On the existence of a limit value in some non expansive optimal control problems.

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Abstract

We investigate the limit of the average value of an optimal control problem when the horizon converges to infinity. For this aim, we suppose suitable nonexpansive-like assumptions which does not imply that the limit is independent of the initial state as it is usually done in the literature.

1 Introduction

We consider the following optimal control denoted $\Gamma_t(y_0)$:

$$V_t(y_0) := \inf_{u \in \mathcal{U}} \frac{1}{t} \int_{s=0}^{t} h(y(s,u,y_0), u(s))ds, \tag{1}$$

where $s \mapsto y(s,u,y_0)$ denotes the solution to

$$y'(s) = g(y(s), u(s)), \quad y(0) = y_0. \tag{2}$$

Here $\mathcal{U}$ is the set of measurable controls from $\mathbb{R}_+$ to a given non empty metric space U. Throughout the paper, we will suppose Lipschitz regularity of $g : \mathbb{R}^d \times U \to \mathbb{R}^d$ which implies that for a given control u in $\mathcal{U}$ and a given initial condition y_0, equation (2) has a unique absolutely continuous solution.

The main goal of the paper consists in studying the asymptotic behaviour of $V_t(y_0)$ when t tends to ∞. This problem has been considered in several papers concerning ergodic control. There is a huge literature concerning this topic, we refer the reader to [1, 6, 7, 8, 9, 14, 15, 16, 22, 24, 27] and the references therein. In the ergodic control literature, most of papers concerns with cases where the assumptions do imply that the limit of $V_t(y_0)$ as $t \to +\infty$ exists and is independent

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of y_0. In the present paper, on the contrary, attention will be focussed on cases where the limit exists and may depend on y_0 (we exhibit several examples with such a behavior). Our aim is to obtain a general result which contains in particular the more easy to state following result (here and throughout the paper, $< \cdot, \cdot >$ will stand for the canonical scalar product and B will be the associated closed unit ball).

Proposition 1.1. *Assume that g is Lipschitz, that there exists a compact set N which is - forward - invariant by the control system (2) and that h is a continuous function which does not depend on u. Assume moreover that :*

$$\forall (y_1, y_2) \in N^2,\ \sup_{u\in U} \inf_{v\in U} < y_1 - y_2, g(y_1, u) - g(y_2, v) > \leq 0. \tag{3}$$

Then problem (1) has a value when t converges to $+\infty$ i.e. there exists $V(y_0) := \lim_{t\to+\infty} V_t(y_0)$.

Condition (3) means a non expansive property of the control system, while the condition

$$\forall (y_1, y_2) \in N^2,\ \sup_{u\in U} \inf_{v\in U} < y_1 - y_2, g(y_1, u) - g(y_2, v) > \leq -C\|y_1 - y_2\|^2$$

expresses a dissipativity property of the control system. The above dissipativity condition does imply that the limit is independent of y_0 (one can deduce this for instance from [2]).

The value function (1) can also be characterized through - viscosity - solution of a suitable Hamilton-Jacobi equation. In several articles initiated by the pioneering work [19] the limit of $V_t(y_0)$ is obtained by "passing to the limit" on the Hamilton-Jacobi equation. This required coercivity properties of the Hamiltonian which could be implied by controlability and/or dissipativity of the control system but which are not valid in the nonexpansive case (3). We refer to [10] for a two dimensional case with non coercive (and non convex) Hamiltonian. Moreover the PDE approach is out of the scope of the - long enough - present article.

Definition 1.2. *The problem $\Gamma(y_0) := (\Gamma_t(y_0))_{t>0}$ has a limit value if $\lim_{t\to\infty} V_t(y_0)$ exists. Whenever it exists, we denote this limit by $V(y_0)$.*

Our main aim consists in giving one sufficient condition ensuring the existence of the limit value. As a particular case of our main results we will obtain Proposition (1.1).

It is also of interest to know if approximate optimal controls for the value $V_t(y_0)$ are still approximate optimal controls for the limit value. This leads us to the following definition.

Definition 1.3. *The problem $\Gamma(y_0)$ has a uniform value if it has a limit value $V(y_0)$ and if :*

$$\forall \varepsilon > 0, \exists u \in \mathcal{U}, \exists t_0, \forall t \geq t_0, \frac{1}{t} \int_{s=0}^{t} h(y(s, u, y_0), u(s))ds \leq V(y_0) + \varepsilon.$$

Whenever the uniform value exists, the controller can act (approximately) optimally independently of the time horizon. On the contrary, if the limit value exists but the uniform value does not, he really needs to know the time horizon before choosing a control. We will prove that our results do imply the existence of a uniform value. We will be inspired by a recent work in the discrete time case [23]. Previous related works in dynamic programming include [18], [21], [17].

Let us explain now, how the paper is organized. The second section contains some preliminaries and discussions of limit behaviors in examples. In the third section, we state and prove our main result for the existence of the uniform value.

2 Preliminaries

We now consider the optimal control problems $(\Gamma_t(y_0))_t$ described by (1) and (2).

2.1 Assumptions and Notations

We now describe the assumptions made on g and h.

$$(4) \quad \begin{cases} \text{The function } h : I\!R^d \times U \longrightarrow I\!R \text{ is Borel measurable and bounded} \\ \text{The function } g : I\!R^d \times U \longrightarrow I\!R^d \text{ is Borel measurable} \\ \exists L \geq 0, \forall (y, y') \in I\!R^{2d}, \forall u \in U,\ \|g(y,u) - g(y',u)\| \leq L\|y - y'\| \\ \exists a > 0, \forall (y,u) \in I\!R^d \times U,\ \|g(y,u)\| \leq a(1 + \|y\|) \end{cases}$$

With these hypotheses, given u in $\mathcal{U}$ equation (2) has a unique absolutely continuous solution $y(\cdot, u, y_0) : I\!R_+ \to I\!R^d$.

Since h is bounded, we will assume without loss of generality from now on that h takes values in $[0,1]$.

We denote by $G(y_0) := \{y(t,u,y_0), t \geq 0, u \in \mathcal{U}\}$ the reachable set (i.e. the set of states that can be reached starting from y_0).

We denote the average cost induced by u between time 0 and time t by :

$$\gamma_t(y_0, u) = \frac{1}{t}\int_0^t h(y(s,u,y_0), u(s))ds$$

The corresponding Value function satisfies $V_t(y_0) = \inf_{u \in \mathcal{U}} \gamma_t(y_0, u)$.

2.2 Examples

We present here basic examples. In all these examples, the cost $h(y,u)$ only depends on the state y. We will prove later that the uniform value exists in examples 2, 3 and 4.

• Example 1 : here y lies in $I\!R^2$ seen as the complex plane, there is no control and the dynamic is given by $g(y,u) = i\ y$, where $i^2 = -1$. We clearly have :

$$V_t(y_0) \xrightarrow[t\to\infty]{} \frac{1}{2\pi}\int_0^{2\pi} h(|y_0|e^{ri\theta})d\theta,$$

and since there is no control, the value is uniform.

• Example 2 : in the complex plane again, but now $g(y,u) = i\ y\ u$, where $u \in U$ a given bounded subset of $I\!R$, and h is continuous in y.

• Example 3 : $g(y,u) = -y + u$, where $u \in U$ a given bounded subset of $I\!R^d$, and h is continuous in y.

