# Bayesian Social Learning from Consumer Reviews*

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**Abstract**

Motivated by the proliferation of user-generated product-review information and its widespread use, this note studies a market where consumers are heterogeneous in terms of their willingness-to-pay for a new product. Each consumer observes the binary reviews (like or dislike) of customers who purchased the product in the past and uses Bayesian updating to infer the product quality. We show that the learning process is successful as long as the price is not prohibitive and therefore at least some consumers, with sufficiently high idiosyncratic willingness-to-pay, will purchase the product irrespective of their posterior quality estimate. We conclude with a few structural properties of the dynamics of the posterior beliefs.

## 1 Introduction

Online review sites are playing an increasingly large role in consumers' purchasing decisions. A recent survey by Tripavdvisor, a review site for the hospitality industry, shows that 90% of hoteliers think that reviews are very important for their business and 81% check their reviews at least weekly. Other industries such as online retail, motion pictures, and restaurants have seen customers' decisions increasingly influenced by reviews. The proliferation of smartphones is making access to such review sites easier than ever. This note studies the problem of Bayesian social learning from online reviews.

Specifically, a monopolist seller introduces a new product with unknown quality to a market of heterogenous consumers who have access to product reviews generated by consumers that purchased the product in the past, and where: (a) consumers use Bayesian updating to infer from past reviews the product quality; (b) the mechanism by which consumers report their reviews resembles that of online review sites, albeit in a simplified way; and (c) consumers have heterogeneous preferences

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(willingness-to-pay) for the product. The new product (or service) features observed attributes, e.g., location, and unobserved attributes that we denote with the term *quality*, which, to facilitate the Bayesian inference, can either be High (H) or Low (L). The quality experienced by a consumer who purchases the product is equal to the true quality of the product plus some random perturbation, e.g., due to variability in the service delivery process.

Consumers differ in their preferences for the observed product attributes, which is akin to having a heterogenous willingness-to-pay for a base good of low quality. Consumers arrive sequentially over time and make a Bayesian inference about the product quality based on the information available in the market, as described at the end of this paragraph. Following that, consumers make a once and for all decision of whether to purchase or to forgo the product, depending on their posterior quality distribution and on their idiosyncratic preference for the observed attributes that jointly determine their willingness-to-pay. Specifically, if a consumer perceives that the expected quality is $\hat{Q}$, her idiosyncratic preference for the other attributes is $\Theta$ and the price is $p$, then the consumer purchases the product if her expected net utility is non-negative, i.e., if $\Theta + \hat{Q} - p \geq 0$; we assume that the no-purchase option produces zero utility. The heterogeneity in preferences is captured by consumers' types that are private information. Buyers submit a binary review that takes the form of a "like" if their ex-post net-utility was non-negative, and a "dislike" if their ex-post net-utility was negative. The review is based on the true quality of the product plus the effect of the random perturbation experienced upon consumption of the product. It is only partially informative due to the heterogeneity in preferences that remains unseen and the fluctuations in the experienced quality of the product. Each agent observes the ordered sequence of consumer reviews.

We assume that the price is static and non-prohibitive, in the sense that even if consumers perceived the quality to be low, there are still some consumers with sufficiently high type that will choose to purchase and write a review. Proposition 1 shows that the conditional beliefs of the above-mentioned Bayesian learning process converge to a point mass distribution on the true state of the world, i.e., eventually consumers learn the true quality of the product almost surely. No-purchase decisions are not observed, but as we show, however, they do not contribute any information to the learning process, and, as such, their absence does not affect the decisions of the following consumers.

Contrasting to the literature on social learning, and, specifically, Banerjee (1992) and Bikhchandani, Hirshleifer, and Welch (1992), in our model consumers have no private information (signals) on the state of the world, and they are heterogeneous in their preferences; see also Chamley (2003). Information does not aggregate by observing the sequence of actions and making inferences about past beliefs and signals, but instead information accumulates through the effect of online reviews. In common to these papers, consumers are Bayesian and the unknown quality parameter takes on two possible values, namely $H$ or $L$. Our assumption on the price and type distribution that ensures that some consumers will always purchase and generate new reviews, ultimately drives the learning result; It is analogous to the one introduced in Smith and Sørensen (2000), who show that asymptotic learning holds if agents' signals have unbounded strength (i.e., try the product irrespective of the observed actions of all predecessors). Heterogeneous willingness-to-pay is what drives experimentation that will drive learning as opposed to assuming that a sufficient fraction of the market has sufficiently (and arbitrarily) accurate quality information so as to always follow their own signal in their decision. Moreover, in the model of signals, learning implies that all consumers make the same decision. This is not true in our model, where after the market has learned, there will be a set of consumers that choose to purchase and a set of consumers that will choose not to purchase, and even among the purchasers, there will be some fraction that will ultimately dislike the product due to the quality noise when they experience the good.

Focusing on the learning trajectory, Proposition 5 shows that the posterior after the review sequence (like, dislike) will be greater than the posterior after the review sequence (dislike, like), highlighting that the order of reviews affects the dynamics of consumers' beliefs and the importance of early reviews on the product's demand. As a corollary, one can show that the likelihood that the next review will be positive is decreasing with the belief, or in other words that reviews tend to be negative following high quality expectation.

Most papers on Bayesian social learning assume a structure where consumers are endowed with private signals on the unknown quality of the product. Acemoglu, Dahleh, Lobel, and Ozdaglar (2011) consider agents who are embedded on a general social network and find conditions on the social network and signal structure for asymptotic learning. Herrera and Hörner (2013) consider a model where, like in this paper, agents make a binary choice—"buy" and "not buy"—but only the "buy" decision can be observed by predecessors. They show that asymptotic learning occurs when signals have unbounded strength and agents observe the time elapsed since the product launch. Acemoglu, Bimpikis, and Ozdaglar (2014) where agents can collect information by forming costly communication links or by delaying their irreversible action. Goeree, Palfrey, and Rogers (2006) consider a model where agents make choices sequentially and their payoff depends not only on the state of the world and their action, but also on an idiosyncratic privately observed shock.

