Optimal Taxation and Quasi-Hyperbolic Preferences

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Abstract

The paper investigates the implications of quasi-hyperbolic preferences for capital taxation. The analysis considers the Ramsey growth model, the Barro growth model, and an overlapping generations economies. It concludes that quasi-hyperbolic preferences are not significant if consumers act as if they have infinite lifespans. With finite lives quasi-hyperbolic preferences impact upon the design of the optimal tax policy.

PRELIMINARY AND VERY INCOMPLETE

1 Introduction

The paper considers the implications of quasi-hyperbolic preferences for capital taxation. It builds on two generally agreed facts. First, that quasi-hyperbolic preferences lead to "under-saving" so justify intervention to incentivise an increase in saving. Second, that the Chamley-Judd optimal tax result demonstrates capital should not be taxed. The purpose of the paper is to explore which of these apparently conflicting facts is true in a variety of growth models.

The papers considers the outcomes when quasi-hyperbolic preferences are embedded within a Ramsey growth model, a Barro growth model, and an overlapping generations model. These models are used to consider the implications of the quasi-hyperbolic preferences for capital taxation. In particular, do the preferences significantly affect existing arguments?

The Mirrlees Review of the UK tax system placed considerable emphasis on the tax treatment of capital income. The motive for this were a set of *equivalence results* between different tax systems. The most notable of these being that a consumption tax is equivalent to an income tax with exemption for interest income. This practical relevance of this equivalence is that it permits a consumption tax to be implemented while retaining the existing administrative structure for an income taxes. From an analytical perspective these results are based on studying budget sets, so are not dependent upon preferences. This gives them widespread applicability.

The equivalences effectively reduce the relevant policy choices to: a comprehensive income tax; a tax on labour income with tax exemption for capital income; a tax on labour income with a tax at a different rate on capital income. The Chamley-Judd results supports the second option in Ramsey growth models. The third option can be supported in overlapping generations economies where there is the potential for dynamic inefficiency.

The first section introduces the different choice problems for non-exponential preferences that depend on whether the consumer is committed or naive. The general results are then explored in more detail for quasi-hyperbolic preferences with logarithmic felicity. The second section focuses on the Ramsey growth model with quasi-hyperbolic preferences. The Barro growth model is analyzed in the third section. The fourth section analyses an overlapping generations economy with three-period lives. The main conclusion is that the effect of quasi-hyperbolic preferences on tax policy is highly dependent on the assumed length of lifespan. Only with finite lives is there a clear and significant effect.

2 Preferences and Types

We consider the class of additively time separable preferences

$$U = B_0 u(c_0) + B_1 u(c_1) + B_2 u(c_2) + \ldots + B_3 u(c_T). \tag{1}$$

Two special cases of these preferences are exponential discounting

$$U = u(c_0) + \delta u(c_1) + \delta^2 u(c_2) + \ldots + \delta^T u(c_T),$$

and quasi-hyperbolic (or "β, δ") discounting

$$U = u(c_0) + \beta\delta u(c_1) + \beta\delta^2 u(c_2) + \ldots + \beta\delta^T u(c_T).$$

It has been known since Strotz (1955) that only exponential discounting leads to intertemporal consistency in choice. The potential for inconsistency creates the need to consider in more detail the decision making "type" of the consumer. Correspondingly, alongside these preferences it is possible to define three different types of consumer:

1. Committed

The sequence $\{c_0, \ldots, c_T\}$ is chosen at time 0 and the plan is followed without revision from that time onwards.

2. Naive

The sequence $\{c_0^0, \ldots, c_T^0\}$ is chosen at time 0, the sequence $\{c_1^1, \ldots, c_T^1\}$ is chosen at 1, and a revised sequence is chosen in every time period until $T-1$.

3. Sophisticated

The sequence $\{c_0, \ldots, c_T\}$ is chosen at time 0 taking into account how future plans will be affected.

In general, each type of consumer generates a different intertemporal consumption path. For the cases of naive and sophisticated it has become commonplace to talk of present and future selves who face distinct objectives and make different decisions. A sophisticated consumer is aware of how the future selves will behave and takes this into account. A naive consumer is not aware. The issues the paper addresses are the implications that these different types have for capital taxation as well as the effect of quasi-hyperbolic preferences. The first step is to contrast the choices made by the three types to obtain some basic results prior to placing the preferences within a specific economic model. These results will explain the later findings on tax policy.

To illustrate the argument that we are going to make assume that the felicity function is logarithmic and that the lifespan of the consumer is three periods. This is the minimum lifespan necessary for a difference to emerge between committed, naive, and sophisticated. Under these assumptions the *commitment problem* is described by the optimization

$$\max_{\{x_0,x_1,x_2\}} U = \tilde{B}_0 \ln(x_0) + \tilde{B}_1 \ln(x_1) + \tilde{B}_2 \ln(x_2), \tag{2}$$

subject to the budget constraint

$$W_0 = p_0 x_0 + \frac{p_1 x_1}{1+r} + \frac{p_2 x_2}{(1+r)^2} \tag{3}$$

Define $q_i = p_i/(1+r)^i$. Then the budget constraint can be written

$$W_0 = q_0 x_0 + q_1 x_1 + q_2 x_2. \tag{4}$$

The necessary conditions are

$$\tilde{B}_i \frac{1}{x_i} - \lambda q_i = 0, \quad i = 0, ..., 2,$$

which can be substituted into the budget constraint to give

$$W_0 = \frac{1}{\lambda}\left[\tilde{B}_0 + \tilde{B}_1 + \tilde{B}_2\right].$$

Hence, consumption in period i is given by

$$x_i = \frac{\tilde{B}_i}{\tilde{B}_0 + \tilde{B}_1 + \tilde{B}_2}\frac{W_0}{q_i}, \quad i = 0, ..., 2.$$

This construction can be repeated for the *naive consumer*. In period 0 the decision problem is

$$\max_{\{x_0,x_1,x_2\}} U = B_0 \ln(x_0) + B_1 \ln(x_1) + B_2 \ln(x_2)$$

subject to the budget constraint (4). The necessary conditions are

$$B_i \frac{1}{x_i} - \lambda q_i = 0, \quad i = 0, ..., 2$$

which give

$$x_0 = \frac{B_0}{B_0 + B_1 + B_2} \frac{W_0}{q_0}.$$

In period 1 the decision problem becomes

$$\max_{\{x_1, x_2\}} U = B_0 \ln(x_1) + B_1 \ln(x_2)$$

and the budget constraint

$$\begin{aligned} W_1 &= p_1 x_1 + \frac{p_2}{1+r} x_2 \\ &= \tilde{q}_1 x_1 + \tilde{q}_2 x_2. \end{aligned}$$

The solution becomes

$$x_i = \frac{B_{i-1}}{B_0 + B_1} \frac{W_1}{\tilde{q}_i}, \quad i = 1, 2.$$

However,

$$\begin{aligned} W_1 &= (W_0 - q_0 x_0)(1+r) \\ &= \frac{B_1 + B_2}{B_0 + B_1 + B_2} W_0 (1+r), \end{aligned}$$

which gives

$$x_i = \frac{B_1 + B_2}{B_0 + B_1} \frac{B_{i-1}}{B_0 + B_1 + B_2} \frac{W_0}{\tilde{q}_i} (1+r), \quad i = 1, 2.$$

