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Computation of Fisher-Gale equilibrium by auction

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Abstract

We study the Fisher model of a competitive market from the algorithmic perspective. For that, the related convex optimization problem due to Gale and Eisenberg [10] is used. The latter problem is known to yield a Fisher equilibrium under some structural assumptions on consumers' utilities, e.g. homogeneity of degree 1, homotheticity etc. Our goal is to examine the applicability of the convex optimization framework by departing from these traditional assumptions. We just assume the concavity of consumers' utility functions. For this case we suggest a novel concept of Fisher-Gale equilibrium by using consumers' utility prices. The prices of utility transfer the utility of a consumption bundle to a common numéraire. We develop a subgradient-type algorithm from Convex Analysis to compute a Fisher-Gale equilibrium via Gale's approach. In order to decentralize prices, we additionally implement the auction design, i.e. consumers settle and update their individual prices and producers sell at the highest offer price. Our price adjustment is based on a tâtonnement procedure, i.e. the prices change proportionally to consumers' individual excess supplies. Historical averages of consumption are shown to clear the market of goods. Our algorithm enjoys a convergence rate. In worst case, the number of price updates needed to achieve the ε-tolerance is proportional to $\frac{1}{\varepsilon^2}$.

Keywords: Fisher equilibrium, computation of equilibrium, price adjustment, convex optimization, subgradient methods, decentralization of prices, auction

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1 Introduction

The concept of Fisher equilibrium for a competitive market dates back to 1891, see e.g. [2]. Due to Fisher's model, consumers buy goods by spending given wealths in order to maximize their utility functions. There are fixed amounts of supplied goods available at the market. Fisher equilibrium comprises of optimal consumption bundles and equilibrium prices which clear the market of goods. Aiming at the efficient computation of a Fisher equilibrium, a related convex optimization problem has been proposed in [10]. This so-called Gale's problem consists of maximizing an aggregated logarithmic utility function subject to market feasibility constraints. The feasibility constraints ensure that the aggregated consumption does not exceed the fixed amounts of supplied goods. The solutions of Gale's problem give equilibrium allocations for the Fisher market. Moreover, the Lagrange (or dual) multipliers for its feasibility constraints yield equilibrium prices. It is crucial to point out that the solutions of Gale's problem provide Fisher equilibrium mainly if the wealths are fully spent within the budget constraints. To guarantee the latter fact some structural assumptions on the consumers' utility functions have been made in the literature. In [10] the case of linear utility functions for Fisher market has been considered. Later, the Gale's approach has been extended for concave and homogeneous utility functions of degree one in [11]. The convex optimization framework has been applied in [15] in order to handle homothetic and quasi-concave utilities. Recently in [3], the particular case of concave and non-homogeneous utility functions in potential or logarithmic form has been successively studied.

The goal of the present paper is to examine the applicability of Gale's approach by departing from the structural assumptions on the consumers' utilities. In what follows, we just assume the concavity of consumers' utility functions. In case of general concave utility functions, we cannot guarantee the full spending of wealths within the budget constraints. This is the main reason why under our concavity assumption the concepts of Fisher and Gale equilibrium may come apart. To explain this feature, we generalize both concepts of Fisher and Gale equilibrium by using the so-called utility prices attributed to consumers. They play the role of trade-offs between consumers' budget spending and utility maximization. Prices of utility allow to dynamically transfer the utility of a consumption bundle to a common numéraire. Using this transferable utility, we introduce a novel concept of Fisher-Gale equilibrium. Here, consumers maximize their revenues as the differences of transferred utilities and expenditures expressed in a numéraire (see Definition 3 for details). It turns out that Fisher and Gale equilibria can be viewed as Fisher-Gale equilibrium (see Theorem 1). In particular, for Fisher equilibrium the utility prices are inverse shadow prices (or Lagrange multipliers associated to budget constraints). For Gale equilibrium, the utility prices appear as ratios of wealths to achieved utilities. The latter gives rise to the efficient computation of a Fisher-Gale equilibrium by following the Gale's approach. We revise some previous attempts to solve the Gale's convex optimization problem known in the literature. Already in [13] the ellipsoid method has been applied for that. In [9], a polynomial time algorithm based on a primal-dual scheme has been proposed to tackle the Gale's problem. An interior-point method for Gale's problem is developed in [32]. For an algorithm based on the excess demand function we refer to [6]. An auction-based algorithm for Fisher model has been suggested in [14]. A distributed algorithm via gradient descent for Fisher market with linear and spending constraint utilities has been suggested in [1]. In [12], a decentralized algorithm with the tâtonnement price adjustment has been constructed using the indirect utility functions. We also mention [20] where a simultaneous ascending auction is used to construct a decentralized price adjustment. For comprehensive surveys on the computational issues of economic equilibria see [8, 29].

In this paper we develop a subgradient-type algorithm to compute a Fisher-Gale equilibrium by Gale's approach. Its convergence properties are crucially based on Convex Analysis. The price adjustment corresponds to the quasi-monotone subgradent method for nonsmooth convex minimization, recently suggested in [23]. As objective function for the latter method we take the total logarithmic revenue of the market. Equilibrium prices can be then characterized as its minimizers. By doing so, we independently rediscovered the framework recently proposed in [5]. In [5] the minimization of the total logarithmic revenue has been studied in the smooth setting by using gradient method. For that, the authors concentrate on Leontief utilities and complementary Constant Elasticity of Substitution (CES) utilities which induce a smooth total logarithmic revenue. As to be expected for the usual gradient method, their analysis provides $\frac{1}{\varepsilon}$ rate of convergence for the ε-tolerance. In contrast to this study, we minimize the total logarithmic revenue in the nonsmooth setting assuming just the concavity of consumers' utility functions. In this general case, the total logarithmic revenue need not to be smooth, as already the example with linear additive utilities shows (see Example 2).

In order to decentralize prices, we additionally implement the auction design:

consumers settle and update their individual prices,
and producers sell at the highest offer price.

It is crucial for our approach that the introduction of the auction design preserves convexity of the total logarithmic revenue. Moreover, its convex subgradients w.r.t. a consumer's price become his individual excess supplies, which are easily observable. This is used by consumers to successively update prices by themselves rather than by relying on a central authority. Our price adjustment is based on a tâtonnement procedure, i.e. the prices change proportionally to consumers' individual excess supplies. While our algorithm proceeds, the market clearance is achieved on average. The latter means that during the price adjustment supply meets demand statistically. In mathematical terms, average consumption bundles approach the solution of the Gale's (or adjoint) problem for the minimization of the total logarithmic revenue. Altogether, the sequence of highest offer prices, historical averages of consumption bundles and historical averages of utility prices generated by our algorithm, converges to the set of Fisher-Gale equilibria (see Theorem 5). Moreover, our algorithm is able to guarantee a convergence rate of this process. In the worst case, the number of price updates needed to achieve the ε-tolerance is proportional to $\frac{1}{\varepsilon^2}$. Note that this rate of convergence is optimal for nonsmooth convex minimization, cf. [22]. From the economic perspective, this result explains why competitive markets adjust in efficient way, moreover, it quantifies the worst-case efficiency. Note that relatively low accuracy of price adjustment processes usually suffices for markets. Consequently, our complexity result of $\frac{1}{\varepsilon^2}$ is quite reasonable.

The article is organized as follows. In Section 2 we introduce and discuss the concept of Fisher-Gale equilibrium. In Section 3 we describe the decentralization of prices by the auction. We prove the convergence of our decentralized subgradient-type algorithm toward the set of Fisher-Gale equilibria in Section 4. Appendix is devoted to the mathematical justification of quasi-monotone subgradient schemes.

Notation. Our notation is quite standard. We denote by $\mathbb{R}^n$ the space of n-dimensional column vectors $x = (x^{(1)}, \ldots, x^{(n)})^T$, and by $\mathbb{R}^n_+$ the set of all vectors with nonnegative components. $\mathbb{R}_{++}$ stand for the set of positive real numbers. For x and y from $\mathbb{R}^n$, we introduce the standard scalar product and the Hadamard product

$$\langle x, y \rangle = \sum_{i=1}^{n} x^{(i)} y^{(i)}, \quad x \circ y = \left(x^{(i)} y^{(i)} \right)_{i=1}^{n} \in \mathbb{R}^n.$$

For the vectors $p_1, \ldots, p_I \in \mathbb{R}^n$, we denote by $\max\limits_{i=1,\ldots,I} p_i \in \mathbb{R}^n$ the vector with coordinates

$$\left(\max_{i=1,\ldots,I} p_i \right)^{(j)} = \max_{i=1,\ldots,I} p_i^{(j)}, \quad j = 1, \ldots, n.$$

2 Fisher-Gale equilibrium

We start with the classical concept of Fisher equilibrium, see e.g. [2]. Consider a market with I consumers, which are able to buy n divisible goods. The i-th consumer has to decide on the consumption bundle $x_i \in X_i$, where the consumption set $X_i \subset \mathbb{R}^n_+$ is assumed to be nonempty and convex. Given a vector of prices $p \in \mathbb{R}^n_+$, the i-th consumer maximizes the concave utility function $u_i : \mathbb{R}^n_+ \to \mathbb{R}$ with respect to the so-called budget constraint. The latter says that the acquired consumption bundle cannot cost more than the available wealth $w_i \in \mathbb{R}_+$ of the i-th consumer. We assume that the utility function u_i is positive on the topological interior of the consumption set $\mathrm{int}(X_i)$, i.e. $u_i(x_i) > 0$ for all $x_i \in \mathrm{int}(X_i)$. On the production side of the market there are K producers. Each of them supplies fixed amounts of goods as given by the vectors $e_k \in \mathbb{R}^n_+$, $k = 1, \ldots, K$. The aggregate supply of goods is thus $e \stackrel{\mathrm{def}}{=} \sum_{k=1}^{K} e_k \in \mathbb{R}^n_+$. Finally, equilibrium prices ensure the market clearing condition, i.e. the aggregate consumption never exceeds the available amounts of supplied goods, and the markets of goods with positive prices are perfectly cleared.

Definition 1 (Fisher equilibrium, [2]) *The vector of prices and consumption bundles* $\left(p^*, (x_i^*)_{i=1}^{I} \right)$ *is called Fisher equilibrium, if*

(i) consumers maximize utilities w.r.t. budget constraints, i.e.

$$x_i^* \in \mathit{arg} \max_{\substack{x_i \in X_i \\ \langle p^*, x_i \rangle \le w_i}} u_i(x_i), \quad i = 1, \ldots, I, \tag{1}$$

(ii) the market clearing condition holds, i.e.

$$p^* \ge 0, \quad e - \sum_{i=1}^{I} x_i^* \ge 0, \quad \left\langle p^*, e - \sum_{i=1}^{I} x_i^* \right\rangle = 0. \tag{2}$$

In order to compute Fisher equilibrium, the following convex optimization problem has been proposed in [10, 13]:

$$\max_{\substack{x_i \in X_i \\ i=1,\ldots,I}} \sum_{i=1}^{I} w_i \ln u_i(x_i) \quad \text{such that} \quad \sum_{i=1}^{I} x_i \leq e. \tag{3}$$

The objective function in (3) may be viewed as a socially aggregated utility, i.e. the sum of consumers' wealths assessed by logarithmic utility factors. The feasibility constraint in (3) means that the aggregate consumption never exceeds the available amounts of supplied goods. Market prices appear naturally as Lagrange multipliers for the feasibility constraint. Indeed, due to the duality of convex optimization, we obtain for (3):

$$\max_{\substack{x_i \in X_i \\ i=1,\ldots,I}} \min_{p \geq 0} \sum_{i=1}^{I} w_i \ln u_i(x_i) + \left\langle p, e - \sum_{i=1}^{I} x_i \right\rangle =$$

$$\min_{p \geq 0} \sum_{i=1}^{I} \max_{x_i \in X_i} w_i \ln u_i(x_i) - \langle p, x_i \rangle + \langle p, e \rangle .$$

The latter saddle-point problem can be interpreted economically as follows. Given the vector of prices $p \in \mathbb{R}^n_+$, the i-th consumer maximizes his logarithmic revenue, i.e. he solves

$$LR_i(p) \stackrel{\text{def}}{=} \max_{x_i \in X_i} w_i \ln u_i(x_i) - \langle p, x_i \rangle . \tag{4}$$

