The Drawbacks of Loyalty: a Story of Switching Costs and Implicit Contracts*

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Abstract

We study the circumstances under which firms can engage into tacit collusion when customers have switching costs. In addition to the level of switching costs and the players' patience, market characteristics, such as its granularity or the consumers' sophistication and lifespan, greatly impact the implementability of implicit contracts. When facing two long-lived consumers, firms are less able to collude than when facing short-lived ones. However, this conclusion does not hold for a continuum of long-lived consumers or when these two long-lived consumers are also short-sighted.

Keywords: Implicit contracts, switching costs, collusion, coordination game
JEL: C73, D43, L13

1 Introduction

We develop a duopoly model in which a homogeneous product is sold to two homogeneous consumers over an infinite number of periods. At the beginning of each period, each customer is linked to a firm and endures a switching cost if she buys from its competitor. Contrary to the usual assumption in the literature, consumers are infinitely-lived, which greatly reduces the firms' ability to sustain collusion. This results does not hold for long-lived but short-sighted consumers.

Figure (1) summarizes our results on the implementability of tacit collusion in the case of two consumers. One salient feature of our findings is that players' patience positively affects the feasibility of tacit collusion for short-sighted or short-lived consumers but not for long-lived ones. The underlying mechanism driving our conclusion is that consumers are better-off in an asymmetric market in which high prices are somehow more difficult to sustain. Forward-looking consumers anticipate this and switch even for high deviation prices. This intuition is confirmed by our analysis of the game with heterogeneous discount factors.

This story does not apply to a continuum of consumers. In this case, consumers can free-ride, i.e. benefit from a price decrease after deviation without enduring the switching costs. To ensure against this, the deviating firm must post lower deviations prices, which can make collusion more appealing. We find evidence

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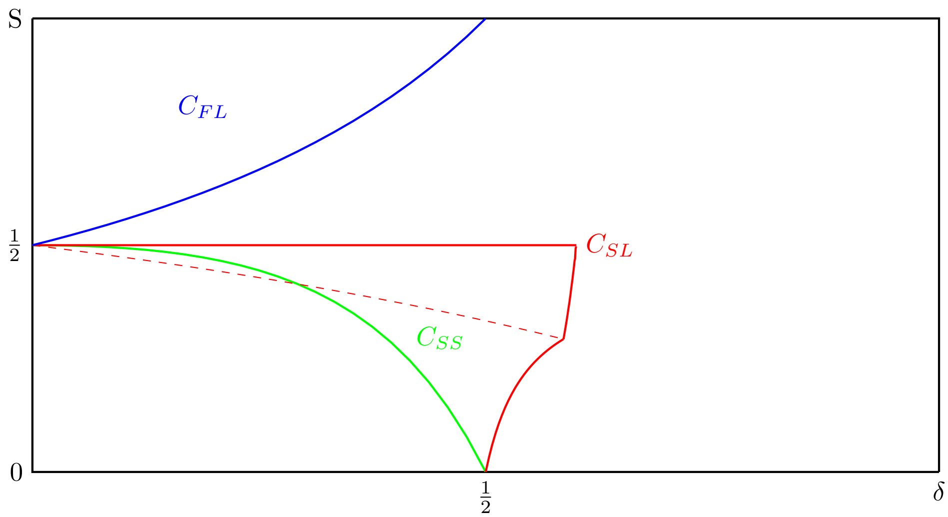

Figure 1: Implementability of full collusion given consumers' sophistication and lifespan (2 consumers) — Each plain line represents the lower boundary of the set over which full collusion is implementable, for a given consumer specification. The blue line stands for forward-looking consumers, the green line for short-sighted ones, and the red line for short-lived ones (the dashed line represents an alternative equilibrium).

that collusion for a continuum of consumers is easier to sustain than in the case of short-lived or short-sighted consumers if the consumers are pessimistic, i.e. if they do not believe that the switching strategy is followed by other consumers. If consumers are able to coordinate on equilibria which are not Pareto dominated, the impact of consumers' sophistication on collusion is more ambiguous. Our result however relies on strong limitations on the firms' strategy sets and must be interpreted with caution. Conversely, the number of short-sighted or short-lived customers do not affect the likelihood of collusion.

There is large empirical evidence that consumers do not always respond to price cuts the way classical models of Bertrand competition predict they should. Our paper contributes to the strand of the literature which explains this phenomenon by assuming the existence of some switching costs associated to a change of supplier. The Office of Fair Trade gives the following definition: *"Switching costs can be defined as the real or perceived costs that are incurred when changing supplier but which are not incurred by remaining with the current supplier"*. Such costs are particularly present in the IT sector, whether they arise from the differences in offers between platforms (*Will my favorite software be available in other operating systems?*), the use of proprietary formats and the resulting lack of compatibility across platforms, or the effort needed to learn how to use a new technology. Switching costs are even inherent to some of the industry products. For instance, in some video games, such as massively multiplayer online role-playing games, the player controls an avatar whose abilities and characteristics evolve with the player's effort. The company's profits are generated by the sales of the product and possible subsequent extensions, monthly subscriptions, or the sales of services enhancing the gaming experience.[1] The switching costs in this case stem from the time cost associated to the development of a skilled avatar.

However, switching costs do not only apply to IT sector: they can be thought more generally as the savings

[1]The leading product in this sector, *World of Warcraft* produced four extensions since its release, and gets a monthly subscription from its ten million users.

generated by synergies resulting from a long-term relationship between a supplier and a retailer. Hence, if a city council wants to change the company which operates their transportation network, it must take into account all the possible inefficiencies generated by the transition. Whichever its type, switching costs create frictions on customers' purchasing decisions. Therefore, our main research questions is to investigate whether such purchasing frictions facilitate tacit collusion.

The answer to this question depends on the type of consumers we consider. If the interaction between the firms and the consumers only last one period, switching costs can be thought of as transportation costs. At each period, firms equally share a new group of consumers who leave at the end of the period. We say that our customers are short-lived players and have a passive role in the game. This framework is useful as it allows us to center our attention to firms' behavior. But firms and consumers can also have infinitely repeated interactions. Our long-lived consumers in this case can either anticipate how their purchasing decisions of the current period will affect the strategies of all players in the future, or they focus on their current period profit. These consumers are respectively forward-looking or short-sighted. The short-sighted case gives us a good benchmark, when compared to the forward-looking case, to assess the impact of consumers' sophistication on collusion. The forward-looking case introduces many new challenges: consumers are strategic and can coordinate to break collusion, but deviations of firms also depend on the consumers' expectation of their future utility when switching. Hence, in each period, after prices are posted, a coordination game is played across consumers who must decide between switching or staying loyal. As these games are challenging to solve and usually display a multiplicity of equilibria, most of the forward-looking part is restricted to a two firms - two consumers case. But, we also find promising evidence that a large number of consumers facilitate tacit collusion. Indeed, to attract all the market, the deviating firm has to compensate consumers for the earnings they would have had by free-riding, i.e. let everyone switch but themselves. This decreases the deviation prices, hence makes collusion easier.

We work under a duopolistic setting where firms produce a homogeneous good and customers incur the same switching costs if they decide to change supplier. We do not allow firms to price discriminate between customers. Therefore, at each period, each firm faces the trade-off between setting a low price to attract the customers of the rival or a high price to extract rents from the captive customers. Cabral (2012) gives an analysis of the case in which firms can price discriminate between captive and non-captive consumers.

This trade-off is present in many two-period models of switching costs such as Klemperer (1987) and Padilla (1992). However, a two-period setting is inadequate to study the firms' incentives to tacitly collude. In this regard, our model is closer to models with an infinite number of interactions: Farrell & Shapiro (1988), Beggs & Klemperer (1992), Padilla (1994) and Anderson at al. (2004). The last two works study the likelihood of implementing tacit agreements between firms when customers have switching costs. However, our analysis differs form the theirs in the following aspects: i) our work consider both short-run and long-run customers; ii) instead of working with an overlapping-generation model, our long-run customers live infinitely, which considerably changes the set of parameters over which collusion is sustainable, iii) and we draw implications regarding the level of sophistication of our long-run customers. The feature of infinite interactions between firms and consumers in the forward-looking case closely relates our analysis to Biglaiser and Crémer (2011) and Biglaiser, Crémer and Dobos (2012). However, these papers investigate different issues, such as the incumbency problem and the impact of heterogeneous switching costs. Our analysis of customers as short-run players also links our paper to the literature on mixed pricing in oligopoly, e.g.

Shilony (1977) and Narashiman (1988) as the profit function of the firm fails to be continuous. We applied Shilony's methodology to recover our results but were also able to find an additional equilibrium in mixed strategies in which indifferent customers choose to switch when they are indifferent. In equilibrium, we find that less switching occurs in equilibrium compared to the tie-breaking rule assumed by Shilony's where in the case of indifference customers decide not to switch.

The remaining of the paper is organized as follows. Section 2 presents the model. Section 3 deals with two forward-looking customers whereas section 4 introduces a continuum of consumers. Section 5 studies alternative specifications for the customers, namely their sophistication, lifespan, and the possibility of increasing switching costs. Finally, section 6 concludes.

2 Model

Two firms A and B produce a non durable homogeneous good at zero cost. On the demand side, two customers $j = 1, 2$ wish to buy a single unit of the good at each period and obtain a gross utility of 1. We consider that each customer is captive with degree $S \in [0, 1]$ to either firm A or B. Captivity in our model can be considered as the magnitude of switching costs. If a customer captive to one firm buys from the other she incurs an additional cost equal to S. Captivity of a consumer j at date t is denoted C_j^t. We assume that at the beginning of the game, firms equally share the market. Without loss of generality, we suppose that customer 1 is captive to firm A and customer 2 to firm B, hence $C_1^1 = A$ and $C_2^1 = B$.
The state of the world ξ_i^t represents the number of consumers captive to firm i at the beginning of period t . In our model, firms interact for an infinite number of periods $t = 1, \dots, T$, and $T = \infty$. Each period has two stages, in stage 1 there is a Bertrand-type competition where firms set their price simultaneously $p_i^t \in \mathbb{R}$ for $i = A, B$. In stage 2, customers effectuate their purchasing decisions $d_j^t = \{A, B, \emptyset\}$.[2] Our game is one of complete information as firms and customers are able to observe the full history of prices and purchasing decisions.

Customers' behavior

Customer behavior depends on the type of customers that we are considering. In the following, we omit long-lived when we talk about forward-looking or short-sighted consumers. When customers are short-run or short-sighted, they buy from the firm offering the best bargain once discounted for switching costs. Conversely, forward-looking customers anticipate that their purchasing decisions affects future equilibrium price, accordingly, they behave strategically and they take current purchasing decisions anticipating future prices and switching costs. While the former type of customers maximize current utility, the later maximizes the present discounted utility with a discount factor δ^{co}. Moreover, the analytical difference between short-run and long-run customers lies in the way captivity changes across periods. In the former case, at the beginning of each period customer 1 is captive to firm A and 2 to B, regardless of past actions. We can interpret it as old consumers leaving the market at the end of each period, replaced by new ones the next period. In the latter case, the captivity of a consumer changes according to her previous purchasing decision, i.e. $C_j^t = d_j^{t-1}$.[3]

[2] Where $d_j = \emptyset$ stands for not purchasing.
[3] If the customer did not effectuate any purchase at time t, i.e, $d_j^t = \emptyset$, then $C_j^t = C_j^{t-1}$.

Firms' behavior

In each period, firms compete simultaneously and non cooperatively in prices. We assume that firms do not observe the identity of customers who are captive, hence do not discriminate between captive and prospective customers.[4] We also consider that firms are not capacity constrained and they are able to sell to both consumers at any time. Therefore, at the beginning of each period, each firm simultaneously posts its price to maximize its discounted expected profit:

$$\Pi_i = \sum_{t=1}^{\infty} \delta^{t-1} \pi_i^t(p^t, d^t) \quad i = A, B, \tag{2.1}$$

where p^t is the firms' action profile and d^t the consumers' choice at time t. The discount factor $\delta^f \in (0, 1]$ is the same for both firms.

Simple strategy profiles

To obtain the equilibrium sequence of prices of the game, we will restrict attention to simple strategy profiles. In our game, a simple strategy profile consists of an initial prescribed outcome $\mathbf{p}(c)$ and a punishment outcome $\mathbf{p}(i, \xi_i)$ for each firm $i = A, B$ and each state of the world $\xi = 0, 1, 2$, where $\mathbf{p}$ is an outcome path $(p^1, p^2, ...)$. We focus on symmetric stationary equilibria, hence:

- $\mathbf{p}(A, \xi) = \mathbf{p}(B, \xi)$ for all ξ and we can omit the firm index.
- If firm i is in state ξ_i, firm $-i$ is necessarily in state $\xi_{-i} = 2 - \xi_i$, hence plays $\mathbf{p}(2 - \xi_i)$.
- In any path $\mathbf{p}(\xi)$, the firm indefinitely posts a price $p(\xi)$.

