The Kolm tax, the tax credit, and the flat tax*

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Sept. 2007; rev. April 2008

1 Introduction

In several of his recent contributions, most notably Kolm (2004), Serge-Christophe Kolm has developed a solution to the macro-justice problem which he calls Equal Labor Income Equalization (ELIE). It consists in a particular labor income taxation scheme that he advocates as the ideal compromise between freedom and equality requirements.

The ELIE proposal is, in essence, a first-best taxation scheme involving a parameter, k, which can be thought of as the share of every individual's labor time which is equally shared within society. At an ELIE allocation, earning ability is taxed in such a way that the net income an individual would get, should he choose to work precisely k, is equalized among individuals. If an individual's earning ability is s, he pays the tax ks and receives

*We would like to thank participants in the conference organised in Caen in honour of Serge-Christophe Kolm, and Serge Kolm himself for numerous stimulating discussions. This paper has also benefited from comments by M. Salles, J. Weymark and a referee.

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a universal grant g. Therefore, if he works exactly k at the wage rate s, his net income is g, independently of s. Individuals choosing to work more than k are paid marginally at their wage rate, that is, the marginal tax on earnings is zero. Individuals choosing to work less than k have to "buy" their leisure, and have to do so at its marginal value as well.

Implementing the ELIE scheme requires the earning ability of each individual to be observable. If the earning ability is not observable, then Kolm's proposal needs to be adapted into a second-best version. This is what we study in this paper.

In order to refine the ELIE scheme for the second-best context, we will begin by defining a social ordering function compatible with ELIE. A social ordering function defines a complete ranking of the set of allocations for each profile of the population characteristics. The social ordering function, which we define below and which we axiomatize, rationalizes ELIE in the sense that the ELIE allocations maximize the social ordering function in the special case in which the information is complete. Moreover, we believe it incorporates the basic fairness principles underlying ELIE and thereby extends the thrust of the ELIE idea to the comparison of arbitrary allocations.

Then, we use this social ordering function to derive taxation schemes under different information settings. First, we look at the case where the earning ability cannot be observed but incomes and labor times are observable. Consequently, the wage rate can be deduced from the observables, but individuals may still decide to take jobs at wage rates lower than their actual earning ability. This is the same informational framework as in Dasgupta and Hammond (1980). We prove that under certain assumptions the resulting taxation scheme is similar to Kolm's proposal regarding incomes earned by individuals working more than k, but differs substantially for the others.

Second, we look at the case where only income, but not labor time nor the wage rate, is observable. This is the typical case considered in the optimal income taxation literature, following Mirrlees (1971). We derive some insights about the optimal income tax scheme, in particular that taxation of incomes at a constant marginal tax rate equal to k appears as an important benchmark. We therefore establish a surprising connection between ELIE and the flat tax.

This connection is loose, however, for low incomes. Indeed, we also show that in both informational contexts studied here, one feature of the first-best version of ELIE is preserved at the optimal second-best tax: low incomes up to the lowest earning ability should have a marginal tax of zero. This is a noticeable result in the light of recent reforms of the welfare state in which efforts have been made to reduce the marginal tax on low incomes.[1]

A related analysis is made by Simula and Trannoy (2009) who observe that if all individuals work more than k at the ELIE first-best allocation, then it is incentive-compatible when labor time is observable. For the case in which only income is observable, they suggest to seek an incentive-compatible allocation that is as close as possible to ELIE. There are three main differences with our approach. First, we study economies with heterogeneity in skills and preferences, whereas Simula and Trannoy examine economies with heterogeneity in skills only. Second, we use a different social welfare function (more specifically a different social ordering function), to which we give axiomatic foundations. Third, we do not restrict attention to situations in which all individuals work more than k. When some individuals work less than k, the first-best ELIE allocation is not always incentive-compatible even when labor time is observable, as we will show below.

In Section 2, we present the model and define our social ordering function. In Section 3, we give some axiomatic justification to it. In Section 4, we study the optimal tax scheme

[1] This practical evolution has already found echoes in economic theory. See, e.g., Choné and Laroque (2005), Boadway, Marchand, Pestieau and Racionero (2002), and Fleurbaey and Maniquet (2006, 2007).

when both labor times and incomes are observable. In Section 5, we examine the optimal tax scheme when only incomes are observable. Section 6 offers concluding comments.

2 The model

Our model is identical to the one Kolm used to develop his ELIE proposal. This model was introduced in the axiomatic literature on fairness by Pazner and Schmeidler (1974). It generalizes the model of Mirrlees (1971) by allowing individuals to have different preferences, not only different earning abilities. There are two goods, labor time (l) and consumption (c). The population of an economy is finite. If the set of individuals in an economy is N, each individual $i \in N$ has a *production skill* $s_i \geq 0$ enabling him to produce the quantity $s_i l_i$ of the consumption good with labor time l_i. This individual can also choose to work at a lower productivity (wage rate) $w_i \leq s_i$, in which case his earnings are equal to $w_i l_i$. Agent i also has *preferences* represented by an ordering R_i over bundles $z_i = (l_i, c_i)$ where $0 \leq l_i \leq 1$ and $c_i \geq 0$. Let $X = [0,1] \times \mathbb{R}_+$ denote the individual's labor-consumption set. We assume that the wage rate w_i does not directly affect the individual's satisfaction.

We study the domain $\mathcal{E}$ of economies defined as follows. Let $\mathcal{N}$ denote the set of non-empty finite subsets of the set of positive integers $\mathbb{N}_+$ and $\mathcal{R}$ denote the set of continuous, convex and strictly monotonic (negatively in labor, positively in consumption) orderings over X. An *economy* $e = (s_N, R_N)$ belongs to the domain $\mathcal{E}$ if $N \in \mathcal{N}$, $s_N \in \mathbb{R}_+^N$, and $R_N \in \mathcal{R}^N$, that is,

$$\mathcal{E} = \bigcup_{N \in \mathcal{N}} (\mathbb{R}_+^N \times \mathcal{R}^N).$$

Let $e = (s_N, R_N) \in \mathcal{E}$. An *allocation* is a vector $x_N = (w_i, z_i)_{i \in N} \in \mathbb{R}_+^N \times X^N$. It is *feasible* for e if $w_i \leq s_i$ for all $i \in N$ and

$$\sum_{i \in N} c_i \leq \sum_{i \in N} w_i l_i.$$

Since w_i does not affect i's satisfaction, we will generally restrict attention to bundle-allocations $z_N = (z_i)_{i \in N} \in X^N$ in the context of social evaluation.[2] Bundle-allocations will also be called allocations for short when there is no risk of confusion. Let $Z(e)$ be the subset of X^N such that $l_i = 0$ for all $i \in N$ such that $s_i = 0$. We will restrict attention to this subset, as it does not make sense in any first-best or second-best context to make an individual work when his productivity is nill. This restriction is useful because it simplifies the presentation of our social ordering in the next paragraphs.

A *social ordering* for an economy $e = (s_N, R_N) \in \mathcal{E}$ is a complete ordering over the set $Z(e)$ of (bundle-)allocations. A *social ordering function* (SOF) $\mathbf{R}$ associates every economy $e \in \mathcal{E}$ with a social ordering $\mathbf{R}(e)$. We write $z_N \, \mathbf{R}(e) \, z'_N$ to denote that allocation z_N is at least as good as z'_N in e. The corresponding strict social preference and social indifference relations are denoted $\mathbf{P}(e)$ and $\mathbf{I}(e)$, respectively. Following the social choice tradition initiated by Arrow (1951), we require a social ordering to rank all allocations in $Z(e)$, not just the feasible allocations.[3] We depart, however, from Arrow's legacy by letting the social ordering depend on s_N, not just on R_N. This is because fairness principles may recommend treating individuals differently depending on their earning ability. As it turns out, this happens with ELIE.

Our next objective in this section is to define the social ordering function that we consider associated to Kolm's ELIE proposal. This requires introducing some terminology. Let $e = (s_N, R_N) \in \mathcal{E}$ and let $i \in N$. For $A \subseteq X$, let $m(R_i, A) \subseteq A$ denote the set of

[2] At the cost of heavier notations, we could always deal with allocations x_N and deduce from the Pareto principle the fact that only z_N really matters.