Here, the logarithmic revenue is given as the difference between i-th consumer's logarithmically assessed wealth and his expenditures. Finally, the equilibrium prices are characterized by minimizing the total logarithmic revenue of consumers and producers:

$$TLR(p) \stackrel{\text{def}}{=} \sum_{i=1}^{I} LR_i(p) + \langle p, e \rangle .$$

Motivated by the forgoing discussion, we define

Definition 2 (Gale equilibrium, [10, 13]) *The vector of prices and consumption bundles* $\left(p^*, (x_i^*)_{i=1}^I\right)$ *is called Gale equilibrium, if it solves the saddle point problem*

$$\min_{p \geq 0} \sum_{i=1}^{I} \max_{x_i \in X_i} w_i \ln u_i(x_i) - \langle p, x_i \rangle + \langle p, e \rangle .$$

Namely,

(i) consumers maximize logarithmic revenues, i.e.

$$x_i^* \in arg \max_{x_i \in X_i} w_i \ln u_i(x_i) - \langle p^*, x_i \rangle , \quad i = 1, \ldots, I, \tag{5}$$

(ii) the market clearing condition holds, i.e.

$$p^* \geq 0, \quad e - \sum_{i=1}^{I} x_i^* \geq 0, \quad \left\langle p^*, e - \sum_{i=1}^{I} x_i^* \right\rangle = 0. \tag{6}$$

It is well-known in the literature under which conditions the concepts of Fisher and Gale equilibrium coincide. In case of $X_i = \mathbb{R}^n_+$ and linear utility functions $u_i(\cdot)$, $i = 1, \ldots, I$, the equivalence of Fisher and Gale equilibrium has served as a starting point for the seminal paper [10]. In [11], the equivalence result has been generalized for concave and homogeneous utility functions of degree 1. The convex optimization framework (3) has been applied in [15] in order to handle homothetic and quasi-concave utilities. Recently in [3], the case of concave and non-homogeneous utility functions in potential or logarithmic form has been successively tackled. It is worth to mention that the equivalence of Fisher and Gale concepts crucially relies on the full spending of wealths within the budget constraints. It turns out that the structural assumptions on the utilities provide the latter fact. The goal of the present paper is to examine the applicability of the convex optimization approach (3) by departing from the structural assumptions on the consumers' utilities. We merely assume that the utility functions $u_i(\cdot)$, $i = 1, \ldots, I$, are concave. Moreover, as a novelty we introduce general compact consumption sets X_i with $0 \in X_i$, $i = 1, \ldots, I$ rather than $X_i = \mathbb{R}^n_+$ as in the previous literature. The compactness assumption on X_i refers to the fact that the consumption is bounded. Naturally, taking into account there are physical limits to what people can consume and want to consume in order to satisfy their needs. The bounded consumption can also be justified by ecological reasons. The unbounded desire for wealth is not an issue here, since the wealth w_i is a primitive in Fisher's model (confer the discussion on this assumption in [27]). In case of general concave utility functions and compact consumption sets, we cannot guarantee the full spending of wealths within the budget constraints. This is the main reason why under our assumptions the concepts of Fisher and Gale equilibrium need not to coincide in general. To explain this feature, we generalize both concepts of Fisher and Gale equilibria by using the so-called utility prices $q_i \in (0, \infty]$ attributed to the i-th consumer. Prices of utility q_i allow to dynamically transfer the utility $u_i(x_i)$ of a consumption bundle x_i to a common numéraire by $q_i u_i(x_i)$. For the discussion on the concept of transferable utility we refer e.g. to [18].

Definition 3 (Fisher-Gale equilibrium) *The vector of prices and consumption bundles* $\left(p^*, (x_i^*)_{i=1}^I\right)$ *is called Fisher-Gale equilibrium w.r.t. utility prices* $(q_i)_{i=1}^I$, *if*

(i) consumers maximize revenues fulfilling budget constraints, i.e.

$$x_i^* \in arg \max_{x_i \in X_i} \; q_i u_i(x_i) - \langle p^*, x_i \rangle, \; and \; \langle p^*, x_i^* \rangle \leq w_i, \quad i = 1, \ldots, I, \tag{7}$$

(ii) the market clearing condition holds, i.e.

$$p^* \geq 0, \quad e - \sum_{i=1}^{I} x_i^* \geq 0, \quad \left\langle p^*, e - \sum_{i=1}^{I} x_i^* \right\rangle = 0. \tag{8}$$

Note that the utility price $q_i = \infty$ in (7) means that $x_i^* \in \arg \max\limits_{x_i \in X_i} u_i(x_i)$.

In what follows, we discuss the novel concept of Fisher-Gale equilibrium in detail. First, note that utility prices $(q_i)_{i=1}^I$ from Definition 3 play the role of trade-offs between consumers' budget spending and utility maximization. By properly choosing utility prices the consumers may keep some budget unspent. The latter will cause the sacrifices in their achieved utility. Next Example 1 highlights this issue. Here, we examine the consumer's revenue maximization (7) for homogeneous utility functions of degree $\gamma \in (0,1)$.

Example 1 (Homogeneity of degree $\gamma \in (0,1)$) *Let us consider the consumer's revenue maximization as in (7):*

$$\max_{x \geq 0} \; qu(x) - \langle p, x \rangle, \;\; \textit{and} \; \langle p, x \rangle \leq w, \tag{9}$$

where the utility function u is homogeneous of degree $\gamma \in (0,1)$, i.e.

$$u(tx) = t^\gamma u(x) \quad \textit{for all } x, t \geq 0.$$

Substituting $x = ty$ into (9), we have

$$\max_{x \geq 0} \; qu(x) - \langle p, x \rangle = \max_{y, t \geq 0} \; qt^\gamma u(y) - t\langle p, y \rangle. \tag{10}$$

Maximizing first w.r.t. t for a fixed y, we obtain

$$t = \left[\frac{\gamma q u(y)}{\langle p, y \rangle} \right]^{\frac{1}{1-\gamma}}.$$

Substituting this formula into (10), we get

$$\max_{y \geq 0} \; \left[\frac{qu(y)}{\langle p, y \rangle^\gamma} \right]^{\frac{1}{1-\gamma}} \gamma^{\frac{\gamma}{1-\gamma}} (1-\gamma).$$

Due to the homogeneity of $u(\cdot)$ of degree γ, this maximization problem is equivalent to

$$\max_{\substack{\langle p, y \rangle = 1 \\ y \geq 0}} \; u(y). \tag{11}$$

Here, one unit of the numéraire is spent optimally w.r.t. the usual utility maximization. Having a solution y^* of (11), we obtain a solution of (10):

$$x^* = ty^* = \left[\gamma q u(y^*)\right]^{\frac{1}{1-\gamma}} y^*.$$

The optimal budget spending and achieved utility are

$$\langle p, x^* \rangle = \left[\gamma q u(y^*)\right]^{\frac{1}{1-\gamma}}, \quad u(x^*) = \left[\gamma q u(y^*)\right]^{\frac{\gamma}{1-\gamma}} u(y^*).$$

Further, note that from $\langle p, x^* \rangle \leq w$, we deduce that the utility price need to satisfy:

$$q \leq \frac{w^{1-\gamma}}{\gamma u(y^*)}.$$

From here we see that utility prices compromise both the budget spending and the achieved utility. In particular, if $q \to 0$, *then* $\langle p, x^* \rangle \to 0$ *and* $u(x^*) \to 0$; *if* $q \to \frac{w^{1-\gamma}}{\gamma u(y^*)}$, *then* $\langle p, x^* \rangle \to w$ *and* $u(x^*) \to w^\gamma u(y^*)$.

Further, it turns out that by setting particular utility prices in (9) we recover Fisher's utility maximization (1) and Gale's logarithmic revenue maximization (5). Indeed, if $q = \frac{w^{1-\gamma}}{\gamma u(y^*)}$, *then* $\langle p, x^* \rangle = w$, *and* $x^* = wy^*$. *Here, the whole budget is spent, and we have the optimal consumption of Fisher's utility maximization (1). If* $q = \frac{w^{1-\gamma}}{\gamma^\gamma u(y^*)}$, *then* $\langle p, x^* \rangle = \gamma w$, *and* $x^* = \gamma w y^*$. *The latter gives us the optimal consumption of Gale's logarithmic revenue maximization (5).* $\square$

Next Theorem 1 shows in general that equilibria due to Fisher and Gale are particular cases of Fisher-Gale equilibrium. For Fisher equilibrium the utility prices arise as inverse shadow prices (or Lagrange multipliers associated to budget constraints). For Gale equilibrium the utility prices can be found as ratios of wealths to achieved utility values.

Theorem 1

(a) If $\left(p^*, (x_i^*)_{i=1}^I\right)$ *is a Fisher equilibrium with Lagrange multipliers* λ_i^* *associated to budget constraints in (1), then* $\left(p^*, (x_i^*)_{i=1}^I\right)$ *is a Fisher-Gale equilibrium w.r.t. utility prices* $\left(\frac{1}{\lambda_i^*}\right)_{i=1}^I$.

(b) If $\left(p^*, (x_i^*)_{i=1}^I\right)$ *is a Gale equilibrium, then* $\left(p^*, (x_i^*)_{i=1}^I\right)$ *is a Fisher-Gale equilibrium w.r.t. utility prices* $\left(\frac{w_i}{u_i(x_i^*)}\right)_{i=1}^I$.

Proof:
(a) Let $\left(p^*, (x_i^*)_{i=1}^I\right)$ be a Fisher equilibrium according to Definition 1. Optimality conditions for (1) read

$$\langle \nabla u_i(x_i^*) - \lambda_i^* p^*, x_i^* - y_i \rangle \geq 0 \quad \text{for all } y_i \in X_i, \tag{12}$$

$$\lambda_i^* \geq 0, \langle p^*, x_i^* \rangle \leq w_i, \lambda_i^* \left(w_i - \langle p^*, x_i^* \rangle\right) = 0,$$

Due to concavity of utility functions $u_i(\cdot)$, $i = 1, \dots, I$, we have

$$\langle \nabla u_i(x_i^*), x_i^* - y_i \rangle \leq u_i(x_i^*) - u_i(y_i) \quad \text{for all } y_i \in X_i. \tag{13}$$

Together with (12) we obtain

$$u_i(x_i^*) - u_i(y_i) - \lambda_i^* \langle p^*, x_i^* - y_i \rangle \geq 0 \quad \text{for all } y_i \in X_i,$$

thus, if $\lambda_i^* \neq 0$,

$$\frac{1}{\lambda_i^*} u_i(x_i^*) - \langle p^*, x_i^* \rangle \geq \frac{1}{\lambda_i^*} u_i(y_i) - \langle p^*, y_i \rangle \geq 0 \quad \text{for all } y_i \in X_i.$$

If $\lambda_i^* = 0$, then the utility price is formally set to $\frac{1}{\lambda_i^*} = \infty$, and

$$u_i(x_i^*) \geq u_i(y_i) \quad \text{for all } y_i \in X_i.$$

(b) Let $\left(p^*, (x_i^*)_{i=1}^I\right)$ be a Gale equilibrium according to Definition 2. Optimality conditions for (5) read

$$\left\langle \frac{w_i}{u_i\left(x_i^*\right)} \nabla u_i(x_i^*) - p^*, x_i^* - y_i \right\rangle \geq 0 \quad \text{for all } y_i \in X_i, \tag{14}$$

Again using (13), we obtain

$$\frac{w_i}{u_i\left(x_i^*\right)} u_i(x_i^*) - \langle p^*, x_i^* \rangle \geq \frac{w_i}{u_i\left(x_i^*\right)} u_i(y_i) - \langle p^*, y_i \rangle \quad \text{for all } y_i \in X_i.$$

Moreover, setting $y_i = 0$ in (14) and in (13), we have

$$\langle p^*, x_i^* \rangle \leq w_i \frac{\langle \nabla u_i(x_i^*), x_i^* \rangle}{u_i\left(x_i^*\right)} \leq w_i \frac{u_i(x_i^*) - u_i(0)}{u_i\left(x_i^*\right)} \leq w_i.$$

Overall, the assertions (a) and (b) follow. □

Using Theorem 1, we relate the Fisher-Gale equilibrium to the well-known Negishi's approach to exchange equilibria from [21].