The paper is organized as follows. Section 2 introduces the Bayesian learning model from reviews, and Section 3 establishes the asymptotic learning result and studies the structural properties of the learning trajectory.

**Notation.** Given any sequence $\{X_t\}$ of i.i.d. random variables, the distribution function of $X_1$ is denoted by $F_X$, its survival function by $\overline{F}_X$, its density by $f_X$. The symbol $\mathbb{1}_{\{A\}}$ denotes the indicator of the event $A$.

## 2 The model

A monopolist introduces a product or service of unknown quality to a market of heterogeneous consumers who try to learn about this quality through a social learning mechanism and will make their respective purchase decisions accordingly. Specifically, the monopolist introduces a product of intrinsic quality $Q$ that for simplicity is assumed to take one of two possible values $L$ or $H$, where $H > L$. The intrinsic quality of the product is determined through a random draw at time $t = 0$, and takes value $H$ with probability $\pi_0$ and value $L$ with probability $1 - \pi_0$. The realization of $Q$ is assumed to be unknown to the potential consumers.

Consumers arrive sequentially and are indexed by their arrival time $t \in \{1, 2, \ldots\}$. They are heterogeneous with respect to their preference for the product. Consumer $t$'s preference is represented by her type $\Theta_t$. Types are i.i.d. random variables with a strictly increasing continuous distribution function $F_\Theta$. The type $\Theta_t$ is known to consumer $t$, but not to the other consumers. A consumer $t$ who purchases the product will experience a quality level $Q_t = Q + \varepsilon_t$, where $\varepsilon_t$ is a random fluctuation around the nominal and initially unknown quality level $Q$. This fluctuation could be the result of variations in the product itself, or even variations in the way individuals experience or perceive quality. The random variables $\varepsilon_t$ are i.i.d. with a continuous, zero mean distribution function $F_\varepsilon$, independent of the types $\Theta_t$.

Each consumer $t$ makes a once-and-for-all purchase decision denoted by $B_t \in \{0, 1\}$: she either buys the product ($B_t = 1$) or does not buy it ($B_t = 0$). If a consumer buys the product, her payoff is given by the following simple additive form $V_t := \Theta_t + Q_t - p$, where $\$p$ is the price of the product, which is assumed to be constant over time. If she chooses to forgo the product, her payoff is given by 0, without loss of generality. That is, the payoff of consumer $t$ is given by $B_t V_t$. Whatever the purchase decision is, consumers do not revisit it in later periods.

If $B_t = 1$, once consumer $t$ has bought the product and experienced its quality, she publicly posts a review $R_t$, where

$$R_t = \begin{cases} \text{👍} & \text{if } B_t = 1, \text{ and } V_t \geq 0, \\ \text{👎} & \text{if } B_t = 1, \text{ and } V_t < 0, \\ \text{✗} & \text{if } B_t = 0. \end{cases}$$

Depending on the consumer's ex-post net-utility, a review is either positive or negative.[1] Although consumers who do not buy the product do not review it, it is useful to suppose that they report a blank review ✗. We will show that, in contrast to models with private signals, in our model ✗'s are not informative.

Define the time indices of consumers who choose to purchase the product

$$\tau_1 = \min(t \mid B_t = 1) \quad \text{and} \quad \tau_k = \min(t \mid t > \tau_{k-1}, B_t = 1)$$

and let the corresponding review history be, for $\tau_k \leq t < \tau_{k+1}$,

$$h_t = (R_{\tau_1}, \ldots, R_{\tau_k}),$$

[1] All our results extend to the case when each buying consumer writes a review with fixed positive probability, as long as this is independent of the reviews of past consumers and the experienced utility $V_t$.

Consumer $t$ observes history $h_{t-1}$. Let $\mathscr{H}_t$ be the set of all histories $h_t$. Note that the realization of $\Theta_t$ and $\varepsilon_t$ is never revealed to consumers different from $t$. Here consumers *generate* information about the quality of the product when they review it, whereas in the literature on social learning with signals, they *reveal* privately held information when making a purchase decision.

The form of the utility function, review decision, information structure, and all the distributions of the relevant random variables are assumed to be common knowledge.

Consumer $t$ chooses either to buy or forgo the product to maximize her expected payoff

$$B_t \mathsf{E}[V_t | \Theta_t, h_t].$$

Note that $V_t$ is independent of the actions of the other consumers, including the ones taken by consumers $1, \ldots, t-1$; past actions affect player $t$'s inference, not her payoff. We call $B = (B_1, B_2, \ldots)$ the sequence of all consumers' decisions.

## 3 Asymptotic learning

Given history $h_t$, define

$$\pi(h_t) := \mathsf{P}(Q = H | h_t).$$

We frequently use the shorthand notation $\pi_t := \pi(h_t)$. The belief determines the buyers' purchase decision. Consumer $t$ will buy the product if and only if the expected net utility from buying is greater than zero:

$$\mathsf{E}[V_t | h_{t-1}, \Theta_t] = \Theta_t + \mathsf{E}[Q_t | h_{t-1}] - p = \Theta_t + \pi_{t-1} H + (1 - \pi_{t-1}) L - p \geq 0,$$

or, alternatively, if and only if $\Theta_t \geq \theta(\pi_{t-1}, p)$, where

$$\theta(\pi, p) := p - (\pi H + (1 - \pi) L) = p - \mathsf{E}_\pi[Q]. \tag{1}$$

Note that $\varepsilon_t$ does not affect the purchase decision since it has zero mean and is independent of the history and $\Theta_t$. Given the purchase criterion, in each period $t$ the seller faces an expected demand function $\overline{F}_\Theta(\theta(\pi_{t-1}, p))$. The following assumption will hold throughout the paper.