[3] Although we think it is less justified, we could have restricted the definition of social ordering functions to feasible allocations and still derive the same results.

bundles (if any) that are the best in A for preferences R_i, that is,

$$m(R_i, A) = \{z_i \in A \mid \forall z_i' \in A,\ z_i \, R_i \, z_i'\}.$$

For $z_i = (l_i, c_i) \in X$, $s_i \in \mathbb{R}_+$, let $B(z_i, s_i) \subseteq X$ denote the *budget set* obtained with s_i and such that z_i is on the budget frontier:

$$B(z_i, s_i) = \{(l_i', c_i') \in X \mid c_i' - s_i l_i' \leq c_i - s_i l_i\}.$$

In the special case in which $s_i = 0$ and $c_i = 0$, we adopt the convention that

$$B(z_i, s_i) = \{(l_i', c_i') \in X \mid c_i' = 0,\ l_i' \geq l_i\}.$$

Let ∂B denote the upper frontier of set B. Also, let $IB(z_i, s_i, R_i) \subseteq X$ denote the *implicit budget* at z_i, that is, the budget set with slope s_i having the property that i is indifferent between z_i and his preferred bundle in that budget set:

$$IB(z_i, s_i, R_i) = B(z_i', s_i) \text{ for any } z_i' \text{ such that } z_i' \, I_i \, z_i \text{ and } z_i' \in m(R_i, B(z_i', s_i)).$$

By strict monotonicity of preferences, this definition is unambiguous. See Figure 1 for an illustration of this notion. Notice that bundle z_i need not belong to the implicit budget. Also note that implicit budgets provide a set representation of the preferences of individual i in e in the sense that

$$z_i \, R_i \, z_i' \Leftrightarrow IB(z_i, s_i, R_i) \supseteq IB(z_i', s_i, R_i).$$

We are now equipped to define the SOFs that will be used in the subsequent discussion. We consider a family of SOFs, parameterized by a coefficient $k \in [0, 1]$. For each $k \in [0, 1]$, the corresponding SOF will be denoted $\mathbf{R}^k$. Each SOF in the family is based on a specific utility representation of the preferences, and it compares allocations by applying the leximin aggregation rule to the utility vectors derived from these numerical representations.

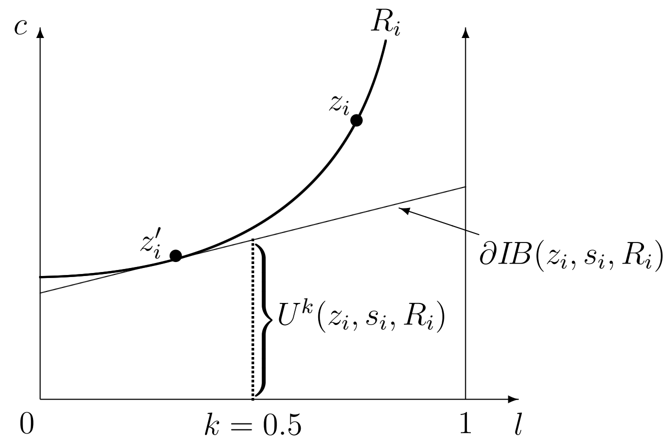

Figure 1

Assuming that a parameter $k \in [0, 1]$ has been chosen, let us begin by defining the utility functions. Let $e = (s_N, R_N) \in \mathcal{E}$ and let $i \in N$. The utility function $U^k(\cdot, s_i, R_i)$ is constructed as follows: $U^k(z_i, s_i, R_i)$ is the vertical coordinate of the bundle with abscissa k on the frontier of the implicit budget of individual i at z_i. Formally,

$$U^k(z_i, s_i, R_i) = u \Leftrightarrow (k, u) \in \partial IB(z_i, s_i, R_i).$$

This construction is illustrated in Figure 1 for the case $k = 0.5$. Observe that this construction works only for bundles such that $(k, 0) \in IB(z_i, s_i, R_i)$ when $s_i > 0$. Let $Y^k(e)$ denote the subset of $Z(e)$ such that this condition is satisfied for all $i \in N$.

Such utility indexes depend on k and also on s_i, not just on R_i. This is justified by the fact that the philosophy of ELIE is not welfarist. These utility indexes in fact measure how well-off an individual is in terms of budget opportunities, not in terms of subjective satisfaction or happiness. Even though these indexes are ordinally consistent with each individual's preferences, the interpersonal comparisons they generate are basically resourcist, not welfarist. Moreover, the principles underlying ELIE stipulate that individuals are partly (depending on k) entitled to enjoy the benefits of their own productivity, so it is normal for the corresponding indexes to be sensitive to k and to individual skills.

It must be emphasized that the axiomatic justification that is offered in the next section provides a joint derivation of the social aggregation rule and of these utility indexes from basic principles.

The social ordering $\mathbf{R}^k(e)$ on $Y^k(e)$ is obtained by applying the leximin criterion to vectors of U^k utility levels. The definition of the leximin criterion is the following. For two vectors of real numbers u_N, u'_N, one says that u_N is weakly better than u'_N for the leximin criterion, which will be denoted here by

$$u_N \geq_{lex} u'_N,$$

when the smallest component of u_N is not lower than the smallest component of u'_N, and if they are equal, the second smallest component is not lower, and so forth.

k-**Leximin** $(\mathbf{R}^k)$: For all $e = (s_N, R_N) \in \mathcal{E}$, all $z_N, z'_N \in Y^k(e)$,

$$z_N \, \mathbf{R}^k(e) \, z'_N \Leftrightarrow u_N \geq_{lex} u'_N$$

where, for all $i \in N$, $u_i = U^k(z_i, s_i, R_i)$ and $u'_i = U^k(z'_i, s_i, R_i)$.

This SOF is illustrated in Figure 2, for a two-individual economy $e = ((s_1, s_2), (R_1, R_2)) \in \mathcal{E}$. We see in the figure that $s_1 < s_2$ and that the preferences R_1 are less leisure oriented preferences than R_2. The allocations $z_N = (z_1, z_2)$ and $z'_N = (z'_1, z'_2)$ have to be compared. First, the implicit budgets associated with the four bundles are identified. Then, on the frontier of each budget, the bundle with abscissa k is identified. The vertical coordinates of these bundles are denoted by u_1, u_2, u'_1, u'_2 on the figure (corresponding respectively to $U^{0.5}(z_1, s_1, R_1)$, $U^{0.5}(z_2, s_2, R_2)$, $U^{0.5}(z'_1, s_1, R_1)$ and $U^{0.5}(z'_2, s_2, R_2)$). We observe that $u'_1 < u_2 < u_1 < u'_2$. These inequalities imply that $z_N \, \mathbf{P}^{0.5}(e) \, z'_N$.

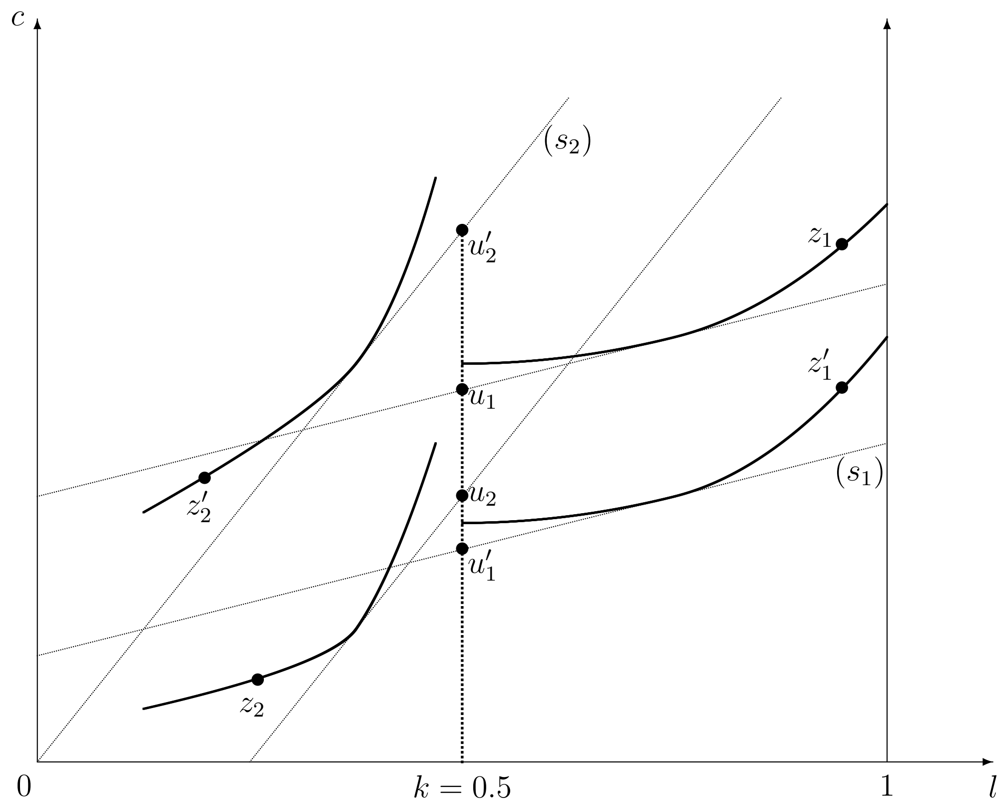

Figure 2

We do not study the extension of $\mathbf{R}^k(e)$ to $Z(e) \backslash Y^k(e)$, as this is of little consequence for the study of taxation. It is easy to find reasonable extensions. For instance, when $(k, 0) \notin IB(z_i, s_i, R_i)$ and $s_i > 0$, one can define

