Intergenerational Risk Sharing under Endogenous Labor Supply*

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Abstract

This paper evaluates intergenerational risk-sharing in the context of a pre-funded social security scheme. The central feature of the model is that the welfare costs from labor-market distortions from risk-sharing transfers are explicitly taken into account. Equity risk manifests itself in the form of implicit taxes and subsidies on the labor earnings of participants. The labor-supply choices of participants are assumed to be elastic with respect to wage-differentials, implying that risk-sharing results in labor-market distortions. I show that labor-supply effects impede the pension fund from taking advantage of intergenerational risk-sharing. The analysis thereby provides an economic justification for solvency rules that require financial losses to be levied primarily upon currently-living generations.

Keywords: intergenerational risk-sharing, labor-supply distortions, pre-funded social security scheme

JEL classification: D91, G11, G23, H55

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1 Introduction

The inability of current generations to share risks with generations that are not born yet causes financial markets to be incomplete and thus inefficient[1]. This implies that there is a role for a long-lived social planner (i.e. the government) to reallocate risk across generations. The government‘s power of taxation gives it a unique ability to make commitments on behalf of future generations. Intergenerational risk-sharing can be facilitated in various ways. This paper evaluates risk sharing in the context of a pre-funded social security scheme. The central feature of the model is that the welfare costs from labor-market distortions from risk-sharing transfers are explicitly taken into account. Equity risk manifests itself for participants in the form of implicit taxes and subsidies on their labor earnings. A drop in the value of pension fund assets can lead to a rise in the pension contribution rate, a decline in the value of pension entitlements, or a combination of the two. By deviating the contribution rate from accrual rate, the pension fund induces a wage-differential upon its working participants. It is assumed that the labor-supply choices of participants are elastic with respect wage-differentials, implying that risk-taking and risk-sharing distorts the labor-supply choices of workers. This paper shows that labor-supply effects impede the pension fund from taking advantage of intergenerational risk-sharing.

Examples of nation-wide pre-funded pension funds include the Social Security Trust Funds in the United States[2], the Japan Government Pension Investment Fund, the Canada Pension Plan and the ATP fund in Denmark. Some pre-funded retirement schemes, such as the US social security trust funds, have been put in place as a buffer against demographic shocks and are expected to decline in size in the coming decades. Other pre-funded pension schemes, such as the Canada Pension Plan, are permanent in nature and are expected to grow in size in the coming decades.

Several countries that have set up a funded tiers in their pension system in the form of IRAs, including Australia, Ireland and Estonia. Risk-sharing between non-overlapping generations is not possible in financial markets and is thus not facilitated in a pre-funded pension system with individual retirement account (IRA). A collective pension fund has the potential to outperform a system with IRAs. If designed properly, intergenerational

[1]This point was made by Diamond (1977), Merton (1983) and Gordon and Varian (1988). More recent contributions include Shiller (1999), Gottardi and Kubler (2008), Cui, de Jong, and Ponds (2007), Bohn (2006), Smetters (2006), Ball and Mankiw (2007) and Gollier (2008).

[2]While most pre-funded social security funds are diversified with respect to asset class as well as internationally, the US trust funds are fully invested in government bonds. Proposals to invest government funds in private securities can be controversial, as illustrated in the debates during the Clinton-administration about investing social security trust funds in the stock market. See White (1996), ACSS (1997), GAO (1998), Greenspan (1999) and Greenspan (1999).

risk-sharing contracts lead to a Pareto-improvement for all generations from an ex-ante point of view. However, ex-post realizations may be disadvantageous for some unlucky generations. A feasible risk-sharing solution therefore requires participation to be mandatory. Intergenerational risk sharing leads to better time-diversification of the risk that comes with investments in high-yielding long-lived assets. The improved time-diversification increases the appetite for risk-taking and allows individuals to take better advantage of the equity premium in financial markets.

Without exception, the previous work on pre-funded pension schemes assumes a non-distortionary implementation of intergenerational risk-transfers. The assumption of non-distortionary transfers, better known as lump-sum transfers, is unrealistic in the context of pension schemes[3]. For example, it is unrealistic to assume that pension funds are able to provide new entrants with pension rights that have a negative value when recouping previous losses upon them. Instead, a pension fund is able to extract quasi-rents from workers by requiring participation in the fund to be mandatory and by inducing a wedge between the contribution rate and the value of pension entitlements received in return. Future generations can thus be committed to share in current financial shocks, but only through implicit taxes and subsidies on their labor earnings. Risk-taking and risk-sharing in pension funds thereby inherently induces distortions in labor markets[4]. Throughout the analysis it is assumed that the implicit taxes and subsidies induced by the pension fund are proportional to labor earnings. This assumption comes from the common-place observation that pension contribution and benefit levels are proportional to labor earnings. Pension contributions are typically a certain percentage of labor earnings while the benefit formulas of pension funds are usually some function that is linear in past labor earnings.

The welfare analysis in this paper is based upon an overlapping generations model. I adopt a partial equilibrium framework in which the factor prices for labor and capital are exogenously determined on international markets. As in Beetsma and Bovenberg (2009), the model for the pension fund is 'stand-alone' in the sense that there is no risk-absorbing sponsor in the form of the government or corporations. I take the perspective of a social planner who

[3] Lump-sum risk sharing transfers in a pension fund are not only unrealistic, but also unfair. A pension fund does not observe the earnings capacity of participants, so that a participant with a low earnings capacity contributes just as much to the recovery process of the fund as his or her counterpart with a high earnings capacity if the pension fund applies uniform lump-sum risk-sharing transfers. This leads to *intra*generational unfairness.

[4] Notice that not all the financial gains and losses in a pension fund manifest themselves for participants in the form of taxes and subsidies. If the pension fund recovers from a financial loss through an unanticipated cut in benefit levels, then retirees will experience this as a lump-sum transfer. However, if the benefit cut is permanent in nature, then workers will anticipate lower benefit levels in the future, implying that they will attach a lower value to their pension entitlements and have less incentives to supply labor.

maximizes the ex-ante welfare of participants by optimizing the contribution, investment and payout policy of the pension fund. The discount factor used by the social planner to weigh the welfare of different generations is chosen such that all generations are equally well off from an ex-ante perspective. Since all generations have identical properties, the social surplus from intergenerational risk-sharing is divided equally among all generations. In the special case where labor supply is inelastic, there are no distortions in labor supply choices and the model adopts an analytical solution[5]. The general case in which labor supply is elastic is solved using numerical solution techniques. The overlapping generations model is preceded by a stylized setting with two-agents. This simplified model allows me to explain the main intuition of the paper in a simple way. However, the assumption of a two-agent setting is not innocuous. The quantitative results in the overlapping generations model differ substantially from those in the two-agent setting.

The four most important findings of this paper are as follows. First, I find that distortions erode a large fraction of the ex-ante welfare gains from intergenerational risk sharing. For the benchmark parameters in this paper[6], 46% of the welfare gain is eroded. If the wage-elasticity of labor supply exceeds 1.2, the welfare costs from distortions dominate the welfare gains from risk sharing. In this case, the pension fund is not welfare improving anymore and workers are better off in a system with individual retirement accounts.

As a second finding, there is a trade-off between consumption smoothing on the one hand and minimizing distortions in labor markets on the other hand. The principle of consumption smoothing implies that financial shocks should be smoothed over the consumption levels of as many generations as possible. That is: all future consumption levels are adjusted proportionally equally as a result of financial gain or loss at present. However, the principle of consumption smoothing causes consumption levels to follow a random walk as all adjustments in the consumption are persistent in nature[7]. This implies that contribution rates can rise

[5] The same holds true if the pension fund would be able to levy taxes and subsidies in lump-sum form.

[6] As a benchmark parameter for the wage elasticity of labor supply I choose 0.5. There is a large empirical literature that studies the wage elasticity of labor-supply choices of workers. The consensus in the literature (e.g. Blundell and MaCurdy (1999), Alesina, Gleaser, and Sacerdote (2005)) is that the labor-supply elasticity at the intensive margin (i.e. choices about hours of work or weeks of work) is close to zero for male workers. There is a large variation in the estimates found for female workers, but the median estimate is close to one. Labor-supply choices at the extensive margin (i.e. labor force participation and employment choices) are important as well (e.g. Heckman (1993) and Saez (2002)). In particular, there is a large literature that finds the retirement decisions of individuals to be quite responsive to financial incentives in pension schemes (e.g. Stock and Wise (1990), Samwick (1998) and Gruber and Wise (1999)).

[7] Random-walk consumption is a familiar result in the literature. The random-walk result for consumption has been found by Merton (1969) and Samuelson (1969) in the setting of a consumption-investment problem, by Hall (1987) for the case of an infinitely-lived consumer, by Gollier (2008) in a setting where a social planner chooses consumption for different generations and in Ball and Mankiw (2007) in a setting where non-overlapping generations trade with each other in a fictitious financial market.

to high levels in the situation of a succession of negative returns on investments. The marginal costs from distortions in such a bad scenario become very high, as further increases in the contribution rate become very costly. As a result, the pension fund will not find it optimal anymore to smooth financial shocks over all future generations. Instead, it is optimal to recover from previous losses, and thus let the contribution rate fall, to restore its capacity to take risks in the future. This implies that financial shocks are levied primarily upon currently-living generations. This result stands in striking contrast with the existing literature that finds that governments should set their debt policies in a way that taxes are smoothed over time, see e.g. Barro (1979), Lucas and Stockey (1983) and Bohn (1990).

The third finding of the paper is that labor-supply distortions allows me to obtain a risk sharing solution that is more likely to be politically sustainable. The analysis in Gollier (2008) has pointed out that risk-sharing contracts are '*hardly politically sustainable if a succession of negative shocks on financial markets arises early in the life of the fund*'. Risk-sharing solutions can be welfare improving for all generations from an ex-ante perspective, but some unlucky generations may lose from an ex-post perspective. I show that recognizing labor-supply effects leads to a risk-sharing solution that are less likely to cause political tensions. Solutions that are sustainable from an economic point of view are thus also more likely to be sustainable from a political point of view. The pension fund recovers from financial gains and losses relatively quickly, restoring its capacity for future risk taking. The solution in this paper is consistent with solvency regimes that require pension funds levy financial shocks primarily upon currently living generations[8].

The fourth finding of this paper is that the ability of workers to vary their labor supply can reduce welfare. This result stands in striking contrast to the existing literature on portfolio choice with flexible labor-supply initiated by Bodie, Merton, and Samuelson (1992) and further developed by, among others, Farhi and Panageas (2007), Choi, Shim, and Shin (2008) and Gomes, Kotlikoff, and Viceira (2008). All these papers take the perspective of an individual investor in which flexible labor supply is used as a buffer against income shocks. For an individual investor, a negative wealth shock causes the marginal utility from working to increase and hence agents increase labor supply. In other words, income effects cause labor-supply behavior to become more counter-cyclical, enabling an individual investors to take greater advantage of the equity premium in financial markets. In contrast, this paper takes the perspective of pension fund asset management rather than the portfolio choices at the individual level. If financial shocks are levied upon participants through taxes and subsidies, the financial gains and losses from risk taking not only induce income effects in labor supply

[8]The Dutch regulator requires pension funds that are underfunded to be fully funded within 3 years and to have restored their financial buffer for risk-taking within 15 years.

(as in the analysis of Bodie, Merton, and Samuelson (1992)) but also substitution effects which work in the opposite direction. Financial incentives in pension plans may therefore result in pro-cyclical labor supply behavior, thereby reducing the appetite for risk-taking and reducing welfare. Pro-cyclical labor-supply behavior induced by substitution-elasticity in labor supply causes intergenerational risk-sharing to become less effective. Income-elasticity in labor supply on the other hand increases the effectiveness of intergenerational risk-sharing.

Many papers have studied the risk sharing properties of pay-as-you-go pension schemes, e.g. Bohn (1998), Krueger and Kubler (2002) and Gottardi and Kubler (2008). There is also a large literature on the welfare effects from a shift from an unfunded towards a funded pension system (see Lindbeck and Persson (2003) and Shiller (2003) for broad perspectives). A shift towards funding simply reallocates resources between generations when all economic variables are deterministic and one abstracts from distortions in capital and labor markets. Under these assumptions, no Pareto-improvement exists because no resources are created once the 'winners' from the reform have fully compensated the 'losers'. However, a shift towards funding can reduce risk sharing (only free market possibilities remain) but it can also reduce distortions in labor and capital markets. Some studies find that the welfare gains from risk sharing are larger than the welfare losses from distortions, e.g. Nishiyama and Smetters (2007) and Fehr and Habermann (2008). Others find that distortions dominate, implying that there exists a Pareto improving path towards funding, e.g. Krueger and Kubler (2006), Fuster, Imrohoroglu, and Imrohoroglu (2007). All these papers restrict themselves to a transition towards a system with individual retirement accounts, and thus ignore the potential for risk sharing in pre-funded pension schemes.

Intergenerational risk-sharing is more attractive in pre-funded pension systems for two reasons. In contrast to the case of pay-as-you-go schemes, there is no capital crowding-out effect in pre-funded schemes given that pension savings are invested in the financial market. In addition, pre-funded schemes feature a close link between pension contributions and pension benefits whereas this link may be weaker in pay-as-you-go schemes[9]. Only few papers have studied the risk-sharing aspects of pre-funded pension schemes. Teulings and de Vries (2006), Ball and Mankiw (2007) and Gollier (2008) have examined how pension funds are able to facilitate risk-sharing with unborn generations[10]. However, these papers ignore the effects of risk sharing on labor and capital markets. Beetsma and Bovenberg (2009) examine the effects of risk sharing in a pre-funded scheme on capital markets but do

[9]Not all pay-as-you-go systems feature a weak link between contributions and benefits, see for instance Sweden's notional defined-contribution scheme.

[10]Smetters (2006) points out that an appropriate chosen capital tax can also facilitate risk sharing across generations, implying that intergenerational risk-sharing not require direct government ownership of equities.

not focus on labor market effects[11].

Notice that the analysis in this paper is not applied to corporate pension schemes. In a perfectly competitive labor market, a wage differential induced by the pension plan forces an employer to offer a compensating wage-differential to prevent an influx or outflow of workers as a result of the actuarial unfairness of the pension plan. Under perfect labor-market competition, it is thus the employer who is on the hook for shortfalls[12], not the employees. The model therefore primarily applies to nation-wide pension funds[13]. Arguably, the model also applies to the case of an industry-wide pension fund[14], in which it is more difficult for participants to evade the pension policy by switching employers. Participants cannot evade the pension contract by switching employers *within* the industry. Switching to an employer *outside* the industry can be unattractive due to the accumulated industry-specific human capital. The opportunities for switching jobs are thus reduced in the case of an industry-wide pension fund, allowing a pension fund to extract quasi-rents from its workers. This paper points out that intergenerational risk-sharing in industry-wide funds can become unattractive if the fund induces labor-supply movements across sectors[15]

The structure of the remainder is as follows. Section 2 examines labor-supply effects in a stylized risk-sharing framework with two agents. Arrow-Pratt approximations are used to derive analytical results. Chapter 3 extends the analysis to an overlapping generations framework and is solved by using numerical solution techniques. Finally, section 4 concludes.

[11] Boonenkamp and Westerhout (2009) also examine the labor-supply distortions from intergenerational risk sharing in the context of a funded pension scheme. Their analysis is restricted to the case of a two-agent model and provides analytical results only for the case of Cobb-Douglas preferences over consumption and leisure. Their quantitative results are consistent with the welfare losses of the two-agent model of chapter 2 of this paper: 10-25% of the social surplus from risk sharing is eroded by distortions. As noted earlier, the assumption of a two-agent setting is not innocuous.

[12] Rauh (2006) provides empirical evidence that the investment decisions of employers are distorted if they share in the funding risk of their corporate pension plan.

[13] In a sense, the model also applies to state-sponsored pension funds for civil servants in which tax payers are eventually on the hook for shortfalls (see Novy-Marx and Rauh (2009)). However, this application is not fully consistent with the setting of the paper because the pension fund induces labor supply distortions upon all workers in the state, not only the civil servants in the pension fund. Novy-Marx and Rauh (2009) point out that citizens may find ways to evade such taxes. They argue that if a state invests heavily in equity and the market performs poorly, then some of its taxpayers, facing larger future tax bills, may leave for states that performed better. Similar intuition helps explain the phenomenon of suburban flight (away from urban areas), which was at least in part driven by citizens voting with their feet for lower taxation (Papke (1987) and Ladd and Bradbury (1988)).

