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Competitive Equilibrium from Equal Incomes for Two-Sided Matching*

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Abstract

Using the assignment of students to schools as our leading example, we study many-to-one two-sided matching markets without transfers. Students are endowed with cardinal preferences and schools with ordinal ones, while preferences of both sides need not be strict. Using the idea of a competitive equilibrium from equal incomes (CEEI, Hylland and Zeckhauser (1979)), we propose a new mechanism, the Generalized CEEI, in which students face different prices depending on how schools rank them. It always produces fair (justified-envy-free) and *ex ante* efficient random assignments and stable deterministic assignments if both students and schools are truth-telling. We show that each student's incentive to misreport vanishes when the market becomes large, given all others are truthful. The mechanism is particularly relevant to school choice as schools' priority orderings over students are usually known and can be considered as their ordinal preferences. More importantly, in settings like school choice where agents have similar ordinal preferences, the mechanism's explicit use of cardinal preferences may significantly improve efficiency. We also discuss its application in school choice with group-specific quotas and in one-sided matching.

Keywords: *Two-Sided Matching, Weak Preferences, School Choice, Efficiency, Fairness, Stability, Incentive Compatibility, Competitive Equilibrium from Equal Incomes*

JEL Codes: C78, D82, I29

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1 Introduction

Two-sided matching in which indivisible positions at a set of institutions are assigned to agents with unit demand is commonly observed in real life. Examples include (i) many-to-one matching such as school choice (student placement in public schools), college admission, and firms' employment of workers; and (ii) one-to-one matching, e.g., office assignment and college dormitory allocation. The former includes the latter as a special case where each institution has one position available.

Taking school choice as our leading example where monetary transfers are precluded, this paper studies how to match agents with institutions efficiently and fairly in many-to-one matching. We assume that students are endowed with von Neumann-Morgenstern (vN-M) utilities and schools with possibly weak ordinal preferences. Students thus belong to different preference groups at each school. As the literature on two-sided matching usually considers strict ordinal preferences on both sides, our approach opens another dimension for possible efficiency gain, in particular when schools' ranking over students is not strict.[1]

Each seat at a school is treated as a continuum of probability shares, and we focus on random assignments that are probability distributions over deterministic allocations. Every random assignment can be resolved into deterministic assignments with some lotteries (Birkhoff (1946), von Neumann (1953), and Kojima and Manea (2010)).

Building upon a competitive equilibrium from equal incomes (CEEI, Hylland and Zeckhauser (1979)), we propose a Generalized CEEI mechanism (G-CEEI).[2] It elicits ordinal preferences from schools and cardinal ones from students, and then computes a random assignment as an equilibrium outcome from a pseudo competitive market such that at the market-clearing prices, the random assignment of each student is a solution to her expected-utility maximization problem if students, as price-takers, are endowed with an equal artificial income. Given the random assignment, the mechanism conducts a lottery and implements a deterministic assignment.

The unique feature of the G-CEEI mechanism is that students in different preference groups at a school face different prices of that school. Fix a school, if some students in a given preference group at that school can obtain positive probability shares of that school, the school must be free to those who are more preferred by that school. In addition, if a school has been consumed completely by students who are more preferred by that school, a student must face an infinite price of that

[1] For a textbook treatment of and a survey of the literature on two-sided matching, please see Roth and Sotomayor (1990).

[2] Hylland and Zeckhauser (1979) consider the original CEEI mechanism in a setting of one-to-one, one-sided matching.

school. Therefore, the mechanism respects schools' preferences in the sense that more preferred students at a school have the right to be assigned to that school earlier.

We establish the existence of equilibrium prices given any reported preferences in the mechanism and further reveals its interesting links with two widely used mechanisms: the Boston (Abdulkadiroglu and Sonmez (2003)) and the Gale-Shapley mechanisms (Gale and Shapley (1962)).[3,4] Given a random assignment prescribed by the G-CEEI, if every student has strict preferences and consumes a bundle which includes at most one school with a positive and finite price, the assignment must also be a Nash equilibrium outcome of the Boston mechanism. When both sides have strict preferences, the assignment of the Gale-Shapley are always equilibrium outcomes of the G-CEEI.

We further consider several commonly used criteria to evaluate the performance of the G-CEEI mechanism. A key property in the two-sided matching literature is **stability**. A deterministic assignment is stable if and only if there is no student-school pair in which both can be strictly better off by leaving the current match and being re-matched together. Our mechanism always delivers stable deterministic assignments with respect to stated preferences.

Another desirable property is Pareto efficiency. In the context of strict preferences, it is well-known that stability is a sufficient condition for Pareto efficiency (Erdil and Ergin (2006)). Given the possibly weak preferences on both sides, we instead define *ex ante* **(Pareto) efficient** random assignments as those which are not Pareto dominated by any other assignment with respect to both students' vN-M utilities and schools' ordinal preferences.[5] We also consider fairness criterion in terms of **justified-envy-freeness,** i.e., no one envies those who are in the same or lower preference groups at all schools. If both sides do report their preferences truthfully, the random assignment prescribed by the G-CEEI mechanism is shown to be *ex ante* efficient and justified-envy-free, and therefore it always leads to a Pareto optimal deterministic assignment.

The incentive compatibility for both sides is another important property in the literature.

[3] In the Boston mechanism which is commonly used in school choice, students submit rank-order lists of schools. Each school first considers students who rank it first, and assigns seats in order of their priority at that school. Then, each school that *still has available seats* considers *unmatched* students who rank it second. This process continues until the market is cleared.

[4] The Gale-Shapley mechanism, also known as the deferred-acceptance mechanism, can be student-proposing or school-proposing, while both are similarly defined. In the former, students apply to their most-preferred schools, while schools hold the most preferred applicants up to their capacities and reject the rest. In the second round, those who are rejected apply to their second-preferred schools, and schools pool them with those who are held from the previous round and again only keep the most preferred up to their capacities. This process continues until there is no new rejections, and then the matching is finalized.

[5] A student is better off if there is an increase in her expected utility, and schools' welfare is evaluated in terms of first-order stochastic dominance: A school is better off if students matched with this school in the new assignment first-order stochastically dominate those of the old one. For schools' preferences, our approach is the same as ordinal efficiency introduced by Bogomolnaia and Moulin (2001).

Unfortunately, the G-CEEI mechanism is not strategy-proof, i.e., stating the true preferences is not always a dominant strategy for every agent.[6] Roth (1982) shows there is no strategy-proof mechanism that always produces stable matching. For example, schools may have incentives to misreport their capacity and/or preferences, as Roth (1982) and Sonmez (1997) show under the Gale-Shapley mechanism. In the following, we assume schools are always truthful and focus on the strategic behavior of students. We show that when the market becomes large, each student's incentive to misreport goes to zero, given that everyone else is truth-telling. More precisely, the market grows in the sense that the number of students goes to infinity, while the number of seats at each school goes to infinity proportionally, given a constant number of schools.

Given these desirable properties, the G-CEEI mechanism is particularly applicable to school choice and other resource allocation problems based on priorities. For example, housing allocation may give priorities to current tenants, and schools usually have priority ordering over students. For example, in the Boston public school system, there are four priority groups in the following order (a) students who have siblings at the school (siblings) and are in the school's reference area (walk zone), (b) siblings, (c) walk zone, (d) other students. The priority structure is usually determined by government or local laws and requires that higher-priority students obtain that school first, *ceteris paribus*.

Although schools' priority rankings over students are not their preferences, we may treat them as ordinal preferences and the G-CEEI mechanism respects the priority structure appropriately: It gives students with higher priorities at a school the right to be assigned to that school earlier. Moreover, given how the priorities are determined, the G-CEEI leaves little room for schools to misreport.

The mechanism can be also applied to the case where some schools have group-specific quotas. This corresponds to the "controlled choice" constraints in school choice under which schools must balance their student bodies in terms of gender, ethnicity, socioeconomic status, or test scores. For example, the Racial Imbalance Law discourages schools from having a minority enrollment that is "substantially" above or below the level of that of the school district overall. Given these group-specific quotas, we can split the schools into sub-schools corresponding to each group and give the group the highest priority/preference. With these sub-schools, we may apply the G-CEEI and the properties of its random assignments are similar to the above.

[6] Hylland and Zeckhauser (1979) give an example showing there are sometimes incentives for students to misreport their preferences when schools do not have preferences.

One-sided matching is a special case of our two-sided setting. If schools do not rank students, the G-CEEI mechanism is equivalent to the CEEI mechanism as proposed by Hylland and Zeckhauser (1979). The *ex ante* efficiency now only considers students' welfare since schools have no preferences, and justified-envy-freeness is strengthened to be envy-free as all students have equal "rights" at any school. Hylland and Zeckhauser conjecture that students' incentive to misreport vanishes when the market grows, but no proof is provided. Our results therefore fill this gap, as the asymptotic incentive compatibility still holds in one-sided matching.

In the following, we give a brief review of related literature in Section 2. Section 3 sets up the model for two-sided matching, and Section 4 defines the G-CEEI mechanism and investigates its properties. In Section 5, we consider applications of the G-CEEI in school choice and one-sided matching. The case of group-specific quotas is also discussed in this section. The paper concludes in Section 6 where we also point out some potential concerns and open questions regarding the mechanism.

2 Literature Review

Our analyses extend the two-sided matching literature in two directions: (i) weak preferences on both sides are allowed, and (ii) cardinal preferences are explicitly considered in the matching process.

In the two-sided matching literature, it has been a standard assumption that both sides have strict ranking over the other side, despite the existence of weak orderings in various real-life settings. Moreover, matching mechanisms usually elicit ordinal preferences of agents. Recently, it has been noted in the literature that when preferences are weak, some issues arise; for example, stability no longer implies Pareto efficiency (Erdil and Ergin (2006)). In several school choice districts in the United States, the student proposing Gale-Shapley mechanism is applied after schools using exogenous tie-breakers to form strict priority ordering over students. Although such a tie-breaking procedure artificially makes preferences/priorities strict, it adversely affects the welfare of the students since it introduces artificial constraints. Abdulkadiroglu, Pathak, and Roth (2009) empirically document the extent of potential efficiency loss associated with stability, while Erdil and Ergin (2008) propose an algorithm for the computation of student-optimal stable matching when priorities are weak. Noting that students may differ in their cardinal preferences, Miralles (2008), Abdulkadiroglu, Che, and Yasuda (2008), and Abdulkadiroglu, Che, and Yasuda (2011) emphasize the importance of eliciting signals of cardinal preferences of students in matching mechanisms.

