WORKING PAPERS

N° TSE-513

July 2014

"Implementation in Weakly Undominated Strategies, with Applications to Auctions and Bilateral Trade"

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Implementation in Weakly Undominated Strategies, with Applications to Auctions and Bilateral Trade*

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July 20, 2014

Abstract

We study the mechanism-design problem of guaranteeing desirable performances whenever agents are rational in the sense of not playing weakly dominated strategies. We first provide an upper bound for the best performance we can guarantee among all feasible mechanisms. We then prove the bound to be tight under certain conditions in auction and bilateral-trade applications. In particular, we find that

*I am grateful to Ilya Segal, Matthew Jackson, Jonathan Levin, Paul Milgrom, Andrzej Skrzypacz, Stephen Morris, Dirk Bergemann, Gabriel Carroll, Koichi Tadenuma, Hideshi Itoh, Eve Ramaekers, Olivier Tercieux, Tilman Börgers, Philippe Jehiel, Jacques Crémer, Christian Hellwig, Thomas Mariotti, Jean Tirole, and seminar participants at Stanford University, Hitotsubashi University, Yokohama National University, Duke University, University of Rochester, École Polytechnique, California Institute of Technology, Toulouse School of Economics, Boston University, University of California Davis, Paris School of Economics, University of Michigan, University of Copenhagen, and University of Glasgow for the valuable comments on various stages of the paper. I would also like to thank the editor and three anonymous referees for their helpful comments and suggestions. I gratefully acknowledge the financial support of the B.F. Haley and E.S. Shaw Fellowship for Economics through the Stanford Institute for Economic Policy Research.

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a second-price auction is optimal in revenue with interdependent values, which is neither dominant-strategy nor ex post incentive compatible, but satisfies the novel incentive compatibility introduced in this analysis.

1 Introduction

In mechanism design, a typical assumption is that agents play a Bayesian equilibrium, having a common prior over their payoff-relevant private information. However, these assumptions are sometimes considered too strong and are criticized in the literature (e.g., Wilson (1987)). For example, in anonymous online trading environments, the parties may not know each other very well, and hence, it may be too demanding to assume that the parties correctly predict each other's strategies (in order for them to play a Bayesian equilibrium), and that they share the same beliefs over their values.

In this paper, we study the problem of designing a mechanism in a more "robust" manner, so that, even if these assumptions concerning the agents' possible behaviors or beliefs do not hold, the mechanism can guarantee a desirable level of performance (e.g., revenue or surplus). More specifically, we consider a situation where each agent is *rational* in the sense that he does not play any strategy that is weakly dominated,[1] but he may play any strategy that is not weakly dominated (called an *admissible* strategy).[2]

[1] In the literature on decision theory and game theory, admissibility is often considered a reasonable assumption for an individual's "rationality". See, for example, Kohlberg and Mertens (1986). In the literature on implementation theory, several studies, including Börgers (1991), Jackson (1992), Börgers and Smith (2012b), and Yamashita (2012) examine implementation in admissible strategies in various contexts. Note that admissibility allows only one round of elimination of weakly dominated actions, and in this sense, no mutual or common knowledge of rationality is assumed.

[2] For example, they may not play a Bayesian equilibrium, because an agent may choose his strategy as a best response to his conjecture about the opponents' choices, but the conjecture may simply be wrong, and hence his actual play may not be a best response to the others' actual plays.

Our goal is to characterize the highest level of performance that can be guaranteed given whatever admissible strategies are played, and the mechanism that achieves this objective, which we refer to as the *worst-case optimal* mechanism. To be specific, we assume that a mechanism designer exists who has a probability assessment for the agents' payoff-relevant private information, and who aims to maximize the expected value of his own utility (e.g., revenue or surplus). However, the designer does not know which admissible strategy each agent plays, and therefore, he evaluates each mechanism according to his expected utility that is guaranteed (or the worst case) among all the admissible strategy profiles of the agents. Such a "pessimistic" approach may be reasonable in situations where the agents do not know each other very well, or where a mechanism must be designed far in advance so that it is difficult to predict the agents' knowledge or beliefs about each other at the actual time of playing the mechanism.

One of the main challenges in this approach is that it is not generally possible (or at least straightforward) to invoke a revelation principle in order to focus on revelation mechanisms in seeking a desirable mechanism.[3] There is no *a priori* restriction on the number of messages or their dominance relations that the desirable mechanism should exhibit, and hence, the optimization problem among all mechanisms could be intractable. Nevertheless, we propose a procedure to solve for this optimization problem under certain conditions. Our approach is that, instead of attempting to characterize such an optimal mechanism, which is potentially very complicated, we first provide an upper bound for the highest level of performance that can be guaranteed among all feasible mechanisms (Theorem 1). An advantage of this approach is that the upper bound is given by a maximization problem where standard

[3]A problem of designing a mechanism where *all* admissible strategy profiles induce desirable outcomes in the sense of guaranteeing certain performance has a qualitatively similar feature to a *full* implementation problem, which aims to make all possible outcomes desirable in the sense of a social choice correspondence. As in full implementation, a desirable mechanism is not necessarily a revelation mechanism, but rather may need to have larger message spaces in order to eliminate some undesirable outcomes.

techniques developed in the literature could be applicable. In fact, we could interpret the upper-bound problem as maximizing the designer's objective among all "revelation mechanisms" that satisfy certain incentive compatibility, which, with a continuous payoff-type space, induces integral envelope expressions. In this sense, Theorem 1 may be interpreted as establishing a version of revelation principle (not for the highest performance guarantee but for its upper bound).

Although this upper bound is not necessarily tight, we provide the conditions under which the bound is tight, and moreover, the worst-case optimal mechanism is characterized. Precise conditions on primitives that imply the tight bounds and their interpretations vary across applications. Therefore, in the second part of the paper, we examine three applications.

The first application is the worst-case maximization of a weighted sum of revenue and surplus in a private-value auction setting. We show that, under a version of the monotone virtual-value condition in Myerson (1981), the upper-bound level of this objective is guaranteed by a (version of a) second-price auction (Theorem 2), a dominant-strategy incentive-compatible mechanism. The observation that desirable mechanisms in a certain robustness sense sometimes take the form of dominant-strategy or ex post incentive-compatible mechanisms (as in this and our third application) appears in several studies, but in different contexts. For example, Ledyard (1979), Bergemann and Morris (2005), and Börgers and Smith (2012a) consider implementation of social choice functions or correspondences of a particular class, while we consider (worst-case) maximization of the designer's objective function. Chung and Ely (2007) consider revenue maximization in private-value auction environments, and this study lies closer to our problem in this respect. We discuss the relationship in greater detail in Section 4.1.

The second application is revenue maximization in an interdependent-value auction, and the main result is worst-case optimality of a (version of a) second-price auction, under a similar condition as in the first application (Theorem 3). Although this result is qualitatively similar to the first application, it has a very different interpretation, because with interdependence, a second-price auction is generally neither dominant-strategy nor ex post incentive compatible. Thus, this is an instance where our upper bound implies the worst-case optimal mechanism that is neither dominant-strategy nor ex post incentive compatible. In fact, we introduce a novel incentive condition, *incentive compatibility for value revelation*, which a second-price auction satisfies, and we argue that this is a key incentive condition in our problem with interdependent values.

We can view this incentive compatibility for value revelation as a generalization of dominant-strategy incentive compatibility in the context of interdependent values, but in a different way from ex post incentive compatibility. In a mechanism that satisfies this incentive condition, (i) each agent is asked to report his valuation, instead of his payoff type, and (ii) the trading rule is designed so that, *if an agent knows his willingness to pay upon solely observing his payoff type*, then truth-telling of such willingness to pay is weakly dominant. Hence, in a private-value environment, this second condition immediately implies dominant-strategy incentive compatibility. However, in an interdependent-value environment, each agent may have multiple admissible messages, depending on his "belief" about the other agents' payoff types.[4] This new class of mechanisms could be useful in more general "robust" mechanism-design problems with interdependent values, because, as Jehiel, Moldovanu, Meyer-ter-Vehn, and Zame (2006) show, in a generic environment with interdependent values, only a *constant* objective can be ex post implementable.[5] Conversely, a mechanism with incentive compatibility

[4]To provide an intuition, imagine a bidder in an interdependent-value auction setting whose private signal indicates that his value for the object is between one and two. In a second-price auction, any bid below one and above two is weakly dominated (by bidding one or two), but any bid between one and two may be admissible.

