Cooperative decision-making for the provision of a locally undesirable facility

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Abstract

We consider the decentralized provision of global public good with local externalities in a spatially explicit model. Communities (municipalities, cities, countries) decide on the localization of a facility that benefits to all but exhibits costs to the host and its immediate neighbors. They share the costs through transfers. We examine the cooperative game associated with this so-called NIMBY problem ("Not In My Back-Yard") with communities located along a line. A core solution is constrained by individual participation of communities and exclusion of all potentially polluted neighbors. The former constraint set lower bounds on individual welfare while the later defines upper bounds on the welfare of small coalitions. We provide necessary and sufficient conditions for a core solution to exist. Next we generalize the game with local externalities in a network. We show that the exclusion of the immediate neighbors of the facility still constrains the core welfare distributions.

Keywords: NIMBY, externality, pollution, core, cooperative game, waste, spatial model.
JEL codes: C71, D62, Q53, R53.

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1 Introduction

1.1 The NIMBY problem

In June 2008, the French government call for candidates for hosting a site for low-level radioactive waste storage among 3115 technically eligible municipalities. About 40 of them, all located in North Eastern France, showed their interest in the project. On June 2009, 24th, after detailed investigations, the Government announced the selection of two municipalities for complete investigations: Pars-les-Chavanges and Auxon. However, facing strong public opposition in surrounding municipalities and pressure from intermediary political levels[1], Pars-les-Chavanges withdrew from the process on July, 4th and Auxon followed on August, 11th. The ANDRA stated that "consistently to the approach chosen by the Government and the Andra, based on the voluntary participation of municipalities, the municipalities resorted to their right to withdraw from the project"[2]. The process for site selection was then in a dead end.

In the Haute-Pyrénées département in Southwestern France, waste used to be stored in three landfills that will be closed by July 2013. A public administration proposed three potential new sites which would have the desired geological characteristics. One month before the announcement an association was created by the 60 municipalities to oppose the decision of the administration. It started organizing demonstration and was soon backed by all the local representatives of selected areas. On December 2011, just before the closure of the second site of Lourdes-Mourles, all the mayors had officially left the negotiations [3]. At present, there remains little hope that any eligible municipality would accept the project and, by 2013, the department is likely to be exporting all its wastes to adjacent areas.

The two above cases are illustrations of the *Not In My Backyard* (NIMBY) phenomenon: a

[1] A letter from the President of the Region Champagne-Ardennes to the Minister issued on the 6th of March 2009 emphasizes the threat that such a project would impose on the Champagne economic activities, emphasizing then a potentially significant externality from the candidating municipalities on the whole region.

[2] `http://www.andra.fr/pages/fr/menu1/les-solutions-de-gestion/etudier-une-solution-de-gestion-pour-les-dechets-favl/la-recherche-de-site-en-2008-et-2009-6717.html` Accessed on March 2013, 4th.

[3] They mainly justifies their rejection on the ground of social costs and risks they deem have not been considered seriously but also "because [their] territory encompasses a dense meshing of small villages what makes impossible to construct such a project without having several municipalities impacted by the nuisances associated (odors, noise, pests...)". Whereas not firmly grounded, this point emphasizes the idea that externalities are perceived as spread across several municipalities in this example.

rejection that usually arises for facilities such as waste treatment plants, airports, prison, windmills, nuclear or coal power plants. They share a common feature: although many agents (municipalities, cities, consumers, states, countries,...) benefit from the use of the facility, few of them suffer from local negative externalities such as air, water, noise or "landscape" pollution. This is the reason why the localization of the facility is a sensitive issue. To make it accepted by municipalities, it should be related to some form of compensation to the host of the facility and its immediate neighbors (that are suffering from the negative externalities). The paper examines the problem of localizing an undesirable facility and designing compensation for the negatively impacted agents simultaneously. It relies on cooperative game theory applied to a spatially explicit model of public good provision with localized negative externalities. We investigate the determinants of cooperation for a successful solution to the problem.

The NIMBY problem is first embedded in the more simple spatial structure: a line. A set of n communities are designing an *agreement* to localize an undesirable facility and perform transfers among them. The damages from running the facility are incurred by the host and it immediate neighbors. Each of them incur a damage of δ of the cost of hosting. Partial agreements by a subgroup of communities can be designed or no agreement at all. This defines the outside option of coalitions of communities in the negotiation described as a cooperate game associated to the problem. We distinguish between excludable and non-excludable facilities. We show that the cooperative game exhibits non-standard properties. It is a cooperative game with externalities in the sense that the welfare that a group of communities can enjoy depends on the cooperative behavior of communities outside the group as well as the localization of the facility they build on their own. Yet externalities can be negative or positive: a group of communities can benefit or suffer from the cooperation of others. Nevertheless, the best that can happen for a coalition of communities in the non-excludable case is that the other communities are not cooperating. The highest welfare that a coalition of communities can achieve if the others are not cooperating is called the "collapse in outside cooperation" (CIOC) value. It is defined by the corresponding CIOC characteristic function. We provide necessary and sufficient conditions for a core solution to the NIMBY problem to exists with the CIOC characteristic function. It relies on two forces. First, individual rationality requires that the communities located close the the facility enjoy a net benefit from it. It forces the externality parameter δ (share of the hosting cost incurred by the neighboring communities) to be lower than the ratio of the benefit of using the facility over the hosting cost. Second, communities are tempted to exclude the neighbors of the host of the facility to avoid compensating them from the pollution damages. This deviation strategy by coalitions of size $n-1$ or $n-2$ sets also an upper bound on the externality parameter δ. Next we generalize our model to richer spatial structures. Communities are embedded in a graph. The links capture the spatial externalities: all communities connected to the host of the facility suffers from a share of the cost of hosting the facility. We show that creating more links in the graph without changing the total cost of the facility at the optimal location enlarges the core. Intuitively, it means that when a facilities is hurting more neighboring communities if it is located elsewhere than in the optimal location, reaching an agreement to solve the NIMBY problem is easier. We found similar driving forces for the existence of the core in the graph than in the line: individual rationality and exclusion of some neighboring communities.

1.2 Related literature

1.2.1 The core and the uncapacited facility location problem

Our paper considers the location choice and cost sharing of a single locally undesirable public project. Such problem echoes to the uncapacited facility location problem (UFLP) in operational research. A basic version of the UFLP is presented in Goemans and Skutella (2000). They consider N individuals. The cost of opening a facility at i is y_i and any customer j have to be connected to a facility to get an access to the service. Cost of connection between i and j is c_{ij}. The first problem is to locate facilities and allocate access such that each customer has access to one facility and total costs are minimized. A second problem is to allocate the total costs such that core constraints are met i.e. no subgroup of agents has an incentive to leave and design its own facility problem. For this class of problems, the authors derive a necessary and sufficient condition for the non-emptiness of the core and provide instances where such condition applies. However they show that the second problem is NP-complete in the general case.

A noteworthy illustration of such results can be found in Lebreton and Weber (2003). Formally, they consider a distribution of agents over a line and set opening cost to g and connexion costs c_{ij} to an increasing function of the distance between agents $d(|i-j|)$. In such setting, they show that the core is never empty for efficient projects and that core allocations require partial equalization in the sense that agents with high connection cost bear some, but not the whole, amount of these costs.

We see that transportation costs are central in such problems. Due to them, the core consists in the choice of a central location and a partial compensation of remote agents. However, when locating a locally undesirable public project, an new force may play in the opposite direction, agents may want to be connected to the facility but would rather have it located far from them. In order to isolate the implication of such feature, we depart from the UFLP by setting connexion/transportation costs to zero and introduce a nuisance cost that plays in the opposite direction. Such a cost differs from connection cost in the sense that it is borne whether or not agents are connected. We further allow for the option of no project and introduce a non-rival benefit b to any connected agent. In the formal model of Le Breton and Weber, preference are single peaked in the sense that, in the absence of transfers, agents are worse off the further from them the project. This feature is crucial for getting their non-emptiness result. In our model, preferences over project location are single dipped in the sense that the further the project the better off the agent is. With such modifications, existence of core allocations is no longer guaranteed. We indeed show that non-emptiness is no longer guaranteed.

When applied to locating undesirable projects, the UFLP problem implicitly adopts a representation of a project which yields costs to a single community, implicitly assuming externalities are concentrated within its territory. Such assumption may be doubtful as soon as we think of risks such as watershed pollution, as communities are small or as we note that such projects may be located at the borders of communities[4]. Following this criticism, we explicitly introduce space in our model in order to emphasize the difficulties that may arise as soon as the costs are spread over more than a single community.

1.3 Public good provision, externalities and the empty core

Most of the literature on the provision of a public good emphasizes the free-riding problem. Cooperation is undermined because individuals or small coalitions can benefit from public good provision (e.g. pollution abatement) without paying the cost (see e.g. Carraro and Siniscalco, 1993; Yi, 1997 or Ray and Vohra, 2001 for recent analysis). Whereas the free-riding problem still arises for the non-excludable facility case, it is absent in the excludable case. Our paper highlights another obstacle to cooperation: the exclusion of the neighbors of the facility host from the agreement to avoid to pay

[4] Although - to our knowledge - no work documents a statistically significant uneven distribution of locally undesirable land use on the borders of jurisdictions, anecdotal evidence are pervasive ("Approximately 2/3 of landfills in Pennsylvania are located at or near county or state boundaries" (Ingberman, 1995, p.S24)). As an additional piece of evidence, Helland et Whitfort (2003) show that US facilities located in counties that borders other states have significantly higher levels of toxic release into the air and the water what suggests that pollution across national borders are not internalized by local communities.

for the damages caused in the neighborhood. It is present and condition the core solution in both cases. The instability we disentangle from free-riding is a reminiscence of the one found in Shapley and Shubik (1969) 's garbage game for which a nice simple example is given in Moulin (2003, p.233). In this game, three players have some garbage to drop somewhere. They can freely dispose their garbage at other's. In such a setting, two-player coalitions are beneficial since they coordinate their dropping strategy to the third player's. As a result, the grand coalition never forms and the core is always empty. Similarly, in our set-up the exclusion of the polluted neighbors of the host is the obstacle to cooperation at the global level. Nevertheless, a core solution that includes all players does sometime exist.

The effect of externalities within the cooperative framework raises specific methodological issues. In such setting, the worth of a coalition a priori depends on the cooperative behavior as well as on the non-cooperative interaction among coalitions. A possible approach would be to exogenously specify the cooperative behavior of outside member in response to the formation of a blocking coalition. This gives rise to a specific value function and an associated notion of the core. With positive external effects, this is for instance the approach adopted by Chander and Tulkens (1997) in the context of pollution abatement negotiations. They adopts the view that outside members stop cooperating and play their individual best reply strategy when a blocking coalition forms (γ-core concept) and study how stable coalitions could emerge in such setting. Their study, as well as in Barrett (1994) and Carraro and Siniscalco (1993), focuses on free-riding as a source of instability. They show that such force may be overcome to some extent.