**Assumption 1.** (a) $\text{supp}(\varepsilon_t) = \mathbb{R}$ and (b) given the fixed product price \$$p$, $\overline{F}_\Theta(p - L) > 0$.

Assumption 1 (a) assures that there is always a positive probability that a consumer will derive positive or negative net utility from buying the product, irrespective of whether the intrinsic quality $Q$ is high or low and of the value of the consumer type. Assumption 1 (b) will prove necessary for social learning to occur. It assures that under the firm's price, there will be some consumers who have sufficiently high idiosyncratic type $\Theta$ and therefore will choose to buy regardless of the product true quality. The assumption trivially holds when the distribution of $\Theta_t$ is unbounded, and it is reminiscent of Assumption 3 in Goeree et al. (2006).

The belief $\pi_t$ is a random variable in $[0, 1]$. Lemma 6 demonstrates some basic properties of the belief. First, starting from any history $h_t$, the belief increases after a 👍 and decreases after a 👎,

$$\pi((h_t, \text{👎})) \leq \pi(h_t) \leq \pi((h_t, \text{👍})),$$

and this inequality is weak only when $\pi(h_t) \in \{0, 1\}$. This is expected, since the ex-post net utility of a buyer is higher when the intrinsic quality is high, which itself increases the probability of a positive review. The opposite result holds in case of 👎.

Following a no-buy, the posterior will not change, since the sequence of reviews remains unchanged. However, this would have been the case even if ✗ reviews had been observable. A no-buy decision of consumer $t + 1$ merely reveals that her type is lower than $\theta(\pi(h_t))$. This carries no information about the quality of the product. This observation is in sharp contrast with the literature on social learning from signals, where any action can be informative by revealing the agent's private signal. Our main result is the following:

**Proposition 1.** *Let Assumption 1 hold. If $\pi_0 \in (0, 1)$, then $\pi_t \to \mathbb{1}_{\{Q=H\}}$ with probability 1.*

As long as consumers purchase the product, the drift of the belief process is positive when the quality is high and negative when it is low. As the number of reviews grows large, the posterior will converge and correctly identify the intrinsic quality $Q$ of the product.

Although the learning result appears intuitive, it is not trivial, due to the fact that learning happens only when consumers buy and the section of consumers who buy is not a random representative of the whole population of consumers. The bias that the purchasing induces has to be filtered out in order to achieve asymptotic learning. Moreover the observations are not exchangeable and no finite-dimensional sufficient statistic exists. Although the observations are the result of the censoring, due to the purchasing decision of the consumers, the usual models of inference under censored observations cannot be applied since the censoring is endogenous and not independent of the types. The following corollaries will clarify the conditions for learning to occur.

**Corollary 2.** *If $\overline{F}_\Theta(p - L) = 0$, then with positive probability consumers stop buying at some finite time $t$, so no learning occurs.*

Corollary 2 shows that it is possible that consumers do not learn the quality of the object, even asymptotically. This is due to the fact that at some point they stop buying. The reason for this is that their type distribution is bounded above, so when the probability that the quality is high goes below a certain level, nobody has an incentive to buy.

There is a connection and a difference with respect to the classical literature on social learning based on private signals à la Smith and Sørensen (2000). In both cases the boundedness of a distribution makes the difference between learning and not learning. In the private-signal model not learning corresponds to customers getting stuck on a dominated action. In our model it just implies that customers do not update their posterior any longer and they stop buying. Nevertheless not buying may be the right decision. This will never be discovered.

**Corollary 3.** *If $\varepsilon_t = 0$ for all $t$, then $\pi(h_t) = 0$ if $h_t$ contains at least one* 👎*. Otherwise $\pi(h_t)$ converges monotonically to* 1*.*

Corollary 3 says that when the quality of the object is exactly revealed to buyers, learning either happens fast or is monotone. If a customer posts a 👎, then learning is immediate, since this betrays that the quality is low. Otherwise the probability that the quality is high converges to 1 monotonically and every customer makes the right decision.

**Corollary 4.** *If $\mathsf{P}(-\eta \leq \varepsilon_1 \leq \eta) = 1$, with $\eta < H - L$, then, if $Q = L$, there exist a finite $t$ and a history $h_t$ such that $\mathsf{P}(h_t) > 0$ and $\pi(h_t) = 0$.*

Corollary 4 deals with the case where the quality is revealed almost exactly. Here the phenomenon of immediate learning (of the bad quality) can happen with positive probability.

We conclude this section with a couple of structural properties of the dynamics of the learning process. Specifically, in the spirit of comparative statics analysis, we compare outcomes of the social learning process under a single change in model parameters or review histories.

Identical reviews may carry different information, because the reviewers observed different past review histories. Thus, reviews are not exchangeable random variables and potentially carry different weights on the posterior distributions of a future consumer. Do earlier or later reviews carry more weight in forming a posterior belief? How does self-selection bias due to a low belief (after negative reviews) or a high belief (after many positive reviews) affect the likelihood of positive or negative reviews, respectively? The next proposition answers these two questions. It compares the belief resulting from two histories: one where a positive review is followed by a negative one, $h_{\text{👍},\text{👎}} := (\text{👍}, \text{👎})$ and the reverse sequence $h_{\text{👎},\text{👍}} := (\text{👎}, \text{👍})$. The resulting structural result then offers insight onto the effect of self-selection bias.