[5]Note that there are notable subclasses of nongeneric (but economically important) environments with interdependence where their result does not apply. One of these is a one-dimensional, single-crossing environment as in Maskin (1992) and Dasgupta and Maskin (2000). Another is a private-good environment as in Bikhchandani (2006).

for value revelation, e.g., a second-price auction, can "robustly" implement more nontrivial objectives in such an environment.[6]

The third application in the paper is surplus maximization in a (private-value) bilateral-trade setting (Myerson and Satterthwaite (1983), Hagerty and Rogerson (1987)). We show that, under a novel condition, which we refer to as the *monotone weighted surplus* condition, the upper-bound level of the expected surplus is guaranteed by a posted-price mechanism (Theorem 4). A posted-price mechanism sets a trading price in advance, and the agents trade if and only if both agree to this price. This mechanism is clearly dominant-strategy incentive compatible.

2 Definitions and notation

There is a set of N agents, $I = \{1, \ldots, N\}$. We consider a quasilinear setting, including auctions and bilateral trades as applications. Specifically, an allocation is denoted by $x = (q_i, p_i)_{i \in I} \in X$, where $q_i \in \mathbb{R}$ represents the (one-dimensional) "assignment" to agent i, and $p_i \in \mathbb{R}$ represents his payment. X may incorporate feasibility constraints. For example, in an auction, $X = \{(q_i, p_i)_{i \in I} | \forall i,\ q_i \in [0, 1],\ \sum_i q_i \leq 1\}$.

Each agent i has a payoff-relevant signal $\theta_i \in \Theta_i$, where Θ_i is a measurable space. We denote a signal profile by $\theta = (\theta_i)_{i \in I} \in \Theta = \prod_i \Theta_i$. Agent i's valuation for the assignment is $v_i(\theta) \in V_i = [0, 1]$, which can vary with θ_{-i} as well as with θ_i, and thus, this environment exhibits interdependent values. His utility given a signal profile θ and an allocation $x = (q_i, p_i)_{i \in I}$ is $u_i(x, \theta) = v_i(\theta) q_i - p_i$.

Even though the allocation is one-dimensional for each agent, it does not mean that Θ_i must also be one-dimensional. For example, in an auction of

[6]The idea that some mechanisms that are not ex post incentive compatible may still achieve desirable outcomes in a certain robustness sense appears in Jehiel, Moldovanu, Meyer-ter-Vehn, and Zame (2006) and Meyer-ter-Vehn and Morris (2011). In this paper, we also identify a necessary condition for implementable objectives in a class of problems in the form of an upper bound for the performance level we can potentially guarantee.

an oil tract, agent i's signal may be two-dimensional, say $\theta_i = (c_i, d_i) \in \mathbb{R}^2$, where c_i is a noisy signal of the amount of oil in the tract, and d_i is an idiosyncratic component, such as the cost of digging the well, refining the oil, etc. Then, i's value may be given by $v_i(\theta) = \pi_i(c_1, \ldots, c_N) + d_i$ with an increasing function $\pi_i(\cdot)$ representing the estimated amount of oil in the tract for each signal profile.[7]

Given agent i's signal θ_i, let $V_i(\theta_i) = \{v_i(\theta) | \theta_{-i} \in \Theta_{-i}\}$ denote the set of i's possible valuations given θ_i. Throughout the paper, we assume that $V_i(\theta_i)$ is a compact interval. We say that an environment is a *private-value environment* if, for all i, we have $\Theta_i = V_i$ and $V_i(v_i) = \{v_i\}$ for each $v_i \in V_i$.

The designer has a utility function $w : X \times \Theta \to \mathbb{R}$. For example, for revenue maximization, we have $w(x, \theta) = \sum_i p_i$, and for surplus maximization, we have $w(x, \theta) = \sum_i v_i(\theta) q_i$. He also has a prior distribution over Θ, denoted by $\Phi \in \Delta(\Theta)$. However, we do not assume that the agents share the same prior. Rather, each agent may have a very different prior from the designer (and from the other agents).

A *mechanism* is denoted by $\Gamma = \langle M, g \rangle$, where each M_i is agent i's message space, $M = \prod_i M_i$, and $g : M \to X$ is an outcome function. Given a message profile $m = (m_i)_{i \in I} \in M$, we denote the induced allocation by $g(m) = (q_i^g(m), p_i^g(m))_{i \in I}$. A mechanism is *feasible* if each M_i (i) is finite, and (ii) contains a message that corresponds to "opt-out" or "nonparticipation", m^{out}, such that $q_i^g(m^{\text{out}}, m_{-i}) = p_i^g(m^{\text{out}}, m_{-i}) = 0$ for any $m_{-i} \in M_{-i}$.[8]

In any given mechanism, each agent i of type θ_i may play any message that is *admissible* (i.e., not weakly dominated) in a mechanism.

[7] As in Jehiel, Moldovanu, Meyer-ter-Vehn, and Zame (2006), a "robustness" concept in the literature, namely ex post implementation, has a very limited set of implementable objectives in such a multidimensional, interdependent-value environment.

[8] Some mechanisms discussed in this paper, such as a (continuous-version of a) second-price auction, violate the finiteness, and hence, is infeasible. However, whenever we claim that such a mechanism is "optimal", we identify a sequence of feasible mechanisms that converges (in an appropriate sense) to such a mechanism, and in this sense, we treat such a mechanism as "approximately" feasible. See Section 4.1 and 4.2 for the detail.

Definition 1. In a mechanism $\Gamma = \langle M, g \rangle$, a message $m_i \in M_i$ is *admissible for* θ_i, if there exists some $\theta_{-i} \in \Theta_{-i}$ such that no message weakly dominates m_i if the agent's value is $v_i(\theta_i, \theta_{-i})$, i.e., there is no $m_i' \in M_i$ that satisfies (i) $u_i(g(m_i', m_{-i}), \theta_i, \theta_{-i}) \geq u_i(g(m_i, m_{-i}), \theta_i, \theta_{-i})$ for any $m_{-i} \in M_{-i}$, and (ii) $u_i(g(m_i', m_{-i}), \theta_i, \theta_{-i}) > u_i(g(m_i, m_{-i}), \theta_i, \theta_{-i})$ for some $m_{-i} \in M_{-i}$.

Because i's preference can vary with θ_{-i}, a message m_i is said to be admissible for θ_i if m_i is not weakly dominated given *some* θ_{-i}. The implicit idea is that i may possess any "belief" for θ_{-i} and $m_{-i} \in M_{-i}$. Thus, many messages may be admissible for θ_i, especially when $V_i(\theta_i)$ is large.

We denote by $M_i^A(\theta_i) \subseteq M_i$ the set of all admissible messages for θ_i, and by $M^A(\theta) = \prod_i M_i^A(\theta_i)$ the set of all admissible message profiles in state θ.

We evaluate a mechanism according to its guaranteed performance level given whatever admissible strategies are played. In this sense, we assume that the designer is uncertainty-averse with respect to the agents' (admissible) strategies.

Definition 2. The *performance guarantee* of a mechanism $\Gamma = \langle M, g \rangle$ is

$$W(\Gamma) = \int_\theta \left[\min_{m \in M^A(\theta)} w(g(m), \theta) \right] d\Phi.$$

The main goal of this paper is to characterize the highest performance guarantee among all feasible mechanisms, i.e., $\sup_\Gamma W(\Gamma)$, under certain conditions.

3 An upper bound for the performance guarantee

One of the main challenges is that it is not generally possible (or at least straightforward) to invoke a revelation principle in order to focus on revelation mechanisms in seeking a desirable mechanism.[9] Because there is no *a*

[9] In the sense that we aim to design a mechanism where *all* admissible strategy profiles induce desirable outcomes (in the sense of guaranteeing certain performances), it has a

priori restriction on the number of messages or their dominance relations that the desirable mechanism should exhibit, the optimization problem among all mechanisms could be intractable.

Instead of attempting to characterize such a potentially complicated optimal mechanism, we therefore first provide an upper bound for $\sup_\Gamma W(\Gamma)$.

Theorem 1. For any mechanism Γ,

$$W(\Gamma) \;\le\; \overline{W} = \sup_{f=(q,p):V\to X} \int_\theta \Big[\inf_{v\in V(\theta)} w(f(v),\theta)\Big] d\Phi$$
$$\text{sub.to} \quad v_i q_i(v) - p_i(v) \ge \int_0^{v_i} q_i(\tilde{v}_i, v_{-i})\ d\tilde{v}_i,\ \forall i, v.$$

Therefore, $\overline{W}$ is an upper bound for $\sup_\Gamma W(\Gamma)$. We prove the theorem in Section 3.1. In the remainder of this section, we provide an informal interpretation of this result. We first consider the private-value case, and then the interdependent-value case.

With private values (i.e., $\Theta_i = V_i$ for each i, and $V_i(v_i) = \{v_i\}$ for each $v_i \in V_i$; see page 7), the upper bound has a simpler expression, as follows.