Remark 1 (Negishi's approach and Fisher-Gale equilibrium) *In [21] Negishi aims at characterizing exchange equilibria as welfare maximizers by appropriately choosing utility prices. In order to apply Negishi's approach, we equivalently reformulate Fisher equilibrium in terms of exchange. For that, we assign to every consumer a fraction of the producers' supplied goods proportional to his wealth. We call the vector of prices and consumption bundles* $\left(p^*, (x_i^*)_{i=1}^I\right)$ *exchange equilibrium, if*

(i) consumers maximize utilities w.r.t. modified budget constraints, i.e.

$$x_i^* \in arg \max_{\substack{x_i \in X_i \\ \langle p^*, x_i \rangle \leq \left\langle p^*, \frac{w_i}{\sum_{i=1}^I w_i} e \right\rangle}} u_i(x_i), \quad i = 1, \ldots, I,$$

(ii) the market clearing condition holds, i.e.

$$p^* \geq 0, \quad e - \sum_{i=1}^I x_i^* \geq 0, \quad \left\langle p^*, e - \sum_{i=1}^I x_i^* \right\rangle = 0.$$

It is straightforward to see that the concepts of Fisher and exchange equilibria are equivalent. Namely, if $\left(p^*, (x_i^*)_{i=1}^I\right)$ *is a Fisher equilibrium, then it is also an exchange equilibrium. Vice versa, if* $\left(p^*, (x_i^*)_{i=1}^I\right)$ *is a an exchange equilibrium, then* $\left(\frac{\sum_{i=1}^I w_i}{\langle p^*, e\rangle} p^*, (x_i^*)_{i=1}^I\right)$ *is a Fisher equilibrium. Here, we use the invariance of the exchange equilibrium under the scaling of prices.*

The Negishi's welfare maximization is

$$\max_{\substack{x_i \in X_i \\ i=1,\dots I}} \left\{ \sum_{i=1}^{I} q_i u_i(x_i) \,\middle|\, \sum_{i=1}^{I} x_i \le e \right\}, \tag{15}$$

where q_i, $i = 1, \dots, I$, are positive utility prices. Due to [21], if $\left(p^*, (x_i^*)_{i=1}^I\right)$ is an exchange equilibrium, then there exist utility prices $(q_i)_{i=1}^I$ such that $(x_i^*)_{i=1}^I$ maximizes Negishi's welfare with dual (or Lagrange) multipliers p^* w.r.t. the market feasibility constraints $\sum_{i=1}^I x_i \le e$. As we have seen in Theorem 1 (a), these utility prices can be taken as inverse shadow prices corresponding to the budget constraints. Also, the converse statement is given in [21]. Namely, fixing some utility prices in (15), its welfare maximizer with corresponding dual prices of goods forms an exchange equilibrium, but in general with redistributed wealths. Hence, in order to find a Fisher equilibrium, it is sufficient to determine utility prices for (15) such that initial budget constraints are fulfilled. However, this task is as challenging as to compute a Fisher equilibrium itself. E.g., in [17] an adjustment of utility prices according to the consumers' savings is studied. In our approach, we relax the concept of Fisher equilibrium by imposing budget constraints for Negishi's welfare maximizers:

$$\begin{gathered} (x_i^*)_{i=1}^I \in arg \max_{\substack{x_i \in X_i \\ i=1,\dots I}} \left\{ \sum_{i=1}^{I} q_i u_i(x_i) \,\middle|\, \sum_{i=1}^{I} x_i \le e \right\}, \\ and \ \langle p^*, x_i^* \rangle \le w_i, \quad i = 1, \dots, I, \end{gathered} \tag{16}$$

where prices p^* are dual multipliers w.r.t. the market feasibility. After a moment of reflection we see that $\left(p^*, (x_i^*)_{i=1}^I\right)$ from (16) is a Fisher-Gale equilibrium. □

Theorem 1 clarifies how consumers may settle utility prices in a meaningful way, i.e. in consistency with their economic behavior. At least two possibilities are

- inverse shadow prices,
- wealth/utility ratios.

In this paper we examine the adjustment of utility prices according to the wealth/utility relation by following the Gale's approach. For that, we assume that the i-th consumer is able to compute an optimal consumption bundle x_i by maximizing the logarithmic revenue (4), i.e.

$$x_i \in \arg \max_{x_i \in X_i} \ w_i \ln u_i(x_i) - \langle p, x_i \rangle, \tag{17}$$

for a fixed vector of prices $p \in \mathbb{R}^n_+$. Let us provide algorithmic and economic justifications for this assumption.

1) Algorithmic justification for logarithmic revenue maximization

As we have seen in the proof of Theorem 1 (b), a consumption bundle x_i from (17)

- satisfies the budget constraint, i.e. $\langle p, x_i \rangle \le w_i$,
- maximizes revenue with the utility price $q_i = \frac{w_i}{u_i(x_i)}$, i.e.

$$q_i u_i(x_i) - \langle p, x_i \rangle \ge q_i u_i(y_i) - \langle p, y_i \rangle \qquad \text{for all } y_i \in X_i. \tag{18}$$

In accordance with this interpretation, we always associate the utility price $q_i = \frac{w_i}{u_i(x_i)}$ with the consumption bundle x_i. The maximization of the logarithmic revenue as in (17) can be performed unintentionally by subgradient dynamics

$$\dot{x}_i \in \frac{w}{u_i(x)} \nabla u_i(x) - p. \tag{19}$$

In order to form the subgradient $\frac{w}{u_i(x)} \nabla u_i(x) - p$ of the logarithmic revenue, the consumer estimates the marginal utility $\nabla u_i(x)$ which is further reassessed by the utility price $\frac{w}{u_i(x)}$. Here, the utility price is taken as the wealth/utility ratio. Finally, the comparison of this reassessed marginal utility $\frac{w}{u_i(x)} \nabla u_i(x)$ with the prices of goods p is performed. Thus, it is reasonable to assume that $\frac{w}{u_i(x)} \nabla u_i(x) - p$ can be used in (19) by the i-th consumer. Note that there is evidence from behavioral economics that consumer's choices need not be consistent with the maximization of a preference relation (see [16] and references therein). The reason for that is usually referred to as consumers' bounded rationality. Classic examples include status-quo biases, attraction, compromise and framing effects, temptation and self-control, consideration sets, and choice overload. Within our approach, the consumption based on the maximization of the logarithmic revenue is consistent with the concept of transferable utility (cf. also [4]). Further, we mention that the discretization of (19) leads to subgradient schemes for nonsmooth convex optimization. Those are known to enjoy guaranteed rates of convergence [22]. This explains how consumers efficiently maximize the logarithmic revenue by successively using its subgradients $\frac{w}{u_i(x)} \nabla u_i(x) - p$.

2) Economic justification for logarithmic revenue maximization

Let us define a quasilinear utility function for the i-th consumer as follows

$$U(x_i, \tau_i) \stackrel{\text{def}}{=} w_i \ln u_i(x_i) + \tau_i, \tag{20}$$

where $\tau_i \in \mathbb{R}_+$ denotes the unspent numéraire. We consider the i-th consumer's quasilinear utility maximization [18]:

$$\max_{\substack{\langle p, x_i \rangle + \tau_i \le w_i \\ x_i \in X_i, \tau_i \ge 0}} U_i(x_i, \tau_i).$$

Due to (20), the budget constraint $\langle p, x_i \rangle + \tau_i \le w_i$ is tight for optimal consumption bundles x_i, and we get

$$\max_{\substack{\langle p, x_i \rangle + \tau_i \le w_i \\ x_i \in X_i, \tau_i \ge 0}} U_i(x_i, \tau_i) = \max_{\substack{\langle p, x_i \rangle + \tau_i = w_i \\ x_i \in X_i, \tau_i \ge 0}} w_i \ln u_i(x_i) + \tau_i$$

$$= w_i + \max_{\substack{\langle p, x_i \rangle \le w_i \\ x_i \in X_i}} w_i \ln u_i(x_i) - \langle p, x_i \rangle = w_i + \max_{x_i \in X_i} w_i \ln u_i(x_i) - \langle p, x_i \rangle .$$

The latter equality follows from the proof of Theorem 1 (b), since $\langle p, x_i \rangle \leq w_i$ always holds for optimal consumption bundles of the logarithmic revenue maximization. Thus, the constraint $\langle p, x_i \rangle \leq w_i$ is superfluous here. In case of a general quaislinear utility this budget constraint is usually neglected by an assumption of no income effects [18]. No income effects mean that the available wealth w_i does not affect consumption. This assumption has been questioned in the framework of Marshallian partial equilibrium analysis in [19, 30]. In [30] the setting of a variable number of commodities is suggested. Sufficient conditions for a neo-classical utility function to induce small income effects are provided if the number of commodities is sufficiently large. In [19] a special class of quasilinear functions with restrictions for large income levels is studied. Due to our approach, an assumption on income effects is not needed. Overall, the logarithmic revenue maximization (17) is equivalent to the maximization of the particular quasilinear utility function (20) with respect to the budget constraint. $\square$

Finally, we compare the optimal budget spending for Fisher and Gale equilibria. In case of Fisher equilibrium consumers face the full budget spending under standard monotonicity assumptions. In turn, for Gale equilibrium the optimal budget spending is adjustable and depends on elasticities of utility functions.

Remark 2 (Budget spending) *Assuming $X_i = \mathbb{R}^n_+$, we consider the Fisher's utility maximization*

$$\max_{\substack{x_i \geq 0 \\ \langle p, x_i \rangle \leq w_i}} \quad u_i(x_i).$$

If $u_i(\cdot)$ is strictly monotone, then the optimal budget spending $\langle p, x_i \rangle$ amounts to the available wealth w_i. In this context, the budget spending is fixed at the wealth level w_i which should be known a priori as the amount of numéraire surely spent on the market under consideration. Moreover, every strictly monotone transformation of $u_i(\cdot)$ also induces the full budget spending (cf. Example 1). Hence, the optimal budget spending is not affected by the elasticities of the utility function u_i [18]:

$$\varepsilon_j(x_i) \overset{\text{def}}{=} \frac{\partial_{x_j} u_i(x_i)}{u_i(x_i)} \cdot x^{(j)}, \quad j = 1, \ldots, n.$$

In case of Gale's logarithmic revenue maximization

$$\max_{x_i \geq 0} \; w_i \ln u_i(x_i) - \langle p, x_i \rangle$$

the situation is different. In fact, assuming $u_i(\cdot)$ to be differentiable, we have necessary optimality conditions

$$\frac{w}{u(x_i)} \nabla u(x_i) = p$$

for an optimal consumption bundle $x_i \in \mathbb{R}^n_{++}$. Multiplying by x_i, we get

$$\langle p, x_i \rangle = w_i \frac{\langle \nabla u(x_i), x_i \rangle}{u_i(x_i)} = w_i \sum_{j=1}^{n} \varepsilon_j(x_i).$$

This formula says that the ratio of the optimal budget spending $\langle p, x_i \rangle$ to the available wealth w_i is the sum of elasticities $\varepsilon_j(x_i)$, $j = 1, \ldots, n$. Here, the optimal budget spending is adjusted depending on utility elasticities and is not known a priori. The available wealth w_i has a role of its upper bound which may or may not be reached. Recall from the proof of Theorem 1 (b) that

$$\sum_{j=1}^{n} \varepsilon_j(x_i) \le 1$$

for a concave utility function u_i. □

3 Auction design

Theorem 1 (b) suggests that for finding a Fisher-Gale equilibrium we may solve the following saddle point problem:

$$\min_{p \ge 0} \sum_{i=1}^{I} \max_{x_i \in X_i} \; w_i \ln u_i(x_i) - \langle p, x_i \rangle + \langle p, e \rangle .$$

First, we concentrate on the Fisher-Gale equilibrium prices as minimizers of the total logarithmic revenue, cf. [5]:

$$TLR^* \stackrel{\text{def}}{=} \min_{p \in \mathbb{R}^n_+} TLR(p), \qquad (\mathbf{P})$$

where

$$TLR(p) = \sum_{i=1}^{I} LR_i(p) + \langle p, e \rangle , \quad LR_i(p) = \max_{x_i \in X_i} \left[w_i \ln u_i(x_i) - \langle p, x_i \rangle \right] .$$

In order to ensure solvability in ($\mathbf{P}$), we assume that the market is productive. The productivity of the market says that there exist $\bar{x}_i \in X_i$ with $u_i(\bar{x}) > 0$, $i = 1, \ldots, I$, such that the supply of goods strictly exceeds the aggregate demand, i.e.

$$\sum_{i=1}^{I} \bar{x}_i < e.$$

Actually, the market productivity can be viewed as the standard Slater condition for the logarithmic welfare maximization (3). It is well-known that Slater condition implies the existence and boundedness of Lagrange multipliers (e.g., [25]), which are equilibrium prices in our context. Hence, at productive markets the set of equilibrium prices (or, equivalently, minimizers of the total logarithmic revenue TLR) is nonempty and bounded. From now on, let us assume the market productivity throughout.