**Proposition 5.**

(a) *If $f_\varepsilon$ is log-concave, then for any histories $h' \in \mathscr{H}_{t'}$ and $h'' \in \mathscr{H}_{t''}$ with $t', t'' \in \mathbb{N} \cup \{0\}$ we have*

$$\pi(h', h_{\text{👍},\text{👎}}, h'') \geq \pi(h', h_{\text{👎},\text{👍}}, h'').$$

(b) *For any $\pi \in [0,1]$ and $q \in \{L, H\}$ we have that $\mathsf{P}\big(R_{t+1} = \text{👍} \,|\, (B_{t+1}, \pi_t, Q) = (1, \pi, q)\big)$ is decreasing in $\pi$ and $\mathsf{P}\big(R_{t+1} = \text{👎} \,|\, (B_{t+1}, \pi_t, Q) = (1, \pi, q)\big)$ is increasing in $\pi$.*

Part (a) shows that under some weak assumption on the distribution of $\varepsilon_t$, the earlier review is more influential[2]. This result holds for any distribution of types $\Theta$. Proposition 5(a) should be

[2] As will become clear in its proof, this result could be stated under more general assumptions. We present it this way for the sake of simplicity.

understood in the context of consumers' self-selection given their unobservable types (e.g., see Li and Hitt, 2008). As reviews vary over time, the corresponding cutoff for purchase, $\theta(\pi_t, p)$ varies as well. Holding $Q$ fixed, a consumer who purchases when the cutoff $\theta(\pi_t, p)$ is low is more likely to be disappointed than a consumer who purchases when the cutoff is high, since the latter is more likely to be a high type. As a result, 👎 is a weaker negative signal in $h_{\text{👍},\text{👎}}$ than in $h_{\text{👎},\text{👍}}$. Similarly, a purchasing consumer with a low cutoff is less likely to be satisfied, because on average her type is low, resulting is a stronger positive effect of the 👍 in $h_{\text{👍},\text{👎}}$ than in $h_{\text{👎},\text{👍}}$. In conclusion, earlier reviews have a higher effect on the posterior belief.

Part (b) shows another effect of self-selection bias: namely, that the likelihood that the next review will be positive is decreasing with the belief, or with the expected quality. Actually it is decreasing with $\pi_{t-1}H + (1 - \pi_{t-1})L - p$. This shows that not only the likelihood of a consumer writing a good review is decreasing in $\pi$, it is also increasing in the price $p$. When wither $\pi$ goes up or $p$ goes down, the probability that a customer buys increases. With this the probability that a customer will be unsatisfied increases. All the above is in the sense of comparative statics, that is, *ceteris paribus* if in situation A the probability $\pi_t$ is higher (or the price $p$ is lower) than in situation B, then the probability of seeing a favorable review from consumer $t$ is lower in A than in B.

## 4 Proofs

Given random variables $X$, its failure rate is denoted by $\lambda_X$ and its reverse failure rate by $\rho_X$, that is,

$$\lambda_X(t) = \frac{f_X(t)}{\overline{F}_X(t)} \quad \text{and} \quad \rho_X(t) = \frac{f_X(t)}{F_X(t)}.$$

Notice that

$$\mathsf{P}(R_t = r | \pi(h_t) = \pi, Q = q) = \begin{cases} \int_{\theta(\pi,p)}^{\infty} \overline{F}_\varepsilon(p - q - x)\ \mathrm{d}F_\Theta(x) & \text{for } r = \text{👍}, \quad (2a) \\ \int_{\theta(\pi,p)}^{\infty} F_\varepsilon(p - q - x)\ \mathrm{d}F_\Theta(x) & \text{for } r = \text{👎}, \quad (2b) \\ F_\Theta(\theta(\pi, p)) & \text{for } r = \text{✗}. \quad (2c) \end{cases}$$

where $\theta(\pi, p)$ is defined as in (1). We can therefore define the shorthand notation

$$G(r, \pi, q) := \mathsf{P}(R_t = r | \pi_t = \pi, Q = q), \tag{3}$$

$$G(r, \pi) := \pi G(r, \pi, H) + (1 - \pi) G(r, \pi, L). \tag{4}$$

Lemma 6 illustrates some properties of the Bayesian updating,

**Lemma 6.** *Let Assumption 1 hold.*

(a) *For all* $\pi \in [0,1]$, $G(\text{👍}, \pi, H) > G(\text{👍}, \pi, L)$.

(b) *For all* $\pi \in [0,1]$, $G(\text{👎}, \pi, H) < G(\text{👎}, \pi, L)$.

(c) *There exist* $\underline{G}, \overline{G} \in (0,1)$ *such that* $\underline{G} \leq G(r, \pi, q, p) \leq \overline{G}$ *for* $r \in \{\text{👍}, \text{👎}\}$, $q \in \{H, L\}$, $\pi \in [0,1]$.

(d) *For all* $q \in \{H, L\}$ *and* $\pi \in [0,1]$ *we have* $G(\text{👎}, \pi, q) + G(\text{👍}, \pi, q) = \overline{F}_{\Theta}(\theta(\pi, p))$.

(e) *For any history* $h_{t-1}$, *we have* $\pi(h_{t-1}, \text{✗}) = \pi(h_{t-1})$.