Corollary 1. Assume that $\Theta_i = V_i$ for each i, and that $V_i(v_i) = \{v_i\}$ for each $v_i \in V_i$. For any mechanism Γ,

$$W(\Gamma) \;\le\; \overline{W}^{\mathrm{PV}} = \sup_{f=(q,p):V\to X} \int_v w(f(v),v) d\Phi$$
$$\text{sub.to} \quad v_i q_i(v) - p_i(v) \ge \int_0^{v_i} q_i(\tilde{v}_i, v_{-i})\ d\tilde{v}_i,\ \forall i, v.$$

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qualitatively similar feature to a *full* implementation problem. As in full implementation, a desirable mechanism is not necessarily a revelation mechanism, but rather may need to have larger message spaces in order to eliminate some undesirable outcomes. However, contrary to the popular approach in the literature on full implementation, our result does not rely on the use of mechanisms with "integer-game" or "tail-chasing" structures (see Jackson (1991), Abreu and Matsushima (1992)) because only finite mechanisms are feasible in our setup.

Note that $f = (q, p) : V \to X$ is an allocation rule or a revelation mechanism in a standard sense. The objective, $\int_v w(f(v), v)d\Phi$, is the designer's expected utility given that the agents report their values truthfully. The constraint is, as shown in the proof, obtained by the "local and downward" incentive compatibility of the following kind: if agent i has value $v_i \in V_i$, then he would not be better off by pretending to have a slightly lower value than v_i, regardless of the other agents' choices. In other words, the constraint is a "partial" incentive condition for truth-telling being weakly dominant. In this sense, we may interpret this result as a sort of revelation principle (not for $\sup_\Gamma W(\Gamma)$ but for its upper bound) based on the local and downward incentive compatibility.

Theorem 1 (or Corollary 1) would be most useful when the upper bound is in fact a tight bound, i.e., $\sup_\Gamma W(\Gamma) = \overline{W}^{\mathrm{PV}}$. In Section 4, in a private-value auction (Section 4.1) and in bilateral trade (Section 4.3), we provide sufficient conditions for each of these applications under which the solution, say f^*, to the upper-bound problem given in Corollary 1 has the property that $M_i^A(\theta) = \{\theta_i\}$ for every i and for (Φ-)almost every θ_i, i.e., truth-telling is the only admissible message. With a caveat treated more formally in Section 4, this suggests that the truth-telling performance, $\int_v w(f^*(v), v)d\Phi$, is guaranteed in the revelation mechanism f^*, and hence, the worst-case optimality of f^* (and the tightness of the bound) is implied.[10]

Now we consider the interdependent-value case. Recall that

$$\overline{W} = \sup_{f=(q,p):V\to X} \quad \int_\theta \Big[\inf_{v\in V(\theta)} w(f(v), \theta)\Big] d\Phi$$

$$\text{sub.to} \quad v_i q_i(v) - p_i(v) \geq \int_0^{v_i} q_i(\tilde{v}_i, v_{-i})d\tilde{v}_i, \ \forall i, v.$$

Here, f is no longer an allocation rule or a revelation mechanism in a

[10] The caveat is that f^* is not necessarily a feasible mechanism because it may have infinitely many messages, e.g., a second-price auction in Section 4.1. In that case, we construct a sequence of feasible mechanisms, $\{\Gamma_k\}_{k=1}^\infty$, such that their performance guarantees converge to that of f^*, i.e., $\lim_{k\to\infty} W(\Gamma_k) = \int_v w(f^*(v), v)d\Phi$, and we interpret f^* as "approximately" feasible.

standard sense, because the domain of f is V, rather than Θ. Nevertheless, we can interpret f in an analogous way as in the private-value case.

To see this, we first introduce the following novel incentive compatibility condition.

Definition 3. $f = (q, p) : V \to X$ is *incentive compatible for value revelation (in admissible strategies)* if, for each i and $\theta_i \in \Theta_i$, we have $M_i^A(\theta_i) = V_i(\theta_i)$.

Suppose that the designer uses this "value-revelation" mechanism f that is incentive compatible for value revelation. For agent i with θ_i, for any report v_i' outside his possible values $V_i(\theta_i)$, there is another report $v_i \in V_i(\theta_i)$ that is always better than v_i' regardless of the others' signals θ_{-i} and the others' reports v_{-i}. By abuse of terminology, any report $v_i \in V_i(\theta_i)$ is said to be *truthful*, and any report $v_i' \notin V_i(\theta_i)$ is said to be *untruthful*. The concept of incentive compatibility for value revelation has a similar spirit as that of dominant-strategy incentive compatibility in that any untruthful reports are weakly dominated.

If each agent i with θ_i never reports $v_i' \notin V_i(\theta_i)$, then the designer's expected utility in this value-revelation mechanism is at least

$$\int_\theta \Big[\inf_{v \in V(\theta)} w(f(v), \theta)\Big] d\Phi.$$

As in the private-value case, the constraint of the upper-bound problem can be interpreted as "partial" incentive compatibility conditions: *if agent i knows (or believes) that his value is $v_i \in V_i$ for sure,* then he would not be better off by pretending to have a slightly lower value than v_i. In this sense, we may interpret Theorem 1 as a sort of revelation principle (not for $\sup_\Gamma W(\Gamma)$ but for its upper bound) based on the local and downward incentive compatibility for value revelation.

In the next section, in an interdependent-value auction (Section 4.2), we provide sufficient conditions under which the solution f^* to the upper-bound problem is incentive compatible for value revelation. With the same caveat as in the private-value case, this suggests that the truth-telling performance is guaranteed in the value-revelation mechanism f^*, and hence, the worst-case optimality of f^* (and the tightness of the bound) is implied.[11]

3.1 Proof of Theorem 1

This subsection is devoted to the proof of Theorem 1.

Fix an arbitrary feasible mechanism $\Gamma = \langle M, g\rangle$. For each i, let $\tilde{\Theta}_i = \Theta_i \cup V_i$ be an augmented set of agent i's types, such that each type $v_i \in V_i$ has $V_i(v_i) = \{v_i\}$ and is called a *private-value type*. The designer's prior assigns $\Phi(\Theta) = 1$. Augmenting the type space is useful in simplifying the proof, but the result holds true even without the augmentation. In the mechanism Γ, for each $v_i \in V_i$, let $M_i^A(v_i)$ denote the set of admissible messages for private-value type v_i.

The proof consists of several lemmas. The first lemma shows a connection between the set of admissible messages for θ_i and that for each private-value type v_i such that $v_i \in V_i(\theta_i)$.

Lemma 1. For each i and $\theta_i \in \Theta_i$, we have $M_i^A(\theta_i) \supseteq \bigcup_{v_i \in V_i(\theta_i)} M_i^A(v_i)$.

Proof. Let $v_i \in V_i(\theta_i)$. By assumption, there exists θ_{-i} such that $v_i = v_i(\theta_i, \theta_{-i})$. Then, each $m_i \in M_i^A(v_i)$ is admissible for θ_i as well. Therefore, $M_i^A(\theta_i) \supseteq M_i^A(v_i)$. Because v_i is arbitrary, we obtain $M_i^A(\theta_i) \supseteq \bigcup_{v_i \in V_i(\theta_i)} M_i^A(v_i)$. $\square$

As an implication of finiteness of Γ, for each i, $V_i = [0, 1]$ is finitely partitioned into $\{V_i^k\}_{k=1}^K$ so that any two types $v_i, v_i' \in V_i^k$ have the same ordinal preference over $\{g(m) | m \in M\}$. This implies that $M_i^A(v_i) = M_i^A(v_i')$. Also, each V_i^k is connected, as in the following lemma.

Lemma 2. For each i, k, if $v_i, v_i' \in V_i^k$, then for any $\alpha \in (0, 1)$, $\alpha v_i + (1 - \alpha) v_i' \in V_i^k$.

[11] Again, the caveat is that f^* is not necessarily feasible because it may have infinitely many messages. In that case, we construct a sequence of feasible mechanisms, $\{\Gamma_k\}_{k=1}^{\infty}$, such that their performance guarantees converge to that of f^*, i.e., $\lim_{k\to\infty} W(\Gamma_k) = \int_v w(f^*(v), v)d\Phi$.

Proof. Let $(q,p),(q',p') \in \{g(m)|m \in M\}$. For $v_i, v_i' \in V_i^k$, without loss of generality, we assume

$$\begin{aligned} v_i q_i - p_i &\geq v_i q_i' - p_i', \\ v_i' q_i - p_i &\geq v_i' q_i' - p_i'. \end{aligned}$$

This implies that, for $\alpha \in (0,1)$,

$$(\alpha v_i + (1-\alpha)v_i')q_i - p_i \geq (\alpha v_i + (1-\alpha)v_i')q_i' - p_i',$$

and thus, $\alpha v_i + (1-\alpha)v_i' \in V_i^k$. □

We assume $V_i^k \leq V_i^{k+1}$ (in a natural set order) for each k without loss of generality. Let $\underline{v}_i^k = \inf V_i^k$. Note that $\underline{v}_i^1 = 0$. The next lemma is also immediate given the finiteness of Γ.

