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Simple Centrifugal Incentives in Downsian Dynamics

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Abstract

The main purpose of this short paper is to examine how traditional Downsian dynamics (convergence of the parties to the median of the distribution) are altered by the introduction of centrifugal incentives arising from the fact that any motion towards the center induces a lost of votes at the extremes of the electorate. Our analysis provides a new rationale for platform differentiation. It also yields new insights in the case when centripetal incentives are dominant on one side of the political spectrum while centrifugal incentives take over on the other side. This may apply for instance to the 2012 French elections.

JEL classification codes: D71, D72.

Keywords: Electoral Competition; Mixed equilibria; Centrifugal incentives.

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1 Introduction

The main purpose of this short paper is to explore the equilibrium behavior of two parties in a one dimensional spatial model of electoral competition, say $[0, 1]$, similar to Hotelling (1929) and Downs (1957) except[1] for the fact that we introduce some centrifugal incentives. To quote Cox (1990) to whom we borrow this terminology: "*Centripetal incentives lead political parties (or candidates) to advocate centrist policies; centrifugal incentives, on the other hand, lead to the advocacy of more of less extreme positions*". In this paper, by centrifugal incentives, me mean, more broadly, incentives for each party to move in the direction of the other rather than moving towards the center as the notion of center is not always very well defined. Centrifugal incentives may arise from several sources. Here, we have in mind the situation where the two competing parties face the presence of two extreme parties (with fixed positions) at the extreme left and right of the ideological spectrum. Some of the left (right) voters may decide to vote for the extreme left (right)party if they find the conventional left (right) candidate moving too much towards the center. Our model introduces these incentives in the simplest conceivable way. This model may well describe the situation of the 2012 French presidential election where the two conventional left and right candidates (Hollande and Sarkozy) were facing Melenchon at the extreme left and Le Pen at the extreme right. Political observers agree on the fact that the presence of the two extremes had strong implications on the nature of the electoral competition between the two main contenders.

Three types of configurations may happen.

The first one describes the case where for both parties, centripetal incentives dominate centrifugal incentives. In that case, the Downsian logic of *minimal differentiation* where the two parties converge towards a point which may be more or less close to the center depending on the degree of asymmetry between the centrifugal incentives of the two parties applies. For the uniform distribution, the point of convergence will belong to the interval $\left[\frac{1}{3}, \frac{2}{3}\right]$.

In contrast, the second one describes a case where for both parties, centrifugal incentives dominate centripetal incentives. Not surprisingly, in such cases each party converges towards its extreme position. We have a situation of *maximal differentiation*. However, as we will see this case raises some interesting coordination problems if the parties are totally opportunistic i.e. without real attachment to the left or the right. If such a coordination mechanism is absent we prove existence of a symmetric equilibrium in mixed strategies. The

[1]We refer to Duggan (2012) and Osborne (1995) for nice surveys of the multiplicity of variants of the Hotelling-Downs's model which have studied in the literature.

expected degree of platform differentiation in this equilibrium is increasing in the degree of the centrifugal incentives.

The third configuration[2] describes an interesting situation where the balance between the two types of incentives is not the same for the two parties. For one party (say the one on the right) centrifugal incentives dominate centripetal incentives while the domination is reversed for the other party (the one on the left). In such case, the type of dynamics (out of equilibrium) that we should observe is the right party moving to the extreme right in order to (re)conquer part of this electorate and the left party chasing after it to conquer its more moderate voters lost on the way, even at the expense of losing some of the voters at the extreme left as the lost is by assumption not too severe. This may well be the situation that has occurred in the last rounds of the 2012 campaign of French presidential elections. Sarkozy has been advocating policies targeted to the extreme right while Hollande was gaining voters at the center without the fear of losing too much voters on the left. This configuration is interesting as there is no equilibrium in pure strategies : the best reply dynamics are chaotic. We will demonstrate the existence of an equilibrium in mixed strategies. One interesting feature of the equilibrium is that the *support is distant the more extreme plies on the left and that no party plays extreme left policies but the density is decreasing which means that a significant fraction of the probability mass is concentrated around the leftist policy in the support.*

Our analysis provides a rationale for platform differentiation when parties are purely office-motivated and identical in every possible dimension. Platform differentiation in a uni-dimensional setting is usually understood in the literature as the result of some kind of asymmetry between the two competing parties. The predominant explanations[3] combine *a)* uncertainty about the preferences of the voters, *b)* asymmetric policy preferences[4] (the two competing parties are not purely office motivated and they have conflicting policy preferences) and *c)* asymmetric valence characteristics[5] (one of the two parties is perceived to have a non-policy advantage over the other). The present approach indicates that platform differentiation could be totally independent of the characteristics of the two parties and be due to other elements of the political environment. Our approach relates to Palfrey (1984)

[2]There is some similarity between the game in that configuration (when $\alpha = 1$ and $\beta = 0$) and the uni-directional Hotelling-Downs 's model analysed in Cancian, Bergström and Bills (1995),Gabszewicz, Laussel and Le Breton (2008) and Xefteris (2012).

[3]See Calvert (1985) and Roemer (2001).

[4]Notable examples are Wittman (1977), Ortuno-Ortin (1997), Llavador (2006) and Roemer (1994, 1997).

[5]Aragonès and Palfrey (2002), Laussel and Le Breton (2002), Hummel (2010) and Aragonès and Xefteris (2012) show that in such cases pure strategy equilibria do not exist and, thus, the expected degree of policy differentiation is not degenerate.

who shows that two office-motivated candidates may differentiate in equilibrium if they expect entry of a third candidate; each of the two established candidates is afraid that if she approaches the other established candidate too much then the entrant will locate such that she will loose all extreme voters at her side. That is, Palfrey (1984) considers that an established candidate either takes all the extreme voters at her side or none while in our model each candidate may take any arbitrary fraction of the extreme voters; our approach allows us to analyze any possible degree of centrifugal incentives.

The paper is organized as follows. In section 2, we describe the game. Then, in section 3, we provide some general results. In section 4 we expose the equilibrium analysis in the case where the electorate is uniformly distributed over $[0,1]$. Finally in section 5, we discuss some side issues including extensions, interpretations of our setup and equilibrium analysis.