As the maximum of linear functions the total logarithmic revenue $TLR(p)$ is convex w.r.t. the price p. However, the total logarithmic revenue is in general nonsmooth, even in case of homogeneous utilities. We illustrate this by examining markets with Leontief, Cobb-Douglas and linear additive utilities.

Example 2 (Leontief, Cobb-Douglas and linear additive utilities, cf. [5])

a) Let consumers apply Leontief utility functions

$$u_i(x_i) = \min_{1 \le j \le n} \frac{x_i^{(j)}}{b_i^{(j)}},$$

where $b_i^{(j)}$ *are positive scaling coefficients. This case corresponds to complementary goods. Assuming* $X_i = \mathbb{R}_+$*, we obtain after simple computations:*

$$LR_i(p) = \max_{x_i \ge 0} \left[w_i \ln \left(\min_{1 \le j \le n} \frac{x_i^{(j)}}{b_i^{(j)}} \right) - \langle p, x_i \rangle \right] = -w_i \ln \langle p, b_i \rangle + w_i \left(\ln w_i - 1 \right),$$

and, thus,

$$TLR(p) = -\sum_{i=1}^{I} w_i \ln \langle p, b_i \rangle + \langle p, e \rangle + \sum_{i=1}^{I} w_i \left(\ln w_i - 1 \right).$$

In case of Leontief utilities the total logarithmic revenue turns out to be smooth.

b) Let consumers apply Cobb-Douglas utility functions

$$u_i(x_i) = \alpha \prod_{j=1}^{n} \left(x_i^{(j)} \right)^{\alpha_i^j},$$

where α_i^j *are positive elasticities with* $\sum_{j=1}^{n} \alpha_i^j = 1$*, and* α *is a positive scaling coefficient. This case also corresponds to complementary goods. Assuming* $X_i = \mathbb{R}_+$*, we obtain after simple computations:*

$$\begin{aligned} LR_i(p) &= \max_{x_i \ge 0} \left[w_i \ln \left(\alpha \prod_{j=1}^{n} \left(x_i^{(j)} \right)^{\alpha_i^j} \right) - \langle p, x_i \rangle \right] \\ &= -w_i \ln p^{(j)} + w_i \left(\ln(\alpha w_i) + \sum_{j=1}^{n} \alpha_i^j \ln \alpha_i^j - 1 \right), \end{aligned}$$

and, thus,

$$TLR(p) = -\sum_{i=1}^{I} w_i \ln p^{(j)} + \sum_{i=1}^{I} w_i \left(\ln \alpha w_i + \sum_{j=1}^{n} \alpha_i^j \ln \alpha_i^j - 1 \right).$$

In case of Cobb-Douglas utilities the total logarithmic revenue turns out to be also smooth.

c) Let consumers apply linear additive utility functions

$$u_i(x_i) = \langle a_i, x_i \rangle = \sum_{j=1}^{n} a_i^{(j)} x_i^{(j)},$$

where $a_i^{(j)}$ are positive scaling coefficients. This case corresponds to substitutionary goods. Assuming $X_i = \mathbb{R}_+$, we obtain after simple computations:

$$LR_i(p) = \max_{x_i \geq 0} \left[w_i \ln \langle a_i, x_i \rangle - \langle p, x_i \rangle \right] = -w_i \ln \left(\min_{1 \leq j \leq n} \frac{p^{(j)}}{a_i^{(j)}} \right) + w_i \left(\ln w_i - 1 \right),$$

and, thus,

$$TLR(p) = -\sum_{i=1}^{I} w_i \ln \left(\min_{1 \leq j \leq n} \frac{p^{(j)}}{a_i^{(j)}} \right) + \langle p, e \rangle + \sum_{i=1}^{I} w_i \left(\ln w_i - 1 \right).$$

In case of linear additive utilities the total logarithmic revenue is nonsmooth. Hence, we emphasize that the total logarithmic revenue need not to be smooth. Also note that in all cases a), b) and c) Gale equilibrium coincides with Fisher equilibrium, since Leontief, Cobb-Douglas and linear additive utilities are monotone and homogeneous. □

Note that in [5] the minimization of the total logarithmic revenue has been studied in the smooth setting by assuming Leontief utilities and complementary Constant Elasticity of Substitution (CES) utilities, such as Cobb-Douglas utilities for example. We present a nonsmooth treatment for the case of general concave utility functions. Our goal is to explain how agents can efficiently tackle the nonsmooth convex minimization problem (**P**) by successively updating prices. It is crucial for our approach that the updates of prices correspond to subgradient-type schemes for solving (**P**).

Theorem 2 (Subdifferential of TLR, cf. [5]) *For $p \in \mathbb{R}^n_+$ it holds:*

$$\partial TLR(p) = e - \sum_{i=1}^{I} arg \max_{x_i \in X_i} \left[w_i \ln u_i(x_i) - \langle p, x_i \rangle \right].$$

Proof:
We apply [25, Theorem 23.8] on the subdifferential of the sum of convex functions in order to obtain

$$\partial TLR(p) = e - \sum_{i=1}^{I} \partial LR_i(p).$$

Due to [31, Theorem 2.4.18] on the convex subdifferential of a max-type function, we also have

$$\partial LR_i(p) = -\arg \max_{x_i \in X_i} \left[w_i \ln u_i(x_i) - \langle p, x_i \rangle \right], \quad i = 1, \ldots, I.$$

Overall, the assertion follows. □

Due to Theorem 2, the subgradients of TLR represent the excess supply, i.e.

$$\nabla TLR(p) = e - \sum_{i=1}^{I} x_i \in \partial TLR(p), \tag{21}$$

where $x_i \in \arg\max_{x_i \in X_i} \; w_i \ln u_i(x_i) - \langle p, x_i \rangle$. This gives rise to use the subgradients $\nabla TLR(p)$ for the iterative minimization of TLR. E.g., the change of prices Δp can be taken proportional to the current excess demand:

$$\Delta p \sim -\nabla TLR(p).$$

However, as it can be seen from (21), the subgradients of TLR are known neither to consumers nor to producers. Indeed, $\nabla TLR(p)$ represents the *aggregate* excess supply. For getting access to its value, one would assume the existence of a manager who collects the information about all consumption bundles x_i, producers' fixed supplies e_k and aggregates them over the whole market. Recall that $e = \sum_{k=1}^{K} e_k$. Here, the full information about consumption and production over the market must be available to the manager. Besides, the prices need to be updated by the manager, thus, leading to price regulation. Clearly, these assumptions can be justified only within a centrally planned economy. Aiming to avoid this restriction, we decentralize prices.

The decentralization of prices can be implemented by the introduction of the auction design:

i-th consumer settles and updates his individual prices p_i,
and producers sell at the highest offer price $\max\limits_{i=1,\ldots,I} p_i$.

Note that for vectors $p_1, \ldots, p_I \in \mathbb{R}^n$, we denote by $\max\limits_{i=1,\ldots,I} p_i \in \mathbb{R}^n$ the vector with coordinates

$$\left(\max_{i=1,\ldots,I} p_i \right)^{(j)} = \max_{i=1,\ldots,I} p_i^{(j)}, \quad j = 1, \ldots, n.$$

Now, the total logarithmic revenue depends on the consumers' prices $(p_i)_{i=1}^I$ as follows:

$$TLR(p_1, \ldots, p_I) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{I} LR_i\left(p_i\right) + \left\langle \max_{i=1,\ldots,I} p_i, e \right\rangle =$$

$$\sum_{i=1}^{I} \max_{x_i \in X_i} \; w_i \ln u_i(x_i) - \langle p_i, x_i \rangle + \sum_{k}^{K} \left\langle \max_{i=1,\ldots,I} p_i, e_k \right\rangle. \tag{22}$$

The decentralization of prices makes the corresponding subdifferential information about excess demands available to consumers. In fact, note that the total logarithmic revenue TLR from (22) is convex in the variables $(p_i)_{i=1}^I$. Let us obtain an expression for its convex subgradients $\nabla_{p_i} TLR(p_1, \ldots, p_I)$ w.r.t. p_i:

$$\nabla_{p_i} TLR(p_1, \ldots, p_I) = \sum_{k}^{K} \mu_{ik} \circ e_k - x_i, \quad i = 1, \ldots, I. \tag{23}$$

Here, $x_i \in \arg\max\limits_{x_i \in X_i} \; [w_i \ln u_i(x_i) - \langle p_i, x_i \rangle]$ is the demand of i-th consumer w.r.t. his individual price p_i. Further, $\mu_{ik}^{(j)}$ denotes the share of k-th producer's supply e_k to i-th consumer for good j. Indeed, the shares $\mu_{ik}^{(j)}$ for good j sum up to 1 over all consumers

$i = 1, \dots, I$. Moreover, the share $\mu_{ik}^{(j)}$ vanishes if the i-th consumer's price $p_i^{(j)}$ is less than the highest offer price $\max\limits_{i=1,\dots,I} p_i^{(j)}$ for good j.

Thus, we write

$$(\mu_{ik})_{i=1}^I \in M(p_1, \dots, p_I),$$

where

$$M(p_1, \dots, p_I) \stackrel{\text{def}}{=} \left\{ (\mu_i)_{i=1}^I \in [0,1]^{n \times I} \;\middle|\; \begin{array}{c} \sum\limits_{i=1}^I \mu_i^{(j)} = 1, \\ \mu_i^{(j)} = 0 \text{ if } p_i^{(j)} \neq \max\limits_{i=1,\dots,I} p_i^{(j)} \\ j = 1, \dots, n, i = 1, \dots, I \end{array} \right\}.$$

We claim that the subdifferential information in (23) is known to i-th consumer. First, note that x_i is his consumption bundle. Despite of the fact that the shares μ_{ik} and the supplies e_k cannot be estimated by i-th consumer, their aggregate product $\sum_k^K \mu_{ik} \circ e_k$ is perfectly available to him. Indeed, $\sum_k^K \mu_{ik} \circ e_k$ forms the bundle of goods supplied by all producers to i-th consumer independently from each other. Altogether, the subgradients $\nabla_{p_i} TLR(p_1, \dots, p_I)$ represent the individual excess of i-th consumer's supply over his demands. Overall, we obtain:

Theorem 3 (Producers' excess supply and TLR)

$$\partial_{p_i} TLR(p_1, \dots, p_I) = \sum_k^K \mu_{ik} \circ e_k - arg \max_{x_i \,\in\, X_i} \left[w_i \ln u_i(x_i) - \langle p_i, x_i \rangle \right], \quad i = 1, \dots, I,$$

with demand shares $(\mu_{ik})_{i=1}^I \in M(p_1, \dots, p_I)$.

Due to Theorem 3, the subdifferential of $TLR(p_1, \dots, p_I)$ is completely available to i-th consumer. This fact suggests to adjust prices by solving the minimization problem

$$\min_{p_1, \dots, p_I \in \mathbb{R}^n_+} TLR(p_1, \dots, p_I). \qquad (\mathbf{PD})$$

Note that the minimization problem (**PD**) is stated w.r.t. the decentralized consumers' prices $(p_i)_{i=1}^I$, while previously in (**P**) one minimizes over the common prices p.