The solution for the sophisticated consumer at time 1 is the same as that of the naive consumer. At time 0, however, the sophisticated consumer takes into account of the dependence of later choices on x_0. Since

$$x_i = \frac{B_{i-1}}{B_0 + B_1} \frac{W_1}{\tilde{q}_i} = \frac{B_{i-1}}{B_0 + B_1} \frac{(W_0 - p_0 x_0)}{\tilde{q}_i}, \quad i = 1, 2,$$

the optimization at time 0 is

$$\max_{\{x_0\}} U = B_0 \ln(x_0) + B_1 \ln\left(\frac{B_0}{B_0 + B_1} \frac{(W_0 - q_0 x_0)}{\tilde{q}_i}\right) + B_2 \ln\left(\frac{B_1}{B_0 + B_1} \frac{(W_0 - q_0 x_0)}{\tilde{q}_i}\right),$$

which has solution

$$x_0 = \frac{B_0}{B_0 + B_1 + B_2} \frac{W_0}{q_0}.$$

It can be seen that this outcome is the same as for the naive consumer. This is a consequence of the logarithmic utility function and is not a general result.

It is an interesting question to consider whether it is possible to determine the type of consumer from the observed demands. The construction will now show that this is not possible unless the discount factors $\{B_i\}$ are known. In other words, a committed consumer with discount factors $\left\{\tilde{B}_i\right\}$ can be mistaken for a naive or a sophisticated consumer with discount factors $\{B_i\}$.

The idea behind this argument is to set $\left\{\tilde{B}_0, \tilde{B}_1, \tilde{B}_2\right\}$ so that the commitment problem gives the same solution as the naive and sophisticated (in the case of log preferences) problems for $\{B_0, B_1, B_2\}$. Putting the solutions side-by-side (with added superscripts to denote the solution)

$$
\begin{aligned}
x_0^C &= \frac{\tilde{B}_0}{\tilde{B}_0+\tilde{B}_1+\tilde{B}_2}\frac{W_0}{q_0}, & x_0^N &= \frac{B_0}{B_0+B_1+B_2}\frac{W_0}{q_0},\\
x_1^C &= \frac{\tilde{B}_1}{\tilde{B}_0+\tilde{B}_1+\tilde{B}_2}\frac{W_0}{q_1}, & x_1^N &= \frac{B_1+B_2}{B_0+B_1}\frac{B_0}{B_0+B_1+B_2}\frac{W_0}{(1+r)q_1},\\
x_2^C &= \frac{\tilde{B}_2}{\tilde{B}_0+\tilde{B}_1+\tilde{B}_2}\frac{W_0}{q_2}, & x_2^N &= \frac{B_1+B_2}{B_0+B_1}\frac{B_1}{B_0+B_1+B_2}\frac{W_0}{(1+r)q_2}.
\end{aligned}
$$

Let the $\left\{\tilde{B}_i\right\}$ be chosen so that

$$
\begin{aligned}
\tilde{B}_0 &= B_0 \ (=1 \text{ typically}),\\
\tilde{B}_1 &= B_0\frac{B_1+B_2}{B_0+B_1},\\
\tilde{B}_2 &= B_1\frac{B_1+B_2}{B_0+B_1}.
\end{aligned}
$$

These imply

$$
\begin{aligned}
\tilde{B}_0+\tilde{B}_1+\tilde{B}_2 &= B_0 + B_0\frac{B_1+B_2}{B_0+B_1} + B_1\frac{B_1+B_2}{B_0+B_1}\\
&= B_0+B_1+B_2.
\end{aligned}
$$

What this construction has shown is that the naive and sophisticated solutions for the objective function

$$U = B_0 \ln(x_0) + B_1 \ln(x_1) + B_2 \ln(x_2),$$

are identical to the commitment solution for the objective function

$$U = B_0 \ln(x_0) + B_0\frac{B_1+B_2}{B_0+B_1}\ln(x_1) + B_1\frac{B_1+B_2}{B_0+B_1}\ln(x_2).$$

The two objective functions conform to the general structure of (1). This demonstrates that observing the consumption path does not reveal the type of the consumer in the absence of prior knowledge of the discount factors. In addition, it allows the naive and sophisticated solutions to be derived very simply as a commitment solution without the need for repeated optimization.

The following theorem shows that this construction can be extended to any additively separable utility function.

Theorem 1 *There exist $\tilde{B}_t$, $t = 1, ..., T$, $\tilde{B}_t > 0$ all t,such that the naive optimum for the objective function $U = \sum_{t=0}^{T} B_t u(x_t)$ is identical to the commitment optimum for objective $U = \sum_{t=0}^{T} \tilde{B}_t u(x_t)$.*

Proof. The optimum for the commitment is described by the set of necessary conditions

$$\tilde{B}_t u'(x_t) = \lambda q_t,\ t = 0, ..., T$$

and the solution for the naive by

$$\begin{aligned} B_0 u'(x_0) &= \lambda \tilde{q}_0, \\ B_0 u'(x_t) &= \mu_t \tilde{q}_t,\ t = 1, ..., T-1, \\ B_1 u'(x_T) &= \mu_{T-1} \tilde{q}_T. \end{aligned}$$

where

$$\begin{aligned} \tilde{q}_0 &= q_0, \\ \tilde{q}_t &= (1+r)^t q_t,\ t = 1, ..., T-1, \\ \tilde{q}_T &= (1+r)^{T-1} q_T \end{aligned}$$

Since the sequence $\{x_0, ..., x_T\}$ is the same for both solutions and $\tilde{q}_0 = q_0$ then it follows that

$$\tilde{B}_0 = B_0$$

Now set

$$\begin{aligned} \mu_t &= \frac{\tilde{B}_0}{\tilde{B}_t} \frac{\lambda}{(1+r)^t}, t = 1, ..., T-1, \\ \mu_{T-1} &= \frac{\tilde{B}_0}{\tilde{B}_t} \frac{\lambda}{(1+r)^{T-1}}. \end{aligned}$$

The remainder of the values are then calculated using

$$\frac{\tilde{B}_t u'(x_t)}{\tilde{B}_0 u'(x_0)} = \frac{q_t}{q_0} = \frac{\tilde{B}_t u'(x_t)}{\tilde{B}_0 u'(x_0)}$$

hence

$$\tilde{B}_t = B_0 \frac{u'(x_0)}{u'(x_t)}.$$

Since $u'(x) > 0$ it follows that $\tilde{B}_t > 0$ all t. ∎

This analysis can be used to make two further observations. First, $\{B_i\}$ and $\left\{\tilde{B}_i\right\}$ are identical for exponential discounting which provides an alternative perspective on consistency of decision making in this case. Second, if $\{B_i\}$ has a quasi-hyperbolic structure then $\left\{\tilde{B}_i\right\}$ does not. To show the first point, assume that

$$B_0 = 1, B_1 = \delta, B_2 = \delta^2.$$

Then

$$\begin{aligned}
\tilde{B}_1 &= B_0 \frac{B_1 + B_2}{B_0 + B_1} = 1\frac{\delta + \delta^2}{1 + \delta} = \delta, \\
\tilde{B}_2 &= B_1 \frac{B_1 + B_2}{B_0 + B_1} = \delta\frac{\delta + \delta^2}{1 + \delta} = \delta^2.
\end{aligned}$$