In the context of negative externalities, several contributions emphasize the potential relevance of the concept of the core for the problem of locating a locally undesirable facility. Lejano and Davos (2001) argue for coalition formation considerations in the design of compensation schemes. With a simple numerical example of a cooperative game with transferable utility, they argue that a compensation scheme that leaves the host indifferent may fail to be a core allocation. Such numerical example implicitly steams from a cooperative game with externalities across communities. We further emphasize that the core may actually be empty. Our argument is related to the criticism originally addressed by Aivazian and Callen (1981) to the Coase theorem. Their argument lies on the idea that, even with transferable utility, a cooperative game involving more than 2 agents may have an empty core. From this simple observation, Aivazan and Callen conclude that Coase bargaining may not occur or may lead to an inefficient and endless recontracting process. This argument is later developed by Stearns (1993) in the context of the siting of a hazardous waste facility. The author constructs a simple example in which a Condorcet cycle arises in a situation where three communities have to collectively decide where to site a noxious facility. Our argument follows these lines but, instead of relying on ad hoc examples, we construct a case which shows that emptiness of the core arises naturally when it comes to site a facility with inter-community negative externalities.

The rest of the paper is organized as follows. Section 2 introduces the NIMBY model over the line. It defines partition function and characteristic functions from our set-up. It then identifies some of their properties. Section 3 deals with the core and some notions of restricted core. In Section 4, we investigate how our results extend to more general spatial structures. Section 5 discusses policy insights. We conclude the paper in Section 6.

2 The NIMBY cooperative game on a line

2.1 The Model

We consider the linear NIMBY problem $\sigma = (N, b, c, \delta, q)$ where $N = \{1, ..., n\}$ is a set of communities or agents (land owners, municipalities, cities, regions, countries,...) located along a line. They might decide to launch a project consisting on building and running a public facility (e.g. waste treatment plant), a utility (nuclear or coal power plant) or a polluting factory. Each community $i = 1, ..., n$ enjoys a benefit b from using the facility. However, the facility creates local nuisances. Only the host and its immediate neighbors suffer from the inconvenience caused by the facility (pollution, risk of accident or contamination,...). The damage for the host plus the technical cost of the project amounts to c and the host's neighbors face an additional damage δc with $\delta > 0$[5]. We

[5]The interpretation of δ is the share of a neighbor's pollution cost as compared to total costs at the host's. Formally, total cost at the host's is the sum of a technical cost c_t (construction, management, etc.) and a pollution cost c_p. Denoting by α the multiplicative change in pollution cost at the immediate neighbors', the additional cost at each of them's is αc_p. Then we get $\delta = \alpha \frac{c_p}{c_t + c_p}$. So δ captures both the evolution of pollution costs with distance and the share of pollution costs in total costs at the host's.

distinguish between excludable ($q = e$) and non-excludable ($q = ne$) facilities. In the first case, the communities which build the facility can exclude the others from using it: the benefit b is enjoyed only by the owners. The facilities is then a club good: a non-rival but excludable good with private costs. In the second case, they cannot exclude. The benefit is b for all communities provided that at least one facility is built. The facility is a pure public good: non-rival and non-excludable. For clarity reasons, we propose to focus on excludable facilities first. We will consider non-excludability of benefits in a second time in order to assess the interplay between free-riding and the source of instability we emphasize. Σ_L denotes the set of such linear problems and Σ_L^e and Σ_L^{ne} denote the respective restrictions of the problem to the excludable and non-excludable case.

The total welfare induced by a facility shared by a set of k communities is $kb - c$, $kb - (1+\delta)c$ or $kb - (1 + 2\delta)c$ depending on the set and the location choice. A facility should be built if the benefits exceed the costs when located at the optimal site. We assume $b < c$ (a single community does not build) and $nb > (1 + \delta)c$ (it is efficient to build a facility located efficiently in the grand coalition). Of course, since a facility is non-rival, it is efficient to build only one of them and share it among all communities and, to minimize the environmental damage, the facility is efficiently located at one extreme of the line.

The communities agree on a location of the facility and a way to share the net benefit of using it. It can be defined equivalently in terms of transfers among communities to pay for the total costs of the facility (including environmental damage). A global agreement is an agreement among all communities. In the excludable case, it is efficient to reach a global agreement and to build the facility at one end of the line. Assume, without loss of generality, that it is built at 1. The total welfare generated with such agreement is thus $nb - (1 + \delta)c$. A **global agreement** is a distribution $\boldsymbol{x} = (x_i)_{i=1,...,n}$ of the total welfare where x_i denotes community i's welfare with:

$$\sum_{i \in N} x_i = nb - (1 + \delta)c.$$

An agreement can also be defined in terms of budget-balanced transfers $\boldsymbol{t} = (t_i)_{i=1,...,n}$ with $\sum_{i=1}^{n} t_i = 0$. The host 1 enjoys a welfare of $x_1 = b - c + t_1$ where t_1 is the compensation received from hosting the facility. Its only neighbor 2 obtains $x_2 = b - \delta c + t_2$. It is thus paid t_2 for the nuisances. Each community $i = 3, ..., n$ gets $x_i = b + t_i$, thereby paying $-t_i$ to finance the compensations $t_1 + t_2$.

Communities can sign partial agreements to build and share a facility or no agreement at all. A partial agreement is an agreement among a subset of communities $S \subset N$.We need to figure out how much agents in a coalition S can achieve by signing their own agreement. The welfare that a coalition $S \subset N$ achieves depends not only on its own behavior but also on the behavior of the communities outside coalition S. By agreeing to build a facility close to the members of S, the communities outside S exert a negative externality on S which reduced the value of S. Technically, the NIMBY cooperative game exhibit *externalities*: the worth or value of a coalition S depends on the behavior of communities outside S. For instance, if the communities outside S agree to build a facility, a member of S that is neighbor to the facility might suffer from a *negative externality* then S would experience a welfare loss. Moreover, in the non-excludable case, all members of S can benefit from the facility built by an outside coalition then experiencing a *positive externality*. In this case, positive externalities are global while negative externalities are local. To summarize, the value of any coalition S is determined by the cooperative behavior of communities outside S and the related building strategies. The former concept is summarized by a partition $\mathcal{P}$ of N. The later should be a Nash equilibrium building strategy of the non-cooperative game defined by the partition $\mathcal{P}$. We successively detail each object.

Partial agreements for building facilities define a partition $\mathcal{P}$ of the set of communities N. Let us denote by $\mathbb{P}(N)$ the set of all partitions of N. Members of each $S \in \mathcal{P}$ are cooperating in their facility building strategy. They jointly decide on whether to build a facility in S and its location. Let us denote S's building strategy by its location $l \in S \cup \{0\}$ where $l = 0$ if no facility is built. In a partition $\mathcal{P} = \{S_1, ..., S_m\}$, each coalition of communities $S_i \in \mathcal{P}$ picks its best building strategy l_i given the building strategies of others coalitions S_j, $j \neq i$. A Nash equilibrium in partition $\mathcal{P}$ is a vector $\boldsymbol{l} = (l_1, ..., l_m)$ where each strategy l_i is a best reply to the strategies played by others l_{-i} for $i = 1, ..., m$. Let us denote by $\mathcal{E}(\mathcal{P})$ the set of Nash equilibria in the non-cooperative game induced by $\mathcal{P}$.

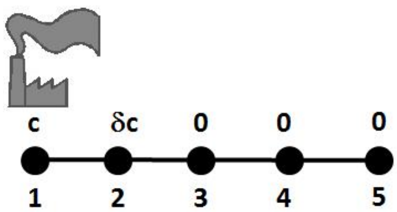


Figure 1: 5 agent case with location at 1 and no compensation

For a given partition, the non-cooperative building game may exhibit multiple equilibria as shown in the following example. First, communities can be indifferent between two locations of the facility. In the 5-community case represented on figure 1. Let us consider the partition $\mathcal{P} = \{\{1,3\}\{2,4,5\}\}$. For any building decision of coalition $\{2,4,5\}$, coalition $\{1,3\}$ would build a facility as long as $2b > c$. Yet it is indifferent between locating the facility in 1 and 3. However, such a decision impact the welfare of coalition $\{2,4,5\}$: the externality cost is δc only on agent 2 while it is also on agent 4 in the second case. Second, communities can coordinate on different Nash equilibria when benefits are non-excludable. For instance, the partition $\{\{1,3\};\{2,4,5\}\}$ exhibits three Nash equilibria $\mathcal{E}(\mathcal{P}) = \{(1,0);(3,0);(0,2)\}$ when $2b > c$. In the first two equilibria $(1,0)$ and $(3,0)$, coalition $\{2,4,5\}$ does not build because it benefits from the facility built by coalition $\{1,3\}$ who is indifferent between locating it at 1 or 3. Symmetrically, in the last case, coalition $\{1,3\}$ free-rides on the facility built by coalition $\{2,4,5\}$ which locates it at 2.

A partition function (Thrall and Lucas, 1963) is a function that assigns to every coalition S, partition $\mathcal{P}$ of N and equilibrium $\boldsymbol{l} \in \mathcal{E}(\mathcal{P})$ a real number. For any problem $\sigma \in \Sigma_L$, we denote such function by $v_\sigma(S,\mathcal{P},\boldsymbol{l})$. It is the welfare achieved by coalition S embedded in the partition $\mathcal{P}$ with the Nash equilibrium $\boldsymbol{l}$ of the non-cooperative building game defined by $\mathcal{P}$.

A coalition S in a partition $\mathcal{P}$ blocks a global agreement $\boldsymbol{x}$ if it can achieve a higher welfare in some Nash equilibrium $\boldsymbol{l} \in \mathcal{E}(\mathcal{P})$. A global agreement is in the **core** of the cooperative game associated to the NIMBY problem if it is not blocked by any coalition of N. Formally, the core is defined as follows.

Definition 1. *Let $\sigma \in \Sigma_L$. An agreement $\boldsymbol{x}$ is in the core $\mathcal{C}(\sigma)$ iff it satisfies the following core lower bounds:*

$$\sum_{i \in S} x_i \geq v_\sigma(S,\mathcal{P},\boldsymbol{l}) \tag{1}$$

for all $\mathcal{P} \in \mathbb{P}(N)$, all $S \in \mathcal{P}$ and all $\boldsymbol{l} \in \mathcal{E}(\mathcal{P})$.

The above definition of the core is somehow restrictive. The core lower bounds are defined for any potential configurations in terms of partial agreements (or partition) and equilibrium decisions. Yet some of those configurations might appear irrelevant because they are unlikely. Or they can be discarded because they do not bind core lower bound. To address those issues we need to investigate more the partition functions v_σ.

2.2 Restricted characteristic functions and restricted cores

A significant limit of our notion of a core appears with the following remark: if a blocking coalition is to form, it may anticipate that the instability its very formation demonstrate, may also occur among its own or among outside members. This is the criticism originally formulated by Ray (2007). As an illustration in the 5-agent case in figure 1, if agent 2 is not compensated for the externalities it would potentially bear, she gets $b - \delta c$. If $b - \delta c < 0$ and she expects outside members to stop cooperating after her withdrawal, she would do so as nobody would build in such a configuration what would result in a higher payoff 0. However, its leaving implies that the cost of the project perceived among outside members dwindles to c which favors the hypothesis that they would build the project anyway. Then, agent 2 may discard such expectation and stay in the project.

The intuition described above can be formally captured by restricting the partitions considered to compute the value of a coalition. In a problem $\sigma \in \Sigma_L$, we will define two characteristic functions based on specific restrictions over possible partitions[6]. First, the non-cooperative or collapse in outside cooperation (CIOC) characteristic function v_σ^c defines the highest welfare a coalition can achieve if non-members fail to cooperate: $v_\sigma^c(S) = v_\sigma(S, \{S, \{i\}_{i \in N \backslash S}\}, \boldsymbol{l})$ for any $S \subset N$. When deciding to deviate from the global agreement, a coalition S considers its non-cooperative or CIOC value $v_\sigma^c(S)$ if it expects that the communities outside S will not cooperate to build facilities. Note that since communities outside S do not build, the equilibrium building strategy $\boldsymbol{l}$ boils down to a single facility located inside S. Although they might be multiple optimal localization of the facility in S, all those localizations lead to a unique value $v_\sigma^c(S)$. Such approach corresponds to the $\gamma-$core based on the idea that outside members play their individual best reply strategies (Chander and Tulkens, 1997).

