Uncertainty about Uncertainty in Communication

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Abstract

I study how higher order uncertainty affect communication outcomes when the sender's preference is uncertain. I identify a '*richness*' condition on the players' type spaces, under which in every equilibrium, there exists some good senders who cannot fully reveal their information, even though they have no conflict of interest with the receiver, and they never send the same message as the bad senders. Moreover, under a '*contagion condition*', no good sender can fully reveal her information in any equilibrium. By applying my model to organizational design, I show that the principal is more inclined to delegate decision rights when the agent faces higher order uncertainty.

Keywords: higher order uncertainty, cheap talk, separating equilibrium

JEL Codes: D82, D83

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1 Introduction

When experts communicating with decision makers, the latter is rarely sure about the former's underlying motives. In ancient dynasties, emperors suspect their ministers' unrighteousness and loyalty. In stock markets, investors doubt about the analysts' hidden incentives. In bureaucracies, clerks are cautious about the potential conflicts with their colleagues. In these situations, the expert will feel uncertain about the decision maker's belief if the latter has no credible device to signal trust.[1]

I study the impact of higher order uncertainty to cheap talk communication (Crawford and Sobel [1982]) when the sender can have two possible preferences: either she is good, i.e. her preference is perfectly aligned with the receiver's; or she is bad, and simply aims to maximize the receiver's action. The receiver can have various beliefs and higher order beliefs on the sender's preference, and the sender can have various higher order beliefs over the receiver's beliefs. I adopt the universal type space formulation (Mertens and Zamir [1985]) and show that when the two players' type spaces satisfy a '*richness condition*', then in every equilibrium, there is a positive probability that the good sender cannot fully transmit her information, even though she does not pool with the bad ones (good senders pool within themselves). Furthermore, under a '*contagion condition*', no good sender can fully reveal her information in any equilibrium.

My model has a broad range of economic implications, especially in the situation where communication is harassed by rumors. In the above examples, rumors can be deliberately spread by political enemies, competing analysts, discontented former co-workers, etc. They may also arise inadvertently among the public. The subtle thing is, whether the decision maker trust the rumor or not mainly depends on his subjective evaluation, other than the informativeness of the rumor on the sender's true preference.

My '*no full communication*' result is in sharp contrast with the case in which the sender does not face higher order uncertainty (Morgan and Stocken [2003]): a good sender can fully communicate her information as long as she can fully separate herself from the bad ones. However, when injecting higher order uncertainty into the game, the marginal good sender's expected loss is strictly positive in every equilibrium, since she is unsure about the receiver's action after receiving her equilibrium message. Therefore, the largest equilibrium action below the marginal type must be bounded away from it, and the receiver's sequential rationality condition forces some good senders to pool within

[1] In my model, the receiver cannot credibly convey his prior belief through cheap talk.

themselves.

Under the richness condition which leads to my main result, the players' posterior beliefs cannot be originated from the same prior, i.e. they '*agree to disagree*' (Aumann [1976]). Intuitively, in context of rumors, my condition implies that:

1. Rumors have an impact, even when they are unfounded. Historical evidence ranges from the Great Purge of Joseph Stalin, to...

2. Rumors affect everyone, and almost equally. If it is possible that a good sender has a kind of belief, then it is also possible for a bad one to have that belief. This means, even when the sender is congruent, it does not give her more confidence that the receiver will not trust the rumor. When rumors prevail, everyone has a reason to fear that he or she is going to be distrusted.

3. Anticipating that the receiver has heard a rumor, the sender is very uncertain about what is in the receiver's mind, i.e., the receiver may have a potentially rich set of beliefs, and such beliefs cannot be communicated credibly to the sender via cheap talk.

As a direct application to my model, I revisit delegation problem when a better informed agent faces higher order uncertainty. I show that higher order uncertainty decreases the receiver's expected welfare, and there is no cross-type compensation. Delegating decision rights to the agent brings an additional benefit to the principal: eliminating the welfare losses caused by higher order uncertainty, since the principal's belief and higher order beliefs are no longer relevant when decision rights are being delegated.

Related Literature: This paper revisits the main insight of Crawford and Sobel (1982), that in general, the sender's information can be fully transmitted via cheap talk messages if and only if her preference is fully aligned with the receiver's. In a complementary paper, Pei (2012) shows that when the sender needs to acquire information at a cost, then she must communicate all her information in any equilibrium, regardless of the conflict of interests between the players. This paper examines the case in which the sender cannot fully communicate her information, even though there is no conflict of interest.

This paper is related to two strands of literature: strategic communication when the receiver has private information; and when the sender's preference is uncertain. Starting from Watson (1996), many papers analyzed situations where the receiver has private information on the state of the world, and examine how this private information improves or hinders communication. Olszewski (2004) shows that when the sender has honesty concerns, then information can be fully revealed when the receiver also receives a private signal. In contrast, many recent papers describe various environments in which the receiver's private information makes communication less informative.[2] In contrast, my paper focus on the receiver's private information on the sender's preference, and allows for more general forms of uncertainty and higher order uncertainty.

On the uncertain sender-preference side, this paper is closest to Morris (2001) in formulating the senders' preferences, where the good sender's preference is fully aligned with the receiver's, and the bad sender simply wishes to maximize the receiver's action. The main differences are: whether or not the receiver has private information, and whether the game lasts for 1 period or 2 periods. Another related paper is Blume and Board (2012), where they assume both the sender and the receiver are uncertain about each other's language competence. They show that informative equilibrium still exists, and the optimal communication protocol will try to make use of all the messages available.

2 A Motivating Example

An emperor (receiver) needs his minister's (sender) advice on how much to spend on a construction project. Let θ be the most appropriate amount, which is distributed uniformly on $[0, 1]$, and a be the emperor's decision. The minister knows the θ, and can be either *upright* (good) or *corrupted* (bad). An upright minister's preference is fully aligned with the emperor's:

$$U^g = U^r = -(a - \theta)^2$$

A corrupted minister would always like the emperor to spend more, so that he can divert more money into his own pocket.

$$U^b = a$$

[2] For example, Galeotti et.al. (2009) studies communication in networks, and shows that a sender is less likely to report information truthfully to a receiver if the latter has too many other sources of information. Lai (2009) studies a context where the amateur receiver can tell the difference between high and low states, but the cut-off threshold is private information. Chen (2009), Moreno de Barreda (2010), Ishida and Shimizu (2012) identify situations where the receiver receives a private signal about the fundamentals, and examine how it hinders communication and make some equilibria not monotone. Goltsman and Pavlov (2011) examine the case in which the sender talks to multiple receivers and characterize the equilibria of the game.

I assume that the emperor does not know the minister's type, and believes that the minister is corrupted with probability η. Let us consider the following two cases separately:

1. The minister knows the emperor's belief, i.e., there is only '*1st order uncertainty*'.

2. The minister feels unsure about how the emperor perceives of him, perhaps he is uncertain about whether the emperor has believed in the rumors his political enemies have spread, or he does not know whether the emperor is suspicious or credulous. In this case, the minister also faces '*2nd order uncertainty*'.

In the first case, there exists an equilibrium where the good minister fully reveals θ if and only if:

$$\theta < \frac{1}{1+\sqrt{\eta}} \tag{2.1}$$

When information is fully revealed, the emperor takes the most appropriate action. If θ is above this threshold, the good minister pools with the bad one, which induces action $a = \frac{1}{1+\sqrt{\eta}}$. In this example, the good minister fully reveals his information with probability $\frac{1}{1+\sqrt{\eta}}$.

In the second case, suppose the minister believes that $\eta \in \{\eta_1, \eta_2\}$ and $\eta = \eta_1$ with probability p_1 and $\eta = \eta_2$ with probability p_2, where $p_1 \in (0,1)$, then the qualitative features of the equilibrium completely changes. Next, I show the following claim, which explains the qualitative differences between the equilibria with or without higher order uncertainty.

Claim 1. *The good minister can never fully reveal his information.*

Proof of the Claim: For any message m^b sent by the bad type with positive probability in equilibrium:

$$p_1 a^1(m^b) + p_2 a^2(m^b) \in \arg\max_{m \in M} \left\{ p_1 a^1(m) + p_2 a^2(m) \right\}$$

where M is the set of messages. Define a subset M^b as:

$$M^b \equiv \left\{ m \middle| p_1 a^1(m) + p_2 a^2(m) = \max_{m' \in M} \left\{ p_1 a^1(m') + p_2 a^2(m') \right\} \right\}$$

the bad type will always send a message in M^b.