• Example 4 : in $I\!R^2$. The initial state is $y_0 = (0,0)$ and the control set is $U = [0,1]$. For a state $y = (y_1, y_2)$ and a control u, the dynamic is given by $y'(s) = g(y(s), u(s)) = \begin{pmatrix} u(s)(1-y_1(s)) \\ u^2(s)(1-y_1(s)) \end{pmatrix}$, and the cost is $h(y) = 1 - y_1(1-y_2)$. Notice that for any control, $y_1'(s) \geq y_2'(s) \geq 0$, and thus $y_2(t) \leq y_1(t)$ for each $t \geq 0$. One can easily observe that $G(y_0) \subset [0,1]^2$.

If one uses the constant control $u = \varepsilon > 0$, we obtain $y_1(t) = 1 - \exp(-\varepsilon t)$ and $y_2(t) = \varepsilon y_1(t)$. So we have $V_t(y_0) \xrightarrow[t\to\infty]{} 0$.

More generally, if the initial state is $y = (y_1, y_2) \in [0,1]^2$, by choosing a constant control $u = \varepsilon > 0$ small, one can show that the limit value exists and $\lim_{t\to\infty} V_t(y) = y_2$.

Notice that there is no hope here to use an ergodic property, because

$$\{y \in [0,1]^2, \lim_{t\to\infty} V_t(y) = \lim_{t\to\infty} V_t(y_0)\} = [0,1] \times \{0\},$$

and starting from y_0 it is possible to reach no point in $(0,1] \times \{0\}$.

• Example 5 : in $I\!R^2$, $y_0 = (0,0)$, control set $U = [0,1]$, $g(y,u) = (y_2, u)$, and $h(y_1, y_2) = 0$ if $y_1 \in [1,2]$, $= 1$ otherwise.

We have $u(s) = y_2'(s) = y_1''(s)$, hence we may think of the control u as the acceleration, y_2 as the speed and y_1 as the position of some mobile. If $u = \varepsilon$ constant, then $y_2(t) = \sqrt{2\varepsilon y_1(t)}\ \forall t \geq 0$.

We have $u \geq 0$, hence the speed cannot decrease. Consequently, the time interval where $y_1(t) \in [1,2]$ cannot be longer than the time interval where $y_1(t) \in [0,1)$, and we have $V_T(y_0) \geq 1/2$ for each T.

One can prove that $V_T(y_0) \xrightarrow[T\to\infty]{} 1/2$ by considering the following controls : choose $\hat{t}$ in $(0,T)$ such that $(2/\hat{t}) + (\hat{t}/2) = T$, make a full acceleration up to $\hat{t}$ and completely stop accelerating after : $u(t) = 1$ for $t < \hat{t}$, and $u(t) = 0$ for $t \geq \hat{t}$.

Consequently the limit value exists and is $1/2$. However, for any control u in $\mathcal{U}$, we either have $y(t,u,y_0) = y_0$ for all t, or $y_1(t,u,y_0) \xrightarrow[t\to\infty]{} +\infty$. So in any case we have $\frac{1}{t}\int_0^t h(y(s,u,y_0),u(s))ds \underset{t\to\infty}{\longrightarrow} 1$. The uniform value does not exist here, although the dynamic is very regular.

• Example 6 : in $I\!R^d$. The control set is $U \in B \subset I\!R^q$. $g(y,u) = -Ay$ where A is a positive definite matrix. $h(y,u) = \varphi(y) + \psi(u)$ where φ and ψ are arbitrary Lipschitz function onto $I\!R$.

3 Existence results for the uniform value

3.1 A technical Lemma

Let us define $V^-(y_0) := \liminf_{t\to+\infty} V_t(y_0)$ and $V^+(y_0) := \limsup_{t\to+\infty} V_t(y_0)$. Adding a parameter $m \geq 0$, we will more generally consider the costs between time m and time $m+t$:

$$\gamma_{m,t}(y_0,u) = \frac{1}{t}\int_m^{m+t} h(y(s,u,y_0),u(s))ds,$$

and the value of the problem where the time interval $[0,m]$ can be devoted to reach a good initial state, is denoted by :

$$V_{m,t}(y_0) = \inf_{u\in\mathcal{U}} \gamma_{m,t}(y_0,u).$$

Of course $\gamma_t(y_0,u) = \gamma_{0,t}(y_0,u)$ and $V_t(y_0) = V_{0,t}(y_0)$.

Lemma 3.1. *For every m_0 in $I\!R_+$, we have :*

$$\sup_{t>0}\inf_{m\leq m_0} V_{m,t}(y_0) \geq V^+(y_0) \geq V^-(y_0) \geq \sup_{t>0}\inf_{m\geq 0} V_{m,t}(y_0).$$

Proof : We first prove $\sup_{t>0}\inf_{m\leq m_0} V_{m,t}(y_0) \geq V^+(y_0)$. Suppose by contradiction that it is false. So there exists $\varepsilon > 0$ such that for any $t > 0$ we have $\inf_{m\leq m_0} V_{m,t}(y_0) \leq V^+(y_0) - \varepsilon$. Hence for any $t > 0$ there exists $m \leq m_0$ with $V_{m,t}(y_0) \leq V^+(y_0) - (\varepsilon/2)$. Now observe that

$$V_{m,t}(y_0) = \inf_u \frac{1}{t}\int_m^{m+t} h(y(s,u,y_0),u(s))ds = \frac{1}{t}\inf_u\{\int_0^{m_0+t} h(y(s,u,y_0),u(s))ds$$

$$-\int_{m+t}^{m_0+t} h(y(s,u,y_0),u(s))ds - \int_0^m h(y(s,u,y_0),u(s))ds\} \geq \frac{m_0+t}{t}V_{m_0+t}(y_0) - 2\frac{m_0}{t}.$$

Hence

$$\frac{m_0+t}{t}V_{m_0+t}(y_0) - 2\frac{m_0}{t} \leq V^+(y_0) - (\varepsilon/2).$$

Passing to the limsup when t goes to $+\infty$ we obtain a contradiction.

We now prove $V^-(y_0) \geq \sup_{t>0}\inf_{m\leq 0} V_{m,t}(y_0)$. Assume on the contrary that it is false. Then there exists $\varepsilon > 0$ and $t > 0$ such that $V^-(y_0) + \varepsilon \leq \inf_{m\leq 0} V_{m,t}(y_0)$. So for any $m \geq 0$, we have $V^-(y_0) + \varepsilon \leq V_{m,t}(y_0)$. We will obtain a contradiction by concatenating trajectories. Take $T>0$, and write $T=lt+r$, with l in $I\!N$ and r in $[0,t)$. For any control u in $\mathcal{U}$, we have : $T\gamma_T(y_0,u) = t\gamma_{0,t}(y_0,u) + t\gamma_{t,t}(y_0,u) + ... + t\gamma_{(l-1)t,t}(y_0,u) + r\gamma_{lt,r}(y_0,u) \geq lt(V^-(y_0)+\varepsilon)$. Hence

$$\gamma_T(y_0,u) \geq \frac{T-r}{T}(V^-(y_0)+\varepsilon).$$

So for T large enough we have $V_T(y_0) \geq V^-(y_0)+\varepsilon/2$, hence a contradiction by taking the liminf when $T \to \infty$. □

Remark : it is also easy to show that for each $t_0 \geq 0$, we have $\inf_{m\geq 0}\sup_{t>t_0} V_{m,t}(y_0) \geq V^+(y_0)$.

The following quantity will play a great role in the sequel.