2 The Model

Consider two political parties 1 and 2 competing for an electorate on the interval $[0,1]$. Each voter is described by an ideal point: if a voter votes for one of these two parties, he votes for the party which is the closest to his ideal point. The distribution of ideal points is described by a cumulative distribution function F which is assumed to be absolutely continuous with a density denoted f. We denote by x and y the platforms of parties 1 and 2. For a profile $(x,y) \in [0,1]^2$, we assume that their electoral shares $\pi_1(x,y)$ and $\pi_2(x,y)$ are as follows:

$$\pi_1(x,y) = \begin{cases} \alpha F(x) + (F(\frac{x+y}{2}) - F(x)) \text{ if } x < y \\ \frac{1}{2}(\beta(1-F(x)) + \alpha F(x)) \text{ if } x = y \\ \beta(1-F(x)) + (F(x) - F(\frac{x+y}{2})) \text{ if } x > y \end{cases}$$

and

$$\pi_2(x,y) = \begin{cases} \beta(1-F(y)) + (F(y) - F(\frac{x+y}{2})) \text{ if } x < y \\ \frac{1}{2}(\beta(1-F(x)) + \alpha F(x)) \text{ if } x = y \\ \alpha F(y) + (F(\frac{x+y}{2}) - F(y)) \text{ if } x > y \end{cases}$$

where α and β are parameters in $[0,1]$. The behavioral assumptions behind this algebra are as follows. It is assumed that voters who feel that the political debate offers a "true" (between left and right) choice vote (as in Downs) for the party which is the closest to their ideal point. Only a fraction α of those who consider the menu too rightist vote for the less rightist party and only a fraction β of those who consider the menu too leftist vote for the less leftist party. We may assume that the voters who do not vote either abstain or vote for a minority party located to the extreme left or to the extreme right. A more structural defense of that behavior appears in section 5.

This two-player game displays some features that need to be emphasized. First, we note that unless $\alpha = \beta = 1$, the game is *not zero-sum.* In spite of the fact that the game is competitive, it also contains coordination dimensions. Second, the game is *symmetric* in the sense that $\pi_1(x, y) = \pi_2(y, x)$ for all $(x, y) \in [0, 1]^2$. In particular, $\pi_1(x, x) = \pi_2(x, x)$ for all $x \in [0, 1]$. Third, the game is *discontinuous*: the function is discontinuous on the diagonal of the square. Discontinuous games raise intricate difficulties as they do not necessarily admit equilibria in mixed strategies. In this paper, we prove existence of an equilibrium for the general case using the Dasgupta and Maskin (1986) conditions and we then construct explicitly an equilibrium in mixed strategies for the uniform distribution case.

The conventional Downs-Hotelling model corresponds to $\alpha = \beta = 1$. We assume that the two parties want to maximize their expected electoral supports instead of the probability of winning the election. The standard Downsian specification pays attention exclusively to *centripetal* forces, that is, incentives pushing each party to move in the direction of its opponent. Centrifugal electoral incentives are absent. The party on the left has an incentive to move on the right as the cost of losing electors on its left is equal to 0.

The model that we consider introduces *centrifugal* forces. For the sake of illustration, consider the incentives of party 2 in the case where $x < y$. We have:

$$\frac{\partial \pi_2}{\partial y}(x, y) = (1 - \beta)f(y) - \frac{1}{2}f(\frac{x + y}{2})$$

and therefore:

$$\frac{\partial \pi_2}{\partial y}(x, y) < 0 \text{ if and only if } (1 - \beta)f(y) - \frac{1}{2}f(\frac{x + y}{2}) < 0.$$

If β is close to 1, not surprisingly, the inequality is likely to holds true as the right party does not lose too much voters on its right by moving to the left. At the other extreme, that is, when $\beta = 0$, the condition writes $f(y) < \frac{1}{2}f(\frac{x+y}{2})$. If the density does not decrease too fast, it will not be satisfied. For instance, when F is uniform, it does not hold true. Precisely, when F is uniform, the general inequality holds true if and only if $\beta > \frac{1}{2}$.

3 General Remarks

As already noted, the payoff functions are discontinuous. Let us look at them from the perspective of party 1 when the distribution of voters is uniform..

Let $\min\{\alpha, \beta\} > \frac{1}{2}$. In such a case the centripetal incentives are strong and, thus, the payoff function π_1 is increasing on $[0, y[$, decreasing on $]y, 1]$ and displays a discontinuity at

y iff $y \neq \frac{\beta}{\alpha+\beta}$. The discontinuity is as depicted on figure 1 if $y > \frac{\beta}{\alpha+\beta}$ or as depicted on figure 2 if $y < \frac{\beta}{\alpha+\beta}$.

Insert Figure 1 here

Insert Figure 2 here

Now, let $\max\{\alpha, \beta\} < \frac{1}{2}$. In such case, the centrifugal incentives are strong and, thus,the payoff function π_1 is decreasing on $[0, y[$, increasing on $]y, 1]$ and displays a discontinuity at y iff $y \neq \frac{\beta}{\alpha+\beta}$. Since $\pi_1(0, y) = \frac{y}{2}, \pi_1(1, y) = \frac{1-y}{2}$ and $\pi_1(y, y) = \frac{\alpha y+\beta(1-y)}{2}$, the best response of party 1 to y is 0 if $y > \frac{1}{2}$ and 1 if $y < \frac{1}{2}$. Indeed, since $\max\{\alpha, \beta\} < \frac{1}{2}$, we cannot have both $\frac{y}{2} < \frac{\alpha y+\beta(1-y)}{2}$ and $\frac{1-y}{2} < \frac{\alpha y+\beta(1-y)}{2}$. But the value at one extreme may be smaller than the value at the discontinuity point. The graph of π_1 is as depicted on figure 3 if $y > \frac{\beta}{\alpha+\beta}$ or as depicted on figure 4 if $y < \frac{\beta}{\alpha+\beta}$.

Insert Figure 3 here

Insert Figure 4 here

Finally, let $\alpha > \frac{1}{2} > \beta$. In such case, the centrifugal incentives are strong to the right and the centripetal incentives are strong to the left and, thus, the payoff function π_1 is increasing on $[0, y[$, increasing on $]y, 1]$ and displays a discontinuity at y iff $y \neq \frac{\beta}{\alpha+\beta}$. The graph of π_1 is as depicted on figure 5 if $y > \frac{\beta}{\alpha+\beta}$ or as depicted on figure 6 if $y < \frac{\beta}{\alpha+\beta}$.

Insert Figure 5 here

Insert Figure 6 here

Remark 1 *The discontinuity of $\pi_1(x, y)$ at $y \neq \frac{\beta}{\alpha+\beta}$ and of $\pi_2(x, y)$ at $x \neq \frac{\beta}{\alpha+\beta}$ implies, whenever $\alpha + \beta > 0$, that the two parties' strategies cannot have an atom at the same point $\overline{x} \neq \frac{\beta}{\alpha+\beta}$ because each party would obtain a strictly larger payoff by choosing a platform just to the right or to the left of such an $\overline{x}$. This rules out symmetric pure strategy equilibria other than $x = y = \frac{\beta}{\alpha+\beta}$ as well as symmetric atomic mixed strategy equilibria.*

Equilibrium existence in such discontinuous games is not straightforward. Our first task will be to demonstrate that our game satisfies the Dasgupta and Maskin (1986) conditions which guarantee the existence of an equilibrium in mixed strategies in games with discontinuities.

Proposition 1 *The game admits a Nash equilibrium in mixed strategies for any* $(\alpha, \beta) \in [0,1]^2$ *and any absolutely continuous* F.