Next, a good sender with type θ sends message $m^b \in M^b$ only if:

$$\sum_i p_i (a^i(m^b) - \theta)^2 \leq \arg\max_{m \in M} \sum_i p_i (a^i(m) - \theta)^2 \tag{2.2}$$

which can be re-written as:

$$\frac{1}{2}\Big[\sum_i p_i a^i(m^b)^2 - \sum_i p_i a^i(m)^2\Big] \leq \theta\Big[\sum_i p_i a^i(m^b) - \sum_i p_i a^i(m)\Big] \tag{2.3}$$

So there exists a threshold θ_0, such that the good sender sends a message in M^b if and only if $\theta > \theta_0$.

For any two messages $m, m' \in M^b$, both messages are sent with positive probability by good types if and only if:

$$\sum_i p_i a^i(m')^2 = \sum_i p_i a^i(m)^2$$

If this is not the case, let the LHS of the above equality larger than the RHS, then no good sender sends m', which means it is only sent by the bad types. Hence, it implies that all receivers take the same action upon receiving m'. This contradicts the fact that

$$\sum_i p_i a^i(m') = \sum_i p_i a^i(m)$$

and

$$\sum_i p_i a^i(m')^2 > \sum_i p_i a^i(m)^2$$

we have assumed before. So equality must hold. This implies that all types of senders are indifferent between m and m'.

Suppose there exists a message $m \in M^b$ such that $a^i(m) = \theta_0$ for $i = 1, 2$. Then, good senders with any θ prefer m than other messages in M^b, which implies that all bad senders send m. Since different types of receivers have different beliefs over the sender's distribution, their reactions upon receiving m cannot be the same, which leads to a contradiction. So there does not exist a message which makes the expected loss of the marginal good sender $\theta = \theta_0$ equal to 0. If type $\theta_0 - \varepsilon$ good sender can fully separate himself, then type $\theta_0 + \varepsilon$ prefers to induce $a = \theta_0 - \varepsilon$ than any message in M^b if we make ε small enough. This means, there does not exist a type arbitrary close to θ_0 who can fully separates himself.

Let a_1 be the largest equilibrium action induced by good senders with $\theta < \theta_0$, $\theta_0 - a_1 > 0$. The receiver's sequential rationality condition requires that good senders with $\theta \in [\theta_1, \theta_0]$ induces action a_1, where $\theta_1 \equiv 2a_1 - \theta_0$. Since type θ_1 good sender has to be indifferent between a_1 and the next equilibrium action: a_2, $a_2 - \theta_1 = \theta_1 - a_1$. Iterating this process, the equilibrium is characterized by

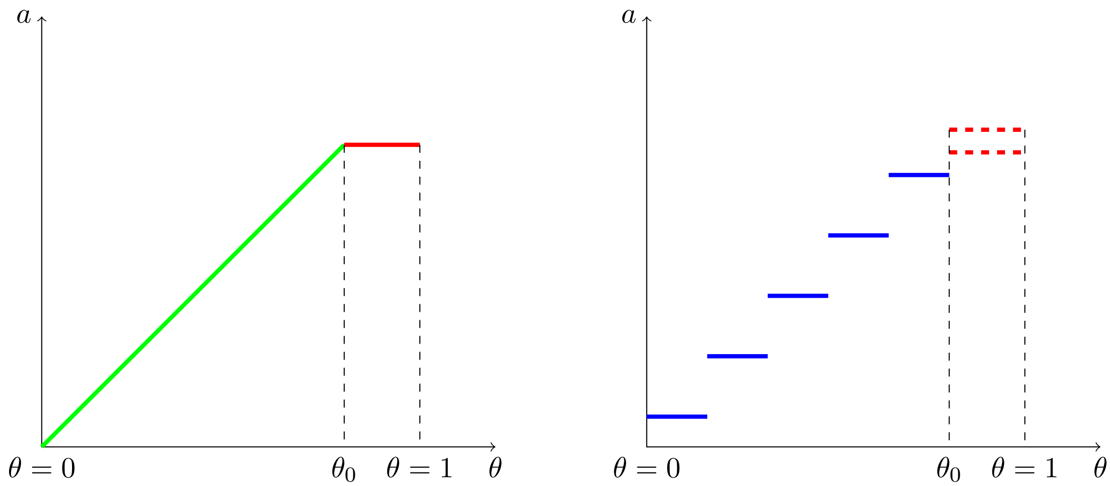

Figure 1: Good sender's equilibrium strategy with or without second order uncertainty.

a partition of equal-lengths intervals within $[0, \theta_0]$. Good senders within each interval pool within themselves, although not being joint by any bad senders. In any equilibrium, no good sender can fully reveal his information. □

The comparison of the good minister's equilibrium strategy with or without second order uncertainty is shown in the graph, where the red line is the range in which good ministers pools with bad one, the green line is the fully separating range, and the blue line is the range where good ministers pool within themselves.

3 Cheap Talk Under Higher Order Uncertainty

Let $\theta \in [0, 1]$ be the state of the world and $a \in A = \mathbb{R}$ be the receiver's action.[3] The sender knows θ and can be either '*good*' or '*bad*'. The good sender's preference is fully aligned with the receiver's: $u(a, \theta)$, which depends on the latter's action and the state of the world. The bad sender's preference is given by $u^b = a$. I assume that $u \in \mathcal{C}^2$ and

$$\frac{\partial^2 u}{\partial a \partial \theta} > 0, \quad \frac{\partial^2 u}{\partial a^2} < 0, \quad u(\theta, \theta) = \max_{a \in A} u(a, \theta)$$

The receiver's prior on θ is $F(\theta)$, which is absolutely continuous, has full support over Θ and adopts pdf $f(\theta)$.

[3]It is obvious that all our results can be carried over to the case where Θ is a convex bounded subset of $\mathbb{R}$.

Remark: I assume there is no conflict of interest between her and the receiver, in order to isolate the effect of higher order uncertainty on full information revelation.

3.1 Types and Beliefs

For higher order beliefs, I adopt the implicit formulation of Mertens and Zamir (1985), in which a '*type*' is characterized by its belief on the joint distribution of the fundamentals and the other players' type.[4] In this paper, the sender and the receiver's type spaces are characterized by $(\Psi, \hat{u}, \hat{\theta}, \hat{\pi})$ and $(\Phi, \hat{\tau})$ respectively. The sender's type is $\psi \in \Psi$, which is described by her preference $\hat{u}(\psi) \in \{g, b\}$, her belief on θ: $\hat{\theta}(\psi) \in \Delta(\Theta)$, and her belief over the receiver's type $\hat{\pi}(\psi) \in \Delta(\Phi)$. The receiver's type is $\phi \in \Phi$, which is described by his belief over the sender's type $\hat{\tau}(\phi) \in \Delta(\Psi)$. For convenience, I assume both Ψ and Φ are finite, and there is *no higher order uncertainty* if Φ is a singleton, and Ψ has only 2 elements.