We relate the minimization problems (**P**) and (**PD**) by exploiting the fact that they have the same adjoint problem (3):

$$\max_{\substack{x_i \,\in\, X_i \\ i = 1, \dots, I}} \left\{ \Phi(x_1, \dots, x_I) \;\middle|\; \sum_{i=1}^I x_i \leq e \right\}, \qquad (\mathbf{A})$$

where

$$\Phi(x_1, \dots, x_I) \stackrel{\text{def}}{=} \sum_{i=1}^I w_i \ln u_i(x_i). \qquad (24)$$

In (**A**) the central authority assigns consumption bundles by maximizing the logarithmic welfare of the society and by ensuring the market feasibility. In order to state (**A**), the central authority needs to know agents' utility functions, consumption sets, etc. Obviously, this information about the consumers is hardly observable to the central authority. Consequently, it cannot be justified in general that the welfare maximization problem is tackled directly. Nevertheless, note that the prices of goods play the role of Lagrange or dual multipliers for the market feasibility constraint

$$\sum_{i=1}^{I} x_i \leq e.$$

Confer already [13, 26] for similar interpretations.

In order to prove that (**A**) is the adjoint problem not only for (**P**), but also for (**PD**), we need the following simple Lemma 1.

Lemma 1 *For $x_i, e \in \mathbb{R}^n_+$, $i = 1, \dots, I$, the inequality*

$$\sum_{i=1}^{I} x_i \leq e \tag{25}$$

is equivalent to

$$\sum_{i=1}^{I} \langle p_i, x_i \rangle \leq \left\langle \max_{i=1,\dots,I} p_i, e \right\rangle \text{ for all } p_i \in \mathbb{R}^n_+, i = 1, \dots, I. \tag{26}$$

Proof:
(i) Let (25) be satisfied. For $p_i \in \mathbb{R}^n_+$, $i = 1, \dots, I$, we have

$$\sum_{i=1}^{I} \langle p_i, x_i \rangle - \left\langle \max_{i=1,\dots,I} p_i, e \right\rangle = \sum_{j=1}^{n} \left(\sum_{i=1}^{I} p_i^{(j)} x_i^{(j)} - \max_{i=1,\dots,I} p_i^{(j)} e^{(j)} \right).$$

For (26) to hold, it is sufficient to show that

$$\sum_{i=1}^{I} p_i^{(j)} x_i^{(j)} - \max_{i=1,\dots,I} p_i^{(j)} e^{(j)} \leq 0 \text{ for all } j = 1, \dots, n.$$

Indeed, setting for fixed $j \in \{1, \dots, n\}$

$$p^{(j)} = \max_{i=1,\dots,I} p_i^{(j)} \text{ and } \mathcal{I}^{(j)} = \left\{ i \in \{1, \dots, I\} \,\middle|\, p_i^{(j)} = p^{(j)} \right\}, \tag{27}$$

we obtain:

$$\sum_{i=1}^{I} p_i^{(j)} x_i^{(j)} - \max_{i=1,\dots,I} p_i^{(j)} e^{(j)} = \sum_{i \in \mathcal{I}^{(j)}} p^{(j)} x_i^{(j)} + \sum_{i \notin \mathcal{I}^{(j)}} p_i^{(j)} x_i^{(j)} - p^{(j)} e^{(j)} =$$

$$= \sum_{i \in \mathcal{I}^{(j)}} p^{(j)} x_i^{(j)} + \sum_{i \notin \mathcal{I}^{(j)}} p_i^{(j)} x_i^{(j)} - p^{(j)} e^{(j)} + \sum_{i \notin \mathcal{I}^{(j)}} p^{(j)} x_i^{(j)} - \sum_{i \notin \mathcal{I}^{(j)}} p^{(j)} x_i^{(j)}$$

$$= p^{(j)} \left(\sum_{i=1}^{I} x_i^{(j)} - e^{(j)} \right) + \sum_{i \notin \mathcal{I}^{(j)}} \left(p_i^{(j)} - p^{(j)} \right) x_i^{(j)}.$$

The last expression is nonpositive due to (25), (27), and $p^{(j)}, x_i^{(j)} \in \mathbb{R}_+$, $i = 1, \ldots, I$.
(ii) Let (26) be satisfied. Setting there $p_i = p \in \mathbb{R}^n_+$, we get

$$\left\langle p, \sum_{i=1}^{I} x_i \right\rangle \leq \langle p, e \rangle \quad \text{for all } p \in \mathbb{R}^n_+.$$

Hence, (25) is fulfilled. $\square$

Theorem 4 *It holds:*

$$\min_{p \in \mathbb{R}^n_+} TLR(p) = \min_{p_1, \ldots, p_I \in \mathbb{R}^n_+} TLR(p_1, \ldots, p_I)$$

$$= \max_{\substack{x_i \in X_i \\ i = 1, \ldots, I}} \left\{ \Phi\left(x_1, \ldots, x_I\right) \;\middle|\; \sum_{i=1}^{I} x_i \leq e \right\}.$$

Proof:

$$TLR(p_1, \ldots, p_I) = \max_{\substack{x_i \in X_i \\ i = 1, \ldots, I}} \left[\Phi\left(x_1, \ldots, x_I\right) - \sum_{i=1}^{I} \langle p_i, x_i \rangle + \left\langle \max_{i=1,\ldots,I} p_i, e \right\rangle \right]. \tag{28}$$

Using this representation (28) of $TLR(p_1, \ldots, p_I)$, we obtain:

$$\min_{p_1, \ldots, p_I \in \mathbb{R}^n_+} TLR(p_1, \ldots, p_I) =$$

$$= \min_{p_1, \ldots, p_I \in \mathbb{R}^n_+} \max_{\substack{x_i \in X_i \\ i = 1, \ldots, I}} \left[\Phi\left(x_1, \ldots, x_I\right) - \sum_{i=1}^{I} \langle p_i, x_i \rangle + \left\langle \max_{i=1,\ldots,I} p_i, e \right\rangle \right]$$

$$= \max_{\substack{x_i \in X_i \\ i = 1, \ldots, I}} \Phi\left(x_1, \ldots, x_I\right) + \min_{p_1, \ldots, p_I \in \mathbb{R}^n_+} - \sum_{i=1}^{I} \langle p_i, x_i \rangle + \left\langle \max_{i=1,\ldots,I} p_i, e \right\rangle \tag{29}$$

$$= \max_{\substack{x_i \in X_i \\ i = 1, \ldots, I}} \left\{ \Phi\left(x_1, \ldots, x_I\right) \;\middle|\; \begin{array}{l} \sum_{i=1}^{I} \langle p_i, x_i \rangle \leq \left\langle \max_{i=1,\ldots,I} p_i, e \right\rangle \\ \text{for all } p_i \in \mathbb{R}^n_+, i = 1, \ldots, I \end{array} \right\}.$$

Applying Lemma 1, the adjoint constraint $\sum_{i=1}^{I} x_i \leq e$ is equivalent to

$$\sum_{i=1}^{I} \langle p_i, x_i \rangle \leq \left\langle \max_{i=1,\ldots,I} p_i, e \right\rangle \quad \text{for all } p_i \in \mathbb{R}^n_+, i = 1, \ldots, I.$$

Overall, (**A**) is the adjoint problem for (**PD**). Analogously, (**A**) is the adjoint problem for (**P**). □

Corollary 1 *Let* $(p_i)_{i=1}^I$ *solve* (**PD**) *and* $(x_i)_{i=1}^I$ *solve its adjoint problem* (**A**). *Then, the highest offer prices together with consumption bundles*

$$\left(\max_{i=1,\dots,I} p_i, (x_i)_{i=1}^I\right)$$

form a Gale equilibrium. Moreover, the i-th consumer's bundle $x_i^{(j)}$ *vanishes if his individual price* $p_i^{(j)}$ *is less than the highest offer price* $\max\limits_{i=1,\dots,I} p_i^{(j)}$ *for good* j, *i.e.*

$$x_i^{(j)} = 0 \text{ if } p_i^{(j)} \neq \max_{i=1,\dots,I} p_i^{(j)}, \quad i = 1,\dots,I, j = 1,\dots,n.$$

Proof:

Due to Theorem 4:

$$0 \leq TLR\left(\max_{i=1,\dots,I} p_i\right) - \Phi(x_1,\dots,x_I) \overset{(22)}{\leq} TLR(p_1,\dots,p_I) - \Phi(x_1,\dots,x_I) = 0.$$

Hence, $\max\limits_{i=1,\dots,I} p_i$ solves (**P**). Due to the fact that (**A**) is the adjoint problem also for (**P**),

$$\left(\max_{i=1,\dots,I} p_i, (x_i)_{i=1}^I\right)$$

is a Gale equilibrium according to Definition 2.

Further, (29) from Theorem 4 yields

$$-\sum_{i=1}^I \langle p_i, x_i\rangle + \left\langle \max_{i=1,\dots,I} p_i, e\right\rangle = -\sum_{i=1}^I \left\langle \max_{i=1,\dots,I} p_i, x_i\right\rangle - \left\langle \max_{i=1,\dots,I} p_i, e\right\rangle = 0.$$

Thus,

$$\sum_{i=1}^I \left\langle \max_{i=1,\dots,I} p_i - p_i, x_i\right\rangle = 0,$$

or, equivalently,

$$\left\langle \max_{i=1,\dots,I} p_i - p_i^{(j)}, x_i^{(j)}\right\rangle = 0, \quad i = 1,\dots,I, \quad j = 1,\dots,n.$$

The latter implies: $x_i^{(j)} = 0$ if $p_i^j \neq \max\limits_{i=1,\dots,I} p_i^{(j)}$. □

4 Algorithm for Fisher-Gale equilibrium

We describe how consumers may efficiently adjust their individual prices $(p_i)_{i=1}^I$ to arrive at a Fisher-Gale equilibrium. This price adjustment corresponds to the quasi-monotone subgradient method (SM) [23], which is described in Appendix for reader's convenience. It is applied to the minimization of the total logarithmic revenue (**PD**):

$$\min_{p_1,\ldots,p_I \in \mathbb{R}^n_+} TLR(p_1,\ldots,p_I).$$

Let i-th consumer choose a sequence of positive confidence parameters $\{\chi_i[t]\}_{t\geq 0}$, $i = 1,\ldots,I$. We consider the following iteration:

Algorithm for Fisher-Gale equilibrium **(AFG)**

1. Consumers determine their current excess supplies $\nabla_{p_i} TLR(p_1[t],\ldots,p_I[t])$:

a) i-th consumer computes an optimal bundle

$$x_i(p_i[t]) \in \arg\max_{x_i\, \in\, X_i} \left[w_i \ln u_i(x_i) - \langle p_i[t], x_i\rangle\right],$$

and the corresponding utility prices

$$q_i(p_i[t]) = \frac{w_i}{u_i\left(x_i(p_i[t])\right)}, \quad i = 1,\ldots,I.$$

b) k-th producers identifies the highest offer prices

$$p[t] = \max_{i=1,\ldots,I} p_i[t],$$

decides on supply shares

$$(\mu_{ik}[t])_{i=1}^I \in M\left(p_1[t],\ldots,p_I[t]\right),$$

and supplies to the i-th consumer the bundle

$$\mu_{ik}[t] \circ e_k, \quad i = 1,\ldots,I.$$

c) i-th consumer computes his current excess supply

$$\nabla_{p_i} TLR(p_1[t],\ldots,p_I[t]) = \sum_k^K \mu_{ik}[t] \circ e_k - x_i(p_i[t]). \tag{30}$$

2. Consumers accumulate their excess supplies

$$z_i[t] = z_i[t-1] + \nabla_{p_i} TLR(p_1[t],\ldots,p_I[t]), \quad z_i[-1] = 0, \quad i = 1,\ldots,I. \tag{31}$$

3. Consumers compute their price forecasts w.r.t. the confidence parameters $\chi_i[t]$

$$p_i^+[t] = \frac{\zeta_i^{(j)}}{\chi_i[t]}\left(-z_i[t]\right)_+, \quad i = 1,\ldots,I. \tag{32}$$

where $\zeta_i^{(j)}$ are positive scaling coefficients.

