Projections and Functions of Nash Equilbria
Preliminary Draft - Not For Circulation

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Abstract

We show that any compact semi-algebraic subset of mixed action profiles on a fixed player set can be represented as the projection of the set of equilibria of a game in which additional binary players have been added. Even stronger, we show that any semi-algebraic continuous function, or even any semi-algebraic upper-semicontinuous correspondence with non-empty values, from a bounded semi-algebraic set to the unit cube can be represented as the projection of an equilibrium correspondence of a game with binary players in which payoffs depend on parameters from domain of the function or correspondence in a multilinear way.

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1 Introduction

As Nash equilibrium is the most fundamental solution concept in game theory, questions about the structure of Nash equilibria have received much attention. It is clear that, given a finite collection of players and action spaces, not every (compact) set of mixed action profiles can arise as the set of Nash equilibria of a game on these players. Hence, it is natural to question which sets *can* arise.

Nash equilibrium was defined by Nash (1950), [9], [10]. In the standard non-cooperative framework, Nash equilibria are those profiles of actions, in the mixed extension of the game (that is, in the extension in which players are allowed to use randomised strategies), against which no player has an incentive to unilaterally deviate. Nash equilibria are always guaranteed to exist (when the player and action spaces are finite), and it follows easily that the the set of Nash equilibria of a game is always compact. In addition, since Nash equilibria are formally defined in terms of polynomial inequalities, the set of equilibria is *semi-algebraic*.

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It is clear from observing the cases of only one or two players that, once the set of players is fixed, not every semi-algebraic and compact subset of mixed action profiles can be the set of Nash equilibria of some game. Even if the player set is larger, once it has been fixed, algebraic techniques can give bounds on the number of components it may possess, or even on the 'complexity' of the individual components. Hence, a notable vein in the literature has been to study the *topological* and/or *algebraic* structures of the set of equilibria. In particular, it is known that every compact connected semi-algebraic set is *homeomorphic* to a connected component of the set of Nash equilibria of some game;[1] see [1] and the references within. Datta had already shown [5] that any algebraic variety (i.e., a set defined via polynomial equalities) is stably isomorphic to the set of completely mixed equilibria of a 3-player game, where this isomorphism notion allows for semi-algebraic homeomorphisms and equivalences of the form $V \times \mathbb{R}^K \sim V$. However, these results leave open questions on whether perhaps any compact semi-algebraic set can arise *precisely* - and not just up to topological or algebraic equivalence - in some way in the universe of Nash equilibria.

This purpose of this paper is indeed to present such a way. More specifically, given a compact semi-algebraic set X of mixed action profiles, we show that one can enlarge the player set by adding finitely many binary players, and define a game G on the larger player set, such that the projection of the set of equilibria of G to the actions of the original players is precisely X. For this purpose, we actually show a stronger result: any semi-algebraic continuous function from a bounded semi-algebraic set to the unit cube can be represented as the projection of an equilibrium correspondence of a game with binary players in which payoffs depend on parameters from the function's domain in a multilinear way. We also generalize this result to upper semi-continuous semi-algebraic correspondences with convex non-empty values.

Section 2 presents the model of games and equilibria, the notion of semi-algebraic sets and some discussion on the restrictions on the set of Nash equilibria. The results are stated in Section 3, along with some discussion, and the proofs are given in Section 4, which begins with an informal outline of the proofs.

2 Games, Algebra, and Equilibria

2.1 Games

For a finite set of players I, with action spaces $(A^i)_{i\in I}$, a game is a mapping $G : \prod_{i\in I} A^i \to \mathbb{R}^I$ which assigns to each action profile a payoff for each player. G extends multi-linearly to action profiles $z \in \prod_{i\in I} \Delta(A^i)$, where $\Delta(A^i)$ denotes the simplex of probability distributions on A^i, by

$$G(z) = \sum_{a=(a^i)_{i\in I} \in \prod_{i\in I} A^i} \Big(\prod_{i\in I} z^i[a^i]\Big) G(a)$$

[1] Semi-algebraic sets always possess finitely many connected components.

We introduce the following notion which will be very useful for us: For $N \in \mathbb{N}$, an $\mathbb{R}^N$*-parametrized game* $G[\cdot](\cdot)$ on a set of players I with action spaces $(A^i)_{i\in I}$ is a game whose payoffs depend on a parameter $x = (x_1, \ldots, x_N)$ in a multi-linear way: I.e., for each action profile $a \in \prod_{i\in I} A^i$, the mapping $(x_1, \ldots, x_N) \to G[x](a)$ is linear in each x_j.