As a result, a simple strategy profile in our duopolistic game is fully summarized by $(p(c), p(\xi)_{\xi=0,1,2})$. The game starts with the initial path $\mathbf{p}(c)$ and players continue with this path until a deviation takes place. We define the initial path as the collusive one. If a deviation by firm A has occurred resulting in the state of the world ξ, the game switches to the path in which firm A plays $p(\xi)$ and firm B plays $p(2 - \xi)$ forever[5].

Assumptions and equilibrium definitions

We finish the set-up of our model by exposing the main assumptions under which we derive our results and by stating our definition of a fully collusive equilibrium of our game.

Assumption 1. ***(Symmetry):*** *All players have the same discount factor* ($\delta^{co} = \delta^f = \delta$).[6]

Assumption 2. ***(Homogeneous switching costs):*** *Switching costs are homogeneous across consumers and time.*

[4]This assumption might seem quite restrictive as nowadays firms can keep track of the customers that have previously purchased. Indeed, some literature studies the so called behavior price discrimination where firms set different prices to captive and prospective customers. However, in this work we want to abstract from any type of asymmetry that might facilitate deviation from a collusive outcome. Furthermore, some justification of the impossibility by firms to price discriminate might be the existence of regulation that bans such a practice, or it can also be that firms do not directly retail the product, e.g. google android vs. iphone.

[5]We therefore restrict attention to collusive agreements supported by grimm trigger strategies.

[6]Observe that the discount factor is relevant only for forward-looking customers. Also, short-sighted customers are just forward-looking with $\delta^{co} = 0$.

Assumption 3. ***(Tie-breaking rule):*** *Any firm deviates from a stationary path if it gets strictly higher profits.*[7]

Definition 1. *Full collusion is implemented at equilibrium by a simple punishment path* $\mathbf{p}$ *if* $p(c) = 1$ *and* $p(\xi)_{\xi=0,1,2}$ *constitute a stationary equilibrium.*

By stationary equilibrium, we mean that in each possible state of the world, either in the collusive path or if firm i plays $p(i, \xi_i)$ and firm $-i$ plays $p(-i, 2 - \xi_i)$, then no firm has any incentive to deviate, but also no customer wants to switch.

3 Long-run consumers

With long-run customers, captivity changes and depends on the firm from which they have last purchased, i.e. $C_j^t = d_j^{t-1}$ if $d_j^{t-1} = A, B$.[8] The current utility they obtain by purchasing from firm i is then:

$$u_j^t(p_i^t) = 1 - p_i^t - (1 - \mathbb{1}(C_j^t = i))S$$

Customers pay switching costs as long as their purchasing decision is different from the last period. The firms' expected discounted profits are expressed in (2.1).

In the following we characterize full collusion, i.e. $p(c) = 1$, when the punishment path $p(\xi = 1)$ is also fully collusive. Whether other equilibria exist with a lower $p(\xi = 1)$ is a question we devote to future research.

3.1 The Incumbency Game

We study the equilibrium of the continuation game when one of the firms sells to both customers and the other is left with none. This situation can arise either because one of the firms have deviated from the collusive outcome and has attracted the customer of the rival, or one of the customers has switched to the other firm without any deviation coming from any firm. Therefore, we want to characterize $\mathbf{p}(\xi)$ for $\xi = 0, 2$. The sub-game when one of the firms has both customers is similar to a market structure with an incumbent and a potential entrant. However, we find different equilibrium prices than in Biglaiser and Crémer (2011) [1]. In particular, we focus on an equilibrium in which the potential entrant posts a negative price. Equilibria in [1] rely on more complex punishment structures than those allowed in our model[9]. The pair of equilibrium prices of this game is stated in the following lemma:

Lemma 1. *An equilibrium of the game in which one firm has both customers and the other is left with none is the pair of prices* $\{p^*(2), p^*(0)\} = \{(1 - \delta)S, -\delta S\}$ *and customers do not switch. This equilibrium corresponds to the situation in which consumers coordinate on switching as soon as it Pareto dominates coordinating on sticking with the incumbent.*

[7] We do not assume any specific tie-breaking rule coming from the customer. Indeed, we will find that depending on whether a customer decides to switch when indifferent or not gives us different equilibria for some parameters of switching costs where customers are short-run players.

[8] If $d_j^{t-1} = 0$, we find the largest τ with $d_j^\tau \neq 0$ and $C_j^t = d_j^\tau$

[9] We restrict our attention to simple strategy profiles in which the punishment phase supports itself as an equilibrium. We leave to future research variations of our model to more complex punishment structures such as the ones in [1].

Proof. Let us call I, as in incumbent, the firm with two captive customers, and E, as entrant, the firm with none. The decision subgame played by the consumers captive to I and facing (p^I, p^E) is the following:

	St	Sw
St	$(\frac{1-p^I}{1-\delta}, \frac{1-p^I}{1-\delta})$	$(1-p^I, 1-p^E-S)$
Sw	$(1-p^E-S, 1-p^I)$	$(1-p^E-S+\delta\frac{1-p^I}{1-\delta}, 1-p^E-S+\delta\frac{1-p^I}{1-\delta})$

In this table, Sw means switch and St stands for stick. Also, as the consumers are forward-looking and rational, they perfectly anticipate the strategy played by each firm in each subsequent states of the world[10].

A quick analysis of the subgame indicates that the only two possible equilibria are (St, St) and (Sw, Sw): consumers never coordinate to a situation of full collusion. When consumers coordinate on the Pareto dominating equilibrium, (Sw, Sw) is selected if and only if $p^E \leq p^I - S$ (Note that if this inequality is satisfied, (Sw, Sw) is indeed and equilibrium). Therefore, by posting $\bar{p}^E = p^I - S - \varepsilon$ with ε close to zero, E ensures that both consumers deviate.

At the equilibrium of the incumbency game, I sets a price so low that E can only obtain negative profits by undercutting and attracting its customers. This is given by the relation:

$$2\bar{p}^E + 2\delta\frac{p^I}{1-\delta} < 0 \Rightarrow p^I < (1-\delta)(S+\varepsilon) \tag{3.1}$$

Therefore, so as to avoid switches by customers, firm I charges $p^I = (1-\delta)S$ and firm E charges $p^E = p^I - S = -\delta S$. Unilateral and coordinated deviations by customers in this case are not profitable. We conclude by setting $(p(0), p(2)) = (p^E, p^I)$. $\square$

Consumers coordinating on the Pareto dominating equilibrium seems a reasonable assumption. For instance, advertising by the entrant could provide a focal point to consumers and make them choose (Sw, Sw). However failure to coordinate is very hurtful to consumers and one can argue that consumers switch only if doing so is an equilibrium in dominant strategies. If so, the equilibrium incumbent and entrant prices change: to attract consumers the entrant needs to post an even lower price and the incumbent can post a higher price. There also exists an equilibrium in which both consumers switch giving them a utility lower than if they coordinate on sticking. Although this subgame equilibrium seems unnatural, it provides the lowest possible price which the incumbent can sustain using our simple strategies[11]. The following lemma derives these two extreme incumbent prices:

Lemma 2. *Given our simple strategy profile, any incumbent price lies in the interval* $\left[\max\left\{S - \frac{\delta}{1-\delta}, 0\right\}, \frac{\delta+S(1-\delta)}{1+\delta}\right]$.

Proof. Re-examining the coordination subgame, we remark that (Sw, Sw) is the unique equilibrium (in dominant strategies) of the game if and only if $p^E < 1 - S - \frac{1-p^I}{1-\delta}$. Therefore, the minimum deviation price the entrant can rationally post is $1 - S - \frac{1-p^I}{1-\delta} - \varepsilon$, and it will do so whenever it expects the consumers to be pessimistic. We say a consumer is pessimistic if she switches only if it is a dominant strategy to do so. In that case p^I is set to $\frac{\delta+S(1-\delta)}{1+\delta}$. The deviation price increases with the probability that given two possible equilibria (Sw, Sw) and (St, St), the consumers coordinate on the first one. The incumbent price decreases

[10] Here, since we focus on fully collusive equilibria and since we impose no particular punishment when $\xi = 1$, we have $p(c) = p(1) = 1$

[11] We know from [1] that there exists equilibria in which the incumbent posts a zero price but these rely on more complex punishment phases.

with the deviation price. At the other extreme, the entrant expects the consumer to be optimistic, i.e. to choose (Sw, Sw) as soon as doing so is an equilibrium. In this case, it posts the highest deviation price such that (Sw, Sw) is an equilibrium, i.e. $p^I - S + \delta\frac{1-p^I}{1-\delta}$ and p^I is set to its lowest possible value: $S - \frac{\delta}{1-\delta}$. $\square$

3.2 Collusive Equilibrium

We now turn back to the collusive path and characterize the conditions under which $p(c) = p(1) = 1$ is implementable at equilibrium. First, let us point out that in our framework, a deviating firm always tries to attract its competitor's customer. As the punishment path when no consumer switches is just the collusive outcome, any deviation incuding no consumer switch is dominated.

To check whether or not a fully collusive price is implementable, we have to check the following incentive constraints:

- No firm wants to deviate from collusion (IC_f).

$$\frac{1}{1-\delta} \geq 2(\bar{p} + \delta\frac{p(2)}{1-\delta})$$

  where $\bar{p}$ is the optimal deviation price.

- No consumer wants to unilateraly deviate drom the collusive path (IC_c).
- A firm deviating from the collusive path charges the highest possible price $\bar{p}$ which attracts the competitor's customer (IC_a).
- After such a deviation, one of the incumbency game equilibria derived in the previous section is played.

IC_a tells us that after a deviation p^d, the customer of the other firm must be enticed to switch to the deviating firm[12]:

$$1 - p^d - S + \delta\frac{1-p(2)}{1-\delta} \geq 0$$

$\bar{p}$ is the price which makes this inequality binding. Substituting $\bar{p}$ in IC_f gives us the following condition for collusion:

$$\frac{1}{1-\delta} \geq 2(1 - S + \delta\frac{1}{1-\delta}) \Leftrightarrow S \geq \frac{1}{2(1-\delta)}$$

Note that $p(2)$ disappears in the previous inequality. Therefore, the prices paid in the punishment phase do not directly matter in the sustainability of collusion: since the consumers are forward-looking, they accept to switch if the deviating firm reimburses the switching cost minus what they save by buying from an incumbent forever. However, our derivation is valid only if the punishment path is a stationary equilibrium and the incumbency price found in the previous section allows us to check that consumers have no incentives to deviate from the collusive path. If IC_c is satisfied for an incumbent price p it is also satisfied for any price greater than p. Lemma 2 gives us the lowest possible value for $p(2)$. It suffices to check that IC_c is satisfied for this value, which is true as long as $\delta \leq 1/2$. All this proves the following proposition:

[12]It is easy to check that the customer of the deviating firm does not want to switch

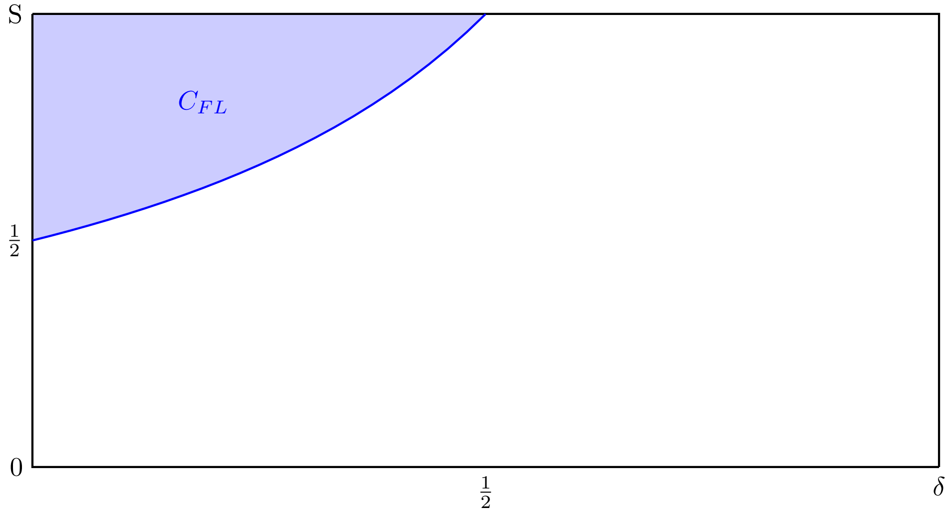

Figure 2: Full collusion in the case of two forward-looking customers. C_{FL} is filled in blue.

Proposition 1. *When customers are forward-looking, full collusion is sustainable if and only if* $S \geq \min\left\{\frac{1}{2(1-\delta)}, 1\right\}$.

As customers are forward-looking, they anticipate that by switching to another firm competition is increased and they will enjoy some positive rents. It is therefore easier for firms to deviate and break full collusion.

Figure (2) displays the set of parameters sustaining full collusion, $C_{FL} : \left\{(\delta, S) | S \geq \min\left\{\frac{1}{2(1-\delta)}, 1\right\}\right\}$.