By using cardinal preferences directly, the G-CEEI mechanism breaks the ties in schools' preferences endogenously. Students with higher cardinal preferences for a school obtain seats at that school before those who are in the same preference group and of the same ordinal preferences. The use of cardinal preferences is particularly important in settings like school choice, where agents usually have similar ordinal preferences.

Recently, the original CEEI mechanism by Hylland and Zeckhauser (1979) has regained some attention in one-sided matching. For example, Miralles (2008) reveals the connection between the CEEI mechanism and the Boston mechanism, given that schools do not rank students.[7] Besides, both Budish (2011) and Budish, Che, Kojima, and Milgrom (Forthcoming) extend the CEEI mechanism to the multi-unit demand setting in which the leading example is assigning course schedules to students. There, every student can register for several courses, and thus they have multi-unit demand. The latter paper also considers additional constraints on the multi-unit demand, such as scheduling and curricular constraints. However, both generalizations are in one-sided matching, i.e., objects do not rank agents and thus agents face the same prices. In contrast, our paper extends the CEEI mechanism to the case of two-sided matching where both sides rank the other side. Moreover, we explicitly study the mechanism's asymptotic incentive compatibility which is omitted in the two papers.

Our proof of asymptotic incentive compatibility is closely related to the classic literature on the price-taking behavior in exchange economy, e.g., Roberts and Postlewaite (1976) and Jackson (1992). Our setting is different in that students have unit demand that restricts their reactions to price changes and, more importantly, in that students may face different prices.

In the matching literature, asymptotic incentive properties of other matching mechanisms also have been studied, e.g., Kojima and Pathak (2009), Che and Kojima (2010), and Kojima and Manea (2010). Besides, as a mechanism design desideratum, Azevedo and Budish (2012) propose a criterion of approximate strategy-proofness which the (Generalized) CEEI mechanism satisfies.

Comparison with Miralles (2011)

In an independent paper, Miralles (2011) studies the same mechanism in a setting with a continuum of students and a finite number of schools. As our paper has a different setting and studies a different set of properties, we believe both papers make contributions in their own rights.

[7] Our result on the relationship between the two mechanisms is stronger and more general, as it considers two-sided matching.

In contrast to our work, strategy-proofness is necessarily satisfied in Miralles' setting, and discussions on incentive properties are therefore omitted. Miralles mainly focuses on priority-based matching problems, such as school choice, and thus his efficiency criterion only takes into account students' welfare. As we study general two-sided matching which sometimes includes priority-based matching as a special case, our efficiency definition takes into account welfare of both sides.

Miralles shows the G-CEEI mechanism satisfies a new concept of fairness, no-unjustified-lower-chances: priorities with respect to one school cannot justify different achievable chances regarding another school.[8] This concept is stronger than justified-envy-freeness, and, however, if justified-envy-freeness is satisfied for any profile of preferences, it implies no-unjustified-lower-chances.

In summary, our results, which are obtained in a finite environment, are particularly relevant when applying the G-CEEI mechanism to real-life problems. In addition to the asymptotic incentive compatibility result, our proof of the existence of equilibrium prices explicitly constructs a price adjustment process which may be used in practice to calculate equilibrium allocations. We also show how to apply the mechanism to school choice with group-specific quotas and note it links to the Boston and Gale-Shapley mechanism.

3 Two-Sided Matching

We consider the following many-to-one matching problem, $\Gamma = \{\mathcal{I}, \mathcal{S}, Q, V, \succsim\}$, where:

(i) $\mathcal{I} = \{i\}_{i=1}^{I}$ is a set of students;

(ii) $\mathcal{S} = \{s\}_{s=1}^{S}$, $S \geq 3$, is a set of schools;[9]

(iii) $Q = [q_s]_{s=1}^{S}$ is a capacity vector, and $q_s \in \mathbb{N}$, $\forall s$; $\sum_{s=1}^{S} q_s = I$, i.e., there are just enough seats to be allocated to students;[10]

(iv) $V = [v_i]_{i \in \mathcal{I}}$, where $v_i = [v_{i,s}]_{s \in \mathcal{S}}$ and $v_{i,s} \in [0, 1]$ is student i's von Neumann-Morgenstern (vN-M) utility associated with school s.

(v) $\succsim = [\geq_s]_{s \in \mathcal{S}}$, where $\geq_s$ is the ordinal preference of school s over individual students. Namely, $i \geq_s j$ means i is at least as preferred as j by s. Moreover, $>_s$ is the strict relation implied by $\geq_s$. According to each school's ordering over students, we further define $k_{s,i}$ as the preference group of student i at school s and $k_{s,i} \in \mathcal{K} \equiv \left\{1, 2, ..., \overline{k}\right\}$ with $\overline{k} \in \mathbb{N}$ and $\overline{k} \leq I$ being the maximum number

[8] Miralles shows that if a random assignment is efficient with respect to students' welfare and is always resolved into stable deterministic assignments, it can be achieved as a G-CEEI assignment under certain regularity conditions. This result indicates that the G-CEEI mechanism may have broader implications for studying two-sided matching.

[9] When there are two schools, the problem is trivial. Many solutions are available to achieve efficient outcomes.

[10] The extension to $\sum_{s=1}^{S} q_s \neq I$ is straightforward.

of preference groups. Therefore, $k_{s,i} \geq k_{s,j}$ if and only if $i \geq_s j$.[11]

This paper assumes complete information in the sense that every student knows the realization of her own preferences, v_i, other students' preferences, v_{-i}, and schools' preferences $\succsim$. The terms, "students" and "schools", can be interpreted more generally as "agents" and "institutions/objects", respectively.

Throughout the paper, we do not rule out the case that $v_{i,s} = v_{i,s'}$ for some i and $s \neq s'$. We also assume that all schools and students are acceptable to the other side, i.e., every school/student is better than the outside option, although the analysis can be extended to the case with unacceptable schools/students. Students are assigned to schools under the unit-demand constraint that each student must be matched with exactly one school.

Similarly, schools' preferences, $\succsim$, need not be strict. Therefore, schools' preferences can be interpreted as their priorities over students, and our setting can naturally be applied to priority-based allocation problems such as school choice in which schools have weak orderings over students. Subsection 5.1 discusses school choice in detail.

If schools do not rank students, or their preferences are not considered, the two-sided problem is then reduced to one-sided matching which is considered in Subsection 5.2.

3.1 Random Assignment and Some Criteria: Definitions

A *random assignment* is a matrix $\Pi = [\pi_i]_{i \in \mathcal{I}} \in \mathcal{A}$, where $\mathcal{A}$ is the space of all possible random assignments; $\pi_i = [\pi_{i,s}]_{s \in \mathcal{S}}$ and $\pi_{i,s} \in [0,1]$ is student i's probability shares in school s, or the probability that student i is matched with school s; $\sum_{s \in \mathcal{S}} \pi_{i,s} = 1$ for all i, and $\sum_{i \in \mathcal{I}} \pi_{i,s} = q_s$ for all s.

If there exists s_i for every i such that $\pi_{i,s_i} = 1$ and $\pi_{i,s} = 0$, $\forall s \neq s_i$, Π is a *deterministic assignment*. Every random assignment can be decomposed into a convex combination of deterministic assignments and can therefore be resolved into deterministic assignments (Kojima and Manea (2010)).[12]

Stability

The arguably most important *ex post* property in two-sided matching that has been studied in the literature is **stability**. A deterministic assignment is stable if and only if there is no student-

[11] Given $\succsim$, the construction of $[k_{s,i}]_{i \in \mathcal{I}, s \in \mathcal{S}}$ may not be unique, but our results remain the same for any given $[k_{s,i}]_{i \in \mathcal{I}, s \in \mathcal{S}}$. Besides, it is innocuous to assume that every school has the same number of preference groups, as there might be no student in a particular preference group at a school.

[12] This result generalizes the Birkhoff-von Newmann theorem (Birkhoff (1946) and von Neumann (1953)). Notice that the convex combination may not be unique in general.

school pair (i, s) such that either (i) student i prefers school s to her current assignment and school s has empty seats, or (ii) student i prefers school s to her current assignment and is more preferred by school s than at least one of the students who are currently matched with s. It has been shown in the literature that stability is a key to a mechanism's success in many two-sided matching problems (Roth (1991)), as it is always possible for students and schools break their current match and rematch with each other.

Efficiency Criteria

A random assignment $\Pi' \in \mathcal{A}$ is **ex ante Pareto dominated** by another random assignment $\Pi \in \mathcal{A}$ if

$$\begin{aligned} \sum_{s \in \mathcal{S}} \pi_{i,s} v_{i,s} &\geq \sum_{s \in \mathcal{S}} \pi'_{i,s} v_{i,s}, \forall i \in \mathcal{I}, \\ \sum_{i \in \{k_{s,i} \geq k\}} \pi_{i,s} &\geq \sum_{i \in \{k_{s,i} \geq k\}} \pi^*_{i,s}, \ \forall s \in \mathcal{S}, \forall k \in \mathcal{K} \end{aligned}$$

and at least one inequality is strict. That is, every student has a weakly higher expected utility in Π, and for each school s the assignment Π first-order stochastically dominates the assignment Π' with respect to $[\geq_s]_{s \in \mathcal{S}}$. A random assignment is **ex ante efficient** if it is not *ex ante* Pareto dominated by any other random assignment.

A deterministic assignment is **Pareto optimal** if it is not Pareto dominated by any other deterministic assignment. Every deterministic assignment in any decomposition of an *ex ante* efficient random assignment is Pareto optimal.

Remark 1 *If one treats schools' preferences as priority constraints as in school choice and only considers students' welfare, the priority need to have a particular structure to achieve efficiency (Ergin (2002) and Kesten (2006)). Unlike others, our definition of efficiency takes into account the welfare of both sides. As schools are endowed with ordinal preferences, we focus on ordinal efficiency for schools as in Bogomolnaia and Moulin (2001). Moreover, McLennan (2002) shows that any ordinally efficient random assignment maximizes the sum of expected utilities for some vector of vN-M utility functions that are consistent with the given ordinal preferences.*

Fairness Criterion

A random assignment Π is **justified-envy-free** if with respect to her expected utility, every student prefers her own random assignment to that of any other student who is weakly less preferred by every school, i.e.,

$$\sum_{s} \pi_{i,s} v_{i,s} \geq \sum_{s} \pi_{j,s} v_{i,s}, \forall i, j, \text{ s.t. } i \geq_s j, \ \forall s \in \mathcal{S}.$$

3.2 Matching Mechanism Given Schools' Preferences

In the following, we assume that schools do not behave strategically and therefore their preferences and capacities can be elicited truthfully. We briefly discuss schools' strategic behavior in Subsection 4.2.