Definition 3.2.

$$V^*(y_0) = \sup_{t>0}\ \inf_{m\geq 0}\ V_{m,t}(y_0).$$

3.2 Main results

Let us state the first version of our main result (which clearly implies Proposition 1.1 stated in the introduction)

Proposition 3.3. *Assume that (4) holds true and furthermore :*
(A) $h(y,u)=h(y)$ only depends on the state, and is continuous on $I\!R^d$.
(B) $G(y_0)$ is bounded,
(C) $\forall (y_1,y_2)\in G(y_0)^2$, $\sup_{u\in U}\inf_{v\in U} < y_1-y_2, g(y_1,u)-g(y_2,v) > \leq 0$.
Then the problem $\Gamma(y_0)$ has a limit value which is $V^*(y_0)$, i.e. $V_t(y_0) \xrightarrow[t\to+\infty]{} V^*(y_0)$. The convergence of $(V_t)_t$ to V^* is uniform over $G(y_0)$, and we have $V^*(y_0) = \sup_{t\geq 1}\ \inf_{m\geq 0}\ V_{m,t}(y_0) = \inf_{m\geq 0}\sup_{t\geq 1}\ V_{m,t}(y_0) = \lim_{m\to\infty, t\to\infty} V_{m,t}(y_0)$. Moreover the value of $\Gamma(y_0)$ is uniform.

Condition (B) can be used to show that (cf Proposition 3.7) : $\forall (y_1,y_2) \in G(y_0)^2$, $\forall \varepsilon>0$, $\forall T\geq 0$, $\forall u \in \mathcal{U}$, $\exists v \in \mathcal{U}$ s.t. : $\forall t \in [0,T]$, $\|y(t,u,y_1) - y(t,v,y_2)\| \leq \|y_1-y_2\|+\varepsilon$. Proposition 3.3 can be applied to the previous examples 1, 2 and 3, but not to example 4. Notice that in example 5, we have $V^*(y_0) = 0 < 1/2 = \lim_t V_t(y_0)$.

We will prove the following generalizations of Proposition 3.3. We put $Z = G(y_0)$, and denote by $\bar{Z}$ its closure in $I\!R^d$.

Theorem 3.4. *Suppose that (4) holds true and furthermore assume that*
(H1) h is uniformly continuous in y on $\bar{Z}$ uniformly in u.
(H2) : There exist a continuous function $\Delta : I\!R^d \times I\!R^d \longrightarrow I\!R_+$, vanishing on the diagonal ($\Delta(y,y)=0$ for each y) and symmetric ($\Delta(y_1,y_2)=\Delta(y_2,y_1)$ for all y_1 and y_2), and a function $\hat{\alpha} :\ I\!R_+ \longrightarrow I\!R_+$ s.t. $\hat{\alpha}(t) \xrightarrow[t\to 0]{} 0$ satisfying :

a) For every sequence $(z_n)_n$ with values in Z and every $\varepsilon > 0$, one can find n such that $\liminf_p \Delta(z_n, z_p) \leq \varepsilon$.

b) $\forall (y_1, y_2) \in \bar{Z}^2$, $\forall u \in U$, $\exists v \in U$ such that

$D \uparrow \Delta(y_1, y_2)(g(y_1, u), g(y_2, v)) \leq 0$ and $h(y_2, v) - h(y_1, u) \leq \hat{\alpha}(\Delta(y_1, y_2))$.

(H3) : For every $(x, y) \in \bar{Z}^2$ and $u \in U$ the set

$$\{(g(x, u), g(y, v), 0)) \mid v \in U,\ h(y, v) - h(x, u) \leq \hat{\alpha}(\Delta(x, y))\},$$

is closed and convex.

Then the problem $\Gamma(y_0)$ has a limit value which is $V^*(y_0)$. The convergence of V_t to V^* is uniform over Z, and we have $V^*(y_0) = \sup_{t\geq 1} \inf_{m\geq 0} V_{m,t}(y_0) = \inf_{m\geq 0} \sup_{t\geq 1} V_{m,t}(y_0) = \lim_{m\to\infty, t\to\infty} V_{m,t}(y_0)$. Moreover the value of $\Gamma(y_0)$ is uniform.

Theorem 3.5. *Suppose that (4)and (H1) holds true and moreover assume that (H2') : The set Z is bounded. There exists a continuous function $\Delta : I\!R^d \times I\!R^d \longrightarrow I\!R_+$, vanishing on the diagonal ($\Delta(y, y) = 0$ for each y) and symmetric ($\Delta(y_1, y_2) = \Delta(y_2, y_1)$ for all y_1 and y_2), and a function $\hat{\alpha} : I\!R_+ \longrightarrow I\!R_+$ s.t. $\hat{\alpha}(t) \underset{t\to 0}{\longrightarrow} 0$ satisfying*

$\forall (y_1, y_2) \in \bar{Z}^2$, $\forall u \in U$, $\exists v \in U$ such that

$D \uparrow \Delta(y_1, y_2)(g(y_1, u), g(y_2, v)) \leq 0$ and $h(y_2, v) - h(y_1, u) \leq \hat{\alpha}(\Delta(y_1, y_2))$.

(H4) for every u the following set-valued map

$$(x, y) \in \bar{Z}^2 \mapsto \{(g(x, u), g(y, v), 0)) \mid v \in U,\ h(y, v) - h(x, u) \leq \hat{\alpha}(\Delta(x, y))\},$$

is Lipschitz continuous.

Then we have the same conclusions as in Theorem 3.4.

Remarks :

• Although Δ may not satisfy the triangular inequality nor the separation property, it may be seen as a "distance" adapted to the problem $\Gamma(y_0)$.

• The assumption H3) could be checked for instance if U is compact and if h and g are continuous with respect to (y, u) and if $g(y, U)$ is a convex set for any y and if h is convex with respect to u.

• $D \uparrow$ is the contingent epi-derivative (cf [5]) (which reduces to the upper Dini derivative if Δ is Lipschitz), defined by : $D{\uparrow}\Delta(z)(\alpha) = \liminf_{t\to 0^+, \alpha'\to\alpha} \frac{1}{t}(\Delta(z + t\alpha') - \Delta(z))$. If Δ is differentiable, the condition $D \uparrow \Delta(y_1, y_2)(g(y_1, u), g(y_2, v)) \leq 0$ just reads : $< g(y_1, u), \frac{\partial}{\partial y_1}\Delta(y_1, y_2) > + < g(y_2, v), \frac{\partial}{\partial y_2}\Delta(y_1, y_2) > \leq 0$.

• Proposition 3.3 will be a corollary of Theorem 3.5. It corresponds to the case where : $\Delta(y_1, y_2) = \|y_1 - y_2\|^2$, and $h(y, u) = h(y)$ does not depend on u (one can just take $\hat{\alpha}(t) = \sup\{|h(x) - h(y)|, \|x - y\|^2 \leq t\}$). In this case the Lipschitz continuity stated in (H4) follows from the Lipschitz continuity of g (cf 4)).

• $H2a$) is a precompacity condition. It is satisfied as soon as $G(y_0)$ is bounded. It is also satisfied when Δ fulfill the triangular inequality and the usual precompacity condition : for each $\varepsilon > 0$, there exists a finite subset C of Z s.t. : $\forall z \in Z, \exists c \in C, \Delta(z,c) \leq \varepsilon$. (see lemma 3.15)

• Notice that $H2$ is satisfied with $\Delta = 0$ if we are in the trivial case where $\inf_u h(y,u)$ is constant.

• Theorem 3.5 can be applied to example 4, with $\Delta(y_1, y_2) = \|y_1 - y_2\|_1$ (L^1-norm). In this example, we have for each y_1, y_2 and u : $\Delta(y_1 + tg(y_1,u), y_2 + tg(y_2,u)) \leq \Delta(y_1,y_2)$ as soon as $t \geq 0$ is small enough.