[14] As an example, there are 78 industry-wide pension funds in the Netherlands covering over 75% of the total number of Dutch working people.

[15] In addition, the tax-base of the pension fund may be affected by decisions at the firm-level. In many industries, it is not always clear which firms belong to the industry and which don't. Newly established firms can therefore decide not to join an industry-wide pension fund if this is not in their interest, i.e. if the scheme is poorly funded.

2 Two agents

Following Gollier (2008), I examine intergenerational risk-sharing in a stylized two-agent setting before turning to the overlapping generations model. Section 2.1 introduces the two-agent model. Section 2.2 presents the autarky solution. Section 2.3 treats the solution for risk sharing under lump-sum transfers, which corresponds to Gollier (2008). Section 2.4 extends the treatment of Gollier (2008) to the case of distortionary transfers. Finally, section 2.5 considers the welfare effects of a suboptimal risk-sharing contract.

2.1 The model

The model features two agents, where first-born agent $i = 1$ is alive during period 1 and the second-born agent $i = 2$ is alive during period 2. The periods 1 and 2 are non-overlapping, so that it is not possible for the two agents to share risks through a financial market. A long-lived pension fund, however, can facilitate intergenerational risk-sharing transfers between the two agents. Risk sharing makes it possible for agent 2 to share in the risks that materialize in period 1. However, it is not possible for agent 1 to share in the risks that realize in period 2 since the realization of these risk occurs after agent 1 has passed away.

The agents supply labor and invest in the financial market during the period in which they are alive. Labor earnings and the proceeds from investments in the financial market are used for consumption. The wage rate w_i against which labor is supplied by agent i (i being equal to 1 or 2) is assumed deterministic. Labor supply is a decision variable of the agent and is denoted by h_i so that the labor earnings of agent i are given by $w_i h_i$. Since only the risk that materializes in period 1 can be shared between the two agents, I abstract from risk taking in the second period[16]. In the first period, the financial market offers two investment opportunities: a riskless asset with zero return and a risky asset with return $\tilde{x}_1$. The mean and variance of the risk $\tilde{x}_1$ are denoted by μ and σ^2 respectively. The consumption level C_1 of agent 1 consists of labor earnings plus the proceeds from investments minus the risk that is transferred to agent 2:

$$C_1 = w_1 h_1 + \alpha x_1 - t(x_1), \tag{2.1}$$

where α denotes the absolute amount[17] invested in the risky asset in period 1 and where $t(x_1)$ is the transfer from agent 1 to agent 2 and is a function of the realization x_1 of the risk

[16]This assumption is harmless when risks are small. However, risk taking in period 2 will decrease the willingness of agent 2 to share in the risks that materialize in the first period if risk exposures are high.

[17]A short-selling constraint (i.e. $\alpha \geq 0$) is not imposed upon the asset allocation because it will follow from equations (2.8), (2.13) and (2.20) that the optimal amount invested in the risky asset is positive as long as the equity premium is positive (i.e. $\mu > 0$).

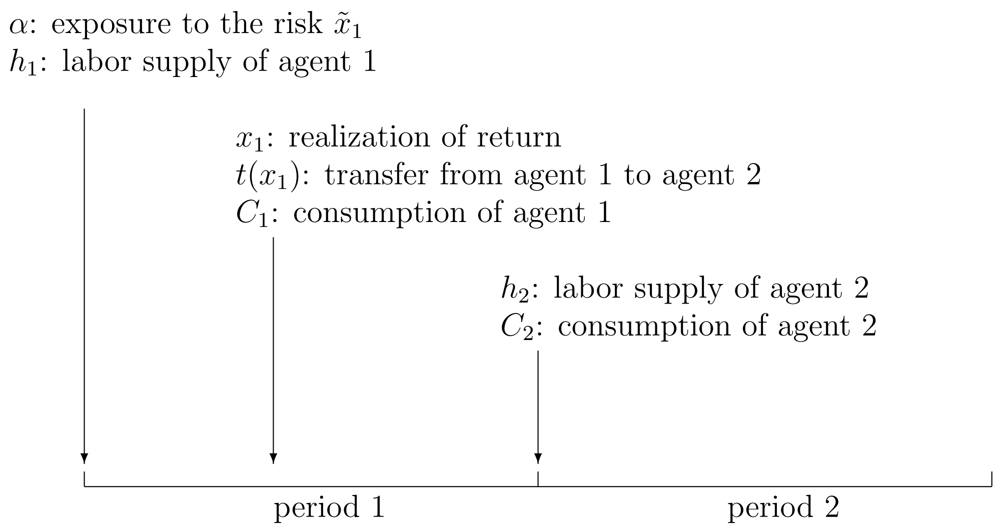


Figure 2.1: *Time-schedule of the two-agent model.*

$\tilde{x}_1$. The consumption level of agent 2 equals labor earnings plus the risk transfer:

$$C_2 = w_2 h_2 + t(x_1). \tag{2.2}$$

The transfer $t(x_1)$ does not accumulate interest between period 1 and 2 because of the assumption of a zero risk-free interest rate.

Figure 2.1 shows the time schedule for the two-agent model. At the beginning of the first period, the risk exposure α is determined and agent 1 takes the labor supply decision. The risk exposure cannot be conditioned on the return on the risky asset, which has not been realized yet at the beginning of the first period. At the beginning of the second period, agent 2 takes the labor-supply decision, which can be conditioned upon the realization of the risk sharing transfer $t(x_1)$.

The preferences of the agents are identical and are given by expected utility over consumption C_i and labor h_i. I restrict my analysis to the case where preferences are such that income effects in labor supply are absent. Income effects in labor supply are found to be small when compared to substitution effects, see Blundell and MaCurdy (1999) and Alesina, Gleaser, and Sacerdote (2005). In any case, the complexity of the analysis is dramatically reduced. The utility U_i of agent i is given by:

$$U_i = \mathbf{E}\left[u(C_i, h_i)\right], \tag{2.3}$$

where

$$u(C_i, h_i) = \frac{1}{1-\gamma}\left(C_i - \frac{\epsilon}{\epsilon+1}(h_i)^{\frac{\epsilon+1}{\epsilon}} + \frac{\epsilon}{\epsilon+1}(h_i^*)^{\frac{\epsilon+1}{\epsilon}}\right)^{1-\gamma}, \tag{2.4}$$

where γ represents the parameter of relative risk aversion with respect to total consumption, i.e. physical consumption and leisure. The parameter ϵ represents the labor supply elasticity with respect to the marginal wage rate. Accordingly, a drop in the wage rate at time t by one percent results in a decline in the labor supply level at time t of ϵ percent. Originating from Greenwood, Hercowitz, and Huffman (1988), the specification in equation (2.3) features no income effects in labor supply. Labor-supply decisions are determined solely by the marginal wage rate against which labor is supplied. In the absence of distortions, the marginal wage rate of agent i equals w_i so that the labor choice of agent i is given by:

$$h_i^* = w_i^{\epsilon}. \tag{2.5}$$

The inclusion of the term $\frac{\epsilon}{\epsilon+1}(h_i^*)^{\frac{\epsilon+1}{\epsilon}}$ in the preference specification of equation (2.3) has two attractive implications. First, preferences simplify into standard CRRA utility over consumption C_i if labor supply levels are undistorted or inelastic (i.e. if labor supply is given by equation (2.5)). Second, it holds that for any choice of labor supply elasticity ϵ, relative risk aversion with respect to consumption C_i of agent i will be around γ if labor supply levels are not too far away from the first-best level h_i^*. This property allows me to examine the effects of a change in the labor supply elasticity ϵ under approximate *ceteris paribus* conditions with respect to relative risk aversion γ.

2.2 Autarky

The optimal solution in autarky (i.e. $t(x_1) = 0$ for any x_1) is well known is and repeated here for the sake of completeness. In autarky, labor-supply choices are not distorted and correspond to equation (2.5) so that preferences simplify into standard CRRA utility over consumption. The optimal exposure α to the risk $\tilde{x}_1$ solves from

$$\max_{\alpha}\left\{\mathbf{E}\left[\frac{1}{1-\gamma}(C_1)^{1-\gamma}\right]\right\} = \max_{\alpha}\left\{\mathbf{E}\left[\frac{1}{1-\gamma}(w_1 h_1^* + \alpha\tilde{x}_1)^{1-\gamma}\right]\right\}, \tag{2.6}$$

where labor supply h_1^* of agent 1 is given by equation (2.5). Under the assumption that the portfolio risk is small, the Arrow-Pratt approximation can be applied (see Appendix A):

$$\mathbf{E}\left[\frac{1}{1-\gamma}(w_1 h_1^* + \alpha\tilde{x}_1)^{1-\gamma}\right] \approx \frac{1}{1-\gamma}\left(w_1 h_1^* + \alpha\mu - \frac{1}{2}\frac{\gamma}{w_1 h_1^*}\alpha^2\sigma^2\right)^{1-\gamma}. \tag{2.7}$$

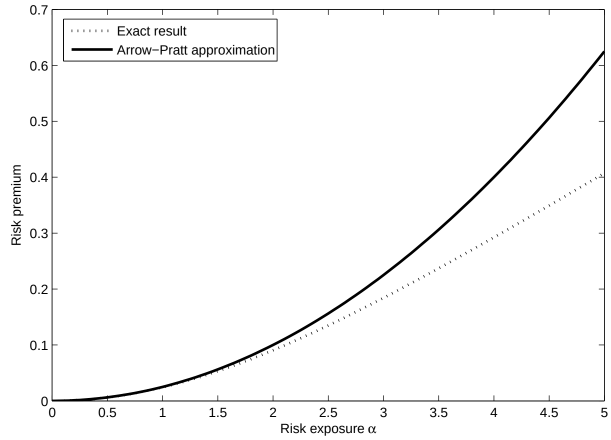


Figure 2.2: *The Arrow-Pratt approximation (solid line) and the exact solution (dotted line) for the risk premium required by an agent with relative risk aversion coefficient* $\gamma = 5$ *and whose earnings are normalized to unity* ($w_1 h_1^* = 1$) *in the situation where the exposure to the risk* $\tilde{x}$ *is equal to* α. *The two possible realizations of* $\tilde{x}$ *are -0.1 and +0.1, both with equal probability, so that* $\mu = 0$ *and* $\sigma = 0.1$. *The exact solution* $g(\alpha)$ *solves the equation* $\mathbf{E}\left[\frac{1}{1-\gamma}\left(w_1 h_1^* + \alpha \tilde{x}_1\right)^{1-\gamma}\right] - \frac{1}{1-\gamma}\left(w_1 h_1^* + \alpha\mu - g(\alpha)\right)^{1-\gamma} = 0$. *The Arrow-Pratt approximation is given by equation* (2.7): $g(\alpha) \approx \frac{1}{2}\frac{\gamma}{w_1 h_1^*}\alpha^2\sigma^2$.

The term $\frac{1}{2}\frac{\gamma}{w_1 h_1^*}\alpha^2\sigma^2$ represents the *risk premium*: the agent is indifferent between paying the risk premium and having an exposure α to a pure risk $\tilde{x} - \mu$. Figure 2.2 illustrates that the Arrow-Pratt approximation is very accurate if the risk exposure is small, but becomes less accurate as the portfolio risk increases. The first-order derivative of equation (2.7) solves the optimal risk exposure α:

$$\alpha^{aut} = \frac{\mu}{\gamma\sigma^2} w_1 h_1^*. \tag{2.8}$$

The agent has an appetite for a positive exposure to equity risk as long as the risk premium is positive ($\mu > 0$) and the agent is not infinitely risk averse ($\gamma < \infty$). If the risk aversion of the agent goes to zero ($\gamma \to 0$), the agent cares only about the expected return so that the risk exposure goes to infinity. Substitution of equations (2.7) and (2.8) into equation (2.6) solves the certainty-equivalent payoff associated with the risk $\tilde{x}_1$:

$$CEQ^{aut} = \alpha^{aut}\mu - \frac{1}{2}\frac{\gamma}{w_1 h_1^*}\left(\alpha^{aut}\right)^2\sigma^2 = \frac{1}{2}\frac{\mu^2}{\gamma\sigma^2} w_1 h_1^*. \tag{2.9}$$

There is a positive welfare gain from the exposure to equity risk as long as the risk premium is positive and the agent is not infinitely risk averse. The welfare gain can be expressed also in terms of the percentage change in the certainty-equivalent consumption level. Substitution of equation (2.8) into equation (2.7) implies that risk taking leads to a percentage increase in the certainty-equivalent consumption level of $\frac{1}{2}\frac{\mu^2}{\gamma\sigma^2}$x100%. Let us apply a quantitative example to this expression. If the average duration of investments of the agent is 30 years, and if stock returns are distributed independent and identically (i.i.d) with a distribution that is close to a lognormal distribution, then it is well-known that the excess mean return over an 30-year period equals 30 times the excess mean return over a 1 year period while the excess volatility over a 30-year period equals $\sqrt{30}$ times the excess volatility over a 1 year period. Assuming a one-year excess mean return and excess volatility of 4.2% and 16.9% respectively[18], their 30-year counterparts are given by $30 \times 0.042 = 1.25$ and $\sqrt{30} \times 0.168 = 0.92$ respectively. Plugging these two values, together with $\gamma = 5$, into the expression above yields an increase in the the certainty-equivalent consumption level of $0.5 \times (1.25^2)/(5 \times 0.92^2)$=18.4%. From this simple calculation we can infer that the welfare gains from risk taking are large for an individual in autarky.

2.3 Lump-sum transfers

The solution for intergenerational risk-sharing with lump-sum transfers is treated by Gollier (2008) and is briefly summarized here for the sake of completeness. Lump-sum transfers do not affect the marginal wage rate against which labor is supplied so that labor-supply choices correspond to equation (2.5) and preferences simplify into standard CRRA utility over consumption. To evaluate the social surplus from risk sharing, let us assume that the two agents decide to share risk $\tilde{x}_1$ together and optimize the total certainty-equivalent payoff (i.e. for the two agents together) that is associated with risk taking in the first period. It turns out that the optimal risk-sharing solution can be Pareto-improving, so that none of the agents becomes worse off from the ex-ante perspective. Following Gollier (2008), let the risk transfer from agent 1 to agent 2 be characterized by a linear function $t(x_1) = t_0 + \eta\alpha x_1$, where α represents the exposure to the risk $\tilde{x}_1$ in period 1. It follows from the Arrow-Pratt approximation in equation (2.7) that the certainty-equivalent payoffs from the the exposure to the risk $\tilde{x}_1$ for agents 1 and 2 are given by:

$$CEQ_1(\alpha,\eta) = -t_0 + (1-\eta)\alpha\mu - \frac{1}{2}\frac{\gamma}{w_1 h_1^*}(1-\eta)^2\alpha^2\sigma^2 \qquad (2.10a)$$

and

$$CEQ_2(\alpha,\eta) = t_0 + \eta\alpha\mu - \frac{1}{2}\frac{\gamma}{w_2 h_2^*}\eta^2\alpha^2\sigma^2. \qquad (2.10b)$$

[18] These values correspond to the parameter values that are used in section 3, enabling me to compare the results of the overlapping generations model to those derived in this section.

Let us assume that the two agents simultaneously decide how much risk to take and how to share it. The optimization problem is then given by:

$$\max_{\alpha,\eta} \{CEQ(\alpha,\eta)\} = \max_{\alpha,\eta} \left\{ \alpha\mu - \frac{1}{2}\frac{\gamma}{w_1 h_1^*}(1-\eta)^2\alpha^2\sigma^2 - \frac{1}{2}\frac{\gamma}{w_2 h_2^*}\eta^2\alpha^2\sigma^2 \right\}. \tag{2.11}$$

Notice that the deterministic transfer t_0 is irrelevant for the optimization problem. In absence of labor-supply distortions, a deterministic transfer between agents does not affect the social surplus from risk sharing so that the term t_0 drops out of the optimization problem. This implies that t_0 can be chosen in such a way that the risk sharing solution is Pareto-improving[19]. The optimal risk sharing rule η^* solve as:

$$\eta^* = \frac{w_2 h_2^*}{w_1 h_1^* + w_2 h_2^*}, \tag{2.12}$$

implying that the equity exposure is allocated according to the relative wealth levels of the two agents. The optimal exposure α to the risk $\tilde{x}_1$ solves as

$$\alpha^* = \frac{\mu}{\gamma\sigma^2}(w_1 h_1^* + w_2 h_2^*) = \alpha^{aut}\left(\frac{w_1 h_1^* + w_2 h_2^*}{w_1 h_1^*}\right). \tag{2.13}$$

Risk sharing increases the demand for the transferrable risk $\tilde{x}_1$ compared to the autarky case by a factor $\frac{w_1 h_1^* + w_2 h_2^*}{w_1 h_1^*}$. For example, the exposure to the risk $\tilde{x}_1$ doubles if the two agents have equal human wealth (in discounted terms). The certainty-equivalent payoff from risk taking increases to:

$$CEQ(\alpha^*,\eta^*) = CEQ^{aut}\left(\frac{w_1 h_1^* + w_2 h_2^*}{w_1 h_1^*}\right). \tag{2.14}$$

If the present discounted value of labor earnings is the same for both agents, the certainty-equivalent return from risk taking increases by 100% as a result of risk sharing. This will be referred to as the *social surplus from risk sharing*. The social surplus is the result of agent 2 being able to take advantage of the risk premium in period 1. The social surplus increases if the unborn agent is more wealthy relative to the agent alive at present.