4. Consumer update

$$p_i[t+1] = \frac{t+1}{t+2} p_i[t] + \frac{1}{t+2} p_i^+[t], \quad i = 1, \ldots, I \tag{33}$$

by combining their previous prices with the forecasts. □

First, we give an interpretation for the price forecast (32). Recall that $z_i[t]$ represents the excess of producers' supply to i-th consumer over his demands for good j accumulated up to time t. If $z_i^{(j)}[t] \geq 0$, i.e. supply exceeds demand, then naturally, the long-term forecast is $p_i^{+(j)}[t] = 0$ for good j. In case of $z_i^{(j)}[t] < 0$, the price forecast $p_i^{+(j)}[t]$ is proportional to the accumulated individual excess demand of i-th consumer with positive scaling coefficients $\zeta_i^{(j)}$. Here, $\chi_i[t]$ plays the role of a confidence parameter. Namely, $\chi_i[t]$'s express to which extent consumers take into account their excess demands while forecasting prices.

Secondly, let us interpret the price update (33):

$$p_i[t+1] = \frac{t+1}{t+2} p_i[t] + \frac{1}{t+2} p_i^+[t].$$

Due to the latter, the next price is a convex combination of the previous price and the price forecast. With time advancing, the proportion of the previous price becomes nearly one, but the fraction of the forecast vanishes. Hence, we conclude that our price update corresponds to a behavior of an experienced consumer. He credits his experience much more than the current forecast. Further, from (33) we have

$$p_i[t+1] = \frac{1}{t+2} \left(p_i[0] + \sum_{r=0}^{t} p_i^+[r] \right). \tag{34}$$

The latter means that the prices generated by (**AFG**) can be viewed as historical averages of preceding forecasts. This averaging pattern is also quite natural to assume for consumer's behavior while adjusting prices.

Next, we produce a feasible sequence for the adjoint problem (**A**) by averaging consumption bundles from (**AFG**). Along with the prices $\{(p_1[t], \ldots, p_I[t])\}_{t \geq 0}$ generated by algorithm (**AFG**), we consider the corresponding historical averages of consumption bundles

$$x_i[t] \stackrel{\mathrm{def}}{=} \frac{1}{t+1} \sum_{r=0}^{t} x_i(p[r]) \in X_i, \quad i = 1, \ldots, I,$$

as well as the corresponding geometric means of utility prices

$$q_i[t] \stackrel{\mathrm{def}}{=} \left(\prod_{r=0}^{t} q_i(p_i[r]) \right)^{\frac{1}{t+1}} = \left(\prod_{r=0}^{t} \frac{w_i}{u_i\left(x_i(p[r])\right)} \right)^{\frac{1}{t+1}}, \quad i = 1, \ldots, I.$$

Next Lemma 2 estimates the dual gap for the minimization problem (**PD**) and its adjoint problem (**A**) evaluated at the historical averages.

For that, we set

$$\begin{array}{lcl} TLR[t] & \stackrel{\text{def}}{=} & TLR(p_1[t],\dots,p_I[t]), \\ \Phi[t] & \stackrel{\text{def}}{=} & \Phi\left(x_1[t],\dots,x_I[t]\right), \\ \Phi_{av}[t] & \stackrel{\text{def}}{=} & \dfrac{1}{t+1}\displaystyle\sum_{r=0}^{t}\Phi\left(x_1(p[r]),\dots,x_I(p[r])\right), \\ F[t] & \stackrel{\text{def}}{=} & \displaystyle\sum_{j=1}^{n}\left(\sum_{i=1}^{I}x_i[t]^{(j)}-e\right)_+^2. \end{array}$$

$TLR[t]$ is the value of the primal problem (**PD**), which is computed at the current prices $(p_1[t],\dots,p_I[t])$. $\Phi[t]$ is the value of the adjoint problem (**A**), which is computed at historical averages $(x_1[t],\dots,x_I[t])$. $\Phi_{av}[t]$ the average value of the adjoint problem (**A**), which is computed at current consumption bundles $(x_1(p[r]),\dots,x_I(p[r]))$. Note that due to the concavity of Φ:

$$\Phi_{av}[t]\le\Phi[t].$$

$F[t]$ is the quadratic penalty for violation of the market feasibility constraint:

$$\sum_{i=1}^{I}x_i[t]\le e.$$

Further, we define the upper and lower remainder terms b_t and d_t:

$$b_t \stackrel{\text{def}}{=} \frac{1}{t+1}\sum_{i=1}^{I}\sum_{r=0}^{t}\frac{1}{\chi_i[r-1]},\quad \chi_i[-1]=\chi_i[0],$$

$$d_t \stackrel{\text{def}}{=} \frac{\sum_{i=1}^{I}\chi_i[t]}{t+1}.$$

Lemma 2 *Let the sequence* $\{p_1[t],\dots,p_I[t]\}_{t\ge0}$ *be generated by (***AFG***) with nondecreasing confidence parameters*

$$\chi_i[t+1]\ge\chi_i[t],\quad t\ge0, i=1,\dots,I.$$

Then, for all $t\ge0$ *it holds:*

$$TLR[t]-TLR^*-C_1d_t\le TLR[t]-\Phi[t]+\frac{C_2}{d_t}F[t]\le TLR[t]-\Phi_{av}[t]+\frac{C_2}{d_t}F[t]\le C_3b_t \quad (35)$$

with some positive constants $C_1,C_2,C_3>0$.

Proof:
The proof of Lemma 2 is based on the application of the quasi-monotone subgradient method for nonsmooth convex minimization from [23]. Its proof is postponed to Appendix for reader's convenience. □

In order to arrive at the equilibrium price, consumers need to appropriately adjust their confidence parameters $\{\chi_i[t]\}_{t\geq 0}$, $i = 1,\ldots,I$. Next Lemma 3 identifies successful adjustment strategies of confidence parameters. Namely, the confidence in the market mechanism increases, but by decreasing increments. This ensures the convergence of the remainder terms b_t, d_t from Lemma 2.

Lemma 3 *Let nondecreasing confidence parameters of the i-th consumer satisfy*

$$\chi_i[t] - \chi_i[t-1] \to 0, \quad \chi_i[t] \to \infty. \tag{36}$$

Then,

$$\frac{\chi_i[t]}{t+1} \to 0, \quad \text{and} \quad \frac{1}{t+1}\sum_{r=0}^{t}\frac{1}{\chi_i[r-1]} \to 0. \tag{37}$$

Moreover, the achievable order of convergence in (37) is $O\left(\frac{1}{\sqrt{t}}\right)$.

Proof:

Since $\chi_i[t]-\chi_i[t-1] \to 0$, it holds by averaging that $\frac{1}{t+1}\sum\limits_{r=0}^{t}\chi_i[r]-\chi_i[r-1] \to 0$. Thus,

$$\frac{1}{t+1}\chi_i[t] = \frac{1}{t+1}\sum_{r=0}^{t}\chi_i[r]-\chi_i[r-1] + \frac{1}{t+1}\chi_i[-1] \to 0.$$

From $\chi_i[t] \to \infty$ we have $\frac{1}{\chi_i[t]} \to 0$, and also by averaging, $\frac{1}{t+1}\sum\limits_{r=0}^{t}\frac{1}{\chi_i[r-1]} \to 0$.

The convergence of the order $O\left(\frac{1}{\sqrt{t}}\right)$ can be achieved in (37) by choosing $\chi_i[t] = O(\sqrt{t})$. In fact, we obtain:

$$\frac{1}{t+1}\sum_{r=0}^{t}\frac{1}{\chi_i[r-1]} = \frac{1}{t+1}\left(\frac{1}{\chi_i[-1]}+\frac{1}{\chi_i[0]}\right) + \frac{1}{t+1}\sum_{r=1}^{t}\frac{1}{\sqrt{r}}.$$

Immediately, we see that $\frac{1}{t+1}\left(\frac{1}{\chi_i[-1]}+\frac{1}{\chi_i[0]}\right) \to 0$ as of the order $O\left(\frac{1}{t}\right)$. Note that for a convex univariate function $\xi(r)$, $r \in \mathbb{R}$, and integer bounds a, b, we have

$$\sum_{r=a}^{b}\xi(r) \leq \int\limits_{a-1/2}^{b+1/2}\xi(s)\mathrm{d}s. \tag{38}$$

Hence, we get

$$\frac{1}{t+1}\sum_{r=1}^{t}\frac{1}{\sqrt{r}} \overset{(38)}{\leq} \frac{1}{t+1}\int\limits_{1-1/2}^{t+1/2}\frac{1}{\sqrt{s}}\mathrm{d}s = \frac{2}{t+1}\sqrt{s}\,\Big|_{1/2}^{t+1/2} = \frac{2}{t+1}\left(\sqrt{t+1/2}-\sqrt{1/2}\right) \to 0.$$

Here, the order of convergence is $O\left(\frac{1}{\sqrt{t}}\right)$. By assuming $\chi_i[t] = O(\sqrt{t})$, the convergence $\frac{\chi_i[t]}{t+1} = \frac{\sqrt{t}}{t+1} \to 0$ is also of the order $O\left(\frac{1}{\sqrt{t}}\right)$. □

Remark 3 *As in the proof of Lemma 3, nondecreasing confidence parameters can be written in the cumulative form:*

$$\chi_i[t] = \sum_{r=0}^{t} h_i[r] + \chi_i[-1]$$

with incremental confidences $h_i[t] \geq 0$. Then, the convergence condition (36) means that incremental confidences tend to zero and sum up to infinity, i.e.

$$h_i[t] \to 0, \quad \sum_{t=0}^{\infty} h_i[t] = \infty.$$

The latter coincides with the usual condition imposed on the step-sizes of the subgradient method for nonsmooth convex minimization (e.g., [22]). However, in our setting $h_i[t]$ play the role of incremental *step-sizes. This gives rise to suppose that confidence parameters $\chi_i[t]$ can be formed by consumers by incremental learning (cf. [28]). In fact, the i-th consumer's confidence in the price adjustment process, $\chi_i[t]$, increases over time, however, by decreasing increments $h_i[t]$. The latter means that consumers properly slow down the pace of their confidence in the market mechanism.* □

Now, we are ready to prove the main convergence result for (**AFG**).

Theorem 5 *Let consumers apply in (**AFG**) confidence parameters satisfying*

$$\chi_i[t] - \chi_i[t-1] \to 0, \quad \chi_i[t] \to \infty, \quad i = 1, \ldots, I.$$

Then, the sequence of highest offer prices and historical averages of consumption bundles

$$\left(\max_{i=1,\ldots,I} p_i[t], (x_i[t])_{i=1}^{I} \right)$$

*from algorithm (**AFG**), converges to the set of Fisher-Gale equilibria w.r.t. the utility prices*

$$\left(\lim_{t\to\infty} q_i[t] \right)_{i=1}^{I}.$$

The achievable rate of convergence is of the order $O\left(\frac{1}{\sqrt{t}}\right)$.