I introduce several definitions on the players' types as well as their type spaces, which are essential to my analysis. First, in order to capture the fact that '*rumors affect everyone almost equally*', I define '*companion type*':

Definition 1 (Companion Type). *ψ and ψ' are 'companions' if and only if $\hat{u}(\psi) \neq \hat{u}(\psi')$ and $\hat{\pi}(\psi) = \hat{\pi}(\psi')$.*

Intuitively, two sender types are companions if their beliefs over the receiver's type are the same, even though their preferences differ. Next, I define the '*richness*' condition on Ψ and Φ:

Definition 2 (Richness). *Ψ is 'rich' if any $\psi \in \Psi$ has a companion type. Φ is 'rich' if for any $\psi' \in \Psi$, $\hat{u}(\psi') = b$, there exists $\psi'' \in \Psi$, $\hat{u}(\psi'') = g$ as well as $\phi, \phi' \in \Phi$, such that $\hat{\pi}(\phi)[\psi] = \hat{\pi}(\phi')[\psi]$ for any $\psi \neq \psi', \psi''$ and $\hat{\pi}(\phi)[\psi] \neq \hat{\pi}(\phi')[\psi]$ for $\psi = \psi'$ or ψ''.*

The richness of Φ implies that the sender faces '*sufficient amount*' of higher order uncertainty, and the richness of Ψ captures the fact that both good senders and bad senders can have the same kind of belief after anticipating a rumor. Under this assumption, I use the following expression for sender and receiver's type space:

$$\Psi = \{g_1, ..., g_n, b_1, ..., b_n\}, \quad \Phi = \{\phi_1, ..., \phi_k\}$$

[4]Mertens and Zamir (1985) show that any implicit type space that has no redundant types and satisfies some topological restrictions is a belief-closed subset of the universal type space. Brandenburger and Dekel (1993) show the relationship between this formulation and the coherency condition on the belief hierarchies.

where g_i and b_i are companions. I make the following assumptions:

Assumption 1. *Ψ and Φ are rich.*

Assumption 2 (Full Support)**.** *$\hat{\pi}(\psi)$ assigns positive probability to all $\phi \in \Phi$. $\hat{\pi}(\phi)$ assigns positive probability to all $\psi \in \Psi$.*

3.2 Equilibrium

The sender's message is denoted by $m \in M$, and her communication strategy is $m : \Psi \times \Delta(\Theta) \to \Delta(M)$. For simplicity, let $m(\psi, \theta)$ be the message sent when her type is ψ and the state is θ. After receiving the message, type ϕ receiver updates his belief, and his action rule is given by:

$$a_\phi(m) \in \arg\max_{a \in A} \int_\theta u(a, \theta) d\mathcal{F}(\theta | m, \phi)$$

where $\mathcal{F}(\cdot | m, \phi)$ is type ϕ receiver's posterior belief on θ after receiving m. The concavity of u ensures that he will always use a pure strategy. The solution concept in this paper is *Perfect Bayesian Equilibrium* (PBE, or 'equilibrium' for short). I define '*fully revealing*' as follows:

Definition 3. *Type g_i sender fully reveals herself at θ if and only if:*

$$a_\phi(m(g_i, \theta)) = \theta$$

for any $\phi \in \Phi$.

Also, as defined in Crawford and Sobel (1982), an equilibrium is '*babbling*' if all types of receivers choose

$$a \in \arg\max_a \int_\theta u(a, \theta) dF(\theta)$$

Apparently, this equilibrium always exits. Next, I introduce two classes of equilibria through the degree of separation they achieve:

Definition 4 (Maximal Separating Equilibrium)**.** *An equilibrium is 'maximal separating' if it is not babbling, and type g_i fully reveals herself at θ if and only if $m(g_i, \theta) \equiv m(b_j)$ for any i, j.*

Definition 5 (Minimal Separating Equilibrium)**.** *An equilibrium is 'minimal separating' if there exists a positive measure set $\Theta_0 \subset \Theta$ and $g_i \in \Psi$, such that type g_i fully reveals herself if $\theta \in \Theta_0$.*

Obviously, if an equilibrium is maximal separating, then it must be minimal separating.

3.3 Benchmark

In this subsection, I prove some benchmark results when there is no higher order uncertainty. I show that a maximal separating equilibrium always exists, and it maximizes the expected welfare of the receiver. Moreover, it takes the form of '*low separating high pooling*' (Kartik [2009]), in which a good sender fully reveals her information when θ is small, and pools with bad senders when θ is large (Morgan and Stocken [2003]).

Lemma 3.1. *When Φ is a singleton, there exists a maximal separating equilibrium in which the good sender fully reveals herself if and only if $\theta < \theta_0$. The bad sender and good senders above θ_0 send a message which induces $a = \theta_0$.*

Let θ_0 be the '*threshold point*', which characterizes a maximal separating equilibrium. Let θ_0^* be the maximum threshold point under u and F, then we have the following welfare property:

Lemma 3.2. *The maximal separating equilibrium characterized by threshold point θ_0^* is the ex ante welfare highest equilibrium for the receiver.*

3.4 Maximal Separating Equilibrium

Obviously, Φ is not rich in the benchmark case. In this subsection, I impose the richness condition on both type spaces, and prove my main result, which shows that rumors make some good senders not being able to fully communicate their information, even though they are not being joint with the bad ones.

Proposition 1. *If Ψ and Φ are rich, there exists no maximal separating equilibrium.*

Proof of Proposition 1: The proof is done by contradiction. Assume that a maximal separating equilibrium exists. Given the equilibrium behavior strategies of each type of sender as well as receiver, two different types of senders agree on what action a given type of receiver will take after receiving a message m. They only disagree on the probability distribution over different types of receivers. If type g_i can fully separate herself at θ by sending message $m(g_i, \theta)$, then $g_{i'}$ $(i' \neq i)$ also can fully separate herself by sending $m(g_i, \theta)$.

So, in any maximal separating equilibrium, there exists $\Theta_1, \Theta_2 \subset \Theta$ such that all types of good senders fully separate if $\theta \in \Theta_1$ and pool with bad ones if $\theta \in \Theta_2$. According to the definition of maximal separating, $\Theta_1 \bigcup \Theta_2 = \Theta$. Let $\overline{\Theta_i}$ denote the closure of a set, then $\overline{\Theta_1} \bigcup \overline{\Theta_2} = \Theta$. Since Θ

is connected under the Euclidean Topology on $\mathbb{R}$, it cannot be written as the union of two disjoint closed sets. So:

$$\overline{\Theta_1} \bigcap \overline{\Theta_2} \neq \emptyset$$

Next, I show that $\overline{\Theta_1} \bigcap \overline{\Theta_2}$ must have a unique element. If not, let $\theta_0, \theta_0' \in \overline{\Theta_1} \bigcap \overline{\Theta_2}$ with $\theta_0 < \theta_0'$. There exists $\{\theta_{ij}\}_{i=1}^{\infty}$ and $\{\theta_{ij}'\}_{i=1}^{\infty}$ $(j = 1, 2)$ such that:

$$\lim_{i\to\infty} \theta_{ij} = \theta_0, \quad \lim_{i\to\infty} \theta_{ij}' = \theta_0'$$

and

$$\{\theta_{ij}\}_{i=1}^{\infty} \subset \Theta_j, \quad \{\theta_{ij}'\}_{i=1}^{\infty} \in \Theta_j$$

So for any $\varepsilon > 0$, there exists N such that $||\theta_{i1} - \theta_{i2}|| < \varepsilon$ when $i > N$. Type θ_{i2} good sender must prefers an action in the pooling range to $a = \theta_{i1}$. Let M^b be the set of messages which are sent with positive probability by bad types in equilibrium. So there exists $m \in M^b$ such that $||a_\phi(m) - \theta_0|| \leq v(\varepsilon)$ where v is strictly increasing with ε and $\lim_{\varepsilon\to 0} v(\varepsilon) = 0$. The same is true for θ_0'. Find ε such that $2v(\varepsilon) < \theta_0' - \theta_0$, then all bad senders strictly prefers to send a message in M^b which induces actions around θ_0' rather than θ_0. So only good senders send message which induces action around θ_0. Let $\theta_0 = \inf \overline{\Theta_1} \bigcap \overline{\Theta_2}$, we get a contradiction.

Let $\{\theta_0\} = \overline{\Theta_1} \bigcap \overline{\Theta_2}$. In a maximal separating equilibrium, the good sender fully separates if and only if $\theta < \theta_0$. Consider good sender with $\theta = \theta_0 + \varepsilon$, when ε is small enough, she has no incentive to deviate to induce action $a = \theta_0 - \varepsilon$ (in the fully separating range) unless there exists a message $m^* \in M^b$ such that $a_\phi(m^*) = \theta_0$ for any $\phi \in \Phi$; or there exists a sequence of messages $\{m_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} ||a_\phi(m_i) - \theta_0|| = 0$$

for any $\phi \in \Phi$. I rule out these two possibilities through contradiction, and once this is done, the proposition is proved.