Remark 1. *For* $S < \frac{1}{2(1-\delta)}$*, a fully collusive price is clearly impossible to sustain. We can wonder whether firms can collude on a lower price. However, as long as* $\mathbf{p}(\xi = 1)$ *is similar to the initial collusive path, such a stationary equilibrium is impossible. Indeed, suppose firms collude on* $p < 1$*, nothing prevents one of the firm to post* $p + \varepsilon$*: no consumer switches and we are still in a state of the world in which firms share the market. The question of collusion under the full collusive price, still open, can be solved using* $p(1) \neq 1$*.*

Remark 2. *When the discount factor is the same for both firms and customers* $\delta^{co} = \delta^f$*, the likelihood of a collusive outcome does not depend on the price of the punishment path* $p(\xi)$ *emerging after a deviation.*[13] *This is because the deviation price takes into account future rents enjoyed by customers and both effects cancel out and we have a neutral result. Conversely, when the discount factors are different, the way customers and firms value the future have opposite effects on the likelihood to maintain a collusive outcome. While a high discount of customers generates a large deviation price, the higher the firms value the future, the lower the deviation price and continuation profits after deviation. Therefore, "ceteris paribus" a collusive outcome is easier to sustain the lower the weight customers put on the future. We treat this case in detail in the following section.*

We conclude this section by examining an extension in which $\delta^{co} \neq \delta^f$. In the next section, we generalize our model to an continuum of consumers and study the sustainability of full collusion under strong restrictions on the firms' strategy sets.

[13] Although the collusive outcome implicitly depends on p^* being lower than p^c.

3.3 Heterogeneous Discount Factors

In the base model, we consider that the discount factors of the customers and the firm coincide. With the same discount factors, we have obtained that the likelihood to maintain collusion does not depend on the equilibrium price arising after deviation.

Here we consider the case in which the discount factor of customers - δ^{co} - and the one of the firms - δ^f - do not coincide. Observe that with short-lived or bounded rational customers, we are in the extreme case where customers do not put any weight to the future i.e. $\delta^{co} = 0$. The analysis is only interesting when customers have a positive discount factor. In this case we obtain that a change in the discount factor of customers have a positive effect on the deviation price and it is neutral on the continuation payoff after deviation. Hence, the larger is the customers discount factor the harder will be to sustain collusion. Indeed collusion is an equilibrium of the game if

$$\delta^f \geq \frac{-p(c) + 2S + \sqrt{p(c)(p(c) - 2(1-\delta^{co})S)}}{2S} = \bar{\delta}^f(p(c), \delta^{co}, S) \Leftrightarrow S \geq \frac{1+\delta^{co} - 2\delta^f}{2(1-\delta^f)^2}$$

Moreover, with sophisticated customers, we have to consider possible deviations from the collusive outcome. We find that no customer switches if the discount factor is low enough.

$$\delta^{co} < \frac{S}{p(c) + \delta^f S} = \tilde{\delta}^{co}(p(c), \delta^f, S) \Leftrightarrow S \geq \frac{\delta^{co}}{1 - \delta^{co}\delta^f}$$

The upper-bound on the customers' discount factor is increasing with the switching cost and decreasing with the firms' discount factor.[14] Hence, full collusion is achieved if $\delta^f \in \left(\bar{\delta}^f(1, \delta^{co}, S), 1\right)$ and $\delta^{co} \in \left(0, \bar{\delta}^{co}(1, \delta^f, S)\right)$. In Figure (3), we draw the parameter sets sustaining a full collusive outcome in this case. The sets always lie above the blue curve. Observe that as long as the discount factor of customers increase the full collusive region is reduced. We can also remark that in some instances, collusion is feasible for very low and high firms' discount factors but not for intermediate value. Indeed, since the optimal deviation price depends positively on this discount factor, it might be too much an effort for a very impatient firm to try to attract the other customer and collusion is implemented.

[14] $\frac{\partial \tilde{\delta}^{co}}{\partial S} = \frac{p(c)}{(p(c)+\delta^f S)^2} > 0 \qquad \frac{\partial \tilde{\delta}^{co}}{\partial \delta^f} = -\frac{S^2}{(p(c)+\delta^f S)^2} < 0.$

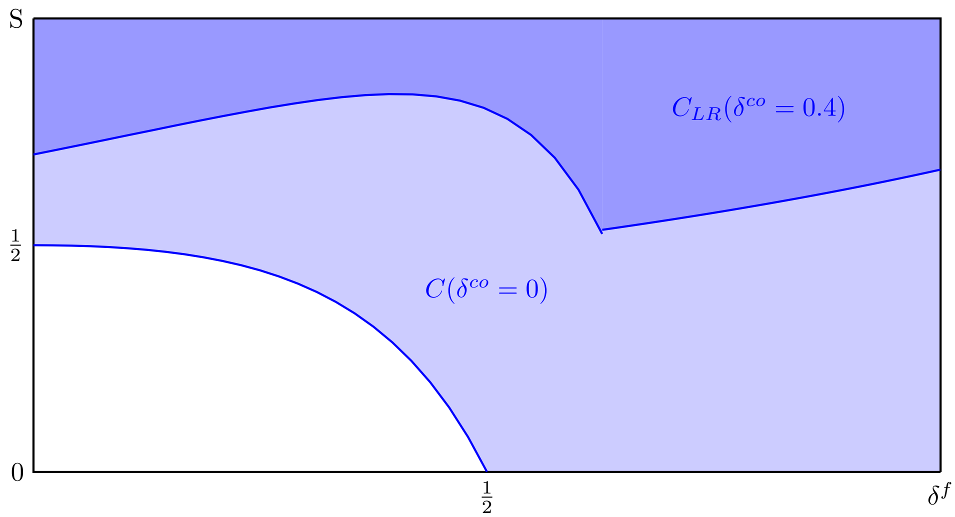

Figure 3: Full collusion in the case of two forward-looking customers when $\delta^{co} \neq \delta^f$.

4 Variation in the Market Granularity: the Case of a Continuum of Consumers

We now generalize the model to a continuum of forward-looking consumers and look for conditions supporting a fully collusive equilibrium. We find that collusion in this case is much easier to sustain than when the consumer market is concentrated. Our result however depends on strong restrictions on the firms' strategy sets and should be taken as preliminary. We first present the generalized model and additional assumptions then move to the resolution of the model.

4.1 Description of the Game

The structure and timing of the game is similar as in section 3 except that we now have a continuum of consumers of mass 1, and initially each firm is endowed with half of the market. As before, we focus on stationary symmetric equilibria and our simple strategy profile can be summarized by an initial collusive outcome $p(c)$ and punishment outcomes $p(\eta) : \eta \in [0,1] \to [-\infty, 1]$. Notice that the state of the world is now a continuous variable which represents the fraction or the mass of customers detained by a firm at the beginning of the period. As a result, the punishment phase is a function of the initial customer base of the firm. We look for equilibria of the following form: each firm initially indefinitely posts $p(c)$. In the case of a deviation resulting in firm i serving a mass η of customers, the firms move to the stationary punishment phase in which firm i indefinitely posts $p(\eta)$ and firm $-i$ posts $p(1-\eta)$.

We introduce two assumptions which are necessary at this point for us to solve the model as well as our definition of a collusive equilibrium.

Assumption 4. ***(Continuity):*** *$p(.)$ is continuous over $[0,1]$.*

Assumption 5. ***(Restriction over the strategy sets):*** *When deviating, firms want to capture the whole market. Any other deviation is suboptimal.*

We justify the first assumption by the fact that firms cannot monitor infinitesimal deviations from consumers. Therefore their price functions must be continuous. The second assumption is used to solve the model and obviously does not necessarily stand in reality: a firm might find it least costly to attract part of the market as the reaction of the competitor might be less harsh in that case. Our future efforts will aim at proving that there exists a price function $p(.)$ such that it is not the case.

Definition 2. *Full collusion is implemented at equilibrium for a continuum of consumers by a simple punishment path* $\mathbf{p}$ *if* $p(c) = 1$ *and* $p(\eta) : \eta \in [0, 1] \to [-\infty, 1]$ *constitute a stationary equilibrium.*

In a purpose of tractability, we look at equilibria in which $p(c) = p(1/2) = 1$, i.e. if the game moves back to a situation in which firms equally share the market, full collusion takes place.

4.2 The Incumbency Game

Let p^{dI} denote the optimal deviation price by the entrant such that all consumers switch. The condition for all consumers switching to be an equilibrium is given by:

$$1 - p^{dI} - S + \delta \frac{1 - p(1)}{1 - \delta} \geq 1 - p(1) + \delta \frac{1 - p(0)}{1 - \delta} \tag{4.1}$$

In this equation, the right-hand side represents what a free-rider could get by sticking to her firm: her continuation value is given by our continuity assumption. Notice also that if this condition is satisfied and $p(0) > p(1)$, then coordinating on switching yields higher utility than coordinating on sticking. According to our assumption 5, the entrant chooses the highest deviation price such that inequality 4.1 is satisfied so:

$$\bar{p}^{dI} = p(1) - S + \frac{\delta}{1 - \delta}(p(0) - p(1))$$

Substituting this deviation in the constraint on negative profits for the deviating entrant yields:

$$0 \geq \bar{p}^{dI} + \frac{\delta}{1 - \delta} p(1) \Leftrightarrow p(1) \leq S - \frac{\delta}{1 - \delta} p(0) \tag{4.2}$$

Finally, the constraint on no unilateral deviation from customers needs to be satisfied:

$$\frac{1 - p(1)}{1 - \delta} \geq 1 - p(0) - S + \delta \frac{1 - p(0)}{1 - \delta} \Leftrightarrow p(1) \leq p(0) + S(1 - \delta) \tag{4.3}$$

Let us call IC_e the incentive constraint stating that the entrant has no profitable deviation and given by inequility 4.2. Inequality 4.3 gives us the incentive constraint stating that no consumer unilateraly deviates, IC_u. Note that if both constraints hold but IC_u is not binding, there is an upward profitable deviation for the incumbent: suppose both firms post prices such that IC_e holds with equality, $p(1)$ is not a best response to $p(0)$. By raising $p(1)$, the incumbent does not lose any customers since IC_u is not binding. IC_e merely states that stable prices in the incumbency game are impervious to deviation by the entrant. We therefore need IC_e to hold, but we also need IC_u to hold with equality. A quick analysis shows that IC_u implies IC_e as long as $p(0) \leq \delta(1 - \delta)S$. This value of the entrant price determines the highest incentive compatible incumbent price, $p(1) = S(1 - \delta^2)$.

Moreover, considering our continuity assumption, taking $p(0) < 0$ would violate the participation constraint of the entrant. Indeed, by continuity, there exists a mass $\eta > 0$ of consumers such that $p(\eta) < 0$ and the firm makes negative profits. Therefore, the lowest value $p(0)$ can take is zero, and the incumbent price in this case is $p(1) = S(1-\delta)$.

This leads to the following proposition whose proof is direct:

Proposition 2. *Any pair $(p(0), p(1))$ satisfying incentive constraints IC_e and IC_u respectively given by inequalities 4.2 and 4.3 such that IC_u is binding constitutes a stationary equilibrium of the incumbency game. Moreover, $p(0) \in [0, \delta(1-\delta)S]$ and $p(1) \in \left[S(1-\delta), S(1-\delta^2)\right]$.*

The difference with the equilibrium of the incumbency game with two consumers is that in the latter there is no binding constraint for unilateral deviation from customers. The only relevant constraint refers to the behavior of the entrant and the form of the optimal deviation price is such that it completely determines the maximum incumbent price. Now, the optimal deviation price is a function of both the incumbent and the entrant prices and an extra constraint is added.

Remark 3. *Our proof relies of course heavily on the assumption that the deviating new entrant tries to attract the whole market. Also, the continuity assumption plays a great role here since it helps us to easily model what happens in case of a unilateral deviation. In contrast, intermediate models with N consumers are harder to solve as we need to explicitly specify the pricing strategies in all states of the world.*

4.3 Implementation of collusion

Suppose that a firm wants to break the collusive outcome. To do so and according to our assumptions, it posts the highest possible price such that all consumers switch. For all the consumers to switch in the coordination subgame suppose that such an action profile is an equilibrium. Hence, it is better to switch than to free-ride:

$$1 - p^d - S + \delta\frac{1-p(1)}{1-\delta} \geq 1 - p(c) + \delta\frac{1-p(0)}{1-\delta}$$

Hence, the optimal deviation price is given by $\bar{p} = 1 - S + \frac{\delta}{1-\delta}(p(0) - p(1))$ and firms do not switch from full collusion as long as:

$$\frac{1}{2}\frac{1}{1-\delta} \geq 1 - S + \frac{\delta}{1-\delta}(p(0) - p(1)) + \frac{\delta}{1-\delta}p(1) \Leftrightarrow \frac{1}{2}\frac{1}{1-\delta} \geq 1 - S + \frac{\delta}{1-\delta}p(0) \tag{4.4}$$

It is easy to check that unilateral switching from consumers is never profitable, i.e. in the coordination subgame sticking to the collusive path is always an equilibrium for consumers. Also, the condition for full collusion only depends on the parameters and $p(0)$. The higher this price the harder it is to sustain collusion. Hence, without further restrictions, any price $p(0)$ implementing an incumbency game equilibrium defines an set of parameters under which full collusion is implementable.