Proof. Dasgupta and Maskin (1986) show that if *a)* the strategy space for each player is represented by a closed interval, *b)* the payoff functions are continuous except on a set of measure zero, *c)* the players' cumulative payoff function is upper semi-continuous, *d)* the range of the payoff function of each player is bounded and *e)* the payoff function of each player is weakly lower semi-continuous for any given strategy of the other player, then the game admits an equilibrium in mixed strategies. Therefore, to prove that our game admits a Nash equilibrium in mixed strategies the only thing that we have to do is to show that all these five conditions are met.

a) The strategy space for each player is $[0,1]$; a closed interval.

b) $\pi_1(x,y)$ and $\pi_2(x,y)$ are continuous except for the main diagonal, that is, except for $x = y$. This line obviously represents a measure zero of all the possible pure strategy profiles which are given by $[0,1]^2$.

c) $\pi_1(x,y)+\pi_2(x,y) = \alpha F(\min\{x,y\})+F(\max\{x,y\})-F(\min\{x,y\})+\beta(1-F(\max\{x,y\}))$ is obviously a continuous function. That is, it is upper semi-continuous as well.

d) $0 \le \pi_1(x,y) \le 1$ and $0 \le \pi_2(x,y) \le 1$ for any $(x,y) \in [0,1]^2$. That is, the players' payoffs are bounded.

e) $\pi_1(x,y)$ is weakly lower semi-continuous in x if $\forall x \in [0,1]$, $\exists \lambda \in [0,1]$ such that for $y = x$,

$$\lambda \liminf_{x \to^- y} \pi_1(x,y) + (1-\lambda) \liminf_{x \to^+ y} \pi_1(x,y) \ge \pi_1(x,y).$$

If $y \in (0,1)$ then observe that for $y = x$, $\pi_1(x,y) = \frac{\alpha F(y)+\beta(1-F(y))}{2}$, $\liminf_{x \to^- y} \pi_1(x,y) = \alpha F(y)$ and $\liminf_{x \to^+ y} \pi_1(x,y) = \beta(1-F(y))$. It is evident that for $\lambda = \frac{1}{2}$ the required weak inequality becomes an equality for any $(\alpha,\beta) \in [0,1]^2$ and any absolutely continuous F and, thus, always holds. If $y \in \{0,1\}$ (say for example that $y = 0$) then for $y = x$, $\pi_1(x,y) = \frac{\beta}{2}$ and $\liminf_{x \to^+ y} \pi_1(x,y) = \beta$. In this case the definition of weak lower semi-continuity requires that $\liminf_{x \to^+ y} \pi_1(x,y) \ge \pi_1(x,y)$ which holds for any $(\alpha,\beta) \in [0,1]^2$ and any absolutely continuous F. That is, $\pi_1(x,y)$ (and equivalently $\pi_2(x,y)$) is weakly lower semi-continuous and the game admits a Nash equilibrium in mixed strategies. $\square$

To have a better understanding of the behavior of parties in this model we explicitly characterize an equilibrium for the uniform case and any parameter values $(\alpha, \beta) \in [0,1]^2$ in the next section.

4 Equilibria in the Case of a Uniform Distribution

In this section, we explore the Nash equilibria of the game which has been defined in section 2 in the case where F is uniform on $[0,1]$. In such case a game is described by the vector of parameters (α, β). We will partition the unit square into three different areas. Since the game is symmetric, this amounts to considering the following three cases, which correspond to subsets of parameters of non-zero measure.

Case 1 (centripetal incentives):$\max\{\alpha,\beta\} > \frac{1}{2}$ and $\min\{\alpha,\beta\} \geq \frac{1}{2}$

Case 2 (mixed incentives): $\max\{\alpha,\beta\} > \frac{1}{2} > \min\{\alpha,\beta\}$

Case 3 (centrifugal incentives): $\max\{\alpha,\beta\} \leq \frac{1}{2}$ and $\min\{\alpha,\beta\} < \frac{1}{2}$.

The section is divided into two subsections. In the first subsection, we examine the set of pure strategy Nash equilibria and the second one, we turn our attention to mixed strategy Nash equilibria.

4.1 Pure Strategy Equilibria

We distinguish three broad cases according to the nature of incentives, centripetal, mixed or centrifugal. But we first deal with the limit case where the centrifugal and centripetal incentives balance exactly, i.e. $\alpha = \beta = \frac{1}{2}$. Clearly this point represents a measure zero of all possible couples of parameters values in $[0,1]^2$.

Let first $\alpha = \beta = \frac{1}{2}$, then the game is as follows:

$$\pi_1(x,y) = \begin{cases} \frac{y}{2} \text{ if } x < y \\ \frac{1}{4} \text{ if } x = y \\ \frac{1-y}{2} \text{if } x > y \end{cases}$$

It is easy to check that there is a continuum of Nash equilibria. Precisely, up to interchangeability, (x,y) with $x \leq y$ is a Nash equilibrium if and only $x \leq \frac{1}{2}$ and $y \geq \frac{1}{2}$. The Nash equilibria are Pareto ranked: the smaller is x and the larger is y, the larger are the payoffs of both players.

4.1.1 Case 1. (Centripetal Incentives)

We speak of centripetal incentives when the voters who do not abstain or vote for a minority party constitute a majority on both sides and a strict majority at least on one of the two sides.

Proposition 2 If $\max\{\alpha,\beta\} > \frac{1}{2}$ and $\min\{\alpha,\beta\} \geq \frac{1}{2}$ then the pure strategy profile $\{\frac{\beta}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}\}$ is the unique pure strategy equilibrium of the game.

Proof. Without loss of generality, let $\alpha > \frac{1}{2}$ and $\beta \geq \frac{1}{2}$. Consider $x < y$. Then:

$$\tfrac{\partial \pi_1}{\partial x}(x,y) = \alpha - \tfrac{1}{2} > 0 \text{ and } \tfrac{\partial \pi_2}{\partial y}(x,y) = \tfrac{1}{2} - \beta \leq 0$$

That is, there is generically ($\min\{\alpha,\beta\} > \frac{1}{2}$) no pure strategies equilibrium in which candidates offer distinct platforms.

Now consider $x = y = \bar{x}$. In this case we have:

$$\pi_1(\bar{x},\bar{x}) = \pi_2(\bar{x},\bar{x}) = \tfrac{1}{2}(\beta(1-\bar{x}) + \alpha\bar{x}).$$

If $\bar{x} > \frac{\beta}{\alpha+\beta}$ then party 1 by deviating to $\widetilde{x} = \bar{x} - \varepsilon$ gets a payoff of:

$$\pi_1(\widetilde{x},\bar{x}) = \alpha(\bar{x}-\varepsilon) + (\tfrac{2\bar{x}-\varepsilon}{2} - (\bar{x}-\varepsilon)) = \tfrac{1}{2}\varepsilon + \alpha\bar{x} - \alpha\varepsilon \to \alpha\bar{x} \text{ for } \varepsilon \to 0.$$

Observe that $\bar{x} > \frac{\beta}{\alpha+\beta} \Longrightarrow \alpha\bar{x} > \beta(1-\bar{x}) \Longrightarrow \alpha\bar{x} > \frac{1}{2}(\beta(1-\bar{x})+\alpha\bar{x})$. One can see that the equivalent occurs if $\bar{x} < \frac{\beta}{\alpha+\beta}$. Therefore, there is generically ($\max\{\alpha,\beta\} > \min\{\alpha,\beta\} \geq \frac{1}{2}$) no pure strategies equilibrium in which candidates offer identical platforms $x = y \neq \frac{\beta}{\alpha+\beta}$.