Given schools' preferences, a matching mechanism $\mu\left(\cdot\right| \succsim)$ is a mapping from students' reported preferences to the space of random assignments, $\mathcal{A}$. We focus on the case that students' cardinal preferences are elicited, i.e., $\mu\left(u\right| \succsim) : [0,1]^{I \times S} \to \mathcal{A}$, where $u = [u_i]_{i \in \mathcal{I}} = [u_{i,s}]_{i \in \mathcal{I}, s \in \mathcal{S}}$ and $u_{i,s} \in [0,1]$ is student i's reported vN-M utility associated with school s.

A matching mechanism is *strategy-proof*, if it is a weakly dominant strategy for each student to report true cardinal preferences when vN-M utilities are elicited, or true ordinal preferences when ordinal preferences are elicited. Strategy-proofness is a desirable feature. However, it is incompatible with *ex ante* efficiency and envy-freeness. The following lemma is a corollary of the impossibility theorem in Zhou (1990) and its proof is therefore omitted.

Lemma 1 *If $S \geq 3$ and thus $I \geq 3$, no strategy-proof mechanism can always deliver a random assignment that is ex ante efficient and justified envy-free.*

Zhou (1990) shows that strategy-proofness, *ex ante* efficiency, and symmetry are not compatible in one-to-one one-sided matching. Symmetry requires that any two students with the same preferences receive the same level of utility, and thus it is implied by justified-envy-freeness.

4 The Generalized CEEI Mechanism

The Generalized CEEI (G-CEEI) mechanism works as follows:

(i) Schools (truthfully) report their ordinal preferences, $\succsim$, and capacities, $[q_s]_{s \in \mathcal{S}}$.

(ii) Students report their cardinal preferences, u.

(iii) The mechanism calculates a random assignment, $[\pi_i]_{i \in \mathcal{I}}$, following three steps:

   (a) Every student is artificially given an equal income which is normalized to be 1.

(b) Given $P = [p_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}} \in \mathcal{P} \equiv [0,+\infty]^{S\times\overline{k}}$ where $p_{s,k}$ is the price of school s for students in preference group k at school s, the mechanism constructs the demand of student i for school s, $\pi_i(u_i, P)$, $\forall i$, by solving her utility maximization problem,[13]

$$\begin{aligned} \pi_i(u_i, P) &\in \arg\max_{\pi_{i,s}} \sum_{s\in\mathcal{S}} \pi_{i,s} u_{i,s}, \\ s.t. \sum_{s\in\mathcal{S}} \pi_{i,s} &= 1;\ \pi_{i,s} \geq 0, \forall s \in \mathcal{S}; \sum_{s\in\mathcal{S}} p_{s,k_{s,i}} \pi_{i,s} \leq 1. \end{aligned}$$

If there are multiple bundles maximizing her expected utility, the cheapest ones are chosen.

(c) The mechanism finds an equilibrium price matrix P^* such that

$$\sum_{i\in\mathcal{I}} \pi_{i,s}(u_i, P^*) = q_s,\ \forall s \in \mathcal{S},$$

and that $\forall s$, $p^*_{s,k} = 0$ if $\sum_{i\in\mathcal{I},\ k_{s,i}\geq k} \pi_{i,s}(u_i, P^*) < q_s$, and $p^*_{s,k} = +\infty$ for all $k < k'$ if $\sum_{i\in\mathcal{I},\ k_{s,i}\geq k'} \pi_{i,s}(u_i, P^*) = q_s$.

(iv) Given $\left[\pi^*_{i,s}\right]_{i\in\mathcal{I},s\in\mathcal{S}} = [\pi_{i,s}(u_i, P^*)]_{i\in\mathcal{I},s\in\mathcal{S}}$, the mechanism conducts a lottery and implements a deterministic assignment.

Remark 2 *In equilibrium, there must exist $k^*(s)$ for all s such that $p^*_{s,k^*(s)} \in [0,+\infty)$; $p^*_{s,k} = 0$ if $k > k^*(s)$; and $p^*_{s,k} = +\infty$ if $k < k^*(s)$. Therefore, if s is consumed completely by students in preference groups $k^*(s)$ and higher, students in s's preference groups lower than $k^*(s)$ face an infinite price, while those in preference groups higher than $k^*(s)$ face a zero price.*

Remark 3 *In the G-CEEI mechanism, schools' preferences are students' rights to obtain a school at a lower and sometimes zero price. More precisely, whenever some less preferred students can get some shares of a school, a more preferred student can always get it for free. More importantly, students can choose not to exercise the right if they do not like that school, but they cannot trade it. This interpretation is similar to the consent in Kesten (2010) which allows students to consent to waive a certain priority/preference at a school, while it is different from the treatment in a top-trading-cycle mechanism which allows students to trade schools' priority/preference (Abdulkadiroglu and Sonmez (2003)).*

[13] If $p_{s,k} = +\infty$, we define $+\infty \cdot 0 = 0$ and $+\infty \cdot \pi_{i,s} = +\infty$ if $\pi_{i,s} > 0$.

In summary, the G-CEEI mechanism has the following properties.

Theorem 1 *Given any reported preferences, there always exists an equilibrium price matrix in the G-CEEI mechanism. If students are truth-telling, its random assignment is ex ante efficient and justified-envy-free, and any corresponding deterministic assignment is stable.*

Moreover, the G-CEEI mechanism is closely linked with two commonly used mechanisms, the Boston mechanism and the Gale-Shapley mechanism.[14]

Proposition 1 *Given a random assignment prescribed by the G-CEEI when both students and schools are truth-telling, if every student has strict preferences and consumes a bundle which includes at most one school with a positive and finite price, the assignment is also a Nash equilibrium assignment of the Boston mechanism.*

Our conditions in the proposition are also necessary. That is, when students do not have strict preferences, or at least one of them spends her income on more than one schools with positive and finite prices, in general, the G-CEEI assignment is not an equilibrium outcome of the Boston mechanism. The above proposition does not imply that any equilibrium outcome of the Boston mechanism is a G-CEEI outcome, although it may be satisfied almost surely if one imposes some conditions on the joint distribution of students' preferences.

When both sides have strict preferences, it must be that $\overline{k} = I$ and that there is exactly one student in each preference group at any school. In this case, we have the following proposition.

Proposition 2 *If both students and schools have strict preferences, every stable deterministic assignment is an equilibrium assignment of the G-CEEI, so are the student-optimal (school-optimal) stable assignments prescribed by the student-proposing (school-proposing) Gale-Shapley mechanism.*

Given the impossibility result in Lemma 1 and the example in Hylland and Zeckhauser (1979), we know that students may sometimes have incentives to misreport their preferences. In the following, we investigate the mechanism's incentive property.

4.1 Incentive Compatibility for Students

4.1.1 Per Capita Demand for Each Preference Group

We define a sequence of economies and per capita demand functions, while taking into account that students in different preference groups face different prices, and thus the per capita demand

[14] The formal definitions of the two mechanisms are available in Appendix 1.

is preference-group-specific.

Let $F_i(P)$ be the augmented set of feasible consumption bundles for student i,

$$F_i(P) \equiv \begin{cases} \left\{ \pi_i = [\pi_{i,s}]_{s\in\mathcal{S}} \;\middle|\; \begin{array}{c} \pi_{i,s} \geq 0,\ \forall s,\ \sum_{s\in\mathcal{S}} \pi_{i,s} = 1, \\ \text{and } \sum_{s\in\mathcal{S}} \pi_{i,s} p_{s,k_{s,i}} \leq 1 \end{array} \right\}, \text{ if } p_{s,k_{s,i}} \leq 1 \text{ for some } s; \\ \left\{ \pi_i = [\pi_{i,s}]_{s\in\mathcal{S}} \;\middle|\; \begin{array}{c} \pi_{i,s} \geq 0,\ \forall s,\ \sum_{s\in\mathcal{S}} \pi_{i,s} = \frac{1}{\min_{t=1,\dots,S}\left\{p_{t,k_{t,i}}\right\}}, \\ \text{and } \sum_{s\in\mathcal{S}} \pi_{i,s} p_{s,k_{s,i}} \leq 1 \end{array} \right\}, \text{ if } p_{s,k_{s,i}} > 1,\ \forall s. \end{cases}$$

When there is no affordable bundles such that $\sum_{s\in\mathcal{S}} \pi_{i,s} = 1$, the second part of the definition assumes that every student is allowed to spend all their money on the cheapest school(s). $F_i(P)$ is then non-empty, closed, and bounded.[15]

Let $U_i = \sum_{s\in\mathcal{S}} \pi_{i,s} v_{i,s}$ be i's expected utility function. Define $G_i(P, v_i)$ as the set of bundles that i would choose from $F_i(P)$ to maximize U_i. Formally,

$$G_i(P, v_i) = \left\{ \pi_i \in F_i(P) \;\middle|\; \begin{array}{c} \forall \pi'_i \in F_i(P), U_i(\pi_i) > U_i(\pi'_i), \\ \text{or } U_i(\pi_i) \geq U_i(\pi'_i) \text{ and } \sum_{s\in\mathcal{S}} \pi_{i,s} p_s \leq \sum_{s\in\mathcal{S}} \pi'_{i,s} p_s \end{array} \right\}.$$

Since $G_i(P, v_i)$ is obtained from the closed, bounded, and non-empty set $F_i(P)$ by maximizing (and minimizing) continuous functions, $G_i(P, v_i)$ must be non-empty. $G_i(P, v_i)$ is a convex set, because $U_i(\pi_i)$ and $\sum_{s\in\mathcal{S}} \pi_{i,s} p_{s,k_{s,i}}$ are linear functions of π_i.

Define $G(P, v)$ as the set of per capita demand for each preference group of each school that can emerge when prices equal P and each student i chooses a vector in $G_i(P, v_i)$, that is, $\forall P \in \mathcal{P}$:

$$G(P, v) = \left\{ D = [d_{s,k}]_{s\in\mathcal{S}, k\in\mathcal{K}} \;\middle|\; \begin{array}{c} d_{s,k} = \frac{1}{|\mathcal{I}|} \sum_{\{i\in\mathcal{I} | k_{s,i} = k\}} \pi_{i,s},\ \forall s, \forall k \\ [\pi_{i,s}]_{s\in\mathcal{S}} \in G_i(P, v_i) \end{array} \right\}.$$

It can be verified that $G(P, v)$ is also closed, bounded, and upper hemicontinuous.