The welfare gain from intergenerational risk sharing can also be expressed in terms of the percentage change in the certainty-equivalent consumption level: $\frac{\frac{1}{2}\frac{\mu^2}{\gamma\sigma^2} w_2 h_2^*}{\left(1+\frac{1}{2}\frac{\mu^2}{\gamma\sigma^2}\right) w_1 h_1^* + w_2 h_2^*}$x100%. Applying the same parameter values for the return distribution and the parameter of relative risk aversion as in the calculation of section 2.2, risk sharing results in a welfare gain of $(0.5 \times (1.25^2)/(5 \times 0.92^2))/((2 + 0.5 \times (1.25^2)/(5 \times 0.92^2)))$=8.5% if the present discounted value of labor earnings is the same for both agents. From this simple calculation it is inferred

[19]The interval of t_0 for which risk sharing is Pareto improving will be derived in equation (2.15).

that the welfare gains from intergenerational risk sharing are large.

Notice that the risk exposure of agent 1 remains unchanged compared to the autarky case: agent 1 only takes a fraction $1-\eta = \frac{w_1 h_1^*}{w_1 h_1^* + w_2 h_2^*}$ of the total risk exposure that has been increased by a factor $\frac{w_1 h_1^* + w_2 h_2^*}{w_1 h_1^*}$. Thus, the social surplus from risk sharing is fully allocated to agent 2 if t_0 is chosen equal to zero[20]. On the other extreme, the social surplus from risk sharing can be fully allocated to agent 1 by choosing $t_0 = -CEQ^{aut} \frac{w_2 h_2^*}{w_1 h_1^*}$. Risk sharing is Pareto-improving as long as

$$-CEQ^{aut} \frac{w_2 h_2^*}{w_1 h_1^*} \leq t_0 \leq 0. \tag{2.15}$$

2.4 Distortionary transfers

As explained in the introduction, the assumption of lump-sum risk sharing transfers is unrealistic. Let us therefore assume that risk sharing transfers take the form of ex-post taxes or subsidies on labor earnings on the labor earnings of agent 2. The labor earnings of agent 1 remain undistorted.

Similar to the previous section, the transfer from agent 1 to agent 2 takes the form of a linear function $t(x_1) = t_0 + \eta \alpha x_1$ of the realization x_1 of $\tilde{x}_1$. In contrast to the previous section, the transfer t_0 matters for the social surplus since it distorts the labor-supply choices of the agents. To keep the analysis simple, let us set t_0=0, causing the average transfer from agent 1 to agent 2 to be close to zero[21]. It will become clear below that setting t_0=0 implies that risk-sharing is a Pareto-improvement and that the full surplus from risk sharing is allocated to agent 2. Risk sharing transfers are levied upon labor earnings through *proportional* taxes and subsidies. Accordingly, the marginal tax or subsidy on labor earnings is equal to the average tax or subsidy. The marginal tax or subsidy levied upon the labor earnings of agent 2 is thus equal to the absolute size of the transfer divided by the labor earnings of agent 2, i.e. $t(\tilde{x}_1)/w_2 h_2$. The marginal wage rate against which labor is supplied by agent 2 thus equals $w_2(1 + t(\tilde{x}_1)/(w_2 h_2))$, so that equation (2.5) implies that the labor supply h_2 of agent 2 is given by:

$$h_2 = \left(w_2 \left(1 + \frac{t(\tilde{x}_1)}{w_2 h_2} \right) \right)^{\epsilon} = h_2^* \left(1 + \frac{t(\tilde{x}_1)}{w_2 h_2} \right)^{\epsilon}. \tag{2.16}$$

The labor-supply choice h_2 of agent 2 is now a random variable since it depends on the stochastic return $\tilde{x}_1$ on the risky asset in period 1. The labor-supply choice h_2 of agent

[20]The same outcome is obtained in a setting where agent 2 is allowed to trade in the financial market in periods 1 and 2. The outcome can thus not only be viewed as the outcome of a social planner, but also as an equilibrium outcome in markets for risk sharing (see Ball and Mankiw (2007)).

[21]In a more advanced analysis, the parameter t_0 can be determined such that the welfare costs from distortions are minized.

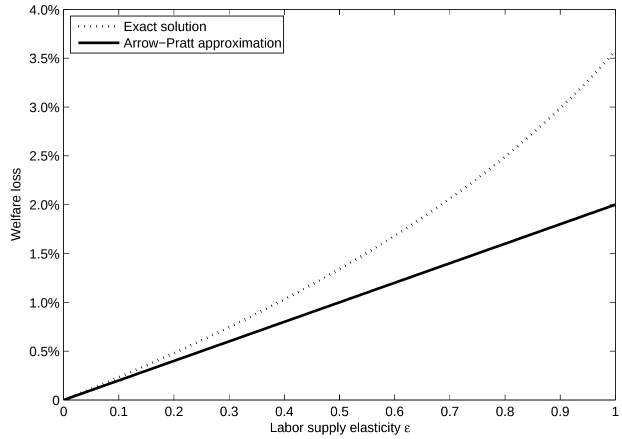


Figure 2.3: *The Arrow-Pratt approximation (solid line) and the exact solution (dotted line) for the welfare loss (as a fraction of undistorted labor earnings* $w_2 h_2^* = 1$*) that results from distortions in labor-supply choices induced by an exposure of 2 the risk* $\tilde{x}_1$ *(i.e.* $\eta\alpha = 2$*). The two possible realizations of* $\tilde{x}$ *are -0.1 and +0.1, both with equal probability, so that* $\mu = 0$ *and* $\sigma = 0.1$*. The relative risk aversion coefficient* γ *is assumed equal to 5. The exact welfare loss* $f(\epsilon)$ *solves* $\mathbf{E}\left[\frac{1}{1-\gamma}\left(w_2 h_2 + \eta\alpha\tilde{x}_1 - \frac{\epsilon}{\epsilon+1}(h_2)^{\frac{\epsilon+1}{\epsilon}} + \frac{\epsilon}{\epsilon+1}(h_2^*)^{\frac{\epsilon+1}{\epsilon}}\right)^{1-\gamma}\right] - \mathbf{E}\left[\frac{1}{1-\gamma}\left(w_2 h_2^* + \eta\alpha\tilde{x}_1 - f(\epsilon)\right)^{1-\gamma}\right] = 0$*, where* h_2 *is given by equation (2.16). The Arrow-Pratt approximation for the welfare loss is given by equation (2.17):* $f(\epsilon) \approx \frac{1}{2}\frac{\epsilon}{w_1 h_1^*}\alpha^2\sigma^2$*.*

2 appears on both sides of equation (2.16) and cannot be solved explicitly. The Arrow-Pratt approximations in this section are therefore derived on the basis of the following approximation of the labor-supply choice:[22]:

$$h_2 = h_2^*\left(1 + \frac{t(\tilde{x}_1)}{w_2 h_2}\right)^{\epsilon} = h_2^*\left(1 + \frac{\eta\alpha\tilde{x}_1}{w_2 h_2}\right)^{\epsilon} \approx h_2^*\left(1 + \frac{\eta\alpha(\tilde{x}_1 - \mu)}{w_2 h_2^*}\right)^{\epsilon}. \qquad (2.16')$$

The approximation in equation (2.17) becomes more accurate as risk transfers are smaller (i.e. if the risk exposure α is small) and if the labor supply h_2 of agent 2 is relatively close to the first-best level h_2^*. Using the approximation for labor-supply choices in equation (2.17), Appendix A shows that an Arrow-Pratt approximation of the expected utility for agent 2 is

[22]Notice that this approximation for labor-supply choices violates the budget constraint for the risk sharing transfer.

given by:

$$\begin{aligned}\mathbf{E}\left[u(C_2,h_2)\right] &= \mathbf{E}\left[\frac{1}{1-\gamma}\left(w_2h_2+\eta\alpha\tilde{x}_1-\frac{\epsilon}{\epsilon+1}(h_2)^{\frac{\epsilon+1}{\epsilon}}+\frac{\epsilon}{\epsilon+1}(h_2^*)^{\frac{\epsilon+1}{\epsilon}}\right)^{1-\gamma}\right] \\ &\approx \frac{1}{1-\gamma}\left(w_2h_2^*+\eta\alpha\mu-\frac{1}{2}\frac{\gamma}{w_2h_2^*}\eta^2\alpha^2\sigma^2-\frac{1}{2}\frac{\epsilon}{w_2h_2^*}\eta^2\alpha^2\sigma^2\right)^{1-\gamma}. \qquad (2.17)\end{aligned}$$

The term $\frac{1}{2}\frac{\gamma}{w_2h_2^*}\eta^2\alpha^2\sigma^2$ represents the risk premium and has been discussed in the previous section. The term $\frac{1}{2}\frac{\epsilon}{w_2h_2^*}\eta^2\alpha^2\sigma^2$ is due to elastic labor supply and represents the welfare loss that results from the labor-supply distortions induced by the risk-sharing transfer.

Under the approximation in equation (2.16'), the welfare costs from distortions are linear in the parameter of labor supply elasticity ϵ. However, Figure 2.3 illustrates that the Arrow-Pratt approximation does a poor job and that the welfare costs from labor-supply distortions are in fact convex. The Arrow-Pratt approximation underestimates the welfare losses from labor-supply distortions because it does not take into account the second-order effects in labor-supply choices that result from the budget constraint: a tax on labor reduces labor supply (the first-order effect) and the resulting reduction in the tax base requires an even higher tax rate (resulting in a second-order effect in labor supply) to prevent the budget constraint from being violated. Figure 2.3 illustrates that this second-order effect causes the welfare costs from distortions to increase *more than proportionally* with the size of distortions, consistent with the intuition of the Harberger triangle. I continue to work with the Arrow-Pratt approximation of equation (2.17), even though we know that it ignores important second order effects for risk compensation (Figure 2.2) and the welfare costs from distortions (Figure 2.3). It will become clear below that both second-order errors cancel out when calculating for the fraction of the social surplus that is eroded by distortions in equation (2.22) (the expression we are most interested in). The Arrow-Pratt approximation in equation (2.17) is thus a very useful one, despite its inaccuracy.

Again, let us assume that the two agents simultaneously decide how much risk to take and how to share it. It follows from equation (2.17) that the optimization problem is given by:

$$\max_{\alpha,\eta}\left\{CEQ(\alpha,\eta)\right\}=\max_{\alpha,\eta}\left\{\alpha\mu-\frac{1}{2}\frac{\gamma}{w_1h_1^*}(1-\eta)^2\alpha^2\sigma^2-\frac{1}{2}\frac{\gamma}{w_2h_2^*}\eta^2\alpha^2\sigma^2-\frac{1}{2}\frac{\epsilon}{w_2h_2^*}\eta^2\alpha^2\sigma^2\right\}. \qquad (2.18)$$

The optimal risk sharing rule η^* becomes:

$$\eta^* = \frac{w_2 h_2^*}{w_1 h_1^* \left(1 + \frac{\epsilon}{\gamma}\right) + w_2 h_2^*}. \tag{2.19}$$

If labor supply is inelastic ($\epsilon = 0$), two agents with equal human wealth share risks equally. If labor supply is elastic ($\epsilon > 0$), this is not the case anymore: agent 1 bears more risk than agent 2. The presence of labor-supply distortions makes it less attractive for agent 2 to bear risks so that it is optimal for two equally wealthy agents to share risks unequally. The optimal equity exposure is given by

$$\alpha^* = \alpha^{aut} \left(\frac{w_1 h_1^* + \frac{1}{1+\frac{\epsilon}{\gamma}} w_2 h_2^*}{w_1 h_1^*} \right) \tag{2.20}$$

and is decreasing in the elasticity of labor supply ϵ. Elastic labor supply thus reduces the appetite for risk taking. As mentioned in the introduction, this result stands in striking contrast to Bodie, Merton, and Samuelson (1992). The exposure to risk is reduced because of two reasons. First, risk taking is accompanied by distortions in labor supply choices, reducing the attractiveness of risky investments. Second, the distortions cause labor supply behavior to become more pro-cyclical, having a destabilizing effect on consumption levels. Substitution of the optimal decision rules yields the surplus from risk sharing:

$$CEQ(\alpha^*, \eta^*) = CEQ^{aut} \left(\frac{w_1 h_1^* + \frac{1}{1+\frac{\epsilon}{\gamma}} w_2 h_2^*}{w_1 h_1^*} \right). \tag{2.21}$$

Equation (2.24) implies that a higher fraction of the social surplus is eroded as labor-supply becomes more elastic. If the present discounted value of labor earnings is the same for both agents, the social surplus from risk sharing is 100% if labor supply is inelastic. If the elasticity of labor supply ϵ increases to 0.5, the surplus drops to 90.9% if the coefficient of relative risk aversion γ equals 5. The fraction of the social surplus that is eroded by distortions is thus 9.1%. More generally, the fraction of the social surplus from risk sharing that is eroded by distortions is given by:

$$\frac{CEQ(\alpha^*, \eta^*)|_{\epsilon=0} - CEQ(\alpha^*, \eta^*)}{CEQ(\alpha^*, \eta^*)|_{\epsilon=0} - CEQ^{aut}} = \frac{\epsilon}{\gamma + \epsilon}. \tag{2.22}$$

Notice that this approximation is *independent* of the distribution parameters μ and σ and independent of the wage levels of the agents. The social surplus from risk sharing is fully preserved if labor supply is inelastic ($\epsilon = 0$). From equation (2.22) we also know that the social surplus is fully eroded if labor supply is infinitely elastic ($\epsilon \to \infty$). Labor-supply distortions are more costly for low levels of the parameter of relative risk aversion γ since these coincide with high levels of risk taking (and thus large risk transfers). If the elasticity of labor supply ϵ equals 0.5, the fraction of the surplus that is eroded by distortions equals $\frac{1}{5}$, $\frac{1}{11}$ and $\frac{1}{21}$ for relative risk aversion levels γ of 2, 5 and 10 respectively.

Figure 2.5 shows that the expression in equation (2.22), which is based on the Arrow-Pratt approximation in equation (2.17), corresponds almost perfectly to the exact value. The ignored second-order effects with respect to the risk premium (Figure 2.2) and the welfare costs from distortions (Figure 2.3) *cancel out* when calculating the expression in equation (2.22). The expression in equation (2.22) is thus not only very simple but also very accurate.