If $\bar{x} = \frac{\beta}{\alpha+\beta}$ one can show using the same formal arguments as before that any deviation to the left or to the right of $\frac{\beta}{\alpha+\beta}$ benefits none of both parties. That is, $\{\frac{\beta}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}\}$ is, generically, the unique equilibrium of the game in pure strategies $\square$

4.1.2 Case 2. (Mixed Incentives)

We speak of mixed incentives when centrifugal forces dominate on one side and centripetal forces on the other. In this case, the following Proposition establishes a general non-existence result.

Proposition 3 *If* $\max\{\alpha,\beta\} > \frac{1}{2} > \min\{\alpha,\beta\}$ *then there is no pure strategy equilibrium.*

Proof. Without loss of generality assume that $\alpha > \frac{1}{2} > \beta$. Consider $x < y$. Then:

$$\frac{\partial \pi_1}{\partial x}(x,y) = \alpha - \frac{1}{2} > 0 \text{ and } \frac{\partial \pi_2}{\partial y}(x,y) = \frac{1}{2} - \beta < 0.$$

That is, there is no pure strategies equilibrium in which candidates offer distinct platforms.

Now consider $x = y = \bar{x} \in (0,1)$. In this case we have:

$$\pi_1(\bar{x},\bar{x}) = \pi_2(\bar{x},\bar{x}) = \frac{1}{2}(\beta(1-\bar{x}) + \alpha\bar{x}).$$

For $\{\bar{x},\bar{x}\}$ to be an equilibrium we must have that $\pi_1(\bar{x},\bar{x}) \geq \pi_1(\bar{x}-\varepsilon,\bar{x})$ for $\varepsilon \to 0$ and $\pi_1(\bar{x},\bar{x}) \geq \pi_1(1,\bar{x})$. These last inequalities are equivalent to $\frac{1}{2}(\beta(1-\bar{x}) + \alpha\bar{x}) \geq \alpha\bar{x} \Longrightarrow \bar{x} \leq \frac{\beta}{\alpha+\beta}$ and to $\frac{1}{2}(\beta(1-\bar{x}) + \alpha\bar{x}) \geq \frac{1-\bar{x}}{2} \Longrightarrow \bar{x} \geq \frac{1-\beta}{1+\alpha-\beta}$. We observe that they may both hold only if $\frac{\beta}{\alpha+\beta} \geq \frac{1-\beta}{1+\alpha-\beta} \Longrightarrow \beta \geq \frac{1}{2}$ which is not true. Finally consider that $x = y = \bar{x} \in \{0,1\}$. in this case it is trivial to see that *a)* if $\bar{x} = 0$ then $\pi_1(0,0) < \pi_1(\varepsilon,0)$ for $\varepsilon \to 0$ and that *b)* if $\bar{x} = 0$ then $\pi_1(1,1) < \pi_1(1-\varepsilon,1)$ for $\varepsilon \to 0$. That is, there is no pure strategies equilibrium when $\max\{\alpha,\beta\} > \frac{1}{2} > \min\{\alpha,\beta\}$ □

4.1.3 Case 3. (Centrifugal Incentives)

We speak of centrifugal incentives when the voters who abstain or vote for a minority party constitute a majority on both sides and a strict majority at least on one of the two sides.

Proposition 4 (i) *If* $\max\{\alpha,\beta\} < \frac{1}{2}$ *then the pure strategy profiles* $\{0,1\}$ *and* $\{1,0\}$ *are the unique pure strategy equilibria of the game.*

(ii) *If* $\max\{\alpha,\beta\} = \frac{1}{2}$ *and* $\min\{\alpha,\beta\} < \frac{1}{2}$*, all pure strategy profiles* $(0,z)$ with $z \geq \frac{1}{2}$, $z = x, y$, *are pure strategy Nash equilibria of the game if* $\alpha < \beta$, *as well as all pure strategy profiles* $(z,0)$ *with* $z \leq \frac{1}{2}$, $z = x, y$, *if* $\alpha > \beta$.

Proof. (i) Consider $x < y$. Then:

$$\frac{\partial \pi_1}{\partial x}(x,y) = \alpha - \frac{1}{2} < 0 \text{ and } \frac{\partial \pi_2}{\partial y}(x,y) = \frac{1}{2} - \beta > 0.$$

That is, if there are, pure strategy equilibria in which candidates offer distinct platforms then they should be such that $x = 0$ and $y = 1$ or $x = 1$ and $y = 0$. Notice that $\pi_1(0,1) = \pi_1(1,0) = \frac{1}{2}$.

Now consider $x = y = \bar{x}$. In this case we have:

$$\pi_1(\bar{x},\bar{x}) = \pi_2(\bar{x},\bar{x}) = \frac{1}{2}(\beta(1-\bar{x}) + \alpha\bar{x}).$$

Since $\max\{\alpha,\beta\} < \frac{1}{2}$ it is obvious that $\pi_1(\bar{x},\bar{x}) < \frac{1}{4}$ for any $\bar{x} \in [0,1]$. Therefore, $\{0,1\}$ and $\{1,0\}$ are, indeed pure strategy equilibria of the game. Moreover we observe that $\max\{\pi_1(1,\bar{x}), \pi_1(1,\bar{x})\} > \frac{1}{4}$ when $\max\{\alpha,\beta\} < \frac{1}{2}$. That is, $\{0,1\}$ and $\{1,0\}$ are the unique pure strategy equilibria of the game

(ii) Consider $\alpha < \beta = \frac{1}{2}$. Then $\pi_1(0,y) = \frac{y}{2}$, $\pi_1(x,y) = \alpha x + \frac{y-x}{2}$ for $x \in (0,y)$, $\pi_1(y,y) = \frac{1}{2}(\alpha y + \frac{1}{2}y)$ and $\pi_1(x,y) = \frac{1-y}{2}$ for $x \in (y,1]$ and symmetrically for π_2. The same argument as above shows that there is no pure strategy equilibrium where the two candidates offer the same platform. The remaining is straightforward. □

4.2 Mixed Strategy Equilibria[6]

4.2.1 Case 1. (Centripetal Incentives)

For this case, the unique pure strategy equilibrium that we identified is a quite robust prediction. Both due to the fact the equilibrium is unique and in pure strategies and because it is symmetric; coordination issues should not interfere with the result. Note that while competitive, it is not strictly competitive[7] (Aumann 1961; Friedman 1983). Indeed, we note that $\pi_1(0,1) = \pi_2(0,1) = \frac{1}{2}$ while $\pi_1(\frac{\beta}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}) = \pi_2(\frac{\beta}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}) = \frac{\alpha\beta}{\alpha+\beta} < \frac{1}{2}$ as $\alpha + \beta > \alpha^2 + \beta^2 \geq 2\alpha\beta$ unless $\alpha = \beta = 1$. This means that we have a prisoner's dilemma like situation: a Nash equilibrium which is Pareto dominated. While not strictly competitive, the game exhibits some competitive features and we conjecture that there is no equilibrium in mixed strategies.