(f) *Whenever* $\pi(h_{t-1}) \in (0,1)$ *we have*

$$\pi(h_{t-1}, \text{👎}) < \pi(h_{t-1}) < \pi(h_{t-1}, \text{👍}).$$

*Proof.* (a) Since $\overline{F}_{\varepsilon}$ is nonincreasing, we have

$$G(\text{👍}, \pi, H) = \int_{\theta(\pi,p)}^{\infty} \overline{F}_{\varepsilon}(p - H - x) \, \mathrm{d}F_{\Theta}(x) > \int_{\theta(\pi,p)}^{\infty} \overline{F}_{\varepsilon}(p - L - x) \, \mathrm{d}F_{\Theta}(x) = G(\text{👍}, \pi, L).$$

(b) Since $F_{\varepsilon}$ is nondecreasing, we have

$$G(\text{👎}, \pi, H) = \int_{\theta(\pi,p)}^{\infty} F_{\varepsilon}(p - H - x) \, \mathrm{d}F_{\Theta}(x) < \int_{\theta(\pi,p)}^{\infty} F_{\varepsilon}(p - L - x) \, \mathrm{d}F_{\Theta}(x) = G(\text{👎}, \pi, L).$$

(c) This follows from Assumption 1, since there exists a fraction of consumers that would always choose to buy the product, and since the support of $\varepsilon$ is large enough.

(d) Just add (2a) and (2b) and consider that given $\pi_t$ the probability of buying is independent of $Q$.

(e) In general, by Bayes' rule,

$$\pi(h_{t-1}, r) = \frac{\mathsf{P}(R_t = r | h_{t-1}, Q = H)\pi(h_{t-1})}{\mathsf{P}(R_t = r | h_{t-1}, Q = H)\pi(h_{t-1}) + \mathsf{P}(R_t = r | h_{t-1}, Q = L)(1 - \pi(h_{t-1}))}.$$

Hence

$$\pi(h_{t-1}, \text{✗}) = \frac{G(\text{✗}, \pi(h_{t-1}), H)\pi(h_{t-1})}{G(\text{✗}, \pi_t, H)\pi(h_{t-1}) + G(\text{✗}, \pi(h_{t-1}), L)(1 - \pi(h_{t-1}))} = \pi(h_{t-1}),$$

since $G(\text{✗}, \pi, q) = F_{\Theta}(\theta(\pi, p))$ for all $q \in \{H, L\}$.

(f) We have

$$
\begin{aligned}
\pi(h_{t-1}, \text{👎}) &= \frac{G(\text{👎}, \pi(h_{t-1}), H)\pi(h_{t-1})}{G(\text{👎}, \pi(h_{t-1}), H)\pi(h_{t-1}) + G(\text{👎}, \pi(h_{t-1}), L)(1-\pi(h_{t-1}))} \\
&< \frac{G(\text{👎}, \pi(h_{t-1}), H)\pi(h_{t-1})}{G(\text{👎}, \pi(h_{t-1}), H)\pi(h_{t-1}) + G(\text{👎}, \pi(h_{t-1}), H)(1-\pi(h_{t-1}))} \\
&= \pi(h_{t-1}) \\
&= \frac{G(\text{👍}, \pi(h_{t-1}), H)\pi(h_{t-1})}{G(\text{👍}, \pi(h_{t-1}), H)\pi(h_{t-1}) + G(\text{👍}, \pi(h_{t-1}), H)(1-\pi(h_{t-1}))} \\
&< \frac{G(\text{👍}, \pi(h_{t-1}), H)\pi(h_{t-1})}{G(\text{👍}, \pi(h_{t-1}), H)\pi(h_{t-1}) + G(\text{👍}, \pi_t, L)(1-\pi(h_{t-1}))} \\
&= \pi(h_{t-1}, \text{👍}).
\end{aligned}
$$

where the first inequality follows from (b) and the second from (a). ☐

The following lemma is needed to prove Proposition 1.

**Lemma 7.** *Define the function*

$$
g(x,y,z) = \log\left(\frac{x}{y}\right)x + \log\left(\frac{z-x}{z-y}\right)(z-x). \tag{5}
$$

*Then,* $0 < x \leq y < z < 1$ *implies* $g(x,y,z) \geq 0$ *with equality if and only if* $x = y$.

*Proof.* All variables in the proof are assumed to satisfy the condition in the statement of the lemma. We will show that $g$ is strictly monotonically decreasing in $x$ and that $g(x,x,z) = 0$. We first show that $g$ is convex with respect to the first argument,

$$
\frac{\partial g(x,y,z)}{\partial x} = \log\left(\frac{x}{y}\right) - \log\left(\frac{z-x}{z-y}\right),
$$

and,

$$
\frac{\partial^2 g(x,y,z)}{\partial x^2} = \frac{1}{x} + \frac{1}{z-x} > 0.
$$

It is easy to see that for $\delta > 0$ small,

$$
\left.\frac{\partial g(x,y,z)}{\partial x}\right|_{x=y} = 0 \quad \text{and} \quad \left.\frac{\partial g(x,y,z)}{\partial x}\right|_{x=x-\delta} < 0.
$$

By convexity we have that $g(x,y,z) \geq g(y,y,z) = 0$, with equality if and only if $x = y$. ☐

*Proof of Proposition 1.* For $t \geq 1$, let $h_t^{\text{full}} := (R_1, \ldots, R_t)$ be the full history including ✗ reviews; recall that in contrast the observed history $h_t$ only includes information from the consumers that bought $\tau_1, \ldots, \tau_k$. Define $\pi_t^{\text{full}} = \pi(h_t^{\text{full}})$. We have

$$\pi_t = \pi_t^{\text{full}} = \mathsf{P}(Q = H | h_t^{\text{full}}) \to \pi_\infty,$$

using the martingale convergence theorem (see, for instance, Karlin and Taylor, 1975). Since $\pi_t \in [0,1]$ for all $t \geq 0$, we further conclude that $\pi_\infty \in [0,1]$, and that $\pi_0 = \mathsf{E}[\pi_t] = \mathsf{E}[\pi_\infty]$.