This is an alternative demonstration that for exponential discounting there is no difference between the choices of committed and naive consumers. Now assume that

$$B_0 = 1, B_1 = \beta\delta, B_2 = \beta\delta^2.$$

Then

$$\begin{aligned}
\tilde{B}_1 &= B_0 \frac{B_1 + B_2}{B_0 + B_1} = \frac{\beta\delta + \beta\delta^2}{1 + \beta\delta}, \\
\tilde{B}_2 &= B_1 \frac{B_1 + B_2}{B_0 + B_1} = \beta\delta\frac{\beta\delta + \beta\delta^2}{1 + \beta\delta},
\end{aligned}$$

which is neither exponential discounting nor quasi-hyperbolic discounting.

The constructions above are dependent on the assumption of a finite lifetime and are changed when lifetime is infinite. This is an important observation for the contrasting results that will be demonstrated for tax policy in the Ramsey growth model and overlapping generations model.

Theorem 2 *The consumption plan for a naive consumer with preferences*

$$U = \sum_{t=0}^{\infty} B_0 u(x_0), \tag{5}$$

where $B_0 = 1$, *is identical to that of a consumer with exponential preferences*

$$U = \sum_{t=0}^{\infty} (B_1)^t u(x_0) \tag{6}$$

Proof. At any time t the naive solves

$$\max_{\{w_{t+1},\dots\}} \sum_{i=0}^{\infty} B_i u(w_{t+i} - w_{t+i+1}),$$

which has necessary conditions for w_{t+1}

$$-B_0 u'(x_t) + B_1 u'(x_{t+1}) = 0.$$

But if the consumption plan matches the plan $\{\tilde{x}_i\}$ of a committed consumer $\left\{\tilde{B}_i\right\}$ then it must be the case that

$$-\tilde{B}_t u'(x_t) + \tilde{B}_{t+1} u'(x_{t+1}) = 0.$$

Since $B_0 = 1$ we have

$$B_1 = \frac{\tilde{B}_{t+1}}{\tilde{B}_t},\ t = 1, \ldots.$$

This condition can only be satisfied if the committed consumer is exponential with

$$\tilde{\delta} = B_1.$$

■

Hence, any naive consumer with an infinite life acts as if they are a committed consumer. This is because the act of choosing a new consumption plan each period ensures that only the first choice of each plan is ever implemented. This result can be illustrated for logarithmic felicity.

For a naive consumer with quasi-hyperbolic preferences and a T period lifetime, the saving path for log utility can be computed as

$$s_{T-1}^n = \frac{\beta\delta}{1+\beta\delta} W_{T-1}, \ldots, s_{T-2}^n = \frac{\beta\left[\delta+\delta^2\right]}{1+\beta\left[\delta+\delta^2\right]} W_{T-2}, \ldots, s_0^n = \frac{\beta\sum_{i=1}^T \delta^i}{1+\beta\sum_{j=1}^T \delta^j} W_0.$$

In the limit as $T \to \infty$

$$s_0^n \to \frac{\beta\frac{\delta}{1-\delta}}{1+\beta\frac{\delta}{1-\delta}} W_0, \quad s_t^n \to \frac{\beta\frac{\delta}{1-\delta}}{1+\beta\frac{\delta}{1-\delta}} W_t. \tag{7}$$

Correspondingly, for a consumer with exponential preferences and discount factor $\tilde{\delta}$ the path of saving is

$$s_{T-1}^e = \frac{\tilde{\delta}}{1+\tilde{\delta}} W_{T-1},\ s_{T-2}^e = \frac{\tilde{\delta}+\tilde{\delta}^2}{1+\tilde{\delta}+\tilde{\delta}^2} W_{T-2}, \ldots, s_0^e = \frac{\sum_{i=1}^T \tilde{\delta}^i}{1+\sum_{j=1}^T \tilde{\delta}^j} W_0$$

So as $T \to \infty$

$$s_0^e \to \frac{\frac{\tilde{\delta}}{1-\tilde{\delta}}}{1+\frac{\tilde{\delta}}{1-\tilde{\delta}}} W_0,\ s_t^e \to \frac{\frac{\tilde{\delta}}{1-\tilde{\delta}}}{1+\frac{\tilde{\delta}}{1-\tilde{\delta}}} W_t.$$

Contrasting (7) and (**??**) it can be seen that the saving path of a naive consumer with quasi-hyperbolic preferences $\{\beta, \delta\}$ is identical to the saving path of a consumer with exponential preferences $\left\{\tilde{\delta}\right\}$ when

$$\tilde{\delta} = \frac{\beta}{\frac{1}{\delta} - 1 + \beta}.$$

Hence, with an infinite lifespan it is not possible to determine is a naive consumer with quasi-hyperbolic preferences or has exponential preferences by looking at the time path of savings.

The saving path for a consumer with commitment and quasi-hyperbolic preferences is given by

$$s^c_{T-1} = \frac{\beta\delta^T}{1+\beta\sum_{j=1}^T \delta^j} W_0,\ s^c_{T-2} = \frac{\beta\left[\delta^{T-1}+\delta^T\right]}{1+\beta\sum_{j=1}^T \delta^j} W_0, ..., s^c_0 = \frac{\beta\sum_{i=1}^T \delta^i}{1+\beta\sum_{j=1}^T \delta^j} W_0.$$

The level of saving at t can alternatively be written as

$$s^c_t = \frac{\delta^t \sum_{i=0}^{T-1-t} \delta^i}{\sum_{j=0}^{T-1} \delta^j} \frac{\beta\sum_{i=1}^T \delta^i}{1+\beta\sum_{j=1}^T \delta^j} W_0.$$

In the limit as $T \to \infty$

$$s^c_0 \to \frac{\beta\frac{\delta}{1-\delta}}{1+\beta\frac{\delta}{1-\delta}} W_0, \quad s^c_t \to \delta^t \frac{\beta\frac{\delta}{1-\delta}}{1+\beta\frac{\delta}{1-\delta}} W_0.$$

The limiting exponential saving plan can be written equivalently as $T \to \infty$

$$s^e_0 \to \frac{\frac{\tilde{\delta}}{1-\tilde{\delta}}}{1+\frac{\tilde{\delta}}{1-\tilde{\delta}}} W_0, \quad s^e_t \to \tilde{\delta}^t \frac{\frac{\tilde{\delta}}{1-\tilde{\delta}}}{1+\frac{\tilde{\delta}}{1-\tilde{\delta}}} W_0.$$

The two are identical with $s^c_0 = s^e_0$ at time 0 if $\tilde{\delta} = \frac{\beta}{\frac{1}{\delta}-1+\beta}$. However, if this is true then $\delta > \tilde{\delta}$ so that $s^e_t > s^c_t$. This gives the surprising result: for parameters such that an exponential and a naive quasi-hyperbolic consumer save at the same level then a committed quasi-hyperbolic consumer saves at a lower level from the first period onward. These observations about intertemporal choices are now applied to the analysis of capital tax policy in growth models.