Lemma 3. For each i, $v_i \in V_i$, and $m_i \in M_i$, there exists $m_i' \in M_i^A(v_i)$ such that, for any $m_{-i} \in M_{-i}$,

$$v_i q_i^g(m_i', m_{-i}) - p_i^g(m_i', m_{-i}) \geq v_i q_i^g(m_i, m_{-i}) - p_i^g(m_i, m_{-i}).$$

Proof. We have either $m_i \in M_i^A(v_i)$ or $m_i \notin M_i^A(v_i)$.

If $m_i \in M_i^A(v_i)$, let $m_i' = m_i$. Then, the inequality is satisfied with equality for any $m_{-i} \in M_{-i}$.

If $m_i \notin M_i^A(v_i)$, then m_i is weakly dominated by some $m_i' \in M_i^A(v_i)$ because M_i is finite. Thus, m_i' satisfies the inequality for any $m_{-i} \in M_{-i}$. □

For each i, take an arbitrary sequence of private-value types, $\{v_i^1, \ldots, v_i^K\}$, such that, for $k = 1, \ldots, K-1$, $v_i^k \in V_i^k$. First, let $m_i^0 = m^{\text{out}} \in M_i$, where m^{out} is the message corresponding to "nonparticipation". For each $k = 1, \ldots, K_i$, given m_i^{k-1}, the previous lemma implies that there is $m_i^k \in M_i^A(v_i^k)$ such that, for any m_{-i},

$$v_i^k q_i^g(m_i^k, m_{-i}) - p_i^g(m_i^k, m_{-i}) \geq v_i^k q_i^g(m_i^{k-1}, m_{-i}) - p_i^g(m_i^{k-1}, m_{-i}).$$

Moreover, for any $v_i \in V_i^k$ and m_{-i}, we have

$$v_i q_i^g(m_i^k, m_{-i}) - p_i^g(m_i^k, m_{-i}) \geq v_i q_i^g(m_i^{k-1}, m_{-i}) - p_i^g(m_i^{k-1}, m_{-i}),$$

which implies, by continuity,

$$\underline{v}_i^k q_i^g(m_i^k, m_{-i}) - p_i^g(m_i^k, m_{-i}) \geq \underline{v}_i^k q_i^g(m_i^{k-1}, m_{-i}) - p_i^g(m_i^{k-1}, m_{-i}).$$

Define $f = (q, p) : V \to X$ so that, if $v = (v_i)_{i \in I} \in \prod_i V_i^{k_i}$ for some $k_1, \ldots, k_N$, then

$$(q(v), p(v)) = (q^g((m_i^{k_i})_{i \in I}), p^g((m_i^{k_i})_{i \in I})).$$

Recall that, for each i and θ_i, we have $M_i^A(\theta_i) \supseteq \bigcup_{v_i \in V_i(\theta_i)} M_i^A(v_i)$. Thus,

$$W(\Gamma) \leq \int_\theta \Big[\inf_{v \in V(\theta)} w(f(v), \theta) \Big] d\Phi.$$

We complete the proof by showing the desired integral envelope condition.

Lemma 4. For each i, $v_i \in V_i$, $v_{-i} \in V_{-i}$,

$$v_i q_i(v) - p_i(v) \geq \int_0^{v_i} q_i(\tilde{v}_i, v_{-i}) \, d\tilde{v}_i.$$

Proof. For an arbitrary $k = (k_1, \ldots, k_N)$, let $v_i \in V_i^{k_i}$. We only show the desired inequality for agent 1.

$$\begin{aligned}
& v_1 q_1(v_1, \ldots, v_N) - t_1(v_1, \ldots, v_N) \\
= \;& v_1 q_1^g(m_1^{k_1}, \ldots, m_N^{k_N}) - p_1^g(m_1^{k_1}, \ldots, m_N^{k_N}) \\
= \;& (v_1 - \underline{v}_1^{k_1}) q_1^g(m_1^{k_1}, \ldots, m_N^{k_N}) + \underline{v}_1^{k_1} q_1^g(m_1^{k_1}, \ldots, m_N^{k_N}) - p_1^g(m_1^{k_1}, \ldots, m_N^{k_N}) \\
\geq \;& (v_1 - \underline{v}_1^{k_1}) q_1^g(m_1^{k_1}, \ldots, m_N^{k_N}) + \underline{v}_1^{k_1} q_1^g(m_1^{k_1 - 1}, m_2^{k_2}, \ldots, m_N^{k_N}) - p_1^g(m_1^{k_1-1}, m_2^{k_2}, \ldots, m_N^{k_N}) \\
\geq \;& (v_1 - \underline{v}_1^{k_1}) q_1^g(m_1^{k_1}, \ldots, m_N^{k_N}) + \sum_{j_1=2}^{k_1} (\underline{v}_1^{j_1} - \underline{v}_1^{j_1 - 1}) q_1^g(m_1^{j_1 - 1}, m_2^{k_2}, \ldots, m_N^{k_N}) \\
& + \underline{v}_1^1 q_1^g(m_1^1, m_2^{k_2}, \ldots, m_N^{k_N}) - p_1^g(m_1^1, m_2^{k_2}, \ldots, m_N^{k_N}) \\
\geq \;& (v_1 - \underline{v}_1^{k_1}) q_1^g(m_1^{k_1}, \ldots, m_N^{k_N}) + \sum_{j_1=2}^{k_1} (\underline{v}_1^{j_1} - \underline{v}_1^{j_1 - 1}) q_1^g(m_1^{j_1 - 1}, m_2^{k_2}, \ldots, m_N^{k_N}).
\end{aligned}$$

Observe that

$$(v_1 - \underline{v}_1^{k_1})q_1^g(m_1^{k_1}, \ldots, m_N^{k_N}) = \int_{\underline{v}_1^{k_1}}^{v_1} q_1(\tilde{v}_1, v_{-1})d\tilde{v}_1,$$

and for each $j_1 = 2, \ldots, k_1$,

$$(\underline{v}_1^{j_1} - \underline{v}_1^{j_1-1})q_1^g(m_1^{j_1-1}, m_2^{k_2}, \ldots, m_N^{k_N}) = \int_{\underline{v}_1^{j_1-1}}^{\underline{v}_1^{j_1}} q_1(\tilde{v}_1, v_{-1})d\tilde{v}_1,$$

and therefore, recalling $\underline{v}_1^1 = 0$,

$$v_1 q_1(v) - p_1(v) \geq \int_0^{v_1} q_1(\tilde{v}_1, v_{-1})d\tilde{v}_1.$$

□

We have shown that, given any Γ, there exists $f = (q, p)$ such that

$$W(\Gamma) \leq \int_\theta \Big[\inf_{v \in \mathrm{int} V(\theta)} w(f(v), \theta)\Big] d\Phi,$$

and for each i, v_i, v_{-i},

$$v_i q_i(v) - p_i(v) \geq \int_0^{v_i} q_i(\tilde{v}_i, v_{-i})\ d\tilde{v}_i.$$

Therefore, for any Γ,

$$\begin{aligned} W(\Gamma) \ \leq \ \overline{W} = \sup_{f=(q,p):V\to X} \quad & \int_\theta \Big[\inf_{v \in V(\theta)} w(f(v), \theta)\Big] d\Phi \\ \text{sub.to} \quad & v_i q_i(v) - p_i(v) \geq \int_0^{v_i} q_i(\tilde{v}_i, v_{-i})\ d\tilde{v}_i, \ \forall i, v. \end{aligned}$$

4 Applications

In this section, we consider three applications: a private-value auction, an interdependent-value auction, and private-value bilateral trade. Under certain conditions in each of these applications, we show that the upper bound characterized in Theorem 1 is tight, and obtain the worst-case optimal mechanism.

4.1 Private-value auction

As the first application, we consider an auction environment with private values (i.e., $\Theta_i = V_i$ for each i, and $V_i(v_i) = \{v_i\}$ for each v_i), where the designer's objective is a weighted sum of revenue and surplus. For $\lambda \in [0,1]$, let

$$w((q_i, p_i)_{i \in I}, v) = \lambda \left(\sum_i p_i \right) + (1-\lambda) \left(\sum_i v_i q_i \right).$$

By Theorem 1 (or Corollary 1), an upper bound for the highest performance guarantee is

$$\begin{aligned} \overline{W} = \sup_{(q,p):V \to X} \quad & \int_v \lambda \left(\sum_i p_i(v) \right) + (1-\lambda) \left(\sum_i v_i q_i(v) \right) d\Phi \\ \text{sub.to} \quad & v_i q_i(v) - p_i(v) \geq \int_0^{v_i} q_i(\tilde{v}_i, v_{-i})\, d\tilde{v}_i, \ \forall i, v. \end{aligned}$$

We assume a generalized version of the "regularity" conditions of Myerson (1981) as in the following Assumptions 1–3.[12]

Assumption 1. (full-support density) Φ is absolutely continuous (with respect to a Lebesgue measure on $\mathbb{R}^N$) with a density ϕ with $\phi(v) > 0$ for all $v \in V$.