Recall the definitions of $G$ in (3) and (4). We have

$$\begin{aligned}
\mathsf{E}[\log \pi_t | \pi_{t-1}, Q = H] &= \log \left( \frac{G(\text{✗}, \pi_t, H)\pi_{t-1}}{G(\text{✗}, \pi_t)} \right) G(\text{✗}, \pi_t, H) \\
&\quad + \log \left( \frac{G(\text{👎}, \pi_t, H)\pi_{t-1}}{G(\text{👎}, \pi_t)} \right) G(\text{👎}, \pi_t, H) \\
&\quad + \log \left( \frac{G(\text{👍}, \pi_t, H)\pi_{t-1}}{G(\text{👍}, \pi_t)} \right) G(\text{👍}, \pi_t, H) \\
&= \log \pi_{t-1} \qquad (6) \\
&\quad + \log \left( \frac{G(\text{👎}, \pi_t, H)}{G(\text{👎}, \pi_t)} \right) G(\text{👎}, \pi_t, H) \\
&\quad + \log \left( \frac{G(\text{👍}, \pi_t, H)}{G(\text{👍}, \pi_t)} \right) G(\text{👍}, \pi_t, H) \\
&= \log \pi_{t-1} + g\left( G(\text{👎}, \pi_t, H), G(\text{👎}, \pi_t), F_\Theta(\theta(\pi_t, p)) \right)
\end{aligned}$$

where the second equality stems from

$$\begin{aligned}
&G(\text{✗}, \pi_t, H, p) = G(\text{✗}, \pi_t, L) \\
&G(\text{✗}, \pi_t, H, p) + G(\text{👎}, \pi_t, H) + G(\text{👍}, \pi_t, H) = 1,
\end{aligned}$$

and Lemma 6(d). The function $g$ was defined in (5).

We can show that there exists $\underline{G} > 0$ such that

$$\underline{G} \leq G(\text{👎}, \pi_t, H) \leq G(\text{👎}, \pi_t) < F_\Theta(\theta(\pi_t, p)) < 1.$$

Indeed, the first inequality follows from Lemma 6(c); the second inequality stems from Lemma 6(b) and (3); the third inequality follows from Lemma 6(d) and (3); the last inequality is a consequence of Assumption 1(b).

Define

$$\gamma(\pi) := g\left( G(\text{👎}, \pi, H), G(\text{👎}, \pi), F_\Theta(\theta(\pi, p)) \right).$$

Using Lemma 6(c) and (3), we see that

$$G(\text{👎}, \pi_t, H) = G(\text{👎}, \pi_t)$$

if and only if $\pi = 1$. Therefore, by Lemma 7,

$$\gamma(\pi) > 0 \quad \text{for } \pi \in [0,1) \quad \text{and} \quad \gamma(\pi) = 0 \quad \text{for } \pi = 1.$$

Assume now, by contradiction, that there exist $\delta, \eta > 0$ such that

$$\mathsf{P}(\pi_\infty < 1 - \eta | Q = H) > 2\delta.$$

Then there exists an integer $T$ such that for all $t > T$

$$\mathsf{P}(\pi_t < 1 - \eta | Q = H) > \delta.$$

The function $\gamma$ is continuous and strictly positive for all $\pi \in [0, 1-\eta]$, therefore

$$\min_{\pi \in [0,1-\eta]} \gamma(\pi) =: \underline{\gamma} > 0.$$

Finally, using (6) iteratively, we get

$$\begin{aligned}
\mathsf{E}[\log \pi_{T+k} | Q = H] &= \log \pi_0 + \sum_{i=1}^{T+k} \mathsf{E}[\gamma(\pi_i) | Q = H] \\
&\geq \log \pi_0 + \sum_{i=T}^{T+k} \mathsf{E}[\gamma(\pi_i) | Q = H] \\
&\geq \log \pi_0 + \sum_{i=T}^{T+k} \mathsf{E}[\gamma(\pi_i) \mathbb{1}_{\{\pi_i \leq 1-\eta\}} | Q = H] \\
&\geq \log \pi_0 + k \underline{\gamma} \delta,
\end{aligned}$$

and we conclude that $\mathsf{E}[\log(\pi_{T+k}) | Q = H] > 0$ by taking $k$ large enough, which contradicts the fact that $\log \pi_t \leq 0$. Therefore, $\mathsf{P}(\pi_\infty < 1 | Q = H) = 0$, or $\mathsf{P}(\pi_\infty = 1 | Q = H) = 1$.

The same argument can be used to prove that $\mathsf{P}(\pi_\infty = 0 | Q = L) = 1$. □

*Proof of Corollary 2.* By Lemma 6(d) we have

$$G(\text{👎}, \pi, q) + G(\text{👍}, \pi, q) = \overline{F}_\Theta(\theta(\pi, p)).$$

Hence if $\overline{F}_\Theta(\theta(\pi, p)) = 0$, then

$$G(\text{✗}, \pi, q) := \mathsf{P}(R_t = \text{✗} | \pi_t = \pi, Q = q) = 1,$$

that is, with probability one no consumer buys. Since the event $\theta(\pi_t, p) = p - L$ has positive probability, buying and therefore learning stop with positive probability. □

*Proof of Corollary 3.* Consumer $t$ buys iff $\Theta_t + \pi_{t-1}H + (1 - \pi_{t-1})L - p \geq 0$. If $\varepsilon_t = 0$, then she dislikes the product iff $\Theta_t + Q - p < 0$. This can happen only if $Q = L$.