$$U^k(z_i, s_i, R_i) = u \Leftrightarrow (k - u, 0) \in \partial IB(z_i, s_i, R_i),$$

which yields $u < 0$. With this extended definition of U^k, $\mathbf{R}^k$ satisfies the axioms introduced in the next section over $Z(e)$.

A SOF is aimed at giving precise policy recommendations as a function of the informational conditions describing the set of tools available to the policy maker. If the informational conditions are those of a first-best world, then maximizing the social ordering $\mathbf{R}^k(e)$ on the set of feasible allocations leads to the ELIE allocations corresponding to parameter k. Indeed, at a (first-best) Pareto-efficient allocation, one has $w_i = s_i$ and

$B(z_i, s_i) = IB(z_i, s_i, R_i)$ for all $i \in N$. A best allocation for $\mathbf{R}^k(e)$ is such that, in addition, the U^k utility levels are equalized, which implies that all budget set frontiers cross at a bundle with abscissa k. An example is given in Figure 3, for the same economy as above.

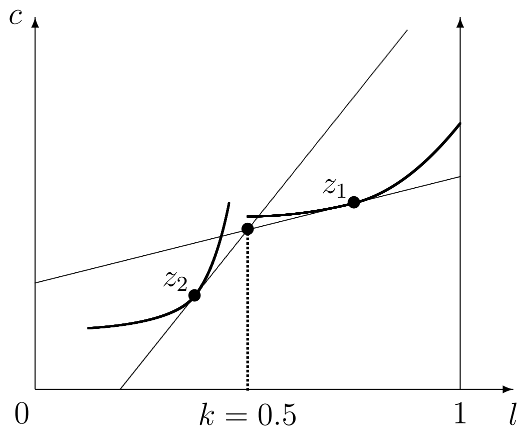

Figure 3

In the next section, we provide axiomatic foundations to this family of SOFs.

3 Axiomatic foundations

The family of SOFs inspired by Kolm's ELIE proposal, which we call the k-Leximin SOFs, satisfy a set of axioms that we define in this section. We also show that every SOF satisfying this set of axioms must satisfy a maximin property, which makes it close to a k-Leximin SOF. The material in this section draws on previous work[4] in which we have provided a similar axiomatic characterization of a family of SOFs containing the $\mathbf{R}^k$. We present a variant of that characterization here in order to highlight the relationship between the k-Leximin SOF on the one hand and Kolm's fairness principles and his justification of ELIE on the other.

[4]See Fleurbaey and Maniquet (2005). For first-best allocation rules, see also Fleurbaey and Maniquet (1996) and Maniquet (1998).

We begin with the key axiom establishing this relationship. This axiom is consistent with Kolm's idea that incomes should be equal among individuals working k. Let $e = (s_N, R_N) \in \mathcal{E}$ and $z_N, z'_N \in Z(e)$. Assume that z_N and z'_N differ only in the bundles of two individuals, say $p, q \in N$, and that at both allocations p and q freely choose to work k in a budget set determined by a lump-sum transfer and their own skill level. Using the notation of the preceding section, it means that for $j = p, q$, one has $l_j = l'_j = k$, $z_j \in m(R_j, B(z_j, s_j))$ and $z'_j \in m(R_j, B(z'_j, s_j))$. Assume, moreover, that p and q do not have the same consumption level in z_N, for instance, $c_p > c_q$. We then regard individual p as relatively richer than individual q. The social situation is not worsened, the axiom says, if $c_p > c'_p > c'_q > c_q$, so that the inequality in consumption between p and q is reduced in z'_N.[5]

k-**Equal Labor Consumption Equalization:** For all $e = (s_N, R_N) \in \mathcal{E}$, all $p, q \in N$, all $z_N, z'_N \in Z(e)$ such that $z_i = z'_i$ for all $i \neq p, q$, if

(i) $l_p = l'_p = l_q = l'_q = k$;

(ii) $z_p > z'_p > z'_q > z_q$;

(iii) for all $j \in \{p, q\}$, $z_j \in m(R_j, B(z_j, s_j))$ and $z'_j \in m(R_j, B(z'_j, s_j))$;

then $z'_N \,\mathbf{R}(e)\, z_N$.

Our next axiom captures the idea that individuals should be held responsible for their preferences and that society should not treat them differently — which, in this context, means that it should not tax them differently — on the sole basis that they have different preferences. This idea is also an important tenet of Kolm's conception of fairness. Consequently, if two individuals have the same skill but possibly different preferences, then they should be given the same treatment, which we interpret as requiring the social

[5]This kind of inequality reduction principle can be traced back to Hammond (1976, 1979). What is specific here is that it is applied to consumption rather than welfare, and for special amounts of labor.

evaluation to focus on the opportunities available to them rather than their particular choice of consumption and labor. Consider two individuals p and q endowed with the same skill s and facing different budget sets $B(z_p, s)$ and $B(z_q, s)$. One set contains the other and the corresponding individual can then be regarded as relatively richer than the other. Assume, now, that we permute their budget sets. By doing so, we may have increased or decreased the observed inequality in consumption or in labor time, depending on these individuals' preferences. Nevertheless, the axiom states that the resulting allocation is equally fair (or equally unfair) as the initial one, because the distribution of budget sets is unchanged.[6] Formally,

Budget Anonymity: For all $e = (s_N, R_N) \in \mathcal{E}$, all $p, q \in N$ such that $s_p = s_q$, all $z_N, z'_N \in Z(e)$ such that $z_i = z'_i$ for all $i \neq p, q$, if

(i) $B(z'_p, s_p) = B(z_q, s_q)$ and $B(z'_q, s_q) = B(z_p, s_p)$;

(ii) for all $j \in \{p, q\}$, $z_j \in m(R_j, B(z_j, s_j))$ and $z'_j \in m(R_j, B(z'_j, s_j))$;

then $z'_N \, \mathbf{I}(e) \, z_N$.

The third axiom is the classical Strong Pareto axiom.

Strong Pareto: For all $e = (s_N, R_N) \in \mathcal{E}$, all $z_N, z'_N \in Z(e)$, if $z_i \, R_i \, z'_i$ for all $i \in N$, then $z_N \, \mathbf{R}(e) \, z'_N$; if, in addition, $z_j \, P_j \, z'_j$ for some $j \in N$, then $z_N \, \mathbf{P}(e) \, z'_N$.

The last axiom is a separability condition. It states that when an individual has the same bundle in two allocations, the ranking of these two allocations should remain the same if this individual were simply absent from the economy. Let $|N|$ denote the cardinality of N.

Separation: For all $e = (s_N, R_N) \in \mathcal{E}$ with $|N| \geq 2$, all $z_N, z'_N \in Z(e)$, if there is $j \in N$

[6] We could as well formulate an axiom that warrants inequality reduction in budget sets for agents with the same skill. The rest of the analysis would follow with little modification.

such that $z_j = z'_j$, then

$$z_N \, \mathbf{R}(e) \, z'_N \Rightarrow z_{N\setminus\{j\}} \, \mathbf{R}(e') \, z'_{N\setminus\{j\}}$$

where $e' = (s_{N\setminus\{j\}}, R_{N\setminus\{j\}}) \in \mathcal{E}$.