• Theorem 3.4 can be applied to example 6 with $\Delta(y_1,y_2) = \|y_1 - y_2\|^2$ and $\hat{\alpha}(t) = kt$ where k is the Lipschitz constant of φ.

• Both assumptions (H2-a) are related to the boundedness of the reachable set $G(y_0)$. This assumption can be weakened because many - non nearly optimal - controls and the corresponding trajectories do not play any role in the definition (1) of V_t. One can think about a subset of the set of controls having the same value function V_t. To be more precise if we suppose that there exists $U' \subset U$ and $\mathcal{U}'$ the set of measurable functions from $I\!R^+$ to U' such that for any $t > 0$ the value

$$V'_t(y_0) := \inf_{u' \in \mathcal{U}'} \frac{1}{t} \int_{s=0}^{t} h(y(s,u',y_0), u'(s)) ds$$

do coincide with $V_t(y_0)$, then one easily obtain generalizations of Theorems 3.4 and 3.5 replacing the reachable set $G(y_0)$ by the reachable set associated with the admissible set of control $\mathcal{U}'$.

There are many cases where a limit value may exists but our approach fails as shown in the following

Counterexample : We consider a one dimensional case where the dynamics (2) reduces to $x'(t) = 1$ and h is given by

$$h(x) := \sum_{k=0}^{+\infty} 1_{[m_{2k}, m_{2k+1}]}(x) \text{ where } m_i = \tfrac{i(i+1)}{2} \text{ for } i = 0,1,2\ldots$$

Then one can check that the limit of $V_t(0)$ as $t \to \infty$ is equal to $\frac{1}{2}$ whereas $\sup_{t>0} \inf_{m\geq 0} V_{m,t}(y_0) = 0$. So our approach, which is based on Lemma 3.1 fails. Note that in this example h is discontinuous but this example could be easily modified by replacing the characteristic functions $1_{[m_{2k}, m_{2k+1}]}$ by suitable C^∞ approximations of $1_{[m_{2k}, m_{2k+1}]}$.

3.3 Proof of Theorems 3.4 and 3.5

We assume in this section that the hypotheses of Theorem 3.4 are satisfied, and we may assume without loss of generality that $\hat{\alpha}$ is non decreasing and upper semicontinuous (otherwise we replace $\hat{\alpha}(t)$ by $\inf_{\varepsilon>0} \sup_{t' \in [0,t+\varepsilon]} \alpha(t')$).

3.3.1 A non expansion property

We start with a proposition expressing the fact that the problem is non expansive with respect to Δ, the idea being that given two initial conditions y_1 and

y_2 and a control to be played at y_1, there exists another control to be played at y_2 such that $t \mapsto \Delta(y(t,u,y_1), y(t,v,y_2))$ will not increase.

Proposition 3.6. *We suppose the hypothesis of Theorem 3.4. Then*

$$
(5) \qquad \begin{cases} \forall (y_1,y_2) \in \bar{Z}^2,\ \forall T \geq 0, \forall u \in \mathcal{U},\ \exists v \in \mathcal{U}, \\ \forall t \in [0,T],\ \Delta(y(t,u,y_1), y(t,v,y_2)) \leq \Delta(y_1,y_2), \\ \ \textit{and for almost every } t \in [0,T], \\ h(y(t,v,y_2), v(t)) - h(y(t,u,y_1), u(t)) \leq \hat{\alpha}(\Delta(y(t,u,y_1), y(t,v,y_2))). \end{cases}
$$

Proof : First fix $y_1, y_2\ \varepsilon > 0$, $T > 0$ and u. Let us consider the following set-valued map $\Phi : I\!R_+ \times \bar{Z} \times \bar{Z} \times I\!R \to I\!R^d \times I\!R^d \times I\!R$

$$\Phi(t,x,y,l) := \{(g(x,u(t)), g(y,v), 0)) \mid v \in U,\ h(y,v) - h(x,u(t)) \leq \hat{\alpha}(\Delta(x,y))\}.$$

Notice that $\Phi(t,x,y,l)$ does not depend on l. Using (4), *H1)* and *H3))*, one can check that Φ is a set valued map which is upper semicontinuous in (x,y,l), measurable in t and with compact convex nonempty values [5, 12].

From the measurable Viability Theorem [13] (cf also [11] section 6.5), condition (H2) b) implies that the epigraph of Δ (restricted to $\bar{Z}^2 \times I\!R$) is viable for the differential inclusion

$$
(6) \qquad (x'(t), y'(t), l'(t)) \in \Phi(t, x(t), y(t), l(t)) \text{ for a. e. } t \geq 0
$$

So starting from $(y_1, y_2, \Delta(y_1,y_2))$, there exists a solution $(x(\cdot), y(\cdot), l(\cdot))$ to (6) which stays for any $t \geq 0$ in the epigraph of Δ namely

$$
(7) \qquad \Delta(x(t), y(t)) \leq l(t) = \Delta(y_1,y_2),\ \forall t \geq 0,
$$

by noticing that $l(\cdot)$ is a constant.

This completes our proof if, from one hand we observe that $x(\cdot) = y(\cdot,u,y_1)$ and from the other hand, we use Filippov's measurable selection Theorem (e.g. Theorem 8.2.10 in [5]) to Φ for finding a measurable control $v \in \mathcal{U}$ such that $y(\cdot) = y(\cdot,v,y_2)$

QED

Proposition 3.7. *We suppose the hypothesis of Theorem 3.5. Then*

$$
(8) \qquad \begin{cases} \forall (y_1,y_2) \in \bar{Z}^2,\ \forall T \geq 0,\ \forall \varepsilon > 0, \forall u \in \mathcal{U},\ \exists v \in \mathcal{U}, \\ \forall t \in [0,T],\ \Delta(y(t,u,y_1), y(t,v,y_2)) \leq \Delta(y_1,y_2) + \varepsilon, \\ \ \textit{and for almost every } t \in [0,T], \\ h(y(t,v,y_2), v(t)) - h(y(t,u,y_1), u(t)) \leq \hat{\alpha}(\Delta(y(t,u,y_1), y(t,v,y_2))). \end{cases}
$$

Proof :

We also denote Ψ the set valued map which values are the closed convex hull of values of Φ (defined in the proof of Proposition 3.6. Using (H1) and (H4), one can deduce that Ψ is Lipschitz in (x, y, l), measurable in t and with compact convex nonempty values.

From the measurable Viability Theorem [13] (cf also [11] section 6.5), conditions (H2) or (H2') imply that the epigraph of Δ (restricted to $\bar{Z}^2 \times I\!R$) is viable for the differential inclusion

$$(x'(t), y'(t), l'(t)) \in \Psi(t, x(t), y(t), l(t)) \text{ for a. e. } t \geq 0 \tag{9}$$

So starting from $(y_1, y_2, \Delta(y_1, y_2))$, there exists a solution $(x(\cdot), y(\cdot), l(\cdot))$ to (9) which stays for any $t \geq 0$ in the epigraph of Δ namely

$$\Delta(x(t), y(t)) \leq l(t) = \Delta(y_1, y_2), \ \forall t \geq 0, \tag{10}$$

by noticing that $l(\cdot)$ is a constant.