By Lemma 6(f) $\pi(h_{t-1}) < \pi(h_{t-1},$ 👍$)$, therefore, if no 👎 appears, then the convergence of $\pi_t$ to 1 is monotone. □

*Proof of Corollary 4.* If every $\varepsilon_t$ falls in the interval $[-\eta, \eta]$ and $R_t =$ 👎, then

$$\begin{aligned} \Theta_t + \pi_{t-1}H + (1 - \pi_{t-1})L - p &\geq 0, \\ \Theta_t + Q + \varepsilon_t - p &< 0, \end{aligned}$$

that is,

$$Q - \eta \leq Q + \varepsilon_t < p - \Theta_t \leq \pi_{t-1}H + (1 - \pi_{t-1})L. \tag{7}$$

If $Q = H$, then (7) holds iff $(1 - \pi_t)(H - L) \leq \eta$. Therefore when $\eta < (1 - \pi_t)(H - L)$, the quality $Q$ can only be $L$. This happens with positive probability. □

Define

$$\begin{aligned} \Gamma(\pi) &= \frac{\pi}{1 - \pi}, \\ \Lambda(r, \pi) &= \frac{G(r, \pi, H)}{G(r, \pi, L)}, \end{aligned}$$

**Definition 1.** We say that the condition ILR (increasing likelihood ratio) holds if

$$\Lambda(r, \pi)$$

is nondecreasing in $\pi$ for $r \in \{$👍, 👎$\}$.

We next discuss some sufficient conditions for ILR, which will depend on the following definitions.

**Definition 2.** (a) The distribution of a random variable $X$ is IFR (increasing failure rate) if its failure rate is nondecreasing.

(b) The distribution of a random variable $X$ is DRFR (decreasing reverse failure rate) if its reverse failure rate is nonincreasing.

**Proposition 8.** *If the distribution of $\varepsilon$ is both IFR and DRFR, then condition ILR holds.*

The proof of Proposition 8 requires some properties of $TP_2$ (total positivity of order two), for which the reader is referred to Karlin (1968) and Karlin and Rinott (1980).

*Proof of Proposition 8.* Notice that $\varepsilon$ is IFR iff its survival function $\overline{F}_\varepsilon$ is log-concave. Write

$$\mathsf{P}(\Theta + \varepsilon > s, \Theta > t) = \int_t^\infty \overline{F}_\varepsilon(s - x)\ \mathrm{d}F_\Theta(x) = \int_{-\infty}^\infty \mathbb{1}_{\{(t,\infty)\}}(x)\overline{F}_\varepsilon(s - x)\ \mathrm{d}F_\Theta(x).$$

Notice that $\overline{F}_\varepsilon$ is log-concave iff $K(s, x) := \overline{F}_\varepsilon(s - x)$ it $\mathrm{TP}_2$. Moreover $L(x, t) := \mathbb{1}_{\{(t,\infty)\}}(x)$ is $\mathrm{TP}_2$. Therefore the convolution

$$\int K(s, x)L(x, t)\ \mathrm{d}F_\Theta(x) = \mathsf{P}(\Theta + \varepsilon > s, \Theta > t)$$

is $\mathrm{TP}_2$. This implies that for $\pi_1 < \pi_2$ we have

$$\begin{aligned}\mathsf{P}(\Theta + \varepsilon > p - H, \Theta > p - \theta(\pi_2))\mathsf{P}(\Theta + \varepsilon > p - L, \Theta > p - \theta(\pi_1)) \geq \\ \mathsf{P}(\Theta + \varepsilon > p - H, \Theta > p - \theta(\pi_1))\mathsf{P}(\Theta + \varepsilon > p - L, \Theta > p - \theta(\pi_2)).\end{aligned}$$

Hence

$$G(\ThumbsUp, \pi_2, H)G(\ThumbsUp, \pi_1, L) \geq G(\ThumbsUp, \pi_1, H)G(\ThumbsUp, \pi_2, L),$$

that is,

$$\frac{G(\ThumbsUp, \pi, H)}{G(\ThumbsUp, \pi, L)}$$

is nondecreasing in $\pi$.

Next, notice that if $\varepsilon$ is DRFR, then its distribution function $F_\varepsilon$ is log-concave.

Write

$$\mathsf{P}(\Theta + \varepsilon \leq s, \Theta > t) = \int_t^\infty F_\varepsilon(s - x)\ \mathrm{d}F_\Theta(x) = \int_{-\infty}^\infty \mathbb{1}_{\{(t,\infty)\}}(x)F_\varepsilon(s - x)\ \mathrm{d}F_\Theta(x).$$

Notice that $F_\varepsilon$ is log-concave iff $K(s, x) := F_\varepsilon(s - x)$ it $\mathrm{TP}_2$. Moreover $L(x, t) := \mathbb{1}_{\{(t,\infty)\}}(x)$ is $\mathrm{TP}_2$. Therefore the convolution

$$\int K(s, x)L(x, t)\ \mathrm{d}F_\Theta(x) = \mathsf{P}(\Theta + \varepsilon \leq s, \Theta > t)$$

is $\mathrm{TP}_2$. This implies that for $\pi_1 < \pi_2$ we have

$$\begin{aligned}\mathsf{P}(\Theta + \varepsilon \leq p - H, \Theta > p - \theta(\pi_2))\mathsf{P}(\Theta + \varepsilon \leq p - L, \Theta > p - \theta(\pi_1)) \geq \\ \mathsf{P}(\Theta + \varepsilon \leq p - H, \Theta > p - \theta(\pi_1))\mathsf{P}(\Theta + \varepsilon \leq p - L, \Theta > p - \theta(\pi_2)).\end{aligned}$$

Hence

$$G(\ThumbsDown, \pi_2, H)G(\ThumbsDown, \pi_1, L) \geq G(\ThumbsDown, \pi_1, H)G(\ThumbsDown, \pi_2, L),$$

that is,

$$\frac{G(\ThumbsDown, \pi, H)}{G(\ThumbsDown, \pi, L)}$$

is nondecreasing in $\pi$, and therefore ILR holds. □

A stronger yet simpler sufficient condition is the following.