2.5 A suboptimal risk-sharing contract

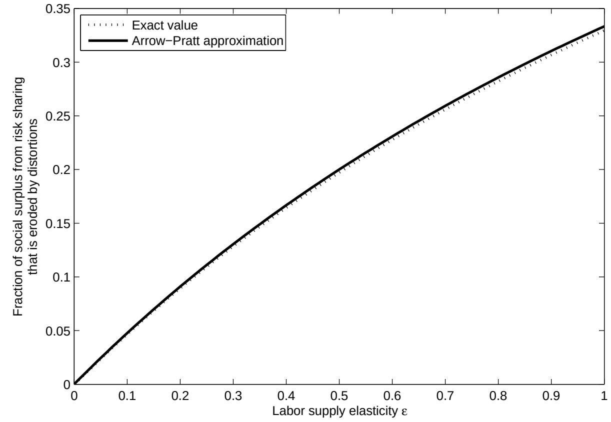
The risk sharing contract was fully optimized in the previous section. In particular, the way in which the gains and losses from risk taking are levied is differentiated between the two agents: non-distortionary lump sum transfers are applied to agent 1 while distortionary taxes and subsidies are applied to agent 2. Pension schemes as they are commonly observed do not feature this type of differentiation. Typically there are no lump sum transfers in a pension fund: all gains and losses from risk taking are levied upon agents through distortionary transfers. This is also what will be assumed in the overlapping generations model in section 3. To be able to compare the results from sections 2 and 3 with each other, let us examine a suboptimal risk sharing contract which applies distortionary transfers only. Thus, a negative (positive) realization x_1 for the return on the risky asset results in a tax (subsidy) on the labor earnings of *both* agents. The optimization problem in equation (2.18) now changes into:

$$\max_{\alpha,\eta} \{CEQ(\alpha,\eta)\} = \max_{\alpha,\eta} \left\{ \alpha\mu - \frac{1}{2}\frac{\gamma+\epsilon}{w_1 h_1^*}(1-\eta)^2\alpha^2\sigma^2 - \frac{1}{2}\frac{\gamma+\epsilon}{w_2 h_2^*}\eta^2\alpha^2\sigma^2 \right\}, \tag{2.23}$$

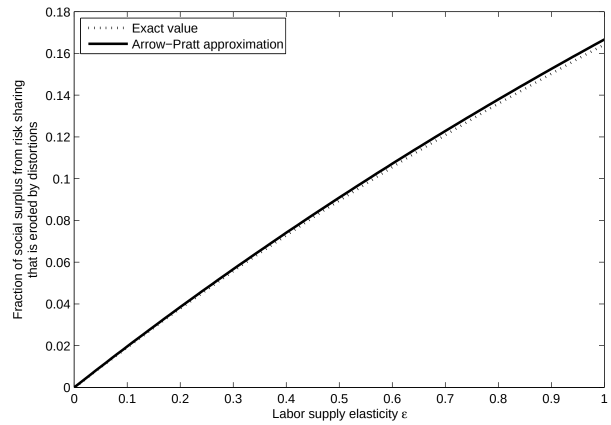
so that the certainty-equivalent payoff from risk taking becomes:[23]:

$$CEQ(\alpha^*,\eta^*) = CEQ^{aut}\frac{\gamma}{\gamma+\epsilon}\left(\frac{w_1 h_1^* + w_2 h_2^*}{w_1 h_1^*}\right). \tag{2.24}$$

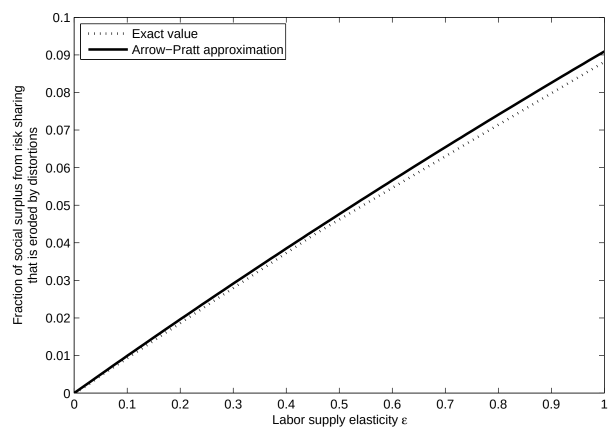
[23]The optimization problem in equation (2.23) is the same as the optimization problem in equation (2.11) in section 2.3, except for the effective relative risk aversion of both agents being equal to $\gamma+\epsilon$ instead of γ.



(a) $\gamma = 2$



(b) $\gamma = 5$



(c) $\gamma = 10$

Figure 2.4: *The exact solution and the Arrow-Pratt approximation for the fraction of the social surplus from risk sharing that is eroded by labor-supply distortions. The realizations of $\tilde{x}$ are -0.08 and +0.12, both with probability 0.5, so that $\mu = 0.02$ and $\sigma = 0.1$.*

The fraction of the social surplus that is eroded by distortions is given by

$$\frac{CEQ(\alpha^*,\eta^*)|_{\epsilon=0} - CEQ(\alpha^*,\eta^*)}{CEQ(\alpha^*,\eta^*)|_{\epsilon=0} - CEQ^{aut}} = \frac{\epsilon}{\epsilon+\gamma}\frac{w_1h_1^* + w_2h_2^*}{w_2h_2^*}. \tag{2.25}$$

Comparing equations (2.22) and (2.25), it follows that the welfare costs from distortions increase by a factor $\frac{w_1h_1^*+w_2h_2^*}{w_2h_2^*}$ as a result of the suboptimal design of the risk sharing contract. That implies that the welfare costs from distortions double if the present discounted value of labor earnings of the two agents are equal. For example, if the parameter of relative risk aversion γ is equal to 5 and the parameter of labor supply elasticity ϵ is equal to 0.5, the sub-optimality of the risk sharing contract causes the welfare loss from distortions to increase from 9.1% to 18.2%. The analysis in section 3 will show that the two-agent framework is not innocuous. Quantitative results for the welfare losses from distortions are substantially larger in an overlapping generations framework: 43%.

The knife-edge case in which the welfare gains from risk sharing exactly equal the welfare costs from distortions is given by

$$\epsilon = \gamma\frac{w_2h_2^*}{w_1h_1^*}. \tag{2.26}$$

If the present discounted value of labor earnings of the two agents are equal, the knife-edge value for ϵ is equal to the parameter of relative risk aversion of the agents. If the discounted value of labor earnings of the unborn agent is small relative to those of the currently-living agent, the welfare gains (time-diversification) from risk sharing are small relative to the welfare costs (distortions). In this situation, the knife-edge value for labor supply elasticity is relatively small. If labor supply becomes more elastic than this knife-edge value, risk sharing becomes welfare decreasing and no Pareto-improving risk sharing solution exists.

3 Overlapping generations

The main virtue of the previous section, its two-agent setting, is also an important limitation. This section evaluates intergenerational risk-sharing in an overlapping generations framework. Section 3.1 introduces the model and section 3.2 describes the autarky problem in which the individual saves and invests on an individual retirement account, in which case the model reduces into the the standard Merton (1969) and Samuelson (1969) model. Section 3.3 discusses the solution in which intergenerational risk-sharing is facilitated by a pension fund.

The model of Gollier (2008) is generalized in three ways. First and most important, the model is extended to the case where labor supply is elastic. Second, the consumption levels of working participants are optimized, instead of being imposed deterministic and constant. Third, the consumption levels of retirees are optimized, instead of being imposed constant for all retirees.

3.1 The model

Consider an overlapping generations model in which each generation works for a period of $n = 40$ years and is subsequently retired for a period of $m = 20$ years. During each period, there are $n + m = 60$ overlapping generations alive. At the beginning of each period, one generation dies and a new generation enters the workforce. Each generation is composed of a fixed number of individuals and this number is normalized to unity. Individuals supply labor during the working period while no labor is supplied during retirement. The annual real wage rate per unit of labor supply, denoted by w, is assumed constant and deterministic and is the same for each generation and in each period in time[24]. The assumption of a deterministic wage rate is consistent with the case of a small open economy in which factor returns are determined on international markets.

The real economy offers two investment opportunities: a riskfree asset and a risky asset. The riskfree asset offers an annual gross return $R = 1.02$. The risky asset offers a return $\tilde{x}_t$ in excess of the risk-free rate in year t. The excess returns $\{\tilde{x}_t\}_{t \geq 0}$ are assumed to be independent and identically distributed and are calibrated to the empirical probability distribution function of the yearly returns on the S&P500 index in excess of the return on US treasury bills over the period 1963-2008. The return distribution features an expected excess return of 4.2% and a standard deviation of 16.9%.

Let the consumption level and the labor-supply level at time t of an individual who enters the labor market at time s be denoted by $C_{s,t}$ and $h_{s,t}$ respectively. Life-time utility U_s of an individual who enters the labor market at time s is given by:

$$U_s = \mathbf{E}_0 \left[\sum_{t=s}^{s+n-1} \beta^{s-t} u\left(C_{s,t}, h_{s,t}\right) + \sum_{t=s+n}^{s+n+m-1} \beta^{s-t} u(C_{s,t}) \right], \tag{3.1}$$

where β represents the subjective time-discount factor of the individual and where the felicity

[24]Benzoni, Collin-Dufresne, and Goldstein (2007) provide empirical evidence that wages and stock returns are cointegrated, and thus highly correlated in the long-run. The effect of long-run correlations between stock and labor markets on the welfare effects of intergenerational risk-sharing are examined in Mehlkopf (2010a).

n	40	Number of working years
m	20	Number of retirement years
R	1.02	Risk free rate
β	1.02^{-1}	Discount factor of individual
γ	5	Parameter of relative risk aversion
ϵ	0.5	Wage-elasticity of labor supply
$\tilde{x}$	Historical distribution	Excess return on stocks

Table 3.1: Default model parameter values.

function u is given by

$$\begin{cases} u(C_{s,t}, h_{s,t}) & = \frac{1}{1-\gamma}\left(C_{s,t} - \frac{\epsilon}{\epsilon+1}\left(h_{s,t}\right)^{\frac{\epsilon+1}{\epsilon}} + \frac{\epsilon}{\epsilon+1}\left(h^*\right)^{\frac{\epsilon+1}{\epsilon}}\right)^{1-\gamma} \\ u(C_{s,t}) & = \frac{1}{1-\gamma}\left(C_{s,t}\right)^{1-\gamma} \end{cases} \tag{3.2}$$

where h^* represents the optimal labor-supply level in absence of distortions to the marginal wage rate:

$$h^* = w^\epsilon. \tag{3.3}$$

As explained in section 2, parameter ϵ represents the wage-elasticity of labor supply[25].. There are no income effects in labor supply. Preferences simplify into time-additive CRRA utility over consumption if labor supply is inelastic or undistorted.

The default model parameters that are used in the remainder of this section are contained in Table 3.1.

3.2 Autarky

As a benchmark I consider the autarky situation where each generation saves and invests for retirement on an individual account. Since preferences, investment opportunities and real wages are constant across time and across individuals, the optimization problem restricts itself to the case of one *single* individual.

3.2.1 Optimization problem

The individual investor enters the labor force at age 25, works up to age 65 and is subsequently retired up to age of 85. During the working period, three decisions have to be made at the beginning of each year: the consumption/savings choice C_{age}, the labor-supply choice h_{age} and the absolute amount amount α_{age} invested in the risky asset. The subscript *age*

[25] In an extension of the model, one could allow the elasticity of labor supply to vary with the business-cycle.

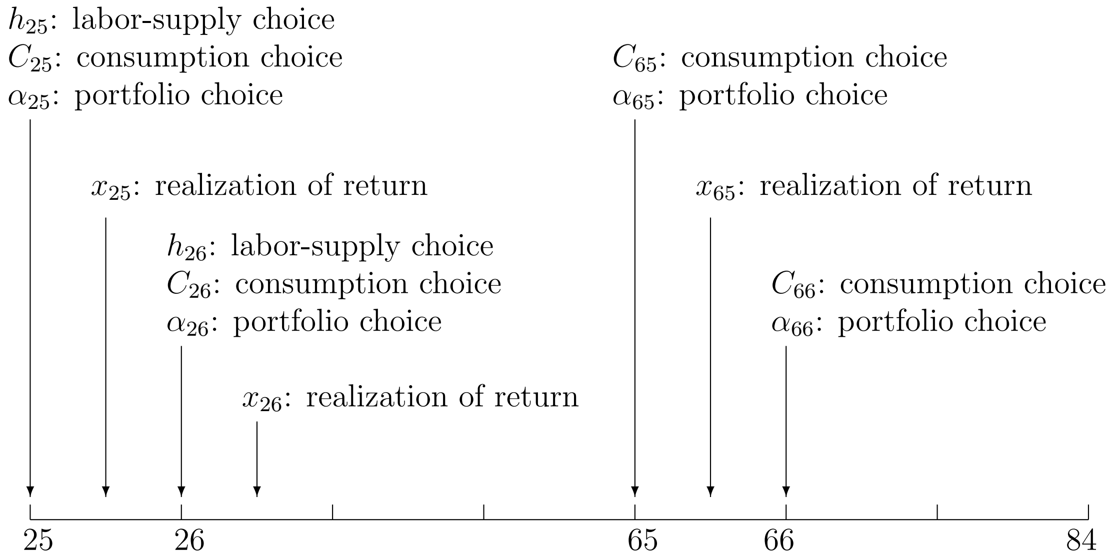


Figure 3.1: *Time-schedule of an individual who saves and invests in an individual retirement savings account.*

indicating the age of the investor. The time-indication is omitted in this section, since the optimization problem in autarky is the same for all cohorts. During the retirement period there is no labor supply anymore, so that only the consumption and investment choices remain.

Figure 3.1 shows the time schedule for individual decision making. The portfolio choice α_{age} is made *before* the return on the risky asset materializes. In autarky, the marginal wage rate against which labor is supplied is undistorted and preferences simplify into standard CRRA utility over consumption. The optimization problem reduces into the well-known problem of Merton (1969) and Samuelson (1969):

$$U = \max_{\alpha,C} \left\{ \mathbf{E} \left[\sum_{age=25}^{84} \beta^{age-25} \frac{1}{1-\gamma} \left(C_{age}\right)^{1-\gamma} \right] \right\}, \tag{3.4a}$$

subject to the budget constraints:

$$W_{age+1} = R\left(W_t + wh^* - C_{age}\right) + \alpha_{age}\tilde{x}_{age} \qquad \text{for } 25 \leq age \leq 64, \tag{3.4b}$$

$$W_{age+1} = R\left(W_{age} - C_{age}\right) + \alpha_{age}\tilde{x}_{age} \qquad \text{for } 65 \leq age \leq 84, \tag{3.4c}$$

$$W_{25} = 0, \tag{3.4d}$$

$$W_{85} = 0, \tag{3.4e}$$

where W_{age} denotes the wealth level in the individual savings account.

3.2.2 Solution

The solution of the dynamic consumption-investment problem equation (3.4) is well-known since Samuelson (1969) and Merton (1969) and is solved in Appendix A.3. The optimal consumption choice C_{age} is characterized by consumption smoothing. Unexpected wealth shocks from risk taking are levied proportionally equally over all remaining consumption levels in the life cycle. Formally it holds that (see Appendix A.3):

$$\frac{d\ log\left(C_s\right)}{d\ log\left(W_{age}+H_{age}\right)}=1 \tag{3.5}$$

for all $25 \leq age < 85$ and for all $age \leq s < 85$. Accordingly, a drop in total wealth by $y\%$ percent results in a decline in all remaining consumption levels by $y\%$ percent, rather than spending it in a few periods. This argument has been proposed by Bovenberg, Nijman, Teulings, and Koijen (2007) to justify the optimality of hybrid pension systems that adjust both contributions and benefits in response to income and wealth shocks. Pension plans that keep contributions fixed (a defined-contribution system) or plans that fix the benefits (a defined-benefit system) are not optimal in their view. The optimal amount α_{age} invested in stocks is given by

$$\alpha_{age}=\eta R\left(W_{age}+H_{age}-C_{age}\right), \tag{3.6}$$

for all age, where H_{age} represents the human wealth of the investor and is defined as the present value of future labor earnings: $H_{age}=\sum_{i=age}^{65-age} R^{-i}wh^*$ if $age < 65$ and $H_{age}=0$ if $age \geq 65$. The scalar η is the unique positive solution of the equation $\mathbf{E}\left[\tilde{x}(1+\eta\tilde{x})^{-\gamma}\right]=0$. For the default parameters and the historically calibrated stock return distribution, parameter η equals 0.2712. Appendix A.4 shows that η adopts an explicit expression in the special case where $\tilde{x}$ is lognormally distributed[26]:

$$\eta \approx \frac{\mu}{\gamma\sigma^2}, \tag{3.7}$$

The historically calibrated stock return distribution is approximately lognormally distributed, so that the approximation in equation (3.7) performs well: the approximation gives yields $\eta \approx 0.042/(5*0.169^2)=0.2935$. Substitution of equation (3.7) into (3.6) implies that the

[26] The approximation applies a log-linearization to the portfolio return as in Campbell and Viceira (2002). In the limit of continuous-time with continuous paths for asset prices, the approximation in equation (3.7) is exact.

portfolio allocation rule is essentially the same as in the two-agent setting of section 2. The optimal amount invested in stocks is a constant share η of total wealth, which equals the sum of the financial wealth W_{age} and human wealth H_{age} for a in the life-cycle investor[27] [28]. As pointed out by Samuelson (1963), there is no time-diversification in the system with individual retirement accounts. The amount invested in equity solely depends on the wealth (human wealth plus financial wealth) of the investor, and thus not on the investment horizon. The portfolio risk taken in the first year is not diversified away by the portfolio risk taken over the $n+m-1$ remaining years. Expression (3.6) implies that it is usually optimal to reduce the share of current financial wealth W_{age} invested in equity when approaching retirement age, since human wealth H_{age} depreciates over the working life. This argument has been proposed by Bodie, Merton, and Samuelson (1992) to justify the standard recommendation to reduce portfolio risk as one approaches the retirement age.