3 Ramsey Growth Model

The observations made in the section above are now applied to understand capital taxation in a Ramsey growth model with quasi-hyperbolic preferences.

It has already been seen that quasi-hyperbolic preferences affect the consumption path when the lifetime of the consumer is finite. This observation carries over to the growth path of a finite economy. To illustrate this point, assume a three-period economy with CRRA utility

$$U = \frac{c_0^{1-\gamma}-1}{1-\gamma} + \beta\delta\frac{c_1^{1-\gamma}-1}{1-\gamma} + \beta\delta^2\frac{c_2^{1-\gamma}-1}{1-\gamma},$$

and the standard production function

$$y_t = Ak_t^\alpha, \quad 0 < \alpha < 1.$$

The growth paths for the committed and sophisticated consumers are shown in table 1. It can be seen from the table that there is a significant difference between the levels of capital accumulated by the committed and the sophisticated consumers. The level for the sophisticated consumer is lower than for the committed, so the quasi-hyperbolic structure does cause a reduction in the level of capital. This provides a motive for intervening with capital taxation.

$\beta = 0.9$				
	Committed		Sophisticated	
γ	k_1	k_2	k_1^*	k_2^*
0.5	1.631	0.589	1.631	0.529
1.0	3.068	0.763	3.068	0.705
1.5	4.253	0.931	4.253	0.876
2.0	5.082	1.066	5.082	1.015
2.5	5.650	1.168	5.651	1.123
3.0	6.050	1.247	6.051	1.206
3.5	6.342	1.309	6.342	1.272

Table 1: A finite growth model

The behaviour in an infinite economy is different to that in a finite economy and can be illustrated by using a logarithmic felicity function. With T periods the objective function is

$$U = \ln(Ak_0^\alpha - k_1) + \beta \sum_{t=1}^{T} \delta^t \ln(Ak_t^\alpha - k_{t+1}),\ k_T = 0.$$

The solution for the committed consumer can be written as

$$k_1^c = \frac{\beta \sum_{i=1}^{T} \alpha^i \delta^i}{1 + \beta \sum_{i=1}^{T} \alpha^i \delta^i} Ak_0^\alpha,$$

and for $1 < t < T - 1$

$$k_t^c = \frac{\sum_{i=1}^{T+1-t} \alpha^i \delta^i}{1 + \sum_{i=1}^{T+1-t} \alpha^i \delta^i} Ak_{t-1}^\alpha.$$

The solution of the naive is a repetition of the first period for the committed which gives

$$k_t^n = \frac{\beta \sum_{i=1}^{T-t+1} \alpha^i \delta^i}{1 + \beta \sum_{i=1}^{T-t+1} \alpha^i \delta^i} Ak_{t-1}^\alpha.$$

It can be seen directly that the path for the committed and the naive differ.

The question is whether the behaviour of the naive consumer provides a motive for taxation. In a finite economy the answer was clearly that it does. But what if the economy is infinite? For the naive consumer the process for capital accumulation is

$$k_t = \frac{\beta \sum_{i=1}^{T-t+1} \alpha^i \delta^i}{1 + \beta \sum_{i=1}^{T-t+1} \alpha^i \delta^i} Ak_{t-1}^\alpha.$$

In the limit as $T \to \infty$,

$$k_t = \frac{\beta \frac{\alpha\delta}{1-\alpha\delta}}{1 + \beta \frac{\alpha\delta}{1-\alpha\delta}} A k_{t-1}^{\alpha}$$

Observe that this is again the choice of an exponential consumer discounting at the rate

$$\tilde{\delta} = \frac{\beta\delta}{(1 - \alpha\delta + \beta\delta\alpha)}.$$

At first sight it appears that the logic of Chamley-Judd result will apply to this economy since the quasi-hyperbolic consumer generates a capital accumulation path that is identical to that of an appropriately defined exponential consumer. However, care is needed in applying this interpretation because of the issues about the objective function and what is to be maximized. These issues are addressed in more detail in the analysis of the overlapping generations economy.

4 Barro Growth Model

The Barro growth model is characterized by the public expenditure being an input into the production function. Output at time t, y_t, is produced according to the production function

$$y_t = A k_t^{\alpha} g_t^{1-\alpha}, \quad 0 < \alpha < 1, \tag{8}$$

where g_t is the public services provided by the government and funded by a tax on output at rate τ. The governemnet budget constraint is given by

$$g_t = \tau y_t.$$

We initially consider a four-period version of the model so the quasi-hyperbolic utility function with logarithmic felicity is

$$U = \ln c_0 + \beta\delta \ln c_1 + \beta\delta^2 \ln c_2 + \beta\delta^3 \ln c_3. \tag{9}$$

Utility is maximized subject to the budget constraints

$$k_1 = (1-\tau) y_0 - c_0,$$

$$k_2 = (1-\tau) y_1 - c_1,$$

$$k_3 = (1-\tau) y_2 - c_2,$$

$$c_3 = (1-\tau) y_3,$$

with a given initial level of private capital, k_0.

The solution of this problem can be found by using backward induction. At time 2, the consumer's problem is described as

$$\max_{\{k_3\}} \quad U = \ln c_2 + \beta\delta \ln c_3 \tag{10}$$

$$s.t. k_3 = (1-\tau)Ak_2^{\alpha}g_2^{1-\alpha} - c_2$$
$$c_3 = (1-\tau)Ak_3^{\alpha}g_3^{1-\alpha}.$$

Substituting the constraints for c_2 and c_3 into (10), the problem can be redefined as

$$\max_{k_3} \quad U = \ln[(1-\tau)Ak_2^{\alpha}g_2^{1-\alpha} - k_3] + \beta\delta\ln[(1-\tau)Ak_3^{\alpha}g_3^{1-\alpha}].$$

The first-order condition is

$$\frac{1}{(1-\tau)Ak_2^{\alpha}g_2^{1-\alpha} - k_3} = \frac{\alpha\beta\delta}{k_3}.$$

So the decision rule for k_3 is given by

$$k_3 = \frac{\alpha\beta\delta}{1+\alpha\beta\delta}(1-\tau)Ak_2^{\alpha}g_2^{1-\alpha}. \tag{11}$$

At time 1, the optimization problem for the consumer is

$$\max_{\{k_2\}} \quad U = \ln[(1-\tau)Ak_1^{\alpha}g_1^{1-\alpha} - k_2] + \beta\delta\ln[(1-\tau)Ak_2^{\alpha}g_2^{1-\alpha} - k_3] + \beta\delta^2[(1-\tau)Ak_3^{\alpha}g_3^{1-\alpha}].$$