From the suppositions made on the dynamics g, the trajectory $(x(\cdot), y(\cdot))$ remains in a compact set (included in some large enough ball $B(0, M)$) on the time interval $[0, T]$. Because Δ is uniformly continuous on $B(0, M) \times B(0, M)$, there exists $\eta \in (0, 1)$ with

$$\forall (x, x', y, y') \in B(0, M+1)^4, \ \|x - x'\| + \|y - y'\| < \eta \Longrightarrow |\Delta(x, y) - \Delta(x', y')| < \varepsilon.$$

Thanks to the Wazewski Relaxation Theorem (cf for instance Theorems 10.4.3 and 10.4.4 in [5]) applied to Ψ, the trajectory $(x(\cdot), y(\cdot), l(\cdot))$ could be approximated on every compact interval by a trajectory to the differential inclusion (6). So there exists $(y_1(\cdot), y_2(\cdot), l(\cdot))$ satisfying

$$(y_1'(t), y_2'(t), l'(t)) \in \Phi(t, y_1(t), y_2(t), l(t)) \text{ for a. e. } t \geq 0$$

such that

$$\|x(t) - y_1(t)\| + \|y(t) - y_2(t)\| < \eta, \ \forall t \in [0, T].$$

From the choice of η and the very definition of Φ we also have for any $t \in [0, T]$

$$\begin{cases} \Delta(y_1(t), y_2(t)) \leq \Delta(x(t), y(t)) + \varepsilon \leq \Delta(y_1, y_2) + \varepsilon \\ h(y_2(t), v(t)) - h(y_1(t), u(t)) \leq \tilde{\alpha}(\Delta(y_1(t), y_2(t))) \end{cases}$$

This completes our proof if, from one hand we observe that $y_1(\cdot) = y(\cdot, u, y_1)$ and from the other hand, we use Filippov's measurable selection Theorem for finding a measurable control $v \in \mathcal{U}$ such that $y_2(\cdot) = y(\cdot, v, y_2)$.

QED

3.3.2 The limit value exists

Since $\hat{\alpha}$ is u.s.c. and non decreasing, we obtain the following consequence of Propositions 3.6 and 3.7.

Corollary 3.8. *For every y_1 and y_2 in $G(y_0)$, for all $T > 0$,*

$$|V_T(y_1) - V_T(y_2)| \leq \hat{\alpha}(\Delta(y_1, y_2)).$$

Define now, for each $m \geq 0$, $G^m(y_0)$ as the set of states which can be reached from x_0 before time m :

$$G^m(y_0) = \{y(t, u, y_0), t \leq m, u \in \mathcal{U}\}, \text{ so that } G(y_0) = \cup_{m\geq 0} G^m(y_0).$$

An immediate consequence of the precompacity hypothesis H2a) is the following

Lemma 3.9. *For every $\varepsilon > 0$, there exists m_0 in $I\!R_+$ such that :*

$$\forall z \in G(y_0), \exists z' \in G^{m_0}(y_0) \text{ such that } \Delta(z, z') \leq \varepsilon.$$

Proof : Otherwise for each positive integer m one can find z_m in $G(y_0)$ such that $\Delta(z_m, z) > \varepsilon$ for all z in $G^m(y_0)$. Use *H2a)* to find n such that $\liminf_m \Delta(z_n, z_m) \leq \varepsilon$. Since $z_n \in G(y_0)$, there must exist k such that $z_n \in G^k(y_0)$. But for each $m \geq k$ we have $z_n \in G^m(y_0)$, hence $\Delta(z_m, z_n) > \varepsilon$. We obtain a contradiction.

QED

We can already conclude for the limit value.

Proposition 3.10. $V_t(y_0) \xrightarrow[t\to\infty]{} V^*(y_0)$.

Proof : Because of lemma 3.1, it is sufficient to prove that for every $\varepsilon > 0$, there exists m_0 such that :

$$\sup_{t>0} \inf_{m\leq m_0} V_{m,t}(y_0) \leq \sup_{t>0} \inf_{m\geq 0} V_{m,t}(y_0) + 2\varepsilon$$

Fix ε, and consider $\eta > 0$ such that $\hat{\alpha}(t) \leq \varepsilon$ as soon as $t \leq \eta$. Use lemma 3.9 to find m_0 such that $\forall z \in G(y_0), \exists z' \in G^{m_0}(y_0)$ *s.t.* $\Delta(z, z') \leq \eta$.

Consider any $t > 0$. We have $\inf_{m\geq 0} V_{m,t}(y_0) = \inf\{V_t(z), z \in G(y_0)\}$, and $\inf_{m\leq m_0} V_{m,t}(y_0) = \inf\{V_t(z), z \in G^{m_0}(y_0)\}$. Let z in $G(y_0)$ be such that $V_t(z) \leq \inf_m V_{m,t}(y_0) + \varepsilon$, and consider $z' \in G^{m_0}(y_0)$ *s.t.* $\Delta(z, z') \leq \eta$. By corollary 3.8, $|V_t(z) - V_t(z')| \leq \hat{\alpha}(\Delta(z, z')) \leq \varepsilon$, so we obtain that

$$\inf_{m\leq m_0} V_{m,t}(y_0) \leq V_t(z') \leq V_t(z) + \varepsilon \leq \inf_m V_{m,t}(y_0) + 2\varepsilon.$$

Passing to the supremum on t, this completes the proof.

QED

Remark 3.11. *Observe that for obtaining the existence of the value, we have used a compactness argument (assumption H2)a)) and condition (8). We did not use explicitly assumption H2)b) which is only used for obtaining (8).*

The rest of the proof is more involved, and is inspired by the proof of Theorem 3.6 in [23].

3.3.3 Auxiliary value functions

The uniform value requires the same control to be good for all time horizons, and we are led to introduce new auxiliary value functions. Given $m \geq 0$ and $n \geq 1$, for any initial state z in $Z = G(y_0)$ and control u in $\mathcal{U}$, we define

$$\nu_{m,n}(z,u) = \sup_{t\in[1,n]} \gamma_{m,t}(z,u), \text{ and } W_{m,n}(z) = \inf_{u\in\mathcal{U}} \nu_{m,n}(z,u).$$

$W_{m,n}$ is the value function of the problem where the controller can use the time interval $[0,m]$ to reach a good state, and then his cost is only the supremum for t in $[1,n]$, of the average cost between time m and $m+t$. Of course, we have $W_{m,n} \geq V_{m,n}$. We write ν_n for $\nu_{0,n}$, and W_n for $W_{0,n}$.

We easily obtain from proposition 3.6, as in corollary 3.8, the following result.

Lemma 3.12. *For every z and z' in Z, for all $m \geq 0$ and $n \geq 1$,*

$$|V_{m,n}(z) - V_{m,n}(z')| \leq \hat{\alpha}(\Delta(z,z')).$$

$$|W_{m,n}(z) - W_{m,n}(z')| \leq \hat{\alpha}(\Delta(z,z')).$$

The following lemma shows that the quantities $W_{m,n}$ are not that high.

Lemma 3.13. $\forall k \geq 1, \forall n \geq 1, \forall m \geq 0, \forall z \in Z,$

$$V_{m,n}(z) \geq \inf_{l\geq m} W_{l,k}(z) - \frac{k}{n}.$$

Proof : Fix k, n, m and z, and put $A = \inf_{l\geq m} W_{l,k}(z)$. Consider any control u in $\mathcal{U}$. For any $i \geq m$, we have

$$\sup_{t\in[1,k]} \gamma_{i,t}(z,u) = \nu_{i,k}(z,u) \geq W_{i,k}(z) \geq A.$$

So we know that for any $i \geq m$, there exists $t(i) \in [1,k]$ such that $\gamma_{i,t(i)}(z,u) \geq A$.