To understand a bit the meaning of an $\mathbb{R}^N$-parametrized game, by denoting $G_t = G(t)$ for each $t \in \{0,1\}^N$, we see that we can express

$$G[x] = \sum_{t\in\{0,1\}^N} \Big(\prod_{k,t^k=1} x^k \prod_{k,t^k=0} (1 - x^k)\Big) G_t$$

Hence, for $x \in [0,1]^N$, one can view the game $G[x]$ as the expected game facing the players as a result of the following process: There are 2^N games, each for one sequence of bits in $\{0,1\}^N$. Nature chooses the N bits independently, the i-th bit with probably $(x_i, 1-x_i)$, and the players simultaneously have to choose their actions; their payoff is then assigned according to the game Nature chose and the actions played.

We adopt several conventions:

If $J \subseteq I$ is a subset of players, $G^J(z)$ denotes the payoffs to the players in J, and z^J denotes the mixed actions of the players in J; formally, $G^J(z) = (G^i(z))_{i\in J}$, $z^J = (z^i)_{i\in J}$.

A *binary player* is a player with two actions, which we think of as 'left' and 'right', and instead of writing a mixed action as $(p, 1-p)$, we denote the mixed action by the single number $p \in [0,1]$.

2.2 Semi-Algebraic Sets and Functions

Let $\mathbb{R}[x_1, \ldots, x_N]$ denote the ring[2] of polynomials in N variables, $x_1, \ldots, x_N$. A semi-algebraic subset of $\mathbb{R}^N$ is a set of the form

$$\cup_{j=1}^{m} \cap_{i=1}^{m_j} \{(x_1, \ldots, x_N) \in \mathbb{R}^n \mid P_{i,j}(x) *_{i,j} 0\} \tag{2.1}$$

for some finite collection $(P_{i,j})_{i,j} \subseteq \mathbb{R}[x_1, \ldots, x_N]$, where for each i, j, $*_{i,j}$ is either $=$ or $>$. In such case, we say that the set is $(P_{i,j})_{i,j}$-semi-algebraic. The semi-algebras sets form an *algebra*: I.e., they are closed under finite unions, finite intersections, and complements.

Equivalently (e.g., [3, Ch. 2]), semi-algebraic sets are those that can be expressed as a formula in first-order logic whose atoms are of the form $P(x) > 0$ or of the form $P(x) = 0$ for some $P \in \mathbb{R}[x_1, \ldots, x_N]$. In particular, we have the Tarski-Seidenberg theorem:

[2] A ring is an algebraic structure with operations of addition and multiplication satisfying certain axioms; we will not need to make use of the specific axioms, which can be found in any introductory text on abstract algebra.

Theorem 2.1. *Let $A \subseteq \mathbb{R}^N$ be semi-algebraic, let $\pi_K : \mathbb{R}^N \to \mathbb{R}^K$ denote the projection to a subset $K \subseteq \{1, \ldots, N\}$ of coordinates. Then $\pi_K(A)$ is semi-algebraic.*

It follows that the composition of semi-algebraic functions is semi-algebraic. We also recall that the closure and boundary of a semi-algebraic set are also semi-algebraic (e.g, [3, Prop. 2.2.2]).

A *semi-algebraic function* $f : A \to \mathbb{R}^K$, where $A \subseteq \mathbb{R}^N$, is one whose graph $Gr(f) := \{(x, y) \in A \times \mathbb{R}^K \mid y = f(x)\}$ is semi-algebraic: It follows that the domain A is semi-algebraic, and that the image / inverse image of a semi-algebraic set under a semi-algebraic function is also semi-algebraic. A *correspondence*, denoted $F : A \Longrightarrow \mathbb{R}^K$, assigns to each $x \in A$ a subset $F(x) \subseteq \mathbb{R}^K$; a correspondence is semi-algebraic if its graph $Gr(F) := \{(x, y) \in A \times \mathbb{R}^K \mid y \in F(x)\}$ is semi-algebraic. Recall also that $F : A \Longrightarrow \mathbb{R}^K$ is called *upper semi-continuous* if $Gr(F)$ is closed in $A \times \mathbb{R}^K$.