It is now natural to consider the worst punishment phase, i.e. phase under which $p(0)$ is minimal. Using proposition 2, we choose $p(0) = 0$ and $p(1) = (1-\delta)S$.

Proposition 3. *In a the case of a continuum of consumers, the largest set under which collusion is implementable is given by $C_\infty = \left\{(\delta, S) | S \geq \max \frac{1-2\delta}{2(1-\delta)}, 0\right\}$.*

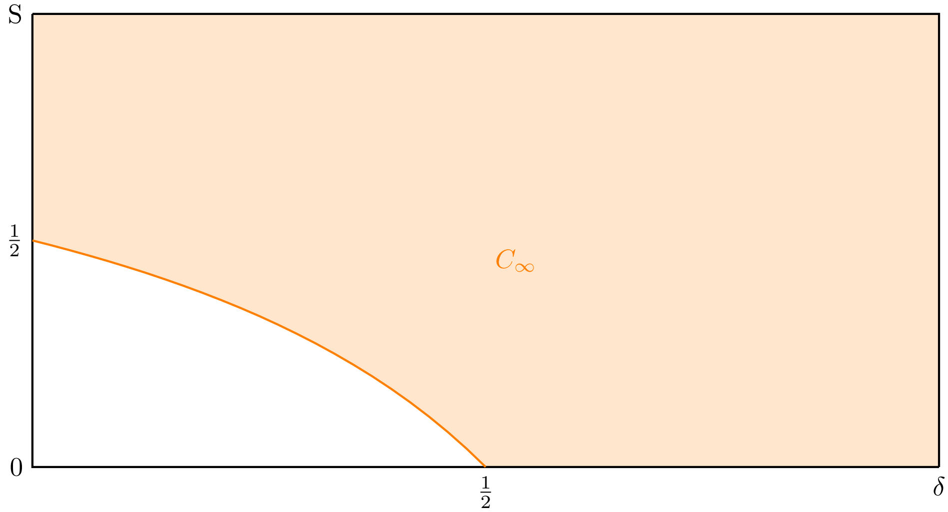

Figure 4: Full collusion in the case of a continuum of customers. C_∞ is filled in orange.

Figure (4) depicts the set sustaining collusion in this case. Clearly the area under which full collusion is possible is wider than in the two consumer case and now both switching costs and discount factors positively affect the implementability of implicit contrats. The reason is that full collusion now only depends on the entrant price, which is the free-rider profit. In the two consumers case, free-riding is equivalent to status quo and brings nothing to consumers. Here, free-riding implies playing the low entrant price, i.e. getting a large surplus. To prevent that the deviating firm must subsidize the consumers by the amount of this surplus, which makes collusion easier.

The next step in our research agenda is to prove that there exists a continuous pricing strategy which enables us to get rid of assumption 5. We now provide an extension in which we study the collusive outcome under refinements of the coordination subgame.

4.4 Collusion with coordinating customers

In our previous analysis, we did not consider equilibrium refinements in which consumers coordinate on the Pareto dominant equilibrium. Our resulting collusive set is so large partly because consumers prefer to stick to their company whereas switching could provide them with a much higher utility, assuming their fellow customers switch too. This might not be realistic in situations where consumer associations or even deviating firms could provide focal points on equilibria leading to an incumbent-new entrant situation. In line with our assumption regarding the strategy sets of firms, we do not allow consumers to consider coordination resulting in states of the world in $(0, 1)$. In that case, consumers decide to stick to their firm if:

$$0 \geq -S + \delta \frac{1 - p(1)}{1 - \delta}$$

This adds another constraint on the possibility to collude. In that case however, choosing the punishment path $p(0) = 0$ might not be the best course of action as it gives the consumers a great incentive to coordinate on switching. By raising $p(1)$ we could relax this constraint and hope to get a larger collusive set, if the effect

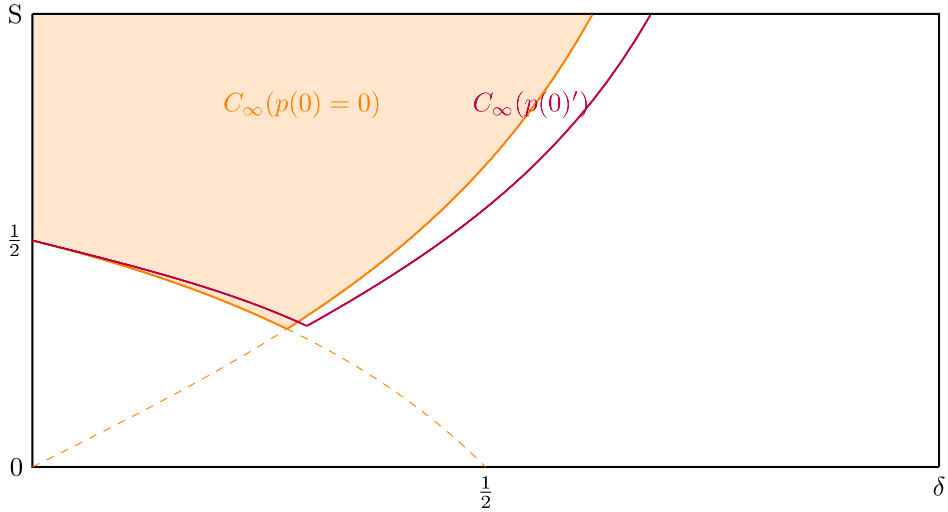

Figure 5: Full collusion in the case of a continuum of customers when they coordinate on the Pareto dominant equilibrium. There is an additional constraint steming from the ability to coordinate. The resulting collusive set is filled in orange. When $p(0)' = \delta(1-\delta)S$, the higher incumbent price relax this constraint a little but gives more bite to the firms' constraint. The lower boundary of the collusive set is represented by the purple curve.

on the incentive constraint of the firms is not too strong. Proposition 2 gives us the maximal $p(1)$ which can be implemented in equilibrium. We represent in Figure (5) the collusive sets when the punishment path is set at its two extreme possible values for $p(1)$. It is clear from the picture that neither one or the other solutions unambiguously increase the implementability of collusion. Notice however that both solutions allow to implement collusion on a broader set of parameters than in the case of two consumers.

In the next section, we depart from the assumption of forward-looking consumers so as to clearly assess the impact of consumer rationality on collusion.

5 Variations in Consumers' Sophistication and Lifespan

5.1 Short-sighted consumers

The case of infinitely-lived but short-sighted customers corresponds to the forward-looking case with ($\delta^{co} = 0$) and $\delta^f = \delta$. Then, firms need not incorporate possible future rents from switching in their deviation prices: customers behave as if they lived for one period only and buy from the firm offering them the best bargain, taking into account any eventual switching cost. Consequently, the optimal deviation price to attract customers is simply the competitor's price minus the switching costs. As before, we look for collusive equilibria in which $p(1) = 1$ and we first need to characterize $p(2)$ and $p(0)$.

We know that $p(0) - \varepsilon = p(2) - S$ since at equilibrium the entrant must play the lowest price such that consumers do not switch (otherwise the incumbent would have some incentives to raise its price). Then, the incumbent will set its highest possible price such that by deviating from $p(0)$ the new entrant only makes negative profits:

$$2(p(0) - \varepsilon) + 2\delta \frac{p(2)}{1-\delta} \leq 0 \Leftrightarrow p(2) \leq (1-\delta)S$$

By deviating the entrant makes zero profit, so by assumption 3 it chooses not to deviate. Hence, $(p(0), p(2)) = (-\delta S, (1-\delta)S)$.

Hence, a collusive outcome is an equilibrium if and only if:

$$\frac{p(c)}{1-\delta} \geq 2\left[(p(c) - S) + \delta \frac{p(2)}{1-\delta}\right]$$

which is equivalent to

$$S \geq \max\left\{0, \frac{1-2\delta}{2(1-\delta)^2}\right\}$$

We denote by C_{SS} the set of parameters (δ, S) sustaining collusion in the short-sighted case. This region is represented in Figure (3).

Notice that in this case consumers do not anticipate that future prices are affected by future states of the world. A consumer looks for the best bargain net of switching costs and her choice is not affected by the choice of the other consumers. Hence, if a firm succeeds in attracting one consumer, it must have attracted all the other ones as well. Notice also that if in the initial period firms equally share the market our proof does not rely on the number of consumers. Our result therefore easily extends to an arbitrary number of consumers (or a continuum).

In a duopsony, collusion is then easier to sustain if consumers are short-sighted. The opposite holds for a continuum of consumers if these do not coordinate on the Pareto dominant outcome.

5.2 Short-lived customers

In this section, we study the case in which customers only live for one period. At each period a new generation of customers arrive and firms equally share the market. As in the case of short-sighted customers, short-run customers shop at the firm maximizing their current utility. Our analysis in this section develops along the lines of Shilony (1977). However, we are able to recover an additional equilibrium in the case where indifferent customers shop at the lowest price firm.

5.2.1 Analysis

Because the captivity for short-lived customers remains fixed, our game is an infinite repetition of the same stage game and the main strategic difference from the previous analysis is that customers' captivity does not depend from their purchasing decisions. The only possible value for the state of the world is $\xi = 1$. Note that this analysis also holds for a continuum of consumers. As in the previous analysis, we restrict attention to simple strategy profiles and because customers live for only one period, the punishment path coincides with the static Nash equilibrium. Therefore, the punishment path for each firm is the Nash reversion $\mathbf{p}(\xi = 1) = \mathbf{p}(N)$. Before characterizing the collusive equilibrium prices sequence, we define the static Nash equilibrium which is introduced in the following Lemma:

Lemma 3. *The static Nash equilibrium is defined by a pair (or distribution) of prices and by the customers' switching decisions. The equilibrium prices depend on the degree of switching costs.*

1. *There exists a pure strategy equilibrium in prices for a level of switching costs $S \geq \frac{1}{2}$ and $S = 0$. In the former, firms set the full collusive price $p(1) = p(c) = 1$ and in the latter the price is set equal to the marginal cost of production $p(1) = 0$. In both cases, no customer switches in equilibrium.*

2. *For a level of switching costs $0 < S < \frac{1}{2}$ the equilibrium price is characterized by price dispersion and customers switch in equilibrium.*

Proof. Point (1) is trivial, if $S \geq \frac{1}{2}$, undercutting the price to attract the customer of the competitor does not constitute a strictly profitable deviation since the price undercut dominates the increase in demand.[15] Conversely, without switching costs, the only equilibrium is one where firm sets prices equal to marginal cost of production. Point (2) which is formally proven in appendix (A) states the trade-off between setting a high price to extract rent from the captive customers and setting a low price in order to attract the customer of the competitor. □

Therefore, when a pure strategy equilibrium in prices fails to exist, firms select a price randomly from a cumulative distribution function $F_i(p)$. Because firms are symmetric and switching costs are homogeneous there exist a symmetric mixed strategy equilibrium.[16] Therefore, both firms set prices according to the same distribution function $F_A(p) = F_B(p) = F(p)$. Suppose also that the support on the equilibrium prices is $\left[\underline{p}, \bar{p}\right]$. When any firm i chooses a price belonging to the support and the other plays according to the cdf $F(p)$, its expected profit is equal to a constant denoted by V. When firm i sets a price p it can happen that: (i) p is the lowest price so that all customers buy from this firm which happens with probability $[1 - F(min(p + S, 1))]$, (ii) p is such that both firms have one customer which happens with probability $[F(min(p + S, 1)) - F(max(0, p - S))]$ and (iii) p is so high that firm i is left with no demand. Hence, firm's i expected profit is

$$\mathbb{E}\pi(p) = p \times [F(min(p + S, 1)) - F(max(0, p - S))] + 2 \times p \times [1 - F(min(p + S, 1))] .$$

In equilibrium, expected profits satisfy

$$\frac{V}{p} = 2 - [F(min(p + S, 1)) + F(max(0, p - S))]. \tag{5.1}$$

Solving the previous expression in p and using the methodology proposed by Shilony (1977), we can obtain the equilibrium distribution of prices. However, departing from the previous work, we have not assumed a tie breaking rule when customers are indifferent between switching or not. Indeed depending on whether customers stays in the firm or switch when they are indifferent gives us two equilibria for some values of switching costs. The following proposition introduces the expected per-period profit of each firm in the punishment path.

Proposition 4. *When customers are short-lived, the Nash reversion of the punishment path is given by:*
i) For low values of switching costs $S \in \left(0, \frac{1}{2+\sqrt{2}}\right]$ the per period expected profit is $V = \left(1 + \sqrt{2}\right) S$ and customers switch with positive probability

[15]In the particular case where the level of switching costs is $S = \frac{1}{2}$ by undercutting the price, the firm gets the same profits as by setting the collusive price.