4.1.2 Sequence of Economies

The following definition is needed to define the sequence of economies.

Definition 1 *A sequence of correspondences $f^{(n)}(P)$ uniformly converge to $f(P)$ if and only if,*

[15] It is important to note that P cannot be an equilibrium whenever the second part of $F_i(P)$'s definition is invoked.

for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that when $n \geq N$,

$$\sup_P d_H \left(f^{(n)}(P), f(P) \right) \leq \varepsilon,$$

where d_H is Hausdorff distance, i.e.,

$$d_H \left(f^{(n)}(P), f(P) \right) = \max \left\{ \begin{array}{c} \sup_{Y \in f(P)} \inf_{Y^{(n)} \in f^{(n)}(P)} \left\| Y^{(n)} - Y \right\|, \\ \sup_{Y^{(n)} \in f^{(n)}(P)} \inf_{Y \in f(P)} \left\| Y^{(n)} - Y \right\| \end{array} \right\},$$

where $\|\cdot\|$ is the Euclidean distance.

Let $\left\{ \Gamma^{(n)} \right\}_{n \in \mathbb{N}}$ be a sequence of matching problems where $\Gamma^{(n)} = \left\{ \mathcal{I}^{(n)}, \mathcal{S}, q^{(n)}, v^{(n)}, \succcurlyeq^{(n)} \right\}$ and $\forall n \in \mathbb{N}$:

(i) $\mathcal{I}^{(n)} \subset \mathcal{I}^{(n')}$ and $q_s^{(n)} < q_s^{(n')}$ for all s if $n < n'$; $\left| \mathcal{I}^{(n)} \right| = \sum_{s \in \mathcal{S}} q_s^{(n)}$; and $q_s^{(n)} / \left| \mathcal{I}^{(n)} \right| = q_s / I$;

(ii) $\succcurlyeq^{(n)}$ is such that the associated preference groups satisfy that $\left| \left\{ i \in \mathcal{I}^{(n)} | k_{s,i} = k \right\} \right| / \left| \mathcal{I}^{(n)} \right| = C_{s,k}$, for all k and s, where $C_{s,k}$ is a constant.[16]

(iii) the number of schools, $S = |\mathcal{S}|$, is constant;

(iv) the corresponding per capita demand $G^{(n)} \left(P, v^{(n)} \right) \to g(P)$ uniformly as $n \to \infty$.

Remark 4 *$g(P)$ is a convex-valued, closed, bounded, and upper hemicontinuous correspondence, since $G^{(n)} \left(P, v^{(n)} \right)$ has these properties. Similar but usually stronger convergence restrictions are assumed in the literature on proving incentive compatibility in exchange economies, e.g., Roberts and Postlewaite (1976) and Jackson (1992).*

Remark 5 *This definition includes two special cases: (i) a sequence of replica economies and $G^{(n)} \left(P, v^{(n)} \right) = g(P)$, for all $n \in \mathbb{N}$; and (2) a sequence of economies in which students' preferences and the associated preference groups are i.i.d. draws from a joint distribution of students' and schools' preferences, while holding constant the relative size of each preference group at each school.*

4.1.3 Results

Strategy-proofness requires truth-telling being a dominant strategy for every student, which means there is no restriction on other students' reports. In the exchange economy, Roberts and Postlewaite (1976) provide an example in which the incentive to misreport does not vanish as the economy grows.

[16] Note that for each $\succcurlyeq^{(n)}$, the construction of corresponding preference groups may not be unique. We assume that the same rule is used to construct the groups given $\succcurlyeq^{(n)}$ for all n.

To study the incentive property, this literature has imposed some restrictions on the sequence of economies and focused on Nash equilibrium. For example, Roberts and Postlewaite (1976) and Jackson (1992) assume some convergence conditions on others' reported demand functions. The following result is in the same spirit and shows that truth-telling is asymptotically a Nash equilibrium.

Proposition 3 *Given* $\left\{\Gamma^{(n)}\right\}_{n\in\mathbb{N}}$*, if other students are truth-telling, each individual's incentive to misreport in the G-CEEI mechanism goes to zero when* $n \to \infty$*.*

Remark 6 *The above proposition says that it is asymptotically a Nash equilibrium if everyone is truth-telling. This is different from the criterion, strategy-proof in the large (SP-L), proposed by Azevedo and Budish (2012). A mechanism is SP-L if, for any agent, any probability distribution of the other agents' reports, and any* $\epsilon > 0$*, in a large enough market the agent maximizes his expected payoff to within* ϵ *by reporting his preferences truthfully. Equivalently, the distribution of other agents' reports is known, but not their realizations. The (Generalized) CEEI mechanism satisfies this criterion. However, in our setting, each student has complete information on other students' preferences and they find their best responses to opponents' realized reports.*

4.2 Schools' Incentives

In the above analysis, we assume schools do not behave strategically, but this assumption is not satisfied in general. Under the student-optimal stable mechanism, Kojima and Pathak (2009) show that the fraction of schools with incentives to misrepresent their preferences when others are truthful approaches zero as the market becomes large. Unlike our setting, their market grows in the sense that the number of schools increases. Under the G-CEEI mechanism, more work need to be done to study the asymptotic incentive properties of both students and schools simultaneously.

5 Applications: School Choice and One-Sided Matching

5.1 School Choice: Priority as Schools' Ordinal Preferences

School choice and other resource allocation problems based on priorities are common in real life. Schools usually have priority ordering over students, and the priority structure is usually determined by government or local laws, and requires that higher-priority students obtain that school first, *ceteris paribus.*. Very often schools' priority ranking over students is not strict. Therefore, if one treats schools' priority as their ordinal preferences, the G-CEEI mechanism can be applied to school choice naturally, and it gives students with higher priorities at a school the right to be assigned to that school earlier.

In the previous literature, it is noted that schools' priority ranking over students is not their preferences and is usually treated as exogenous. However, we may interpret priorities as preferences because (i) priority structure may be correlated with schools' or governments' preferences and (ii) students' priorities at the schools are endogenous. For instance, school districts usually have rules that give high priorities to students who live close to school (neighborhood priority), which can be justified as governments' objective to minimize transportation costs. Priority is also usually given to those who have higher test scores, and this can be interpreted as schools' preferences for better performing students. More importantly, given any priority structure, students endogenously choose their behavior. For example, under the neighborhood priority rule, students choose where to live in order to get access to their preferred school, and thus it creates a market for students to compete (Tiebout (1956)). If rules on priority have been stable over time, schools' priority ordering over students can be highly correlated with students' preferences.

Our explicit use of students' cardinal preferences make the mechanism particularly attractive to school choice. As Abdulkadiroglu, Che, and Yasuda (2011) point out, in settings as school choice, students usually have similar ordinal preferences. Therefore, without information on cardinal preferences, the efficiency that a mechanism can achieve is very limited, and sometimes one cannot do better than a pure random allocation. The G-CEEI opens another avenue for efficiency gain in these settings.

Besides, as the priority structure of schools is normally determined by government or local law, the scope for schools to misreport their "preference" is very limited in school choice.

5.1.1 Matching with Group-Specific Quotas

In some real-life applications, there may be constraints on the allocation of school quotas. For example, schools may have group-specific quotas. The G-CEEI mechanism can be applied in this case as well, and the results are readily extended. Budish, Che, Kojima, and Milgrom (Forthcoming) study this case under the assumption that schools do not rank students besides the group-specific quotas, whereas we allow schools to rank students.

We divide each school into multiple sub-schools each of which has a quota equal to the one for the corresponding group and gives that group the highest priority. Other students' priorities at these sub-schools are determined by the pre-specified rules. With these sub-schools and their priority rankings, our analyses are straightforward to extend.

Note that we rule out the case that some schools cannot fill some group-specific quotas even when their prices for those students are zero. In another words, we assume that the schools with group-specific quotas are attractive enough to that group of students. Although it is possible to relax this assumption and instead impose that every group-specific quota always has to be fulfilled, in our opinion, this is against the principle of school choice, as it may sometimes force some students to attend certain schools.[17] More importantly, it would incur significant efficiency loss to do so.

5.2 One-Sided Matching: Schools Do Not Rank Students

If schools do not rank students, our setting is reduced to one-sided matching, and the G-CEEI mechanism is equivalent to the CEEI mechanism (Hylland and Zeckhauser (1979)).[18] All students then face the same price of each school.

Theorem 3 still holds but can be interpreted differently. The *ex ante* efficiency only considers students' welfare since schools have no preferences; justified-envy-freeness is strengthened to be envy-free as all students have equal "rights" at all schools; and similarly, the asymptotic incentive compatibility still holds in one-sided matching. It therefore proves Hylland and Zeckhauser's conjecture that the incentive to misreport vanishes when the market grows.

The following proposition, which is a corollary of Theorem 1 and Proposition 3, summarizes the properties of the CEEI mechanism, and its proof is therefore omitted but available upon request.

Proposition 4 *In one-sided matching, given any reported preferences, there always exists equilibrium prices in the CEEI mechanism. If students are truth-telling, its random assignment is ex ante efficient and envy-free. In a sequence of economies à la Proposition 3, if other students are truth-telling, each individual's incentive to misreport in the CEEI mechanism goes to zero as the market grows.*

[17] With techniques developed in Budish, Che, Kojima, and Milgrom (Forthcoming), it is possible to extend our analyses to the case that all group-specific quotas always have to be met.

[18] The CEEI mechanism is originally proposed for one-to-one matching. It is straightforward to extend it to many-to-one matching, as long as each student has unit demand.

6 Concluding Remarks

In a many-to-one setting, this paper studies the problem of matching students with schools, or more generally matching agents with institutions, when monetary transfers are not possible. Each side has a possibly weak ranking over agents on the other side, and students are endowed with cardinal preferences. Each school seat is viewed as a continuum of probability shares, and students have unit demand.

We use the idea of pseudo-market and define a new mechanism, the G-CEEI mechanism in which students who are more preferred by a school faces a lower and sometimes zero price of that school. In other words, if a student is more preferred by a school, she has the right to be assigned to that school earlier.

We establish the existence of equilibrium prices given any reports of preferences, and we show that it is asymptotically incentive compatible for students. Moreover, when both students and schools are truth-telling, the mechanism delivers random assignment that is justified-envy-free and *ex ante* efficient with respect to both sides' preferences. The corresponding deterministic assignments are always stable. All the results hold true in school choice, school choice with group-specific quota, and one-sided matching. In particular, as the mechanism explicitly uses students' cardinal preferences, it may significantly improve efficiency in settings like school choice where agents have similar ordinal preferences.