Proof:
From Lemma 2 we obtain:

$$TLR[t] - TLR^* - C_1 d_t \leq TLR[t] - \Phi[t] + \frac{C_2}{d_t} F[t] \leq C_3 b_t. \tag{39}$$

This inequality is composed by the objective function TLR of the primal problem (**PD**), computed at the current prices $(p_1[t], \ldots, p_I[t])$, objective function Φ of its adjoint problem

($\mathbf{A}$), computed at historical averages $(x_1[t], \ldots, x_I[t])$, and the quadratic penalty $F[t]$ for violation of the market feasibility constraint:

$$\sum_{i=1}^{I} x_i[t] \leq e.$$

Due to the choice of confidence parameters $\chi_i[t]$, $i = 1, \ldots, I$, Lemma 3 provides:

$$b_t \to 0, \text{ and } d_t \to 0.$$

Using Theorem 4, $(p_i[t])_{i=1}^I$ converges toward the solution set of ($\mathbf{PD}$), and $(x_i[t])_{i=1}^I$ converges toward the solution set of ($\mathbf{A}$) by order $O\left(\frac{1}{\sqrt{t}}\right)$. We apply Corollary 1 to conclude that the sequence of highest offer prices together with historical averages of consumption bundles

$$\left(\max_{i=1,\ldots,I} p_i, (x_i[t])_{i=1}^I\right)$$

converges to the set of Gale equilibria (cf. Definition 2). In order to get the additional convergence to the set of Fisher-Gale equilibria, we apply Theorem 1 (b). For that, it is enough to show that the sequence of geometric means of utility prices

$$q_i[t] \stackrel{\text{def}}{=} \left(\prod_{r=0}^{t} q_i(p_i[r])\right)^{\frac{1}{t+1}} = \left(\prod_{r=0}^{t} \frac{w_i}{u_i\left(x_i(p[r])\right)}\right)^{\frac{1}{t+1}},$$

and the sequence of utility prices corresponding to the average consumption

$$\frac{w_i}{u_i\left(x_i[t]\right)}, \quad i = 1, \ldots, I,$$

have the same limit. From Lemma 2 we know that $\Phi_{av}[t]$ and $\Phi[t]$ have the same limit. Recalling the definitions of $\Phi_{av}[t]$ and $\Phi[t]$, see also (24), the sequences

$$\frac{1}{t+1}\sum_{r=0}^{t} \ln u_i\left(x_i(p[r])\right) \text{ and } \ln u_i\left(x_i[t]\right), \quad i = 1, \ldots, I,$$

have the same limit. Applying exponential and inversion to the latter, the assertion follows. □

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Appendix

Appendix is devoted to the proof of Lemma 2. For that, we first present the quasi-monotone subgradient method for nonsmooth convex minimization from [23]. As already mentioned, the price adjustment (**AFG**) corresponds to this quasi-monotone subgradient method. Using this fact, we prove Lemma 2 in the second part of Apppendix.

Quasi-monotone subgradient methods

We consider the following minimization problem:

$$\min_{x \in X} f(x), \tag{40}$$

where $X \subset \mathbb{R}^n$ is a closed convex set with nonempty interior $\operatorname{int} X$, and f is a convex function on $\mathbb{R}^n$. Moreover, let f be representable as a maximum of concave functions, i.e.

$$f(x) = \max_{a \in A} \Phi(a) + \varphi(x, a), \tag{41}$$

where $A \subset \mathbb{R}^m$ is a closed convex set, $\varphi(\cdot, a)$ is a convex function on $\mathbb{R}^n$ for every $a \in A$, and Φ, $\varphi(x, \cdot)$ are concave functions on $\mathbb{R}^m$ for every $x \in X$. Denote by $a(x)$ one of the optimal solutions of the maximization problem in (41). Then,

$$\nabla f(x) \stackrel{\text{def}}{=} \nabla_x \varphi(x, a(x)) \tag{42}$$

is a subgradient of f at x. Recall that for an arbitrary subgradient $\nabla f(x)$ at $x \in X$ of a convex function f we have:

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle, \quad y \in X. \tag{43}$$

Using the representation (41), we also have:

$$\min_{x \in X} f(x) = \min_{x \in X} \max_{a \in A} \left[\Phi(a) + \varphi(x, a)\right] = \max_{a \in A} \left[\Phi(a) + \min_{x \in X} \varphi(x, a)\right].$$

The latter maximization problem

$$\max_{a \in A} \left[\Phi(a) + \min_{x \in X} \varphi(x, a)\right] \tag{44}$$

is called adjoint for (40) with the adjoint variable $a \in A$.

For the set X, we assume to be known a prox-function $d(x)$.

Definition 4 $d : X \mapsto \mathbb{R}$ *is called a prox-function for* X *if the following holds:*

- $d(x) \geq 0$ *for all* $x \in X$ *and* $d(x[0]) = 0$ *for certain* $x[0] \in X$*;*
- d *is strongly convex on* X *with convexity parameter one:*

$$d(y) \geq d(x) + \langle \nabla d(x), y - x \rangle + \frac{1}{2}\|y - x\|^2, \quad x, y \in X, \tag{45}$$

  where $\|\cdot\|$ *is a norm on* $\mathbb{R}^n$*.*

- *Auxiliary minimization problem*

$$\min_{x \in X} \{\langle z, x \rangle + \chi d(x)\} \tag{46}$$

  is easily solvable for $z \in \mathbb{R}^n, \chi > 0$*.*

As a simple consequence of Definition 4, we have for $x \in X$:

$$d(x) \geq d(x[0]) + \langle \nabla d(x[0]), x - x[0] \rangle + \frac{1}{2}\|x - x[0]\|^2 \geq \frac{1}{2}\|x - x[0]\|^2. \tag{47}$$

For a sequence of positive parameters $\{\chi[t]\}_{t \geq 0}$, we consider the following iteration:

Quasi-monotone Subgradient Method

$$\begin{array}{l}
\textbf{1. } \text{Take a current subgradient } \nabla f(x[t]) = \nabla_x \varphi(x[t], a(x[t])). \\
\textbf{2. } \text{Accumulate subgradients } z[t] = z[t-1] + \nabla f(x[t]),\ z[-1] = 0. \\
\textbf{3. } \text{Compute the forecast } x^+[t] = \arg\min_{x\in X} \left\{\langle z[t], x\rangle + \chi[t] d(x)\right\}. \\
\textbf{4. } \text{Update by combining } x[t+1] = \dfrac{t+1}{t+2} x[t] + \dfrac{1}{t+2} x^+[t].
\end{array} \tag{SM}$$

Note that from (SM) we have

$$z[t] = \sum_{r=0}^{t} \nabla f(x[r]), \quad x[t+1] = \frac{1}{t+2}\left(x[0] + \sum_{r=0}^{t} x^+[r]\right).$$

Next Theorem 6 is crucial for the convergence analysis of the quasi-monotone subgradient method (SM). It estimates the dual gap for the minimization problem (40) and its adjoint problem (44) evaluated at the historical averages.

For that, we define the penalty term h_t and the remainder term ρ_t, $t \geq 0$, as follows:

$$\begin{aligned}
h_t(a) &\stackrel{\text{def}}{=} \min_{x\in X}\left\{\varphi(x,a) + \frac{\chi[t]}{t+1} d(x)\right\}, \quad a \in A, \\
\rho_t &\stackrel{\text{def}}{=} \frac{1}{t+1}\sum_{r=0}^{t} \frac{1}{2\chi[r-1]} \left\|\nabla f(x[r])\right\|_*^2, \quad \chi[-1] = \chi[0].
\end{aligned}$$

Here, $\|\cdot\|_*$ is the conjugate norm to $\|\cdot\|$, i.e.

$$\|s\|_* \stackrel{\text{def}}{=} \max_{s\in\mathbb{R}^n} \left\{\langle s, x\rangle : \|x\| \leq 1\right\}, \quad s \in \mathbb{R}^n. \tag{48}$$

Note that $\Phi + h_t$ is a smoothed version of the objective function in (44).

Further, we define the average adjoint state

$$a[t] \stackrel{\text{def}}{=} \frac{1}{t+1}\sum_{r=0}^{t} a(x[r]), \quad t \geq 0.$$

Note that $a[t] \in A$, since A is convex. Let us write

$$\Phi_{av}[t] \stackrel{\text{def}}{=} \frac{1}{t+1}\sum_{r=0}^{t} \Phi(a(x[r])), \quad t \geq 0,$$

for the average value of the adjoint problem computed along (SM).

Theorem 6 is motivated by the estimate sequence technique (e.g., Section 2.2.1 in [22]) and is due to [23]. We decided to present its proof for readers' convenience.

Theorem 6 (cf. [23]) *Let the sequence $\{x[t]\}_{t\geq 0}$ be generated by (SM) with nondecreasing parameters*

$$\chi[t+1] \geq \chi[t], \quad t \geq 0. \tag{49}$$

Then, for all $t \geq 0$ it holds:

$$f(x[t]) - \Phi(a[t]) - h_t(a[t]) \leq f(x[t]) - \Phi_{av}[t] - h_t(a[t]) \leq \rho_t. \tag{50}$$

Proof:
We define the average linearization terms ℓ_t and ψ_t for f:

$$\begin{aligned}
\ell_t(x) &\overset{\text{def}}{=} \sum_{r=0}^{t} f(x[r]) + \langle \nabla f(x[r]), x - x[r] \rangle, \\
\psi_t &\overset{\text{def}}{=} \min_{x\in X} \{\ell_t(x) + \chi[t] d(x)\}.
\end{aligned}$$

First, we show by induction that for all $t \geq 0$ it holds:

$$f(x[t]) - \frac{\psi_t}{t+1} \leq \rho_t. \tag{51}$$

Let us assume that condition (51) is valid for some $t \geq 0$. Then,

$$\begin{aligned}
\psi_{t+1} &= \min_{x\in X} \{\ell_t(x) + f(x_{t+1}) + \langle \nabla f(x[t+1]), x - x[t+1] \rangle + \chi[t+1] d(x)\} \\
&\overset{(49)}{\geq} \min_{x\in X} \{\ell_t(x) + \chi[t] d(x) + f(x[t+1]) + \langle \nabla f(x[t+1]), x - x[t+1] \rangle\} \\
&\overset{(45)}{\geq} \min_{x\in X} \left\{ \psi_t + \frac{1}{2}\chi[t] \left\| x - x^+[t] \right\|^2 + f(x[t+1]) + \langle \nabla f(x[t+1]), x - x[t+1] \rangle \right\} \\
&\overset{(51)}{\geq} \min_{x\in X} \left\{ \begin{array}{l} (t+1) f(x[t]) - (t+1)\rho_t \\ + \frac{1}{2}\chi[t] \|x - x^+[t]\|^2 + f(x[t+1]) + \langle \nabla f(x[t+1]), x - x[t+1] \rangle \end{array} \right\} \\
&\overset{(43)}{\geq} \min_{x\in X} \left\{ \begin{array}{l} (t+1) \left[f(x[t+1]) + \langle \nabla f(x[t+1]), x[t] - x[t+1] \rangle \right] - (t+1)\rho_t \\ + \frac{1}{2}\chi[t] \|x - x^+[t]\|^2 + f(x[t+1]) + \langle \nabla f(x[t+1]), x - x[t+1] \rangle \end{array} \right\}.
\end{aligned}$$

Since $(t+2)x[t+1] = (t+1)x[t] + x^+[t]$, we obtain

$$\begin{aligned}
\psi_{t+1} &\geq (t+2) f(x[t+1]) - (t+1)\rho_t \\
&+ \min_{x\in X} \left\{ \left\langle \nabla f(x[t+1]), x - x^+[t] \right\rangle + \frac{1}{2}\chi[t] \left\| x - x^+[t] \right\|^2 \right\} \\
&\geq (t+2) f(x[t+1]) - (t+1)\rho_t - \frac{1}{2\chi[t]} \|\nabla f(x[t+1])\|_*^2. \\
&= (t+2) f(x[t+1]) - (t+2)\rho_{t+1}.
\end{aligned}$$

It remains to note that

$$\psi_0 = \min_{x\in X} \{f(x[0]) + \langle \nabla f(x[0]), x - x[0] \rangle + \chi[0] d(x)\} \overset{(47)}{\geq} f(x[0]) - \rho_0.$$

Now, we relate the term $\dfrac{\psi_t}{t+1}$ from (51) to the adjoint problem (44). It holds due to convexity of $\varphi(\cdot, a)$, $a \in A$:

$$
\begin{aligned}
& f(x[r]) + \langle \nabla f(x[r]), x - x[r] \rangle = \\
\overset{(41),(42)}{=} \; & \Phi\left(a(x[r])\right) + \varphi\left(x[r], a(x[r])\right) + \left\langle \nabla_x \varphi\left(x[r], a(x[r])\right), x - x[r] \right\rangle \\
& \leq \Phi\left(a(x[r]) + \varphi\left(x, a(x[r])\right)\right).
\end{aligned}
$$

Hence, we obtain due to concavity of $\varphi(x, \cdot)$, $x \in X$:

$$
\ell_t(x) \leq \sum_{r=0}^{t} \Phi\left(a(x[r]) + \varphi\left(x, a(x[r])\right)\right) \leq (t+1)\left[\Phi_{av}[t] + \varphi\left(x, a[t]\right)\right].
$$