[16]For a proof of existence of equilibrium for this type of games we refer to the theoretical work of Dasgupta and Maskin (1986).

ii) For intermediate values of switching costs $S \in \left(\frac{1}{2+\sqrt{2}}, \frac{1}{2}\right)$ *two Nash equilibria* (σ^I, σ^{II}) *exist depending on whether customers switch or not when they are indifferent. In both equilibria customers switch with positive probability and per period expected profits are*

- *When customers switch* σ^I *then* $V = 1 - S$
- *When customers do not switch* σ^{II} *then* $V = \frac{S+\sqrt{S(4+S)}}{2}$

iii) For large values of switching costs $S \geq \frac{1}{2}$ *the per period profit is* $V = 1$ *and no customer switches.*

Proof. This is a particular case of Shilony's result when the number of firms is equal to two. With no assumption on the tie-breaking rule of customers allows us obtain two equilibria for the range of switching costs $S \in \left(\frac{1}{2+\sqrt{2}}, \frac{1}{2}\right)$. The methodology used is very close to the one in Shilony and the proof is relegated to appendix (A). ☐

With the expected profits obtained in the punishment path we have that the initial outcome $\mathbf{p}(c)$ constitutes an equilibrium of our infinitely repeated game if the present discounted profits of playing the initial path are higher than any deviation. Therefore, firm i plays according to the initial path if the following incentive constraint is satisfied

$$\Pi_i(\mathbf{p}(c)) \geq \pi_i(p_i^d, p_{-i}(c)) + \delta\mathbb{E}\left(\Pi_i(\mathbf{p}(N))\right) \qquad (IC_C^f) \tag{5.2}$$

where the optimal deviation price is:

$$p_i^d = \arg\max_{p_i} \pi_i(p_i, p^1(c)).$$

We find that the optimal deviation price is arbitrarily close to $(p(c) - S)$ for $S < \frac{p(c)}{2}$ and equal to $p(c)$ otherwise. Finally, the sequence of equilibrium prices depending on S and δ, is characterized by the following proposition

Proposition 5. *When customers are short-lived, we have:*

1. *For* $S = 0$ *full collusion is an equilibrium for* $\delta \geq \frac{1}{2}$*.*
2. *For* $S \in \left(0, \frac{1}{2+\sqrt{2}}\right]$ *full collusion is an equilibrium for* $\delta \geq \frac{1-2S}{2-(3+\sqrt{2})S} = \bar{\delta}(S)$*.*
3. *For* $S \in \left(\frac{1}{2+\sqrt{2}}, \frac{1}{2}\right)$ *we have two possible equilibria:*
   - *If* σ^I*: full collusion is an equilibrium for* $\delta \geq \frac{1-2S}{1-S} = \bar{\delta}(S)$ *and partial collusion* $\left(p^c = \frac{\delta - S(2-\delta)}{2\delta - 1}\right)$ *for* $S \in \left(\frac{1}{3}, \frac{1-\delta}{2-\delta}\right)$*.*
   - *If* σ^{II}*: full collusion is an equilibrium for* $\delta \geq \frac{2(1-2S)}{4(1-S)-(S+\sqrt{S(4+S)})} = \bar{\delta}(S)$*.*
4. *For* $S \geq \frac{1}{2}$ *full collusion is an equilibrium* $\forall\delta$*.*

Proof. See Appendix A. ☐

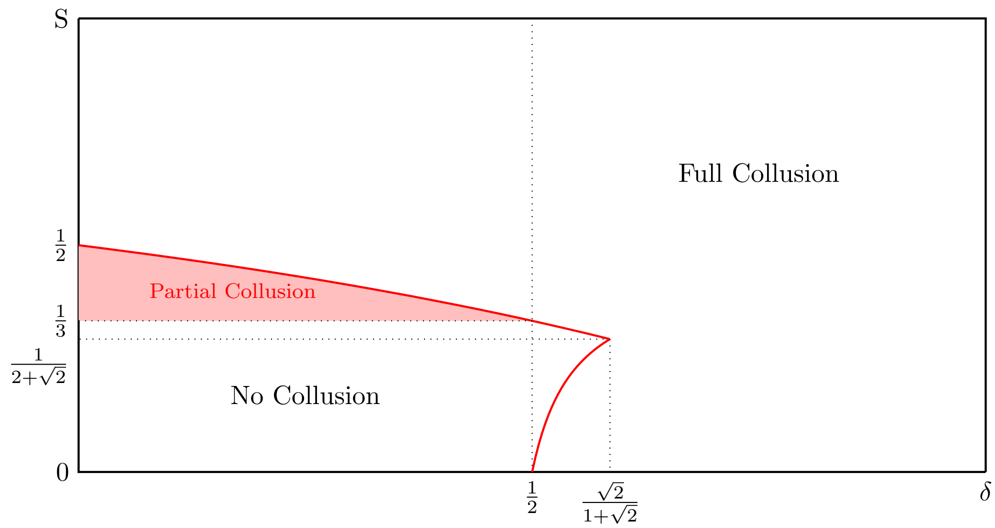

Figure 6: Collusion area with short-run customers when σ^I

In Figures (6) and (7), we represent the regions where a collusive outcome is sustained in equilibrium. Observe that in the regions where collusion is not an equilibrium, we have an equilibrium sequence of random prices where customers switch with a positive probability.

Figure (6) shows the area where a collusion outcome can be sustained when customers switch when indifferent. When the magnitude of switching costs is between $S \in \left(0, \frac{1}{3}\right)$, the full collusive outcome is harder to sustain compared to industries where customers do not have switching costs. Conversely, collusion is easier for the values of the parameter between $S \in \left[\frac{1}{3}, 1\right]$ and full collusion is an equilibrium $\forall \delta$ when $S \in \left[\frac{1}{2}, 1\right]$.[17] Furthermore, the value of the discount factor $\underline{\delta}(S)$ above which collusion is an equilibrium, is not monotone with the level of switching costs. For $S \in \left(0, \frac{1}{2+\sqrt{2}}\right]$ the profits in the punishment path increase with the level of switching costs, while they are decreasing for $S \in \left(\frac{1}{2+\sqrt{2}}, \frac{1}{2}\right)$. It is also the case that the price undercut necessary to attract the customer of the rival is increasing with the level of switching costs. Furthermore, for this range of switching costs we find an area where the only collusive outcome that firms can maintain is one where the collusive price is strictly lower than customer's valuation.

Figure (7) shows the area where a collusion outcome can be sustained when customers do not switch when indifferent. When the magnitude of switching costs is $S \in \left(0, \frac{1}{2}\right)$ the collusive outcome is harder to sustain compared to industries where customers do not have switching costs. As in the previous case, for $S \in \left[\frac{1}{2}, 1\right]$ collusion is always an equilibrium. Moreover, the value of the discount factor $\bar{\delta}(S)$ above which collusion is sustained is increasing but not monotone with the level of switching costs. It is concave for $S \in \left(0, \frac{1}{2+\sqrt{2}}\right]$, but convex when $S \in \left(\frac{1}{2+\sqrt{2}}, \frac{1}{2}\right)$. Convexity is shown in appendix (A). The monotonicity of $\bar{\delta}(S)$ means that the increase of expected profits in the continuation game dominates the decrease of profits

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$$\frac{\partial \underline{\delta}(S)}{\partial S} = \frac{-3+4S}{(1-S)^2} < 0. \qquad \text{The extreme values are: } \begin{cases} \lim_{S \to \frac{1}{2+\sqrt{2}}} \underline{\delta}(S) = \frac{\sqrt{2}}{1+\sqrt{2}}, \\ \lim_{S \to \frac{1}{2}} \underline{\delta}(S) = 0. \end{cases}$$

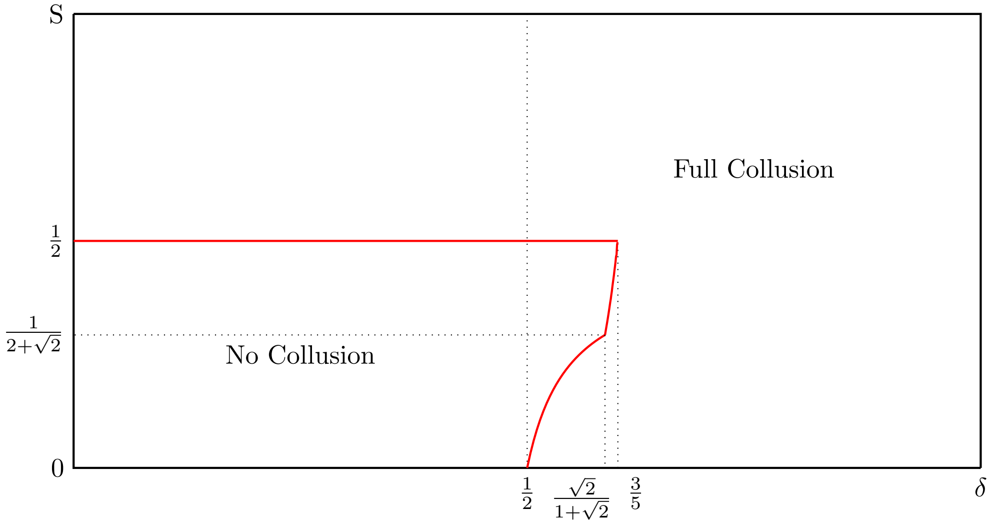

Figure 7: Collusion area with short-run customers when σ^{II}

due to a higher price undercut to attract the customers of the rival.[18] Finally, we define the area of full collusion as $C_{SL} : \{\delta, S | p^c = 1\}$, and this region is obtained easily from proposition (5).

Remark 4. *When customers are more prone to switch (equilibrium σ^I) the full collusive outcome can be sustained in a large region and less switching occurs in equilibrium.*

Notice that short-lived customers are more able to prevent collusion than short-sighted ones (if σ^{II} is played[19]). Since the immediate cost of deviation in both cases is similar (both kinds of consumers switch as soon as $p^d < p(c) - S$), it must be that the future gains from deviation are lower when consumers are myopic. Hence, although the incumbent indefinitely enjoys the whole market in the short-sighted case, its price is so low that a one time deviation in the short-lived case is more profitable. A similar analysis holds for a continuum of forward-looking consumers who do not coordinate on Pareto-dominating equilibria.

5.3 Increasingly Loyal Customers

In the base model we took the case of homogeneous customers in the sense that all customers pay the same switching cost if they decide to change supplier. Moreover, these switching costs are invariant over time. Here we consider the case in which switching costs increase with the number of periods a customers remains captive. Such a model is useful to capture situations of increasing synergies between the seller and the buyer, or the phenomenon of brand loyalty.

We denote the initial value of switching costs as S_0 and switching costs follows a function $f(S_t)$ that is increasing with the number of periods -t- a customer remains captive to a firm. Here the possibility

[18]

$$\text{The limiting values for } \underline{\delta}(S) \text{ are: } \begin{cases} \lim_{S\to \frac{1}{2+\sqrt{2}}} \underline{\delta}(S) & = \frac{\sqrt{2}}{1+\sqrt{2}}, \\ \lim_{S\to \frac{1}{2}} \underline{\delta}(S) & \overset{\left[\frac{0}{0}\right]}{\underset{\mathrm{H}}{=}} \lim_{S\to\frac{1}{2}} \frac{1}{\frac{\partial V}{\partial S}+1} = \frac{3}{5}. \end{cases}$$

[19]If σ^I is played, there is a small area on which collusion is possible for short-lived consumers, but not short-sighted ones

of heterogeneous switching costs arises as switchers have a lower switching costs that the ones remaining captive. Once a customer switches, her level of switching costs goes back to the initial level S_0.[20]

Observe that the results remain unchanged in the case of short-run customers. Therefore, the analysis is only interesting if we consider the case of long-lived customers. For simplicity we just treat here the case of short-sighted customers. We leave the case of forward-looking customers to future research. To simplify the analysis, we introduce the following assumption:

Assumption 6. *The function $f(S_t)$ is bounded to the customer's reservation price, that is,* $\lim_{t\to\infty} f(S_t) \leq 1$.

Then, we obtain that the punishment path is given by the following proposition

Proposition 6. *With short-sighted consumers, the equilibrium punishment path when one firm has both customers and the other is left with none, changes over time and it is given by:*

$$(p_t(\xi = 2), p_t(\xi = 0)) = (S_{t-1} - \delta(1-\delta)V(S), -\delta(1-\delta)V(S)), \tag{5.3}$$

and no customer switches.

The proof is relegated to appendix (B) but notice that the result is fairly intuitive. As the switching costs of customers increase with captivity so does the equilibrium punishment price. Observe that this punishment path is formed by a function that strictly increases over time and a constant, $p_t(2) = f(S_{t-1}) + C$ where $C = -\delta(1-\delta)V(S)$. At the beginning, after deviation from the collusive path have occurred, the price is low in order to retain customers at the firm, this is similar to an investing phase.[21] Later, as the switching costs of customers increase due to the periods of captivity, the firm increase prices, this is the harvesting phase.