The first-order condition is

$$\frac{1}{(1-\tau)Ak_1^{\alpha}g_1^{1-\alpha} - k_2} = \frac{\alpha\beta\delta(1+\alpha\delta)}{k_2},$$

where the decision rule for k_3 in (11) is used. So the decision rule for k_2 is

$$k_2 = \frac{\alpha\beta\delta(1+\alpha\delta)}{1+\alpha\beta\delta(1+\alpha\delta)}(1-\tau)Ak_1^{\alpha}g_1^{1-\alpha}. \tag{12}$$

Going back to time 0, the consumer faces the problem

$$\max_{\{k_1\}} \quad U = \ln[(1-\tau)Ak_0^{\alpha}g_0^{1-\alpha} - k_1] + \beta\delta\ln[(1-\tau)Ak_1^{\alpha}g_1^{1-\alpha} - k_2] + \beta\delta^2[(1-\tau)Ak_2^{\alpha}g_2^{1-\alpha} - k_3] + \beta\delta^3[(1-\tau)Ak_3^{\alpha}g_3^{1-\alpha}].$$

The first-order condition is

$$-\frac{1}{(1-\tau)Ak_0^{\alpha}g_0^{1-\alpha} - k_1} + \beta\delta\frac{\alpha(1-\tau)Ak_1^{\alpha-1}g_1^{1-\alpha} - \frac{\partial k_2}{\partial k_1}}{(1-\tau)Ak_1^{\alpha}g_1^{1-\alpha} - k_2}$$
$$+\beta\delta^2\frac{\left[\alpha(1-\tau)Ak_2^{\alpha-1}g_2^{1-\alpha} - \frac{\partial k_3}{\partial k_2}\right]\frac{\partial k_2}{\partial k_1}}{(1-\tau)Ak_2^{\alpha}g_2^{1-\alpha} - k_3} + \beta\delta^3\frac{\alpha(1-\tau)Ak_3^{\alpha-1}g_3^{1-\alpha}\frac{\partial k_3}{\partial k_1}}{(1-\tau)Ak_3^{\alpha}g_3^{1-\alpha}} = 0.$$

Noting that

$$\frac{\partial k_2}{\partial k_1} = \Gamma(1-\tau)\alpha Ak_1^{\alpha-1}g_1^{1-\alpha} = \alpha\frac{k_2}{k_1},$$
$$\frac{\partial k_3}{\partial k_2} = \Pi(1-\tau)\alpha Ak_2^{\alpha-1}g_2^{1-\alpha} = \alpha\frac{k_3}{k_2},$$

where $\Gamma = \frac{\alpha\beta\delta(1+\alpha\delta)}{1+\alpha\beta\delta(1+\alpha\delta)}$ and $\Pi = \frac{\alpha\beta\delta}{1+\alpha\beta\delta}$, (**??**) can be simplified as

$$-\frac{1}{(1-\tau)Ak_0^\alpha g_0^{1-\alpha} - k_1} + \frac{\beta\delta\alpha}{k_1} + \frac{\beta\delta^2\alpha^2}{k_1} + \frac{\beta\delta^3\alpha^3}{k_1} = 0.$$

So the decision rule for k_1 is

$$k_1 = \frac{\alpha\beta\delta(1+\alpha\delta+\alpha^2\delta^2)}{1+\alpha\beta\delta(1+\alpha\delta+\alpha^2\delta^2)}(1-\tau)Ak_0^\alpha g_0^{1-\alpha}. \tag{13}$$

Suppose that there is now a government that has the same objective as the consumer in each period. In the final period, the government has the objective function and the constraint defined as

$$\max_{\{\tau_3\}} U = \ln c_3$$

$$s.t. \quad c_3 = (1-\tau_3)Ak_3^\alpha g_3^{1-\alpha}.$$

Since $g_3 = \tau_3^{1/\alpha} A^{1/\alpha} k_3$, the problem is

$$\max_{\{\tau_3\}} U = \ln(1-\tau_3)\tau_3^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} k_3.$$

The solution for this problem is

$$\tau_3^* = 1-\alpha.$$

At time 2, the government faces the problem

$$\max_{\{\tau_2\}} U = \ln c_2 + \beta\delta \ln c_3$$

$$s.t. c_3 = (1-\tau_3)Ak_3^\alpha g_3^{1-\alpha}$$
$$c_2 = (1-\tau_2)Ak_2^\alpha g_2^{1-\alpha} - k_3$$
$$k_3 = \frac{\alpha\beta\delta}{1+\alpha\beta\delta}(1-\tau_2)Ak_2^\alpha g_2^{1-\alpha}$$
$$\tau_3 = 1-\alpha.$$

Note that

$$c_3 = (1-\tau_3)\tau_3^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} k_3 = \frac{\alpha\beta\delta}{1+\alpha\beta\delta}\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}} A^{\frac{2}{\alpha}} (1-\tau_2)\tau_2^{\frac{1-\alpha}{\alpha}} k_2,$$

$$c_2 = \frac{1}{1+\alpha\beta\delta}(1-\tau_2)\tau_2^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} k_2.$$

Then the optimal tax rate at time 2 is also defined as $\tau_2^* = 1-\alpha$ and by repeating this process back to time 0, we obtain the same optimal tax rate, $\tau^* = 1-\alpha$, for every period. This is the standard optimal tax rate in the Barro growth model.

By extending the model into T-periods, we can observe that the decision rule for k_1 will have the form

$$k_1 = \frac{\alpha\beta\delta \sum_{t=0}^{T-2} (\alpha\delta)^t}{1 + \alpha\beta\delta \sum_{t=0}^{T-2} (\alpha\delta)^t} (1-\tau) A k_0^{\alpha} g_0^{1-\alpha}. \tag{14}$$

Thus, with infinite T, the decision rule for capital can be defined as

$$k_{t+1} = \frac{\alpha\beta\delta}{1-\alpha\delta(1-\beta)} (1-\tau) A k_t^{\alpha} g_t^{1-\alpha}. \tag{15}$$

Since $y_t = \tau^{(1-\alpha)/\alpha} A^{1/\alpha} k_t$,

$$k_{t+1} = \frac{\alpha\beta\delta}{1-\alpha\delta(1-\beta)} (1-\tau) \tau^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} k_t. \tag{16}$$

This model still shows a balanced growth path with a growth rate of

$$\gamma = \frac{\alpha\beta\delta}{1-\alpha\delta(1-\beta)} (1-\tau) \tau^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}}.$$

The tax rate that maximizes γ is given by the standard result

$$\tau = 1 - \alpha.$$

5 Overlapping generations

It has already been shown that the effects of quasi-hyperbolic preferences are only significant for the paths of savings when lifetime is finite. This makes the overlapping generations economy an ideal vehicle for exploring the how preference structure affects capital accumulation. Since preferences are important, it then follows that there will be a link between preferences and tax policy.

It is assumed that the lifespan of each consumer is three periods. This is the minimum lifespan necessary for the quasi-hyperbolic preferences to have an impact on saving choices. The consumer work in the first two periods of life and is retired in the third period.