4.2.2 Case 2. (Mixed Incentives)

As we saw above, there is no pure strategy equilibrium in this case. Assume without loss of generality that $\alpha > \frac{1}{2} > \beta$. then the following is true.

Proposition 5 *There exists a unique symmetric absolutely continuous mixed equilibrium* $\{G, G\}$ *defined as follows:*

[6] In this subsection, we use a concept of mixed strategy equilibria which is more restrictive than in Proposition 1 since it excludes pure strategy equilibria.

[7] A game is strictly competitive if all possible outcomes are Pareto-optimal.

$$G(x)=\begin{cases} \frac{1}{2}\frac{1-2\alpha}{1-\alpha-\beta}+\frac{\alpha^{1-\frac{1}{\alpha+\beta}}(2\beta-1)}{2(\alpha+\beta-1)}(x(\alpha+\beta)-\beta)^{-1+\frac{1}{\alpha+\beta}} \text{ if } x\in\left[\frac{\beta+(\frac{\alpha^{1-\frac{1}{\alpha+\beta}}(2\beta-1)}{1-2\alpha})^{\frac{\alpha+\beta}{\alpha+\beta-1}}}{\alpha+\beta},1\right] \\ 0 \text{ if } x\in\left[0,\frac{\beta+(\frac{\alpha^{1-\frac{1}{\alpha+\beta}}(2\beta-1)}{1-2\alpha})^{\frac{\alpha+\beta}{\alpha+\beta-1}}}{\alpha+\beta}\right] \end{cases} \quad \text{when } \alpha+\beta\neq 1$$

and

$$G(x)=\begin{cases} 1+\frac{1-2\beta}{2}\ln\frac{x-\beta}{1-\beta} \text{ if } x\in\left[e^{\frac{2-(1-2\beta)\ln(1-\beta)}{2\beta-1}}+\beta),1\right] \\ 0 \text{ if } x\in\left[0,e^{\frac{2-(1-2\beta)\ln(1-\beta)}{2\beta-1}}+\beta)\right] \end{cases} \quad \text{when } \alpha+\beta=1$$

Proof. If there exists a symmetric Nash equilibrium (G,G) in mixed strategies such that G is absolutely continuous with positive density g on the interval $[\underline{g},\bar{g}]\subseteq[0,1]$ (this implies $G(\underline{g})=0$ and $G(\bar{g})=1$), it must be the case that:

$$\pi_1(x,G)=\int_{\underline{g}}^{x}[(x-\tfrac{x+y}{2})+\beta(1-x)]dG(y)+\int_{x}^{\bar{g}}[(\tfrac{x+y}{2}-x)+\alpha x]dG(y)=\pi \text{ for } x\in[\underline{g},\bar{g}]$$

and that $\pi_1(x,G)\leq\pi$ for $x\notin[\underline{g},\bar{g}]$. If there is such a G then $\frac{\partial\pi_1}{\partial x}(x,G)=0$ for $x\in[\underline{g},\bar{g}]$. That is:

$$(\beta-\alpha x-\beta x)g(x)+(1-\alpha-\beta)G(x)+\alpha-\tfrac{1}{2}=0 \text{ for } x\in[\underline{g},\bar{g}].$$

The general solution of this first order differential equation is:

$$G(x)=\frac{1}{2}\frac{1-2\alpha}{1-\alpha-\beta}+C\ (x(\alpha+\beta)-\beta)^{\frac{1}{\alpha+\beta}-1} \text{ when } \alpha+\beta\neq 1 \text{ and}$$

$$G(x)=C+\frac{2\alpha-1}{2}\ln(x-\beta) \text{ when } \alpha+\beta=1.$$

where C is a constant of integration. We notice that $\bar{g}$ must be equal to 1. This is because when $x<y$ we have that $\frac{\partial\pi_2}{\partial y}(x,y)=\frac{1}{2}-\beta>0$. So if $\bar{g}<1$ then $\frac{\partial\pi_2}{\partial y}(G,y)=\int_{\underline{g}}^{\bar{g}}(\frac{1}{2}-\beta)dG(x)=(\frac{1}{2}-\beta)>0$ for any $y>\bar{g}$. That is $\bar{g}$ is not a best response to G and, thus, a symmetric equilibrium $\{G,G\}$ with $\bar{g}<1$ is not possible. From this observation we get that $G(1)=1$. Using this information to compute C in both expressions, we get:

$$
\begin{aligned}
G(x) &= \frac{1}{2}\frac{1-2\alpha}{1-\alpha-\beta}+\frac{\alpha^{1-\frac{1}{\alpha+\beta}}(2\beta-1)}{2(\alpha+\beta-1)}(x(\alpha+\beta)-\beta)^{-1+\frac{1}{\alpha+\beta}} \text{ when } \alpha+\beta\neq 1 \text{ and} \\
G(x) &= 1+\frac{2\alpha-1}{2}\ln\frac{x-\beta}{1-\beta} \text{ when } \alpha+\beta=1.
\end{aligned}
$$

Next we set $G(\underline{g})=0$ and we find:

$$
\begin{aligned}
\underline{g} &= \frac{\beta+\left(\frac{\alpha^{1-\frac{1}{\alpha+\beta}}(2\beta-1)}{1-2\alpha}\right)^{\frac{\alpha+\beta}{\alpha+\beta-1}}}{\alpha+\beta} > 0 \text{ when } \alpha+\beta\neq 1 \text{ and} \\
\underline{g} &= e^{\frac{2-(1-2\beta)\ln(1-\beta)}{2\beta-1}}+\beta>\beta>0 \text{ when } \alpha+\beta=1.
\end{aligned}
$$

We know that when $x>y$ we have that $\frac{\partial\pi_2}{\partial y}(x,y)=\alpha-\frac{1}{2}>0$. So if $\underline{g}>0$ then $\frac{\partial\pi_2}{\partial y}(G,y)=\int_{\underline{g}}^{1}(\alpha-\frac{1}{2})dG(x)=(\alpha-\frac{1}{2})>0$ for any $y<\underline{g}$. That is, if party 1 plays G then party 2 strictly prefers $\underline{g}$ to any $y<\underline{g}$ and is indifferent among any of the policies in $[\underline{g},1]$; playing G is a best response of 2 to party 1 playing G. This concludes the argument □

It is important to note that the density is decreasing and therefore that the CDF is concave. Examples of such mixed strategies are provided on figures 7 and 8.

Insert Figure 7 here

Insert Figure 8 here

4.2.3 Case 3. (Centrifugal Incentives)

As we saw above, there is no symmetric pure strategy equilibrium in this case. Therefore, in absence of a coordination mechanism the two mirror pure strategy equilibria that we identified are not very robust.