1. If there does not exist m^*, but exists $\{m_i\}_{i=1}^{\infty}$, since each message must be sent by some types of bad senders with positive probability, then there exist a bad type: b_1, which sends infinite amount of messages within this set: $\{m_{1,i}\}_{i=1}^{\infty}$ $(\subset \{m_i\})$. Let $(p_1^1, ..., p_k^1)$ be her belief vector over Φ, then

$$m_{1,i} \in \arg\max_{m\in M} \sum_{j=1}^{k} p_j^1 a_{\phi_j}(m) \geq \theta_0$$

for any i. From the definition of $\{m_i\}$:

$$\lim_{i\to\infty}\sum_{j=1}^{k} p_j^1 a_{\phi_j}(m_i) = \lim_{i\to\infty}\sum_{j=1}^{k} p_j^1 a_{\phi_j}(m_{1,i}) = \theta_0$$

for any belief vector. So $\sum_{j=1}^{k} p_j^1 a_{\phi_j}(m_{1,i}) = \theta_0$.

Next, I claim that her companion type: g_1, cannot send any message in $\{m_{1,i}\}$ in equilibrium, no matter what is the value of θ. This is because:

$$\sum_{j=1}^{k} p_j^1 u(a_{\phi_j}(m), \theta) < u\Big(\sum_{j=1}^{k} p_j^1 a_{\phi_j}(m), \theta\Big) = u(\theta_0, \theta)$$

$u(\theta_0, \theta) - \sum_{j=1}^{k} p_j^1 u(a_{\phi_j}(m), \theta)$ must have a strictly positive lower bound in $\theta \in [\theta_0, 1]$ due to compactness. So there exists $\varepsilon_0 > 0$, such that when $||a_\phi(m_0) - \theta_0|| < \varepsilon_0$,

$$\sum_{j=1}^{k} p_j^1 u(a_{\phi_j}(m_0), \theta) > \sum_{j=1}^{k} p_j^1 u(a_{\phi_j}(m), \theta)$$

for any $\theta \in [\theta_0, 1]$. Since for any ε_0, there exists a message in $\{m_{1,i}\}$ which satisfies this. This mean, for any message in $\{m_{1,i}\}$, there always exists another message in $\{m_{1,i}\}$ which g_1 sender strictly prefers as long as $\theta \in [\theta_0, 1]$.

Then, I claim that g_1 cannot send any other messages either as long as $\theta > \theta_0$. For any message, if:

$$\sum_{j=1}^{k} p_j^1 u(a_{\phi_j}(m), \theta) \le \theta_0$$

then:

$$\sum_{j=1}^{k} p_j^1 u(a_{\phi_j}(m), \theta) < u\Big(\sum_{j=1}^{k} p_j^1 a_{\phi_j}(m), \theta\Big) \le u(\theta_0, \theta)$$

for any $\theta \in [\theta_0, 1]$. So g_1 sender always has a profitable deviation in $\{m_{1,i}\}$. If

$$\sum_{j=1}^{k} p_j^1 u(a_{\phi_j}(m'), \theta) > \theta_0$$

for $m' \notin \{m_{1,i}\}$, this is also a contradiction since:

$$\sum_{j=1}^{k} p_j^1 u(a_{\phi_j}(m'), \theta) \leq \sum_{j=1}^{k} p_j^1 u(a_{\phi_j}(m_{1,i}), \theta) = \theta_0$$

This implies that such an infinite sequence of messages cannot exists once Ψ is rich.

2. If there exists m^*, then, all types of good senders must send m^* with positive probability. This is because otherwise, there exist a sequence of messages described above, which can lead to a contradiction. Also, there exists at least one type of bad sender who sends m^* with positive probability. From the richness of Φ, it can never be the case that all types of receivers choose $a = \theta_0$ upon receiving m^*, which leads to a contradiction.

□

The main message from Proposition 1 is that higher order uncertainty is another reason for information not to be fully transmitted in cheap talk communication. When the sender is uncertain about the receiver's belief about herself, some good types must pool together even though they were not joint with the bad ones. The companion type assumption is critical for my reasoning, since this rule out the existence of an infinite sequence of messages, where the actions induced by them converges to the marginal type θ^*.

To illustrate this point, I construct a maximal separating equilibrium when this condition is violated. Let η be the probability that the receiver assigns to a bad sender. $\eta \in \{\eta_1, \eta_2\}$, where $\eta_1 < \eta_2$. Good sender believes that $\eta = \eta_1$ with probability p_g and bad sender believes that $\eta = \eta_2$ with probability p_b, with $p_g > p_b$. When p_g and p_b are common knowledge, players' beliefs can be originated from a common prior if and only if:

$$\frac{p_b \eta_2}{(1-p_b)\eta_1} = \frac{p_g(1-\eta_2)}{(1-p_g)(1-\eta_1)} \tag{3.1}$$

Proposition 2. *In the setting described above, there exists a maximal separating equilibrium.*

The idea of the construction is described below (the detailed proof is in the Appendix): From the proof of Proposition 1, there exists a maximal separating equilibrium if and only if there exists an infinite sequence of messages $\{m_i\}_{i=1}^{\infty}$ such that for any $\varepsilon > 0$, there exists N such that for any $i > N$

$$||a_j(m_i) - \theta_0|| < \varepsilon$$

where θ_0 is unique intersection between the fully separating range and the pooling with bad type range. From the bad type's indifference condition:

$$a_1(\varepsilon_i) = \theta_0 + (1 - p_b)\varepsilon_i$$

$$a_2(\varepsilon_i) = \theta_0 - p_b\varepsilon_i$$

Each message in the sequence is sufficiently characterized by ε_i where $\varepsilon_i > \varepsilon_{i+1}$ and $\lim_{i\to\infty} \varepsilon_i = 0$. Let γ_i be the threshold type who is indifferent between ε_i and ε_{i+1}, which satisfies:

$$p_g(a_1(\varepsilon_i) - \gamma_i)^2 + (1 - p_g)(a_2(\varepsilon_i) - \gamma_i)^2 = p_g(a_1(\varepsilon_{i+1}) - \gamma_i)^2 + (1 - p_g)(a_2(\varepsilon_{i+1}) - \gamma_i)^2 \tag{3.2}$$

which gives:

$$\gamma_i - \theta_0 = x(\varepsilon_i + \varepsilon_{i+1}) \tag{3.3}$$

where

$$x \equiv \frac{p_g(1 - p_b)^2 + (1 - p_g)p_b^2}{2(p_g - p_b)} \tag{3.4}$$

Good senders between $[\gamma_i, \gamma_{i-1}]$ sends message ε_i, where $\gamma_0 = 1$. The length of the interval which induces ε_i for any $i \geq 2$ is $x(\varepsilon_{i-1} - \varepsilon_{i+1})$. Let q_i be the probability of the bad sender sending message ε_i. The receiver's sequential rationality constraint is given by:

$$a_1(\varepsilon_i) = \frac{\frac{1}{2}\eta_1 q_i + (1 - \eta_1)x(\varepsilon_{i-1} - \varepsilon_{i+1})(\theta_0 + \frac{x}{2}(\varepsilon_{i-1} + 2\varepsilon_i + \varepsilon_{i+1}))}{\eta_1 q_i + (1 - \eta_1)x(\varepsilon_{i-1} - \varepsilon_{i+1})} \tag{3.5}$$

$$a_2(\varepsilon_i) = \frac{\frac{1}{2}\eta_2 q_i + (1 - \eta_2)x(\varepsilon_{i-1} - \varepsilon_{i+1})(\theta_0 + \frac{x}{2}(\varepsilon_{i-1} + 2\varepsilon_i + \varepsilon_{i+1}))}{\eta_2 q_i + (1 - \eta_2)x(\varepsilon_{i-1} - \varepsilon_{i+1})} \tag{3.6}$$

The two equations above give rise to the '*consistency condition*'.