We note that the mechanism is not strategy-proof for schools, and we leave it as a future research topic to study schools' asymptotic incentive properties. However, since schools' priority/preference structure is known in school choice, this makes the mechanism more attractive in this setting.

Another concern with the mechanism is that it might be difficult to elicit cardinal preferences from students. For instance, Bogomolnaia and Moulin (2001) argue that agents participating in the mechanisms may have the limited rationality and thus do not know exactly their cardinal preferences. To address this issue theoretically, one may consider the case in which students know their true ordinal preferences while knowing their cardinal preferences with some errors. One can then compare the performance of the G-CEEI mechanism with those of other mechanisms.

From a very different point of view, one may consider the requirement of reporting cardinal preferences as an incentive for student to investigate if the school is a good fit for her. Empirically, He (2012) documents that students in Beijing pay different levels of attention to school quality under the Boston mechanism in which signals of cardinal preferences are elicited. This kind of attention which is related to information acquiring is likely to be welfare-improving.

Bogomolnaia and Moulin also point out that there is convincing experimental evidence that the representation of preferences over uncertain outcomes by vN-M utility functions is inadequate. While how to model decision under uncertainty is beyond the scope of our paper, the G-CEEI can still be applied as long as students' objective function is well defined.

The above and potentially many other concerns about the G-CEEI mechanism call for future research efforts in related fields.

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Appendix 1: The Two Popular Mechanisms

This appendix gives the formal definition of the two popular mechanisms: the Boston mechanism and the Gale-Shapley mechanism.

The **Boston mechanism** asks students to submit rank-ordered lists, uses pre-defined rules to determine schools' ranking over students, and has multiple rounds:

Round 1. Each school considers all the students who rank it first and assigns seats in order of their priority at that school until either there is no seat left or no such student left.

Generally, in:

Round k. The kth choice of the students who have not yet been assigned is considered. Each school that still has available seats assigns the remaining seats to students who rank it as kth choice in order of their priority at that school until either there is no seat left or no such student left.

The process terminates after any round k when every student is assigned a seat at a school, or if the only students who remain unassigned listed no more than k choices. Unassigned students are then matched with available seats randomly.

The **Gale-Shapley mechanism**, which is also known as the deferred-acceptance mechanism, can be student-proposing or school-proposing. In the former, schools announce their enrollment quota and students submit rank-ordered lists of schools. The matching process has several rounds:

Round 1. Every student applies to her first choice. Each school rejects the least preferred students in excess of its capacity and *temporarily holds* the others.

Generally, in:

Round k. Every student who is rejected in Round $(k-1)$ applies to the next choice on her list. Each school pools new applicants and those who are held from Round $(k-1)$ together and rejects the least preferred students in excess of its capacity. Those who are not rejected are *temporarily held* by the schools.

The process terminates after any Round k when no rejections are issued. Each school is then matched with students it is currently holding.

The school-proposing Gale-Shapley mechanism is similarly defined.

Appendix 2: Proofs

Proof of Theorem 1. (i) Existence.

First, we transform the price space from $\mathcal{P} \equiv [0, +\infty]^{S \times \overline{k}}$ to $\mathcal{Z} \equiv [0, \pi/2]^{S \times \overline{k}}$ such that $\forall P \in \mathcal{P}$, there is a $Z \in \mathcal{Z}$ and $Z = [z_{s,k}]_{s \in \mathcal{S}, k \in \mathcal{K}} = [\arctan(p_{s,k})]_{s \in \mathcal{S}, k \in \mathcal{K}}$, with $\arctan(+\infty) \equiv \pi/2$ and $\tan(\pi/2) \equiv +\infty$.[19] Since arctan is a positive monotonic transformation, the reverse statement is also true such that $\forall Z \in \mathcal{Z}$, there is a $P \in \mathcal{P}$ and $P = \mathcal{TAN}(Z) \equiv [\tan(z_{s,k})]_{s \in \mathcal{S}, k \in \mathcal{K}}$.

A price-adjustment process for Γ is defined as,

$$\begin{aligned} & H\left[Z, G\left(\mathcal{TAN}(Z), u\right)\right] \\ \equiv \; & \left\{ Y = [y_{s,k}]_{s \in \mathcal{S}, k \in \mathcal{K}} \;\middle|\; \begin{array}{c} y_{s,k}\left([d_{s,k}]_{k \in \mathcal{K}}\right) = \min\left\{\frac{\pi}{2}, \max\left[0, z_{s,k} + \left(\sum_{\kappa=1}^{k} d_{s,\kappa} - \frac{q_s}{I}\right)\right]\right\} \\ \forall\, [d_{s,k}]_{s \in \mathcal{S}, k \in \mathcal{K}} \in G\left(\mathcal{TAN}(Z), u\right) \end{array} \right\}. \end{aligned}$$

[19] Here and in the following, with some abuse of notation, π, without subscript, is the mathematical constant, i.e., the ratio of a circle's circumference to its diameter.

where $u = (u_1, \cdots, u_I)$ are students' reports, and $G(\mathcal{TAN}(Z), v)$ is per capita demand for each preference group at each school.

Since G is the average demand, it is then upper hemicontinuous and convex-valued, and thus $H[Z, G]$ has the same properties. $H[Z, G]$ satisfies all the conditions of Kakutani's fixed-point theorem, and there must exist a fixed point Z^* such that $Z^* \in H[Z^*, G(\mathcal{TAN}(Z^*), u)]$. Given Z^*, there also exists $[d_{s,k}]_{s \in \mathcal{S}, k \in \mathcal{K}} \in G$ such that $\forall s$, $\forall k$, $z^*_{s,k} = \min\left\{\frac{\pi}{2}, \max\left[0, z^*_{s,k} + \left(\sum_{\kappa=1}^{k} d_{s,\kappa} - \frac{q_s}{I}\right)\right]\right\}$.

This implies that $\forall s$, $\sum_{\kappa=1}^{\overline{k}} d_{s,\kappa} = q_s/I$ and there exists a unique $k^*(s)$ for each s such that $\sum_{\kappa=1}^{k^*(s)} d_{s,\kappa} = \frac{q_s}{I}$, $z^*_{s,k^*(s)} \in \left[0, \frac{\pi}{2}\right)$, and $d_{s,k^*(s)} > 0$; if $k < k^*(s)$, $\sum_{\kappa=1}^{k} d_{s,\kappa} < \frac{q_s}{I}$ and $z^*_{s,k} = 0$; and if $k > k^*(s)$, $d_{s,k} = 0$, and $z^*_{s,k} \in \left[0, \frac{\pi}{2}\right]$.

Moreover, if $d_{s,k} = 0$, and $z^*_{s,k} \in [0, \frac{\pi}{2})$ for some $k > k^*(s)$, there must exist another Z^{**} such that $Z^{**} \in H[Z^{**}, G(\mathcal{TAN}(Z^{**}), u)]$ and that if $k \leq k^*(s)$, $z^{**}_{s,k} = z^*_{s,k}$; and if $k > k^*(s)$, $z^{**}_{s,k} = \frac{\pi}{2}$.

In summary, $\mathcal{TAN}(Z^{**})$ satisfies the form of equilibrium prices and indeed clears the market. Therefore, an equilibrium price vector $P^{**} = \mathcal{TAN}(Z^{**}) \in \mathcal{P}$ exists.

(ii) Efficiency and Fairness.

We define the following rules regarding infinity:

$$0 * +\infty = 0, +\infty \geq +\infty$$

Suppose the G-CEEI random assignment, $\left[\pi^*_{i,s}\right]_{i \in \mathcal{I}, s \in \mathcal{S}}$, is *ex ante Pareto dominated* by another random assignment $[\pi_{i,s}]_{i \in \mathcal{I}, s \in \mathcal{S}}$, i.e.,

$$\sum_{s \in \mathcal{S}} \pi_{i,s} v_{i,s} \geq \sum_{s \in \mathcal{S}} \pi^*_{i,s} v_{i,s}, \forall i \in \mathcal{I}, \tag{1}$$

$$\sum_{i \in \{k_{s,i} \geq k\}} \pi_{i,s} \geq \sum_{i \in \{k_{s,i} \geq k\}} \pi^*_{i,s}, \ \forall s \in \mathcal{S}, \forall k \in \left\{1, 2, ..., \overline{k}\right\}, \tag{2}$$

and at least one inequality is strict.

For any student whose most preferred school is free or of price less than one, she obtains that school for sure, and there is no other assignment to make her better off. If for student i, $\sum_{s \in \mathcal{S}} \pi_{i,s} v_{i,s} > \sum_{s \in \mathcal{S}} \pi^*_{i,s} v_{i,s}$, it must be such that $\sum_{s \in \mathcal{S}} p_{s,k_{i,s}} \pi_{i,s} > 1$ and $\sum_{s \in \mathcal{S}} \pi^*_{i,s} p_{s,k_{i,s}} = 1$. For other students, it must be that $\sum_{s \in \mathcal{S}} p_{s,k_{j,s}} \pi_{j,s} \geq \sum_{s \in \mathcal{S}} p_{s,k_{j,s}} \pi^*_{j,s}$, since $\left[\pi^*_{j,s}\right]_{s \in \mathcal{S}}$ is the cheapest among bundles delivering the same expected utility. Therefore,

$$\sum_{s \in \mathcal{S}} p_{s,k_{i,s}} \pi_{i,s} + \sum_{j \neq i} \sum_{s \in \mathcal{S}} p_{s,k_{j,s}} \pi_{j,s} > \sum_{s \in \mathcal{S}} p_{s,k_{j,s}} \pi^*_{i,s} + \sum_{j \neq i} \sum_{s \in \mathcal{S}} p_{s,k_{j,s}} \pi^*_{j,s}.$$

However, equation (2) implies that:

$$\sum_{j \in \mathcal{I}} \sum_{s \in \mathcal{S}} p_{s,k_{j,s}} \pi^*_{j,s} \geq \sum_{j \in \mathcal{I}} \sum_{s \in \mathcal{S}} p_{s,k_{j,s}} \pi_{j,s},$$

because prices are higher for students in lower preference group. This leads to a contradiction.