Define now by induction $i_1 = m$, $i_2 = i_1 + t(i_1)$,..., $i_q = i_{q-1} + t(i_{q-1})$, where q is such that $i_q \leq n+m < i_q + t(i_q)$. We have $n\gamma_{m,n}(z,u) \geq \sum_{p=1}^{q-1} t(i_p)A \geq nA - k$, so $\gamma_{m,n}(z,u) \geq A - \frac{k}{n}$. Taking the infimum over all controls, the proof is complete.

QED

We know from Proposition 3.10 that the limit value is given by V^*. We now give other formulas for this limit.

Proposition 3.14. *For every state z in Z,*

$$\inf_{m\geq 0} \sup_{n\geq 1} W_{m,n}(z) = \inf_{m\geq 0} \sup_{n\geq 1} V_{m,n}(z) = V^*(z) = \sup_{n\geq 1} \inf_{m\geq 0} V_{m,n}(z) = \sup_{n\geq 1} \inf_{m\geq 0} W_{m,n}(z).$$

Proof of proposition 3.14 : Fix an initial state z in Z. We already have $V^*(z) = \sup_{t>0} \inf_{m\geq 0} V_{m,t}(z) \geq \sup_{t\geq 1} \inf_{m\geq 0} V_{m,t}(z)$. One can easily check that $\inf_{m\geq 0} V_{m,t}(z) \leq \inf_{m\geq 0} V_{m,2t}(z)$ for each positive t. So

$$V^*(z) \geq \sup_{t\geq 1} \inf_{m\geq 0} V_{m,t}(z) \geq \sup_{t\geq (1/2)} \inf_{m\geq 0} V_{m,t}(z) \geq \dots \sup_{t>0} \inf_{m\geq 0} V_{m,t}(z) = V^*(z).$$

Consequently $V^*(z) = \sup_{t\geq 1} \inf_{m\geq 0} V_{m,t}(z)$. Moreover because $V_{m,t} \leq W_{m,t}$ we have also $V^*(z) \leq \sup_{t\geq 1} \inf_{m\geq 0} W_{m,t}(z)$.

We now claim that $V^*(z) = \sup_{t\geq 1} \inf_{m\geq 0} W_{m,t}(z)$. It remains to show $V^*(z) \geq \sup_{t\geq 1} \inf_{m\geq 0} W_{m,t}(z)$. From Lemma 3.13, we know that for all $k \geq 1$, $n \geq 1$ and $m \geq 0$, we have $V_{m,nk}(z) \geq \inf_{l\geq 0} W_{l,k}(z) - \frac{1}{n}$, so $\inf_m V_{m,nk}(z) \geq \inf_{l\geq 0} W_{l,k}(z) - \frac{1}{n}$. By taking the supremum on n , we obtain

$$V^*(z) = \sup_{n\geq 1} \inf_{m\geq 0} V_{m,n}(z) \geq \sup_{n\geq 1} \inf_{m\geq 0} V_{m,nk}(z) \geq \inf_{l\geq 0} W_{l,k}(z).$$

Since k is arbitrary, we have proved our claim.

Since the inequalities

$$\inf_{m\geq 0} \sup_{n\geq 1} \; W_{m,n}(z) \geq \inf_{m\geq 0} \sup_{n\geq 1} \; V_{m,n}(z) \geq \sup_{n\geq 1} \; \inf_{m\geq 0} V_{m,n}(z) = V^*(z)$$

are clear, to conclude the proof of the proposition it is enough to show that $\inf_{m\geq 0} \sup_{n\geq 1} \; W_{m,n}(z) \leq V^*(z)$.

Fix $\varepsilon > 0$. We have already proved that $V^*(z) = \sup_{n\geq 1} \inf_{m\geq 0} W_{m,n}(z)$, so for each $n \geq 1$ there exists $m \geq 0$ such that $W_{m,n}(z) \leq V^*(z) + \varepsilon$. Hence for each n, there exists z'_n in $G(z)$ such that $W_{0,n}(z'_n) \leq V^*(z) + \varepsilon$. We know from Lemma 3.9 that there exists $m_0 \geq 0$ such that : $\forall z' \in G(z), \exists z'' \in G^{m_0}(z) \; s.t. \; \Delta(z', z'') \leq \varepsilon$. Consequently, for each $n \geq 1$, there exists z''_n in $G^{m_0}(z)$ such that $\Delta(z'_n, z''_n) \leq \varepsilon$, and by lemma 3.12 this implies that

$$W_n(z''_n) \leq W_n(z'_n) + \hat{\alpha}(\varepsilon) \leq V^*(z) + \varepsilon + \hat{\alpha}(\varepsilon).$$

Up to now, we have proved that for every $\varepsilon' > 0$, there exists m_0 such that :

$$\forall n \geq 1, \exists m \leq m_0 \text{ s.t. } W_{m,n}(z) \leq V^*(z) + \varepsilon'.$$

Since all costs lie in $[0, 1]$, it is easy to check that $|W_{m,n}(z) - W_{m',n}(z)| \leq |m - m'|$ for each n, m, m'. Hence there exists a *finite* subset F of $[0, m_0]$ such that : $\forall n \geq 1, \exists m \in F$ s.t. $W_{m,n}(z) \leq V^*(z) + 2\varepsilon'$. Considering $\hat{m}$ in F such that the set $\{n \text{ positive integer}, W_{\hat{m},n}(z) \leq V^*(z) + 2\varepsilon'\}$ is infinite, and noticing that $W_{m,n}$ is non decreasing in n, we obtain the existence of a unique $\hat{m} \geq 0$ such that $\forall n \geq 1$, $W_{\hat{m},n}(z) \leq V^*(z) + 2\varepsilon'$. Hence ε' being arbitrary, $\inf_{m\geq 0} \sup_{n\geq 1} \; W_{m,n}(z) \leq V^*(z)$, concluding the proof of Proposition 3.14.

QED

We now look for uniform convergence properties. By the precompacity condition $H2a$), it is easy to obtain that :

Lemma 3.15. *For each $\varepsilon > 0$, there exists a finite subset C of Z s.t. : $\forall z \in Z, \exists c \in C, \Delta(z,c) \leq \varepsilon$.*

We know that $(V_n)_n$ simply converges to V^* on Z. Since $|V_n(z) - V_n(z')| \leq \hat{\alpha}(\Delta(z,z'))$ for all n, z and z', we obtain by lemma 3.15 :

Corollary 3.16. *The convergence of $(V_n)_n$ to V^* is uniform on Z.*

We can proceed similarly to obtain other uniform properties. We have

$$V^*(z) = \sup_{n\geq 1} \inf_{m \geq 0} W_{m,n}(z) = \lim_{n \to +\infty} \inf_{m\geq 0} W_{m,n}(z)$$

since $\inf_{m\geq 0} W_{m,n}(z)$ is not decreasing in n. Using lemmas 3.12 and 3.15, we obtain that the convergence is uniform, hence we get :

$$\forall \varepsilon > 0, \exists n_0, \forall z \in Z, \;\; V^*(z) - \varepsilon \leq \inf_{m\geq 0} W_{m,n_0}(z) \leq V^*(z).$$

By Lemma 3.13, we obtain :

$$\forall \varepsilon > 0, \exists n_0, \forall z \in Z, \forall m \geq 0, \forall n \geq 1, V_{m,n}(z) \geq V^*(z) - \varepsilon - \frac{n_0}{n}.$$