As can be easily checked, the k-Leximin SOFs presented in the previous section satisfy our four axioms. Moreover, any SOF satisfying these axioms must rank allocations exactly like a k-Leximin SOF whenever the lowest levels of utility U^k differ in the allocations being compared.

Proposition 1 *For all* $k \in [0, 1]$:

(i) On $Z(e)$, *the* k*-Leximin SOF satisfies* k*-Equal Labor Consumption Equalization, Budget Anonymity, Strong Pareto and Separation.*

(ii) If a SOF satisfies k*-Equal Labor Consumption Equalization, Budget Anonymity, Strong Pareto and Separation, then it satisfies the following property: for all* $e = (s_N, R_N) \in \mathcal{E}$, *all* $z_N, z'_N \in Y^k(e)$, *if*

$$\min_{i \in N} U^k(z_i, s_i, R_i) > \min_{i \in N} U^k(z'_i, s_i, R_i)$$

then $z_N \, \mathbf{P}(e) \, z'_N$.

The proof of (ii) is in the Appendix.

4 Second best: Observable labor time

We now turn to second-best situations. In this section, we assume that the planner only observes earnings $w_i l_i$ and labor time l_i, so that she can deduce wage rates w_i, but she does not observe the individuals' earning abilities s_i and any individual can choose to work at a lower wage rate than his maximum possible, $w_i < s_i$, if this is in his interest.[7]

[7] Observe that the first-best ELIE allocation can be implemented when s_i is observable, even if individual preferences are private information.

Observe that, in Figure 4 (a variant of Figure 3 with two more individuals), individual 2 would prefer to have individual 3's bundle rather than his own. It would be advantageous for him to work at the same wage rate as individual 3 because this would give him access to individual 3's bundle. Therefore, the ELIE first-best allocation is not, in general, incentive-compatible in this context.[8]

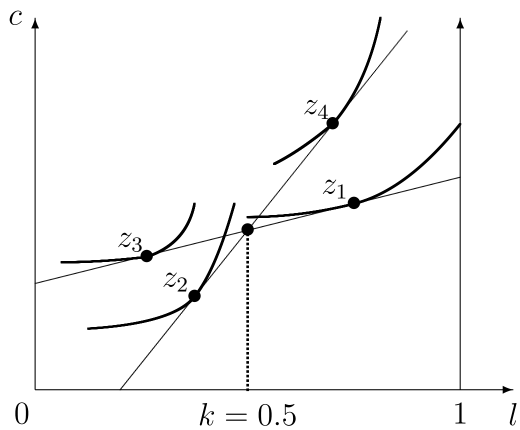

Figure 4

Because the wage rates w_i are (indirectly) observed, the planner can offer a tax function on earnings, $\tau_w : [0, w] \to \mathbb{R}$, that is specific to each value of w. Individual i will then choose w_i and (l_i, c_i) maximizing his satisfaction subject to the constraint that $w_i \leq s_i$ and $c_i \leq w_i l_i - \tau_{w_i}(w_i l_i)$. One can have $\tau_{w_i}(w_i l_i) < 0$, in which case the tax turns into a subsidy. When $c_i = w_i l_i - \tau_{w_i}(w_i l_i)$ for all $i \in N$, the allocation is feasible if and only if $\sum_{i \in N} \tau_{w_i}(w_i l_i) \geq 0$.

[8] Anticipating on notions introduced in this section, we may note that the allocation depicted in Figure 3 is incentive-compatible, because neither individual would want to mimic the other. But this allocation cannot be implemented by the menu of budget sets shown on Figure 3, because individual 2 would like to be able to choose from individual 1's budget set. Another menu must be offered in order to implement the allocation (e.g., offer a menu such that for the two skill levels of these individuals, the post-tax budget set is the same and coincides with the intersection of the budget sets of the figure).

An *incentive-compatible* allocation $x_N = (w_N, z_N)$, in this context, is obtained when no individual envies the bundle of any other individual working at a wage rate he could earn: for all $i, j \in N$, if $s_i \geq w_j$ then $(l_i, c_i)\, R_i\, (l_j, c_j)$. This definition does not refer to tax menus, but there is a classical connection between incentive-compatibility and taxes, which unfolds as follows in the current context. First, every allocation obtained by offering a menu $\{\tau_w\}$ and letting the individuals choose their wage rate and bundle subject to the skill constraint $w_i \leq s_i$ and the budget constraint $c_i \leq w_i l_i - \tau_{w_i}(w_i l_i)$ is incentive-compatible.

Conversely, every incentive-compatible allocation can be obtained by offering a menu of tax functions $\{\tau_w\}$ and letting the individuals choose their wage rate and bundle subject to the skill constraint $w_i \leq s_i$ and the budget constraint $c_i \leq w_i l_i - \tau_{w_i}(w_i l_i)$. For instance, the tax function τ_w can be defined so that the graph of $f_w(l) = wl - \tau_w(wl)$ in the (l, c) space is the lower envelope of the indifference curves of all individuals i such that $w_i \geq w$. For further reference, let this menu of taxes be called the "envelope menu". Note the following fact: when $w > w'$, the set of individuals i for whom $w_i \geq w'$ contains the set of individuals for whom $w_i \geq w$, so that the lower envelope of the indifference curves of the former set is nowhere above the lower envelope for the latter set. In other words, the envelope menu $\{\tau_w\}$ satisfies the following "nesting property": for all $w > w'$, all $l \in [0, 1]$, $wl - \tau_w(wl) \geq w'l - \tau_{w'}(w'l)$. In conclusion, every incentive-compatible allocation can be implemented by a tax menu $\{\tau_w\}$ satisfying the nesting property.

Because w_i does not affect i's satisfaction directly, it is inefficient to let him work at a wage $w_i < s_i$. Fortunately, it is always possible to replace an incentive-compatible allocation $x_N = (w_N, z_N)$ by another $x'_N = (s_N, z_N)$ in which every individual works at his potential and obtains the same bundle as in x_N. This is because for all $i, j \in N$, $s_i \geq s_j$ implies $s_i \geq w_j$, so that if for all $i, j \in N$, $(l_i, c_i)\, R_i\, (l_j, c_j)$ whenever $s_i \geq w_j$, then one also has $(l_i, c_i)\, R_i\, (l_j, c_j)$ whenever $s_i \geq s_j$, which is equivalent to saying that x'_N is incentive-compatible. Therefore, from now on we will focus on bundle-allocations z_N and simply assume that $w_i = s_i$ for all $i \in N$.

Let us now fix $e = (s_N, R_N) \in \mathcal{E}$. Let $\underline{s}$ denote the lowest component in s_N. Our first result is that an optimal allocation for $\mathbf{R}^k(e)$ can be obtained by a menu $\{\tau_w\}$ such that the individuals with the lowest skill face a zero marginal tax. This result is obtained under the following assumption.

Restriction 1 *For all $i \in N$, there is $j \in N$ such that $s_j = \underline{s}$ and $R_j = R_i$.*

This restriction is quite natural for large populations. It is satisfied when preferences and skills are independently distributed, but it is much weaker than that.

Proposition 2 *Assume that earnings and labor time (but not skill) are observable. Let $e = (s_N, R_N) \in \mathcal{E}$ satisfy Restriction 1. Every second-best optimal allocation for $\mathbf{R}^k(e)$ can be obtained by a menu $\{\tau_w\}$ such that $\tau_{\underline{s}}$ is a non-positive constant-valued function.*

Proof. At a laissez-faire allocation z_N^L, $U^k(z_i^L, s_i, R_i) = ks_i$ for all $i \in N$. Therefore, at this allocation, $\min_{i\in N} U^k(z_i^L, s_i, R_i) = k\underline{s}$.

Let z_N^* be an optimal allocation for $\mathbf{R}^k(e)$. The structure of the argument is the following. If z_N^* cannot be obtained by a menu $\{\tau_w\}$ such that $\tau_{\underline{s}}$ is a non-positive constant-valued function, then it is possible to define a new menu $\{\hat{\tau}_w\}$ such that $\hat{\tau}_{\underline{s}}$ is a non-positive constant-valued function, with a corresponding allocation $\hat{z}_N$ such that $\min_{i\in N} U^k(\hat{z}_i, s_i, R_i) \geq \min_{i\in N} U^k(z_i^*, s_i, R_i)$ and such that there is a budget surplus, which proves that z_N^* is not optimal (the budget surplus can be redistributed so as to increase the welfare of all agents).