Figure 3.2 represents a graphical illustration of the solution[29]. The solid lines show the 5%, 50% and 95% percentiles of the individual's consumption level C_{age}, the wealth level W_{age}, the amount α_{age} invested in the risky asset and the (constant) labor supply level h^*. Since shocks are smoothed out over the remaining life-cycle (see equation (3.5)), the confidence interval for consumption diverges over the life-cycle. More shocks are passing by as the investor grows older, causing consumption levels to become more volatile.

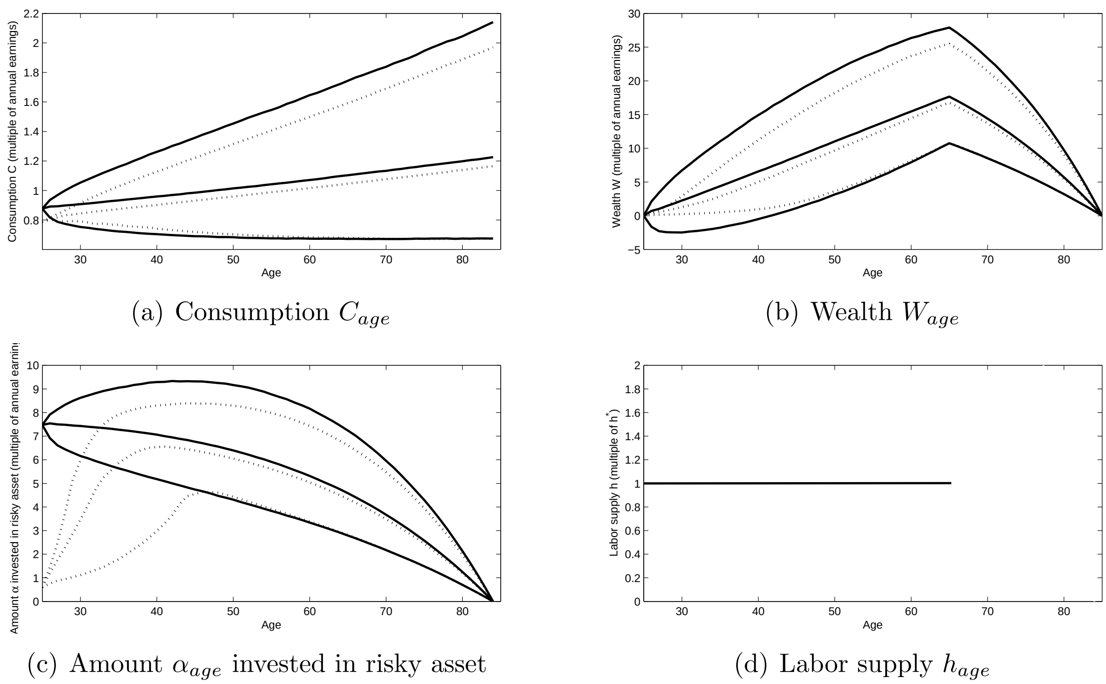
The welfare gain from risk taking is expressed in terms of the percentage change in the certainty-equivalent consumption level[30]. Under the optimal decision rules, the certainty-equivalent consumption level equals 0.901 for the default parameters. In absence of invest-

[27] Due to the discrete character of the model, the amount invested in stocks is not exactly equal to η times the total wealth of the individual, since consumption at time t is subtracted and next-period riskfree interest gain on total wealth is added before multiplication by η. In the continuous-time limit of the model these two terms disappear.

[28] In the setting of Gollier (2008) where the savings rate during the working period is constant and deterministic over time, the expression for the optimal portfolio choice of an individual investor is the same, except that the discounted value of future labor earnings H_t is replaced by the discounted value of deterministic future *savings* during the remaining working period. Since future savings are a lot smaller than future earnings, the risk bearing capacity in the analysis of Gollier (2008) is heavily reduced compared to this paper.

[29] During the beginning of the life-cycle, the amount α_{age} invested in stocks exceeds the financial wealth level W_{age}. This implies that the first-best solution strategy requires the individual to be able to borrow against future labor earnings when folowing the optimal solution strategy. Due to problems with moral hazard and adverse selection, borrowing against human wealth is not possible in real-life situations. The dotted lines in Figure 3.2 show the optimal solution in presence of a borrowing constraint (i.e. $W_{age} \geq 0$ for all age). The borrowing constraint reduces the risk bearing capacity of the individual in the first half of the working period during which financial wealth levels are relatively low. The borrowing constraint causes the certainty-equivalent consumption level to fall by about 2%. This welfare loss is small compared to the other welfare effects discussed in this paper.

[30] Formally, the certainty-equivalent consumption level C^{ce} is defined as the solution of the equation: $U \equiv \sum_{age=25}^{84} \beta^{age-25} u\left(C^{ce}\right)$



(a) Consumption C_{age}

(b) Wealth W_{age}

(c) Amount α_{age} invested in risky asset

(d) Labor supply h_{age}

Figure 3.2: *The 5%, 50% and 95% percentiles of the individual's consumption level C_{age}, the wealth level W_{age}, the amount α_{age} invested in the risky asset and the labor supply level h^* in the absence (solid lines) and in the presence (dotted lines) of a borrowing constraint. The calculations are based upon the default model parameters in Table 3.1. The optimization problem under a borrowing constraint does not have an analytical solution and is be solved numerically by using backward induction, discretization of the state space and cubic interpolation.*

ments in the risky asset (i.e. α_{age}=0 for all age), consumption is constant at a level of 0.787. This implies that the welfare gain from risk taking is equal to (0.901-0.787)/0.787=14.4%[31] .

3.3 Risk Sharing

This section treats the case where intergenerational risk-sharing between non-overlapping is facilitated by a pension fund.

3.3.1 Optimization problem

The overlapping generation framework features $n+m$ overlapping generations in each period of time. At the beginning of each year t, the pension fund collects contributions from the n working generations and pays out benefits to the m retired generations. Let the value of the assets of the pension fund be denoted by W_t. The contribution rate π_t denotes the fraction of labor earnings pledged by workers to the pension fund at time t and is defined as:

$$\pi_t = \pi(W_t), \tag{3.8}$$

where $\pi(\cdot)$ is a time-independent policy function of the pension fund that governs the relationship between contribution rates and the value of pension fund assets W_t. Let benefit level $b_{s,t}$ denote the absolute amount of benefits received from the pension fund at time t by a retired individual who entered the pension fund at time s (s+n≤t<s+n+m). Benefit levels are assumed to be a function of the value of pension fund assets W_t and past labor supply $h_{s,s+i}$ ($0 \leq i < n$):

$$b_{s,t} = b(W_t) \sum_{i=0}^{n-1} \left(\frac{R^{-i}}{\sum_{j=0}^{n-1} R^{-j}} \frac{h_{s,s+i}}{h^*} \right), \tag{3.9}$$

where $b(\cdot)$ is a time-independent policy function of pension fund that governs the relationship between benefit levels $b_{s,t}$ and the value of pension fund assets W_t. The variable $h_{s,t}$ denotes the labor supply level at time t of an individual who started working at time s (s≤t<s+n). The benefit rule in equation (3.9) is characterized by three properties. First, the benefit

[31] This result is in the same order-of-magnitude as the 18.4% welfare gain that was obtained in the stylized two-agent setting of section 2.2. Recall that the calculation in the two-agent setting was based upon an investment duration of 30 years. This duration approximates the duration of investments in this section quite well, since the average saving is made in the middle of the working period (at time $t = 20$) while the average dissaving is made 30 years later in the middle of the retirement period (at time $t = 50$).

rule in equation (3.9) simplifies into $b_{s,t} = b(W_t)$ if labor supply is undistorted or inelastic (i.e. if $h_{s,t} = h^*$ for all s, t). As the second characteristic, the benefit level of a retiree is proportional to all the labor supply levels in his or her working period. This implies that at any time during the working period, the value of pension entitlements that is accrued is proportional to the number of hours worked. As the third characteristic, labor supply in the early working life results in more pension benefits than labor supply later in the working life. This property reflects the time-value of money. Compared to labor supplied at time $i + 1$, labor supplied at time i yields more[32] pension benefits during retirement by a factor R.

The amount α_t invested in stocks by the pension fund at time t is defined as:

$$\alpha_t = \alpha(W_t), \tag{3.10}$$

where $\alpha(\cdot)$ is a time-independent policy function of the pension fund that governs the relationship between the asset management and pension wealth W_t.

The policy specification of the pension fund in equations (3.8), (3.9) and (3.10) are restrictive in two ways. First, the pension fund does not differentiate contributions, benefits and portfolio choices with respect to cohorts. This assumption is consistent with commonly observed policies policies that are simple but implementable. In practice, it is difficult to differentiate pension policies with respect to cohorts due to political or legal constraints[33]. The second way in which the pension fund policy is restrictive is that the functions π, b and α depend on the value of pension fund assets W_t only, while past labor-supply choices are also state variables of the model as they determine future benefit levels. Appendix (A.6) shows that both restrictions are not binding in the special case where labor supply is inelastic. However, in the case with elastic labor supply these restrictions on the policy design become binding. Mehlkopf (2010b) solves the optimal policy of a pension fund under elastic labor supply in absence of these restrictions.

The time-schedule for decision making is illustrated in Figure 3.3. Together, the benefit rule $b(\cdot)$, the contribution rule $\pi(\cdot)$ and the investment rule $\alpha(\cdot)$ form the time-invariant policy rules of the pension fund that is announced at the initial time $t = 0$. At the beginning of every time $t \geq 0$, the working cohorts determine their labor-supply choices $h_{t-n+1,t}$, ..., $h_{t,t}$.

Let $C_{s,t}$ define the consumption level at time t of a participant who started working at time s ($s \leq t < t+n$). My analysis does not allow individuals to save or invest outside the

[32]Alternatively, one can apply the stochastic discount factor of the individual to discount future pension benefits, as in the specification for the utility value of pension entitlements in equation (3.13). The stochastic discount factor of the individual follows endogenously from the model.

[33]The differentiation of pension fund policies with respect to cohorts is referred to as 'generational accounting' and has been introduced by Teulings and de Vries (2006).

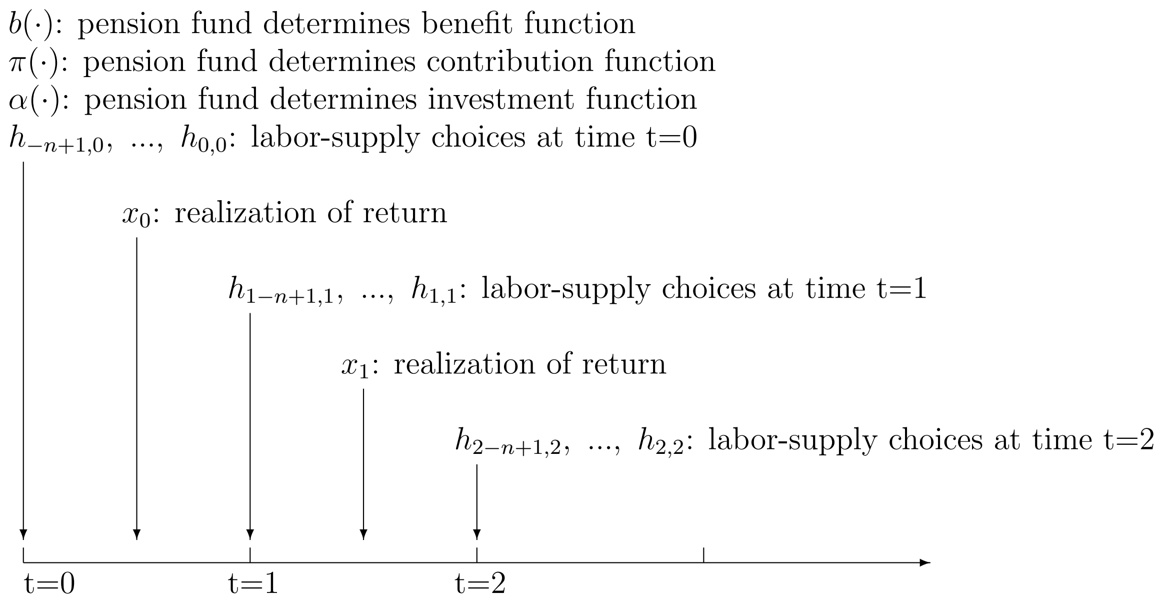


Figure 3.3: *Time-schedule of decision making in the pension fund in the overlapping generations model.*

pension fund, implying that workers consume labor earnings minus pension contributions whereas retirees consume their pension benefits:

$$C_{s,t} = \begin{cases} (1-\pi_t)\, w h_{s,t} & \text{if } s \leq t < s+n. \\ b_{s,t} & \text{if } s+n \leq t < s+n+m. \end{cases} \tag{3.11}$$

Labor-supply choices $h_{s,t}$ follow from the first-order derivative of expected utility U_s (as defined in equation (3.1)) with respect to the labor-supply choice. Appendix A.5 shows that:

$$h_{s,t} = \left(w - w\pi_t + w\psi_{s,t}\right)^{\epsilon} = h^*(1-\pi_t+\psi_{s,t})^{\epsilon}, \tag{3.12}$$

where $w\psi_{s,t}$ is given by[34]:

$$w\psi_{s,t} = \frac{R^{s-t}}{\sum_{j=0}^{n-1} R^{-j}} \frac{1}{h^*} \sum_{i=s+n+1}^{s+n+m} \beta^{i-t} \mathbf{E}_t \left[\frac{u'(C_{s,i})}{u'(C_{s,t}, h_{s,t})} b(W_i) \right]. \tag{3.13}$$

The variable $\psi_{s,t}$ is referred to as the 'accrual rate', as $w\psi_{s,t}$ represents the utility value[35] of

[34]In equation (3.13), $u'(C_{s,t}, h_{s,t}))$ is a short notation for $\left(C_{s,t} - \frac{\epsilon}{\epsilon+1}\left(h_{s,t}\right)^{\frac{\epsilon+1}{\epsilon}} + \frac{\epsilon}{\epsilon+1}\left(h^*\right)^{\frac{\epsilon+1}{\epsilon}}\right)^{-\gamma}$

[35]Notice that the expression in equation (3.13) represents the *utility* value of pension accruals, not the

accrued pension entitlements per unit of labor supply at time t of an individual who entered the labor force at time s. In equation (3.13), the term $b_i(W_i)\left(\frac{R^{s-t}}{\sum_{j=0}^{n-1} R^{-j}}\frac{1}{h^*}\right)$ represents the increase in the benefit level at time i in the retirement period that results from an additional unit of labor supply at time t in the working period. The ratio $\frac{u'(C_{s,i})}{u'(C_{s,t},h_{s,t})}$ represents the stochastic discount factor that yields the utility value at time t of a unit-increase in consumption at time i. Equation (3.12) states that labor-supply choices are fully determined by effective wage rate of a pension fund participant, which is the wage rate w offered by the employer minus pension contributions $w\pi_t$ plus the utility value of pension accruals $w\psi_{s,t}$ [36]. The pension fund distorts labor supply choices if the contribution rate deviates from the accrual rate ($\pi_t \neq \psi_{s,t}$) and if labor supply is elastic ($\epsilon > 0$). A net tax is levied upon labor earnings if the contribution rate exceeds the accrual rate (i.e. $\pi_t > \psi_{s,t}$), a net subsidy is provided in the opposite case. Equation (3.12) implies that the contribution rate should be equal to the accrual rate at all times in the case where labor supply elasticity is infinitely elastic ($\epsilon \rightarrow \infty$).