Suppose instead that for school s, equation (2) is satisfied for all k, and $\exists \underline{k} \in \left\{2, ..., \overline{k}\right\}$, such that

$$\sum_{i \in \{k_{s,i} \geq \underline{k}\}} \pi_{i,s} > \sum_{i \in \{k_{s,i} \geq \underline{k}\}} \pi^*_{i,s}.$$

This implies,

$$\sum_{j\in\mathcal{I}} p_{s,k_{j,s}}\pi_{j,s} < \sum_{j\in\mathcal{I}} p_{s,k_{j,s}}\pi^*_{j,s},$$

because prices are higher for students in lower preference group. Aggregating over all schools,

$$\sum_{j\in\mathcal{I}}\sum_{s\in\mathcal{S}} p_{s,k_{j,s}}\pi_{j,s} < \sum_{j\in\mathcal{I}}\sum_{s\in\mathcal{S}} p_{s,k_{j,s}}\pi^*_{j,s}.$$

However, equation (1) implies that $\sum_{s\in\mathcal{S}} p_{s,k_{j,s}}\pi_{j,s} \geq \sum_{s\in\mathcal{S}} p_{s,k_{j,s}}\pi^*_{j,s}$, $\forall j \in \mathcal{I}$, and thus,

$$\sum_{j\in\mathcal{I}}\sum_{s\in\mathcal{S}} p_{s,k_{j,s}}\pi_{j,s} \geq \sum_{j\in\mathcal{I}}\sum_{s\in\mathcal{S}} p_{s,k_{j,s}}\pi^*_{j,s}.$$

This leads to another contradiction.

Therefore, $\left[\pi^*_{i,s}\right]_{i\in\mathcal{I},s\in\mathcal{S}}$, must be *ex ante efficient.*

To prove $\left[\pi^*_{i,s}\right]_{i\in\mathcal{I},s\in\mathcal{S}}$ is justified-envy-free, suppose that students i and j are such that $k_{s,i} \geq k_{s,j}$, $\forall s \in \mathcal{S}$. Then i faces the same or a lower price at each school than j does. $\left[\pi^*_{j,s}\right]_{s\in\mathcal{S}}$ is also affordable to i. Therefore, i will never envy j's assignment.

(iii) Stability.

Given our setting, stability means that there is no student-school blocking pair (i, s), that is, student i prefers school s to her current assignment and is more preferred by school s than at least one of the students who are currently matched with s.

Under the G-CEEI mechanism, for any student i, we can group all S schools into two categories: 0-price schools, denoted as $\mathcal{O}_i$, i.e., $p_{s,k_{i,s}} = 0$, $\forall s \in \mathcal{O}_i$; and positive-price school(s), denoted as $\mathcal{X}_i$, i.e., $p_{s,k_{i,s}} \in (0, +\infty]$, $\forall s \in \mathcal{X}_i$. $\mathcal{O}_i$ or $\mathcal{X}_i$ may be empty but $\mathcal{O}_i \cup \mathcal{X}_i = \mathcal{S}$, $\forall i \in \mathcal{I}$.

Given a random assignment prescribed by the G-CEEI mechanism, $\forall i \in \mathcal{I}$, denote $\mathcal{V}_i \subseteq \mathcal{S}$ as the set of schools that student i has positive probability shares. In a deterministic assignment generated from this random assignment, student i matches with some s_i in $\mathcal{V}_i$.

Therefore, student i must not prefer any school in $\mathcal{O}_i$ to s_i, and this means that $v_{i,s_i} \geq v_{i,s}$, $\forall s \in \mathcal{O}_i$. If not, suppose $\exists s' \in \mathcal{O}_i$ such that $v_{i,s_i} < v_{i,s'}$. She then can always profitably replace her probability share at s_i with the same amount of probability share at s', because s' is free. This contradicts the condition that the random assignment prescribed by the mechanism maximizes everyone's expected utility.

Alternatively, suppose student i prefers some $s \in \mathcal{X}_i$ to s_i. It must be that the students matched with school s must be in the same or higher preference group than i at school s. This is because $p_{s,k_{j,s}} = +\infty$ for any student j in lower preference groups at s, given $p_{s,k_{i,s}} > 0$.

Hence, there is no blocking pair in any deterministic assignment generated by a random assignment prescribed by the G-CEEI mechanism. ■

Proof of Proposition 1. Let P^* be the equilibrium price vector in the G-CEEI mechanism. Suppose that $s_{i,1}$ is the non-free school on which student i spends income, and that $s_{i,2}$ is her most preferred school among all free ones. Since each student has strict preferences over schools, $s_{i,2}$ is unique.

Her random assignment $\left\{\pi^*_{i,s}\right\}_{s\in\mathcal{S}}$ must be such that:

$$\pi^*_{i,s_{i,1}} = \min\left\{1/p^*_{s_{i,1},k_{s_{i,1},i}}, 1\right\}, \ \pi^*_{i,s_{i,2}} = 1 - \pi^*_{i,s_{i,1}}, \text{ and } \pi^*_{i,s} = 0, \ \forall s \neq s_{i,1}, \neq s_{i,2},$$

or, if i does not spend any money on any non-free schools,

$$\pi^*_{i,s_{i,2}} = 1, \text{ and } \pi^*_{i,s} = 0 \ \forall s \neq s_{i,2}.$$

Therefore,

$$\left\{\pi^*_{i,s}\right\}_{s\in\mathcal{S}} \in \arg\max_{\pi_{i,s}} \left\{ \begin{array}{c} \sum_{s=1}^{S} \pi_{i,s} v_{i,s}, \\ s.t. \sum_{s\in\mathcal{S}} \pi_{i,s} = 1, \ \pi_{i,s} \geq 0, \forall s \in \mathcal{S}, \ \sum_{s\in\mathcal{S}} p^*_{s,k}\pi_{i,s} \leq 1 \end{array} \right\}.$$

Consider that student i's rank-order list in the Boston mechanism is instead $C^*_i = (s_{i,1}, s_{i,2}, \phi, ..., \phi)$ or $C^*_i = (s_{i,2}, \phi, ..., \phi)$ if she does not spend any income at all. Here, ϕ denotes that there is no school ranked. It can be verified that given these rank-order lists, the Boston mechanism clears the market in two rounds and delivers the same random assignment as the G-CEEI mechanism. The only thing left to check is that this is a Nash equilibrium, i.e.,

$$C^*_i \in \arg\max_{C_i} B\left(C_i, C^*_{-i}\right)(v_{i,1}, ..., v_{i,S})',$$

where $B\left(C_i, C^*_{-i}\right) = (\pi_{i,s})_{s\in\mathcal{S}}$ is the vector of probabilities that student i is assigned to each school given $\left(C_i, C^*_{-i}\right)$.

Suppose $\left(C^*_i, C^*_{-i}\right)$ is not a Nash equilibrium and there is another $C'_i = (s'_1, ..., s'_S) \neq C^*_i$. It suffices to show that any assignment resulted from any given deviation, C'_i, is affordable to i in the G-CEEI mechanism.

If $C^*_i = (s_{i,1}, s_{i,2}, \phi, ..., \phi)$, given C^*_{-i}, the schools of which i may obtain some probability shares when ranking them as third or later choices are only $s_{i,1}$ and $s_{i,2}$, while the only non-free school that may be available in the second round is $s_{i,1}$. Therefore, in addition to $s_{i,1}$, i may obtain some probability shares of at most one other non-free school by ranking it top.

Consider random assignments of the following form: $\pi'_{i,s_{i,1}} \geq 0$ and $\pi'_{i,s'_1} > 0$ where $\pi^*_{i,s_{i,1}} + \pi'_{i,s'_1} \leq 1$ and $s_{i,1} \neq s'_1$ are not free in the G-CEEI equilibrium; if $\pi^*_{i,s_{i,1}} + \pi'_{i,s'_1} < 1$, i obtains some shares of free schools to meet the unit demand assumption. This assignment may be obtained by deviating to $C'_i = (s'_1, s_{i,1}, s, \phi, ..., \phi)$, $C'_i = (s'_1, s, s_{i,1}, \phi, ..., \phi)$, or other payoff equivalent strategies, where s is a school that is free for i in the G-CEEI mechanism. Given C^*_{-i}, the most expensive one among all assignments that can be obtained by deviating from C^*_i to C'_i must be of this form. Otherwise, it must be cheaper than the assignment $\left\{\pi^*_{i,s}\right\}_{s\in\mathcal{S}}$.

Moreover, given C^*_{-i} and the rules of the Boston mechanism, we can derive

$$\pi'_{i,s'_1} = \frac{q_{s'_1} - \sum_{j\in\left\{j\in\mathcal{I}:k_{s'_1,j}>k_{s'_1,i}\right\}} \pi^*_{j,s'_1}}{p^*_{s'_1,k_{s'_1,i}}\left(q_{s'_1} - \sum_{j\in\left\{j\in\mathcal{I}:k_{s'_1,j}>k_{s'_1,i}\right\}} \pi^*_{j,s'_1}\right) + 1},$$

where $q_{s'_1} - \sum_{j\in\left\{j\in\mathcal{I}:k_{s'_1,j}>k_{s'_1,i}\right\}} \pi^*_{j,s'_1}$ is the remaining quota at s'_1 after those who are in higher preferences groups claim their shares; and $p^*_{s'_1,k_{s'_1,i}}\left(q_{s_1} - \sum_{j\in\left\{j\in\mathcal{I}:k_{s'_1,j}>k_{s'_1,i}\right\}} \pi^*_{j,s'_1}\right)$ is the total expenditure on s'_1 by students who are in the same preference as i at s'_1, and more importantly it is the total number of such students other than i who have ranked s'_1 as first choice given C^*_{-i}.

Note that the bundles $\left[\pi'_{i,s'_1}, \pi'_{i,s_{i,1}} \leq \left(1-\pi'_{i,s'_1}\right)\right]$, are always affordable in the G-CEEI, as long as $0 < p^*_{s_{i,1},k_{s_{i,1},i}} \leq 1$. If instead, $p^*_{s_{i,1},k_{s_{i,1},i}} > 1$, $s_{i,1}$ must not be available if i ranks it second or lower. This proves any assignment resulted from a deviation, C'_i, is affordable in the G-CEEI mechanism, if $C^*_i = (s_{i,1}, s_{i,2}, \phi, ..., \phi)$.