Second, the cooperative or Rational hostile outside cooperation (RHOC) characteristic function v_σ^r defines the welfare a coalition can achieve if all the non-members cooperate to sign a partial agreement. The partition expected by coalition S is thus $\{S, N\backslash S\}$. Moreover, coalition S expect that if coalition $N\backslash S$ builds a facility, it will locate it at the worst place for S providing that it is a Nash equilibrium (so that this strategy is credible). Formally, the RHOC characteristic function for an arbitrary coalition S is defined by $v_\sigma^r(S) = min_{\boldsymbol{l} \in \mathcal{E}(\{S, N\backslash S\})} v_j(S, \{S, N\backslash S\}, \boldsymbol{l})$.

We now establish some properties of the partition function and the characteristic functions in the excludable case.

[6] We develop a more general definition in the proof of proposition 3 in appendix.

Proposition 1. *For any problem $\sigma \in \Sigma_L^e$, for any $S \subset N$, $\mathcal{P} \ni S$ and $\boldsymbol{l} \in \mathcal{E}(\mathcal{P})$,*

(i) $v_\sigma^c(S) \geq v_\sigma(S, \mathcal{P}, \boldsymbol{l})$

(ii) We might have $v_\sigma^r(S) > v_\sigma(S, \mathcal{P}, \boldsymbol{l})$ or $v_\sigma^r(S) < v_\sigma(S, \mathcal{P}, \boldsymbol{l})$ depending on $\mathcal{P}$

(iii) We might have $v_\sigma(S, \mathcal{P}, \boldsymbol{l}) > v_\sigma(S, \mathcal{P}', \boldsymbol{l})$ or $v_\sigma(S, \mathcal{P}, \boldsymbol{l}) < v_\sigma(S, \mathcal{P}', \boldsymbol{l})$ when $\mathcal{P}'$ is a finer partition of N including S.

The proof of (i) is provided in the appendix.

First, with excludable facilities, the CIOC value is the highest possible value that a coalition can obtain by deviating from the global agreement. It is because, in the excludable case, a coalition gets only the nuisances (if any) and not the benefits from facilities built by outsiders. So the best that can happen for a coalition is that the outsiders do not build any facility which holds under CIOC. In contrast, in the non-excludable case, the CIOC value might be lower than the value based on another partition because a coalition can benefit from the facility built by others.

Second, the RHOC value can be lower or higher than the value with other partitions. An illustration of such result can be provided in the 5 agent case represented in figure 1 with $b < c \leq b + \delta c$[7]: the lowest value for coalition $S = \{2\}$ would be achieved with $\mathcal{P} = \{\{2\}; \{1,4\}; \{3,5\}\}$ and Nash equilibrium $(0,1,3)$ because S would undergo the externalities linked to 2 projects instead of a single one in the case $\mathcal{P} = \{\{2\}; \{1,4,3,5\}\}$. This remark emphasizes the fact that full cooperation of outside members is not the situation which would favor cooperation the most. It also emphasizes that, in the excludable case, despite only negative externalities can arise across coalitions, the cooperative game features ambiguous externalities from an increase in cooperation among outside members.

Third, a coalition does not necessarily benefit from the union of two other coalitions. For instance, S could experience a negative externality when two former non-building coalitions merge and start building at S border. This would be the case in the 5 agent case represented in figure 1 for $2b < c \leq 4c$ and $\mathcal{P} = \{\{2\}; \{1,4\}; \{3,5\}\}$. The merging to $\mathcal{P} = \{\{2\}; \{1,4,3,5\}\}$ would induce the construction of a project at 1 and make the worth of $\{2\}$ decrease. As a consequence, cooperative game externalities in the sense of De Clippel and Serrano (2008) can be positive or negative.

The CIOC and RHOC characteristic functions define the corresponding core concepts. A welfare distribution $\boldsymbol{x}$ is in the CIOC core (resp. RHOC core) if it satisfies the core lower bounds defined by the CIOC (resp. RHOC) characteristic function for all coalitions $S \subseteq N$.

[7] Such assumption implies a coalition facing cost c would build if and only only if it gathers more than 2 agents.

Definition 2. *Let $\sigma \in \Sigma_L$ An agreement $\boldsymbol{x}$ is in the CIOC (resp. RHOC) core $\mathcal{C}^c_\sigma(\sigma)$ (resp. $\mathcal{C}^r_\sigma$) if it satisfies the following core lower bounds:*

$$\sum_{i \in S} x_i \geq v^h_\sigma(S) \tag{2}$$

with $h = c$ (resp. $h = r$) for all $S \subseteq N$.

The (unrestricted) core is always included in restricted cores. As discussed previously, the restricted core allows us to capture two ideas. First, not all possibilities may be credible in the evaluation of the worth of a coalition what justifies to discard them. Second, our definition of the (unrestricted) core assumes that a coalition is extremely optimistic in the sense that among the remaining credible behaviors, it selects the most favorable. Restricted core by allowing us to discard some most favorable outcomes, allows us to consider more permissive cores.

3 The core

3.1 The central result

The first result provides necessary and sufficient condition for the non-emptiness of the (unrestricted) core for excludable facilities.

Proposition 2. *Let $\sigma \in \Sigma_L^e$ such that $n \geq 4$ and $2b \leq c \leq (n-2)b$.*
The core $\mathcal{C}(\sigma)$ is empty if and only if one of the following conditions is met:

1. $\delta > \frac{b}{c}$
2. $\delta > \bar{\delta}(n)$ *where* $\bar{\delta}(n) = \begin{cases} \frac{2}{n-2} & \text{if } n = 4k, k \in \mathbb{N} \\ \frac{2}{n-1} & \text{if } n = 4k+1, k \in \mathbb{N} \\ \frac{2}{n} & \text{if } n = 4k+2, k \in \mathbb{N} \\ \frac{2}{n-1} & \text{if } n = 4k+3, k \in \mathbb{N} \end{cases}$

The proof is developed in appendix. The first condition comes from binding individual rationality constraints while the second captures the following: as soon as nuisance are spread across the borders of the host community, coalitions can save on the cost of the nuisance. Then, the total cost of the project in the grand coalition should be shared in a way such that everyone would bear some cost to the extent of the nuisance they would bear would the project be located at their neighbors. However, because the extent of the nuisance is not directly linked to the cost of the project, this latter may come to be much lower than the payments that should be imposed to avoid the opportunity for some coalitions to externalize part of the costs. In such a case, the core is empty. This problem becomes sharper as the number of agents increases. Such instability can be seen as a version of the garbage game of Shapley and Shubik (1969).

The reason why the domain of (b, c) on which Proposition 2 holds is restricted comes from the fact that construction is endogenous in our model. Such hypothesis assumes that small coalitions would not impact the larger coalition formed by excluding some agents while a coalition formed by excluding all the neighbors of its host is still large enough to beneficially build a project. Such hypothesis is reasonable as long as agents are not too connected what is the case in a linear setting.

1. When $c < 2b$, two-agent coalitions also build a project and we can check that no saving would be made by excluding them. Formally, the second condition is changed to $\delta > 1$ so the identified condition remains necessary but is no more sufficient.

2. When $c > (n-2)b$ coalitions which exclude two agents does not build anymore, and their rationality constraints are met if and only if the first condition is met. This relaxes the second condition and, again, the identified condition remains necessary but is no more sufficient.

Overall, the identified condition holds in setting where there is an asymmetry between small coalitions which would not credibly build a project what creates a possibility for larger coalitions to save on the cost of the project. Such coalitions, however should remain large enough to implement the project. It is in such situation that conditions for non-emptiness are the most stringent.

3.2 Restricted cores

A usual critic for our notion of the core is that a coalition which deviates may not consider all possible deviations among outside members (or, equivalently, may not be optimistic to the point of taking its decision regarding the best outcome only). In such a case, the core could be larger. It is then interesting to question how our emptiness result is modified for the RHOC restriction we defined previously. More especially, we would like to know whether the conditions identified in proposition 2 still hold as necessary conditions for non-emptiness. The following proposition establishes the robustness of one of them to the RHOC restriction on the core lower bounds.

Proposition 3. *Let $\sigma \in \Sigma_L^e$ such that $n \geq 4$ and $2b \leq c \leq (n-2)b$.*
The RHOC-core $\mathcal{C}^r(\sigma)$ is empty if $\delta > \bar{\delta}(n)$

The idea of the proof is the following: when the condition is not met, we know there exists a coalition of size (n-2) or (n-1) which benefits to form under CIOC. When $2b < c$, two-agent coalitions would never rationally build. Then, the value of coalitions of size (n-2) or (n-1) is the same *whatever the expected cooperative behavior of outside members, provided they act rationally.* Then a profitable blocking coalition under CIOC would also be profitable under any cooperative behavior of outside members, for instance, under the RHOC restriction. A formal proof is provided in appendix allowing for any arbitrary restriction.

This result calls for three additional comments:

1. Only the second condition of Proposition 2 is robust to restrictions over the behavior of outside members. Under RHOC, the first condition in Proposition 2 which stems from binding individual rationality constraints is relaxed.

2. As established in the proof, the proposition can actually be generalized to any restriction of expectations over the partitions. However, the fact that 2-agent coalitions, which are too small to benefit from the project, actually do not build is essential for our results. Then, our result is robust to expectations over rational behaviors of outside members only (formally, we restrict our attention to the Nash equilibria arising in a given partition) and does not hold for restrictions of expectations over irrational behaviors. Indeed, it is straightforward to see that the α-core defined by the restriction to the worst possible case among all possible partitions *and strategies* is always non-empty as soon as we allow coalitions to build more than a single facility.[8]

3. The proposition we emphasize only tackles the issue of the credibility of outside members' behavior. Another consistency requirement would be that only *credible* blocking coalition should be considered in the definition of the core. Yet, we note that the binding blocking coalition which lead to the proposition are coalitions of size (n-2) or (n-1). In the excludable case, such coalitions include an isolated agent so they are credible in the sense that they don't feature the instability which led them to form.

3.3 A note on the non-excludable case

When benefits are non-excludable, cooperation may even be more compromised due to the free-rider problem. The following proposition shows that free-riding may be an additional source of concerns for restricted cores:

Proposition 4. *Let $\sigma^e = (N, b, c, \delta, e)$ and $\sigma^{ne} = (N, b, c, \delta, ne)$ be two NIMBY cooperative games with excludable and non-excludable facilities respectively. We have,*

- $\mathcal{C}^c(\sigma^{ne}) = \mathcal{C}^c(\sigma^e)$
- $\mathcal{C}^r(\sigma^{ne}) \subseteq \mathcal{C}^r(\sigma^e)$

Free-riding does not appear under CIOC as no blocking coalition can rely on outside members' building hence the core remains the same. Under the RHOC hypothesis however, the blocking coalition can take advantage of outside members' project. Because rationality constraints with excludable benefits can only be more stringent as compared to the constraints with excludable facilities, we have $\mathcal{C}^r(\sigma) \subseteq \mathcal{C}^r(\sigma)$. A formal proof is provided in appendix.

[8]This comment echoes to a discussion of the garbage game by Laffont (1977).