2.3 Nash Equilibria

The Nash equilibria of G are those $z \in \prod_{i \in I} \Delta(A^i)$ satisfying

$$G^j(z) \geq G^j(b, z^{-j}), \ \forall j \in I, b \in A^j$$

where $z^{-j} = (z_i)_{i \neq j}$. It is easy to see that the set of Nash equilibria of a game G with action sets $(A^i)_{i \in I}$ are a compact semi-algebraic set; indeed, it is the collection of $z \in \prod_{i \in I} \mathbb{R}^{A^i}$, such that:

$$z^j[b] \geq 0, \ \forall j \in I, b \in A^j$$

$$\sum_{b \in A^j} z^j[b] = 1, \ \forall j \in I$$

$$\sum_{a \in \prod(A^i)_{i \in I}} (\prod_{i \in I} z^i[a^i]) G^j(a) \geq \sum_{a \in \prod(A^i)_{i \in I}, a^j = b} \Big(\prod_{i \in I, i \neq j} z^i[a^i] \Big) G^j(a), \ \forall j \in I, b \in A^j$$

It's easy to see similarly that if G is an $\mathbb{R}^N$-parametrized game, then the correspondence $E_G : \mathbb{R}^N \Longrightarrow \prod_{i \in I} \Delta(A^i)$, where $E_G(x)$ is the Nash equilibria of $G[x]$, is semi-algebraic and upper semi-continuous.

2.4 Sets Which Are Not Sets of Equilibria

To motivate our results, we first discuss limitations to the 'complexity' the set of Nash equilibria can have (given a collection of players).

Clearly, given a finite set of players I with finite action spaces $(A^i)_{i \in I}$, not every compact semi-algebraic subset X of $\prod_{i \in I} \Delta(A^i)$ can be the set of equilibria E of some game. For example, if I consists of a single player, then E must be the convex hull of pure strategies. If I consists of two players, then it is easy to show that E must be the finite union products of the form $S_1 \times S_2$ with

S_j being a convex polytype in $\Delta(A_j)$, [6]. Even for more players, not every compact semi-algebraic subset of the space of mixed actions need be the set of equilibria of some game. First of all, there is the issue of the number of connected components.

Proposition 2.1. *There is a function $\phi : \mathbb{N}^3 \to \mathbb{N}$ such that the set of solutions of r polynomial equalities and inequalities in N variables of degrees*[3] *at most $d \geq 2$ has at most $\phi(r, N, d)$ connected components.*

Although we will not need it, we remark that a crude bound is $\phi(r, N, d) = d(2d-1)^{N+r-1}$, [4]; a better bound of $r^N \cdot O(d)^N$ is given in [12].

In particular, we deduce from Section 2.3, since the set of Nash equilibria are defined via $|I| + 2\sum_{j\in I} |A^j|$ inequalities and equalities, that:

Corollary 2.2. *There is a function $\psi : \mathbb{N}^{I+1} \to \mathbb{N}$ such that the number of connected components of the set of Nash equilibria is at most $\psi(|I|, (|A^i|)_{i\in I})$.*

We can also always find *connected* sets which *cannot* be the set (or even a component) of the set of Nash equilibria for the players I with action spaces $(A^i)_{i\in I}$. Again applying Proposition 2.1, we can deduce:

Corollary 2.3. *There is a function $\psi' : \mathbb{N}^{I+1} \to \mathbb{N}$ such that if E is the set of Nash equilibria and P is an affine space of co-dimension* 1,[4] *then $E \cap P$ has at most $\psi'(|I|, (|A^i|)_{i\in I})$ components.*

Indeed, the restriction to P requires adding a single additional equality.

Hence, for example, for $k \in \mathbb{N}$, define the function $f_k : [0,1] \to [0,1]$ by

$$f_k(x) = 2k \cdot \min\{|x - \frac{1}{k} \cdot n| \mid n \in \mathbb{Z}\}$$

and let $L = \{(x,y) \in \mathbb{R}^2 \mid y = \frac{1}{2}\}$. (See Figure 1.) Let $V : \mathbb{R}^2 \to \prod_{i\in I} \mathbb{R}^{A^i}$ be an injective linear map such that $V([0,1]^2) \subseteq \prod_{i\in I} \Delta(A^i)$. Then for $k > \frac{1}{2}\psi'(|I|, (|A^i|)_{i\in I})$, $V(Gr(f_k))$ cannot be the set of equilibria of any game, since $V(Gr(f_k)) \cap P$ has $2k$ components (all singletons) for any affine space P such that $V(\mathbb{R}^2) \cap P = V(L)$.