5.1 Choices

The preferences of a consumer born at time t are given by

$$U^t = \ln(w_t - s_t^t) + \beta\delta \ln(w_{t+1} + (1+r_{t+1}) s_t^t - s_{t+1}^t) + \beta\delta^2 \ln((1+r_{t+2}) s_{t+1}^t).$$

In every case, the consumer at time $t+2$ just consumes accumulated savings with interest $(1+r_{t+2}) s_{t+1}^t$. The difference in choices emerges at t and $t+1$.

5.1.1 Committed

The committed consumer makes a decision at the start of life and follows through the chosen path. The optimization

$$\max_{\{s_t^t, s_{t+1}^t\}} U^t = \ln(w_t - s_t^t) + \beta\delta \ln(w_{t+1} + [1 + r_{t+1}]\, s_t^t - s_{t+1}^t) + \beta\delta^2 \ln([1 + r_{t+2}]\, s_{t+1}^t)$$

The solution is

$$\begin{aligned} s_t^t &= \frac{1}{\left(1 + \beta\delta + \beta\delta^2\right)} \left(\left(\beta\delta + \beta\delta^2\right) w_t - \frac{1}{1 + r_{t+1}} w_{t+1} \right), \\ s_{t+1}^t &= \frac{\beta\delta^2}{1 + \beta\delta + \beta\delta^2} \left((1 + r_{t+1})\, w_t + w_{t+1} \right). \end{aligned}$$

5.1.2 Naive

The first-period solution for the naive is the same as the committed. In the second period the naive consumer takes s_t^t as given so solves

$$\max_{\{s_{t+1}^t\}} U^t = \ln(w_{t+1} + (1 + r_{t+1})\, s_t^t - s_{t+1}^t) + \beta\delta \ln((1 + r_{t+2})\, s_{t+1}^t)$$

This provides the solution

$$s_t^t = \frac{1}{\left(1 + \beta\delta + \beta\delta^2\right)} \left(\left(\beta\delta + \beta\delta^2\right) w_t - \frac{1}{(1 + r_{t+1})} w_{t+1} \right)$$

$$s_{t+1}^t = \frac{\beta\delta}{1 + \beta\delta} \frac{\beta\delta + \beta\delta^2}{1 + \beta\delta + \beta\delta^2} \left((1 + r_{t+1})\, w_t + w_{t+1} \right).$$

5.1.3 Sophisticated consumer

The sophisticated consumer makes the first-period choice taking into account the choices of later selves. The consumer at time $t+1$ takes s_t^t as given and chooses s_{t+1}^t. The objective function for this version of the consumer born at t is

$$U^{t+1} = \ln(w_{t+1} + (1 + r_{t+1}) s_t^t - s_{t+1}^t) + \beta\delta \ln((1 + r_{t+2}) s_{t+1}^t).$$

This gives the chosen level of saving

$$s_{t+1}^t = \frac{\beta\delta}{1 + \beta\delta} \left[w_{t+1} + (1 + r_{t+1}) s_t^t \right]. \tag{17}$$

The consumer at time t takes the solution (17) into account so faces the optimization

$$\begin{aligned} \max_{\{s_t^t\}} U^t &= \ln(w_t - s_t^t) + \beta\delta \ln\left(w_{t+1} + (1 + r_{t+1}) s_t^t - \frac{\beta\delta}{1 + \beta\delta} \left[w_{t+1} + (1 + r_{t+1}) s_t^t \right] \right) \\ &\quad + \beta\delta^2 \ln\left((1 + r_{t+2}) \frac{\beta\delta}{1 + \beta\delta} \left[w_{t+1} + (1 + r_{t+1}) s_t^t \right] \right). \end{aligned}$$

The chosen level of saving is

$$s_t^t = \frac{1}{1+\beta\delta+\beta\delta^2}\left(\left(\beta\delta+\beta\delta^2\right)w_t - \frac{w_{t+1}}{1+r_{t+1}}\right).$$

Using this solution it follows that

$$s_{t+1}^t = \frac{\beta\delta}{1+\beta\delta}\frac{\left(\beta\delta+\beta\delta^2\right)}{1+\beta\delta+\beta\delta^2}\left(w_{t+1}+(1+r_{t+1})w_t\right),$$

which again matches the naive because of the log form.

These different saving patterns imply different capital accumulation levels. The total labor supply is the sum of supply from consumers in the first- and second-periods of life

$$L_t = H_t + H_{t-1}.$$

The production function is assumed to have constant returns to scale in the two factors capital, K_t, and labour, L_t,

$$Y_t = F(K_t, L_t).$$

The capital-labour ratio is found by dividingK_t by $H_t + H_{t-1}$, so that

$$y_t = f\left(k_t\right),$$

where

$$y_t = \frac{Y_t}{H_t+H_{t-1}},\ k_t = \frac{K_t}{H_t+H_{t-1}}.$$

5.2 Capital Accumulation

The saving functions can be used to construct the time paths of the capital stock for the different types of consumers. These can then be contrasted to understand the implications of the quasi-hyperbolic preferences.

In every case, the time path for capital accumulation is determined by the relation

$$K_{t+1} = H_t s_t^t + H_{t-1}s_t^{t-1}.$$

Using the saving functions for a committed consumer gives

$$\begin{aligned} K_{t+1} &= H_t\frac{1}{\left(1+\beta\delta+\beta\delta^2\right)}\left(\left(\beta\delta+\beta\delta^2\right)w_t - \frac{1}{1+r_{t+1}}w_{t+1}\right) \\ &\quad + H_{t-1}\frac{\beta\delta^2}{1+\beta\delta+\beta\delta^2}\left(\left(1+r_t\right)w_{t-1}+w_t\right). \end{aligned}$$

This can be expressed in per capita terms by dividing by $H_t + H_{t-1}$

$$\begin{aligned} \frac{K_{t+1}}{H_t+H_{t-1}} &= \frac{H_t}{H_t+H_{t-1}}\frac{1}{\left(1+\beta\delta+\beta\delta^2\right)}\left(\left(\beta\delta+\beta\delta^2\right)w_t - \frac{1}{1+r_{t+1}}w_{t+1}\right) \\ &\quad + \frac{H_{t-1}}{H_t+H_{t-1}}\frac{\beta\delta^2}{1+\beta\delta+\beta\delta^2}\left(\left(1+r_t\right)w_{t-1}+w_t\right), \end{aligned}$$

or

$$
\begin{aligned}
(1+n)k_{t+1} &= \frac{1+n}{2+n}\frac{1}{\left(1+\beta\delta+\beta\delta^2\right)}\left(\left(\beta\delta+\beta\delta^2\right)w_t-\frac{1}{1+r_{t+1}}w_{t+1}\right) \qquad (18)\\
&+\frac{1}{2+n}\frac{\beta\delta^2}{1+\beta\delta+\beta\delta^2}\left(\left(1+r_t\right)w_{t-1}+w_t\right). \qquad (19)
\end{aligned}
$$