After deviation and given that the punishment path in (5.3) constitutes an equilibrium, we can get the present discounted profits from deviation. The first thing to note is that as switching costs increase with captivity, a firm will necessary deviate from the collusive outcome at the first period by setting a price $p^d = p(c) - S_0$. Hence, it obtains a discounted profits of $\Pi^d = 2 \times (p(c) - S_0) + \delta\Pi(2)$. And we obtain that with the equilibrium prices:

$$\begin{aligned}
\delta\Pi(2) &= 2\delta\left(S_0 - \delta(1-\delta)V(S)\right) + 2\delta^2\left(S_1 - \delta(1-\delta)V(S)\right) + ... + 2\delta^t\left(S_{t-1} - \delta(1-\delta)V(S)\right) + ... \\
&= 2 \times \sum_{t\geq 0} \delta^{t+1}\left(S_t - \delta(1-\delta)V(S)\right) = 2 \times \delta(1-\delta)V(S)
\end{aligned}$$

By adding this last expression to Π^d we get that $\Pi^d = 2 \times (p(c) - S_0) + 2 \times \delta(1-\delta)V(S)$. To compare whether collusion is harder or easier to sustain we have to compare the previous expected profits to the case where customers have homogeneous switching costs. Remember than when customers have homogeneous switching costs the discounted profits from deviation are $\Pi^d_H = 2 \times (p(c) - S) + 2\delta S$. Therefore, collusion is

[20] Some recent studies [2] have considered the case where customers differ in their switching costs. However, in their analysis heterogeneity of switching costs is assumed, while heterogeneity in our model is endogenous.)

[21] Prices could even be negative at the beginning as for some $t < \bar{t}$, it might be that $f(S_{t-1}) < \delta(1-\delta)V(S)$.

harder when switching costs increase in captivity if

$$\begin{aligned}\Pi^d \geq \Pi_H^d &\iff 2 \times (p(c) - S_0) + 2 \times \delta(1-\delta)V(S) \geq 2 \times (p(c) - S) + 2\delta S \\ &\iff V(S) \geq \frac{S(\delta - 1) + S_0}{\delta(1-\delta)}\end{aligned}$$

Collusion then will be harder to sustain in an environment when switching costs increase with captivity the lower the initial level of switching costs and the higher the discounted value of switching costs. This inequality leads us to the following proposition, whose proof is direct:

Proposition 7. *For comparable values of switching costs, i.e.* $(1-\delta)V(S) = S$ *and short-sighted consumers, increasing switching costs make collusion harder to implement compared to constant ones.*

6 Discussion

In the case of two consumers, we have shown that maintaining a fully collusive outcome is easier when customers are short-lived or short-sighted rather than when they are forward-looking. The latter type of customers anticipate future rents coming from switching and the deviating firm does not need to reduce its price much in order to attract customers. As the profits from deviation are higher, implementing a collusive outcome becomes more difficult. We also find that when customers are sophisticated switching never occurs in equilibrium — in the region where full collusion is not sustainable, firms reduce the collusive price. In this regard, social welfare is higher as customers do not incur a wasteful cost of switching.

This result considerably changes when we consider a continuum of consumers. Now, coordination among consumers plays an crucial role at each stage of the game. If consumers are somehow pessimistic and believe the rest of the crowd will stay put, collusion becomes very easy to implement. However, if consumers are more optimistic and coordinate on the equilibrium maximizing their utility, additional constraints enter into consideration, and as in the case of two consumers, high discount factors prevent collusion. It is also noteworthy that the granularity of the market, i.e. whether it is only composed of two consumers or of a continuum of them, only plays a role if consumers are forward-looking. But in that case, more consumers seems to imply more collusion.

Finally, comparing the results we obtained with the case without switching costs, we have that with short-lived or short-sighted customers the collusive outcome without switching costs is sustained if the firm puts sufficient weight to the future, i.e. $\delta \geq \frac{1}{2}$. With short-run customers this lower bound is increased with intermediate values of switching costs and lowered with large switching costs. When customers are short-sighed any positive amount of switching costs reduces this bound and collusion is facilitated. When customers are forward-looking, the fully collusive outcome is never sustained without switching costs. The rents that customers will obtain after deviation are so large that they will always break the collusive outcome. Therefore, the existence of sufficiently large switching costs makes full collusion possible.

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Appendices

A Proofs for short-run consumers

Proof of lemma (3) : To show the non existence of pure strategy equilibrium when $0 < S < \frac{1}{2}$, we analyze different cases:

- No pair of prices (p_A, p_B) with $p_A = p_B$ is an equilibrium. Firm A obtains larger profits by setting a price $p'_A = \min\left\{p_A + \frac{S}{2}, 1\right\}$, since it supplies to its captive customers and it sets a higher price. In the case where $p_A = p_B = 1$, then firm A obtains larger profits by setting a price $p'_A = p_B - S$, in this case, and since indifferent customers buy from the lowest price firm, firm A obtains profits equal to $\pi_A(p'_A, p_B) = 2(1-S) > 1 = \pi_A(p_A, p_B)$.

- No pair of prices (p_A, p_B) with $p_A \neq p_B$ is an equilibrium.
  Let's take the case where $p_A > p_B$.

  $p_A > \min\{p_B + S, 1\}$. This cannot be an equilibrium because at this price firm A serves no consumer. It gets higher profits by setting a price $p'_A = \min\{p_B + S, 1\}$. If $p'_A = 1$, profits for firm A are $\pi_A(p'_A, p_B) = 1 > 0 = \pi_A(p_A, p_B)$. If $p'_A = p_B + S$, it obtains the same profits as before as indifferent customers buy from the lowest price firm.

  $p_A = \min\{p_B + S, 1\}$. If $p_A = p_B + S < 1$, firm A has a profitable deviation by setting a price $p_A = p_B + S - \varepsilon$, in this case firm A keeps its captive customer and gets profits equal to $\pi_A(p'_A, p_B) = p_B + S - \varepsilon > 0 = \pi_A(p_A, p_B)$. If $p_A = 1 \leq p_B + S$, firm A does not have any profitable deviation, but firm B obtains strictly higher profits by setting a price $p'_B = 1 - S$. Firm A does not sell and we are in the same case as before.

  $p_A \in (p_B, p_B + S)$. This cannot be an equilibrium because firm B have incentives to increase its price. Take that $p_A = p_B + \gamma$ for $\gamma \in (0, S)$. Then, firm B gets strictly higher profits by setting a price $p'_B = p_B + \varepsilon$ for $\varepsilon \in (0, S - \gamma)$ as it supplies to its captive customer but with a higher price.

  The same reasoning applies when $p_A < p_B$. *Q.E.D.*

Proof of Proposition (4) : Shilony (1977) proves that there exist a unique equilibrium of the game where the range of the support is $\bar{p} - \underline{p} \leq 2S$. We solve the problem by considering different ranges of p belonging to the support of the game

$\underline{p} + S \leqslant p < \bar{p}$. With p in this range, we get that $F(min(p+S, R)) = 1$ and $F(max(0, p-S)) = F(p-S)$. Therefore, we obtain $F(p-S) = 1 - \frac{V}{p}$ which is equivalent to:

$$F(p) = 1 - \frac{V}{p+S} \quad \text{for} \quad \underline{p} \leqslant p < \bar{p} - S.$$

$\underline{p} \leqslant p < \bar{p} - S$. With p in this range, we get that $F(max(0, p-S)) = 0$ and $F(min(p+S, 1)) = F(p+S)$. Therefore, we obtain $F(p+S) = 2 - \frac{V}{p}$ which is equivalent to:

$$F(p) = 2 - \frac{V}{p-S} \quad \text{for} \quad \underline{p} + S \leqslant p < \bar{p}.$$

$\bar{p} - S < p < \underline{p} + S$. The difference between any price in this range and any point in the support is lower than the value of switching costs. As a result, no firm loses or gain customers when setting this price. Accordingly, no firm plays with positive probability in this range of prices. Since the distribution function is monotone and increasing with p we have that the distribution function in this range is:

$$F(p) = 1 - \frac{V}{\bar{p}} \quad \text{for} \quad \bar{p} - S < p < \underline{p} + S.$$

Therefore, the distribution function is then:

$$F(p) = \begin{cases} 0 & \text{if } p < \underline{p}, \\ 1 - \frac{V}{p+S} & \text{if } \underline{p} \leqslant p < \bar{p} - S, \\ 1 - \frac{V}{\bar{p}} & \text{if } \bar{p} - S < p < \underline{p} + S, \\ 2 - \frac{V}{p-S} & \text{if } \underline{p} + S \leqslant p < \bar{p}, \\ 1 & \text{if } p \geqslant \bar{p}. \end{cases} \tag{A.1}$$

If the distribution function in expression (A.1) constitutes a mixed strategy equilibrium, then the expected profit of any firm is the same regardless at which point of the support it plays, that is, $\pi_i(p_i, \sigma_j^*) = V \ \forall p_i \in [\underline{p}, \bar{p}]$.

Consider firm i sets a price $p_i = \bar{p} - S - \varepsilon$ for any $\varepsilon \in (0, \bar{p} - \underline{p} - S]$. By setting this price, the firm never loses its captive customer and attracts the customer of the other firm when it charges a price $p_j \geq \bar{p} - \varepsilon$. Hence, the expected profit for firm i is:

$$(\bar{p} - S - \varepsilon)\,(1 + [1 - F(\bar{p} - \varepsilon)]) = (\bar{p} - S - \varepsilon) \times \left(\frac{V}{\bar{p} - S - \varepsilon}\right) = V.$$

Consider that firm i sets a price $p_i = \underline{p} + S + \varepsilon$ for any $\varepsilon \in (0, \bar{p} - \underline{p} - S]$. In this case, it will lose its customer if the other firm sets a price $p_j \leq \underline{p} + \varepsilon$. Hence the expected profit of the firm is:

$$(\underline{p} + S + \varepsilon)\left(1 - 1 \times \left[F(\underline{p} + \varepsilon)\right]\right) = \left(\underline{p} + S + \varepsilon\right) \times \left(\frac{V}{\underline{p} + S + \varepsilon}\right) = V.$$

Furthermore, if no firm plays with a positive mass the extremes of the support i.e $A(\underline{p}) = 0$ and $A(\bar{p}) = 0$. If this is the case, we obtain that no firm i loses its captive customer nor gains the customer of the rival by setting a price $p_i = \underline{p} + S$ or $p_i = \bar{p} - S$ respectively[22].

If $p_i = \underline{p} + S$, the firm gets $\underline{p} + S$ and this should be equal to V. This give the condition:

$$\underline{p} = V - S. \tag{A.2}$$

We obtain the same condition if and only if the distribution function in (A.1) has no atom at the lower bound of the support, $\lim\limits_{p \downarrow \underline{p}} F(\underline{p}) = 1 - \frac{V}{\underline{p}+S} = 0$.

If $p_i = \bar{p} - S$, the firm gets $\bar{p} - S$ and this should be equal to V. This gives the condition:

$$\bar{p} = V + S. \tag{A.3}$$

The same condition is obtained if and only if the distribution function in (A.1) has no atom at the upper bound of the support, $\lim\limits_{p \uparrow \bar{p}} F(p) = 2 - \frac{V}{\bar{p}-S} = 1$

Combining condition (A.2) and (A.3), we obtain that $\bar{p} - \underline{p} = 2S$ and the expected profit of the firm is $V = \underline{p} + S$. As a result, expression (A.1) can be written only as a function of the lower bound of the support $\underline{p}$.

$$F(p) = \begin{cases} 0 & \text{if } p < \underline{p}, \\ 1 - \frac{\underline{p}+S}{p+S} & \text{if } \underline{p} \leqslant p < \underline{p} + S, \\ 2 - \frac{\underline{p}+S}{p-S} & \text{if } \underline{p} + S \leqslant p < \underline{p} + 2S, \\ 1 & \text{if } p \geqslant \underline{p} + 2S. \end{cases} \tag{A.4}$$

To obtain the final expression of the distribution function, we make explicit use of the following lemma

Lemma 4. *The distribution function F(p) in expression (A.4) is atomless.*

Proof. We proceed by construction. First, using our tie-breaking rule, we calculate the profits that any firm obtains if it plays at the upper bound of the support. This allows us to obtain the value for the lower bound of the support. Finally, we use this value to prove that no atom exists at $\underline{p} + S$.

If a firm sets a price equal to $p = \underline{p} + 2S.$, then, by the tie-breaking rule, the firm loses customers if the other firm sets a price equal or below $\underline{p} + S$ and this happens with probability $\frac{\underline{p}-S}{\underline{p}}$. Therefore, by setting this price, the firm obtains profits:

$$\pi_i(\underline{p} + 2S, \sigma_j) = (\underline{p} + 2S) \times \left(1 - \frac{\underline{p} - S}{\underline{p}}\right) \iff (\underline{p} + 2S) \times \frac{S}{\underline{p}} = V$$
$$\iff (\underline{p} + 2S) \times \frac{S}{\underline{p}} = \underline{p} + S \iff \underline{p} = \sqrt{2}S$$

[22] Firms will only lose or gain customers by setting this price if firms played with a positive probability at the extremes of the support.