Because $z_N^*\, \mathbf{R}^k(e)\, z_N^L$, necessarily $\min_{i\in N} U^k(z_i^*, s_i, R_i) \geq k\underline{s}$. Let $\{\tau_w\}$ be the envelope menu implementing z_N^*. By construction, the graph of $f_{\underline{s}}(l) = \underline{s}l - \tau_{\underline{s}}(\underline{s}l)$ is the lower envelope of the indifference curves of all $i \in N$ at z_N^*. Consider now the lower envelope of the indifference curves of individuals $i \in N$ such that $s_i = \underline{s}$, and suppose that it lies above $f_{\underline{s}}(l)$ for some $l \in [0,1]$. By Restriction 1, this implies that there are i, j such that $R_i = R_j$, $s_i = \underline{s}$, $s_j > \underline{s}$ and the indifference curve of i at z_i^* is above that of j at z_j^*, in contradiction with incentive-compatibility. Therefore, the graph of $f_{\underline{s}}(l)$ is also the lower envelope of the indifference curves of individuals $i \in N$ such that $s_i = \underline{s}$.

Let $a = \max \tau_{\underline{s}}$. One must have $a \leq 0$ for the following reason. For any individual $i \in N$,

$$U^k(z_i^*, s_i, R_i) = \min \{c - s_i l \mid (l, c) \, R_i \, z_i^*\} + k s_i.$$

Because the graph of $f_{\underline{s}}$ is the envelope curve of the indifference curves of individuals i such that $s_i = \underline{s}$, necessarily there is one such i for whom

$$\min \{c - s_i l \mid (l, c) \, R_i \, z_i^*\} = -\max \tau_{\underline{s}}.$$

For this individual, then, $U^k(z_i^*, s_i, R_i) = -a + k\underline{s}$. Recall that $U^k(z_i^*, s_i, R_i) \geq k\underline{s}$. Therefore, $a \leq 0$.

Let $\hat{\tau}_{\underline{s}}(\underline{s}l) = a$ for all $l \in [0,1]$. The menu $\{\tau_w \mid w > \underline{s}\} \cup \{\hat{\tau}_{\underline{s}}\}$ (i.e., $\hat{\tau}_{\underline{s}}$ replaces $\tau_{\underline{s}}$) still satisfies the nesting property. In every allocation $\hat{z}_N$ obtained with this new menu, all $i \in N$ such that $s_i = \underline{s}$ receive the subsidy $-a$ and have $U^k(\hat{z}_i, s_i, R_i) = -a + k\underline{s}$. For $i \in N$ such that $s_i > \underline{s}$, $U^k(\hat{z}_i, s_i, R_i) = U^k(z_i^*, s_i, R_i)$. Therefore $\min_{i \in N} U^k(\hat{z}_i, s_i, R_i) \geq \min_{i \in N} U^k(z_i^*, s_i, R_i)$. Suppose that for some $i \in N$ such that $s_i = \underline{s}$, z_i^* is no longer in the budget set. This implies that $\tau_{\underline{s}}(s_i l_i^*) < a$ and that when choosing from the new menu, i gets a lower subsidy (namely, $-a$). If there is such an individual, the new menu generates a budget surplus. This budget surplus can be redistributed so as to increase[9]

[9]For the proof that a budget surplus always makes it possible to obtain another incentive-compatible allocation in which every individual is strictly better-off, see Fleurbaey and Maniquet (2006, Lemma 3).

$\min_{i \in N} U^k(z_i^*, s_i, R_i)$, in contradiction with the assumption that z_N^* was optimal for $\mathbf{R}^k(e)$. In conclusion, z_N^* must still be implementable with the new menu. ■

By a similar argument one can show that every optimal allocation can be obtained by a tax menu such that the graph of $f_w(l) = wl - \tau_w(wl)$ lies in the dashed area depicted in Figure 5.

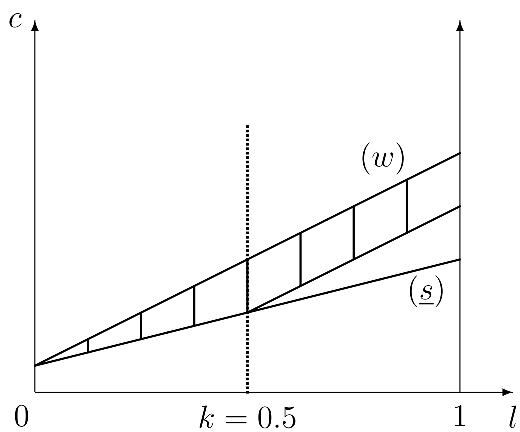

Figure 5

Our next result focuses on the case when the lower bound of such areas is binding. This makes the configuration of budget lines the closest possible, under incentive-compatibility constraints, to the first-best ELIE configuration that was shown in Figures 3–4. It is illustrated in Figure 6.

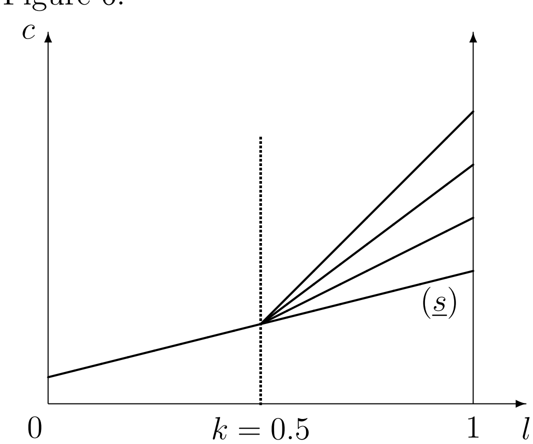

Figure 6

Proposition 3 *Assume that earnings and labor time (but not skill) are observable. Let $e = (s_N, R_N) \in \mathcal{E}$ satisfy Restriction 1 and let $b \geq 0$. For each w, let τ_w be defined by: $\tau_w(wl) = (w - \underline{s})\, l - b$ for $l \leq k$ and $\tau_w(wl) = (w - \underline{s})\, k - b$ for $l \geq k$. If an allocation obtained with the menu $\{\tau_w\}$ is second-best Pareto-efficient, then it is second-best optimal for $\mathbf{R}^k(e)$.*

Proof. Let z_N^* be an allocation obtained with $\{\tau_w\}$. By construction one has

$$\min_{i \in N} U^k(z_i^*, s_i, R_i) = b + k\underline{s}.$$

Assume that z_N^* is Pareto-efficient among all incentive-compatible feasible allocations. Let z_N be another feasible and incentive-compatible allocation such that for some $i \in N$,

$$U^k(z_i, s_i, R_i) > U^k(z_i^*, s_i, R_i).$$

This implies $z_i \, P_i \, z_i^*$. Because z_N^* is Pareto-efficient, there is another $j \in N$ for whom $z_j^* \, P_j \, z_j$.

Two cases must be distinguished.

First case: $l_j^* < k$. By Restriction 1, there is $i_0 \in N$ such that $s_{i_0} = \underline{s}$ and $R_{i_0} = R_j$. Because $l_j^* < k$, z_j^* is in i_0's budget set (see Fig. 6) and $z_{i_0}^* \, I_{i_0} \, z_j^*$. The fact that $z_j^* \, P_j \, z_j$ and that, by incentive-compatibility, $z_j \, R_j \, z_{i_0}$ implies $z_{i_0}^* \, P_{i_0} \, z_{i_0}$, or equivalently,

$$U^k(z_{i_0}, s_{i_0}, R_{i_0}) < U^k(z_{i_0}^*, s_{i_0}, R_{i_0}).$$

As $U^k(z_{i_0}^*, s_{i_0}, R_{i_0}) = b + k\underline{s} = \min_{i \in N} U^k(z_i^*, s_i, R_i)$, this implies that

$$\min_{i \in N} U^k(z_i, s_i, R_i) < \min_{i \in N} U^k(z_i^*, s_i, R_i).$$

Second case: $l_j^* \geq k$. By construction one then has

$$U^k(z_j^*, s_j, R_j) = b + k\underline{s} = \min_{i \in N} U^k(z_i^*, s_i, R_i),$$

which implies again that

$$\min_{i \in N} U^k(z_i, s_i, R_i) < \min_{i \in N} U^k(z_i^*, s_i, R_i).$$

In conclusion, z_N cannot be better than z_N^* for $\mathbf{R}^k(e)$. ■

This result may seem to have limited scope because it is generally unlikely that the menu $\{\tau_w\}$ as defined in the proposition generates a second-best efficient allocation. But one can safely conjecture that if the allocation obtained with this menu is not too inefficient, then the optimal tax menu is close to $\{\tau_w\}$. Note that for $k = 0$, this menu corresponds to the laissez-faire policy (one must then have $b = 0$), which yields an efficient allocation and is indeed optimal for $\mathbf{R}^0(e)$. The likelihood that the optimal menu is close to $\{\tau_w\}$ therefore increases when k is smaller.