Finally, we have to determine how the pension fund is initialized at time t=0. For this, I adopt the approach of Gollier (2008), taking the perspective of a pension reform at time t=0. The reform is implemented as follows. Suppose that we are in the model of section 3.2 where there are n+m generations saving and investing on an individual account according to the optimal decision rules (in absence of a borrowing constraint) as specified in equations (3.5) and (3.6). At a certain point in time, normalized to t=0, there is a pension reform and all generations agree to transfer the wealth in their individual savings accounts to a pension fund that would be allowed to reallocate risks across generations. Let Y_0 denote the value of this initial transfer. In return for their initial transfer, the generations that are alive during the transition date receive pension accumulations according to equation (3.9) for the labor (at the autarky level h^*) that has been supplied by these generations in the past. The value of the variable Y_0 depends on the wealth in the individual accounts of the generations that are alive at the time the transition. There are thus many different scenarios for the reform possible, depending on the realized returns on investments during the $n+m$-year period preceding

market value. The market value of pension accruals is not uniquely defined in the model, since the dimension of the next-period state space (the 46 outcomes from the empirically calibrated stock returns from the S&P500 data in the period 1963-2008) is larger than the number of assets in the economy (two).

[36] In practice, an increase in the contribution rate can have a different labor-supply response than a decline in the pension accrual rate. After all, a cut in pension contributions affects the disposable income of households, whereas a cut in pension rights will not have an impact on consumption levels until retirement. People may not even be aware of the benefit policy of the pension fund. However, empirical evidence shows that old workers are very well aware of the financial incentives of their pension plan when making their retirement decisions. Younger workers on the other hand will have a lower level of awareness. This implies that it can be attractive to differentiate policy instruments across age. Also, benefit cuts may be a more attractive way to recover from financial losses than a decrease in the contribution rate.

the transition. Let us consider only one specific scenario for the reform, namely the scenario in which the size of the transfer Y_0 is equal to the unconditional expectation of the wealth in the individual savings accounts. By focussing on this single case, the variable Y_0 becomes *deterministic* and equals $Y_0 = 573wh^*$ for the default parameters, where wh^* represents annual undistorted labor earnings. Thus, the initial value of assets ($W_0 = Y_0 = 573wh^*$) is about 14 times as large as the annual earnings of working participants ($nwh^* = 40$)[37] [38].

The optimization problem of the pension fund is given by:

$$U = \max_{\alpha,\pi,b} \mathbf{E}_0 \left[\sum_{t=0}^{\infty} \left\{ \delta^t \left(\sum_{i=0}^{n-1} u\left(C_{t-i,t}, h_{t-i,t}\right) + \sum_{i=n}^{n+m-1} u(C_{t-i,t}) \right) \right\} \right], \tag{3.14a}$$

subject to

$$W_{t+1} = R\left(W_t + \sum_{i=0}^{n-1} \pi_t w h_{t-i,t} - \sum_{i=n}^{n+m-1} b_{t-i,t} \right) + \alpha_t \tilde{x}_t, \tag{3.14b}$$

$$W_0 = Y_0, \tag{3.14c}$$

$$W_t > -K \qquad \text{for all } t, \tag{3.14d}$$

where felicity function u is specified in equation (3.2). Parameter δ represents the time-discount rate of the social planner (i.e. the pension fund) to weight the importance of consumption at different points in time. The parameter δ is set such that all generations born after the transition at time t=0 are equally well off from an ex-ante point of view[39].

The constraint in equation (3.14d) is adopted from Gollier (2008), where parameter K is a scalar that represents the present discounted value of future labor earnings under inelastic

[37]This number of roughly consistent with real-life observations. For example, the ABP Pension Fund for Dutch civil servants had 216 billion Euro in assets at the end of 2007. During 2007, it received 6.7 billion in contributions while applying a contribution rate of 19%. The total wage earnings of participants were thus equal to 6.7/0.19=35 billion, implying that assets are 216/35=6.2 times labor earnings. Given that the Dutch pension system is roughly 50% funded and 50% social security (third-pillar private pension savings are relatively small in the Netherlands), assets would have been 12.4 times annual labor earnings if the pension system were fully funded.

[38]In absence of investments in the retirement period and with a fixed savings rate, as in Gollier (2008), the value of the transfer at the date of the reform becomes much smaller: $296wh^*$.

[39]This definition for parameter δ implies that the ex-ante welfare level of the generations that are alive at the time of the transition is inbetween the autarky-welfare level and the welfare level of pension fund participants. More precisely, the generations that are young during the time of transition receive a relatively large fraction of the social surplus from risk sharing, as they spend a relatively long period of their life in the pension fund. They are almost as good off as generations that enter after the transition. On the other hand, the generations that are old at the time of transition spend only a few years in the pension fund, being only slightly better off compared to autarky. Of course, it is possible to give generations alive during the transition a larger share of the social surplus. This can be achieved by providing them more pension entitlements at the time of transition.

labor supply: $K = \sum_{i=0}^{\infty} R^{-1} nwh^* = \frac{R}{R-1} nwh^*$. Equation (3.14d) states that the pension fund can potentially collateralize the future labor earnings of pension fund participants, so that financial wealth levels are allowed to go negative. For the default parameters, the value of future labor earnings $K = 2040wh^*$. The budget constraint in equation (3.14d) is non-binding for the default parameters of the model: the shadow costs from labor elasticity bind stronger than equation (3.14d).

Discounted future labor earnings K consist of the discounted labor earnings of currently-living generations, denoted by K_1, and those of the unborn generations, denoted by K_2 ($K = K_1 + K_2$). For the default parameters we obtain $K_1 = \sum_{i=0}^{n-1} \sum_{j=0}^{i} R^{-j} = 645wh^*$ and $K_2 = K - K_1 = 2040 - 645 = 1395wh^*$, implying that the unborn generations own the majority of the discounted value of future labor earnings. At time t=0, the wealth of currently-living generation equals $Y_0 + K_1 = 573 + 645 = 1218wh^*$ whereas the unborn generations own $K_2 = 1395wh^*$. In terms of present discounted values, the wealth of currently-living cohorts is thus approximately equal to the wealth of all unborn cohorts, consistent with the assumption made for the numerical calculations in section 2.

3.3.2 Solution in case of inelastic labor supply $\epsilon = 0$

With inelastic labor supply, preferences of individuals simplify into standard CRRA preferences over consumption only and the optimization problem adopts an analytic solution[40]. Inelastic labor supply implies that the benefit specification in equation (3.9) reduces into $b_{s,t} \equiv b_t = b(W_t)$ and thus becomes cohort-independent: at any time t all retired individuals have the same benefit level since they have supplied labor at the same undistorted level h^* during their working periods. Appendix A.6 derives the optimal consumption rule and shows that the consumption levels of working participants are equal to those of retired participants at each point in time (i.e. $C_{s,t} \equiv C_t$) and given by:

$$(1 - \pi_t)\, wh^* = b_t \equiv C_t, \tag{3.15}$$

for all t and for all $s \leq t < s + n + m$. Similar to the autarky consumption rule in equation (3.5), the consumption rule of the pension fund in the equation (3.15) is characterized by consumption smoothing. Unexpected wealth shocks from risk taking are levied proportionally equally over all future time periods. Formally, it holds that (see Appendix A.6):

[40] The analytical solution presented in this section holds for any choice for the time-discount parameter δ, not only the value for δ that ensures that utility in all periods of time is of equal importance.

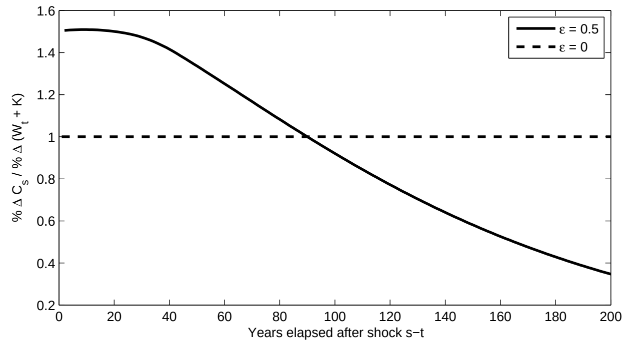


Figure 3.4: *The effect of a wealth shock $\Delta(W_t + K)$ at time t on consumption levels for all $s \geq t$. Consumption levels at time s for the case with elastic labor supply represent the average consumption level of workers: $\frac{1}{n}\sum_{i=0}^{n-1}(1-\pi_s)wh_{s-i,s}$.The dotted line refers to the case with inelastic labor supply ($\epsilon = 0$) while the solid line refers to elastic labor supply ($\epsilon = 0.5$). Calculations are based upon the default parameters in Table 3.1.*

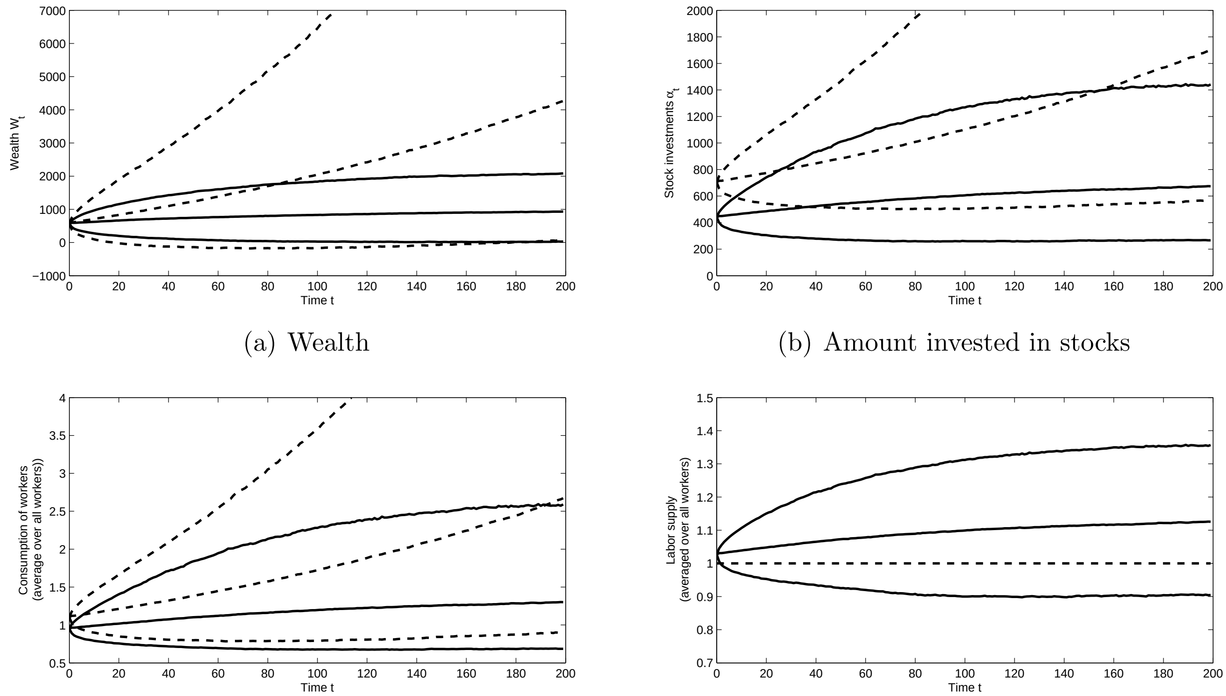
$$\frac{d\ log\,(C_s)}{d\ log\,(W_t + K_t)} = 1 \tag{3.16}$$

for all $s > t$, where K represents the present discounted value of future labor earnings. Accordingly, a drop in total wealth $W_t + K$ by $y\%$ percent results in a decline in consumption in all future periods by $y\%$ percent, rather than being absorbed in a few periods. Equation (3.16) is graphically represented by the dashed line in Figure 3.4.

Notice from equation (3.15) that consumption levels are uniform in age. This solution property is rather convenient, as it is difficult for pension funds to let ex-post welfare levels of one group of individuals (say, retired people) deviate substantially from the ex post welfare level of another group (say, working people). Such deviations could potentially result in an intergenerational conflict, even if all generations are equal in ex-ante terms. Appendix A.6 shows that the optimal investment rule of the pension fund is given by:

$$\alpha_t = \eta R\,(W_t + K - (n+m)C_t)\,, \tag{3.17}$$

where η is defined in equation (3.6) and is approximated by equation (3.7). The portfolio rule is consistent with the result in the two-agent setting in equation (2.13). The optimal amount invested in the risky asset at time t is a time-invariant fraction η of the wealth of currently-living generations $W_t + K_1$ and unborn generations K_2. As pointed out by Gollier (2008), intergenerational risk-sharing thus does not reduce risk if the portfolio choice is endogenous. Comparing equations (3.6) and (3.17), it follows that risk sharing increases the risk-bearing capacity by the relative wealth of the unborn generations. Given that currently-living and



(a) Wealth

(b) Amount invested in stocks

(c) Consumption

(d) Labor supply

Figure 3.5: *The simulated 5%, 50% and 95% quantiles for pension fund assets W_t, the amount α_t invested in risk assets, the consumption level and the labor supply level. The dotted lines refer to the case with inelastic labor supply ($\epsilon = 0$) while the solid lines refer to elastic labor supply ($\epsilon = 0.5$). For inelastic labor supply, the quantiles for consumption at time t represent C_t while for elastic labor supply they represent the average consumption level of workers: $\frac{1}{n}\sum_{i=0}^{n-1}(1-\pi_t)wh_{t-i,t}$. For inelastic labor supply, the quantiles for labor supply at time t represent h^* while for elastic labor supply they represent the average labor supply level of workers: $\frac{1}{n}\sum_{i=0}^{n-1}h_{t-i,t}$. Calculations are based upon the default parameters in Table 3.1.*

unborn generations are approximately equally wealthy, the amount invested in stocks roughly doubles at the date of the transition at time t=0. In absolute terms, the amount invested in stocks increases from $319wh^*$ to $704wh^*$. The initial amount invested in stocks by the pension fund at time t=0 exceeds the amount of pension fund assets (α_0/W_0=123%). The pension fund thus has to borrow against the riskfree rate to implement the optimal investment strategy.

The dashed lines in Figure 3.5 show the modeling outcomes in terms of the 5%, 50% and 95% quantiles for pension fund assets W_t, the amount α_t invested in risk assets, the contribution rate π_t, the benefit level b_t and the (constant) labor supply level h^* for the first 200 years after the transition. Similar to the autarky situation, consumption and wealth levels are diverging over time, which is again due to the consumption-smoothing property of

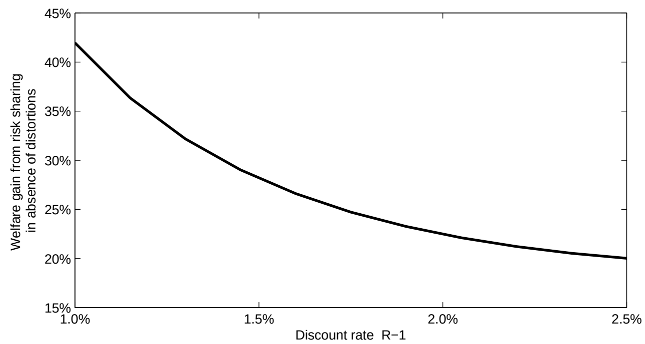


Figure 3.6: *the welfare gain from intergenerational risk-sharing as a function of the discount rate R. In the model, the riskfree rate and the discount rate are equal to each other, so that the riskfree rate changes as well. The subjective discount factor β of the individual remains unaltered.*

the solution. Financial shocks are smoothed proportionally equally all future consumption levels, causing consumption and wealth to adopt a random walk.

Under the optimal intergenerational risk sharing rules, the certainty-equivalent consumption level is equal to 1.103, which corresponds to an increase of (1.103-0.9005)/0.9005=22.5% in comparison to autarky[41]. Figure 3.6 shows the sensitivity of the welfare gain from risk sharing with respect to the discount rate R. Lowering the discount rate from 2% to 1% causes the welfare gain from risk sharing to increase from 22.5% to 41.9%. Recall that the two-agent setting provided the insight that the welfare gains from risk sharing become larger if the relative wealth of unborn generations increases. A lower discount rate causes the present discounted wealth of unborn generations to become higher relative to the wealth of currently-living generations.