$\square$

Finally, with the lower bound $\underline{p} = \sqrt{2}S$, there exist no atom at $\underline{p} + S$. From the picture above, we see that the atom is equal to $A(\underline{p} + S) = \frac{\underline{p}-S}{\underline{p}} - \frac{S}{\underline{p}+2S}$ and it is easy to check that $A(\underline{p} + S) = 0$ when $\underline{p} = \sqrt{2}S$ and the distribution function is equal to

$$\sigma^{II} = F(p) = \begin{cases} 0 & \text{if } p < \sqrt{2}S, \\ 1 - \frac{(1+\sqrt{2})S}{p+S} & \text{if } \sqrt{2}S \leq p < \left(1+\sqrt{2}\right)S, \\ 2 - \frac{(1+\sqrt{2})S}{p-S} & \text{if } \left(1+\sqrt{2}\right)S \leq p < \left(2+\sqrt{2}\right)S, \\ 1 & \text{if } p \geq \left(2+\sqrt{2}\right)S, \end{cases} \tag{A.5}$$

and the expected profit of firms playing this equilibrium is $V = \left(1+\sqrt{2}\right)S$ which is the one introduced in the proposition.

(ii) If switching costs are $S \in \left(\frac{1}{2+\sqrt{2}}, \frac{1}{2}\right)$, there are two symmetric mixed strategy equilibria where firms randomize over a support $[\underline{p}, 1]$ according to $F(p)$ given by

(a) and each firm's expected profit is equals to

$$V = 1 - S$$

(b) and each firm's expected profit is equals to

$$V = \frac{S + \sqrt{S(4+S)}}{2}$$

From expression (A.5), we see that the upper-bound is an increasing function of the level of switching costs S. Since no firm will set a price above the customers reservation price, the distribution function that we obtained is only valid for a value of switching costs $S \in \left(0, \frac{1}{2+\sqrt{2}}\right]$. Consequently, when $S \in \left(\frac{1}{2+\sqrt{2}}, \frac{1}{2}\right)$ the upper-bound of the support is equal to 1 and the distribution function is:

$$F(p) = \begin{cases} 0 & \text{if } p < \underline{p}, \\ 1 - \frac{V}{p+S} & \text{if } \underline{p} \leqslant p < 1-S, \\ 1 - V & \text{if } 1-S < p < \underline{p}+S, \\ 2 - \frac{V}{p-S} & \text{if } \underline{p}+S \leqslant p < 1, \\ 1 & \text{if } p \geqslant 1. \end{cases} \tag{A.6}$$

The shape of the distribution function as well as the expected profit would depends on which end of the support firms plays with a positive probability.

If $F(\underline{p}) = 0$. Firms do not put mass at the lower bound of the support and the distribution function is given by

$$F(p) = \begin{cases} 0 & \text{if } p < V-S, \\ 1 - \frac{V}{p+S} & \text{if } V-S \leqslant p < 1-S, \\ 1 - V & \text{if } 1-S < p < V, \\ 2 - \frac{V}{p-S} & \text{if } V \leqslant p < 1, \\ 1 & \text{if } p \geqslant 1, \end{cases} \tag{A.7}$$

We proceed by construction. First, we define the lower bound and show that for our value of switching costs, the distribution function must have a mass at the upper bound of the support. Finally, the expression of the expected profits is obtained by the absence of atom at price $\underline{p} + S$.

When firms do not play with positive probability at the lower-bound of the support, then we obtain $\underline{p} = V - S$. If we assume that there is no atom at the upper bound, the expected profit is equal to $V = 1 - S$. However, if this is the case, we obtain that the atom at price $\underline{p} + S$ is equal to $A(\underline{p} + S) = \frac{1-2S(2-S)}{1-2S}$, and this is negative for some of the values of switching costs - a contradiction.

Therefore, the distribution function must have an atom at the upper bound of the support which is $A(1) = \frac{S-1+V}{1-S}$. Furthermore, when setting a price equal to $\underline{p} + S$ no firm attracts nor loses any customer.[23] Hence, no firm should play such a price with positive probability. The expected profitability is then obtained from expression $A(\underline{p} + S) = 0$. This implies that:

$$A(\underline{p} + S) = 0 \iff 1 - V = 2 - \frac{V}{V-S} \iff V = \frac{S}{V-S}$$

Observe that by our tie breaking rule, a firm setting a price $1 - S$ never loses its captive customer and attracts the customer of the rival when it sets a price equal to 1. This happens with probability equal to the atom $A(1)$. Therefore, by setting this

[23] This is the case as the range of the support is two times lower than the value of switching costs.

price, the firm obtains profits:

$$\pi_i(1-S,\sigma_j) = (1-S)\times(1+A(1)) \iff (1-S)\left(1+\frac{S-1+V}{1-S}\right) = V$$

Finally, it is left to prove that no firm wants to deviate by setting a price out of the support. Clearly, no firm deviates by setting a price above the upper bound of the support as it makes no sales.

if $F(1)=0$. If firms do not put mass at the upper bound of the support, the distribution function is given by expression

$$\sigma^I = F(p) = \begin{cases} 0 & \text{if } p < \frac{1-S}{2-S}, \\ 1-\frac{1-S}{p+S} & \text{if } \frac{1-S}{2-S} \le p < 1-S, \\ S & \text{if } 1-S \le p < \frac{1+S(1-S)}{2-S}, \\ 2-\frac{1-S}{p-S} & \text{if } \frac{1+S(1-S)}{2-S} \le p < 1, \\ 1 & \text{if } p \ge 1. \end{cases} \tag{A.8}$$

Again we proceed by construction. First, we obtain the expression of the expected profit by the assumption of no atom at the upper bound. By the same reasoning as before, there should be a mass at the lower bound. Finally, the expression for the lower bound is obtained by the absence of atom at the price $\underline{p}+S$.
When firms do not play with positive probability at the upper-bound of the support, then we obtain $V=1-S$. We reach the same contradiction as before by assuming that there is no atom at the lower bound of the support. Therefore, the distribution function must have an atom at the lower bound of the support and it is given by $A(\underline{p}) = \frac{\underline{p}+2S-1}{\underline{p}+S}$. Furthermore, as there is a positive mass at the lower bound of the support, no firm sets with positive probability the price $\underline{p}+S$. Then, the lower bound of the support is obtained from expression $A(\underline{p}+S)=0$

$$A(\underline{p}+S)=0 \iff 1-V = 2-\frac{V}{\underline{p}} \iff \underline{p} = \frac{1-S}{2-S}$$

Observe that by our tie-breaking rule, a firm setting a price equal to $\underline{p}+S$ never attracts the customer of the rival and it loses its customer when the other firm sets a price of $\underline{p}$, which happens with probability $A(\underline{p})$. Therefore, by setting this price, the firm obtains profits equal to:

$$\pi_i(\underline{p}+S,\sigma_j) = (\underline{p}+S)\times\Big(1-A(\underline{p})\Big) \iff (\underline{p}+S)\times\left(1-\frac{\underline{p}+2S-1}{\underline{p}+S}\right) = 1-S$$

Finally, it is left to prove that no firm wants to deviate by setting a price out of the support. As before, no firm has any incentive to set a price above 1 as it makes no sales. If a firm sets a price below the lower bound, the firm attracts the customer of the rival

Proof of Proposition (5).

Part I: In this first part, we prove the case of full collusion. Point (1) of the proposition is the standard case where customers do not have any switching cost, and we refer to any textbook of Microeconomic Theory for a formal proof. To prove points (2),(3),(4) we define first the present discounted profit of the initial path with full collusion. Because customers do not switch in the initial path, firms obtain the same profit at each period. Therefore, the present discounted profits are equal to:

$$\Pi_i(\mathbf{p}(0)) = \frac{1}{(1-\delta)}.$$

If we introduce this result in expression (5.2) together with the optimal deviation price p^d and the present discounted profit of the punishment path obtained in proposition (4), we obtain that for a level of switching costs $S \in \left(0, \frac{1}{2+\sqrt{2}}\right]$, the initial path constitutes an equilibrium if:

$$\Pi_i(\mathbf{p}(0)) = \frac{1}{1-\delta} \ge 2\times(1-S) + \delta\left(\frac{(1+\sqrt{2})S}{(1-\delta)}\right) = p_i^d + \delta\mathbb{E}(\Pi_i(\mathbf{p}(N)))$$
$$\iff \delta \ge \frac{1-2S}{2-(3+\sqrt{2})S} = \bar{\delta}(S)$$

For a level of switching costs $S \in \left(\frac{1}{2+\sqrt{2}}, \frac{1}{2}\right)$, whether the initial path is sustained or not depends on which punishment path firms coordinate. If there is coordination on the mixed strategy equilibrium σ^I, the initial path is sustained if:

$$\Pi_i(\mathbf{p}(0)) = \frac{1}{1-\delta} \geq 2 \times (1-S) + \delta\left(\frac{1-S}{1-\delta}\right) = p_i^d + \delta\mathbb{E}(\Pi_i(\mathbf{p}(N)))$$
$$\iff \delta \geq \frac{1-2S}{1-S} = \bar{\delta}(S),$$

and with coordination on σ^{II}, the initial path is sustained if:

$$\Pi_i(\mathbf{p}(0)) = \frac{1}{1-\delta} \geq 2 \times (1-S) + \frac{\delta V}{1-\delta} = p_i^d + \delta\mathbb{E}(\Pi_i(\mathbf{p}(N)))$$
$$\iff \delta \geq \frac{2(1-2S)}{4(1-S)-(S+\sqrt{S(4+S)})} = \bar{\delta}(S)$$

Finally, point (4) is trivial as the punishment path coincides with the initial path and full collusion is always sustained for any value of the discount factor.

Part II:

Now we consider the region where a partial collusive equilibrium is possible. When $S \geq 1/2$, we have already shown that $p^c = 1$ is sustainable. Therefore, we only consider there case where the magnitude of switching costs is $S < 1/2$. For a collusive price to be sustainable, we need it to be such that no profitable deviation by the firm occurs (condition 1). We also need the collusive price be higher than the expected profit in the static Nash reversion (condition 2). Because our Nash reversion is one with mixed strategy equilibrium we assume that firms in our model are risk neutral.[24]

We proceed to derive both conditions. Condition (2) is easily computed

$$p^c \geq (1-\delta) \times \mathbb{E}\left(\Pi\left(\mathbf{p}(N)\right)\right) = V \qquad (2)$$

Condition (1) will depend on the level of switching costs. If $S \leq \frac{p^c}{2}$ we have that the optimal deviation price is one where it attracts the customers of the rival. Therefore, no firm deviates form the collusive price p^c if

$$\frac{p^c}{1-\delta} \geq 2 \times (p^c - S) + \delta \times \mathbb{E}\left(\Pi\left(\mathbf{p}(N)\right)\right)$$
$$\iff (2\delta-1)p^c \geq \delta(1-\delta) \times \mathbb{E}\left(\Pi\left(\mathbf{p}(N)\right)\right) - 2(1-\delta)S \qquad (1)$$
$$(2\delta-1)p^c \geq \delta V - 2(1-\delta)S$$

When $S \in \left(\frac{p^c}{2}, \frac{1}{2}\right)$ the optimal deviation price is one arbitrarily close to p^c and the firm will not attract the customers of the rival, therefore we obtain that the firm does not deviate form the collusive outcome if

$$\frac{p^c}{1-\delta} \geq p^c + \delta \times \mathbb{E}\left(\Pi\left(\mathbf{p}(N)\right)\right) \iff p^c \geq (1-\delta) \times \mathbb{E}\left(\Pi\left(\mathbf{p}(N)\right)\right) = V \qquad (1)$$

and this coincides with condition (2).

The proof is easy and we just verify that both conditions (1) and (2) are satisfied for different values of switching costs. Indeed, in the main text we found the region where a full collusion outcome is sustainable, in this section we look whether partial collusion can be an equilibrium of the game.

$S \in \left(0, \frac{1}{2+\sqrt{2}}\right)$. In this case both equilibria σ^I and σ^{II} have the same discounted profit $V = (1+\sqrt{2})S$. By introducing it to condition (1) we get

$$p^c(2\delta-1) \geq S\left[(3+\sqrt{2})\delta - 2\right] \qquad (1)$$
$$\Leftrightarrow \begin{cases} p^c \geq \frac{S\left[(3+\sqrt{2})\delta-2\right]}{2\delta-1} \ (=\bar{p}) & \text{for } \delta > 1/2 \\ p^c \leq \bar{p} & \text{for } \delta < 1/2 \\ S = 0 & \text{for } \delta = 1/2 \end{cases}$$

and condition (2) is

$$p^c \geq (1+\sqrt{2})S$$

[24] This assumption is crucial to be able to compare profits of the collusive path with Nash

- For $\delta > 1/2$: if $\bar{p} > 1$, any form of collusion is impossible, firms win the expected profit earned by playing the Nash reversion. $\bar{p} \leq 1$ implies

$$\frac{S\left[(3+\sqrt{2})\delta - 2\right]}{2\delta - 1} \leq 1 \iff \delta(S(3+\sqrt{2}) - 2) \leq 2\delta - 1$$
$$\iff \delta \geq \frac{2S - 1}{S(3+\sqrt{2}) - 2} = f(S)$$

and the area where there is no full collusion in equilibrium we find that $\bar{p} > 1$.[25]

- For $\delta < 1/2$: $\bar{p}$ satisfies condition (2) if $\bar{p} > (1+\sqrt{2})S \Leftrightarrow \delta > 1$. So the maximum collusive price in that case is always below the expected gain in Nash reversion and condition (2) is violated.