The results we emphasize in the RHOC case actually hold for any restriction of expectations over partitions. Unsurprisingly, when possible, free-riding narrows the core. The following proposition shows that it may actually lead to an empty core under rather general conditions on parameters in the RHOC case:

Proposition 5. *Let* $\sigma^{ne} = (N, b, c, \delta, ne) \in \Sigma_L^{ne}$ *and* $k_1 \in \mathbb{N}$ *such that* $(k_1 - 1)b < c + \delta c \leq k_1 b$. *We have:* $min(k_1, n - k_1) > 3 \Rightarrow \mathcal{C}^r(\sigma^{ne}) = \emptyset$

Proof. cf. Appendix □

It is worth noting that free-riding occurs due to the incentive for small coalitions to withdraw while our instability arose due to the incentives of large coalitions to exclude the potential neighbors of the facility. While the former would reasonably expect their complementary to build, the latter may not. Then, in the non-excludable case, both instability can be expected to apply independently.

4 A general NIMBY cooperative game

4.1 The setting

We propose here to extend the results we got over a line to a more general framework: let N be a set of n agents, $\boldsymbol{b} = (b_i)_{i\in N}$ be individual benefits from access to a project, c be the cost of the project when located at any agent's, $\boldsymbol{T} = (\tau_{ij})_{(i,j)\in N^2} \in \mathcal{M}_n(\mathbb{R}_+)$ be a matrix of closeness between agents (taking into account geographical characteristics such as distance, streams, hills, etc.) and $\delta \in \mathbb{R}_+$ the geographical extent of the impact of the project such that, when the project is located at i's, agent j bears a cost $c_{ij} = \delta\tau_{ij}c$. We focus on excludable projects ($q = e$).

A problem is now denoted by $\sigma = (N, \boldsymbol{b}, c, \boldsymbol{T}, \delta)$ and Σ is the set of such problems[9]. Two restricted class of problem are of special interest. The restriction to simple graphs, first, correspond problems in which D is the adjacency matrix of a simple graph i.e. a symmetric matrix such that $\forall (i, j) \in N^2, \tau_{ij} \in \{0, 1\}$. With such representation, we allow for the project located at some agents' to negatively affect several other agents to the same extent. We denote by Σ_G the restriction of Σ to such problems. Finally, if we restrict the graph to the line, we end up with the class of problems Σ_L discussed previously.

In the general case, the optimal location of the project at $S \subseteq N$ is solution to $min_{j\in S} \sum_{i\in S} \tau_{ij}$. We denote by $\underline{\tau}(S)$ the solution to such program and $h(S)$ an optimal site in S. Within a partition $\mathcal{P} \in \mathbb{P}(N)$, and Nash equilibrium of the game $\boldsymbol{l} \in \mathcal{E}(\mathcal{P})$, we can compute the value of any coalition $S \in \mathcal{P}$ given the cooperative behavior of outside members. We will denote it by $v_\sigma(S, \mathcal{P}, \boldsymbol{l})$. The core can be defined from such function as previously:

Definition 3. *Let $\sigma \in \Sigma$. The core $\mathcal{C}(\sigma)$ is the set of agreements $\boldsymbol{x}$ which satisfy the following core lower bounds:*

$$\forall \mathcal{P} \in \mathbb{P}(N)\ \forall S \in \mathcal{P}\ \forall \boldsymbol{l} \in \mathcal{E}(\mathcal{P})\ \sum_{i\in S} x_i \geq v_\sigma(S, \mathcal{P}, \boldsymbol{l}) \tag{3}$$

Restricted value functions v_σ^c and v_σ^r are defined in a similar way as previously.

[9] Such representation is not the most general as it assumes that municipalities are points in the space what prevent any possibility for coalitions to choose a location within i's boundary in order to minimize the pollution which falls within its boundaries and costs are uniform. However, it generalizes the previous argument so we consider worth to develop it

4.2 A necessary condition for non-emptiness of the core

In a similar way as in proposition 2, a necessary condition emerges out of the temptation to exclude some neighbors in order to externalize part of the costs.

Proposition 6. *For any problem* $\sigma = (N, \boldsymbol{b}, c, \boldsymbol{T}, \delta)$*, there exists a threshold* $\bar{\tau}(\boldsymbol{T})$ *such that*

$$\delta > \bar{\tau}(\boldsymbol{T}) \textit{ implies } \mathcal{C}(\sigma) = \emptyset$$

The proof is provided in appendix. The computation of $\bar{\tau}(\boldsymbol{T})$ is an hard combinatorial problem in general and few general results can be obtained. Figure 2 illustrates what higher bound on δ would come as a necessary condition for non-emptiness of the core on several simple graphs with 6 agents.

4.3 Comparative statics

Such problem yields interesting comparative statics. First, we derive the counterintuitive property that the more the neighbors are affected by the project at each location, the larger the set of core allocations[10]:

Proposition 7. *Let* $\sigma = (N, \boldsymbol{b}, c, \boldsymbol{T}, \delta)$ *and* $\sigma' = (N, \boldsymbol{b}, c, \boldsymbol{T'}, \delta)$ *such that* $\boldsymbol{T'} \geq \boldsymbol{T}$*, we have:*
$min_j(\sum_i \tau_{ij}) = min_j(\sum_i \tau'_{ij}) \Rightarrow \mathcal{C}(\sigma) \subseteq \mathcal{C}(\sigma')$

Proof. cf. Appendix □

Figure 2 provides an illustration of proposition 7 for simple graphs: on graphs A to F and K to Q, we observe that, when a link is added while keeping the minimum degree constant, requirements on δ can only be relaxed. Some additional examples suggest that the structure of the graph does influence quite a lot the extent of the problem (measured by the stringency of the condition on δ). The lax condition obtained for the complete graph S can easily be extended to all complete graphs[11] what emphasizes that our argument mainly stands for local pollutions.

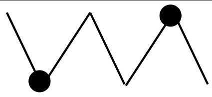
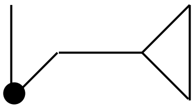
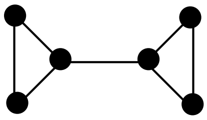
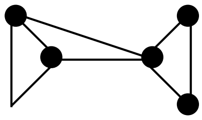
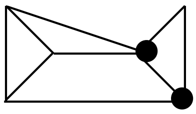
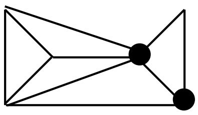
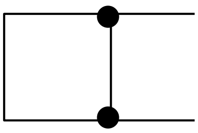
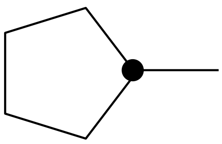
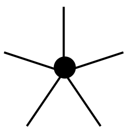
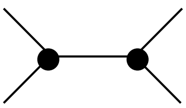
Another comparative statics could be obtained related to the benefits in this framework. The following proposition states that an increase in the benefits leads the set of core allocations to expand:

Proposition 8. *Let* $\sigma = (N, \boldsymbol{b}, c, \boldsymbol{T}, \delta)$ *and* $\sigma' = (N, \boldsymbol{b'}, c, \boldsymbol{T}, \delta)$ *such that* $\boldsymbol{b} \leq \boldsymbol{b'}$*, we have:*

$$\mathcal{C}(\sigma) \neq \emptyset \Rightarrow \mathcal{C}(\sigma') \neq \emptyset$$

[10] $\forall (T, T') \in \mathcal{M}_n(\mathbb{R}_+) \times \mathcal{M}_n(\mathbb{R}_+)$, we write $T' \geq T$ if and only if $\forall (i,j) \in N^2\ \tau'_{ij} \geq \tau_{ij}$

[11] The reader could easily check that in core allocations, agents should all pay δc.

Graph T	$\underline{\tau}(N)$	$\overline{\tau}(T)$
A 	1	1/3
B 	1	1/2
C 	2	1/4
D 	2	1/3
E 	2	1/2
F 	2	1
G 	1	1
H 	1	1/2
I 	1	1
J 	1	1

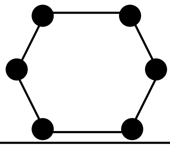
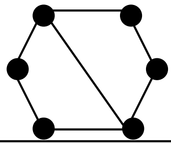
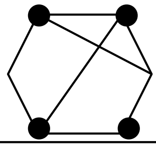
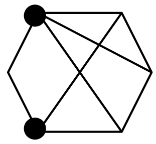
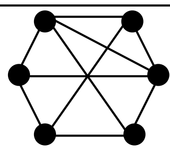
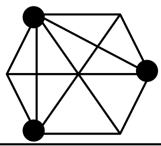
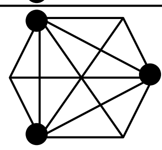
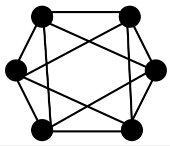
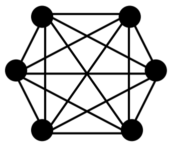
Graph T	$\underline{\tau}(N)$	$\overline{\tau}(T)$
K 	2	1/4
L 	2	1/4
M 	2	1/3
N 	2	1/2
O 	3	1/3
P 	3	1/2
Q 	3	1
R 	4	1/2
S 	5	1

Figure 2: Critical value of δ for non-emptiness of the core for different graphs with 6 agents. We recall a necessary condition for non-emptiness of the core is $\delta \leq \bar{\tau}(\boldsymbol{T})$. The algorithm used to compute such bounds is provided in appendix.

Proof. cf. Appendix □

Here, the benefits play a role in the rationality constraint of non-building coalitions: for such coalitions, the more the benefits, the more costly it is to block a global agreement as it leads to abandon the project under CIOC.

Interestingly enough concerning the relevance of the core concept as a positive one, proposition 7 leads to original empirical predictions. For instance, it suggests that, when controlling by the characteristics of the optimal sites, the density of a zone could have a positive effect on the size of the core (and maybe then the outcome of negotiations) by limiting the opportunity to dismiss neighbors concerns. This suggests a way to test our theoretical point.

5 A positive interpretation

On the positive side, we are lead to doubt the possibility of agreeing on the localization and the compensation scheme that should come with locally undesirable projects as soon as there exists significant externalities across sovereign communities. Provided the insights we got "like augurs divining the future by minute inspection of the entrails of a goose"[12] actually capture an actual feature of negotiations around NIMBY issues[13], we can look for ways to get out of such an emptiness result. The following part focuses on this positive interpretation.

5.1 Cooperation under the polluter-pay principle

The polluter-pay principle imposes that the costs of pollution should be borne by the entity which causes pollution. In our context, such principle would amount to impose the project implementer to systematically compensate the victims for the externality they bear due to the project, whether or not they belong to the host community[14]. Then, in the excludable case, we note the behavior of outside members does not matter because any externality they impose would be internalized. Formally a specific value function v_σ^p and an associated notion of a core $\mathcal{C}^p(\sigma)$ can be defined (cf. appendix). Such principle alleviates the problem as shown by the following statement:

Proposition 9. *Let $\sigma \in \Sigma_L$, $\mathcal{C}^p(\sigma) \neq \emptyset$*

Proof. In appendix, we check that the equal sharing rule always lies in the core. $\square$

Such observation concurs with the Aivazen and Callen's (1981) criticism of Coase theorem: because cooperation is more likely to emerge under some allocation of property rights, such allocation may matter, even for a regulator concerned only with efficiency.

5.2 Investment in protection and norms

Such possibility results from the observation that investing in protection may have the effect of increasing the cost of the project while decreasing its externalities which is formally equivalent to a

[12]This image was used by Coase (1981) in reply to the Aivazian and Callen (1981) criticism which stands in a zero transaction cost world. Actually, such instability may be less constraining in a world with transaction costs. Indeed, adding a cost t of forming a blocking coalition would higher by such amount the critical value of δc above which we established core allocations would not exist.