In practice, a menu like $\{\tau_w\}$ is easy to enforce (assuming that labor time or wage rates are observable), and one can then proceed to check if it generates large inefficiencies.

5 Second best: Unobservable labor time

We now turn to a different second-best context, in which we assume that the planner only observes earned incomes $y_i = w_i l_i$ and is unable to identify the individuals' wage rates, as in the classical literature following Mirrlees (1971). Therefore, redistribution is now made via a single tax function τ. Observe that, in this context, it is always best for every individual $i \in N$ to earn any given gross income by working at his maximal wage rate $w_i = s_i$, because, as tax depends on y_i and not on w_i or l_i, this minimizes l_i for a fixed level of consumption. We can therefore focus on bundle-allocations z_N and simply assume that $w_i = s_i$ for all $i \in N$.

Under this kind of redistribution, individual i's budget set is defined by (see Fig. 7a):

$$B^\tau(s_i) = \{(l, c) \in X |\ c \leq s_i l - \tau(s_i l)\}.$$

It is convenient to focus on the earnings-consumption space, in which the budget set is defined by (see Fig. 7b; we retain the same notation B^τ as no confusion is possible):

$$B^\tau(s_i) = \{(y, c) \in [0, s_i] \times \mathbb{R}_+ \mid c \leq y - \tau(y)\}.$$

In Fig. 7b, the indifference curve is re-scaled so that the choice of labor time in $[0, 1]$ become equivalent to a choice of earnings in $[0, s_i]$.

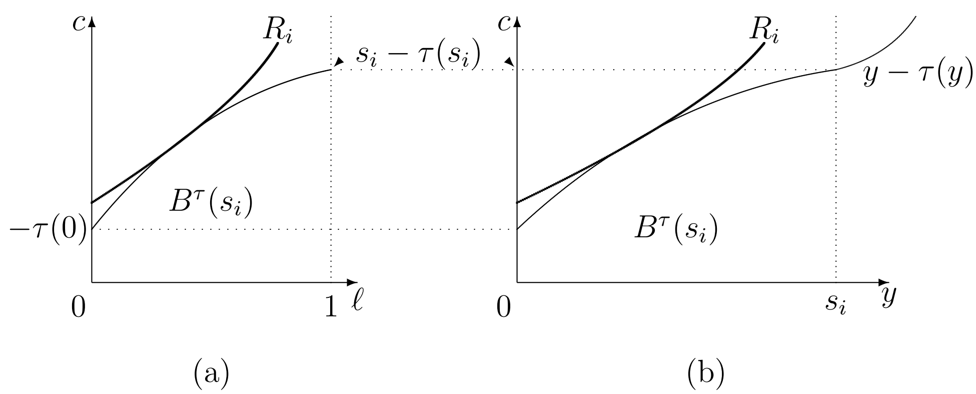

Figure 7

An incentive-compatible allocation z_N, in this context, is obtained when no individual envies the *earnings*-consumption bundle of any other individual earning a level of gross income he could earn: for all $i, j \in N$, if $s_i \geq y_j$ then $(l_i, c_i)\, R_i\, (y_j/s_i, c_j)$.

Every allocation obtained by offering budget sets defined by a tax function τ is incentive-compatible. Conversely, every incentive-compatible allocation can be obtained by offering a tax function τ and letting every individual $i \in N$ choose his bundle in the budget set $B^\tau(s_i)$. For instance, the tax function τ can be defined so that the graph of $f(y) = y - \tau(y)$ in the (y, c) space is the lower envelope of the indifference curves of all agents in this space at the allocation under consideration.

For any $e = (s_N, R_N) \in \mathcal{E}$, let $S = \{s_i \mid i \in N\}$, $\underline{s} = \min S$, and $\overline{s} = \max S$. As in the previous section, our reasoning below does not require a specific number of individuals, but it is clear that we think of economies with a finite but large number of individuals. In particular, we will impose a restriction on the population of individuals so that the pre-tax income cannot be too informative a signal. According to this restriction, over any interval of earnings $[0, s]$ it is impossible, by only looking at preferences over *earnings* and consumption restricted to that interval of earnings, to identify individuals with greater productivity than s and distinguish them, on the basis of their preferences, from individuals with productivity s.

Restriction 2 *For all $i \in N$, all $s \in S$ such that $s < s_i$, there is $j \in N$ such that $s_j = s$ and for all $(y, c), (y', c') \in [0, s] \times \mathbb{R}_+$:*

$$\left(\frac{y}{s_j}, c\right) R_j \left(\frac{y'}{s_j}, c'\right) \Leftrightarrow \left(\frac{y}{s_i}, c\right) R_i \left(\frac{y'}{s_i}, c'\right).$$

Our first result is that the zero marginal tax result still holds for low-skilled individuals.

Proposition 4 *Assume that only earnings are observable. Let $e = (s_N, R_N) \in \mathcal{E}$ satisfy Restriction 2. Every second-best optimal allocation for $\mathbf{R}^k(e)$ can be obtained by a tax function τ that is constant over $[0, \underline{s}]$.*

Proof. This is a corollary of Theorem 3 in Fleurbaey and Maniquet (2007), since $\mathbf{R}^k(e)$ coincides with "Equivalent-Budget" social preferences (defined in that paper) for reference preferences $\tilde{R}$ with indifference curves having cusps at the vertical of k and a marginal rate of substitution at any point $(l, c) \neq (k, c)$ which is lower than $\underline{s}$ if $l < k$ and greater than $\overline{s}$ if $l > k$. ■

We now focus on a particular kind of tax function which satisfies this property. It is defined as follows:

(i) for all $y \in [0, \underline{s}] : \tau(y) = \tau(0) \leq 0$;

(ii) for all $s, s' \in S$ such that $s < s'$ and $s < s_i < s'$ for no $i \in N$, all $y \in [s, s']$,

$$\tau(y) = \min\{\tau(0) + k(s - \underline{s}) + (y - s), \tau(0) + k(s' - \underline{s})\}.$$

A tax function of this kind will be called a k-type tax. This formula calls for some explanations. The tax function is piece-wise linear. The segment on low incomes $[0, \underline{s}]$ is constant, with a fixed subsidy $-\tau(0)$. Then comes a segment, $[\underline{s}, y^1]$, for some y^1 between $\underline{s}$ and the next element s^1 of S, where the rate of taxation is a hundred percent. Of course, no individual is expected to earn an income in this interval. The next segment covers the interval $[y^1, s^1]$, and has a zero marginal tax rate. Then, the function continues with successive pairs of intervals, one with a hundred percent of marginal tax and the other with a zero marginal tax. The key feature is that the points $(s, \tau(s))$, for $s \in S$, are aligned, and the slope of the line is precisely k, that is, for all $s, s' \in S$,

$$\frac{\tau(s) - \tau(s')}{s - s'} = k.$$

When S is a large set with elements spread over the interval $[\underline{s}, \overline{s}]$, the tax function is therefore approximately a flat tax (constant marginal tax rate), except for the $[0, \underline{s}]$ interval where it is constant.

The corresponding budget set delineated by $y - \tau(y)$ is illustrated in Figure 8, where the indifference curves of five individuals (rescaled so as to fit into (y, c)-space) are also depicted.