As a result, for this range of switching costs there is no possible outcome with partial collusion

$S \in \left(\frac{1}{2+\sqrt{2}}, \frac{1}{2}\right)$ For the equilibrium σ^{II} we know that $V = \frac{S+\sqrt{S(4+S)}}{2}$. By introducing this to condition (1) we get

$$p^c(2\delta - 1) \geq \frac{5\delta S - 4S + \delta\sqrt{S(4+S)}}{2} \qquad (1)$$
$$\Leftrightarrow \begin{cases} p^c \geq \frac{5\delta S - 4S + \delta\sqrt{S(4+S)}}{2(2\delta-1)} \ (= \bar{p}) & \text{for } \delta > 1/2 \\ p^c \leq \bar{p} & \text{for } \delta < 1/2 \\ 0 \geq \frac{5\delta S - 4S + \delta\sqrt{S(4+S)}}{2} \Leftrightarrow S = 0 \text{ or } S \geq \frac{1}{2} & \text{for } \delta = 1/2 \end{cases}$$

and condition (2) is

$$p^c \geq \frac{S + \sqrt{S(4+S)}}{2}$$

- For $\delta > 1/2$: if $\bar{p} > 1$, any form of collusion is impossible, firms win the expected profit earned by playing the Nash reversion. $\bar{p} \leq 1$ implies that

$$\frac{5\delta S - 4S + \delta\sqrt{S(4+S)}}{2(2\delta - 1)} < 1 \iff 5\delta S - 4S + \delta\sqrt{S(4+S)} < 2(2\delta - 1)$$
$$\iff \delta \geq \frac{4S - 2}{5S - 4 + \sqrt{S(4+S)}} = f(S)$$

and the area where there is no full collusion in equilibrium we find that $\bar{p} > 1$.

- $\delta < 1/2$: $\bar{p}$ satisfies condition (2) if $\Leftrightarrow \bar{p} \geq V$

$$\frac{5\delta S - 4S + \delta\sqrt{S(4+S)}}{2(2\delta - 1)} < \frac{S + \sqrt{S(4+S)}}{2}$$
$$\iff -3S + \sqrt{S(4+S)} \leq \delta \times \left(-3S + \sqrt{S(4+S)}\right) \iff \delta \geq 1$$

and the profits obtained by Nash reversion are higher than the ones of maximal collusion.[26] We find then that, in the equilibrium σ^{II} there exist no equilibrium with partial collusion.

For the equilibrium σ^I we know that $V = 1 - S$. By introducing this to condition (1) we get

$$p^c(2\delta - 1) \geq \delta - S\left[2 - \delta\right] \qquad (1)$$
$$\Leftrightarrow \begin{cases} p^c \geq \frac{\delta - S[2-\delta]}{2\delta - 1} \ (= \bar{p}) & \text{for } \delta > 1/2 \\ p^c \leq \bar{p} & \text{for } \delta < 1/2 \\ S \geq 1/3 & \text{for } \delta = 1/2 \end{cases}$$

and condition (2) is

$$p^c \geq 1 - S$$

[25] Observe that the last inequality comes from the fact that $S < \frac{2}{3+\sqrt{2}}$ which is the case for the value of switching costs that we are considering here.

[26] Observe that $\sqrt{S(4+S)} > 3S \iff S < \frac{1}{2}$ which is the case that we are considering.

- $\delta > 1/2$: if $\bar{p} > 1$, any form of collusion is impossible, firms earn the expected profit earned by playing the Nash reversion. $\bar{p} \leq 1$ implies that
$$\frac{\delta - S\left[2-\delta\right]}{2\delta - 1} \leq 1 \iff 1 - 2S \leq (1-S)\delta \iff \delta \geq \frac{1-2S}{1-S} = f(S)$$
and the area where there is no full collusion in equilibrium we find that $\bar{p} > 1$.

- $\delta < 1/2$: Full collusion is possible if $\bar{p} \geq 1 \Leftrightarrow \delta \geq f(S)$ with $f(S) = \frac{1-2S}{1-S}$. $f(1/3) = 1/2$ and $f(1/2) = 0$. When $\delta < f(S)$, full collusion is not possible but partial partial collusion is possible if $p^c < 1$ is. To check that, we need to verify that $\bar{p}$, i.e. the maximum collusive price, satisfies condition (2): $\bar{p} \geq 1 - S$
$$\frac{\delta - S\left[2-\delta\right]}{2\delta - 1} \geq (1-S) \iff \delta(3S-1) \leq (3S-1)$$
and the above expression is $\delta \geq 1$ if $S \leq \frac{1}{3}$ and $\delta \leq 1$ if $S \geq \frac{1}{3}$. Therefore, for $S \geq 1/3$, the collusion price is higher than the expected profit in Nash reversion players play the highest collusive price possible $p^c = \bar{p} = \frac{\delta - S(2-\delta)}{2\delta - 1}$. For $S < 1/3$, collusion is not possible and players play the Nash reversion.

Concavity of $\underline{\delta}(S)$ in σ^{II} for $S \in \left(\frac{1}{2+\sqrt{2}}, \frac{1}{2}\right)$: Here we show that the function $\underline{\delta}(S)$ is concave. Remember that we have obtained the expression:
$$\underline{\delta}(S) = \frac{1-2S}{2\left(1 - S - \frac{V}{2}\right)},$$
and solving for the expected profit $V^* = \frac{S}{V^* - S}$ and restricting to the case of a positive solution $V^* > 0$, we obtain:
$$V^* = \frac{S + \sqrt{S(4+S)}}{2}.$$
Finally, combining both expressions we get:
$$\underline{\delta}(S) = \frac{2(1-2S)}{4 - 5S - \sqrt{S(4+S)}},$$
and computing the first and second-order conditions:
$$\frac{\partial \underline{\delta}(S)}{\partial S} = \frac{1}{4S(1+S) + \sqrt{S(4+S)}(5S+4)} > 0$$
$$\frac{\partial^2 \underline{\delta}(S)}{\partial S^2} = -\frac{2\left(4 + 17S + 5S^2 + \sqrt{S(4+S)}(8+4S)\right)}{\sqrt{S(4+S)}\left(4S(4+S) + \sqrt{S(4+S)}(4+5S)\right)^2} < 0,$$
we obtain that the function $\underline{\delta}(S)$ is concave with respect to the level of switching costs.

B Proof for the extensions

Proof of proposition (6): We start by obtaining a candidate equilibrium in the sub-game where one of the firms have both customers. Later, we check that no-firm wants to deviate. Therefore, we work with the sub-game when one of the firms have both customers, by deviating at the first period, and for $t-1$ periods there has not been any switch by any customer. Therefore, at period t there are two types of customers in the economy. The high switching costs customer with switching costs S_t and the low switching costs customer with switching costs S_{t-1}. We define by $\Pi(2) = \sum_{t\geq 0} \delta^t \left(2 \times p_t\right)$ and $\Pi(1) = \sum_{t \geq 0} \delta^t \times p_t$ the expected discounted profits of having two or one customer respectively and the prices p_t stand for the equilibrium price. The firm having no customer obtains zero discounted profits and as in the main text, we use this condition to obtain the equilibrium profit. However, when customers differ on the level of switching costs two possibilities arise. The firm having no customers can undercuts the price to attract both customers, or it sets a price to attract the customer with low switching costs. We proceed to analyze both possibilities.

Attract both customers : The firm having no customers have zero discounted profits, by undercutting the price by S_t it attracts both customers. In the continuation game as there is no further switch it gets a discounted profit of $\delta\Pi(2)$. Therefore, the equilibrium price is obtained by expression
$$2 \times (p_t - S_t) + \delta \Pi(2) = 0 \tag{B.1}$$

by doing some algebra we obtain

$$2 \times (p_t - S_t) \times \delta^t = -\delta\Pi(2) \times \delta^t \iff \sum_{t\geq 0}\Big(2 \times (p_t - S_t) \times \delta^t\Big) = \sum_{t\geq 0}\Big(-\delta\Pi(2) \times \delta^t\Big)$$
$$\iff \Pi(2) - 2 \times V(S) = -\frac{\delta}{1-\delta} \times \Pi(2) \iff \Pi(2) = 2(1-\delta)V(S),$$

and by introducing the last result to equation (B.1) we obtain that the punishment equilibrium prices is $p_t = S_t - \delta(1-\delta)V(S)$, and the pair of prices or punishment path is then

$$\mathbf{p}(i) = \{S_t - \delta(1-\delta)V(S), -\delta(1-\delta)V(S)\}\,.$$

However, with this punishment path, the low switching costs customer switches to the firm having no customers. Hence, if the deviant wants to ensure that no customer switches, the difference in prices has to be equal to the magnitude of the low switching cost customer. Therefore, a punishment path where no consumer switches is given expression (5.3) in the proposition.

Attract low switching costs customer : If the firm with no customers wants to attract only the low switching costs customers, it needs to undercut the price of the rival by S_{t-1}. The continuation game that follows, and assuming that there is no further switch, is one where each firm has a customer each and the discounted profits are given by $\delta\Pi(1)$. Therefore, the equilibrium price is obtained by expression

$$(p_t - S_{t-1}) + \delta \times \Pi(1) = 0 \tag{B.2}$$

and by doing a similar algebra as before, we obtain

$$(p_t - S_{t-1}) \times \delta^t = -\delta\Pi(1) \times \delta^t \iff \sum_{t\geq 0}\Big((p_t - S_{t-1}) \times \delta^t\Big) = \sum_{t\geq 0}\Big(-\delta\Pi(1) \times \delta^t\Big)$$
$$\iff \Pi(1) - \delta \times V(S) = -\frac{\delta}{1-\delta} \times \Pi(1) \iff \Pi(1) = \delta(1-\delta)V(S).$$

If we introduce this result into (B.2) we obtain that the equilibrium price is equal to $p_t = S_{t-1} - \delta^2(1-\delta)V(S)$ and the punishment path arising is defined by

$$p_t(i) = \{S_{t-1} - \delta^2(1-\delta)V(S), -\delta^2(1-\delta)V(S)\}$$

However, with this punishment path no customer switches as the difference between prices is equal to the magnitude of the low switching costs consumers. But then, the way we have obtained this candidate equilibrium price is false as the candidate price equilibrium p_t is different from the one in expression $\Pi(1)$.

Now, we need to prove that there is no profitable deviation from the proposed punishment path. Observe that as customers are short-sighted, no customer switches as the price difference is equal to the value of low switching costs customers. We start by considering possible deviations by the firm with no customers.

$p_{-i} = -\delta(1-\delta)V(S) - \epsilon$. There are two possible situations

$\epsilon = S_t - S_{t-1}$. This way the firm attracts both consumers and its expected discounted profits are

$$2 \times (S_{t-1} - S_t - \delta(1-\delta)V(S)) + \delta\Pi(2) \iff 2 \times (S_{t-1} - S_t) - 2\delta(1-\delta)V(S) + 2\delta(1-\delta)V(S)$$
$$\iff 2 \times (S_{t-1} - S_t) < 0 \qquad \text{as} \quad S_t > S_{t-1}$$

$\epsilon > 0$. This way the firm attracts only the low switching costs customers and its expected discounted profits are

$$-\delta(1-\delta)V(S) + \delta\Pi(1) \iff -\delta(1-\delta)V(S) + \delta^2(1-\delta)V(S) \iff -\delta(1-\delta)^2V(S) < 0$$

$p_{-i} = -\delta(1-\delta)V(S) + \epsilon$. This is not a profitable deviation as the firm does not attract any customer by setting this price and its expected discounted profits are zero.

Finally, It is left to check that the firm having both customers does not want to deviate from the proposed punishment path.

$p_i = S_{t-1} - \delta(1-\delta)V(S) - \epsilon$. It is easy to see that this is not a profitable deviation as the firm has the same demand and it sets a lower price.

$p_i = S_{t-1} - \delta(1-\delta)V(S) + \epsilon$. For $\epsilon = S_t - S_{t-1}$. This does not constitute a profitable deviation if

$$2\times\left(S_{t-1} - \delta(1-\delta)V(S)\right) + \delta\Pi(2) \geq S_t - \delta(1-\delta)V(S) + \delta\Pi(1)$$
$$\iff 2S_{t-1} - S_t \geq \delta\left(\Pi(1) - (1-\delta)V(S)\right)$$

And it depends on the rate of growth of the switching costs and the expected profit obtained when each firm has a customer each. Hence, by assuming a certain form of the function $f(S_t)$, the following is fulfilled and the proposed punishment path constitutes an equilibrium.