Let $\phi_i(\cdot)$ denote the marginal density for v_i, and for each v_{-i}, let $\phi_i(\cdot|v_{-i})$ denote the conditional density for v_i given v_{-i}. Let $\Phi_i(\cdot)$ and $\Phi_i(\cdot|v_{-i})$ denote their CDFs.

Let $h_i(v) = v_i - \lambda \frac{1-\Phi_i(v_i|v_{-i})}{\phi_i(v_i|v_{-i})}$ denote the *virtual value* of agent i given v.

Assumption 2. (symmetry) For each v and its permutation v' (i.e., there exists a bijection $\pi : I \to I$ such that $v_i = v'_{\pi(i)}$ for each i), we have $\phi(v) = \phi(v')$.

Assumption 3. (monotone virtual values) For each i and v, $h_i(v)$ is strictly increasing in v_i, and nonincreasing in v_{-i}.

[12] See Segal (2003) and Chung and Ely (2007).

When v is independently distributed according to the designer's prior, this condition corresponds to the monotone virtual-value condition in Myerson (1981). As in Chung and Ely (2007), if v is affiliated in the sense of Milgrom and Weber (1982), then the condition is also satisfied. Given i and v_{-i}, let $r_i^*(v_{-i}) = \inf\{v_i | h_i(v_i, v_{-i}) > 0\}$ (let $r_i^*(v_{-i}) = 1$ if the set on the right-hand side is empty). By Assumption 3, such $r_i^*(v_{-i})$ uniquely exists for each i and v_{-i}, and is nondecreasing in v_{-i}, which implies that r_i^* is continuous at almost every v_{-i}.[13] By Assumption 2, $r_i^*(\cdot) = r_j^*(\cdot)$ for each i, j, and hence, we denote this by $r^*(\cdot)$ in the following.

A *second-price auction with a reserve-price function* $r^*(\cdot)$ is a revelation mechanism $f = (q, p) : V \to X$ such that, for each i and v, (i) $q_i(v) = 1$ if and only if $v_i > v_{-i}^{(1)} = \max_{j \neq i} v_j$ and $v_i > r^*(v_{-i})$, and (ii) $p_i(v) = q_i(v) \cdot \max\{v_{-i}^{(1)}, r^*(v_{-i})\}$. This is dominant-strategy incentive compatible, and the designer's expected utility under the agents' truth-telling can be written as follows.

$$\sum_i \int_{v | v_i > \max\{v_{-i}^{(1)}, r^*(v_{-i})\}} \left[\lambda \max\{v_{-i}^{(1)}, r^*(v_{-i})\} + (1-\lambda) v_i\right] \, d\Phi.$$

Lemma 5. Under Assumptions 1, 2, and 3, we have

$$\overline{W} = \sum_i \int_{v | v_i > \max\{v_{-i}^{(1)}, r^*(v_{-i})\}} \left[\lambda \max\{v_{-i}^{(1)}, r^*(v_{-i})\} + (1-\lambda) v_i\right] \, d\Phi.$$

Proof. Without loss of generality, we assume that all the constraints are satisfied with equality (otherwise we can increase the payment from the corresponding agent without decreasing the objective). Then, by a standard procedure based on integration by parts,

$$\begin{aligned} \overline{W} &= \sup_{q: V \to X} \int_v \sum_i \left[\lambda\left(v_i q_i(v) - \int_0^{v_i} q_i(\tilde{v}_i, v_{-i}) d\tilde{v}_i\right) + (1-\lambda) v_i q_i(v)\right] \, d\Phi \\ &= \sup_{q: V \to X} \int_v \sum_i \left[v_i - \lambda \frac{1 - \Phi_i(v_i | v_{-i})}{\phi_i(v_i | v_{-i})}\right] q_i(v) \, d\Phi \\ &= \sup_{q: V \to X} \int_v \sum_i h_i(v) q_i(v) \, d\Phi. \end{aligned}$$

[13] See Lavrič (1993) for the proof.

Thus, the pointwise maximization of $\sum_i h_i(v)q_i(v)$ implies that the solution to the right-hand side problem is $q^*(v)$ such that $q_i^*(v) = 1$ if $v_i > \max\{v_{-i}^{(1)}, r^*(v_{-i})\}$, and $q_i^*(v) = 0$ if $v_i < \max\{v_{-i}^{(1)}, r^*(v_{-i})\}$. Therefore,

$$\overline{W} = \sum_i \int_{v | v_i > \max\{v_{-i}^{(1)}, r^*(v_{-i})\}} \left[\lambda \max\{v_{-i}^{(1)}, r^*(v_{-i})\} + (1-\lambda)v_i\right] d\Phi.$$

□

If the second-price auction with a reserve-price function $r^*(\cdot)$ is feasible, then this lemma would be the desired tight-bound result. However, we do not consider this feasible in this paper, because it has infinitely many messages. Nevertheless, we can find a sequence of feasible mechanisms, denoted by $\{\Gamma^K\}_{K=1}^{\infty}$, that converges to this mechanism in an appropriate sense, and hence, we treat this second-price auction as "approximately" feasible.

Specifically, define $\Gamma^K = \langle M^K, (q^K, p^K)\rangle$ as a finite version of a second-price auction such that for each i, (i) $M_i^K = \{\frac{k}{K} | k = 0, \ldots, K\}$, and (ii) for each $v \in M^K$, $(q_i^K(v), p_i^K(v)) = (1, \max\{v_{-i}^{(1)}, r^*(v_{-i})\})$ if $v_i > \max\{v_{-i}^{(1)}, r^*(v_{-i})\}$, and $q_i^K(v) = p_i^K(v) = 0$ otherwise.

Theorem 2. Under Assumptions 1, 2, and 3,

$$\lim_{K\to\infty} W(\Gamma^K) = \overline{W},$$

which, in particular, implies $\sup_\Gamma W(\Gamma) = \overline{W}$.

Proof. Fix $K \in \mathbb{N}$, and consider the mechanism Γ^K. For each i and $v_i \in V_i$, let $\underline{V}_i(v_i)$ be the maximum of $\frac{k}{K}$, $k = 1, \ldots, K$, such that $\frac{k}{K} \leq v_i$, and $\overline{V}_i(v_i)$ be the minimum of $\frac{k}{K}$, $k = 1, \ldots, K$, such that $\frac{k}{K} \geq v_i$. Obviously, in this mechanism Γ^K, $M_i^A(v_i) = \{\underline{V}_i(v_i), \overline{V}_i(v_i)\}$, and therefore, given whatever admissible messages are played, agent i wins for sure in state v if $v_i - \frac{1}{K} > v_{-i}^{(1)} + \frac{1}{K}$ and $v_i - \frac{1}{K} > r^*(\overline{V}_{-i}(v_{-i}))$, where $\overline{V}_{-i}(v_{-i}) = (\overline{V}_j(v_j))_{j\neq i}$.