A *maximal separating equilibrium* is characterized by the following three elements:

- A strictly decreasing positive sequence $\{\varepsilon_n\}_{n=1}^{\infty}$, with $\varepsilon_n \to 0$.
- A positive sequence $\{q_n\}_{n=1}^{\infty}$ where $\sum_n q_n = 1$.
- $\theta_0 \in (\frac{1}{2}, 1)$.

which satisfies $\varepsilon_1 < \frac{1-\theta_0}{1-p_b}$ and the consistency condition. To find such an equilibrium we introduce the following algorithm:

- Fix $(\theta_0, \varepsilon_1)$. This pins down $a_1(m_1)$ and $a_2(m_1)$;
- Find (q_1, ε_2) such that $q_1 > 0$, $0 < \varepsilon_2 < \varepsilon_1$ and the consistency condition with respect to $a_1(m_1)$ and $a_2(m_1)$ are satisfied.
- Iterate the process, find (q_n, ε_{n+1}) such that $q_n > 0$, $0 < \varepsilon_{n+1} < \varepsilon_n$ and the consistency condition with respect to $a_1(m_n)$ and $a_2(m_n)$.
- Check whether $\varepsilon_n \to 0$ and $\sum_n q_n = 1$. If not, modify $(\theta_0, \varepsilon_1)$ and repeat the above procedure.

3.5 Minimal Separating equilibrium

In this section, I identify a condition under which no minimal separating equilibrium exists. For simplicity, I focus on the case where there is only second order uncertainty, where the receiver's assigns probability η to a bad sender. The sender knows F and $\eta \in \{\eta_1, ..., \eta_n\}$, but does not know true value of η.[5] She believes that $\eta = \eta_i$ with probability p_i, where $\sum_{i=1}^n p_i = 1$. p_i is common knowledge among the players. So uncertainty ends at the second order. The receiver is of '*type i*' if his belief is $\eta = \eta_i$.

I also assume that u is symmetric, which means that there exists function v such that: $u(a, \theta) = v(|a - \theta|)$. The next shown below is based on the assumptions just made:

Proposition 3. *When u is symmetric and $f(\theta)$ is non-decreasing in θ, then there exists no minimal separating equilibrium.*

Proposition 3 identifies a '*contagion property*' of the distribution function $F(\theta)$, i.e. under this family of prior distributions, as long as there is pooling in an arbitrary small interval, it makes the good senders of different types to pool within each other throughout the support of the distribution. This completely rule out full revealing, even for θ arbitrarily small (the example in section 2).

4 Delegation under Higher Order Uncertainty [unfinished..]

In this section, I apply my model to study the delegation problem. To ensure tractability, I adopt the setup in Section 2, except that $\eta \in \{\eta_1, ..., \eta_n\}$, where $\eta_1 < ... < \eta_n$. The agent believes that $\eta = \eta_i$ with probability p_i. The following benchmark result states that the principal never has any incentive to delegate decision rights without higher order uncertainty:

[5] For simplicity, we assume the support of η is a finite set. All our results can be carried over if the support of η is a non-singleton measurable set.

Lemma 4.1. *When there is no uncertainty over uncertainty, the principal never has an incentive to delegate.*

Proof. When $\eta \in (0,1)$, under delegation, principal's ex ante expected loss when he has belief η is $\frac{\eta}{3}$, which is strictly larger than his expected loss under the welfare highest equilibrium in centralization:

$$L(\eta) = \frac{1}{3}\Big(\frac{\sqrt{\eta}}{1+\sqrt{\eta}}\Big)^2$$

$\square$

Back to the '*information loss*' versus '*loss of control*' trade-off in Dessein (2002), the former effect is being strictly dominated by the latter one in my setting.

5 Discussions

In this section, I discuss the robustness of the results in various situations. For convenience, I only examine the case where there is only second order uncertainty. First, I illustrate why our results on the non-existence of maximal separating equilibria can be carried over when the sender also faces higher order uncertainty on the distribution of θ. Then, I extend the basic model to a more general multi-dimensional setting and show the non-existence of maximal separating equilibria. Finally, I discuss the welfare implications of higher order uncertainty and the potential applications of my model to the theory of delegation.

5.1 General Uncertainty about Uncertainty

In this subsection, I assume that the sender is also uncertain about F: the receiver's prior belief on θ. $F \in \{F_1, ..., F_k\}$. The following condition on the support of F is useful in our discussions later:

Condition 5.1 (Clearly Ranked)**.** *The set* $\{F_1, ..., F_k\}$ *is 'clearly ranked' if there exists* $k_1, k_2 \in \{1, 2, ..., k\}$ *such that for any* $\theta_1 > \theta_2$*:*

$$\frac{f_{k_1}(\theta_1)}{f_{k_1}(\theta_2)} > \frac{f_{k_2}(\theta_1)}{f_{k_2}(\theta_2)}$$

This is just the '*Monotone Likelihood Ratio Property*'. Without loss of generality, we assume that F_1 dominates F_2 in likelihood ratio once we use this condition.

In general, the receiver's type can be summarized by (i, k), and the distribution of types is given by $\psi \in \Delta(I \times K)$. Under the following assumption, Proposition 1 still holds if $n \geq 2$:

Condition 5.2 (Full Support). *ψ has full support if for any (i, k), $\psi(i, k) > 0$.*

This conclusion is obvious. The next Corollary provides a condition under which no maximal separating equilibria exists even if $n = 1$:

Proposition 4. *If u is symmetric and $\{F_1, ..., F_k\}$ is clearly ranked, then there exists no maximal separating equilibrium.*

The intuition of this Proposition is that when there exists two beliefs, one dominates the other in likelihood ratio, then the actions taken by the two types of receivers cannot be both equal to θ_0 when they receive a message in M^b. When the sender faces uncertainty at this margin, it prevents him from fully disclose his information when $\theta < \theta_0$.

5.2 Multi-Dimensional Cheap Talk

Let $\theta \in \Theta \in \mathbb{R}^n$ and $a \in A = \mathbb{R}^n$. Θ is compact and convex.[6] Let $||\cdot||$ be the length (L^2 norm) of a vector in $\mathbb{R}^n$. The bad sender's preference is given by:

$$U^b = -||a - \theta^*||$$

where $\theta^* \in \Theta$ is his 'ideal point'. The good sender and the receiver's preferences are given by:

$$U^g = U^r = u(||a - \theta||)$$

while u is strictly concave and strictly decreasing with $||a - \theta||$.[7]

Let η be the receiver's belief on the sender being bad and $F(\cdot)$ is his prior belief about θ. F is a full support distribution and $\eta \in \{\eta_1, ..., \eta_n\}$.

I examine whether maximal separating equilibrium exists in a multi-dimensional setting. Let Θ_1 be the set where the good sender fully separates and Θ_2 be the set where they pool with bad senders.

[6] This implies that Θ is connected under the Euclidean Topology.

[7] Remember in the 1-dimensional case, I only require that $u(a, \theta)$ is convex and reaches its maximum at $a = \theta$. But now, I require that the utility depends only on the distance between the action and the state.

Lemma 5.1. *In any maximal separating equilibrium, there exists $r > 0$ such that the good sender fully reveals himself if and only if $||\theta - \theta^*|| > r$. Furthermore, for any $\theta \in \Theta$ such that $||\theta - \theta^*|| = r$ and $\varepsilon > 0$, there exists $m(\theta, \varepsilon) \in M^b$ such that $||a^i(m_\varepsilon) - \theta|| < \varepsilon$ for any i.*

According to Lemma 5.1, the boundary for 'pooling with bad types' is an '*equi-distance ball*' around the bad sender's ideal point θ^*. Also, there exists message in which every action induced by this message is located within a small neighborhood of the boundary point: θ_0. Next, I state the main result in this subsection:

Proposition 5. *If $n \geq 2$, then there exists no maximal separating equilibrium.*

Proposition 5 extends the non-existence of maximal separating equilibria to a multi-dimensional setting with any arbitrary ideal point, without imposing the restriction that $\theta^* \in \partial\Theta$. Still, it is true that for any θ such that the good sender fully separates himself, there exists a minimum distance between θ and any '*boundary point*' θ_0. So, there exists a positive measure set such that good senders pool only within themselves. This '*non-fully separation*' is again, caused by the higher order uncertainty faced by the good sender.

6 Conclusion

This paper points out that higher order uncertainty faced by the sender is another reason for information not to be fully transmitted in a cheap talk game...

A Appendix I: Proof in Section 3

Proof of Lemma 3.1: The receiver's action when he believes that m is sent by bad senders as well as good senders with $\theta > \theta_0$ is:

$$a(\theta_0) \in \arg\max_a \left\{ \eta \int_0^1 u(a,\theta)dF(\theta) + (1-\eta) \int_{\theta_0}^1 u(a,\theta)dF(\theta) \right\} \tag{A.1}$$

where η is the receiver's belief on the sender's probability of being bad.