[13]Proposition 7 suggests an empirical test of such interpretation.

[14]Note that such requirement is weaker than endowing all polluted communities with a clear veto power.

decrease in δ hence making more likely for core allocations to exist. This comment could justify the use of stringent standards to foster cooperation.

Consider a problem in Σ_L where coalitions' strategy set is enriched so that a coalition can actually choose between a polluting project δ^+ at lower cost c^- and a clean project δ^- at a higher cost c^+ ($\delta^- < \delta^+$,$c^- < c^+$). Then, it is straightforward to see that a the necessary condition for non-emptiness of the core discussed previously is $\delta^+ \leq \bar{\delta}(n)$ which only involves δ^+. Then, the problem seems rather alleviated by stringent norms (which discard high δ from the strategy sets) rather than by the mere existence of cleaner technologies[15].

5.3 Subsidies and/or Penalties

The source of instability comes from the possibility for coalitions to save on the cost of the project by excluding some agents. Then, subsidizing the project in the grand coalition or fining some deviating coalitions would alleviate the problem; How much is needed? The following proposition characterizes the least core value i.e. the lowest amount to grant to the grand coalition or the least penalty that have to be charged for defection[16] so that the core is non-empty for problems in Σ_L:

Proposition 10. *Let $\sigma \in \Sigma_L^e$, the least core value is $(\delta - \bar{\delta}(N))c$*

The proof is provided in appendix. Unsurprisingly, the least core value is less than δc and increases with the discrepancy between δ and the higher bounds previously defined.

[15] We however recall that c encompasses the pollution costs at the host's so coalition have an incentive to invest in cleaner technology but to an inefficiently low level.

[16] Although not exactly equivalent, Stearns (1993) reports an interesting example of an attempt to use fines in the context of a locally undesirable facility: "In, New York, the Court held the take-title provisions of the Low-Level Radioactive Waste Policy Amendment Act of 1985 unconstitutional because the amendments penalized states that failed either to enter into regional pacts for low-level radioactive waste disposal or otherwise to become self-sufficient within a preset time frame." [18]. This fine was eventually judged as an unconstitutional infringement of the federal state over local sovereignty.

6 Conclusion

In this paper, we highlight a specific obstacle to cooperation for the provision of a locally undesirable facility. We find conditions under which core agreements fail to exist in the context of inter-community externalities. The rationale for such a result is the following: if the nuisance on neighboring communities is large, the benefits of excluding some neighbors are high. Then, no sharing rule will prevent all coalitions to contemplate building a project on its own while saving on the cost of the project by leaving the neighbors uncompensated.

This result has both normative and positive implications. Normatively, our emptiness results leads to question the relevance of core solutions for projects with significant negative externalities across communities. On the positive side, such a result suggests that cooperation for the provision of a locally undesirable facilities may fail to emerge. However, some arguments may lead us to question the relevance of such interpretation. First, we emphasize that cooperation may fail to emerge but we are not able to predict any specific outcome. Second, such instability is less likely in a world with transaction costs. Third, we do not consider the counterbalancing force of the transportation costs we discussed in the UFLP. It may also ease the problem. These observations lead us to question the positive interpretation. However, its theoretical robustness appeals to further investigations. It would be interesting to test the empirical relevance of our argument. Our work may prove useful to provide original implications which could be empirically tested on a reduced form model. For instance, proposition 7 suggests that, when controlling by the characteristics of the optimal sites, the density of a zone could have a positive effect on the size of the core (and maybe then the outcome of negotiations) by limiting the opportunity to dismiss neighbors concerns.

References

[1] Varouj A. Aivazian and Jeffrey L. Callen. The coase theorem and the empty core. *Journal of Law and Economics*, 24(1):175–181, 1981.

[2] Scott Barrett. Self-enforcing international agreements. *Oxford economic papers*, 46:878–894, 1994.

[3] Carlo Carraro and Domenico Siniscalco. Strategies for the international protection of the environment. *Journal of Public Economics*, 52:309–328, 1993.

[4] Parkash Chander and Henry Tulkens. The core of an economy with multilateral environmental externalities. *International Journal of Game Theory*, 26:379–401, 1997.

[5] Ronald H. Coase. The coase theorem and the empty core: A comment. *Journal of Law and Economics*, 24(1):183–187, 1981.

[6] Geoffroy De Clippel and Roberto Serrano. Marginal contributions and externalities in the value. *Econometrica*, 76(6):1413–1436, 2008.

[7] Ray Debraj and Rajiv Vohra. Coalitional power and public goods. *Journal of Political Economy*, 109(6):1355–1384, 2001.

[8] Michel X. Goemans and Martin Skutella. Cooperative facility location games. *Journal of Algorithms*, 50:194–214, 2004.

[9] Eric Helland and Andrew B. Whitfort. Pollution incidence and political juridiction: Evidence from the tri. *Journal of Environmental Economics and Management*, 46:403–424, 2003.

[10] Daniel E. Ingberman. Siting noxious facilities: Are market efficient? *Journal of Environmental Economics and Management*, 29:S20–S33, 1995.

[11] Jean-Jacques Laffont. *Effets externes et théorie économique.* collection Monographies du Séminaire d'Econométrie, 1977.

[12] Raul P. Lejano and Climis A. Davos. Siting noxious facilities with victim compensation: n-person games under transferable utility. *Socio-Economic Planning Sciences*, 35:109–124, 2001.

[13] Jirí Matousek and Berndt Gärtner. *Understanding and using linear programming.* Springer, 2007.

[14] Hervé Moulin. *Fair division and collective welfare.* MIT press, 2003.

[15] Bezalel Peleg and Peter Sudhölter. *Introduction to the theory of cooperative games.* Kluwer Academic Publishers, 2003.

[16] Debraj Ray. *A Game-Theoretic Perspective on Coalition Formation.* Oxford University Press, 2007.

[17] Lloyd S. Shapley and Martin Shubik. On the core of an economic system with externalities. *The American Economic Review*, 4(1):678–684, 1969.

[18] Maxwell L. Stearns. The misguided renaissance of social choice. *The Yale Law Journal*, 103:1219–1292, 1993.

[19] R. M. Thrall and W. F. Lucas. N-person games in partition function form. *Naval Research Logistics Quarterly*, 10(1):281–298, 1963.

[20] Sang-Seung Yi. Stable coalition structures with externalities. *Games and Economic Behavior*, 20:201–237, 1997.

Proof of Proposition 1

Let $\sigma = (N, b, c, \delta, e) \in \Sigma_L^e$.

Let us define k_0, k_1 and k_2 as the natural integers such that:

$(k_0 - 1)b < c \leq k_0 b$

$(k_1 - 1)b < c + \delta c \leq k_1 b$

$(k_2 - 1)b < c + 2\delta c \leq k_2 b$

These numbers corresponds to the minimum number of agents a coalition should gather to implement the project given the cost it fronts. We can naturally rank the three numbers: $k_0 \leq k_1 \leq k_2$.

The CIOC characteristic function for excludable facility is defined as:

$$v_\sigma^c(S) = \begin{cases} |S|b - c - \delta c & \text{if } |S| \geq k_2 \wedge S \notin S_1 \\ |S|b - c & \text{if } |S| \geq k_1 \wedge S \in S_1 \\ 0 & \text{if } |S| < k_1 \vee (|S| < k_2 \wedge S \notin S_1) \end{cases}$$

where $S_1 = \{S / \exists i \in S i - 1 \notin S \wedge i + 1 \notin S\}$ is the set of blocking coalitions with costs c.[17]. Such value only includes the benefits due to S own project. As soon as we consider other possible partitions, because benefits are excludable, S can only face higher costs so its worth would always be (weakly) lower.

Proof of Proposition 2

Let $\sigma = (N, b, c, \delta, e) \in \Sigma_L^e$ such that $n \geq 4$ and $2b \leq c \leq (n-2)b$. From proposition 1 we get the identity of $\mathcal{C}(\sigma)$ and $\mathcal{C}_\sigma^c(\sigma)$ so we can focus on the value function v_σ^c defined above. Let $\boldsymbol{x} \in \mathcal{C}_\sigma^c(\sigma)$ then:

$$\forall S \subset N, \sum_{i \in S} x_i \geq v_\sigma^c(S) \tag{4}$$

$$\sum_{i \in N} x_i = v_\sigma^c(N) \tag{5}$$

Simplification of the system of inequalities 4

1 - Rationality of building coalitions

(n-1)-coalitional rationality constraints

[17] By convention, $\forall S \subseteq N, 0 \notin S$ and $n + 1 \notin S$

Let $j \in N$. Given that $2b \leq c \leq (n-2)b$, the constraint associated to the rationality of coalition $N\backslash\{j\}$ writes:

$$\sum_{i \in N\backslash\{j\}} x_i \geq v_\sigma^c(N\backslash\{j\}) = nb - c - \delta c \,\mathbf{1}\{N\backslash\{j\} \notin S_1\}$$

Besides, we know from equation 5 that $x_j = nb - c - \delta c - \sum_{i \in N\backslash\{j\}} x_i$. Therefore

$$x_j \leq b - \delta c \,\mathbf{1}\{N\backslash\{j\} \in S_1\}$$

which can be restated:

$$x_i \leq \begin{cases} b - \delta c \text{ if } i = 2 \text{ or } i = n-1 \\ b \text{ otherwise} \end{cases}$$

This constraint shows there exists strong pressure for the potential neighbors of efficiently located facilities not to be compensated: in core allocations, both agents 2 and n-1 should at least contribute to the extent of the externality of such facility whether or not it is located close to them. Besides, no agent could be subsidized[18].

(n-2)-coalitional rationality constraints

The same reasoning leads to the following requirement $\forall (i,j) \in N^2$:

$$(x_i + x_j) \leq 2b - \delta c \,\mathbf{1}\{N\backslash\{i,j\} \in S_1\}$$

Which is equivalent to $\forall (i,j) \in N^2/i < j$:

$$(x_i + x_j) \leq \begin{cases} 2b - \delta c \text{ if } j = i+2 \\ 2b \text{ otherwise} \end{cases}$$

This constraint traduces the strong incentives to exclude the neighbors of the facility. Because any two individuals separated by an agent i may allow the grand coalition to ignore the externality δc, we should make sure that their contribution is at least of this amount. Otherwise, it becomes beneficial for the other agents to exclude both individuals and build the project in between them.

Other coalitional rationality constraints

We now show that if $\boldsymbol{x}$ meets (n-1) and (n-2)-coalitional rationality constraints, then $\boldsymbol{x}$ also meets all other constraints for building coalitions. Let's assume $\boldsymbol{x}$ meets both (n-1) and (n-2)-coalitional rationality constraints and let S be a coalition such that $|S| < n-2$.

[18] This result would change would the removal of the optimal site increase the costs of the project. Then the host could be subsidized in core allocations.

1. If $S \in S_1$, we know $\exists \bar{S} \in S_1 / |\bar{S}| = n-2 \wedge S \subset \bar{S}$. Because $\boldsymbol{x}$ meets both (n-1) and (n-2)-coalitional rationality constraints, we know:

$$\begin{cases} \sum_{i \in \bar{S}} x_i \geq |S|b - c & \\ -x_i \geq -b & \forall i \in \bar{S} \backslash S \end{cases}$$

   Where the first line comes from the rationality of $\bar{S}$ and the second from accurate (n-1)-coalitional rationality constraints. Adding these inequations, we get the following:

$$\sum_{i \in S} x_i \geq |S|b - c = v_\sigma^c(S)$$

   So $\boldsymbol{x}$ meets coalitional rationality constraints for all building coalitions S in S_1.