Thus, Γ^K guarantees

$$\begin{aligned}
W(\Gamma^K) &= \int_v \min_{m\in M^A(v)} \left[\lambda\left(\sum_i p_i^K(m)\right) + (1-\lambda)\left(\sum_i v_i q_i^K(m)\right)\right] d\Phi \\
&\geq \int_v \lambda\left(\min_{m\in M^A(v)} \sum_i p_i^K(m)\right) + (1-\lambda)\left(\min_{m\in M^A(v)} \sum_i v_i q_i^K(m)\right) d\Phi \\
&\geq \sum_i \int_{\left\{v \middle| v_i - \frac{1}{K} > \max\{v_{-i}^{(1)} + \frac{1}{K}, r^*(\overline{V}_{-i}(v_{-i}))\}\right\}} \\
&\quad \left[\lambda\left(\max\{v_{-i}^{(1)} + \frac{1}{K}, r^*(\overline{V}_{-i}(v_{-i}))\}\right) + (1-\lambda)v_i\right] d\Phi.
\end{aligned}$$

Because r^* is continuous at almost every v_{-i}, for each $\varepsilon > 0$, there exists $K(\varepsilon)$ such that, for any $K > K(\varepsilon)$,

$$\begin{aligned}
W(\Gamma^K) &\geq \sum_i \int_{v|v_i > \max\{v_{-i}^{(1)}, r^*(v_{-i})\}} \left[\lambda\left(\max\{v_{-i}^{(1)}, r^*(v_{-i})\}\right) + (1-\lambda)v_i\right] d\Phi - \varepsilon \\
&= \overline{W} - \varepsilon.
\end{aligned}$$

Therefore,

$$\lim_{K\to\infty} W(\Gamma^K) = \overline{W}.$$

$\square$

For revenue maximization (i.e., $\lambda = 1$), the result is qualitatively similar to Theorem 1 in Chung and Ely (2007) ("maxmin foundation" of a dominant-strategy auction), although we use different solution concepts: they consider Bayesian implementation with a universal type space (Mertens and Zamir (1985)), while we consider admissible strategies.[14] Despite the differences, there appears to be some conceptual relationship in our arguments. Very

[14] At a more technical level, they consider a finite value space with single-crossing virtual values, while we consider a continuous value space with monotone virtual values. Neither result implies the other.

roughly, in their Bayesian incentive-compatible mechanism, for each payoff-type of each agent, Chung and Ely (2007) identify a belief type whose incentive compatibility is binding. Then, they show that the optimal mechanism under these (Bayesian) incentive compatibility conditions is equivalent to that under dominant-strategy incentive compatibility. In our framework, for each payoff type of each agent, we identify an incentive condition implied by admissibility, and show that the optimal mechanism under this set of incentive conditions is equivalent to that under dominant-strategy incentive compatibility.[15]

4.2 Interdependent-value auction

The second application is an interdependent-value auction, where the designer's objective is revenue, i.e.,

$$w((q_i, p_i)_{i \in I}, \theta) = \sum_i p_i.$$

By Theorem 1, the upper bound for the highest expected revenue we can guarantee is

$$\begin{aligned} \overline{W} = \sup_{(q,p):V \to X} \quad & \int_\theta \left[\inf_{v \in V(\theta)} \sum_i p_i(v) \right] d\Phi \\ \text{sub.to} \quad & v_i q_i(v) - p_i(v) \geq \int_0^{v_i} q_i(\tilde{v}_i, v_{-i}) \; d\tilde{v}_i, \; \forall i, v. \end{aligned}$$

Let $\underline{V}_i(\theta_i) = \min V_i(\theta_i)$ denote the minimum possible valuation that agent i with signal θ_i could possess, among all θ_{-i}. We then obtain another upper

[15] Yamashita (2013b) further examines the formal relationship between Bayesian mechanism design with "large" type spaces (as in Chung and Ely (2007), Bergemann and Morris (2005), Börgers and Smith (2012a)) and mechanism design with admissible strategies. In general, those two approaches yield a similar set of implementable objectives in private-value environments, but not generally in interdependent-value environments.

bound, $\overline{W}'$, that is (weakly) even higher than $\overline{W}$:

$$\begin{aligned}\overline{W}' = \sup_{(q,p):V\to X} \quad & \int_\theta \sum_i p_i(\underline{V}(\theta))d\Phi \\ \text{sub.to} \quad & v_i q_i(v) - p_i(v) \geq \int_0^{v_i} q_i(\tilde{v}_i, v_{-i})\ d\tilde{v}_i,\ \forall i, v.\end{aligned}$$

Intuitively, $\int_\theta \sum_i p_i(\underline{V}(\theta))d\Phi$ represents the expected revenue in a value-revelation mechanism (q,p) under truth-telling behavior when each i of θ_i believes that his value is $\underline{V}_i(\theta_i)$.

Let Ψ denote the probability distribution over V such that, for each measurable $E \subseteq V$, $\Psi(E) = \Phi(\{\theta|\underline{V}(\theta) \in E\})$. That is, Ψ is the probability measure over the minimum possible values of the agents induced by Φ. Then, we have

$$\int_\theta \sum_i p_i(\underline{V}(\theta))d\Phi = \int_v \sum_i p_i(v)d\Psi,$$

and therefore,

$$\begin{aligned}\overline{W}' = \sup_{(q,p):V\to X} \quad & \int_v \sum_i p_i(v)d\Psi \\ \text{sub.to} \quad & v_i q_i(v) - p_i(v) \geq \int_0^{v_i} q_i(\tilde{v}_i, v_{-i})\ d\tilde{v}_i,\ \forall i, v.\end{aligned}$$

We can interpret this problem as a revenue-maximization problem in a private-value setting. Thus, similar conditions as found in the previous section are useful for characterizing the worst-case optimal mechanism.

Assumption 4. (full-support density) Ψ is absolutely continuous (with respect to a Lebesgue measure on $\mathbb{R}^N$) with a density ψ with $\psi(v) > 0$ for all $v \in V$.

Let $\psi_i(\cdot)$ denote the marginal density for v_i, and for each v_{-i}, let $\psi_i(\cdot|v_{-i})$ denote the conditional density for v_i given v_{-i}. Let $\Psi_i(\cdot)$ and $\Psi_i(\cdot|v_{-i})$ denote their CDFs.

Let $h_i(v) = v_i - \frac{1-\Psi_i(v_i|v_{-i})}{\psi_i(v_i|v_{-i})}$ denote the virtual value of agent i given v.

Assumption 5. (symmetry) For each v and its permutation v', we have $\psi(v) = \psi(v')$.

Assumption 6. (monotone virtual values) For each i and v, $h_i(v)$ is strictly increasing in v_i, and nonincreasing in v_{-i}.

For each i and v_{-i}, let $r_i^*(v_{-i}) = \inf\{v_i | h_i(v_i, v_{-i}) > 0\}$, or one if the set on the right-hand side is empty. By Assumption 6, such $r_i^*(v_{-i})$ uniquely exists for each i and v_{-i}, and is nondecreasing in v_{-i} (and hence continuous at almost every v_{-i}). By Assumption 5, $r_i^*(\cdot) = r_j^*(\cdot)$ for each i, j, and hence, we denote it by $r^*(\cdot)$ in the following.

As in the previous section, a second-price auction with a reserve-price function $r^*(\cdot)$ guarantees the upper-bound level of expected revenue, $\overline{W}'$, which is approximated by a sequence of finite versions of second-price auctions.

Theorem 3. Under Assumptions 4, 5, and 6, we have

$$\lim_{K\to\infty} W(\Gamma^K) = \overline{W} = \sum_i \int_{v|v_i > \max\{v_{-i}^{(1)}, r^*(v_{-i})\}} \max\{v_{-i}^{(1)}, r^*(v_{-i})\}\ d\Psi,$$

which in particular imply $\sup_\Gamma W(\Gamma) = \overline{W}$.

We omit the proof because it is immediate from Lemma 5 and Theorem 2 with $\lambda = 1$.

Although the result is qualitatively similar to the first application to private-value auctions, it has a quite different interpretation, because a second-price auction is generally neither dominant-strategy nor ex-post incentive compatible with interdependent values. In fact, the result suggests that the new class of incentive conditions identified in this paper, namely incentive compatibility for value revelation, plays a key role in understanding the highest performance guarantee in interdependent-value settings.

Another difference from the private-value case is that Assumptions 4–6 are on the distribution of the minimum possible valuations $\underline{V}(\cdot)$, rather than on Φ or the actual valuation functions $v(\cdot)$. For example, let $\theta_i =$

$(c_i, d_i) \in \Theta_i \subseteq \mathbb{R}^2$ be i's two-dimensional signal, and $v_i(\theta) = \pi_i(c) + d_i$ be i's valuation function. Recall that c_i is a common component and d_i is a private component. Suppose that, for each i and c_i, $\min_{c_{-i}} \pi_i(c_i, c_{-i}) = 0$ (hence nonnegative interdependence). Then, i's minimum possible value given θ_i is $\underline{V}_i(\theta_i) = d_i$, and thus, Ψ is simply the marginal distribution of Φ over the private components of the agents. In this case, for Theorem 3 to hold true, we do not need any assumption on the distribution over the common components.

4.3 Private-value bilateral trade

The third application is surplus maximization in private-value bilateral trade (Myerson and Satterthwaite (1983), Hagerty and Rogerson (1987)). Specifically, let $I = \{1, 2\}$, where $i = 1$ is a seller and $i = 2$ is a buyer. For each i, $\Theta_i = V_i$, and for each $v_i \in V_i$, $V_i(v_i) = \{v_i\}$.

Let $(q, p) \in [0, 1] \times \mathbb{R}$ represent the trade allocation, where q is the probability that the seller provides the good to the buyer, and p is the payment from the buyer to the seller. The seller's utility given $(q, p) \in X$ is $p - v_1 q$ (hence, v_1 may be interpreted as the seller's opportunity cost), and the buyer's utility given (q, p) is $v_2 q - p$.