From the concavity of u, $a(\theta_0)$ is unique for any θ_0 and is continuous in θ_0. When $\theta_0 = 0$, $a(\theta_0) = \arg\max_a \int_0^1 u(a,\theta)dF(\theta) > \theta_0$; when $\theta_0 = 1$, $a(\theta_0) = \arg\max_a \int_0^1 u(a,\theta)dF(\theta) < \theta_0$. So there exists θ_0^* such that $a(\theta_0^*) = \theta_0^*$. □

Proof of Lemma 3.2: The receiver's expected utility in the maximal separating equilibrium with threshold point θ_0^* is:

$$(1-\eta) \int_0^{\theta_0^*} u(\theta,\theta)dF(\theta) + \left[\eta \int_0^1 u(\theta_0^*,\theta)dF(\theta) + (1-\eta) \int_{\theta_0^*}^1 u(\theta_0^*,\theta)dF(\theta) \right]$$

where η is defined as in the proof of Lemma 3.1.

In any equilibrium, bad types cannot induce an action larger than θ_0^*. Given that $a \in [0, \theta_0]$ where $\theta_0 \leq \theta_0^*$:

$$(1-\eta) \int_{\theta_0^*}^1 u(a,\theta)dF(\theta) \leq (1-\eta) \int_{\theta_0^*}^1 u(\theta_0,\theta)dF(\theta)$$

So

$$\begin{aligned}
& \max_{a \leq \theta_0^*, a(\theta) \leq a} \left\{ \eta \int_0^1 u(a,\theta)dF(\theta) + (1-\eta) \int_{\theta_0^*}^1 u(a(\theta),\theta)dF(\theta) \right\} \\
\leq \quad & \max_{a \leq \theta_0^*} \left\{ \eta \int_0^1 u(a,\theta)dF(\theta) + (1-\eta) \int_{\theta_0^*}^1 u(a,\theta)dF(\theta) \right\} \\
= \quad & \eta \int_0^1 u(\theta_0^*,\theta)dF(\theta) + (1-\eta) \int_{\theta_0^*}^1 u(\theta_0^*,\theta)dF(\theta)
\end{aligned} \tag{A.2}$$

Also

$$\int_0^{\theta_0^*} u(\theta,\theta)dF(\theta) \geq \int_0^{\theta_0^*} u(a(\theta),\theta)dF(\theta)$$

for any $a(\theta)$, so the the maximal separating equilibrium is welfare highest for the receiver. □

Proof of Proposition 2: Now I show the existence of maximal separating equilibrium following the algorithm described in the main text. I start with the Proposition below:

Proposition 6. *For any* $\theta_0 > \frac{1}{2}$*, there exists* $\{(\varepsilon_i, q_i)\}_{i\in\mathbb{N}}$ *such that the consistency conditions are satisfied.*

Proof of Proposition: The consistency condition imply that:

$$\eta_1\big(a_1(\varepsilon_i) - \frac{1}{2}\big)q_i = x(1-\eta_1)(\varepsilon_{i-1} - \varepsilon_{i+1})\Big[\theta_0 + \frac{x}{2}(\varepsilon_{i-1} + 2\varepsilon_i + \varepsilon_{i+1}) - a_1(\varepsilon_i)\Big] \tag{A.3}$$

$$\eta_2\big(a_2(\varepsilon_i) - \frac{1}{2}\big)q_i = x(1-\eta_2)(\varepsilon_{i-1} - \varepsilon_{i+1})\Big[\theta_0 + \frac{x}{2}(\varepsilon_{i-1} + 2\varepsilon_i + \varepsilon_{i+1}) - a_2(\varepsilon_i)\Big] \tag{A.4}$$

Divide the first equation by the second one, I obtain an equation with only one unknown variable: ε_{i+1}.

$$A(\varepsilon_i) = \frac{\theta_0 + \frac{x}{2}(\varepsilon_{i-1} + 2\varepsilon_i + \varepsilon_{i+1}) - a_2(\varepsilon_i)}{\theta_0 + \frac{x}{2}(\varepsilon_{i-1} + 2\varepsilon_i + \varepsilon_{i+1}) - a_1(\varepsilon_i)} \tag{A.5}$$

where for any $i \geq 1$,

$$A(\varepsilon_i) \equiv \frac{\eta_2(1-\eta_1)(a_2(\varepsilon_i) - \frac{1}{2})}{\eta_1(1-\eta_2)(a_1(\varepsilon_i) - \frac{1}{2})} \tag{A.6}$$

I obtain a recursive expression for ε_{i+1}:

$$\begin{aligned}\varepsilon_{i+1} &= \frac{2}{x}\Big(\frac{A(\varepsilon_i)a_1(\varepsilon_i) - a_2(\varepsilon_i)}{A(\varepsilon_i) - 1} - \theta_0\Big) - 2\varepsilon_i - \varepsilon_{i-1} \\ &= \frac{2}{x}\Big(\frac{A(\varepsilon_i)}{A(\varepsilon_i) - 1} - p_b\Big)\varepsilon_i - 2\varepsilon_i - \varepsilon_{i-1}\end{aligned} \tag{A.7}$$

for any $i \geq 2$, and

$$\varepsilon_2 = \frac{2}{x}\Big(\frac{A(\varepsilon_1)}{A(\varepsilon_1) - 1} - p_b\Big)\varepsilon_1 - \varepsilon_1 - \frac{(1-\theta_0)}{x} \tag{A.8}$$

Next, I construct a sequence of $\{\varepsilon_i\}_{i=1}^{\infty}$ which is monotonic decreasing and converges to 0. If $\varepsilon_{i+1} > 0$, then $A(\varepsilon_i) > 1$, which gives rise to:

$$\varepsilon_1 \leq \min\Big\{\frac{1-\theta_0}{1-p_b}, \frac{(p_g - p_b)(\theta_0 - \frac{1}{2})}{p_b(1-p_b)}\Big\} \equiv \overline{\varepsilon} \tag{A.9}$$

Let

$$k_{i+1} \equiv \frac{\varepsilon_{i+1}}{\varepsilon_i}$$

From the expression above, we know that:

$$k_{i+1} = \frac{2}{x}\Big(\frac{A(\varepsilon_i)}{A(\varepsilon_i)-1} - p_b\Big) - 2 - \frac{1}{k_i}$$

Conditional on $A(\varepsilon_i) > 1$, it is easy to verify that k_{i+1} is increasing in ε_i.[8] Since $\varepsilon_i \to 0$, the steady state value, k^*, is given by:

$$k^* + \frac{1}{k^*} + 2 = \frac{2}{x}\Big(\frac{A}{A-1} - p_b\Big) \tag{A.10}$$

where

$$A = \frac{\eta_2(1-\eta_1)}{\eta_1(1-\eta_2)}$$

Now I show the existence of the sequence $\{\varepsilon_n\}$. I begin with the initial condition:

$$k_2 = \frac{2}{x}\Big(\frac{A(\varepsilon_1)}{A(\varepsilon_1)-1} - p_b\Big) - 1 - \frac{1-\theta_0}{x\varepsilon_1}$$

Since conditional on $A(\varepsilon_1) > 1$, the above equation is strictly increasing in ε_1: when $\varepsilon_1 \to \frac{(p_g-p_b)(\theta_0-\frac{1}{2})}{p_b(1-p_b)}$, $k_2 \to +\infty$; when $\varepsilon_1 \to 0$, $k_2 \to -\infty$. When $\varepsilon_1 = \frac{1-\theta_0}{1-p_b}$, we show that $k_2(\varepsilon_1) \geq 1$. This is equivalent to:

$$1 + \frac{1}{A(\frac{1-\theta_0}{1-p_b}) - 1} \geq 1 + \frac{p_b(1-p_g)}{2(p_g-p_b)}$$

and it holds under the CPA. From the arguments above and the continuity of the RHS with respect to ε_i and applying the Intermediate Value Theorem, we know that for any $k \in (p^*, 1)$ there exists ε_1 such that $k_2(\varepsilon_1) = k$.