Proposition 6 *If* $\max\{\alpha,\beta\}\leq\frac{1}{2}$ *and* $0<\min\{\alpha,\beta\}<\frac{1}{2}$, *then there exists unique symmetric absolutely continuous mixed equilibrium* $\{G,G\}$ *given by:*

$$G(x) = \begin{cases} \frac{1}{2}\frac{1-2\alpha}{1-\alpha-\beta} + \frac{(2\alpha-1)(-\beta)^{-1+\frac{1}{\alpha+\beta}}\beta}{2(\alpha+\beta-1)}(x(\alpha+\beta)-\beta)^{\frac{1}{\alpha+\beta}-1} \text{ if } x \in [0, \frac{\beta}{\alpha+\beta}) \\ \frac{1}{2}\frac{1-2\alpha}{1-\alpha-\beta} + \frac{(2\beta-1)\alpha^{1-\frac{1}{\alpha+\beta}}}{2(\alpha+\beta-1)}(x(\alpha+\beta)-\beta)^{\frac{1}{\alpha+\beta}-1} \text{ if } x \in [\frac{\beta}{\alpha+\beta}, 1] \end{cases}$$

Proof. The derivation of the general form of $G(x)$ is performed as before. The big difference here is that, unlike the $\max\{\alpha, \beta\} > \frac{1}{2} > \min\{\alpha, \beta\}$ case, the support $[\underline{g}, \bar{g}] \subseteq [0, 1]$ of a symmetric mixed strategies equilibrium should be such that $[\underline{g}, \bar{g}] = [0, 1]$. This is because, if $\underline{g} > 0$ (one can offer an equivalent argument to exclude the $\bar{g} < 1$ case) then $\frac{\partial \pi_2}{\partial y}(G, y) = \int_{\underline{g}}^{\bar{g}} (\alpha - \frac{1}{2}) dG(x) = (\alpha - \frac{1}{2}) < 0$ for any $y < \underline{g}$. That is, playing $\underline{g}$ is not a best response of party 2 to party 1 playing such a mixed strategy. Therefore, if a symmetric atomless mixed equilibrium exists it should satisfy $[\underline{g}, \bar{g}] = [0, 1]$ $G(0) = 0$ and $G(1) = 1$.Since in this case $\alpha + \beta \neq 1$, the general form of $G(x)$ is:

$$G(x) = \frac{1}{2}\frac{1-2\alpha}{1-\alpha-\beta} + C(x(\alpha+\beta)-\beta)^{\frac{1}{\alpha+\beta}-1}.$$

We know that if $G(0) = 0$ then $C = \frac{(2\alpha-1)(-\beta)^{-1+\frac{1}{\alpha+\beta}}\beta}{2(\alpha+\beta-1)}$ and if $G(1) = 1$ then $C = \frac{(2\beta-1)\alpha^{1-\frac{1}{\alpha+\beta}}}{2(\alpha+\beta-1)}$. So if such an equilibrium exists then $G(x)$ should be a piece wise function. If we define:

$$G_A(x) = \frac{1}{2}\frac{1-2\alpha}{1-\alpha-\beta} + \frac{(2\alpha-1)(-\beta)^{-1+\frac{1}{\alpha+\beta}}\beta}{2(\alpha+\beta-1)}(x(\alpha+\beta)-\beta)^{\frac{1}{\alpha+\beta}-1}$$

and

$$G_B(x) = \frac{1}{2}\frac{1-2\alpha}{1-\alpha-\beta} + \frac{(2\beta-1)\alpha^{1-\frac{1}{\alpha+\beta}}}{2(\alpha+\beta-1)}(x(\alpha+\beta)-\beta)^{\frac{1}{\alpha+\beta}-1}$$

we observe that $G_A(x) = G_B(x)$ if and only if $x = \frac{\beta}{\alpha+\beta}$. Moreover we have that $G_A(0) = 0$, $\frac{\partial G_A}{\partial x} > 0$ for any $x \in [0, \frac{\beta}{\alpha+\beta}]$, $\frac{\partial G_B}{\partial x} > 0$ for any $x \in [\frac{\beta}{\alpha+\beta}, 1]$ and $G(1) = 1$. In other words G is a continuous, strictly increasing cumulative distribution function with full support in $[0, 1]$.

So if party 1 uses this strategy then $\pi_2(G, y) = \bar{\pi}$ for any $y \in [0, 1]$; party 2 playing G is a best response to party 1 playing G $\square$

We observe that the shape of the mixed equilibrium will depend upon the value of $\alpha + \beta$. If $\alpha + \beta$ is less than 1, the probability mass will be more on the extremes with a density first decreasing and then increasing. Different shapes of the density may appear, as illustrated on figures 9 and 10.

Insert Figure 9 here

Insert Figure 10 here

If $\alpha + \beta = 1$, then G is uniform as illustrated on figure 11.

Insert Figure 11 here

Finally, if $\alpha + \beta$ is larger than 1, then the probability mass is more concentrated in the center with a density first increasing and then decreasing, as illustrated on figure 12.

Insert Figure 12 here

Remark 2 *There is a smooth transition between the pure symmetric Nash equilibria and the mixed ones. For the sake of illustration take $\alpha = \beta$. When $\alpha \in \left[0, \frac{1}{4}\right]$, the mass of $\frac{1}{2}$ on both sides of $\frac{1}{2}$ is located around the two extremes and tends to $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$[8] when α tends to 0. That is, for $(\alpha, \beta) = (0, 0)$ a symmetric mixed equilibrium of the game is given by the diagonal profile of mixed strategies (G, G) where $G = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. In contrast, when $\alpha \in \left[\frac{1}{4}, \frac{1}{2}\right]$, the mass of $\frac{1}{2}$ on both sides of $\frac{1}{2}$ is located around $\frac{1}{2}$ and tends to $\delta_{\frac{1}{2}}$ when α tends to $\frac{1}{2}$.*

5 Discussion and Complements

In this section, we discuss some issues to which we have alluded before without providing details. We discuss in turn, non uniform distributions of the voters, the behavioral origins of centrifugal incentives and the equilibrium analysis.