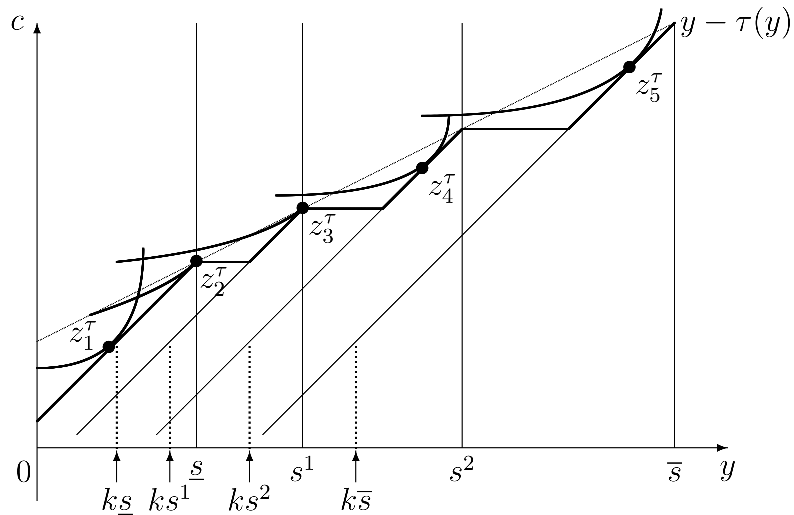

Figure 8

An interesting property of a k-type tax function τ is that for all $i \in N$, if i chooses a bundle that is on the segment with slope 1 (zero marginal tax) just below s_i, then

$$U^k(z_i, s_i, R_i) = -\tau(0) + k\underline{s}.$$

Let us prove this fact (called property P for further reference) by focusing, for clarity but without loss of generality, on individual 4 from Figure 8, assuming that $s_4 = s^2$. From the second term in the definition of τ, one has

$$\tau(y_4) = \tau(0) + k(s^2 - \underline{s}),$$

Bundle z_4 is optimal for R_4 in the budget set for which this level of tax is lump-sum. Therefore one simply has

$$\begin{aligned} U^k(z_4, s_4, R_4) &= -\tau(y_4) + ks^2 \\ &= -\tau(0) + k\underline{s}, \end{aligned}$$

which achieves the proof. Note that the quantity $-\tau(0) + k\underline{s}$ is independent of s_i. This tax function therefore equalizes U^k across individuals who work full time or just below full time. This equality can be seen in Figure 8, where the vertical segments at $k\underline{s}$, ks^1, ks^2 and $k\overline{s}$ between the horizontal axis and the lines of slope 1 corresponding to each of these values of s all have the same length.

From Figure 8 it is also clear that an individual who chooses a bundle on a lower segment of the budget set must have a greater implicit budget, so that $-\tau(0) + k\underline{s}$ is a lower bound for $U^k(z_i, s_i, R_i)$, for all $i \in N$. Note that this lower bound is attained at least by all $i \in N$ such that $s_i = \underline{s}$. Therefore every allocation z_N obtained with a k-type tax is such that $\min_{i \in N} U^k(z_i, s_i, R_i) = -\tau(0) + k\underline{s}$.

This observation is important in order to obtain the next result.

Proposition 5 *Assume that only earnings are observable. Let $e = (s_N, R_N) \in \mathcal{E}$ satisfy Restriction 2. If an allocation obtained with a k-type tax is second-best Pareto-efficient, then it is second-best optimal for $\mathbf{R}^k(e)$.*

Proof. Let z_N be an allocation obtained by a k-type tax function τ. The value of $\min_{i \in N} U^k(z_i, s_i, R_i)$ is $-\tau(0) + k\underline{s}$, as explained in the paragraph preceding the proposition. Assume that z_N is efficient in the set of incentive-compatible feasible allocations.

Let z'_N be another feasible and incentive-compatible allocation (not necessarily obtained by a k-type tax function), such that not all individuals are indifferent between z_i and z'_i. By Pareto-efficiency of z_N, there must be some $i \in N$ such that $z_i \, P_i \, z'_i$, or equivalently, $U^k(z_i, s_i, R_i) > U^k(z'_i, s_i, R_i)$.

Two cases must be distinguished.

First case: y_i is on the last segment of i's budget set (i.e., the segment with slope 1 just below s_i). In this case, by property P, one has

$$U^k(z_i, s_i, R_i) = -\tau(0) + k\underline{s} = \min_{i \in N} U^k(z_i, s_i, R_i),$$

so that $U^k(z_i, s_i, R_i) > U^k(z_i', s_i, R_i)$ implies

$$\min_{i \in N} U^k(z_i, s_i, R_i) > \min_{i \in N} U^k(z_i', s_i, R_i)$$

and therefore $z_N \, \mathbf{P}^k(e) \, z_N'$.

Second case: y_i is on the last segment of the budget set for some $s < s_i$. By Restriction 2 there is j such that $s_j = s$ and j has the same preferences as i over bundles (y, c) such that $y \in [0, s]$. The fact that $z_i \, P_i \, z_i'$ then implies that $z_j \, P_j \, z_j'$ (if j could obtain a bundle (y', c') that were at least as good as (y_j, c_j), i could also have it). As a consequence,

$$U^k(z_j, s_j, R_j) = -\tau(0) + k\underline{s} > U^k(z_j', s_j, R_j),$$

implying that

$$\min_{i \in N} U^k(z_i, s_i, R_i) > \min_{i \in N} U^k(z_i', s_i, R_i)$$

and therefore $z_N \, \mathbf{P}^k(e) \, z_N'$.

In conclusion z_N is optimal for $\mathbf{R}^k(e)$. ■

If no allocation obtained with a k-type tax function is efficient, the k-type tax functions are nonetheless interesting benchmarks because of the property that individuals working full time have the minimum value of $U^k(z_i, s_i, R_i)$, whatever their skill. Since two different k-type tax functions (for the same k) define nested budget sets (one function always dominates the other), it is a valuable exercize to seek the lowest feasible k-type tax function, and we conjecture that the optimal tax function is often close to it for low values of k. (For $k = 0$, the lowest feasible k-type tax function is the laissez-faire policy $\tau \equiv 0$, which is optimal for $\mathbf{R}^0(e)$.)

6 Concluding comments

Kolm's ELIE proposal strikes the imagination because it yields a simple configuration of budget sets. Such a configuration, however, is not compatible with incentives in general when individuals' earning potential is not observable (even when, as Kolm assumes, labor time is observable). Our purpose in this paper has been to extend the ELIE concept into a full social ordering that can serve to rank all allocations, and to examine the optimal tax that one derives from this ordering in the two prominent second-best contexts studied in the optimal taxation literature.

We have seen that one feature of the ELIE first-best configuration is preserved in the two second-best contexts studied here: the low-skilled individuals should face a zero marginal tax rate. This is a rather striking feature of a tax rule. It contrasts substantially with classical results obtained with standard social welfare functions in the optimal taxation literature where individuals are assumed to have the same preferences over labor and consumption and to differ only in their skills.[10] But it is consonant with recent reforms of income tax and income support institutions in countries like the United States or the United Kingdom.

Another, perhaps more surprising, result is the connection between ELIE and the flat tax proposal. For tax functions that bear on total earnings (and do not depend on labor time or the wage rate), ELIE can be roughly summarized as "a zero marginal tax up to the lowest wage and the constant marginal tax rate k beyond that". When the lowest wage rate is zero (for instance because of unemployment), this description boils down to the flat tax at rate k.

A full description of the optimal tax has not been provided in this paper because in a model with multi-dimensional heterogeneity of individuals, it is extremely difficult

[10] See, e.g., Diamond (1998).

to provide a precise description of the set of feasible taxes. This leaves opportunities for future research. But it may be worth stressing that the social ordering function proposed in this paper can easily be used to evaluate any feasible allocation when the distribution of the population characteristics is known, and in previous works we have shown how to make evaluations directly from the budget set generated by the tax function.[11] This can be done as well with $\mathbf{R}^k$. For instance, in the framework of the previous section, one can evaluate an arbitrary tax function by seeking the lowest k-type tax function that lies nowhere below it. When political constraints make the optimal tax out of reach, this kind of criterion can serve to evaluate piecemeal reforms of suboptimal tax functions.

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[11] See Fleurbaey (2006), Fleurbaey and Maniquet (2006).

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Appendix

In this appendix, we prove Proposition 1(ii), that is: for any given $k \in [0, 1]$, if a SOF satisfies k-Equal Labor Consumption Equalization, Budget Anonymity, Strong Pareto and Separation, then it satisfies the following property: for all $e = (s_N, R_N) \in \mathcal{E}$ and all $z_N, z'_N \in Y^k(e)$, if

$$\min_{i \in N} U^k(z_i, s_i, R_i) > \min_{i \in N} U^k(z'_i, s_i, R_i)$$

then $z_N \, \mathbf{P}(e) \, z'_N$.