Considering n large gives :

$$\forall \varepsilon > 0, \exists K, \forall z \in Z, \forall n \geq K, \;\; \inf_{m\geq 0} V_{m,n}(z) \geq V^*(z) - \varepsilon \tag{11}$$

Write now, for each state z and $m \geq 0$: $h_m(z) = \inf_{m' \leq m} \sup_{n\geq 1} W_{m',n}(z)$. $(h_m)_m$ converges to V^*, and as before, by Lemmas 3.12 and 3.15, we obtain that the convergence is uniform. Consequently,

$$\forall \varepsilon > 0, \exists M \geq 0, \forall z \in Z, \exists m \leq M, \;\; \sup_{n \geq 1} W_{m,n}(z) \leq V^*(z) + \varepsilon. \tag{12}$$

3.3.4 On the existence of a uniform value

In order to prove that $\Gamma(y_0)$ has a uniform value we have to show that for every $\varepsilon > 0$, there exist a control u and a time n_0 such that for every $n \geq n_0$, $\gamma_n(y_0, u) \leq V^*(y_0) + \varepsilon$. In this subsection we adapt the proofs of Lemma 4.1 and Proposition 4.2. in [23]. We start by constructing, for each n, a control which : 1) gives low average costs if one stops the play at *any* large time before n, and 2) after time n, leaves the player with a good "target" cost. This explains the importance of the quantities $\nu_{m,n}$. We start with the following

Lemma 3.17. $\forall \varepsilon > 0, \exists M \geq 0, \exists K \geq 1, \forall z \in Z, \exists m \leq M, \forall n \geq K, \exists u \in \mathcal{U}$ *such that :*

$$\nu_{m,n}(z,u) \leq V^*(z) + \varepsilon/2, \text{ and } V^*(y(m+n,u,z)) \leq V^*(z) + \varepsilon. \tag{13}$$

Proof : Fix $\varepsilon > 0$. Take M given by (12), so that $\forall z \in Z, \exists m \leq M,\ \sup_{n \geq 1} W_{m,n}(z) \leq V^*(z) + \varepsilon$. Take $K \geq 1$ given by (11) such that : $\forall z \in Z$, $\forall n \geq K$, $\inf_m V_{m,n}(z) \geq V^*(z) - \varepsilon$.

Fix an initial state z in Z. Consider m given by (12), and $n \geq K$. We have to find u in $\mathcal{U}$ satisfying (13).

We have $W_{m,n'}(z) \leq V^*(z) + \varepsilon$ for every $n' \geq 1$, so $W_{m,2n}(z) \leq V^*(z) + \varepsilon$, and we consider a control u which is ε-optimal for $W_{m,2n}(z)$, in the sense that $\nu_{m,2n}(z,u) \leq W_{m,2n}(z) + \varepsilon$. We have :

$$\nu_{m,n}(z,u) \leq \nu_{m,2n}(z,u) \leq W_{m,2n}(z) + \varepsilon \leq V^*(z) + 2\varepsilon.$$

Denoting $X = \gamma_{m,n}(z,u)$ and $Y = \gamma_{m+n,n}(z,u)$.

$$\begin{array}{lcccc} & & X & & Y \\ \text{time} \quad 0 & m & & m+n & & m+2n \end{array}$$

Since $\nu_{m,2n}(z,u) \leq V^*(z) + 2\varepsilon$, we have $X \leq V^*(z) + 2\varepsilon$, and $(X+Y)/2 = \gamma_{m,2n}(z,u) \leq V^*(z) + 2\varepsilon$. Since $n \geq K$, we also have $X \geq V_{m,n}(z) \geq V^*(z) - \varepsilon$. And $n \geq K$ also gives $V_n(y(m+n,u,z)) \geq V^*(y(m+n,u,z)) - \varepsilon$, so $V^*(y(m+n,u,z)) \leq V_n(y(m+n,u,z)) + \varepsilon \leq Y + \varepsilon$. Writing now $Y/2 = (X+Y)/2 - X/2$ we obtain $Y/2 \leq (V^*(z) + 5\varepsilon)/2$. So $Y \leq V^*(z) + 5\varepsilon$, and finally $V^*(y(m+n,u,z)) \leq V^*(z) + 6\varepsilon$.

QED

We can now conclude the proof of theorem 3.4.

Proposition 3.18. *For every state z in Z and $\varepsilon > 0$, there exists a control u in $\mathcal{U}$ and T_0 such that for every $T \geq T_0$, $\gamma_T(z,u) \leq V^*(z) + \varepsilon$.*

Proof : Fix $\alpha > 0$.

For every positive integer i, put $\varepsilon_i = \frac{\alpha}{2^i}$. Define $M_i = M(\varepsilon_i)$ and $K_i = K(\varepsilon_i)$ given by lemma 3.17 for ε_i. Define also $n_i = \text{Max}\{K_i, \frac{M_{i+1}}{\alpha}\} \geq 1$.

We have : $\forall i \geq 1, \forall z \in Z, \exists\, m(z,i) \leq M_i, \exists u \in \mathcal{U}$, s.t.

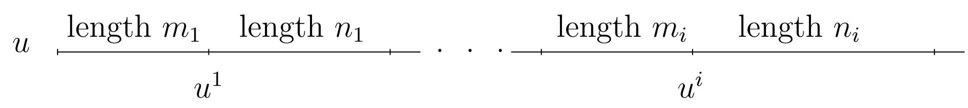
$$\nu_{m(z,i),n_i}(z,u) \leq V^*(z) + \frac{\alpha}{2^{i+1}} \text{ and } V^*(y(m(z,i)+n_i,u,z)) \leq V^*(z) + \frac{\alpha}{2^i}.$$

We now fix the initial state z in Z, and for simplicity write v^* for $V^*(z)$. We define a sequence $(z^i, m_i, u^i)_{i \geq 1}$ by induction :

• first put $z^1 = z$, $m_1 = m(z^1, 1) \leq M_1$, and pick u^1 in $\mathcal{U}$ such that $\nu_{m_1,n_1}(z^1,u^1) \leq V^*(z^1) + \frac{\alpha}{2^2}$, and $V^*(y(m_1+n_1,u^1,z^1)) \leq V^*(z^1) + \frac{\alpha}{2}$.

• for $i \geq 2$, put $z^i = y(m_{i-1}+n_{i-1}, u^{i-1}, z^{i-1})$, $m_i = m(z^i, i) \leq M_i$, and pick u^i in $\mathcal{U}$ such that $\nu_{m_i,n_i}(z^i, u^i) \leq V^*(z^i)+\frac{\alpha}{2^{i+1}}$ and $V^*(y(m_i+n_i, u^i, z^i)) \leq V^*(z^i)+\frac{\alpha}{2^i}$.

Consider finally u in $\mathcal{U}$ defined by concatenation : first u^1 is followed for time t in $[0, m_1 + n_1)$, then u^2 is followed for t in $[m_1 + n_1, m_2 + n_2)$, etc... Since $z^i = y(m_{i-1} + n_{i-1}, u^{i-1}, z^{i-1})$ for each i, we have $y(\sum_{j=1}^{i-1} m_j + n_j, u, z) = z^i$ for each i. For each i we have $n_i \geq M_{i+1}/\alpha \geq m_{i+1}/\alpha$, so an interval with length n_i is much longer than an interval with length m_{i+1}.