Finally, we get

$$
\frac{\psi_t}{t+1} \leq \Phi_{av}[t] + \min_{x \in X} \left\{ \varphi\left(x, a[t]\right) + \frac{\chi[t]}{t+1} d(x) \right\} = \Phi_{av}[t] + h_t(a[t]). \tag{52}
$$

Altogether, (51) and (52) provide the right-hand side of the formula (50). The left-hand side is due to

$$
\Phi_{av}[t] = \frac{1}{t+1} \sum_{r=0}^{t} \Phi(a(x[r])) \leq \Phi\left(\frac{1}{t+1}\sum_{r=0}^{t} a(x[r])\right) = \Phi(a[t]),
$$

which is a consequence of the concavity of Φ. □

Additionally, we need the following result on the quadratic penalty for general convex optimization problems. From now on, let us consider the maximization problem

$$
\Phi^* \overset{\text{def}}{=} \max_{a \in A} \left\{ \Phi(a) \mid g_l(a) \leq 0, l = 1, \ldots, L \right\}, \tag{53}
$$

where $A \subset \mathbb{R}^m$ is a closed convex set, Φ is a concave function, and $g_l(\cdot)$, $l = 1, \ldots, L$ are convex functions on $\mathbb{R}^m$. We assume that the convex feasible set of the maximization problem (53) has a Slater point (e.g., [25]). Let a^* be an optimal solution of (53) with some Lagrange multipliers λ_l^*, $l = 1, \ldots, L$, i.e.

$$
\left\langle \nabla \Phi(a^*) - \sum_{l=1}^{L} \lambda_l^* \nabla g_l(a^*), a^* - a \right\rangle \geq 0, \quad \text{for all } a \in A, \tag{54}
$$

$$
\lambda_l^* \geq 0, \quad g_l(a^*) \leq 0, \quad \sum_{l=1}^{L} \lambda_l^* g_l(a^*) = 0. \tag{55}
$$

Lemma 4 *It holds for $\kappa > 0$:*

$$
\max_{a \in A} \left[\Phi(a) - \frac{\kappa}{2} \sum_{l=1}^{L} \left(g_l(a)\right)_+^2 \right] \leq \Phi^* + \frac{1}{2\kappa} \sum_{l=1}^{L} \lambda_l^*.
$$

Proof:
Due to the concavity of Φ and the convexity of g_l, $l = 1, \ldots, L$, it holds for all $a \in A$:

$$\Phi(a) \leq \Phi(a^*) + \langle \nabla \Phi(a^*), a - a^* \rangle \,, \tag{56}$$

$$g_l(a) \geq g_l(a^*) + \langle \nabla g_l(a^*), a - a^* \rangle \,. \tag{57}$$

We estimate

$$\begin{aligned} \Phi(a) &\overset{(56)}{\leq} \Phi(a^*) + \langle \nabla \Phi(a^*), a - a^* \rangle \overset{(54)}{\leq} \Phi^* + \sum_{l=1}^{L} \lambda_l^* \langle \nabla g_l(a^*), a - a^* \rangle \\ &\overset{(57)}{\leq} \Phi^* + \sum_{l=1}^{L} \lambda_l^* \left(g_l(a) - g_l(a^*) \right) \overset{(55)}{=} \Phi^* + \sum_{l=1}^{L} \lambda_l^* g_l(a), \quad a \in A. \end{aligned}$$

Hence,

$$\begin{aligned} \max_{a \in A} \left[\Phi(a) - \frac{\kappa}{2} \sum_{l=1}^{L} \left(g_l(a)\right)_+^2 \right] &\leq \Phi^* + \max_{a \in A} \sum_{l=1}^{L} \left[\lambda_l^* g_l(a) - \frac{\kappa}{2} \left(g_l(a)\right)_+^2 \right] \\ &\leq \Phi^* + \sum_{l=1}^{L} \max_{b_l \in \mathbb{R}} \sum_{l=1}^{L} \left[\lambda_l^* b_l - \frac{\kappa}{2} \left(b_l\right)_+^2 \right] \\ &= \Phi^* + \sum_{l=1}^{L} \frac{1}{2\kappa} \lambda_l^*. \end{aligned}$$

$\square$

Proof of Lemma 2

We start by proving that the price adjustment (**AFG**) is a variant of the quasi-monotone subgradient method (SM). For that, it suffices to show that

1) the price forecast (32) can be derived by means of Euclidean prox-functions,
2) TLR can be represented as the maximum of concave functions.

Firstly, we define the Euclidean prox-functions:

$$d_i(p) \overset{\text{def}}{=} \frac{1}{2} \sum_{j=1}^{n} \frac{1}{\zeta_i^{(j)}} \left(p^{(j)} \right)^2, \quad i = 1, \ldots, I,$$

where $\zeta_i^{(j)}$ are positive scaling coefficients. The corresponding norms in Definition 4 and their conjugates according to (48) are

$$\|p\|_i^2 = \sum_{j=1}^{n} \frac{1}{\zeta_i^{(j)}} \left(p^{(j)} \right)^2, \quad \|s\|_{i*}^2 = \sum_{j=1}^{n} \zeta_i^{(j)} \left(s^{(j)} \right)^2, \quad i = 1, \ldots, I.$$

For $z_i[t] \in \mathbb{R}^n, \chi_i[t] > 0$ we consider the minimization problem as from step 3. in (SM):

$$\min_{p_1, \ldots, p_I \in \mathbb{R}_+^n} \left\{ \sum_{i=1}^{I} \langle z_i[t], p_i \rangle + \chi_i[t] d_i(p_i) \right\}. \tag{58}$$

Its unique solution is the price forecast (32) as from step 3. in (**AFG**):

$$p_i^{+(j)}[t] = \frac{\zeta_i^{(j)}}{\chi_i[t]} \left(-z_i^{(j)}[t]\right)_+, \quad j = 1, \ldots, n, i = 1, \ldots, I.$$

Secondly, it follows from (28) that the total logarithmic revenue is representable as a maximum of concave functions:

$$TLR(p_1, \ldots, p_I) = \max_{\substack{x_i \in X_i \\ i = 1, \ldots, I}} \Phi(x_1, \ldots, x_I) + \varphi(p_1, \ldots, p_K, x_1, \ldots, x_I),$$

where

$$\varphi(p_1, \ldots, p_I, x_1, \ldots, x_i) = -\sum_{i=1}^{I} \langle p_i, x_i \rangle + \left\langle \max_{i=1,\ldots,I} p_i, e \right\rangle.$$

Overall, we may apply Theorem 6 to get the following inequality:

$$TLR(p_1[t], \ldots, p_I[t]) - \Phi_{av}[t] - h_t(x_1[t], \ldots, x_I[t]) \le \rho_t, \tag{59}$$

where

$$\begin{aligned} h_t(x_1[t], \ldots, x_I[t]) &= \min_{p_1,\ldots,p_I \in \mathbb{R}^n_+} \left\{ \varphi(p_1, \ldots, p_I, x_1[t], \ldots, x_I[t]) + \frac{1}{t+1} \sum_{i=1}^{I} \chi_i[t] d_i(p_i) \right\}, \\ \rho_t &= \frac{1}{t+1} \sum_{i=1}^{I} \sum_{r=0}^{t} \frac{1}{2\chi_i[r-1]} \|\nabla_{p_i} TLR(p_1[t], \ldots, p_I[t])\|_{i*}^2 . \end{aligned}$$

We relate the penalty term h_t to $F[t]$ from Lemma 2. For that, we define the Euclidean prox-function

$$d(p) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{j=1}^{n} \left(p^{(j)}\right)^2.$$

It holds:

$$\begin{aligned} h_t(x_1[t], \ldots, x_I[t]) &\le \min_{p \in \mathbb{R}^n_+} \left\{ \varphi(p, \ldots, p, x_1[t], \ldots, x_I[t]) + \frac{1}{t+1} \sum_{i=1}^{I} \chi_i[t] d_i(p) \right\} \\ &= \min_{p \in \mathbb{R}^n_+} \left\{ \left\langle p, e - \sum_{i=1}^{I} x_i[t] \right\rangle + \frac{1}{t+1} \sum_{i=1}^{I} \chi_i[t] d_i(p) \right\} \\ &\le \min_{p \in \mathbb{R}^n_+} \left\{ \left\langle p, e - \sum_{i=1}^{I} x_i[t] \right\rangle + \frac{\sum_{i=1}^{I} \chi_i[t]}{t+1} \frac{1}{\min_{i,j} \zeta_i^{(j)}} d(p) \right\} \\ &= -\frac{t+1}{\sum_{i=1}^{I} \chi_i[t]} \min_{i,j} \frac{\zeta_i^{(j)}}{2} \sum_{j=1}^{n} \left(\sum_{i=1}^{I} x_i^{(j)}[t] - e \right)_+^2 \\ &= -\frac{C_2}{d_t} F[t], \end{aligned}$$

where $C_2 = \min_{i,j} \frac{\zeta_i^{(j)}}{2}$.

Now, we relate the remainder term ρ_t to b_t from Lemma 2. For that, let the constant $C_3 > 0$ bound the sequence of i-th consumer's excess supplies:

$$\|\nabla_{p_i} TLR(p_1[t], \ldots, p_I[t])\|_{i*}^2 \le 2C_3, \quad t \ge 0, i = 1, \ldots, I, \tag{60}$$

where, due to (30),

$$\nabla_{p_i} TLR(p_1[t], \ldots, p_I[t]) = \sum_{k}^{K} \mu_{ik}[t] \circ e_k - x_i(p_i[t]).$$

The existence of C_3 in (60) follows from the compactness of the consumption sets X_i, $i = 1, \ldots, I$ (see Section 2). Then, it holds:

$$\rho_t = \frac{1}{t+1} \sum_{i=1}^{I} \sum_{r=0}^{t} \frac{1}{2\chi_i[r-1]} \|\nabla_{p_i} TLR(p_1[t], \ldots, p_I[t])\|_{i*}^2 \le C_3 b_t.$$

Altogether, we estimated

$$h_t\left(x_1[t], \ldots, x_I[t]\right) \le -\frac{C_2}{d_t} F[t], \quad \rho_t \le C_3 b_t.$$

Substituting this into (59), we get the right-hand side of (35) in Lemma 2:

$$TLR[t] - \Phi_{av}[t] + \frac{C_2}{d_t} F[t] \le C_3 b_t.$$

Now, we estimate the dual gap in Lemma 2 from below. For that, we apply Lemma 4 and Theorem 4 to obtain

$$\begin{aligned}
\Phi[t] - \frac{C_2}{d_t} F[t] &= \Phi\left(x_1[t], \ldots, x_I[t]\right) - \frac{C_2}{d_t} \sum_{j=1}^{n} \left(\sum_{i=1}^{I} x_i^{(j)}[t] - e \right)_+^2 \\
&\le \max_{\substack{x_i \in X_i \\ i = 1, \ldots, I}} \Phi\left(x_1, \ldots, x_I\right) - \frac{C_2}{d_t} \sum_{j=1}^{n} \left(\sum_{i=1}^{I} x_i^{(j)} - e \right)_+^2 \\
&\le \max_{\substack{x_i \in X_i \\ i = 1, \ldots, I \\ \sum_{i=1}^{I} x_i \le e}} \Phi\left(x_1, \ldots, x_I\right) + \frac{d_t}{4C_2} \sum_{j=1}^{n} p^{*(j)} = TLR^* + C_1 d_t,
\end{aligned}$$

where $C_1 = \dfrac{\sum_{j=1}^{n} p^{*(j)}}{4C_2}$ and p^* is an equilibrium price. The latter exists due to the assumption on market productivity. Note that Lagrange multipliers for the market feasibility constraint in the adjoint problem (**A**) coincide with minimizers p^* of the total logarithmic revenue TLR.

Finally, we estimate

$$TLR[t] - \Phi[t] + \frac{C_2}{d_t} F[t] \ge TLR[t] - TLR^* - C_1 d_t.$$

This is the left-hand side of (35) in Lemma 2. It remains to note that the inequality in the middle of (35) follows due to the concavity of Φ:

$$\Phi_{av}[t] \leq \Phi[t].$$