In the same manner, we can show that i can afford any assignment from deviations when $C^*_i = (s_{i,2}, \phi, ..., \phi)$. This complete the proof that $\left(C^*_i, C^*_{-i}\right)$ is a Nash equilibrium in the Boston mechanism. ■

Proof of Proposition 2. Given a stable matching, for each school s, we may find $\underline{k}_s = \min_{i\in\{j\in\mathcal{S}\ \mid\ j \text{ is matched with } s\}} \{k_{s,i}\}$ which is the lowest preference group at s among those who are matched with s. We may then define the following price system:

$$p_{s,k} = \begin{cases} 0, \text{ if } k_{s,i} \geq \underline{k}_s \\ +\infty, \text{ if } k_{s,i} < \underline{k}_s \end{cases}, \forall s.$$

This price system satisfies the requirement of the G-CEEI mechanism. We need to show that students maximize their expected utility given the prices.

The only possible deviation for a student i is to choose some school s which is free to her. That is, she is in a higher preference group at s than someone who is already accepted by s. If this deviation is profitable to i, it must also be profitable to s. Therefore, (i, s) forms a blocking pair. By the definition of stability, there is no such pair.

This proves that any stable matching is an equilibrium assignment of the G-CEEI mechanism. Since the student-proposing or the school-proposing Gale-Shapley mechanism always delivers stable matchings, their outcomes are necessarily equilibrium assignments of the G-CEEI. ■

To prove Proposition 3, the following two lemmata are useful.

Lemma 2 *If prices are fixed, it is a weakly dominant strategy for each student to report truthfully in the G-CEEI mechanism.*

Proof of Lemma 2. Assume the prices are fixed at $P = [p_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}}$ and $F_i(P)$ is then the augmented set of possible assignments.

Truthful report leads to the assignment:

$$\pi_i(v_i, P) \in \arg\left\{\max_{[\pi_{i,s}]_{s\in\mathcal{S}}\in F_i(P)} \sum_{s\in\mathcal{S}} \pi_{i,s} v_{i,s}\right\} \subseteq F_i(P).$$

If there are multiple solutions, $\pi_i(v_i, P)$ should be interpreted as any element in the set of solutions. Denote $V(P) \equiv \sum_{s\in\mathcal{S}} \pi_{i,s}(v_i, P) v_{i,s}$, i.e., it is the maximized expected utility that i can obtain within $F_i(P)$ and also the expected utility by truth-telling.

Suppose i with preference v_i reports u_i, then the assignment solution is

$$\pi_i(u_i, P) \in \arg\left\{\max_{[\pi_{i,s}]_{s\in\mathcal{S}}\in F_i(P)} \sum_{s\in\mathcal{S}} \pi_{i,s} u_{i,s}\right\} \subseteq F_i(P).$$

Therefore

$$\sum_{s\in\mathcal{S}}\pi_{i,s}\left(u_i,P\right)v_{i,s}\leq V\left(P\right)=\sum_{s\in\mathcal{S}}\pi_{i,s}\left(v_i,P\right)v_{i,s},$$

where the first inequality comes from the fact that $\pi_i\left(u_i,P\right)\in F_i\left(P\right)$. This proves that reporting truthfully is a weakly dominant strategy. ■

Lemma 3 *In the G-CEEI mechanism and in the sequence of economies* $\left\{\Gamma^{(n)}\right\}_{n\in\mathbb{N}}$*, let* $\mathcal{P}_{u_i}^{(n)}\subset[0,+\infty]^{S\times\bar{k}}$ *be the set of equilibrium prices given* $\left(u_i,v_{-i}^{(n)}\right)$*. Then* $\lim_{n\to\infty}d_H\left(\mathcal{P}_{v_i}^{(n)},\mathcal{P}_{u_i}^{(n)}\right)=0$*,* $\forall u_i\in[0,1]^S$*,* $\forall i\in\mathcal{I}^{(n)}$*.*

Proof. This is proven by the following three steps.

(1) Misreporting cannot affect per capita demand by preference groups in the limit.

First, recall that per capita demand of each preference group at each school is $G\left(P,v\right)$ for $P\in[0,+\infty]^{S\times\bar{k}}\equiv\mathcal{P}$ and v is the tuple of all students' preferences.

Since each student can increase or decrease the total demand of a preference group at a school at most by one seat, $\forall\left[d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in G^{(n)}\left(P,\left(u_i,v_{-i}^{(n)}\right)\right)$, there must exist $\left[d'_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in G^{(n)}\left(P,\left(v_i,v_{-i}^{(n)}\right)\right)$, such that, $\forall s$, $\forall k$,

$$d'_{s,k}-\frac{1}{\left|\mathcal{I}^{(n)}\right|}\leq d_{s,k}\leq d'_{s,k}+\frac{1}{\left|\mathcal{I}^{(n)}\right|}.$$

Similarly, $\forall\left[d'_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in G^{(n)}\left(P,\left(v_i,v_{-i}^{(n)}\right)\right)$, there exists $\left[d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in G^{(n)}\left(P,\left(u_i,v_{-i}^{(n)}\right)\right)$, such that $\forall s$, $\forall k$,

$$d_{s,k}-\frac{1}{\left|\mathcal{I}^{(n)}\right|}\leq d'_{s,k}\leq d_{s,k}+\frac{1}{\left|\mathcal{I}^{(n)}\right|}.$$

Therefore, given any P,

$$\sup_{u_i\in[0,1]^S}d_H\left(G^{(n)}\left(P,\left(u_i,v_{-i}^{(n)}\right)\right),G^{(n)}\left(P,\left(v_i,v_{-i}^{(n)}\right)\right)\right)\leq\frac{\sqrt{S\bar{k}}}{\left|\mathcal{I}^{(n)}\right|},$$

which implies, given any P,

$$\lim_{n\to\infty}\sup_{u_i\in[0,1]^S}d_H\left(G^{(n)}\left(P,\left(u_i,v_{-i}^{(n)}\right)\right),G^{(n)}\left(P,\left(v_i,v_{-i}^{(n)}\right)\right)\right)=0. \tag{3}$$

By definition, $G^{(n)}\left(P,\left(v_i,v_{-i}^{(n)}\right)\right)\to g\left(P\right)$ uniformly. Therefore, Equation (3) implies that $G^{(n)}\left(P,\left(u_i,v_{-i}^{(n)}\right)\right)\to g\left(P\right)$ uniformly as $n\to\infty$.

(2) Price Adjustment Process

Similar to the proof for Theorem 1, define $Z\equiv\left[z_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in[0,\pi/2]^{S\times\bar{k}}\equiv\mathcal{Z}$, where $z_{s,k}=\arctan\left(p_{s,k}\right)$, $\forall s$, $\forall k$.

A price adjustment process for $\Gamma^{(n)}$ is defined as,

$$\begin{aligned} & H\left[Z, G^{(n)}\left(\mathcal{TAN}(Z), \left(v_i, v_{-i}^{(n)}\right)\right)\right] \\ \equiv & \left\{ Y = [y_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}} \;\middle|\; \begin{array}{c} y_{s,k}\left([d_{s,k}]_{k\in\mathcal{K}}\right) = \min\left\{\pi/2, \max\left[0, z_{s,k} + \left(\sum_{\kappa=1}^{k} d_{s,\kappa} - q_s/I\right)\right]\right\} \\ \forall [d_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}} \in G^{(n)}\left(\mathcal{TAN}(Z), \left(v_i, v_{-i}^{(n)}\right)\right) \end{array} \right\}, \end{aligned}$$

where, $\mathcal{TAN}(Z) \equiv [\tan(z_{s,k})]_{s\in\mathcal{S},k\in\mathcal{K}}$. It is straightforward to verify that the correspondence H is a mapping from $\mathcal{Z}$ to $\mathcal{Z}$, given $\left(v_i, v_{-i}^{(n)}\right)$. Similarly,

$$\begin{aligned} & H[Z, g(\mathcal{TAN}(Z))] \\ \equiv & \left\{ Y = [y_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}} \;\middle|\; \begin{array}{c} y_{s,k}\left([d_{s,k}]_{k\in\mathcal{K}}\right) = \min\left\{\pi/2, \max\left[0, z_{s,k} + \left(\sum_{\kappa=1}^{k} d_{s,\kappa} - q_s/I\right)\right]\right\} \\ \forall [d_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}} \in g(\mathcal{TAN}(Z)); \end{array} \right\}. \end{aligned}$$

Claim: $H\left[Z, G^{(n)}\left(\mathcal{TAN}(Z), \left(v_i, v_{-i}^{(n)}\right)\right)\right] \to H[Z, g(\mathcal{TAN}(Z))]$ uniformly as $n \to \infty$.

The uniform convergence of $G^{(n)}\left(P, \left(v_i, v_{-i}^{(n)}\right)\right)$ to $g(P)$ means that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that when $n > N$, $\forall P \in \mathcal{P}$, i.e., $\forall Z \in \mathcal{Z}$,

$$\begin{aligned} \sup_{\left[d_{s,k}^{(n)}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \in G^{(n)}\left(P, \left(v_i, v_{-i}^{(n)}\right)\right)} \; \inf_{[d_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}} \in g(P)} \left\| \left[d_{s,k}^{(n)} - d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \right\| &\leq \varepsilon, \text{ and} \\ \sup_{[d_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}} \in g(P)} \; \inf_{\left[d_{s,k}^{(n)}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \in G^{(n)}\left(P, \left(v_i, v_{-i}^{(n)}\right)\right)} \left\| \left[d_{s,k}^{(n)} - d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \right\| &\leq \varepsilon. \end{aligned}$$

By the definition of the Euclidean distance, the first inequality implies that,

$$\begin{aligned} \sup_{\left[d_{s,k}^{(n)}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \in G^{(n)}} \; \inf_{[d_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}} \in g(P)} \left\| \left[\begin{array}{c} \min\left\{\frac{\pi}{2}, \max\left[0, \arctan(p_{s,k}) + \left(\sum_{\kappa=1}^{k} d_{s,\kappa}^{(n)} - \frac{q_s}{I}\right)\right]\right\} \\ - \min\left\{\frac{\pi}{2}, \max\left[0, \arctan(p_{s,k}) + \left(\sum_{\kappa=1}^{k} d_{s,\kappa} - \frac{q_s}{I}\right)\right]\right\} \end{array} \right]_{s\in\mathcal{S},k\in\mathcal{K}} \right\| \\ \leq \varepsilon. \end{aligned}$$

Or, equivalently,

$$\sup_{Y^{(n)} \in H\left[Z, G^{(n)}\left(\mathcal{TAN}(Z), \left(v_i, v_{-i}^{(n)}\right)\right)\right]} \; \inf_{Y \in H[Z, g(\mathcal{TAN}(Z))]} \left\| Y^{(n)} - Y \right\| \leq \varepsilon. \tag{4}$$

Similarly, we have,

$$\sup_{Y \in H[Z, g(\mathcal{TAN}(Z))]} \; \inf_{Y^{(n)} \in H\left[Z, G^{(n)}\left(\mathcal{TAN}(Z), \left(v_i, v_{-i}^{(n)}\right)\right)\right]} \left\| Y^{(n)} - Y \right\| \leq \varepsilon. \tag{5}$$

Since (4) and (5) are satisfied for all $n > N$ and $\forall Z \in \mathcal{Z}$, $H\left[Z, G^{(n)}\left(\mathcal{TAN}(Z), \left(v_i, v_{-i}^{(n)}\right)\right)\right]$ converges to $H[Z, g(\mathcal{TAN}(Z))]$ uniformly.

