\left[\sum_{i=1}^3 \left(\frac{d\ln|\delta_i|}{dp} - \frac{d\ln|\sigma_\delta|}{dp}\right)\boldsymbol{\Xi}_i\right] f_y(y)\, dy
\end{aligned}
$$

Third, I compute the effect of a perturbation dp on the government objective:

$$\frac{d\mathscr{W}}{dp} = \sum_{x\in\{i,s\}} \frac{d}{dp}\left\{\int_0^\infty \int_{-\infty}^\infty \mathcal{V}^*(y)\,\bar{v}_x(\delta)\, f_{y,\delta}^{x,p}(y,\delta)\,dyd\delta\right\}$$
$$= \frac{d}{dp}\left\{\int_0^\infty \int_{-\infty}^\infty \frac{\frac{1}{1-\gamma}\left(\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p}y^{1-p}\right)^{1-\gamma}}{\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2}\bar{v}^p(y,\delta)\, f_{y,\delta}^p(y,\delta)\,dyd\delta\right\}.$$

I now decompose the integral into three parts: $\frac{d\mathscr{W}}{dp} = \frac{1}{dp}(W_1+W_2+W_3+W_4)$, where

$$W_1 = \int_0^\infty \int_{-\infty}^\infty \Bigg[\frac{\frac{1}{1-\gamma}\left(\frac{1+(p+dp)\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p-dp}y^{1-p+dp}\right)^{1-\gamma}}{\rho_1+\rho_2-(1-\gamma)\left(1-\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\mu_c-\frac{1}{2}(1-\gamma)^2\left(1-\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)^2\sigma_c^2}$$
$$-\frac{\frac{1}{1-\gamma}\left(\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p}y^{1-p}\right)^{1-\gamma}}{\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2}\Bigg]\bar{v}^p(y,\delta)\, f_{y,\delta}^p(y,\delta)\,dyd\delta$$

$$W_2 = \int_0^\infty \int_{-\infty}^\infty \frac{\frac{1}{1-\gamma}\left(\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p}y^{1-p}\right)^{1-\gamma}}{\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2}\Bigg\{\bar{v}^p\left(y^{1+\frac{\varepsilon}{1+p\varepsilon}dp},\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\delta\right)\times\ldots$$
$$\ldots\times\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)^2 y^{\frac{\varepsilon}{1+p\varepsilon}dp} f_{y,\delta}^p\left(y^{1+\frac{\varepsilon}{1+p\varepsilon}dp},\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\delta\right)-\bar{v}^p(y,\delta)\, f_{y,\delta}^p(y,\delta)\Bigg\}dyd\delta$$

$$W_3 = \int_0^\infty \int_{-\infty}^\infty \frac{\frac{1}{1-\gamma}\left(\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p}y^{1-p}\right)^{1-\gamma}}{\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2}\Bigg\{\bar{v}^p\left(y^{1+\frac{\varepsilon}{1+p\varepsilon}dp},\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\delta\right)\times\ldots$$
$$\ldots\times\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)e^{\frac{\varepsilon}{1+p\varepsilon}\ln y dp}e^{\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right]\delta dp}\times\ldots$$
$$\ldots\times f_{y,\delta}^p\left(e^{\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\ln y+\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right]\delta dp},\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\delta\right)$$
$$-\bar{v}^p\left(y^{1+\frac{\varepsilon}{1+p\varepsilon}dp},\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\delta\right)\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)^2 y^{\frac{\varepsilon}{1+p\varepsilon}dp} f_{y,\delta}^p\left(y^{1+\frac{\varepsilon}{1+p\varepsilon}dp},\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\delta\right)\Bigg\}dyd\delta$$

$$W_4 = \int_0^\infty \int_{-\infty}^\infty \frac{\frac{1}{1-\gamma}\left(\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p}y^{1-p}\right)^{1-\gamma}}{\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2}\Bigg\{\bar{v}^{p+dp}(y,\delta)\, f_{y,\delta}^{p+dp}(y,\delta)-\ldots$$
$$\ldots-\bar{v}^p\left(y^{1+\frac{\varepsilon}{1+p\varepsilon}dp},\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\delta\right)\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)e^{\frac{\varepsilon}{1+p\varepsilon}\ln y dp}\times\ldots$$
$$\ldots\times e^{\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right]\delta dp} f_{y,\delta}^p\left(e^{\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\ln y+\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right]\delta dp},\left(1+\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}dp\right)\delta\right)\Bigg\}dyd\delta$$

and I compute each term in turn. First, we easily find

$$W_1+W_2 = \lambda\int_0^\infty\left[\partial_p(y)\,\boldsymbol{\omega}(y)+\frac{d\ln\boldsymbol{\rho}}{dp}\hat{\boldsymbol{\omega}}(y)\right]f_y^p(y)\,dy,$$

which are the standard (static and dynamic) terms already present in the frictionless formula. Next, we have