2. If $S \notin S_1$ then $\exists \bar{S} \in 2^N / |\bar{S}| = n-2 \wedge S \subset \bar{S}$. As previously, we know:

$$\begin{cases} \sum_{i \in \bar{S}} x_i \geq |S|b - c - \delta c & \\ -x_i \geq -b & \forall i \in \bar{S} \backslash S \end{cases}$$

   And then:

$$\sum_{i \in S} x_i \geq |S|b - c - \delta c = v_\sigma^c(S)$$

   Along with the previous result, this establishes that $\underline{x}$ meets coalitional rationality constraint for all building coalitions.

2 - Rationality of non-building coalitions

In the following we establish that individual rationality constraints imply others constraints for non-building coalitions. Indeed, assume that individual rationality constraints are met for all agents:

$$\forall i \in N, x_i \geq 0$$

Then, summing up individual rationality constraints, we get:

$$\forall S \in 2^N, \sum_{i \in S} x_i \geq 0$$

This establishes the property.

Derivation of the conditions

A linear program

Previous results allow us to simplify the system of constraints defining the core. The linear programming problem (L) defines the maximum welfare which can be allocated in core allocations[19]:

$$max_{\boldsymbol{x}} \sum_{i \in N} x_i \text{ subject to } \begin{cases} x_2 \leq b - \delta c \\ \forall j \in [1, n-2], x_j + x_{j+2} \leq 2b - \delta c \\ x_{n-1} \leq b - \delta c \\ \forall i \in N, x_i \leq b \\ \forall i \in N, x_i \geq 0 \end{cases}$$

which we restate in matrix form and introducing slack variables $\boldsymbol{s} = (s_i)_{i=1\ldots n}$ for the n first constraints and $\boldsymbol{s'} = (s'_i)_{i=1\ldots n}$ for the n following:

$$max_{\boldsymbol{X}} \{z = \boldsymbol{e}_n^T \boldsymbol{x} : \boldsymbol{M}\boldsymbol{X} = \boldsymbol{b}, \boldsymbol{X} \geq 0\}$$

where $\boldsymbol{M} = \begin{pmatrix} L_n & I_n & 0 \\ I_n & 0 & I_n \end{pmatrix}$, $\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{s} \\ \boldsymbol{s'} \end{pmatrix}$ and $\boldsymbol{b} = b \begin{pmatrix} 1 \\ 2 \\ \vdots \\ 2 \\ 1 \\ \boldsymbol{e_n} \end{pmatrix} - \delta c \begin{pmatrix} \boldsymbol{e_n} \\ \boldsymbol{0_n} \end{pmatrix}$ where $\boldsymbol{I_n}$ is the identity matrix and $\boldsymbol{L_n}$ is the $n \times n$ matrix:

$$\boldsymbol{L_n} = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}$$

[19] Strictly speaking, this program characterizes the dual core (sometimes called the anti-core) of the dual of our cooperative game. It is straightforward to check that this set coincides with the core of the game we consider.

The problem is feasible if and only if $b - \delta c \geq 0$ (then the vector $\boldsymbol{x} = \mathbf{0}$ is a feasible solution). Besides, it is bounded (we have, for instance, $\sum_{i=1}^{n} x_i \leq nb$) so there exists a basic feasible optimal solution. In the following, we adopt the notation of Matousek and Gärtner (2007). We will have to treat two differents cases. In each case, we show a basic solution associated to list of variables B_n and consider the simplex tableau associated:

$$\mathcal{T}(B_n) = \frac{\boldsymbol{x_B} = \boldsymbol{A_B^{-1}} \boldsymbol{b} - \boldsymbol{A_B^{-1}} \boldsymbol{A_N} \boldsymbol{x_N}}{z = z_B + \boldsymbol{r^T} \boldsymbol{x_N}}$$

where $\boldsymbol{r^T} = \boldsymbol{c_N^T} - \boldsymbol{c_B^T} \boldsymbol{A_B^{-1}} \boldsymbol{A_N}$. We check it is optimal (i.e. $\boldsymbol{r} \leq 0$) and compute the optimum value ($z^* = z_B = \boldsymbol{c_B^T} \boldsymbol{A_B^{-1}} \boldsymbol{b}$).

Case i: if n is even

In this case, we consider the basic solution corresponding to the basis $B = (x_1, \cdots, x_n, s'_1, \cdots, s'_n)$. The associated tableau is:

$$\mathcal{T}(B_n) = \frac{\boldsymbol{x_B} = \boldsymbol{A_B^{-1}} \boldsymbol{b} - \boldsymbol{A_B^{-1}} \boldsymbol{x_N}}{z = z_B + \boldsymbol{r^T} \boldsymbol{x_N}}$$

where $\boldsymbol{A_B} = \begin{pmatrix} L_n & 0 \\ I_n & I_n \end{pmatrix}$. When n is even, L_n is of full rank, then A_B is invertible and: $\boldsymbol{A_B^{-1}} = \begin{pmatrix} L_n^{-1} & 0 \\ -L_n^{-1} & I_n \end{pmatrix}$. We compute:

$$\begin{aligned} \boldsymbol{r^T} &= -\boldsymbol{e^T} \boldsymbol{L_n^{-1}} \\ &= \begin{cases} (0, -1, -1, 0, \cdots, 0, -1, -1, 0) \text{ if } n = 4k, k \in \mathbb{N} \\ (-1, -1, 0, 0, -1, -1, \cdots, 0, 0, -1, -1) \text{ if } n = 4k+2, k \in \mathbb{N} \end{cases} \end{aligned}$$

where the dots indicate the repetition of the pattern formed by the 4 preceeding elements. Then $\boldsymbol{r} \leq 0$ and the solution $\boldsymbol{x_B}$ is an optimal basic feasible solution. The computation of z_B yields:

$$z_B = nb - \begin{cases} \frac{n}{2} \delta c \text{ if } n = 4k, k \in \mathbb{N} \\ \frac{n+2}{2} \delta c \text{ if } n = 4k+2, k \in \mathbb{N} \end{cases}$$

Case ii: n is odd

We consider the basic solution corresponding to the basis $B = (x_1, x_2, s_3, x_4, \cdots, x_n, x_3, s'_2, \cdots, s'_n)$. Matrices in the associated tableau can be written as follows:

$$A_B = \left(\begin{array}{cc} \begin{array}{c|c} \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{array} & \begin{array}{cc} 0 & \\ 0 & \mathbf{0} \\ 1 & \end{array} \\ \hline \mathbf{0} & L_{n-3} \end{array} & \begin{array}{c|c} \begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} & \mathbf{0} \\ \hline \begin{array}{ccc} 1 & 0 & 0 \\ & \mathbf{0} & \end{array} & \mathbf{0} \end{array} \\ \begin{array}{c|c} \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} & \mathbf{0} \\ \hline \mathbf{0} & I_{n-3} \end{array} & \begin{array}{c|c} \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} & \mathbf{0} \\ \hline \mathbf{0} & I_{n-3} \end{array} \end{array} \right)$$

Inverting A_B by block, we compute:

$$\boldsymbol{r^T} = \begin{cases} ((-1,0,-1)|(-1,-1,0,0,-1,-1,\cdots,0,0,-1,-1)|(-1,0,0)|\boldsymbol{0^T_{n-3}}) & \text{if } n = 4k+1, k \in \mathbb{N} \\ ((-1,-1,0)|(0,-1,-1,0,\cdots,0,-1,-1,0)|(0,0,0)|\boldsymbol{0^T_{n-3}}) & \text{if } n = 4k+3, k \in \mathbb{N} \end{cases}$$

where the dots indicate the repetition of the pattern formed by the 4 preceeding elements. Then $\boldsymbol{r} \leq 0$ and the solution $\boldsymbol{x_B}$ is an optimal basic feasible solution. The computation of z_B yields:

$$z_B = nb - \frac{n+1}{2}\delta c$$

Summary

Let's assume

$$\delta \leq \frac{b}{c} \tag{6}$$

Then, the maximum welfare which can be allocated in core allocations is:

$$W(n) = \begin{cases} nb - \frac{n}{2}\delta c & \text{if } n = 4k, k \in \mathbb{N} \\ nb - \frac{n+1}{2}\delta c & \text{if } n = 4k+1, k \in \mathbb{N} \\ nb - \frac{n+2}{2}\delta c & \text{if } n = 4k+2, k \in \mathbb{N} \\ nb - \frac{n+1}{2}\delta c & \text{if } n = 4k+3, k \in \mathbb{N} \end{cases}$$

An additional constraint characterizes the core,

$$\sum_{i=1}^{n} x_i = nb - c - \delta c$$

Then a necessary and sufficient condition for non-emptiness is that $W(n) \geq nb - c - \delta c$ which can be restated in terms of a higher bound on δ:

$$\delta \leq \bar{\delta}(n) \text{ where } \bar{\delta}(n) = \begin{cases} \frac{2}{n-2} & \text{if } n = 4k, k \in \mathbb{N} \\ \frac{2}{n-1} & \text{if } n = 4k+1, k \in \mathbb{N} \\ \frac{2}{n} & \text{if } n = 4k+2, k \in \mathbb{N} \\ \frac{2}{n-1} & \text{if } n = 4k+3, k \in \mathbb{N} \end{cases} \tag{7}$$

Conditions 6 and 7 are necessary and sufficient conditions for non-emptiness of the core.

Proof of Proposition 3

In order to prove a more general statement, we introduce the notion of restricted core.

Definition 4. *A* ***restriction*** *on cooperative behaviors is a correspondence* $\mathbb{P}' : \begin{cases} 2^N \longrightarrow \mathbb{P}(N\backslash S) \\ S \longrightarrow \mathbb{P}'(N\backslash S) \end{cases}$ *such that* $\forall S \in N$ $\mathbb{P}'(N\backslash S) \subseteq \mathbb{P}(N\backslash S)$ *with strict inclusion for some* S.

We denote by $\mathcal{R}(N)$ the set of all restrictions.

Definition 5. *For any restriction* $\mathbb{P}' \in \mathcal{R}(N)$*, we define the* ***restricted core*** $\mathcal{C}_l^{P'}(N)$ *as the set of global agreements which satisfy:*

$$\forall S \subset N \ \forall \mathcal{P} \in \mathbb{P}'(N\backslash S) \ \forall \boldsymbol{l} \in \mathcal{N}(\mathcal{P}) \ \sum_{i \in S} x_i \geq v(S, \mathcal{P}, \boldsymbol{l}) \tag{8}$$

We will show the following more general property which states that, as soon as 2-agent coalitions never build, the condition 7 is a necessary condition for non-emptiness *whatever the restriction on the cooperative behavior of outside members.*

Proposition 11. $\forall \sigma \in \Sigma_L^e$ *If* $2b < c$ *then,* $\forall \mathbb{P}' \in \mathcal{R}(\mathcal{N})$ $\delta > \bar{\delta}(n) \Rightarrow \mathcal{C}_e^{P'}(N) = \emptyset$

Let $\sigma = (N, b, c, \delta, e) \in \Sigma_L^e$ such that $2b < c$ and $\mathbb{P}' \in \mathcal{R}(N)$. In the system defining the restricted core $\mathcal{C}_\sigma^{\mathbb{P}'}(N)$, we can show as in the proof of proposition 2 that the constraints for building coalitions can be restricted to the (n-1) and (n-2) coalition rationality constraints. Because $2b < c$, these constraints are the same. The rationality of non-building coalitions, however can no more be restricted to positivity constraints. Formally, we introduce the following linear program (LP'):

$$max_{\boldsymbol{x}} \sum_{i \in N} x_i \text{ subject to } \begin{cases} x_2 \leq b - \delta c \\ \forall j \in [1, n-2], x_j + x_{j+2} \leq 2b - \delta c \\ x_{n-1} \leq b - \delta c \\ \forall i \in N, x_i \leq b \\ \forall S \in \mathcal{S}_{NB}, \sum_{i \in S} x_i \geq v^c_\sigma(S) \end{cases}$$

where $\mathcal{S}_{NB}$ denote the set of non building coalitions in σ. Because $\forall S \in S_{NB}$, $v^c_\sigma(S) \leq 0$, (LP') is a less constrained program than (LP).