Note that this notation is consistent with that introduced in Section 2. Specifically, in the model introduced in Section 2, let $X = \{(q_i, p_i)_{i \in I} | q_2 = -q_1 \in [0, 1],\ p_2 = -p_1 \in \mathbb{R}\}$ be the set of feasible allocations. Then, the seller's utility is $v_1 q_1 - p_1 = p_2 - v_1 q_2$ and the buyer's utility is $v_2 q_2 - p_2$. By identifying (q_2, p_2) as (q, p), we obtain the bilateral-trade model in this subsection.

The designer's objective is surplus, i.e.,

$$w((q, p), \theta) = (v_2 - v_1) q.$$

By Theorem 1, the upper bound for the highest expected surplus we can guarantee is

$$
\begin{aligned}
\overline{W} = \sup_{(q,p):V\to X} \quad & \int_v (v_2 - v_1) q(v)\ d\Phi \\
\text{sub.to} \quad & p(v) - v_1 q(v) \geq \int_{v_1}^{1} q(\tilde{v}_1, v_2)\ d\tilde{v}_1, \forall v, \\
& v_2 q(v) - p(v) \geq \int_{0}^{v_2} q(v_1, \tilde{v}_2)\ d\tilde{v}_2, \forall v.
\end{aligned}
$$

As opposed to the auction environments, a feasible allocation has to satisfy the budget-balance condition, and hence, it is not obvious whether the solution to the upper-bound problem satisfies all the constraints with equality.

In the following, we introduce a sufficient condition for the environment under which the solution to the upper-bound problem is dominant-strategy incentive compatible.

Assumption 7. (monotone weighted surplus) Φ is absolutely continuous (with respect to a Lebesgue measure on $\mathbb{R}^2$) with a density ϕ. For any $v_1 < v_2$, $\mu(v) \equiv (v_2 - v_1)\phi(v)$ is strictly decreasing in v_1 and strictly increasing in v_2.

$\mu(v)$ quantifies the impact on expected surplus of making the agents trade, or in other words, it is the trade surplus in state v weighted by the density of the state, $\phi(v)$. The monotonicity of the weighted surplus μ in $(-v_1, v_2)$ means that more-efficient types (i.e., lower v_1 and higher v_2) have higher impacts on expected surplus.

For example, the condition is satisfied if (i) more efficient types are more likely, i.e., the density $\phi(v)$ is nonincreasing in v_1 and nondecreasing in v_2, or if (ii) ϕ is differentiable and the rate of change in ϕ is sufficiently small so that, for any v, $\left|\frac{\partial\phi(v)/\partial v_1}{\phi(v)/(1-v_1)}\right|, \left|\frac{\partial\phi(v)/\partial v_2}{\phi(v)/v_2}\right| < 1$.[16]

[16] We measure the fractional change in the seller's type with respect to the highest-cost type, so we have $1 - v_1$ instead of v_1 in the denominator. The condition is satisfied for a class of common distributions under appropriate truncation and restrictions on parameter values.

A *posted-price mechanism with price* $r \in [0,1]$ is a revelation mechanism $(q,p) : V \to X$ such that $(q(v),p(v)) = (1,r)$ if $v_2 > r > v_1$, and $(q(v),p(v)) = (0,0)$ otherwise. It is dominant-strategy incentive compatible,[17] and it guarantees the following expected surplus.

$$\int_{v_1=0}^{r}\int_{v_2=r}^{1} \mu(v)\ dv_2 dv_1.$$

Observe that, under Assumption 7, this is strictly convex in r. Thus, the optimal posted price, r^*, uniquely exists, and is characterized by the first-order condition:

$$\int_0^{r^*} \mu(v_1,r^*)\ dv_1 = \int_{r^*}^{1} \mu(r^*,v_2)\ dv_2.$$

Theorem 4. Under Assumption 7, we have

$$\sup_{\Gamma} W(\Gamma) = \overline{W} = \int_{v_1=0}^{r^*}\int_{v_2=r^*}^{1} \mu(v)\ dv_2 dv_1.$$

The theorem states that the upper-bound expected surplus can be guaranteed by the posted-price mechanism with r^*. Note that the posted-price mechanism with r^* itself can be considered to be feasible.[18] Therefore, we obtain the tightness of $\overline{W}$.

Hagerty and Rogerson (1987) show that essentially any dominant-strategy mechanism in this bilateral-trade setting is a (possibly randomized) posted-price mechanism. Our result says that, even if nondominant-strategy mechanisms are allowed, the designer would optimally choose a (deterministic) posted-price mechanism if he aims to guarantee the highest possible expected surplus.

[17] More rigorously, truth-telling is the unique admissible choice for every type of each agent i except for the threshold type $v_i = r$, which occurs with probability zero under Assumption 7.

[18] For example, consider a mechanism $\Gamma^* = \langle M^*, (q^*,p^*)\rangle$ with (i) $M_i^* = \{0,1\}$ for each i, and (ii) $(q^*(m),p^*(m)) = (1,r^*)$ if $m = (1,1)$, and $q^*(m) = p^*(m) = 0$ otherwise. This is feasible, and equivalent to the posted-price mechanism with r^*.

To provide an intuition for how Assumption 7 plays a role for Theorem 4, we ask if the posted-price mechanism with price r^* can be improved by modifying the mechanism. Suppose some improvement was possible, for example, in some states where $v_1 < v_2 < r^*$ (a similar argument holds for the other case where improvement was possible in states where $v_2 > v_1 > r^*$). The trading price is necessarily less than r^*, which makes the buyer with $v_2 \in (r^*, r^* + \varepsilon)$ prefer this new outcome to trading with price r^* for some $\varepsilon > 0$. This implies that, in the worst-case scenario, welfare loss must occur in some states where $v_2 \in (r^*, r^* + \varepsilon)$ and $v_1 < r^*$. However, under the monotone weighted surplus in Assumption 7, this welfare loss is greater than the welfare gain, because this assumption basically states that allowing more trade in states where the buyer's value is higher has a greater impact in the expected surplus than allowing more trade in states where the buyer's value is lower.

Proof. For each $v_2 \in (r^*, 1)$, define $r(v_2)$ such that

$$\int_0^{r(v_2)} \mu(\tilde{v}_1, r^*)\ d\tilde{v}_1 = \int_{v_2}^1 \mu(r^*, \tilde{v}_2)\ d\tilde{v}_2.$$

Because $\mu(v)$ is strictly decreasing in v_1, strictly increasing in v_2, and continuous, $r(v_2)$ uniquely exists for each $v_2 \in (r^*, 1)$. As a function of v_2, r is strictly decreasing and differentiable, where $r'(v_2)\mu(r(v_2), r^*) = -\mu(r^*, v_2)$. Moreover, $r(v_2) \to r^*$ as $v_2 \downarrow r^*$, and $r(v_2) \to 0$ as $v_2 \uparrow 1$.

We decompose the upper-bound problem into infinitely many subproblems. Specifically, for each $v_2 \in (r^*, 1)$, the *subproblem* v_2 is given by

$$\begin{aligned} & W_{v_2} \\ = \quad & \sup_{q:V\to[0,1]} \int_{r(v_2)}^1 \mu(\tilde{v}_1, v_2) q(\tilde{v}_1, v_2) d\tilde{v}_1 + \int_0^{v_2} \mu(r(v_2), \tilde{v}_2) q(r(v_2), \tilde{v}_2)(-r'(v_2)) d\tilde{v}_2 \\ & \text{sub.to} \quad (v_2 - r(v_2)) q(r(v_2), v_2) \geq \int_{r(v_2)}^1 q(\tilde{v}_1, v_2)\ d\tilde{v}_1 + \int_0^{v_2} q(r(v_2), \tilde{v}_2)\ d\tilde{v}_2, \\ & \hfill \forall v_2 \in (r^*, 1) \end{aligned}$$

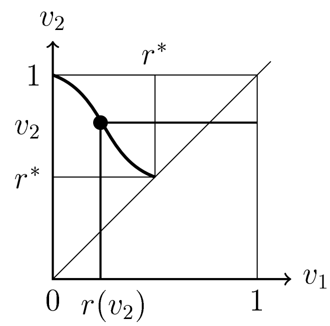

Figure: $r(v_2)$ and the *subproblem* v_2.

Lemma 6.

$$\overline{W} \leq \int_{r^*}^{1} W_{v_2} \; dv_2.$$

Proof. Fix an arbitrary $\varepsilon > 0$. By the definition of $\overline{W}$, there exists (q^o, p^o) : $V \to [0,1] \times \mathbb{R}$ that satisfies all the constraints of the problem of $\overline{W}$, and furthermore, $\int_v (v_2 - v_1) q^o(v) d\Phi \geq \overline{W} - \varepsilon$.