Similarly, the time path for capital accumulation with a naive or a sophisticated consumers is governed by

$$
\begin{aligned}
K_{t+1} &= H^t\frac{1}{\left(1+\beta\delta+\beta\delta^2\right)}\left(\left(\beta\delta+\beta\delta^2\right)w_t-\frac{1}{\left(1+r_{t+1}\right)}w_{t+1}\right)\\
&+H^{t-1}\frac{\beta\delta}{1+\beta\delta}\frac{\beta\delta+\beta\delta^2}{1+\beta\delta+\beta\delta^2}\left(\left(1+r_t\right)w_{t-1}+w_t\right),
\end{aligned}
$$

or, in per capita terms

$$
\begin{aligned}
(1+n)k_{t+1} &= \frac{1+n}{2+n}\frac{1}{\left(1+\beta\delta+\beta\delta^2\right)}\left(\left(\beta\delta+\beta\delta^2\right)w_t-\frac{1}{\left(1+r_{t+1}\right)}w_{t+1}\right) \qquad (20)\\
&+\frac{1}{2+n}\frac{\beta\delta}{1+\beta\delta}\frac{\beta\delta+\beta\delta^2}{1+\beta\delta+\beta\delta^2}\left(\left(1+r_t\right)w_{t-1}+w_t\right). \qquad (21)
\end{aligned}
$$

The use of the factor price conditions

$$
r_t=f'(k_t),\quad w_t=f(k_t)-k_tf'(k_t),
$$

then turn (18) and (20) into second-order difference equations. Given initial values $\{k_{-1},k_0\}$ it is then possible to iterate the equations forward to generate the capital path.

Before the capital paths are analyzed it is first helpful to analyze the steady state of each accumulation condition. To make an explicit computation possible the standard assumption is made that $y=k^\alpha$. Define k_N,k_S,k_C as the steady state capital labour ratios for the naive, sophisticated, and committed consumers. The lemma provides the contrast of steady states.

Lemma 3 *If the steady state is stable then $k_N=k_S<k_C$.*

Proof. The steady state levels solve

$$
\begin{aligned}
(1+n)k_C &= \frac{1}{2+n}\frac{(1-\alpha)\,k_C^\alpha}{\left(1+\beta\delta+\beta\delta^2\right)}\\
&\times\left((1+n)\left(\left(\beta\delta+\beta\delta^2\right)-\frac{1}{1+\alpha k_C^{\alpha-1}}\right)+\beta\delta^2\left(2+\alpha k_C^{\alpha-1}\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(1+n)k_N &= \frac{1}{2+n}\frac{(1-\alpha)\,k_N^\alpha}{\left(1+\beta\delta+\beta\delta^2\right)}\\
&\times\left((1+n)\left(\left(\beta\delta+\beta\delta^2\right)-\frac{1}{1+\alpha k_N^{\alpha-1}}\right)+\frac{\beta\delta}{1+\beta\delta}\left(\beta\delta+\beta\delta^2\right)\left(2+\alpha k_N^{\alpha-1}\right)\right)
\end{aligned}
$$

with $k_N = k_S$. To compare the committed and the naive it is necessary to contrast the two terms in the brackets.

Consider the second term. If

$$\frac{\beta\delta}{1+\beta\delta}\left(\beta\delta+\beta\delta^2\right) > \beta\delta^2,$$

then

$$\beta\delta+\beta\delta^2 > \delta+\beta\delta^2,$$

which is false. So, $\beta\delta^2 < \frac{\beta\delta}{1+\beta\delta}\left(\beta\delta+\beta\delta^2\right)$ and it is not possible to have $k_N = k_C$. Since the right-hand side of each expression is monotonically increasing in k the claim follows. ■

Figure 1 shows the different growth paths for the committed and naive. It can be seen that the committed consumer accumulates capital more quickly and achieves a higher steady state level of capital. This figure captures the general perception that quasi-hyperbolic preferences lead to lower capital accumulation when consumers behave naively.

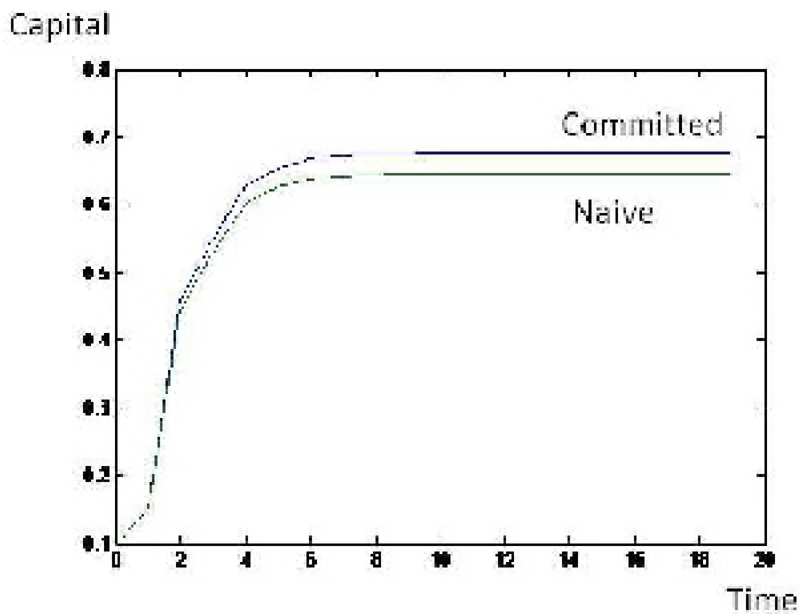

Figure 1: Capital accumulation paths

Figure 2 confirms the effect of quasi-hyperbolic preferences. It demonstrates that the level of capital accumulated in the steady state is lower as present bias increases (lower β).

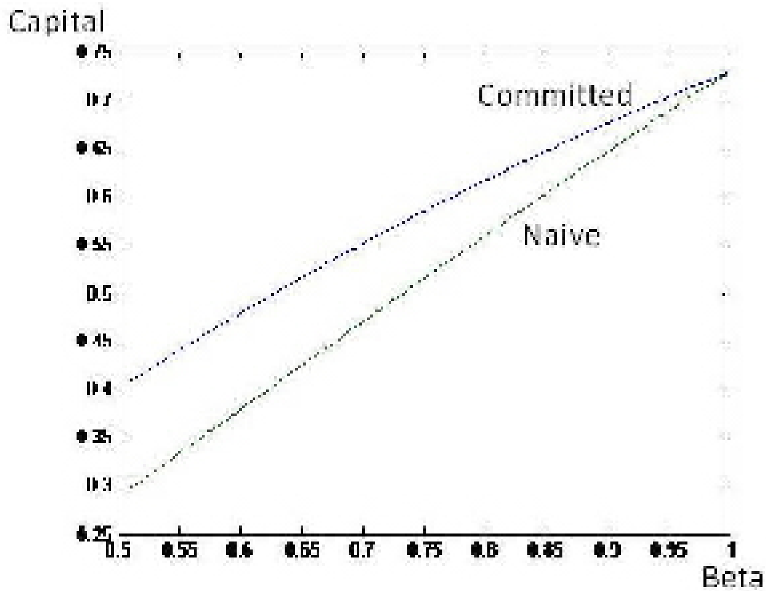

Figure 2: Steady state capital and present bias

THE NEXT STEP IS A GOLDEN RULE ANALYSIS

5.3 Taxation

The taxation of capital and labour income can be introduced into the analysis by the change of variables

$$w_t \rightarrow [1 - \tau_l]\, w_t, \quad r_t \rightarrow [1 - \tau_k]\, r_t.$$

Imposing a balanced budget for the government in each period

$$\tau_l w_t + \tau_k r_t k_t = 0,$$

makes tax policy into a choice of whether to introduce a tax system or not with any capital tax balanced by a subsidy to labour (or vice versa). The balanced budget can be used to eliminate τ_l from explicit consideration and to express growth in terms of τ_k alone. This makes it possible to investigate how the choice of capital tax τ_k affects the growth path for the naive consumer.