3.3.3 Solution in case of elastic labor supply $\epsilon > 0$

In the case of elastic labor-supply, the optimization problem does not adopt an analytical solution and is solved by using numerical solution techniques described in appendix B. Monte-Carlo simulations are used to calculate welfare levels. The accrual rate $\psi_{s,t}$ takes the form of a conditional expection and is derived by applying across-path regressions to the simulation paths. The technique of across-path regressions has been applied by Longstaff

[41]The large welfare gain from risk sharing is consistent with the intuition in the two-agent setting in section 2.3, which found a (sizable, although smaller) 8.5% welfare gain. Recall that the calculation in section 2.3 assumed the present discounted wealth of the currently living agent to be equal to the wealth of the agent alive at present. Indeed, this is roughly the case in the overlapping generations model, since we have that $Y_0 + K_1 = 1218wh^*$ and $K_2 = 1395wh^*$.

and Schwartz (2001) in the context of option pricing and by Brandt, Goyal, Santa-Clara, and Stroud (2005) in the context of dynamic consumption-portfolio choice. The optimal policy functions π, b and α are solved by using a a grid search algorithm. I impose all three functions to be *linear*, so that there are six policy parameters to be solved. The restriction of linear policy rules is not binding in the case of inelastic labor supply, but becomes restrictive in the general case with elastic labor supply.

The solid lines in Figure 3.5 show the modeling outcomes in terms of the 5%, 50% and 95% quantiles for pension fund assets W_t, the amount α_t, the consumption levels and the labor supply levels. Consumption levels represent the average consumption level of workers: $\frac{1}{n}\sum_{i=0}^{n-1}(1-\pi_t)wh_{t-i,t}$ while labor supply represents the average labor supply of workers: $\frac{1}{n}\sum_{i=0}^{n-1} h_{t-i,t}$. Figure 3.5 shows that consumption is a mean-reverting process that yields a stationary distribution in the long-run. The intuition for this result is that the random-walk property for consumption is unsustainable in the case of elastic labor supply. If consumption follows a random walk then wage-differentials follow a random walk as well, implying that the welfare costs from labor-supply distortions diverge as time progresses. Diverging wage-differentials lead to large distortions in labor markets, so that the pension fund departs from the perfect consumption smoothing principle. The pension fund faces a trade-off between consumption smoothing on the one hand and reducing distortions in labor markets on the other hand. The solid line in Figure 3.4 shows that nearby consumption levels are more elastic with respect to a current financial shock relative to far-away consumption levels if labor supply is elastic. Financial shocks are thus levied primarily upon currently-living generations, implying that labor-supply effects impede the the pension fund from taking advantage of intergenerational risk-sharing.

Figures 3.5 shows that the amount invested in stocks at time $t = 0$ drops from from $704wh^*$ to $440wh^*$. This corresponds to a drop in the fraction of financial wealth W_t allocated to stocks (α_0/W_0) from 123% to 77%. There are two effects that explain the decline in the exposure to stock market risk. The first-order effect is that risk taking comes at a price in the form of distortions in labor supply decisions. As a second-order effect, substitution effects in labor supply induced by the wage differentials cause labor supply to become pro-cyclical: the effective wage rate of a pension fund participant falls after a negative wealth shock (when labor earnings are taxed), while increasing after a positive shock. Labor earnings thus to become *more positively correlated to stock returns*, reducing the appetite for risk taking in the investment portfolio.

The risk sharing solution can be welfare improving for all generations from an ex-ante perspective, but some unlucky generations may be worse off ex-post if a succession of negative

	inelastic labor supply	elastic labor supply
Pr[welfare loss > 10%]	8.8%	6.8%
Pr[welfare loss > 20%]	3.9%	2.5%
Pr[welfare loss > 40%]	0.7%	0.0%

Table 3.2: *The probability that the cohort that enters the pension fund at time* $t = 50$ *does not want to join the pension fund. Welfare losses from joining the pension fund are expressed in terms of the percentage reduction in the certainty equivalent consumption level relative to the welfare in autarky treated in section 3.2. The numbers for elastic labor supply refer to the benchmark parameters.*

shocks on financial markets arises early in the life of the fund. In this situation, the risk-sharig solution can become politically unsustainable. Table 3.2 shows that the probability of ex-post political tensions to decline once labor-supply effects are recognized. Financial shocks are being *recouped* over time instead of being spread out, implying that the pension fund *recovers* from financial gains and losses. This result is consistent with solvency rules that require pension funds to levy financial shocks primarily upon currently-living generations. The probability that joining the pension fund at time $t = 50$ reduces welfare levels by more than 40% (compared to autarky) is equal to 0.0% in my model. If labor supply effects are ignored, this scenario cannot be rule out: it has a probability of 0.7%.

The solid line in Figure 3.7 represents the certainty-equivalent consumption level of a pension fund participant as a function of the parameter of labor supply elasticity ϵ. Consistent with the results derived in the two-agent setting, the introduction of labor supply distortions reduces ex-ante welfare levels. For the default parameter ($\epsilon = 0.5$), the certainty equivalent consumption level equals 1.011. This implies that the fraction of the welfare gain from intergenerational risk sharing that is eroded by distortions equals: (1.103-1.011)/(1.103-0.901)=46%[42]. Consistent with the analysis in paragraph 2.6, the certainty-equivalent return on savings drops below the autarky level if labor supply becomes very elastic. If labor supply becomes infinitely elastic, the pension fund must set the contribution rate equal to the accrual rat at all times to prevent the wage-level of participants from being distorted. This implies that pension fund assets cannot be exposed to financial market risk and the solution

[42]Recall that the two-agent setting gave a (sizeable, although smaller) 18.2% fraction. The welfare effects (both benefits as well as costs) from risk-sharing in the overlapping generations framework are substantially larger compared to the two-agent setting, due to the fact that there are more shocks that can be shared across generations. When comparing our results to the two-agent settings, we compare results to the 'suboptimal risk sharing' case discussed in section 2.5. After all, the risk sharing facilitated by the pension fund is 'suboptimal' because it uses distortionary transfers only (and thus no lump-sum transfers). Mehlkopf (2010b) solves the policy of a pension fund without this restriction.

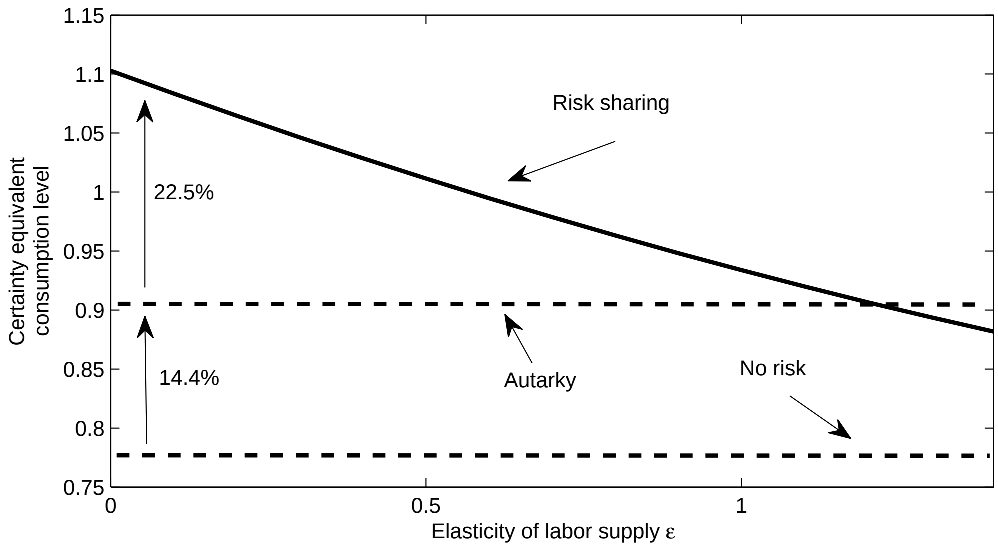


Figure 3.7: *The solid line represents the ex-ante certainty-equivalent consumption level in the pesnion fund for various levels of the compensated wage-elasticity of labor supply ϵ. The dashed line represents the welfare level in the setting with individual retirement accounts as treated in section3.2. The dotted lines represent the welfare level in the case in which there are no investments in stocks ($\alpha_t = 0$ for all t).*

corresponds to the case where to risk is taken. The knife-edge case for the labor-supply elasticity ϵ at which risk sharing is not welfare improving anymore is about $\epsilon = 1.2$. This knife-edge value for ϵ is large as a measure for individual labor-supply choices in the usual interpretation. However, it is not an unrealistic parameter if it is interpreted as a measure for labor-supply effects in the case of an industry-wide pension scheme in which, as discussed in the introduction, intergenerational risk sharing distorts firm-level decisions and labor-supply movements across industries.

Notice that the analysis does not include other taxes on labor supply in the model, such as a general labor tax levied by the government. A subsidy from the pension fund neutralizes the distortions from other taxes, implying that a net subsidy from the pension fund causes labor supply choices to be less distorted instead of more distorted. This would imply that the inclusion of other labor taxes in the model can potentially reduce the welfare costs from distortions. However, in the presence of other taxes on labor, a net tax levied by the pension fund will become more costly, as it adds to already existing distortions in the labor market. The effect of the inclusion of a government tax to the model is thus ambiguous.

4 Conclusion

The analysis in this paper has pointed out that labor supply effects may impede pension funds from taking advantage from intergenerational risk-sharing. In order to prevent excessive distortions in the labor market, it is optimal for a pension fund to recoup financial gains and losses primarily upon currently-living generations. The analysis thereby provides an economic justification for solvency regulations that require pension funds to recover from financial shortfalls in a relatively short period of time. Solvency regulations cannot be justified from the existing literature on intergenerational risk sharing, which finds that shocks should be smoothed over as many generations as possible. Smoothing shocks over many generations is not optimal anymore once the welfare costs from distortions in labor markets are recognized.

The analysis has also shown that the welfare effects from labor-supply flexibility are not unambiguous. Labor-supply flexibility makes it more difficult for governments to commit future generations to share in current financial risks. The analysis in this paper therefore suggests that governments may want to give workers incentives to work full-time and retire at a fixed age, allowing society to take more advantage of intergenerational risk-sharing. Governments are able to reduce labor supply flexibility by using the incentives in social security. The United States social security system currently provides such incentives, as social security income depends on the individuals average earnings in his 35 highest earnings years. As noted by French (2005), this provides incentives to retire at age 65 and to work full-time during the working life.

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A Proofs of equations

A.1 Proof of equation (2.7)

This section derives the Arrow-Pratt approximation along the lines of Gollier (2001). To derive equation (2.7), first notice that the expected utility from consumption of agent 1 can be written as

$$\mathbf{E}\left[\frac{1}{1-\gamma}\left(w_1 h_1^* + \alpha\tilde{x}_1\right)^{1-\gamma}\right] = \mathbf{E}\left[\frac{1}{1-\gamma}\left(w_1 h_1^* + \alpha\mu + \alpha\hat{x}_1\right)^{1-\gamma}\right] \tag{A.1}$$

where

$$\hat{x}_1 = \tilde{x}_1 - \mu \tag{A.2}$$

so that $\hat{x}_1$ is distributed with mean zero and variance σ^2. Let $\pi(w_1 h_1^*, \gamma, \tilde{x}_1\alpha)$ denote the risk premium that is associated with the risk $\alpha\hat{x}_1$:

$$\mathbf{E}\left[\frac{1}{1-\gamma}\left(w_1 h_1^* + \alpha\mu + \alpha\hat{x}_1\right)^{1-\gamma}\right] \equiv \frac{1}{1-\gamma}\left(w_1 h_1^* + \alpha\mu - \pi(w_1 h_1^*, \gamma, \tilde{x}\alpha)\right)^{1-\gamma} \tag{A.3}$$

The equation states that agent 1 is indifferent between bearing the risk $\hat{x}_1$ and paying the fixed risk premium. To simplify notation, let us define a function g as follows

$$g(\alpha) \equiv \pi(w_1 h_1^*, \gamma, \tilde{x}\alpha) \tag{A.4}$$

so that

$$\mathbf{E}\left[\frac{1}{1-\gamma}\left(w_1 h_1^* + \alpha\mu + \alpha\hat{x}_1\right)^{1-\gamma}\right] = \frac{1}{1-\gamma}\left(w_1 h_1^* + \alpha\mu - g(\alpha)\right)^{1-\gamma} \tag{A.5}$$

Assuming the risk to be small, the effect of the expected return $\alpha\mu$ on the risk premium is negligible, equation (A.5) is approximated by:

$$\mathbf{E}\left[\frac{1}{1-\gamma}\left(w_1 h_1^* + \alpha\hat{x}_1\right)^{1-\gamma}\right] \approx \frac{1}{1-\gamma}\left(w_1 h_1^* - g(\alpha_i)\right)^{1-\gamma} \tag{A.6}$$

The function g is approximated by a Taylor expansion around $\alpha = 0$:

$$g(\alpha) \approx g(0) + \alpha g'(0) + \frac{1}{2}\alpha^2 g''(0) \tag{A.7}$$

It follows from equation (A.6) that

$$g(0) = 0 \tag{A.8}$$

Differentiating equation (A.6) with respect to α_i yields

$$\mathbf{E}\left[\hat{x}_1 \left(w_1 h_1^* + \alpha \hat{x}_1\right)^{-\gamma}\right] = -g'(\alpha)\left(w_1 h_1^* - g(\alpha)\right)^{-\gamma} \tag{A.9}$$

from which it follows that

$$g'(0) = 0 \tag{A.10}$$

since $\mathbf{E}[\hat{x}] = 0$. Differentiating again with respect to α yields

$$-\mathbf{E}\left[\hat{x}_i^2 \gamma \left(w_1 h_1^* + \alpha \hat{x}_1\right)^{-\gamma-1}\right] = -g''(\alpha)\left(w_1 h_1^* + g(\alpha)\right)^{-\gamma} + [g'(\alpha)]^2 \gamma \left(w_1 h_1^* + g(\alpha)\right)^{-\gamma-1} \tag{A.11}$$

Evaluating this expression at $\alpha = 0$ yields

$$g''(0) = \frac{\gamma}{w_1 h_1^*}\sigma^2 \tag{A.12}$$

where it is used that $g(0) = 0$ and $g'(0) = 0$ and $\mathbf{E}\left[\hat{x}_1^2\right] = \sigma^2$. Substitution of equations (A.8), (A.10) and (A.12) into equation (A.7) yields

$$g(\alpha) \approx \frac{1}{2}\frac{\gamma}{w_1 h_1^*}\sigma^2 \alpha^2 \tag{A.7'}$$

Substitution of equations (A.6) and (A.7') into equation (A.1) yields equation (2.7).