5.1 Non Uniform Distributions and Rational Origins of Centrifugal Incentives

In our simple model of electoral competition, the strategic calculus of each party is based on marginal rates of substitution. For instance, for the leader of the party located on the left, the relevant question is: how many centrist voters do I gain and how many leftist voters do I lose if my platform becomes infinitesimally more rightist. Consider an arbitrary distribution

[8] δ_z denotes the Dirac mass in z.

of voters described by the density f which will be assumed differentiable, symmetric with respect to (i.e. $f(x) = f(1-x)$ for all $x \in \left[0, \frac{1}{2}\right]$) and strictly increasing on $\left[0, \frac{1}{2}\right[$(and so strictly decreasing on $\left[\frac{1}{2}, 1\right]$). Consider a profile of platforms (x, y) such that: $x \leq \frac{1}{2} \leq y$ and consider the symmetric case where $\alpha = \beta$. Without loss of generality assume that $\frac{x+y}{2} \leq \frac{1}{2}$. For the left party, moving on the right leads to a gain of $\frac{1}{2}f(\frac{x+y}{2})$ voters and to a lost of $(1-\alpha)f(x)$ voters. If $\alpha \geq \frac{1}{2}$, then the gain is always larger than the cost and the equilibrium is defined by $x = y = \frac{1}{2}$. If otherwise $\alpha < \frac{1}{2}$, a marginal equilibrium is obtained when the marginal rate of substitution $\frac{\frac{1}{2}f(\frac{x+y}{2})}{(1-\alpha))f(x)}$ is equal to 1 i.e.

$$\frac{1}{2}f(\frac{x+y}{2}) = (1-\alpha)f(x)$$

i.e.

$$f(x) = \frac{1}{2(1-\alpha)}f(\frac{x+y}{2})$$

Let us test when the symmetric profile $(x, 1-x)$ with $x < \frac{1}{2}$ is a (local) Nash equilibrium[9]. From above, the first order condition writes:

$$f(x) = \frac{1}{2(1-\alpha)}f(\frac{1}{2})$$

If $f(0) = 0$, then the above equation has a unique solution x^*. We may also check that the (local) second order condition is satisfied. Indeed the second derivative at x^*

$$\frac{1}{4}f'(\frac{1}{2}) + (\alpha-1)f'(x^*) = (\alpha-1)f'(x^*)$$

is negative as $\alpha < 1$. For the sake of illustration consider the (symmetric) Beta distribution[10]:

$$f(x) = \frac{\Gamma(2\gamma)}{\Gamma(\gamma)^2}\left(x\left(1-x\right)\right)^{\gamma-1} \text{ over } [0,1]$$

with $\gamma > 1$. In such case, x^* is the solution of the equation:

$$x(1-x) = \frac{1}{4}\left(\frac{1}{2(1-\alpha)}\right)^{\frac{1}{\gamma-1}}$$

We obtain:

[9]Since we have assumed f strictly increasing, this argument does not apply to the uniform distribution. In fact, for the uniform distribution the marginal rate of substitution at any profile (x, y) is equal to $\frac{1}{2(1-\alpha)}$. It is equal to only by accident, precisely when $\alpha = \frac{1}{2}$.

[10]Γ denotes the Gamma function. In particular $\Gamma(n) = (n-1)!$ for any integer n.

$$x^* = \frac{1}{2} - \frac{\sqrt{1 - \left(\frac{1}{2(1-\alpha)}\right)^{\frac{1}{\gamma-1}}}}{2}$$

As expected, we observe that differentiation $x^*(\alpha, \gamma)$ is reduced when the electorate is more concentrated around the center (large values of γ) or when the centrifugal incentives are less intense. This is illustrated on table 1 for $\alpha = 0$ and on table 2 for $\alpha = \frac{1}{4}$.

γ	2	10	100
x^*	0.14645	0.36387	0.45824

Table 1

γ	2	10	100
x^*	0.21132	0.39506	0.46803

Table 2

Of course, we could consider marginal rates of substitution constructed in a totally different manner. In this paper, we have not explicitly derived the behavior of voters to the right of the rightest platform or the left of the more to the left platform from an explicit rational choice model. In the case where the lost votes go to one of the two extreme parties located respectively in 0 and in 1, their behavior can be rationalized in many alternative ways. While the decision to abstain can be motivated by the utility differential between the two platforms, it can also well be the case (as supported by questionnaire evidence in the recent French presidential elections) that extreme voters (on the left and on the right) were displeased by the centrist attitudes of the two leading candidates and decide to abstain or to vote for extremist parties. In the traditional model where all policy dimensions are summarized by the one dimensional ideological axis, the fraction of the electorate between 0 and x should be divided differently. A rational model would suggest the replacement of $\alpha F(x)$ by $F(x) - F(\frac{x}{2})$ for the left party. The payoff of the left party is now equal to:

$$\pi_1(x, y) = F(\frac{x+y}{2}) - F(\frac{x}{2})$$

But we could also assume that no all voters between 0 and $\frac{x}{2}$ vote for the extreme party as while attracted on this specific ideological dimension, they may be repulsed by some other dimensions (there are fixed positions as opposed to those which are coined as pliable by

Grossman and Helpman (2001)). For some voters the attractiveness of the extreme party is dominated by other considerations. Suppose that a voter located at $t \in [0, x]$ vote for 0 iff $t < \theta\,(x - t)$ where θ is a positive preference parameter. The larger is θ, the more willing is voter t to vote for the extreme left party: θ is inversely related to the repulsion created by the other ideological dimensions of the extreme left party. In such a case the payoff of the left party is equal to:

$$F(\frac{x+y}{2}) - F(\frac{\theta x}{1+\theta})$$

When θ tends to 0, we have the standard Downsian model while when θ tends to $+\infty$, we have the case where the left party loses all the voters to its left. In the case where F is uniform, we obtain:

$$\pi_1(x, y) = \frac{y}{2} + x\left(\frac{1}{2} - \frac{\theta}{1+\theta}\right)$$

i.e. the model of this paper with $\alpha = \frac{1}{1+\theta}$.

An alternative rationale for the game which is played by the two main parties results from the following two-stage game. In the first stage two purely vote-share maximizing parties (we will call them "unconstrained") choose their platforms x and y from the discrete policy space $\{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$. In the second stage two parties with heterogeneous ideological constraints (we will call them "constrained") observe the choices of the unconstrained parties and choose their policy platforms from the same policy space in the following manner. One of these two parties is constrained to offer a policy at most as right as $\min\{x, y\}$ and the other is constrained to offer a policy at most as left as $\max\{x, y\}$. These constraints capture the restraints that symbolic politics impose on real political outcomes. A party, for example, with the term "communist" in its name is very hard to choose a policy platform to the right of a party which proclaims to be socialist or centrist even if it is run by the most office-motivated of politicians. Each of these two parties either cares to secure as much votes as possible (without violating the aforementioned ideological constraints) or it aims to preserve its ideological purity by sticking to its traditional political platform. We assume that there is imperfect information about this dimension of the game. That is, the "unconstrained" parties which move first expect that the constrained parties will behave as vote-share maximizers with probability p and that they will stick to their traditional ideological platforms with probability $(1 - p)$. We finally consider that the ideal policies of a continuum of voters is distributed uniformly in the policy space and that a voter votes for the party which offered the policy platform nearer to her ideal policy. In the extreme case in which a voter equally values the policy platform of a constrained party and the policy platform of an unconstrained party we assume that the voter votes for the unconstrained one. That is, we assume that the unconstrained parties enjoy a minimal valence advantage compared to the constrained ones as in Aragonès and Palfrey (2002). If the constrained parties are vote-share maximizers (with probability p) they choose $\max\{0, \min\{x, y\} - \frac{1}{n}\}$ and $\min\{1, \max\{x, y\} + \frac{1}{n}\}$ respectively and if they aim to preserve their ideological purity (with probability $1 - p$) we assume that they choose their traditional platforms 0 and 1 respectively. Notice that for $n \to \infty$ this model converges to the model of this paper with $\alpha = \beta = \frac{1}{2}(1 - p) \leq \frac{1}{2}$; the case of strong centrifugal incentives.