First, (LP') allows for feasible solution in which x_i may be negative then condition 6 is no more a necessary condition for non-emptiness. Second, such relaxation of the positivity constraints in (LP) does not allow to improve on the objective. Indeed, we exhibited in the previous proof basic optimal feasible solutions which encompassed all x_i. This indicates that the reduced cost associated to positivity of the x_i is null so the optimal value of the objective does not change and condition 7 remains necessary. This establishes proposition 11 which implies proposition 3.

Proof of Proposition 4

Let $(\sigma^e, \sigma^{ne}) \in \Sigma^e_L \times \Sigma^{ne}_L$ such that $\sigma^e = (N, b, c, \delta, e)$ and $\sigma^{ne} = (N, b, c, \delta, ne)$.

Formally, Proposition 4 comes from of the comparisons of the systems defining the core. In the case of CIOC, is it straightforward to check that $\forall S \subseteq N, v^c_{\sigma^e}(S) = v^c_{\sigma^{ne}}(S)$ so $\mathcal{C}^c(\sigma^e) = \mathcal{C}^c(\sigma^{ne})$. In the case of RHOC, note that values $v^r_\sigma(S)$ and $v^c_\sigma(S)$ differ if and only if outside members $N \backslash S$ build. Note, in addition that, any coalition that builds in the non-excludable case would have built in the excludable case so the externalities imposed by outside members on blocking coalitions are always higher in the excludable case. From both observations, we get

$$\forall S \subseteq N, v^r_{\sigma^{ne}}(S) \leq v^c_{\sigma^{ne}}(S) \tag{9}$$

Let $\boldsymbol{x} \in \mathcal{C}^r(\sigma^{ne})$, then we have:

$$\begin{cases} \forall S \subset N, \sum_{i \in S} x_i \geq v^r_{\sigma^{ne}}(S) \\ \sum_{i \in N} x_i = v^r_{\sigma^{ne}}(N) \end{cases}$$

Noting that $v^r_{\sigma^{ne}}(N) = v^r_{\sigma^e}(N)$ and using 9, it follows:

$$\begin{cases} \forall S \subset N, \sum_{i\in S} x_i \geq v^r_{\sigma^e}(S) \\ \sum_{i\in N} x_i = v^r_{\sigma^e}(N) \end{cases}$$

So $\boldsymbol{x} \in \mathcal{C}^r(\sigma^e)$. Then, $\mathcal{C}^r(\sigma^{ne}) \subseteq \mathcal{C}^r(\sigma^e)$.

Proof of Proposition 5

Let $\sigma \in \Sigma^e_L$ such that $min(k_1, n-k_1) > 3$. Note first that such assumption implies $n \geq 7$.

Because $k_1 > 3$, we know that 3 individuals would not build a project with cost $c + \delta c$. Because $n - 3 > k_1$, any blocking coalition of 3-agents fronts a complementary who would rationally build. Then, 3-agent coalitions with cost $c + \delta c$ can free-ride on the others. In particular, any $\boldsymbol{x} \in \mathcal{C}^r(\sigma)$ should meet the following three-agent rationality constraints:

$$\begin{cases} x_1 + x_2 + x_3 \geq 3b - \delta c \\ x_4 + x_5 + x_6 \geq 3b - \delta c \end{cases}$$

what sums up to:

$$\sum_{i=1}^{6} x_i \geq 6b - 2\delta c \tag{10}$$

However, $\boldsymbol{x}$ should also meet the following (n-1) and (n-2) rationality constraints[20]:

$$\begin{cases} x_1 + x_3 \leq 2b - \delta c \\ x_4 + x_6 \leq 2b - \delta c \\ x_2 \leq b - \delta c \\ x_5 \leq b \end{cases}$$

which sum up to:

$$\sum_{i=1}^{6} x_i \leq 6b - 3\delta c \tag{11}$$

Conditions 10 and 11 are contradictory then $\mathcal{C}^r(\sigma) = \emptyset$.

As soon as $k_1 \leq 3$ or $n - k_1 \leq 3$, free riding is moderated. In the former case, even small coalitions would build and then are not credible enough to always induce construction from outside members. In the latter case, even a small blocking coalition would lead outside members to abandon the project when forming.

[20]See the proof of proposition 2 for more details about how these rationality constraints are derived

Proof of Proposition 6

Let $\sigma = (N, \boldsymbol{b}, c, \boldsymbol{T}, \delta) \in \Sigma$ and $\boldsymbol{x} \in \mathcal{C}(\sigma)$. We have the property that the value function under collapse in outside cooperation actually corresponds to the maximum value over all possible partitions and Nash equilibria. Then, we can restrict our attention to this restricted value. We know that x pertains to the dual core of the dual game:

$$\{x \in \mathbb{R}^n | \sum_i x_i = v^c(N) \text{ and } \sum_{i \in S} x_i \leq v^*(S)\}$$

where

$$\begin{aligned} v^*(S) &= v^c(N) - v^c(N \backslash S) \\ &= \sum_{i \in N} b_i - c - \underline{\tau}(N)\delta c - max(\sum_{i \in N \backslash S} b_i - c - \underline{\tau}(N \backslash S)\delta c, 0) \\ &= min(\sum_{i \in S} b_i - (\underline{\tau}(N) - \underline{\tau}(N \backslash S))\delta c, \sum_{i \in N} b_i - c - \underline{\tau}(N)\delta c) \end{aligned}$$

We consider the linear program (LP2):

$$max_{\boldsymbol{x}}\{\sum_{i \in N} x_i : \forall S \in 2^N \sum_{i \in S} x_i \leq v^*(S)\}$$

This linear program is bounded (by $\sum_N b_i$, for instance) and feasible (take, for all $i \in N$, $x_i = min_{S \in 2^N} v^*(S)$), then it admits a finite optimum and so its dual (LP2*), which writes:

$$min_{\boldsymbol{\chi}}\{\sum_{S \subseteq N} \chi_S v^*(S) : \forall i \in N \sum_{S: i \in S} \chi_S = 1, \chi_S \geq 0\}$$

Let $\boldsymbol{\chi^*}$ be an optimal feasible solution of (LP2*). We know that a necessary and sufficient condition for non-emptiness of the core is:

$$\sum_{S \subseteq N} \chi^*_S v^*(S) \geq v^c(N)$$

We define $\tilde{v}(S) = \sum_{i \in S} b_i - (\underline{\tau}(N) - \underline{\tau}(N \backslash S))\delta c \geq v^*(S)$ and replace v^* by $\tilde{v}$ in (LP2*). This leads to (LP3):

$$min_{\boldsymbol{\chi}}\{\sum_{i \in N} b_i - \delta c \sum_{S \subseteq N} \chi_S(\underline{\tau}(N) - \underline{\tau}(N \backslash S)) : \forall i \in N \sum_{S: i \in S} \chi_S = 1, \chi_S \geq 0\}$$

Such program is bounded and feasible, then an optimal solution exists. Let denote by $\tilde{\boldsymbol{\chi}}$ an optimal feasible solution of (LP3). Note that it only depends on the spatial structure $\boldsymbol{T}$. We know that the value of (LP3) is higher than that of (LP2) then, a necessary condition for non-emptiness of the core is:

$$\sum_{i\in N} b_i - \delta c \sum_{S\subseteq N} \tilde{\chi}_S(\underline{\tau}(N) - \underline{\tau}(N\backslash S)) \geq v^c(N) = \sum_{i\in N} b_i - c - \underline{\tau}(N)\delta c$$

What leads to the necessary condition:

$$\delta \leq \frac{1}{\sum_{S\subseteq N} \tilde{\chi}_S(\underline{\tau}(N\backslash S) - \underline{\tau}(N))) - \underline{\tau}(N)} = \bar{\tau}(\boldsymbol{T})$$

Proof of Proposition 7

Let $\sigma = (N, \boldsymbol{b}, c, \boldsymbol{T}, \delta) \in \Sigma$. As previously, we have the property that the value function under collapse in outside cooperation actually corresponds to the maximum value over all possible partitions and Nash equilibria. Then, we can restrict our attention to this restricted value v^c. Let $\boldsymbol{x} \in \mathcal{C}(\sigma)$ and $S \subseteq N$, the value of coalition S is:

$$v^c(S) = max(0, \sum_{i_i nS} b_i - c - min_{i\in S} \sum_{j\in S} \tau_{ij}\delta c)$$

And the associated core constraint is:

$$\sum_{i\in S} x_i \geq max(0, \sum_{i_i nS} b_i - c - min_{i\in S} \sum_{j\in S} \tau_{ij}\delta c) \tag{12}$$

To fully characterize the core, an additional equation is needed:

$$\sum_{i\in N} x_i \geq max(0, \sum_{i\in N} b_i - c - \underline{\tau}(N)\delta c) \tag{13}$$

All these constraints fully characterize the core. Let's now consider the problem $\sigma' = (N, \boldsymbol{b}, c, \boldsymbol{T'}, \delta)$ where $\boldsymbol{T'} \in \mathcal{M}_n(\mathbb{R}_+)$ is such that $\boldsymbol{T'} \geq \boldsymbol{T}$ and $min_{j\in N} \sum_i \tau_{ij} = min_{j\in N} \sum_i \tau'_{ij}$. Rationality constraints writes in σ':

$$\sum_{i\in S} x_i \geq max(0, \sum_{i_i nS} b_i - c - min_{i\in S} \sum_{j\in S} \tau'_{ij}\delta c) \tag{14}$$

Because $\boldsymbol{T'} \geq \boldsymbol{T}$, we note that for any blocking coalition S, rationality constraints are more stringent in 12 than in 14 while the constraint 13 does not change. It establishes $\mathcal{C}(\sigma) \subseteq \mathcal{C}(\sigma')$.