For each $v_2 \in (r^*, 1)$, q^o satisfies all the constraints of the subproblem v_2, because

$$\begin{aligned}(v_2 - r(v_2)) q^o(r(v_2), v_2) &= U_1(r(v_2), v_2) + U_2(r(v_2), v_2) \\ &\geq \int_{r(v_2)}^{1} q^o(\tilde{v}_1, v_2) \; d\tilde{v}_1 + \int_0^{v_2} q^o(r(v_2), \tilde{v}_2) \; d\tilde{v}_2,\end{aligned}$$

where the inequality is because q^o satisfies all the constraints in the problem of $\overline{W}$.

Therefore,

$$\begin{aligned}&\overline{W} - \varepsilon \\ \leq\ & \int_{r^*}^{1} \int_{r(v_2)}^{1} \mu(\tilde{v}_1, v_2) q^o(\tilde{v}_1, v_2) \; d\tilde{v}_1 dv_2 + \int_0^{r^*} \int_0^{r^{-1}(v_1)} \mu(v_1, \tilde{v}_2) q^o(v_1, \tilde{v}_2) \; d\tilde{v}_2 dv_1 \\ =\ & \int_{r^*}^{1} \left[\int_{r(v_2)}^{1} \mu(\tilde{v}_1, v_2) q^o(\tilde{v}_1, v_2) \; d\tilde{v}_1 + \int_0^{v_2} \mu(r(v_2), \tilde{v}_2) q^o(r(v_2), \tilde{v}_2)(-r'(v_2)) \; d\tilde{v}_2 \right] dv_2 \\ \leq\ & \int_{r^*}^{1} W_{v_2} \; dv_2,\end{aligned}$$

where the equality is obtained by substituting v_1 with $r(v_2)$.

Because $\varepsilon > 0$ is arbitrary, we obtain the desired inequality. $\square$

Lemma 7. For each $v_2 \in (r^*, 1)$, we have

$$W_{v_2} = \int_{r(v_2)}^{r^*} \mu(\tilde{v}_1, v_2)\ d\tilde{v}_1 + \int_{r^*}^{v_2} \mu(r(v_2), \tilde{v}_2)(-r'(v_2))\ d\tilde{v}_2.$$

Proof. In the subproblem v_2, because both the objective and constraints are linear in each $q(v)$, in the solution, there exists a threshold value $\mu^* \in [0,1]$ that satisfies the following. For each $\tilde{v}_2 \in (0, v_2)$, we have $q(r(v_2), \tilde{v}_2) = 1$ if and only if $\mu(r(v_2), \tilde{v}_2)(-r'(v_2)) > \mu^*$. For each $\tilde{v}_1 \in (r(v_2), 1)$, we have $q(\tilde{v}_1, v_2) = 1$ if and only if $\mu(\tilde{v}_1, v_2) > \mu^*$.

Moreover, Assumption 1 implies that there exist τ_1^*, τ_2^* such that $-r'(v_2)\mu(r(v_2), \tau_2^*) = \mu(\tau_1^*, v_2)$, (i) $q(r(v_2), \tilde{v}_2) = 1$ if and only if $\tilde{v}_2 > \tau_2^*$, and (ii) $q(\tilde{v}_1, v_2) = 1$ if and only if $\tilde{v}_1 < \tau_1^*$. Furthermore, the constraint must be satisfied with equality, i.e., $v_2 - r(v_2) = (\tau_1^* - r(v_2)) + (v_2 - \tau_2^*)$, or $\tau_1^* = \tau_2^*$. The only pair (τ_1^*, τ_2^*) that satisfies these conditions is such that $\tau_1^* = \tau_2^* = r^*$. $\square$

These lemmas imply

$$\overline{W} \leq \int_{r^*}^{1} W_{v_2}\ dv_2 = \int_0^{r^*} \int_{r^*}^{1} \mu(v)\ dv_2 dv_1.$$

However, the right-hand side coincides with the worst-case expected surplus guaranteed by the posted-price mechanism with price r^*. $\square$

5 Concluding remarks

This paper studied the mechanism-design problem of guaranteeing desirable performances whenever the agents are rational in the sense of not playing weakly dominated strategies. In Section 3, we provided an upper bound for the best performance guarantee among all feasible mechanisms. This upper bound is given by the supremum of the truth-telling outcome in a mechanism where each agent reports his own "valuation".

Then, in Section 4, we applied this upper bound to private-value and interdependent-value auctions and private-value bilateral trade. Under certain conditions, we showed that the upper bound is tight, and obtained the worst-case optimal mechanisms (with a limiting argument when the exact optimal mechanism is infeasible). In private-value environments, the optimal mechanisms satisfy dominant-strategy incentive compatibility, the classical notion of "robust" mechanisms. More specifically, in an auction setting, the optimal mechanism for a weighted average of expected revenue and surplus is a second-price auction (with a reserve-price function) under the monotone virtual-value condition. In a bilateral-trade setting, the optimal mechanism for expected surplus is a posted-price mechanism under the monotone weighted-surplus condition. In an interdependent-value auction, we found that the optimal mechanism is a second-price auction, which is neither dominant-strategy nor ex post incentive compatible, but satisfies the novel incentive compatibility introduced in the paper, which we refer to as incentive compatibility for value revelation.

While we identified several environments where the upper bound is tight, we believe that it would also be useful to provide instances where the bound is not tight. For example, in the bilateral-trade application in Section 4.3, imagine that the designer's prior Φ is discrete and $\Phi(v) = \frac{1}{2}$ if $v \in \{(0, \frac{1}{3}), (\frac{2}{3}, 1)\}$ and $\Phi(v) = 0$ otherwise.[19] Then, our upper bound for the highest expected surplus coincides with the first-best trade surplus, $\frac{1}{3}$, because, under the first-best trade rule $q(0, \frac{1}{3}) = q(\frac{2}{3}, 1) = 1$ (and $q(v) = 0$ otherwise), the integral envelope conditions reduce to ex post individual-rationality conditions. However, no feasible mechanism can guarantee the first-best trade surplus.[20]

[19] Although Φ is not absolutely continuous in this example, the argument does not essentially change as long as an (absolutely continuous) Φ is close to the one discussed here in an appropriate sense.

[20] Suppose contrarily that a feasible mechanism $\langle M, g \rangle$ could guarantee the first-best trade surplus. Then, as in Lemma 3, there exist $m_1 \in M_1^A(\frac{2}{3})$ such that m_1 is always weakly better than m^{out} for the seller with $v_1 = \frac{2}{3}$, $m_1' \in M_1^A(0)$ such that m_1' is always weakly better than m_1 for the seller with $v_1 = 0$, $m_2 \in M_2^A(\frac{1}{3})$ such that m_2 is always

Intuitively, the integral envelope condition in Theorem 1 corresponds to local and downward incentive compatibility, and therefore, a solution to the upper-bound problem may not satisfy other constraints such as global incentive constraints, especially when the solution to the upper-bound problem is not monotonic, as in this counterexample.

Finally, even though we focused on linear environments, some concepts and techniques developed in this paper may be useful in more general mechanism design or implementation problems. For example, for some assignment problems with divisible goods, it may be more natural to allow for the agents' utilities being nonlinear in q. The working paper version of this analysis, Yamashita (2013a), establishes a similar upper bound as in Theorem 1 (or Corollary 1) in such a nonlinear environment but with private values. As another example, for some mechanism-design problems without monetary transfers, dominant-strategy incentive-compatible (or strategy-proof) mechanisms are examined in the literature in private-value environments.[21] The counterpart of incentive compatibility for value revelation (or more generally "preference revelation") may naturally be defined in those problems but with interdependence. It may be interesting to see whether mechanisms that satisfy such incentive conditions perform well. One of the challenges would be to extend our upper-bound result to more general environments, which is left for future research.

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weakly better than m^{out} for the buyer with $v_2 = \frac{1}{3}$, and $m_2' \in M_2^A(1)$ such that m_2' is always weakly better than m_2 for the buyer with $v_2 = 1$. These necessarily imply $g(m_1, m_2) = (0, 0)$, $g(m_1', m_2) = (1, p_1)$ with $p_1 \leq \frac{1}{3}$, $g(m_1, m_2') = (1, p_2)$ with $p_2 \geq \frac{2}{3}$, and $g(m_1', m_2') = (q, p)$ for some (q, p) such that $p \in [p_2, q+p_1-1]$. However, $q+p_1-1 \leq \frac{1}{3} < p_2$, and therefore, such a mechanism cannot exist.

[21] See, for example, Moulin (1980) for single-peaked voting problems, and Gale and Shapley (1962), Dubins and Freedman (1981), and Roth (1982) for matching problems.

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