3 Results

The main result of this paper is:

Theorem 3.1. *Let I be a finite set of players with finite sets $(A^i)_{i\in I}$ of actions, and let $X \subseteq \prod_i \Delta(A^i)$ be compact and semi-algebraic. Then there exists a set*

[3] The degree of a monomial is the sum of the degrees from all variables; e.g., the degree of x^3y^2z is 6.

[4] That is, $P \subseteq \prod_{i\in I} \mathbb{R}^{A^i}$ is of the form $u + V$, where V is of dimension one less than the space $\prod_{i\in I} \mathbb{R}^{A^i}$.

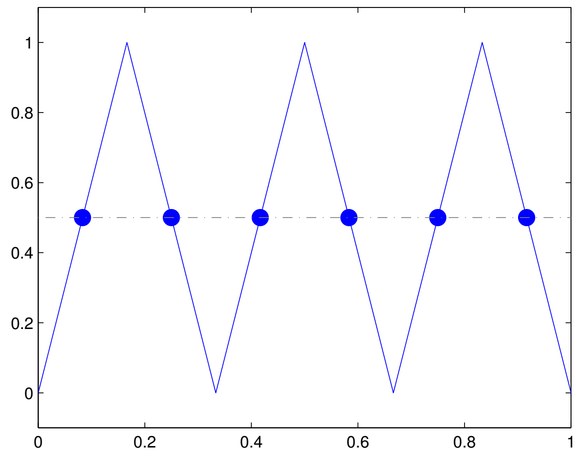

Figure 1: The Function f_3 with the line $L = \{(x, y) \mid y = \frac{1}{2}\}$.

of binary players $\mathcal{P}$, and a game G on the player set $I \cup \mathcal{P}$ such that if E is the set of equilibria of G, then the projection of E to $\prod_i A^i$ is X; more precisely,

$$X = \{(z^i)_{i \in I} \mid z \in \prod_{i \in I} \Delta(A^i) \times \prod_{j \in \mathcal{P}} \Delta(\{left, right\}) \textit{ is an equilibrium of } G\}$$

From the examples and arguments in Section 2.4, we deduce that the size of the set of additional binary players $\mathcal{P}$ in Theorem 3.1 cannot be bounded as a function only of I and the $(A^i)_{i \in I}$, but may be arbitrary large as a function of the given compact semi-algebraic set X (even when X is connected). Our proofs, in attempt to keep things simple (the proofs will be detailed enough as is!), have not made an attempts to derive a bound on the size of $\mathcal{P}$ as a function of X (e.g., on the number of polynomials needed to define X), and we leave this matter to future research.

The main step in proving Theorem 3.1 is the following theorem, which is of interest in itself:

Theorem 3.2. *Let $A \subseteq \mathbb{R}^N$ be compact and semi-algebraic, and let $f : A \to [0, 1]^K$ be a continuous semi-algebraic function. Then there exists an $\mathbb{R}^N$-parametrized game G on a set of binary players $\{\alpha_1, \ldots, \alpha_K\} \cup \mathcal{P}$ such that for each $x \in A$, in any equilibrium z of $G[x]$, we have $(z^{\alpha_1}, \ldots, z^{\alpha_K}) = f(x)$.*

In other words, any semi-algebraic function on a compact semi-algebraic set can be realised as the projection of the equilibrium correspondence of a game with binary players in which payoffs depend multilinearly on coordinates from the function's domain.

Theorem 3.2 was proven by the author in [8] for the case $A = [0,1]$, $K = 1$, and f which is piece-wise linear. We remark that the continuity is necessary, since the equilibrium correspondence is upper-semicontinuous.

We will in fact strengthen Theorem 3.2 (although this is not needed for the proof of Theorem 3.1):

Theorem 3.3. *Let $A \subseteq \mathbb{R}^N$ be bounded and semi-algebraic, let $F : A \Longrightarrow [0,1]^K$ be an upper semi-continuous semi-algebraic correspondence with non-empty convex values (i.e., $\forall x \in A,\ F(x) \neq \emptyset$ and is convex). Then there exists a $\mathbb{R}^N$-parametrized game G on a set of binary players $\{\alpha_1, \ldots, \alpha_K\} \cup \mathcal{P}$ such that for each $x \in A$,*

$$F(x) = \{(z^{\alpha_1}, \ldots, z^{\alpha_K}) \mid z \text{ is an equilibrium of } G[x]\} \tag{3.1}$$

In other words, any semi-algebraic u.s.c. correspondence with non-empty values on a bounded semi-algebraic set can be realised as the projection of the equilibrium correspondence of a game with binary players in which payoffs depend multilinearly on coordinates from the function's domain.