First step. We first prove that for all $e = (s_N, R_N) \in \mathcal{E}$, $p, q \in N$, $z_N, z'_N \in Y^k(e)$ such that $z_i = z'_i$ for all $i \neq p, q$, if

$$U^k(z_p, s_p, R_p) < U^k(z'_p, s_p, R_p) < U^k(z'_q, s_q, R_q) < U^k(z_q, s_q, R_q),$$

then $z'_N \, \mathbf{R}(e) \, z_N$. Let $z^1_N, z^2_N \in Y^k(e)$ be defined by: for all $j \in \{p, q\} : z^1_j \, I_j \, z_j$, $z^2_j \, I_j \, z'_j$, $z^1_j \in IB(z_j, s_j, R_j)$ and $z^2_j \in IB(z'_j, s_j, R_j)$; for all $j \notin \{p, q\} : z^1_j = z_j$, $z^2_j = z'_j$. By Strong Pareto,

$$z_N \, \mathbf{I}(e) \, z^1_N \text{ and } z'_N \, \mathbf{I}(e) \, z^2_N.$$

We now define $M = \{a, b\} \in \mathcal{N}$ with $a, b \notin N$, $s_a = s_p$, $s_b = s_q$, and $R_a = R_b \in \mathcal{R}$ such that, when facing budget sets with slope s_p and s_q, respectively, they choose a labor time equal to k. Let $z^1_a, z^1_b, z^2_a, z^2_b \in X$ be defined by

$$\begin{aligned} z^1_a &\in m(R_a, B(z^1_p, s_a)), \\ z^1_b &\in m(R_b, B(z^1_q, s_b)), \\ z^2_a &\in m(R_a, B(z^2_p, s_a)), \\ z^2_b &\in m(R_b, B(z^2_q, s_b)). \end{aligned}$$

We need to prove that $z'_N \, \mathbf{R}(e) \, z_N$. Assume, on the contrary, that $z_N \, \mathbf{P}(e) \, z'_N$. Then, by Strong Pareto,

$$\left(z_{N \setminus \{p,q\}}, z^1_p, z^1_q\right) \, \mathbf{P}(e) \, (z'_{N \setminus \{p,q\}}, z^2_p, z^2_q).$$

By Separation,

$$\left(z_{N \setminus \{p,q\}}, z^1_p, z^1_q, z^2_a, z^2_b\right) \, \mathbf{P}(e') \, (z'_{N \setminus \{p,q\}}, z^2_p, z^2_q, z^2_a, z^2_b).$$

where $e' = ((s_N, s_a, s_b), (R_N, R_a, R_b))$.[12] By Budget Anonymity, swapping the budgets of individuals p and a, as well as those of q and b, one gets

$$\left(z_{N\setminus\{p,q\}}, z_p^2, z_q^2, z_a^1, z_b^1\right) \mathbf{I}(e') \left(z_{N\setminus\{p,q\}}, z_p^1, z_q^1, z_a^2, z_b^2\right).$$

Observe that $z_a^1 < z_a^2 < z_b^2 < z_b^1$. By k-Equal Labor Consumption Equalization,

$$\left(z_{N\setminus\{p,q\}}, z_p^2, z_q^2, z_a^2, z_b^2\right) \mathbf{R}(e') \left(z_{N\setminus\{p,q\}}, z_p^2, z_q^2, z_a^1, z_b^1\right)$$

By transitivity, if we gather the above relations, we get

$$\left(z_{N\setminus\{p,q\}}, z_p^2, z_q^2, z_a^2, z_b^2\right) \mathbf{P}(e') \left(z'_{N\setminus\{p,q\}}, z_p^2, z_q^2, z_a^2, z_b^2\right),$$

an obvious contradiction (recall that $z_{N\setminus\{p,q\}} = z'_{N\setminus\{p,q\}}$).

Second step. Let $e = (s_N, R_N) \in \mathcal{E}$, and $z_N, z'_N \in Y^k(e)$. Assume that

$$\min_{i\in N} U^k(z_i, s_i, R_i) > \min_{i\in N} U^k(z'_i, s_i, R_i)$$

whereas $z'_N \,\mathbf{R}(e)\, z_N$. Let $p \in N$ be such that $U^k(z'_p, s_p, R_p) = \min_{i\in N} U^k(z'_i, s_i, R_i)$. Let $z_N^1 \in Y^k(e)$ be such that $U^k(z_p^1, s_p, R_p) = U^k(z'_p, s_p, R_p)$ and for all $j \in N \setminus \{p\}$,

$$U^k(z_j^1, s_j, R_j) > \max\left\{U^k(z_j, s_j, R_j), U^k(z'_j, s_j, R_j)\right\}.$$

By Strong Pareto, $z_N^1 \,\mathbf{P}(e)\, z'_N$. Let $z_N^2 \in Y^k(e)$ be such that for all $j \in N \setminus \{p\}$,

$$\min_{i\in N} U^k(z_i, s_i, R_i) > U^k(z_j^2, s_j, R_j) > U^k(z_p^2, s_p, R_p) > U^k(z_p^1, s_p, R_p).$$

By an iterative application of the first step, we can prove that $z_N^2 \,\mathbf{R}(e)\, z_N^1$. That is, for each $j \in N \setminus \{p\}$ we apply the argument to individuals j and p, decreasing the index of

[12]If z_N and z'_N are feasible, then so are z_N^1 and z_N^2, but $\left(z_{N\setminus\{p,q\}}, z_p^1, z_q^1, z_a^2, z_b^2\right)$ and $(z'_{N\setminus\{p,q\}}, z_p^2, z_q^2, z_a^2, z_b^2)$ need not be. This is, therefore, where the proof needs to be changed if we want to restrict the definition of SOF to feasible allocations. But the change is minor: an agent c should be added as well, having a sufficiently large skill, a sufficiently large labor time and low consumption that the resulting allocation is feasible.

j to $U^k(z_i^2, s_i, R_i)$ while slightly increasing that of individual p, calibrating such increases so that p's index reaches $U^k(z_p^2, s_p, R_p)$ when p is paired with the last individual from $N \setminus \{p\}$. By transitivity, we get $z_N^2 \, \mathbf{P}(e) \, z_N$, contradicting Strong Pareto.

7 Re Koichi's comments

1. We focus on incentive-compatibility of allocations rather than menu of budget sets. The case k close to zero is not ethically appealing, but it is important to mention what happens in this area.

2. A new figure (4) clarifies the incentive problem, and it is mentioned that the allocation in Fig. 3 is actually incentive-compatible.

3. The formulations have changed (and the order of statements about the taxation principle reversed), it should now be clear.

4. OK.

5. Reformulated and sorted out.

6. There seems to be a confusion here. The new formulation should clear it out. (We are dealing with second-best optimality)

7. The propositions now parallel each other.

8. OK.

9. OK (except the first one).

8 Re Maurice's comments

1. "finite" is now specified, thanks!

2. We do not work with a single economy, but with many economies of all sizes (because of Separation), in the axiomatic analysis.

9 Re John's comments

1. The problematic statement is true but superfluous in the proof (it is now deleted). To see that it is true, suppose not. Then one can increase the function $\tau_{\underline{s}}$ without changing the value of $\min_{i \in N} U^k$ (because the min is achieved only by agents with greater skill). This generates a budget surplus which can be used to make every agent better-off, thus raising $\min_{i \in N} U^k$, and contradicting the assumption that the allocation was optimal.

2. OK, a paragraph added.

3. It appears a little superfluous to introduce the notion of a consumption function (because the notation should mention its dependency on τ), but the function itself is mentioned wherever useful, with minimal notation. Concerning the suggestion to make tax functions bear on labor, it appears less cumbersome to keep tax functions as functions of earnings throughout (that is, of course, a matter of preference). It is then transparent why when wage rates are not observed, only one such function must be offered.

4. We are not ashamed of Restriction 1, esp. compared to a litterature in which it is assumed that *all* agents have identical preferences.

5. OK
6. OK
7. We have explored this issue at length. Consistency refers to some agents leaving with their bundle while the others retain the rest of the cake. Separation refers to some agents disappearing, and initial transfers between them and the other agents are forgotten. As Separation is an immediate variant of Separability, we prefer this name.
8. OK (see introduction)
9. The diagrams have been improved.
10. Most suggestions have been adopted, and extrapolation to the rest of the paper has been made. In addition, we have made our best to make the arguments easier to follow, with additional steps and explanations.