Now, we examine the difference equation:

$$\Delta k_{i+1} \equiv k_{i+1} - k_i = \frac{2}{x}\Big(\frac{A(\varepsilon_i)}{A(\varepsilon_i)-1} - p_b\Big) - (k_i + 2 + \frac{1}{k_i}) \tag{A.11}$$

Lemma A.1. *$k_i < k^*$ for any $i \in \mathbb{N}$.*

Proof of Lemma: We prove by contradiction. If there exists $k_i \geq k^*$, then:

$$\Delta k_{i+1} > \big(k^* + 2 + \frac{1}{k^*}\big) - \big(k_i + 2 + \frac{1}{k_i}\big) \geq 0$$

---

[8] This is because $\frac{A}{A-1}$ is decreasing in A and

$$\frac{\partial A}{\partial \varepsilon_i} = \frac{-\theta_0 + \frac{1}{2}}{(\theta_0 + (1-p_b)\varepsilon_i - \frac{1}{2})^2} < 0$$

So $k_{i+1} > k^*$, and

$$k_{j+1} - k_j \geq \big(k^* + 2 + \frac{1}{k^*}\big) - \big(k_{i+1} + 2 + \frac{1}{k_{i+1}}\big) \equiv \delta > 0$$

for any $j \geq i+1$. This implies that there exists $j' \in \mathbb{N}$, such that $k_{j'} \geq k^* + (j' - i - 1)\delta > 1$, which leads to a contradiction. □

Lemma A.2. *There exists a trajectory* $\{(k_i, \varepsilon_i)\}_{i=1}^{\infty}$ *with* k_i *is strictly increasing and* $\lim_{i\to\infty} k_i = k^*$*, such that:*

$$k_{i+1} = \frac{2}{x}\Big(\frac{A(\varepsilon_i)}{A(\varepsilon_i) - 1} - p_b\Big) - 2 - \frac{1}{k_i}$$

$$\varepsilon_{i+1} = k_{i+1}\varepsilon_i$$

Proof of Lemma: We begin by defining two functions, $\underline{k}(\varepsilon)$ and $\overline{k}(\varepsilon)$:

$$\underline{k}(\varepsilon) + 2 + \frac{1}{\underline{k}(\varepsilon)} = \frac{2}{x}\Big[\frac{A(\varepsilon)}{A(\varepsilon) - 1} - p_b\Big] \tag{A.12}$$

$$k^* + 2 + \frac{1}{\overline{k}(\varepsilon)} = \frac{2}{x}\Big[\frac{A(\varepsilon)}{A(\varepsilon) - 1} - p_b\Big] \tag{A.13}$$

$\underline{k}(\cdot) \leq \overline{k}(\cdot)$, with equality holds if and only if $\varepsilon = 0$ ($\underline{k}(0) = \overline{k}(0) = k^*$) and both are strictly decreasing with respect to ε.

For any given ε_i, $k_{i+1} \geq k_i$ if and only if $k_i \geq \underline{k}(\varepsilon_i)$, and $k_{i+1} < k^*$ if and only if $k_i \leq \overline{k}(\varepsilon_i)$. Let G_1 be a graph, such that:

$$G_1 \equiv \Big\{(k, \varepsilon)\Big| k \in [\underline{k}(\varepsilon), \overline{k}(\varepsilon)], 0 \leq \varepsilon \leq \overline{\varepsilon}\Big\}$$

I define $\{G_n\}_{n=2}^{\infty}$ iteratively:

$$G_{n+1} = \Big\{(k_i, \varepsilon_i)\Big|(k_{i+1}, \varepsilon_{i+1}) \in G_n\Big\}$$

$G_2 \subset G_1$ since $k_{i+1}(\underline{k}(\varepsilon_i), \varepsilon_i) = \underline{k}(\varepsilon_i) < \underline{k}(\varepsilon_{i+1})$, $\overline{k}(\varepsilon) \leq k^*$ and k_{i+1} is increasing with k_i. $G_{n+1} \subset G_n$ can be proved through induction.

From the continuity of k_{i+1}, we can define a sequence of closed intervals:

$$[\underline{k}^{(n)}(\varepsilon), \overline{k}^{(n)}(\varepsilon)]$$

such that $(k, \varepsilon) \in G_n$ if and only if $k \in [\underline{k}^{(n)}(\varepsilon), \overline{k}^{(n)}(\varepsilon)]$. Also, $\underline{k}^{(n)}(\varepsilon) \leq \underline{k}^{(n+1)}(\varepsilon)$, $\overline{k}^{(n)}(\varepsilon) \geq \overline{k}^{(n+1)}(\varepsilon)$ for any ε and n. For each ε, we construct an infinite sequence of shrinking intervals. According to the Closed Interval Theorem, they have non-empty intersection. Define:

$$G^{\infty} \equiv \bigcap_{n=1}^{\infty} G_n$$

which is non-empty for any intersection of ε. $\square$

Since at the initial point, ε_2 is increasing with k_2, so there exists $(k, \varepsilon) \in G^{\infty}$ which satisfies the consistency condition for k_2 and $\varepsilon_1 \equiv \frac{\varepsilon_2}{k_1}$. $\square$

$\square$

Proof of Proposition 3: Let Θ_1, Θ_2 and Θ_3 be the sets where the good sender fully separates, pools with bad senders and pools only among themselves. Obviously:

$$\Theta_1 \bigcup \Theta_2 \bigcup \Theta_3 = \Theta$$

Let M^b be the set of messages sent by good senders in Θ_2 and M^p be the set of messages sent by good senders in Θ_3. Without loss of generality, I assume $M^b \bigcap M^p = \emptyset$. Following the same argument as the proof of Proposition 1, there exists:

$$\theta_0 \in \overline{\Theta_1} \bigcap \left(\overline{\Theta_2 \cup \Theta_3} \right)$$

To reduce notations, let $a^i(m)$ be the receiver's action when receiving m with belief $\eta = \eta_i$. The rest of the proof is done through the following steps:

Lemma A.3. $\theta_0 \notin \overline{\Theta_2}$.

Proof. Obviously, $\overline{\Theta_1} \bigcap \overline{\Theta_2}$ cannot have more than 1 element. If $\overline{\Theta_1} \bigcap \overline{\Theta_2} = \{\theta_0\}$, then, there exists $m^b \in M^b$ such that $a^i(m^b) = \theta_0$ for any i.

Let $\Theta_3' \equiv \overline{\Theta_3} \bigcap [\theta_0, 1]$, which is a closed set.

1. If $\Theta_3' \neq \emptyset$, then, there exists $\theta_1 \in \Theta_3' \bigcap \Theta_2 \subset [\theta_0, 1]$ such that type θ_1 good sender must induces a message m' such that:

$$\sum_i p_i a^i(m') \geq \theta_0$$

But from the Jensen's Inequality:

$$\sum_i p_i u(a^i(m'), \theta_1) \leq u\Big(\sum_i p_i a^i(m'), \theta_1\Big) \leq u(\theta_0, \theta_1) \tag{A.14}$$

So type θ_1 good sender prefers to induce $a = \theta_0$ by sending m^b, contradiction.

2. If $\Theta_3' = \emptyset$, then, either $\overline{\Theta_2} = [\theta_0, 1]$ or there exists $\theta_1 < \theta_0$ such that type θ_1 good sender pools with bad types. In the former case, we can get similar contradiction as in Proposition 1. In the latter case, type θ_1 good sender prefers to induce the equilibrium action by type $\theta_0 - \varepsilon \in \Theta_1$ good sender instead of his own equilibrium action, which is also a contradiction.

The Lemma is then proved. □

From the Lemma, we know that in any minimal separating equilibrium, there exists $\theta_0 \in (0, 1)$ such that fully revealing exists only below θ_0, and there exists a minimal distance between 'pooling with bad sender range' and 'fully separating range', and in this range, there exists a sequence of messages, such that $|a_i(m) - \theta_0|$ can be arbitrarily small. This ensures the existence of an interval of good senders $[\theta_2, \theta_1]$ above θ_0, such that they pool within each other in equilibrium.