5.2 Missing Equilibria

We have shown that for all $(\alpha, \beta) \in [0, 1]^2$, our game admits an equilibrium in mixed strategies. We have not totally characterized the set of Nash equilibria. The purpose of this section is to elaborate on that and in particular to identify (if any) new symmetric or asymmetric Nash equilibria.

To illustrate the issue, consider the case where $\alpha = \beta = 0$. In such case:

$$\pi_1(x, y) = \begin{cases} \frac{y-x}{2} \text{ if } x \leq y \\ \frac{x-y}{2} \text{if } x \geq y \end{cases}$$

Note that the game is continuous. We already know that $(0, 1)$ and $(1, 0)$ and (G, G) (where $G = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$) are Nash equilibria. Let $x \in [0, 1]$. Consider now the off diagonal profile $\left(\frac{1}{2}, G\right)$. What is the best response of party 2 to $\frac{1}{2}$? Let $y \in [0, 1]$. Since $\pi_2(\frac{1}{2}, y) = \frac{1}{2}\left|y - \frac{1}{2}\right|$, 0 and 1 are the best responses. Therefore $\left(\frac{1}{2}, G\right)$ is also a Nash equilibrium. Finally[11], consider the off diagonal profile (U, G) where U denotes the uniform probability on $[0, 1]$. What is the best response of party 2 to U ? Let $y \in [0, 1]$. Since $\pi_2(U, y) = \int_0^y \frac{y-x}{2}dx + \int_y^1 \frac{x-y}{2}dx = \frac{y^2}{2} - \frac{y}{2} + \frac{1}{4}$ is convex and symmetric around $\frac{1}{2}$,0 and 1 are the best responses. Therefore (U, G) is also a Nash equilibrium. Can we focus on specific Nash equilibria in this plethora of Nash equilibria ? We note that in the case of $(0, 1)$ and $(1, 0)$, $\pi_1 = \pi_2 = \frac{1}{2}$, while for all others $\pi_1 = \pi_2 = \frac{1}{4}$. We have here a real coordination problem[12]. The two parties should occupy the extremes but may fail to do so by lack of coordination. In that respect the mixed strategies G and U are less risky.

This construction extends to the general case. Let $(\alpha, \beta) \in \left]0, \frac{1}{2}\right[^2$ and let G be the mixed strategy $p\delta_0 + (1-p)\delta_1$ where $p \in [0, 1]$. Since $\pi_1(x, G) = p\left[\frac{x}{2} + \beta(1 - x)\right] + (1-p)(\alpha x + \frac{1-x}{2})$

[11] It is clear from the examples provided here that we can construct more Nash equilibria.

[12] If left and right have an intrinsic meaning, note however that the profile $(0, 1)$ stands as the natural equilibrium.

for all $x \in]0,1[$, we obtain that it does not depend upon x iff:

$$p = \frac{1}{2} \times \frac{1-2\alpha}{1-\beta-\alpha}$$

In such case, since $\pi_1(0,G) = p\frac{\beta}{2} + (1-p)\frac{1}{2}$ and $\pi_1(1,G) = p\frac{1}{2} + (1-p)\frac{\alpha}{2}$ are strictly smaller than $\pi_1(x,G)$ for all $x \in]0,1[$, we obtain that any $x \in]0,1[$ is a best response to G. In particular, $\frac{1}{2}$ is a best response to G. Since both 0 and 1 are best responses to $\frac{1}{2}$, we have demonstrated that $\left(\frac{1}{2}, G\right)$ and $\left(G, \frac{1}{2}\right)$ are Nash equilibria. Consider also as before the off diagonal profile (U,G) where U denotes the uniform probability on $[0,1]$. What is the best response of party 2 to U ? Let $y \in [0,1]$. Since $\pi_2(U,y) = \int_0^y \frac{y-x}{2}dx + \int_y^1 \frac{x-y}{2}dx = \left(\frac{1}{2} - \alpha - \beta\right) y^2 + \left(\alpha + \beta - \frac{1}{2}\right) y + \frac{1}{4}$ is convex and symmetric around $\frac{1}{2}$ if $\alpha = \beta$ and $\alpha < \frac{1}{4}$, 0 and 1 are the best responses. Therefore, if $\alpha = \beta$ and $\alpha < \frac{1}{4}$, (U,G) is also a Nash equilibrium. Finally, consider the strategy profile (H,G) where H is the absolutely continuous mixed strategy identified in section 4. We argued that any $x \in]0,1[$ is a best response to G and, thus, H is a best response to G. Moreover we know that any $y \in [0,1]$ is a best response to H. That is, (H,G) is also an equilibrium for $(\alpha,\beta) \in \left]0, \frac{1}{2}\right[^2$.

We have just shown that for all $(\alpha,\beta) \in \left[0, \frac{1}{2}\right[^2$, the game admits at least seven Nash equilibria which are, counting for interchangeability: $(0,1)$,$\left(\frac{1}{2}, G\right)$, (H,G) and (H,H). We have also shown that for some specific configurations of the parameters α and β, there are some other Nash equilibria. In the case where $\alpha = \beta = 0$, the symmetric atomic equilibrium (G,G) replaces the the symmetric atomless equilibrium (H,H) derived when $\left]0, \frac{1}{2}\right[^2$. We conjecture that for all $(\alpha,\beta) \in \left[0, \frac{1}{2}\right[^2$, the game admits a unique symmetric equilibrium in mixed strategies.

6 Concluding Remarks

Contemporary empirical evidence questions the standard Downsian prediction of minimal differentiation[13]; parties are observed to offer distinct and, in some cases, very differentiated policy platforms. This paper introduced centrifugal incentives in the standard Downsian model and analyzed the equilibrium behavior of electoral shares maximizing parties in this context. Our analysis pointed to the fact that the observed differentiation need not be solely attributed to asymmetric characteristics of competing parties. It could very well be attributed to elements of the electoral framework which reduce the appeal of centrist parties to extremist voters.

[13] See Warwick (2004) for a discussion of recent theories and empirical findings regarding the "riddle" of party differentiation.

The equilibria that we characterized have certain very intuitive and some counterintuitive dimensions. Some of the intuitive findings are the following. The symmetric pure strategy equilibrium that we characterized for the strong centripetal incentives case implies that the parties' point of convergence will be in the side of the electoral spectrum in which attraction of extremist voters is easier. Moreover, the symmetric mixed strategy equilibrium that we characterized for the case of strong centrifugal incentives indicates that the extremism of parties' platforms increases when centrifugal incentives increase. Something which is less intuitive though occurs in the case of mixed incentives. When voters to the right of both parties' platforms find it very hard to vote one of these two parties while voters to the left find it easier we have shown that in equilibrium a) (as far as extreme policies are concerned) each of the competing parties will never choose an extreme left platform but will locate at the extreme right with positive probability and b) (as far as moderate platforms are concerned) it is more probable that each of the two parties chooses a moderate leftist platform than a moderate rightist platform.

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