The theorem in particular implies that the compactness of domain in Theorem 3.2 is not needed; only boundedness. Clearly, by the same reasoning above, the upper-semicontinuity is necessary.

We remark that this theorem is not true if the convexity assumption is dropped. Let $F : [0,1] \to [0,1]$ be defined by

$$F(x) = \begin{cases} \{0\} & \text{if } x < \frac{1}{2} \\ \{0,1\} & \text{if } x = \frac{1}{2} \\ \{1\} & \text{if } x > \frac{1}{2} \end{cases}$$

Suppose the conclusion of the theorem held for this F. Then there is a connected component C of the of set of equilibria of $G[\frac{1}{2}]$ which is stable, i.e., for any neighbourhood V of C, there is a neighbourhood U of $\frac{1}{2}$ such that for $y \in U$, $G[y]$ contains equilibria in V; see [7]. Hence, for any $\varepsilon > 0$ there is $\delta > 0$ such that for any x_1, x_2, with $\frac{1}{2} - \delta < x_1 < \frac{1}{2} < x_2 < \frac{1}{2} + \delta$, $G[x_1], G[x_2]$ either must both contain equilibria y with $y^\alpha < \varepsilon$, or must both contain equilibria y with $y^\alpha > 1 - \varepsilon$.

It appears that the convexity assumption may be weakened somewhat, but it is left for future research to determine the full class of correspondences for which the result holds.

4 Proofs

Throughout, we will denote $B^N_M = (-M, M)^N$ for $N \in \mathbb{N}$, $M > 0$.

4.1 Informal Outline of Proof

As mentioned, to prove Theorem 3.1, the main step is to prove Theorem 3.2. First, in Section 4.2, we define those functions and correspondence (in the case of one-dimensional range) for which Theorem 3.2 and 3.3 hold as *exactly representable*, or just *representable* if some affine transformation of it is exactly representable. That section goes on demonstrate that the representable functions (on a given bounded set) include all polynomials, is closed under compositions, and forms an algebra over $\mathbb{R}$.[5] Furthermore, the notion of representability is extended to sets - more precisely, of one set A w.r.t. to another set B - if an appropriate version of the indicator function of A (one which is 'indifferent' on the relative boundary of A in B) is representable. Section 4.3 then shows that to represent a function, all we actually need is to represent sets - specifically, to represent those sets of points above/below the graph of the function (see See Figure 2).

Section 4.4 provides a main tool we will use, that of a cylindrical algebraic decomposition, C.A.D., of the Euclidian space, which allows us to view any semi-algebraic set as the union of *cells* of a well-behaved partition of the entire space. This then facilitates the core of the proof in Section 4.5, which (in somewhat informal terms) shows inductively on the dimension N that: (i) A cell of a C.A.D. of $\mathbb{R}^N$ (which satisfies the additional condition of being *stratified*) can be represented w.r.t. the union of cells of the same or lower dimension. (ii) Any set in $\mathbb{R}^N$ is representable. (iii) From these and Section 4.3, it follows easily that any function on an $N-1$ dimension space (that is, whose graph is contained in $\mathbb{R}^N$) is representable. The case of functions with multi-dimension range is then handled by treating each coordinate independently.

Once Theorem 3.2 has been satisfied, it follows that the set of profiles which is image of a semi-algebraic continuous function on $[0,1]^N$ can be the set of equilibria of some game; one represents the function as in Theorem 3.2 by a $\mathbb{R}^N$-parametrized game $G[x]$, and then adds N auxiliary players who are indifferent and who choose the input coordinates; because of their indifference, their equilibria profiles range over the entire unit cube. (One also needs to transition from binary players to players with more general action spaces; Lemma 4.2 handles this matter.) Since any compact semi-algebraic set is the union of such images, one then represents each function in this way and then have a collection of players who play a game with finitely many equilibria 'choose the function': each equilibrium for these players corresponds to a function.

Follow these, the proof of Theorem 3.3 is shown in two steps: First in Section 4.7 in the case when the range is one-dimensional ($K = 1$), which is accomplished by extending the correspondence to an open set and then using the same trick of representing the sets of points above / below the graph; and then in Section 4.8 for range of arbitrary dimension in an essentially inductive manner.