The first figure shows that growth path of capital for the committed without tax, and for the naive with three different tax rates. In this case, a positive tax on capital moves the economy with the naive consumer closer to the outcome with a committed consumer.

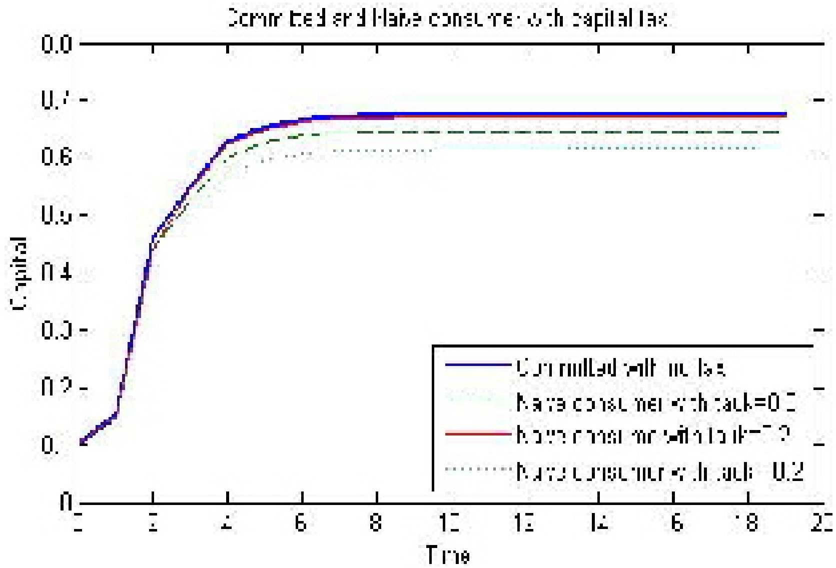

Figure 3: Effect of capital taxation

The second figure shows the steady-state capital level as a function of the tax rate. The steady state level of the capital stock rises as the tax rate increases but the (absolute) difference between committed and naive is smallest for a capital subsidy.

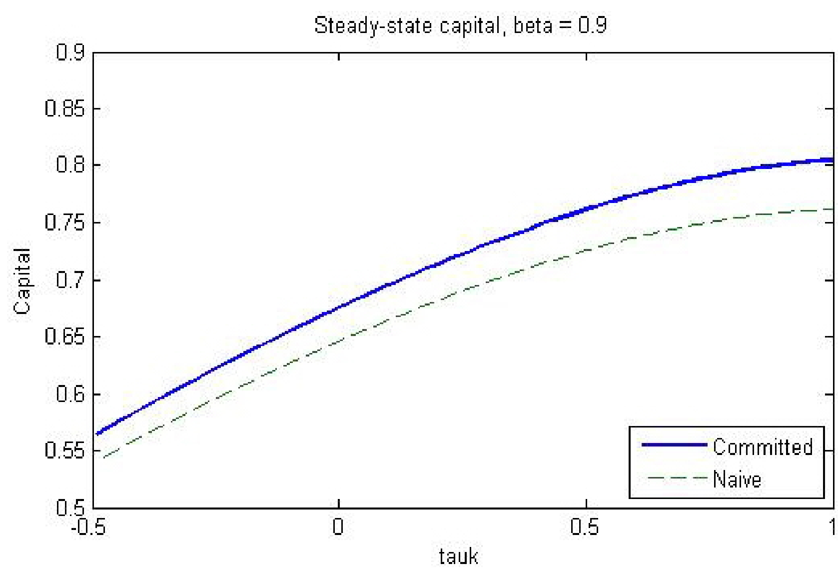

FIgure 4: Steady state capital stock

RELATIONSHIP TO GOLDEN RULE

The next question is to consider how the tax policies be evaluated from a welfare perspective. To do this the standard difficulty must be confronted of how to deal with the multiple selves of the naive. The fact that each of the selves has a different objective function creates a problem of determining what is the welfare evaluation. For example, if the social planner chooses one of the selves as representative the other two selves will object to the choice made. An alternative approach is to consider the selves as distinct (and in the overlapping generations model there are examples of each self alive at every time) and to consider the extent of unanimity between selves on policy.

Felicity is plotted in figure 5 for the multiple selves of the naive. The younger is the self the higher is the preferred tax rate. In the case illustrated the oldest self prefers a capital subsidy. The figure demonstrates the main point that there is no unanimity between the multiple selves about the tax rate. If there was a vote then the single-peaked preferences allow the median voter theorem to be applied. So the rate preferred by median - the self at time 2 in this case - will be the Condorcet winner.

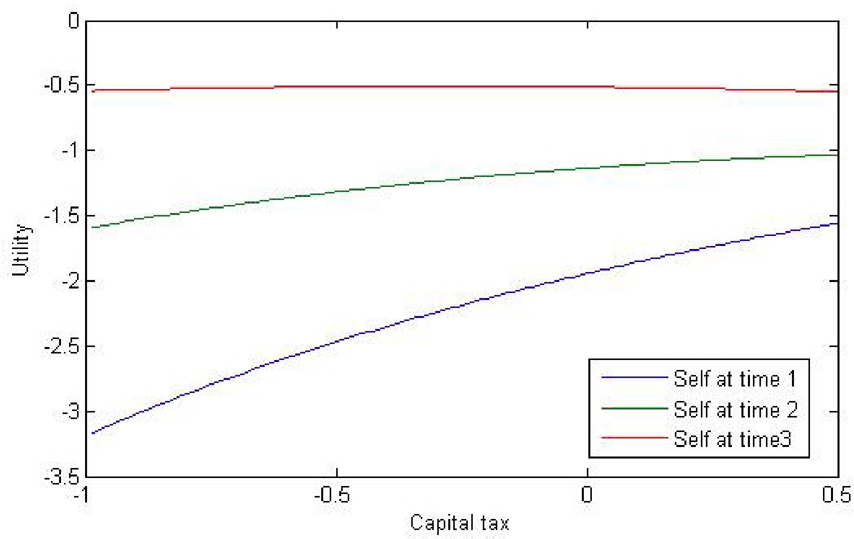

Figure 5: Multiple selves and optimal tax

6 Conclusions

Quasi-hyperbolic preferences distort savings patterns.

But in the long run the initial effect is diminished.

With successive generations the preferences have an effect.

This can motivate a tax intervention which will be unanimously supported by all the multiple selves.

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