$$W_3 = \int_0^\infty \int_{-\infty}^\infty \frac{\frac{1}{1-\gamma}\left(\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p}y^{1-p}\right)^{1-\gamma}}{\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2} \times \dots$$
$$\dots \times \left\{e^{-(1-\gamma)\delta dp}\bar{v}\left(ye^{-\frac{1}{1-p}\delta dp},\delta\right) - \bar{v}(y,\delta)\right\} f^p_{y,\delta}(y,\delta)\,dyd\delta$$
$$= -\frac{dp}{1-p}\int_0^\infty \int_{-\infty}^\infty y\delta\frac{\partial}{\partial y}\left\{\frac{\frac{1}{1-\gamma}\left(\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p}y^{1-p}\right)^{1-\gamma}}{\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2}\bar{v}(y,\delta)\right\} f^p_{y,\delta}(y,\delta)\,dyd\delta$$
$$= -\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right]dp\int_0^\infty\left\{\int_{-\infty}^\infty \delta\frac{\partial\bar{\mathcal{V}}(y,\delta)}{\partial\ln y}f_{\delta|y}(\delta\,|y)\,d\delta\right\}f_y(y)\,dy.$$

Finally, we have

$$W_4 = \int_0^\infty \int_{-\infty}^\infty \frac{\frac{1}{1-\gamma}\left(\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p}y^{1-p}\right)^{1-\gamma}}{\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2}\bar{v}^p(y,\delta)\left\{f^{p+dp}_{y,\delta}(y,\delta)-\dots\right.$$
$$\left.\dots-\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)e^{\frac{\varepsilon}{1+p\varepsilon}\ln y dp}e^{\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right]\delta dp}f^p_{y,\delta}\left(e^{\left(1+\frac{\varepsilon}{1+p\varepsilon}dp\right)\ln y+\left[\frac{1}{1-p}\frac{1+\varepsilon}{1+p\varepsilon}-\frac{\varepsilon}{1+p\varepsilon}\right]\delta dp},\delta\right)\right\}dyd\delta$$
$$+\int_0^\infty \int_{-\infty}^\infty \frac{\frac{1}{1-\gamma}\left(\frac{1+p\varepsilon}{1+\varepsilon}\frac{1-\tau}{1-p}y^{1-p}\right)^{1-\gamma}}{\rho_1+\rho_2-(1-\gamma)\mu_c-\frac{1}{2}(1-\gamma)^2\sigma_c^2}\left\{\bar{v}^{p+dp}(y,\delta)-\bar{v}^p\left(y^{1+\frac{\varepsilon}{1+p\varepsilon}dp},\delta\right)\right\}f^p_{y,\delta}(y,\delta)\,dyd\delta$$
$$=\int_0^\infty \int_{-\infty}^\infty \bar{\mathcal{V}}^p(y,\delta)\sum_{\delta_i\in\{\underline{\delta},\delta^*,\bar{\delta}\}}\left\{\left(\frac{\partial\ln|\delta_i|}{\partial p}-\frac{d\ln\sigma_\delta}{dp}\right)\frac{\partial\ln f_{y,\delta}(y,\delta)}{\partial\ln\delta_i}f_{\delta|y}(\delta\,|y)\right\}f_y(y)\,dyd\delta$$
$$+\int_0^\infty \int_{-\infty}^\infty \bar{\mathcal{V}}^p(y,\delta)\left\{\frac{\partial\ln\bar{v}^p(y,\delta)}{\partial p}-\frac{\varepsilon}{1+p\varepsilon}\ln y\frac{\partial\ln\bar{v}^p(y,\delta)}{\partial\ln y}\right\}f_{\delta|y}(\delta\,|y)\,f_y(y)\,dyd\delta,$$

where the first equality uses the fact that $f^i_{\delta|y}(\underline{\delta}\,|y) = f^i_{\delta|y}(\bar{\delta}\,|y) = 0$, and the second equality is obtained similarly as the extensive elasticity term on government revenue above. □

Figure 6 plots the extensive margin elasticities $\left(\frac{d\ln|\underline{\delta}|}{dp}-\frac{d\ln|\sigma_\delta|}{dp}\right)\underline{\Xi}(y)$ and $\left(\frac{d\ln\bar{\delta}}{dp}-\frac{d\ln|\sigma_\delta|}{dp}\right)\bar{\Xi}(y)$ for $\varepsilon = 0.33$ (left panel) and $\varepsilon = 1$ (right panel).