A.2 Proof of equation (2.17)

Equation (2.17) is derived along the same lines as equation (2.7) was derived, using the Arrow-Pratt approach. The expected utility from consumption of agent 2 can be written as

$$\begin{aligned} & \mathbf{E}\left[\frac{1}{1-\gamma}\left(w_2 \tilde{h}_2 + \alpha \tilde{x}_1 - \frac{\epsilon}{\epsilon+1}(\tilde{h}_2)^{\frac{\epsilon+1}{\epsilon}} + \frac{\epsilon}{\epsilon+1}(h_2^*)^{\frac{\epsilon+1}{\epsilon}}\right)^{1-\gamma}\right] \\ = \; & \mathbf{E}\left[\frac{1}{1-\gamma}\left(w_2 \tilde{h}_2 + \alpha\mu + \alpha \hat{x}_1 - \frac{\epsilon}{\epsilon+1}(\tilde{h}_2)^{\frac{\epsilon+1}{\epsilon}} + \frac{\epsilon}{\epsilon+1}(h_2^*)^{\frac{\epsilon+1}{\epsilon}}\right)^{1-\gamma}\right] \end{aligned} \tag{A.13}$$

where $\tilde{h}_2$ represents the labor-supply choice of agent 2 and is a stochastic variable (and will be defined later in equation (A.19)) and where

$$\hat{x}_1 = \tilde{x}_1 - \mu \tag{A.14}$$

so that $\hat{x}_1$ is a pure risk distributed with mean zero and variance σ^2. Let $\pi(w_2\tilde{h}_2, \gamma, \tilde{x}_1\alpha, \epsilon)$ denote the risk premium that is associated with the risk $\alpha\hat{x}_1$:

$$\begin{aligned} & \mathbf{E}\left[\frac{1}{1-\gamma}\left(w_2\tilde{h}_2 + \alpha\mu + \alpha\hat{x}_1 - \frac{\epsilon}{\epsilon+1}(\tilde{h}_2)^{\frac{\epsilon+1}{\epsilon}} + \frac{\epsilon}{\epsilon+1}(h_2^*)^{\frac{\epsilon+1}{\epsilon}}\right)^{1-\gamma}\right] \\ \equiv \quad & \frac{1}{1-\gamma}\left(w_2h_2^* + \alpha\mu - \pi(w_2\tilde{h}_2, \gamma, \tilde{x}_1, \alpha, \epsilon)\right)^{1-\gamma} \end{aligned} \tag{A.15}$$

Thus, $\pi(w_2\tilde{h}_2, \gamma, \tilde{x}_1, \alpha, \epsilon)$ denotes the risk premium that makes agent 2 indifferent between bearing the pure risk $\hat{x}_1$ on the one hand and paying the fixed risk premium and facing no labor-supply distortions on the other hand. To simplify notation, let us define a function g as follows

$$g(\alpha) \equiv \pi(w_2\tilde{h}_2, \gamma, \tilde{x}_1, \alpha, \epsilon) \tag{A.16}$$

so that

$$\begin{aligned} & \mathbf{E}\left[\frac{1}{1-\gamma}\left(w_2\tilde{h}_2 + \alpha\mu + \alpha\hat{x}_1 - \frac{\epsilon}{\epsilon+1}(\tilde{h}_2)^{\frac{\epsilon+1}{\epsilon}} + \frac{\epsilon}{\epsilon+1}(h_2^*)^{\frac{\epsilon+1}{\epsilon}}\right)^{1-\gamma}\right] \\ \equiv \quad & \frac{1}{1-\gamma}\left(w_2h_2^* + \alpha\mu - g(\alpha)\right)^{1-\gamma} \end{aligned} \tag{A.17}$$

Assuming the risk to be small, the effect of the expected return $\alpha\mu$ on the risk premium is negligible, so that equation (A.17) is approximated by:

$$\begin{aligned} & \mathbf{E}\left[\frac{1}{1-\gamma}\left(w_2\tilde{h}_2 + \alpha\hat{x}_1 - \frac{\epsilon}{\epsilon+1}(\tilde{h}_2)^{\frac{\epsilon+1}{\epsilon}} + \frac{\epsilon}{\epsilon+1}(h_2^*)^{\frac{\epsilon+1}{\epsilon}}\right)^{1-\gamma}\right] \\ \approx \quad & \frac{1}{1-\gamma}\left(w_2h_2^* - g(\alpha)\right)^{1-\gamma} \end{aligned} \tag{A.18}$$

The stochastic variables for respectively the labor-supply choice $\tilde{h}_2$ and the tax or subsidy on labor supply $\tilde{\tau}$ are functions of each other and do not attain an explicit solution. To arrive at an explicit expression, let the labor-supply choice be approximated by

$$\tilde{h}_2 = h_2^*(1+\tilde{\tau})^\epsilon = h_2^*\left(1 + \frac{\alpha\tilde{x}_1}{w_2\tilde{h}_2}\right)^\epsilon \approx h_2^*\left(1 + \frac{\alpha\hat{x}_1}{w_2h_2^*}\right)^\epsilon \tag{A.19}$$

Substitution of equation (A.19) into equation (A.18) yields

$$\mathbf{E}\left[\frac{1}{1-\gamma}\left(w_2 h_2^*\left(1+\frac{\alpha\hat{x}_1}{w_2 h_2^*}\right)^{\epsilon}+\alpha\hat{x}_1-\frac{\epsilon}{\epsilon+1}\left(h_2^*\right)^{\frac{\epsilon+1}{\epsilon}}\left(\left(1+\frac{\alpha\hat{x}_1}{w_2 h_2^*}\right)^{\epsilon+1}-1\right)\right)^{1-\gamma}\right]$$
$$\equiv \frac{1}{1-\gamma}\left(w_2 h_2^*-g(\alpha)\right)^{1-\gamma} \tag{A.20}$$

The function g is approximated by a Taylor expansion around $\alpha = 0$:

$$g(\alpha) \approx g(0)+\alpha g'(0)+\frac{1}{2}\alpha^2 g''(0) \tag{A.21}$$

It follows from equation (A.20) that

$$g(0) = 0 \tag{A.22}$$

Differentiating equation (A.20) with respect to α and evaluating the resulting expression at $\alpha = 0$ yields

$$g'(0) = 0 \tag{A.23}$$

Differentiating equation (A.20) twice with respect to α and evaluating the resulting expression at $\alpha = 0$ yields

$$g''(0) = \frac{\gamma+\epsilon}{w_2 h_2^*}\sigma^2 \tag{A.24}$$

where it is used that $g(0) = 0$ and $g'(0) = 0$ and $\mathbf{E}\left[\hat{x}_1^2\right] = \sigma$. Substitution of equations (A.22), (A.23) and (A.24) into equation (A.21) yields

$$g(\alpha) \approx \frac{1}{2}\frac{\gamma+\epsilon}{w_2 h_2^*}\alpha^2\sigma^2 \tag{A.21'}$$

Substitution of equations (A.17) and (A.21') into equation (A.13) yields equation (2.17).

A.3 Proof of equations (3.5) and (3.6)

By backward induction, the optimization problem of section 3.2 can be written as:

$$v_t(W_t) = \beta^t u(C_t) + \max_{\alpha_t,\ C_t}\left\{\mathbf{E}\left[v_{t+1}\left(R(W_t + wh^* - C_t) + \alpha_t\tilde{x}\right)\right]\right\} \tag{A.25}$$

for $t = 25,\ ...,\ 84$, with $v_{84}(W_{84}) = \frac{1}{1-\gamma}(W_{84})^{1-\gamma}$ and $v_{25}(0) = U$. Let us consider the trial solution $v_t = \lambda_t u(W_t + H_t)$ for the value function. Substitution of the trial solution into equation (A.25) gives:

$$
\begin{aligned}
v_{t+1}\left(R(W_t + wh^* - C_t) + \alpha_t \tilde{x}\right) &= \lambda_{t+1} \frac{1}{1-\gamma} \left(R(W_t + wh^* - C_t) + \alpha_t \tilde{x} + H_{t+1}\right)^{1-\gamma} \\
&= \lambda_{t+1} \frac{1}{1-\gamma} \left(R(W_t + H_t - C_t) + \alpha_t \tilde{x}\right)^{1-\gamma} \qquad \text{(A.26)}
\end{aligned}
$$

The first-order conditions with respect to investments α_t is then given by

$$0 = \lambda_{t+1} \mathbf{E}\left[\tilde{x}\left(R(W_t + H_t - C_t) + \alpha_t \tilde{x}\right)^{-\gamma}\right] \qquad \text{(A.27)}$$

and rewriting yields the optimal portfolio rule in equation (3.6). The first-order conditions with respect to consumption C_t is given by:

$$0 = \beta^t C_t^{-\gamma} - \lambda_{t+1} R \mathbf{E}\left[\left(R(W_t + H_t - C_t) + \alpha_t \tilde{x}\right)^{-\gamma}\right] \qquad \text{(A.28)}$$

which yields the optimal consumption rule:

$$C_t = \kappa_t (W_t + H_t) \qquad \text{(A.29)}$$

where

$$\kappa_t = \frac{\lambda_{t+1}^{\frac{-1}{\gamma}} R^{\frac{-1}{\gamma}} \mathbf{E}\left[(R(1+\eta)\tilde{x})^{-\gamma}\right]^{\frac{-1}{\gamma}}}{\left(\beta^t\right)^{\frac{-1}{\gamma}} + \lambda_{t+1}^{\frac{-1}{\gamma}} R^{\frac{-1}{\gamma}} \mathbf{E}\left[(R(1+\eta)\tilde{x})^{-\gamma}\right]^{\frac{-1}{\gamma}}} \qquad \text{(A.30)}$$

for $25 \leq t < 84$ and $\lambda_{84} = 1$. Substitution of equations (3.6) and (A.29) into equation (A.25) yields:

$$U = \lambda_t \frac{1}{1-\gamma} \left(W_t + H_t\right)^{1-\gamma} \qquad \text{(A.31)}$$

where

$$\lambda_t = \kappa_t^{1-\gamma} + \beta \lambda_{t+1} \left(1 - \kappa_t\right)^{1-\gamma} \mathbf{E}\left[R(1 + \eta \tilde{x}_1)^{1-\gamma}\right] \qquad \text{(A.32)}$$

for $25 \leq t < 84$ and $\lambda_{84} = 1$

It is easy to see that the optimal solution satisfies consumption smoothing. An unexpected wealth shock (in terms of total wealth $W_t + H_t$) at any time t leads to proportionally equally adjustments to all future consumption levels in the life-cycle. To see this, notice that a change in the total wealth at time t by y% results in a drop in consumption C_t at time t according to the optimal consumption rule in equation (A.29). According to the investment rule in equation (3.6), the amount α_t invested in stocks at time t changes by y% as well. This implies that the return on total wealth, and thereby next-period total wealth itself, also changes by y%. Consumption smoothing is now satisfied by recursion.

A.4 Proof of equation (3.7)

Approximation in equation (3.7) can be derived by using the log-linearization approach of Campbell and Viceira (2002). To be written.

A.5 Proof of equation (3.12)

For an individual who entered the pension fund at time s, the partial derivative of utility U with respect to labor supply $h_{s,t}$ at time t ($s \leq t \leq s+n$) is given by:

$$0 = \frac{\partial \mathbf{E_t} \left[\sum_{i=s}^{s+n-1} \beta^{i-s} u\left(C_{s,i}, h_{s,i}\right) + \sum_{i=s+n}^{s+n+m-1} \beta^{i-s} u(C_{s,i}) \right]}{\partial h_{s,t}} \tag{A.33}$$

where $C_{s,t}$ represents the consumption level at time t of the cohort that entered the pension fund at time s:

$$C_{s,t} = \begin{cases} (1-\pi_t) w h_{s,t} & \text{if } s \leq t < s+n \\ b_{s,t} & \text{if } s+n \leq t < s+n+m \end{cases} \tag{A.34}$$

Observing that labor supply at time t only affects utility from consumption and leisure at time t and utility gained from consumption during retirement, the first-order-condition simplifies into:

$$0 = \frac{\partial \left[\beta^{t-s} u\left(C_{s,t}, h_{s,t}\right) \right]}{\partial h_{s,t}} + \frac{\partial \mathbf{E}_t \left[\sum_{i=s+n+1}^{s+n+m} \beta^{i-s} u(C_{s,i}) \right]}{\partial h_{s,t}} \tag{A.35}$$

The first term on the right-hand-side of equation (A.35) can be rewritten:

$$\begin{aligned} & \frac{\partial \left[\beta^{t-s} u\left((1-\pi_t) w h_{s,t}, h_{s,t}\right) \right]}{\partial h_{s,t}} \\ = \; & \beta^{t-s} \left(C_{s,t} - \frac{\epsilon}{\epsilon+1} \left(h_{s,t}\right)^{\frac{\epsilon+1}{\epsilon}} + \frac{\epsilon}{\epsilon+1} \left(h^*\right)^{\frac{\epsilon+1}{\epsilon}} \right)^{-\gamma} \left((1-\pi_t) w - \left(h_{s,t}\right)^{\frac{1}{\epsilon}} \right) \end{aligned} \tag{A.36}$$

Substitution of equation (3.9) allows us to rewrite the second term in equation (A.35) as:

$$\begin{aligned} & \frac{\partial \mathbf{E}_t \left[\sum_{i=s+n+1}^{s+n+m} \beta^{i-s} u(C_{s,i}) \right]}{\partial h_{s,t}} \\ = \; & \mathbf{E}_t \left[\sum_{i=s+n+1}^{s+n+m} \beta^{i-s} \left(C_{s,i}\right)^{-\gamma} b(W_{s+n+1}) \frac{1}{h^*} \frac{R^{s-t}}{\sum_{j=0}^{n-1} R^{-j}} \right] \end{aligned} \tag{A.37}$$

Substitution of equations (A.36) and (A.37) into equation (A.35) and rewriting yields equation (3.12) and (3.13).

A.6 Proof of equations (3.15) and (3.17)

The solutions for the consumption rule in equation (3.15), the investment rule in equation (3.17) and the time-discounting rule in equation (A.41) follow directly from the solution in Gollier (2008). The optimization problem of the overlapping generations model in the case of inelastic labor supply is mathematically equivalent to the optimization problem solved in Gollier (2008), except that in every period there are $n+m$ individuals who consume instead of just one. Given that the utility of each of the $n+m$ individuals is weighted equally, it is optimal for the pension fund to give all agents the same consumption level at each point in time. The solution for the optimal consumption rule is given by:

$$C_t = \frac{\nu}{n+m}\left(W_t + K\right) \tag{A.38}$$

where $\nu = 1 - \left(\delta R^{1-\gamma}\mathbf{E}\left[(1+\eta\tilde{x})^{-\gamma}\right]\right)^{\frac{1}{\gamma}}$, where η is the same as in equation (3.6) and where K represents the present discounted value of all future labor earnings. The optimal investment rule is given by:

$$\alpha_t = \eta R\left(W_t + K - (n+m)C_t\right) \tag{A.39}$$

where η is defined in equation (3.6) and is approximated by equation (3.7). The ex ante utility of any generation (measured from the beginning of the working period) adopts an analytical solution under the optimal decision rules:

$$U = \sum_{t=1}^{n+m}\beta^{t-1}\left(\frac{\nu}{n+m}\right)^{1-\gamma}\frac{1}{1-\gamma}(Y_0+K)^{1-\gamma}, \tag{A.40}$$

Recall that δ is chosen by the social planner such that the ex ante utility gained from consumption and leisure is the same at all periods in time. This the case if

$$\delta = R^{-1}\left(\mathbf{E}\left[(1+a^*\tilde{x})^{-\gamma}\right]\right)^{\frac{1}{\gamma-1}}. \tag{A.41}$$

For the default parameters it follows that $\delta = 0.9747$. The proof that the optimal solution satisfies consumption smoothing is similar to that in Appendix A.3. An unexpected wealth shock (in terms of total wealth $W_t + K_t$) at any time t leads to proportionally equally adjustments to all future consumption levels. To see this, notice that a change in the total wealth at time t by y% results in a drop in consumption C_t at time t according to the optimal consumption rule in equation (3.17). According to the investment rule in equation (3.17), the amount α_t invested in stocks at time t changes by y% as well. This implies that the return on total wealth, and thereby next-period total wealth itself, also changes by y%. Consumption smoothing is now satisfied by recursion.

B Description of numerical solution method

This section describes the numerical solution method that is used to solve the optimization problem for the pension fund with distortionary transfers in section 3.3.3.

The benefit rule f in equation (3.9) as well as the investment rule α and the contribution rule π are approximated by time-invariant first-order polynomials in W_t: $f(W_t) \approx f_0 + f_1 W_t$, $\alpha(W_t) \approx \alpha_0 + \alpha_1 W_t$ and $\pi(W_t) \approx \pi_0 + \pi_1 W_t$. These approximations for the decision rules of the pension fund reduce the number of decision variables of the decision making problem to six scalars which need to be solved: α_0, α_1, f_0, f_1, π_0 and π_1. These parameters are solved using a search algorithm in which Monte-Carlo simulation runs are repeated and where the parameter choices are adjusted at the beginning of every new simulation run. All paths start at the date of the pension reform where the initial pension assets are equal to Y_0. It is assumed that the pension fund arrives in the steady state 200 years after the date of the reform. The parameter choices of the six unknown variables are adjusted by a search algorithm at the beginning of every new simulation run until the welfare of participants reaches its maximum.

Recall that it is imposed in section 3.3.3 that all generations in the pension fund are equally well off: everyone shares equally in the social surplus from risk sharing. This criterion is met by requiring the ex ante welfare of a participant entering at time $t = 0$ to be equal to the ex ante welfare of the generation at enters at time $t = 200$ (if the labor supply elasticity is not too close to zero, the pension fund has reached the steady state at time $t = 200$.). This condition effectively pins down any of the six unknown decision parameters as a function of the other five. The restriction that all generations should be equally well off thereby effectively reduces the number of unknowns decision parameters of the problem from six to five. Applying the search algorithm to the five 'core' decision variables, the sixth decision variable is adjusted at the beginning of every new simulation run until the welfare levels of the initial and the steady state generation converge to each other.

The calculation of the labor-supply choices of workers in equation (3.12) is based upon the value of pension accruals. The value of pension accruals takes the form of a *conditional expectation* of variables (pension fund assets and the consumption level) at a future point in time: the retirement date. Given that simulation paths run *forward* in time, the conditional expectations (and thus the value of accruals) cannot be derived on the basis of the information of the present simulation run and are therefore calculated on the basis upon the previous simulation run (recall that simulation runs are repeated). Since we are working with a large number of simulations, the conditional expectations can be calculated on the basis of *across-path regressions* that are applied to the scenario paths of the previous simulation run. Simulation runs are repeated until the regression estimates (and thereby labor-supply choices) converge.