**Corollary 9.** *If the density $f_\varepsilon$ is log-concave, then ILR holds.*

*Proof.* If the density $f_\varepsilon$ is log-concave, then both the distribution function $F_\varepsilon$ and the survival function $\overline{F}_\varepsilon$ are log-concave, therefore the proof of Proposition 8 can be applied. □

Corollary 9 shows that ILR is a fairly natural assumption on $\varepsilon$ given its interpretation as a mean zero noise around the product's quality. For example, ILR holds if $\varepsilon$ is has a normal distribution or a Gumble distribution. We can now prove our result.

*Proof of Proposition 5. Part (a).* We have

$$\begin{aligned} &\Gamma\left(\pi(h_{\ThumbsUp,\ThumbsDown})\right) - \Gamma\left(\pi(h_{\ThumbsDown,\ThumbsUp})\right) \\ &\quad = \Gamma(\pi_0)\left[\Lambda\left(\ThumbsUp, \pi_0\right)\Lambda\left(\ThumbsDown, \pi(\ThumbsUp)\right) - \Lambda\left(\ThumbsDown, \pi_0\right)\Lambda\left(\ThumbsUp, \pi(\ThumbsDown)\right)\right] \end{aligned} \tag{8}$$

We know from Lemma 6(f) that

$$\pi(\ThumbsDown) \leq \pi_0 \leq \pi(\ThumbsUp).$$

Therefore, by the ILR property (Definition 1),

$$\Lambda\left(\ThumbsUp, \pi_0\right) \geq \Lambda\left(\ThumbsUp, \pi(\ThumbsDown)\right) \quad \text{and} \quad \Lambda\left(\ThumbsDown, \pi(\ThumbsUp)\right) \geq \Lambda\left(\ThumbsDown, \pi_0\right),$$

which implies that the right hand side of (8) is nonnegative. Nonnegativity of the left hand side provides the result $\pi(h_{\ThumbsUp,\ThumbsDown}) \geq \pi(h_{\ThumbsDown,\ThumbsUp})$.

Note that $h'$ is summarized in $\pi_0$ in (8). Since the result holds for all $\pi_0$, it will hold for all prior histories $h'$. By monotonicity of the Bayesian update in the belief, the inequality is preserved after history $h''$. This completes the proof of part (a).

*Part (b).* Note that

$$\mathsf{P}\left(R_{t+1} = \ThumbsUp \,|(B_{t+1}, \pi_t, Q) = (1, \pi, q)\right) = G(\ThumbsUp, \pi, q)/\overline{F}_\Theta(\theta(\pi, p))$$

and

$$\mathsf{P}\left(R_{t+1} = \ThumbsDown \,|(B_{t+1}, \pi_t, Q) = (1, \pi, q)\right) = G(\ThumbsDown, \pi, q)/\overline{F}_\Theta(\theta(\pi, p)).$$

For ease of notation, we omit the arguments of the cutoff function $\theta(\pi, p)$ in this proof and write $\theta$

instead. Consider the derivative

$$
\begin{aligned}
&\frac{\partial \mathsf{P}\left(R_{t+1} = \text{👍} \,|(B_{t+1}, \pi_t, Q) = (1, \pi, q)\right)}{\partial \pi} \\
&\quad = (H - L)(\overline{F}_\Theta(\theta))^{-2} \left[\overline{F}_\varepsilon(p - q - \theta) f_\Theta(\theta) \overline{F}_\Theta(\theta) - f_\Theta(\theta) \int_{\theta(\pi,p)}^{\infty} \overline{F}_\varepsilon(p - q - x) f_\Theta(x) \;\mathrm{d}x\right] \\
&\quad = (H - L) f_\Theta(\theta)(\overline{F}_\Theta(\theta))^{-2} \left[\overline{F}_\varepsilon(p - q - \theta) \overline{F}_\Theta(\theta) - \int_{\theta}^{\infty} \overline{F}_\varepsilon(p - q - \theta) f_\Theta(x) \;\mathrm{d}x\right] \\
&\quad < (H - L) f_\Theta(\theta) \overline{F}_\varepsilon(p - q - \theta)(\overline{F}_\Theta(\theta))^{-2} \left[\overline{F}_\Theta(\theta) - \int_{\theta}^{\infty} f_\Theta(x) \;\mathrm{d}x\right] \\
&\quad = 0,
\end{aligned}
$$

where $(H - L) = -\partial\theta(\pi, p)/\partial\pi$, the inequality follows from the fact that $\overline{F}_\Theta$ is decreasing in $x \in [\theta, \infty)$, and the final equality from the definition of survival function. Note that

$$
\mathsf{P}\left(R_{t+1} = \text{👍} \,|(B_{t+1}, \pi_t, Q) = (1, \pi, q)\right) + \mathsf{P}\left(R_{t+1} = \text{👎} \,|(B_{t+1}, \pi_t, Q) = (1, \pi, q)\right) = 1,
$$

and so

$$
\begin{aligned}
\frac{\partial \mathsf{P}\left(R_{t+1} = \text{👎} \,|(B_{t+1}, \pi_t, Q) = (1, \pi, q)\right)}{\partial \pi} &= -\frac{\partial \mathsf{P}\left(R_{t+1} = \text{👍} \,|(B_{t+1}, \pi_t, Q) = (1, \pi, q)\right)}{\partial \pi} \\
&> 0.
\end{aligned}
$$

□

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