Proof of Proposition 8

Let $\sigma = (N, \boldsymbol{b}, c, \boldsymbol{T}, \delta) \in \Sigma$, $\sigma' = (N, \boldsymbol{b'}, c, \boldsymbol{T}, \delta) \in \Sigma$ such that $\boldsymbol{b} \leq \boldsymbol{b'}$. Given the expression of v^* in σ, the linear program (LP2*) can be written:

$$min_{\boldsymbol{\chi}}\{\sum_{S \subseteq N} \chi_S(\sum_{i \in S} b_i - max((\underline{\tau}(N) - \underline{\tau}(N \backslash S))\delta c; c - \underline{\tau}(N)\delta c - \sum_{i \in N \backslash S} b_i)) : \forall i \in N \sum_{S: i \in S} \chi_S = 1, \chi_S \geq 0\}$$

Then $\mathcal{C}(\sigma)$ is non-empty if and only if

$$max_{\boldsymbol{\chi}}\{\sum_{S \subseteq N} \chi_S max((\underline{\tau}(N) - \underline{\tau}(N \backslash S))\delta c; c - \underline{\tau}(N)\delta c - \sum_{i \in N \backslash S} b_i)) : \forall i \in N \sum_{S: i \in S} \chi_S = 1, \chi_S \geq 0\} \leq c + \underline{\tau}(N)\delta c$$

Similarly $\mathcal{C}(\sigma')$ is non-empty if and only if

$$max_{\boldsymbol{\chi}}\{\sum_{S \subseteq N} \chi_S max((\underline{\tau}(N) - \underline{\tau}(N \backslash S))\delta c; c - \underline{\tau}(N)\delta c - \sum_{i \in N \backslash S} b'_i)) : \forall i \in N \sum_{S: i \in S} \chi_S = 1, \chi_S \geq 0\} \leq c + \underline{\tau}(N)\delta c$$

The comparison of the two conditions gives the result.

Proof of Proposition 9

Let $\sigma = (N, b, c, \delta, e) \in \Sigma_L^e$.

Formally, the polluter pay principle leads to define the following value function[21]:

$$\forall S \subset N,\ v_\sigma^p(S) = \begin{cases} 0 & \text{if } S < k_1 \vee (S < k_2 \wedge (1 \notin S \wedge n \notin S)) \\ |S|b - c - \delta c & \text{if } S \geq k_1 \wedge (1 \in S \vee n \in S) \\ |S|b - c - 2\delta c & \text{if } S \geq k_2 \wedge (1 \notin S \wedge n \notin S) \end{cases}$$

Then we check that the equal sharing rule $\boldsymbol{x^e} / \forall i \in N,\ x_i^e = \dfrac{v_\sigma^p(N)}{n}$ is a core allocation. Let $S \subseteq N$. We have $\forall i \in N$:

$$x_i^e = b - \frac{c + \delta c}{n} > 0$$

Then:

$$\sum_{i \in S} x_i^e = |S|b - \frac{|S|}{n}(c + \delta c)$$

[21] We note that such game is not convex. Indeed let $(S, T, i) \in 2^N \times 2^N \times N$ be such that $S \subset T \subseteq N \backslash \{i\}$. Take $i \in \{1, n\}$ and S such that $|S| \geq k_3 \wedge 1 \notin S \wedge n \notin S$ and T such that $1 \in T \vee n \in T$. The marginal contribution of i is:

$$\forall S \subset N,\ v_\sigma^p(S) = \begin{cases} m_i(S) = v(S \cup \{i\}) - v(S) = b + \delta c \\ m_i(T) = v(T \cup \{i\}) - v(T) = b \end{cases}$$

So $m_i(S) > m_i(T)$ what establishes that the game is not convex.

Given the value function, we observe that, whichever type is coalition S,

$$|S|b - \frac{|S|}{n}(c + \delta c) \geq v_\sigma^p(S)$$

So we have:

$$\begin{cases} \forall S \subset N, \sum_{i \in S} x_i^e \geq v_\sigma^p(S) \\ \sum_{i \in N} x_i^e = v_\sigma^p(N) \end{cases}$$

So $\boldsymbol{x^e} \in \mathcal{C}^p(\sigma)$.

Proof of Proposition 10

Let $\sigma = (N, b, c, \delta, e) \in \Sigma_L^e$ and let τc be a fraction of the cost that is granted to the players conditional on the realization of the project in the grand coalition (or, equivalently, a fine on blocking coalitions). Then, (n-1) coalitional rationality constraints write, under CIOC:

$$x_i \leq \begin{cases} b - (\delta - \tau)c \text{ if } i = 2 \text{ or } i = n-1 \\ b + \tau c \text{ otherwise} \end{cases}$$

And (n-2) coalitional rationality constraints:

$$(x_j + x_l) \leq \begin{cases} 2b - (\delta - \tau)c \text{ if } j = i + 2 \\ 2b + \tau c \text{ otherwise} \end{cases}$$

This lower the minimal amount that has to be collected to:

$$D(n) = \begin{cases} \frac{n}{2}(\delta - \tau)c & \text{if } n = 4k, k \in \mathbb{N} \\ \frac{n+1}{2}(\delta - \tau)c & \text{if } n = 4k+1, k \in \mathbb{N} \\ \frac{n+2}{2}(\delta - \tau)c & \text{if } n = 4k+2, k \in \mathbb{N} \\ \frac{n+1}{2}(\delta - \tau)c & \text{if } n = 4k+3, k \in \mathbb{N} \end{cases}$$

And the associated constraint on δ accordingly:

$$\delta \leq \begin{cases} \frac{2}{n-2} + \tau & \text{if } n = 4k, k \in \mathbb{N} \\ \frac{2}{n-1} + \tau & \text{if } n = 4k+1, k \in \mathbb{N} \\ \frac{2}{n} + \tau & \text{if } n = 4k+2, k \in \mathbb{N} \\ \frac{2}{n-1} + \tau & \text{if } n = 4k+3, k \in \mathbb{N} \end{cases}$$

Individual rationality constraints remain the same but now the welfare of agents 2 and $n-1$ is $b - (\delta - \tau)c$ what imposes the additional constraint:

$$\delta \leq \frac{b}{c} + \tau$$

Then necessary and sufficient conditions for the non-emptiness of the core are derived as formerly from which we derive the least core value $\underline{\tau}$:

$$\underline{\tau} = \begin{cases} max(0; \delta - \frac{2}{n-2}; \delta - \frac{b}{c}) & \text{if } n = 4k, k \in \mathbb{N} \\ max(0; \delta - \frac{2}{n-1}; \delta - \frac{b}{c}) & \text{if } n = 4k+1, k \in \mathbb{N} \\ max(0; \delta - \frac{2}{n}; \delta - \frac{b}{c}) & \text{if } n = 4k+2, k \in \mathbb{N} \\ max(0; \delta - \frac{2}{n-1}; \delta - \frac{b}{c}) & \text{if } n = 4k+3, k \in \mathbb{N} \end{cases}$$

Computation of $\bar{\tau}(\boldsymbol{T})$ for graphs

The algorithm to compute the critical value of delta on a simple graph can be simplified using the following result:

Proposition 12. *Let $\sigma = (N, b, c, \boldsymbol{T}, \delta) \in \Sigma_G$ such that the minimal degree of the graph $\boldsymbol{T}$ is at least 1 and can only be lower after the withdrawal of a single agent. The set of CIOC-core lower bounds can be reduced to the bounds for coalitions in*

$$\mathcal{H} = \{\{N \backslash S | S \subseteq \mathcal{N}(i) : i \in N\} \cup_{i \in N} \{i\}\}$$

where $\mathcal{N}(i)$ is the set of i's neighbors.

Proof. The following proof is a generalization of the first part of the proof of proposition 2.
Let $\sigma = (N, b, c, \boldsymbol{T}, \delta) \in \Sigma_G$ such that the minimal degree of $\boldsymbol{T}$ is at leat 1 and can only be lower after the withdrawal of an agent. Let $\boldsymbol{x}$ be an allocation which satisfies the efficiency condition and the CIOC core lower bounds for coalitions in the collection $\mathcal{H} = \{\{N \backslash S | S \subseteq \mathcal{N}(i) : i \in N\} \cup_{i \in N} \{i\}\}$. We have:

$$\sum_{i \in N} x_i = nb - c - \underline{\tau}(N)\delta c \tag{15}$$

$$\sum_{i \in S} x_i \geq |S|b - c - \underline{\tau}(S)\delta c \; \forall S \in \{N \backslash T | T \subseteq \mathcal{N}(i) : i \in N\} \tag{16}$$

$$x_i \leq b \; \forall i \in N \tag{17}$$

$$x_i \geq 0 \; \forall i \in N \tag{18}$$

We want to prove that $\boldsymbol{x}$ satisfies core bounds for any coalition $S \subset N$.
Let $S \subset N$ then

1. Either S does not build then we shall show $\sum_{i\in S} x_i \geq 0$. It directly stems from the inequalities 18

2. Either S builds then we shall show $\sum_{i\in S} x_i \geq |S|b - c - \underline{\tau}(S)\delta c$ where $\underline{\tau}(S) = min_{j\in S} \sum_{i\in S} \tau_{ij}$

   (a) if $\underline{\tau}(S) < \underline{\tau}(N)$, we consider an optimal site in S $j^* \in argmin_{j\in S} \sum_{i\in S} \tau_{ij}$. The constraint associated to $S^* = N\backslash(\mathcal{N}(j^*) \cap (N\backslash S))$ belongs to the set of constraints 16. It writes:

   $$\sum_{i\in S^*} x_i \geq |S^*|b - c - \underline{\tau}(S^*)\delta c = |S^*|b - c - \underline{\tau}(S)\delta c$$

   Then using inequalities 17 we have, for all $i \in S^*\backslash S$, $-x_i \geq -b$. Adding this inequalities gives the result.

   (b) if $\underline{\tau}(S) \geq \underline{\tau}(N)$ then we get the constraint combining conditions 15 and 17.

□

Such simplication allows us to restrict the number of coalitions to consider to less than $n * 2^{\underline{d}(G)}$ constraints instead of 2^n. We following program in R solves the linear program (LP4):

$$max_{\boldsymbol{\chi}}\{\sum_{S\in\mathcal{H}} \chi_S(\underline{\tau}(N) - \underline{\tau}(N\backslash S)) : \forall i \in N \sum_{S:i\in S} \chi_S = 1, \chi_S \geq 0\}$$

And derive the value of $\bar{\tau}(\boldsymbol{T})$ according to the formula:

$$\bar{\tau}(\boldsymbol{T}) = \frac{1}{\sum_{S\in\mathcal{H}} \tilde{\chi}_S(\underline{\tau}(N\backslash S) - \underline{\tau}(N))) - \underline{\tau}(N)}$$

The following code yields the critical value for graph A in figure 2 (it is written in the R software).

```
#######################################################
#    Finding the critical value of delta on a graph   #
#######################################################
library(linprog)


dmin<-function(M){
min(rowSums(M))
}


constraints<-function(M){
```

```
n<-dim(M)[1]
A<-NULL
b<-NULL
for(i in 1:dim(M)[1]){
neighbors<-which(M[i,]>0, arr.ind=TRUE)
di<-length(neighbors)
for(k in 0:(dmin(M)-1)){
ExcludableCoalitions<-matrix(neighbors[combn(1:di,di-k)],ncol=choose(di,di-k))
for(l in 1:choose(di,di-k)){
constraint<-rep(0,dim(M)[1])
constraint[ExcludableCoalitions[,l]]<-1
A<-rbind(A,constraint)
b<-cbind(b,dmin(M)-k)
}
}
}
rownames(A)<-NULL
cbind(A,t(b))
}

deltac<-function(M){
n<-dim(M)[1]
A<-constraints(M)
A1<-rbind(A[,1:n],diag(1,n))
dndns<-c(A[,n+1],rep(0,n))
chi<-solveLP(dndns,rep(1,n),A1,maximum = TRUE,const.dir = rep("=",n))$solution
1/(chi%*%dndns-dmin(M))
}

##Testing case A in Figure 2
#Case A
```

```
MA<-matrix(c(
0,1,0,0,0,0,
1,0,1,0,0,0,
0,1,0,1,0,0,
0,0,1,0,1,0,
0,0,0,1,0,1,
0,0,0,0,1,0), nrow = 6, ncol = 6)
deltac(MA)
```