Figure 6: Extensive margin elasticities: $\varepsilon = 0.33$ and $\varepsilon = 1$

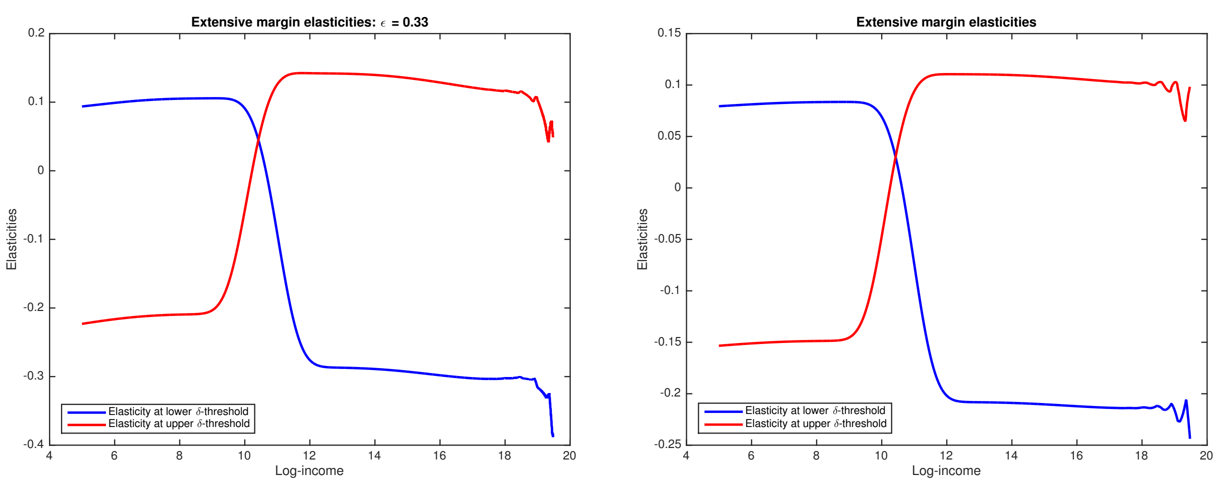


Figure 7 plots the revenue effects of the tax reform disaggregated by income, that is,

$$-T'\left(y\right)\frac{y\boldsymbol{\varepsilon}^{*}\left(y\right)}{1-T'\left(y\right)}\Psi_{p}'\left(y\right)+T\left(y\right)\left[\sum_{i=1}^{3}\frac{d\ln\frac{\left|\delta_{i}\right|}{\left|\sigma_{\delta}\right|}}{dp}\boldsymbol{\Xi}_{i}\left(y\right)\right],$$

for $\varepsilon = 0.33$ (left panel) and $\varepsilon = 1$ (right panel) in the frictionless and the frictional models.

Figure 7: Revenue effects of tax reforms

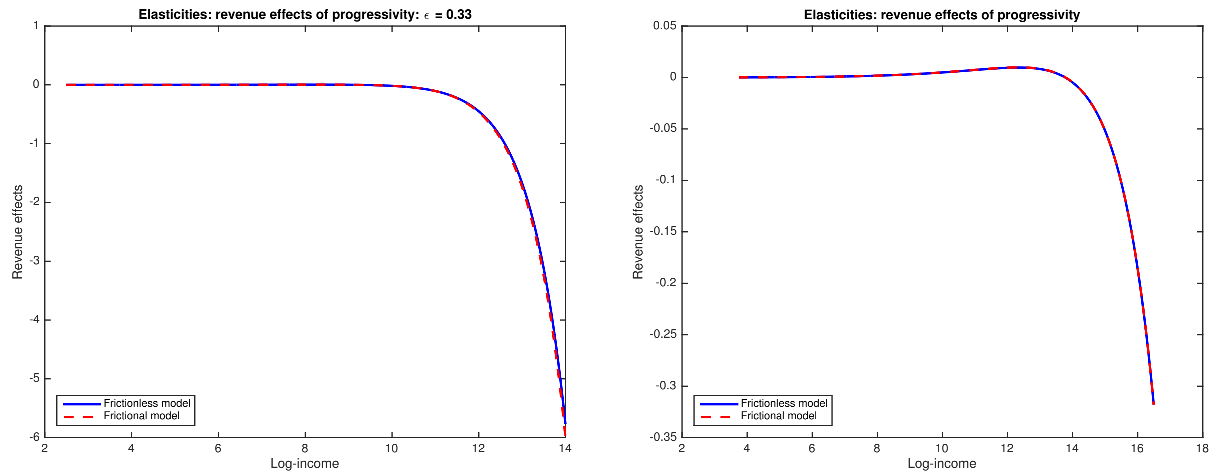


Figure 8 plots the welfare effects of increasing progressivity disaggregated by income for $\varepsilon = 0.33$ and $\varepsilon = 1$, that is, $-\left[\boldsymbol{\omega}\left(y\right)\Psi_{p}\left(y\right)+\frac{d\ln\boldsymbol{\rho}}{dp}\hat{\boldsymbol{\omega}}\left(y\right)\right]+\left[\sum_{i=1}^{3}\frac{d\ln\frac{\left|\delta_{i}\right|}{\left|\sigma_{\delta}\right|}}{dp}\boldsymbol{\Omega}_{i}\left(y\right)+\tilde{\boldsymbol{\Omega}}\left(y\right)\right]$.

Figure 8: Welfare effects of tax reforms