Let $\theta_3 \equiv 2\theta_2 = \theta_1$. I show that:

$$u(a_2^*, \theta_2) > u(a_1^*, \theta_2)$$

where

$$a_2^* \equiv \arg\max_a \int_{\theta_3}^{\theta_2} u(a, \theta) dF(\theta)$$

$$a_1^* \equiv \arg\max_a \int_{\theta_2}^{\theta_1} u(a, \theta) dF(\theta)$$

If the sender weakly prefers a_2^* to a_1^* when $\theta = \theta_2$, then in the equilibrium partition, $|\theta_3' - \theta_2| \geq |\theta_2 - \theta_1|$. If this relationship holds for any $[\theta_2, \theta_1]$, then the lengths of the intervals must be non-decreasing from right to left. If this is the case, then there never exists a positive threshold θ_0.

When the utility function is symmetric, I only need to show that $a_1^* + a_2^* > 2\theta_2$ to get this contradiction. Let $w(y) = \frac{\partial u}{\partial a}|_{|a-\theta|=y}$. Since

$$\int_0^{a_1^* - \theta_2} w(y) dF(a_1^* - y) = \int_0^{a_2^* - \theta_3} w(y) dF(a_1^* + y)$$

and $f(\theta)$ is non-decreasing, $a_1^* - \theta_2 = \theta_2 - a_2^*$, $a_2^* - \theta_3 = \theta_1 - a_1^*$ so:

$$\int_0^{a_1^*-\theta_2} w(y)dF(a_2^* - y) > \int_0^{a_2^*-\theta_3} w(y)dF(a_2^* + y)$$

which concludes our proof. □

B Appendix III: Proof in Section 5

Proof of Proposition 4: Let $m \in M^b$ be the message sent by good senders when $\theta \in [\theta_0, 1]$ and $\pi(\theta)$ be the receiver's probability of sending m when the state is θ. When F_1 dominates F_2 in likelihood ratio. Define

$$F_i'(\theta) \equiv \frac{\int_0^\theta f_i(\vartheta)\pi(\vartheta)d\vartheta}{\int_0^1 f_i(\vartheta)\pi(\vartheta)d\vartheta}$$

So F_1' dominates F_2' in likelihood ratio. Then since $u(a,\theta)$ is symmetric:

$$\int_{\theta_0}^1 \frac{\partial u(\theta_0,\theta)}{\partial a} dF_1(\theta) > \int_{\theta_0}^1 \frac{\partial u(\theta_0,\theta)}{\partial a} dF_2(\theta)$$

$$\int_0^1 \frac{\partial u(\theta_0,\theta)}{\partial a} dF_1'(\theta) > \int_0^1 \frac{\partial u(\theta_0,\theta)}{\partial a} dF_2'(\theta)$$

So the actions taken by type F_1 receiver must be strictly larger than the one taken by F_2 receiver. So they cannot be both equal to θ_0, which is a contradiction. □

Proof of Lemma 5.1: Since Θ is connected and $\overline{\Theta_1} \bigcup \overline{\Theta_2} = \Theta$, similar to the argument in Lemma 3.2, $\overline{\Theta_1} \bigcap \overline{\Theta_2} \neq \emptyset$. Let $\theta_0 \in \overline{\Theta_1} \bigcap \overline{\Theta_2}$, then there exists two sequences $\{\theta_k^1\}_{k=1}^\infty$ and $\{\theta_k^2\}_{k=1}^\infty$ such that:

$$\lim_{k\to\infty} \theta_k^l = \theta_0, \quad \{\theta_k^l\}_{k=1}^\infty \subset \Theta_l$$

For any $\varepsilon > 0$ there exists $m(\varepsilon, \theta_0)$ such that $||a^i(m_\varepsilon) - \theta|| < \varepsilon$ for any i.

Let

$$\Theta^* = \left\{\theta \middle| ||\theta - \theta^*|| = ||\theta_0 - \theta^*||\right\} \bigcap \Theta$$

$$\Theta^{in} = \left\{\theta \middle| ||\theta - \theta^*|| < ||\theta_0 - \theta^*||\right\} \bigcap \Theta$$

$$\Theta^{out} = \left\{\theta \middle| ||\theta - \theta^*|| > ||\theta_0 - \theta^*||\right\} \bigcap \Theta$$

We prove that $\overline{\Theta_1}\bigcap\overline{\Theta_2}=\Theta^*$. If not:

- If there exists
$$\theta' \in \Theta^{in}\bigcap(\overline{\Theta_1}\bigcap\overline{\Theta_2})$$
then bad types have an incentive to induce $\theta'' \in B(\theta',\varepsilon)\bigcap\Theta_1$, contradiction.

- If there exists
$$\theta' \in \Theta^{out}\bigcap(\overline{\Theta_1}\bigcap\overline{\Theta_2})$$
then there will be no bad type inducing action around θ', contradiction.

So $int\Theta_1 = \Theta^{out}$ and $int\Theta_2 = \Theta^{in}$, where $r \equiv ||\theta_0 - \theta^*||$ □

Proof of Proposition 5: Preserving the notations used in the proof of Lemma 5.1, and let $[\theta,\theta']$ denote the interval connecting points θ and θ'.

First, we show that for any $\theta_0 \in \Theta^*$, there exists $m(\theta_0)$ such that $a^i(m(\theta_0)) = \theta_0$ for any i.

Define the following mapping: $L_{\theta_0} : \Theta \to [\theta^*,\theta_0]$, such that: $L_{\theta_0}(\theta) \in [\theta^*,\theta_0]$, and $||\theta-\theta^*|| = ||L_{\theta_0}(\theta)-\theta^*||$. Then, for any $m' \in M^b$ such that $||a^i(m')-\theta_0|| < \varepsilon$ but:

$$\sum_i p_i||a^i(m')-\theta_0|| > 0$$

From the bad sender's incentive constraint:

$$\sum_i p_i||a^i(m')-\theta^*|| = ||\theta_0-\theta^*|| = \sum_i p_i||L_{\theta_0}(a^i(m'))-\theta^*|| \tag{B.1}$$

Since for any $\theta \in [\theta^*,\theta_0]$,
$$||L_{\theta_0}(a^i(m'))-\theta|| \le ||a^i(m')-\theta|| \tag{B.2}$$

So, we can we have:

$$\begin{aligned} u(||\theta_0-\theta||) &\ge \sum_i p_i u(||L_{\theta_0}(a^i(m'))-\theta||) \\ &\ge \sum_i p_i u(||a^i(m')-\theta||) \end{aligned} \tag{B.3}$$

The first inequality comes from (A.15) and the Jensen's Inequality; the second one comes from (A.17). From the concavity of u, we know there exists ε' such that a message m'' with $||a^i(m'')-$

$\theta_0|| \leq \varepsilon'$ for all i is strictly preferred by any good sender with type $\theta \in [\theta^*, \theta_0]$. So there must exists m for any $\theta_0 \in \Theta^*$ such that $a^i(m) = \theta_0$ for any i.

Since Θ^* is a connected $n-1$ dimensional manifold, for any interior point $\theta_0 \in \Theta^*$:

$$\theta_0 \in \arg\max_a \left\{ \eta_i \int_\theta u(||a-\theta||)\pi^b(\theta)dF(\theta) + (1-\eta_i)\int_{\theta\in\Theta^{in}} u(||a-\theta||)\pi^g(\theta)dF(\theta) \right\} \tag{B.4}$$

while $\pi^g(\theta)$ and $\pi^b(\theta)$ is the probability of good (or bad) sender with type θ to induce action θ_0.

From Lemma 5.1, we know:

$$\arg\max_a \left\{ (1-\eta_i)\int_{\theta\in\Theta^{in}} u(||a-\theta||)\pi^g(\theta)dF(\theta) \right\} \in \Theta^{in}$$

so

$$\arg\max_a \left\{ \eta_i \int_\theta u(||a-\theta||)\pi^b(\theta)dF(\theta) \right\} \in \Theta^{out}$$

If the (A.18) is satisfied for η_i, then for $\eta_{i+1} > \eta_i$:

$$\theta_0 = \arg\max_a \left\{ \eta_{i+1} \int_\theta u(||a-\theta||)\pi^b(\theta)dF(\theta) + (1-\eta_{i+1})\int_{\theta\in\Theta^{in}} u(||a-\theta||)\pi^g(\theta)dF(\theta) \right\} \in \Theta^{out} \tag{B.5}$$

which is a contradiction. So there is no maximal separating equilibrium once the possible values of η is no fewer than 2. $\square$

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