For each $i \geq 1$, we have $V^*(z^i) \leq V^*(z^{i-1})+\frac{\alpha}{2^{i-1}}$. So $V^*(z^i) \leq +\frac{\alpha}{2^{i-1}} + \frac{\alpha}{2^{i-2}} ... + \frac{\alpha}{2} + V^*(z^1) \leq v^* + \alpha - \frac{\alpha}{2^i}$. So $\nu_{m_i,n_i}(z^i, u^i) \leq v^* + \alpha$.

Let now T be large.

- First assume that $T = m_1 + n_1 + ... + m_{i-1} + n_{i-1} + r$, for some positive i and r in $[0, m_i]$. We have :

$$\begin{aligned} \gamma_T(z,u) &= \frac{1}{T}\int_0^T h(y(s,u,z),u(s))ds \\ &\leq \frac{1}{T}\left(\sum_{j=1}^{i-1} n_j\right)(v^*+\alpha) + \frac{m_1}{T} + \frac{1}{T}\left(\sum_{j=2}^{i} m_j\right) \end{aligned}$$

But $m_j \leq \alpha n_{j-1}$ for each j, so

$$\gamma_T(z,u) \leq v^* + 2\alpha + \frac{m_1}{T}.$$

- Assume now that $T = m_1 + n_1 + ... + m_{i-1} + n_{i-1} + m_i + r$, for some positive i and r in $[0, n_i]$. The previous computation shows that :

$$\int_0^{T-r} h(y(s,u,z),u(s))ds \leq m_1 + (T-r)(v^* + 2\alpha).$$

Since $\nu_{m_i,n_i}(z^i, u^i) \leq v^* + \alpha$, we obtain :

$$\begin{aligned} T\gamma_T(z,u) &= \int_0^{T-r} h(y(s,u,z),u(s))ds + \int_{T-r}^{T} h(y(s,u,z),u(s))ds, \\ &\leq m_1 + (T-r)(v^*+2\alpha) + r(v^*+\alpha), \\ &\leq m_1 + T(v^*+2\alpha). \end{aligned}$$

Consequently, here also we have :

$$\gamma_T(z,u) \leq v^* + 2\alpha + \frac{m_1}{T}.$$

This concludes the proofs of Proposition 3.18 and consequently, of Theorem 3.4.

QED

Remark : The extension of our result to differential games is a very difficult - and challenging - problem. Indeed even for the discrete time case the approach of [23] (which was the starting point of our work) is not extendable to two players case. An important property satisfied in the 1-player case only is that on each trajectory $(y(s))$ the liminf cost $V^-(y(s))$ can only increase with time, and in the 2-player case both $\sup_{t\geq 1} \inf_{m\geq 0} V_{m,t}(y_0)$ and $\inf_{m\geq 0}\sup_{t\geq 1} V_{m,t}(y_0)$ may differ from $\lim_{t\to\infty} V_t(y_0)$. In a different idea, let us mention that non expansive mappings have been studied in the context of 2-player zero-sum games, because the Shapley operator naturally appears to be non-expansive : here the state variable represents the value function itself, hence typically the state space is infinite dimensionnal [20].

For ergodic differential games, the already obtained results (cf for instance [1] and its bibliography) concerns firstly cases where one player has a a much more important role than the other one : he may drives the system whatever his opponent is doing and secondly cases with a coercive Hamiltonian (or cases which are of a previous form after a change of variable). To the best knowledge of the authors, the case of long time average behavior of differential games with two players is only treated in a specific case in dimension two [10].

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Références

[1] Alvarez, O. ; Bardi, M. (2010) Ergodicity, stabilization, and singular perturbations for Bellman-Isaacs equations. Mem. Amer. Math. Soc. 204.

[2] Z. Artstein, and V. Gaitsgory, (2000) *The value function of singularly perturbed control systems*, Appl. Math. Optim., 41 , 425-445.

[3] Aubin J. P., A. Cellina (1984) Differential Inclusion *Springer.*

[4] Aubin J. P., (1992) Viability Theory *Birkhauser.*

[5] Aubin J. P., Frankowska H. (1990) Set-Valued Analysis *Birkhaüser.*

[6] Arisawa, M. (1998) Ergodic problem for the Hamilton Jacobi Belmann equations II, *Ann. Inst. Henri Poincaré, Analyse Nonlinéaire*, 15 ,1 , 1–24.

[7] Arisawa, M. and P.L. Lions (1998) On ergodic stochastic control. *Com. in partial differential equations*, 23, 2187–2217.

[8] A. Bensoussan, (1988) Perturbation methods in optimal control, Wiley/Gauthiers-Villars, Chichester.

[9] Bettiol, P. (2005) On ergodic problem for Hamilton-Jacobi-Isaacs equations *ESAIM : Cocv*, 11, 522–541.

[10] Cardaliaguet P. Ergodicity of Hamilton-Jacobi equations with a non coercive non convex Hamiltonian in $I\!R^2/Z^2$ preprint [hal-00348219 - version 1] (18/12/2008)

[11] Carja, O., Necula, M., Vrabie, I. (2007) Viability, Invariance and Applications, North-Holland.

[12] K. Deimling (1992) Multivalued Differential Equations, De gruyter Seris in Nonlinear Analysis and Apllications.

[13] Frankowska, H., Plaskacz, S. and Rzezuchowski T. (1995) : Measurable Viability Theorems and Hamilton-Jacobi-Bellman Equation, J. Diff. Eqs., 116, 265-305.

[14] Gaitsgory V (1986) On the use of the averaging method in control problems. (Russian) Differentsialnye Uravneniya 22 , no. 11, 1876ñ1886,

[15] Khasminskii R.Z., (1968) On the Averaging Principle for Ito Stochastic Equations, Kybernetika, 4(3) (1968), 260ñ279.

[16] P.V. Kokotović, H.K. Khalil, and J. O'Reilly, (1986) Singular perturbation methods in control : analysis and design, Academic Press, London.

[17] Lehrer, E. and D. Monderer (1994) : Discounting versus Averaging in Dynamic Programming. Games and Economic Behavior, 6, 97-113.

[18] Lehrer, E. and S. Sorin (1992) : A uniform Tauberian Theorem in Dynamic Programming. Mathematics of Operations Research, 17, 303-307.

[19] Lions P.-L. , Papanicolaou G. , Varadhan S.R.S., Homogenization of Hamilton-Jacobi Equations, unpublished work.

[20] Kohlberg, E. ; Neyman, A. (1981) Asymptotic Behavior of Nonexpansive Mappings in Normed Linear Spaces. Israel journal of Mathematics, 38, 269-275.

[21] Monderer, D. and S. Sorin (1993) : Asymptotic properties in Dynamic Programming. International Journal of Game Theory, 22, 1-11.

[22] Papanicolaou, G.C., Stroock, D., Varadhan, S.R., (1977) Martingale approach to some limit theorems. Duke Univ. Math. Ser. III 1ñ116.

[23] Renault, J. (2007) Uniform value in Dynamic Programming. http ://fr.arxiv.org/abs/0803.2758, to appear in the Journal of the European Mathematical Society.

[24] Stockbridge R.H., (1990) Time-Average Control of a Martingale Problem : Existence of a Stationary Solution, Ann. Probab., 18, 190ñ205.

[25] Tichonov, A. N. (1952) Systems of differential equations containing small parameter near derivatives, Math. Sbornik. 31 575ñ586.

[26] Veliov, V. Critical values in long time optimal control. Unpublished work. Seminar of Applied Analysis in université de Brest (2003).

[27] Yin G.G., Zhang Q., (1997) Continuous-Time Markov Chains and Applications. A Singular Perturbation Approach, Springer, New York.