From the proof for Theorem 1, $H\left[Z, G^{(n)}\right]$ is upper hemicontinuous and convex-valued and thus satisfies all the conditions of Kakutani's fixed-point theorem.

Claim: Given $\left(v_i, v_{-i}^{(n)}\right)$ and any equilibrium price $P \in P$, its positive monotonic transformation $Z \in Z$ is a fixed point of $H\left[Z, G^{(n)}\left(\mathcal{TAN}(Z), \left(v_i, v_{-i}^{(n)}\right)\right)\right]$.

If P^* is an equilibrium price, there must exist a unique $k^*(s) \in \mathcal{K}$ for each s such that, for some $[d_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}} \in G^{(n)}\left(P^*, \left(v_i, v_{-i}^{(n)}\right)\right)$,
(i) $p^*_{s,k^*(s)} \in [0, +\infty)$ and $\sum_{\kappa=1}^{k^*(s)} d_{s,\kappa} = \frac{q_s}{I}$,
(ii) $\sum_{\kappa=1}^{k} d_{s,\kappa} < \frac{q_s}{I}$ and $p^*_{s,k} = 0$ if $k < k^*(s)$, and
(iii) $d_{s,k} = 0$ and $p^*_{s,k} = +\infty$ if $k > k^*(s)$.

Let $P^* = \mathcal{TAN}(Z^*)$, given the same $[d_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}}$, we must have

$$\begin{aligned}
\min\left\{\frac{\pi}{2}, \max\left[0, z^*_{s,k} + \left(\sum_{\kappa=1}^{k} d_{s,\kappa} - \frac{q_s}{I}\right)\right]\right\} &= 0 = z^*_{s,k}, \text{ if } k < k^*(s);\\
\min\left\{\frac{\pi}{2}, \max\left[0, z^*_{s,k} + \left(\sum_{\kappa=1}^{k} d_{s,\kappa} - \frac{q_s}{I}\right)\right]\right\} &= z^*_{s,k}, \text{ if } k = k^*(s);\\
\min\left\{\frac{\pi}{2}, \max\left[0, z^*_{s,k} + \left(\sum_{\kappa=1}^{k} d_{s,\kappa} - \frac{q_s}{I}\right)\right]\right\} &= \frac{\pi}{2} = z^*_{s,k}, \text{ if } k > k^*(s).
\end{aligned}$$

Therefore, $Z^* \in H\left[Z^*, G^{(n)}\left(\mathcal{TAN}(Z^*), \left(v_i, v_{-i}^{(n)}\right)\right)\right]$.

Note that not every fixed point of H is an equilibrium price vector as the proof for Theorem 1 has discussed, while the transformation of any equilibrium price vector is a fixed point.

Similarly, when student i reports u_i, $H\left[Z, G^{(n)}\left(\mathcal{TAN}(Z), \left(u_i, v_{-i}^{(n)}\right)\right)\right]$ has the same properties and converges to $H\left[Z, g\left(\mathcal{TAN}(Z)\right)\right]$ uniformly, since $G^{(n)}\left(P, \left(u_i, v_{-i}^{(n)}\right)\right)$ converges to $g(P)$ uniformly. In the same manner, the transformations of all the equilibrium prices can be found as a fixed point of $H\left[Z, G^{(n)}\left(\mathcal{TAN}(Z), \left(u_i, v_{-i}^{(n)}\right)\right)\right]$.

Denote $\mathcal{P}_{v_i}^{(\infty)}$ as the set of equilibrium prices corresponding to the subset of fixed points of $H\left[Z, g\left(\mathcal{TAN}(Z)\right)\right]$ which all have the desired form.

(3) Asymptotic Equivalence between $\mathcal{P}_{v_i}^{(\infty)}$ and $\mathcal{P}_{u_i}^{(n)}$.

In equilibrium, some prices may be $+\infty$ for some s and k. We supplement the definition of

Euclidean distance by defining the following for $+\infty$:[20]

$$\begin{aligned}
\left|(+\infty)-(+\infty)\right| &= 0;\ \sqrt{+\infty}=+\infty;\ (+\infty)^2=+\infty; \\
\left|(+\infty)-x\right| &= \left|x-(+\infty)\right|=+\infty,\ \forall x\in[0,+\infty)\,; \\
\text{and }\ (+\infty)+x &= +\infty,\ \forall x\in[0,+\infty]\,.
\end{aligned}$$

For any $\widehat{P}^{(n)}\in\mathcal{P}_{u_i}^{(n)}$, by definition, $\exists\left[d_{s,k}^{(n)}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in G^{(n)}\left(\widehat{P}^{(n)},\left(u_i,v_{-i}^{(n)}\right)\right)$, such that $q_s/I=\sum_{\kappa=1}^{\bar{k}}d_{s,\kappa}^{(n)}$, $\forall s$. Since $G^{(n)}\left(P,\left(u_i,v_{-i}^{(n)}\right)\right)\to g(P)$ uniformly as $n\to\infty$,

$$\lim_{n\to\infty}\inf_{\left[d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in g\left(\widehat{P}^{(n)}\right)}\left\|\left[q_s/I\right]_{s\in\mathcal{S}}-\left[\sum_{\kappa=1}^{\bar{k}}d_{s,\kappa}\right]_{s\in\mathcal{S}}\right\|=0,$$

which implies that $Z=\mathcal{TAN}^{-1}\left(\widehat{P}^{(n)}\right)$ has to be fixed point of $H\left[Z,g\left(\mathcal{TAN}(Z)\right)\right]$ in the limit. Therefore for some $P^*\in\mathcal{P}_{v_i}^{(\infty)}$,

$$\lim_{n\to\infty}\left\|P^*-\widehat{P}^{(n)}\right\|=0,$$

which means, more precisely,
(i) when n is large enough, there is $\left[k^*(s)\right]_{s\in\mathcal{S}}\in\mathcal{K}^S$ such that $\forall s$, $0\le p_{s,k^*(s)}^*,\widehat{p}_{s,k^*(s)}^{(n)}<+\infty$; $p_{s,k}^*=\widehat{p}_{s,k}^{(n)}=0$ if $k<k_s^*$; $p_{s,k}^*=\widehat{p}_{s,k}^{(n)}=+\infty$ if $k>k_s^*$;
(ii) $\lim_{n\to\infty}\left\|\left[p_{s,k^*(s)}^*\right]_{s\in\mathcal{S}}-\left[\widehat{p}_{s,k^*(s)}^{(n)}\right]_{s\in\mathcal{S}}\right\|=0$.

Since this is true for $\forall\widehat{P}^{(n)}\in\mathcal{P}_{u_i}^{(n)}$,

$$\lim_{n\to\infty}\sup_{\widehat{P}^{(n)}\in\mathcal{P}_{u_i}^{(n)}}\inf_{P^*\in\mathcal{P}_{v_i}^{(\infty)}}\left\|P^*-\widehat{P}^{(n)}\right\|=0. \tag{6}$$

On the other hand, for any $P^*\in\mathcal{P}_{v_i}^{(\infty)}$, by definition, $\exists\left[d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in g(P^*)$, such that $q_s/I=\sum_{\kappa=1}^{\bar{k}}d_{s,\kappa}$, $\forall s$. Since $G^{(n)}\left(P,\left(u_i,v_{-i}^{(n)}\right)\right)$ converges to $g(P)$ uniformly,

$$\lim_{n\to\infty}\inf_{\left[d_{s,k}^{(n)}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in G^{(n)}\left(P^*,\left(u_i,v_{-i}^{(n)}\right)\right)}\left\|\left[q_s/I\right]_{s\in\mathcal{S}}-\left[\sum_{\kappa=1}^{\bar{k}}d_{s,\kappa}^{(n)}\right]_{s\in\mathcal{S}}\right\|=0,$$

which implies that P^* is an asymptotic equilibrium price for $\left(u_i,v_{-i}^{(n)}\right)$, i.e., $\lim_{n\to\infty}\inf_{\widehat{P}^{(n)}\in\mathcal{P}_{u_i}^{(n)}}\left\|P^*-\widehat{P}^{(n)}\right\|=0$ which means the above two properties (i) and (ii) being satisfied. As this is true for all $P^*\in\mathcal{P}_{v_i}^{(\infty)}$, therefore

$$\lim_{n\to\infty}\sup_{P^*\in\mathcal{P}_{v_i}^{(\infty)}}\inf_{\widehat{P}^{(n)}\in\mathcal{P}_{u_i}^{(n)}}\left\|P^*-\widehat{P}^{(n)}\right\|=0. \tag{7}$$

Combining (6) and (7), we have $\lim_{n\to\infty}d_H\left(\mathcal{P}_{v_i}^{(\infty)},\mathcal{P}_{u_i}^{(n)}\right)=0$, $\forall u_i\in[0,1]^S$ and $\forall i\in\mathcal{I}^{(n)}$.

Furthermore, $\lim_{n\to\infty}d_H\left(\mathcal{P}_{v_i}^{(\infty)},\mathcal{P}_{v_i}^{(n)}\right)=0$ and therefore $\lim_{n\to\infty}d_H\left(\mathcal{P}_{v_i}^{(n)},\mathcal{P}_{u_i}^{(n)}\right)=0$, $\forall u_i\in$

[20] $p_{s,k}=+\infty$ means that there is no supply for the preference group k at school s. It therefore makes sense to define the distance between $+\infty$ and $+\infty$ as 0.

$[0,1]^S$ and $\forall i \in \mathcal{I}^{(n)}$. ■

Proof of Proposition 3. From Lemma 3, if other students are truth-telling, given any reporting of student i, the equilibrium prices converge to the set of prices when the student is truth-telling. From Lemma 2, truth-telling becomes a best response for each student when prices converge. Therefore, given others being truth-telling, the incentive for an individual student to misreport goes to zero as $n \to \infty$ in the G-CEEI mechanism. ■