4.2 Representable Functions & Sets

To facilitate the proofs, we introduce the following concept:

Definition 4.1. *A continuous function* $f : A \to [0, 1]$ *on a semi-algebraic subset*

[5] That is, closed under multiplication by scalar, and closed under addition and multiplication.

$A \subseteq \mathbb{R}^N$ will be called exactly representable if there is an $\mathbb{R}^N$-parametrized game G_f on a set of binary players $\{\alpha_f\} \cup \mathcal{P}_f$ such that for any $x \in A$ and any equilibrium z of $G_f[x]$, $z^{\alpha_f} = f(x)$.
It will be called representable if for some $a, b \in \mathbb{R}$ with $a \neq 0$, $a \cdot f + b$ is exactly representable.
More generally, an u.s.c. correspondence $F : A \Longrightarrow [0,1]$ will be called exactly representable if there is an $\mathbb{R}^N$-parametrized game G_F on a set of binary players $\{\alpha_F\} \cup \mathcal{P}_F$ such that for any $x \in A$, if E_x is the set of equilibria of $G_F[x]$, the projection of E_x to the action of α_F, $\{z^{\alpha_F} \mid z \in E_x\}$, is $F(x)$.
Similarly we define F to be representable if for some such a, b, $a \cdot F + b$ is exactly representable.

Theorem 3.2 states in particular (the case $K = 1$) that every continuous semi-algebraic function from a compact semi-algebraic set to $[0,1]$ is exactly representable. To simplify notation, if $\alpha_1, \ldots, \alpha_n$ are players and $\mathcal{P}_1, \ldots, \mathcal{P}_k$ are sets of players, we write $(\alpha_1, \ldots, \alpha_n, \mathcal{P}_1, \ldots, \mathcal{P}_k)$ instead of $\{\alpha_1, \ldots, \alpha_n\} \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_k$; and when writing $(\alpha_1, \ldots, \alpha_n, \mathcal{P}_1, \ldots, \mathcal{P}_k)$, it is understood that the α_j are individual players while the $\mathcal{P}_j$ are collections of players who we do not have a need to name or single out individually.

Lemma 4.2. *Given a finite set B of actions, there is an $\mathbb{R}^B$-parametrized $(2|B|+1)$-player game, $G_\aleph[x]$, where Player α has action set B and the players $(\beta^j, \gamma^j)_{j \in B}$ are binary, such that for $x \in \Delta(B)$, $G_\aleph[x]$ has a unique equilibrium, z, and it is such that $z^\alpha = x$.*

Proof. First, introduce the following useful $\mathbb{R}$-parametrized game:

$$H(q) = \begin{array}{|c|c|}\hline 1, -1 & 1 - 4q, 3 - 4q \\ \hline 4q - 3, 4q - 1 & 1, -1 \\ \hline \end{array}$$

For $0 < q < 1$, the unique equilibrium is $(q, 1-q) \otimes (q, 1-q)$.

For each $b \in B$, $x \in \Delta(B)$, and action profile y for the players $(\alpha, (\beta_b, \gamma_b)_{b \in B})$, define[6]

$$G_\aleph^{\beta_b, \gamma_b}[x](y) = H(q(x,y)), \quad q(x,y) = \frac{1}{3}(x[b] - y^\alpha[b]) + \frac{1}{2}$$

(Observe that $0 < q(x,y) < 1$.) Also define

$$G_\aleph^\alpha[x](b, y^{-\alpha}) = y^{\beta_b} - \frac{1}{2}$$

Clearly, in any equilibrium z,

$$z^{\beta_b} = z^{\gamma_b} = \frac{1}{3}(x[b] - z^\alpha[b]) + \frac{1}{2} \tag{4.1}$$

Since $z^\alpha, x \in \Delta(B)$, if $z^\alpha \neq x$, then we must have some $b^* \in B$ with $z^\alpha[b^*] > x[b^*]$ and some $b^o \in B$ with $z^\alpha[b^o] < x[b^o]$; but then we would have

$$G_\aleph^\alpha[x](b^o, z^{-\alpha}) = \frac{1}{3}(x[b^o] - z^\alpha[b^o]) > 0 > \frac{1}{3}(x[b^*] - z^\alpha[b^*]) = G_\aleph^\alpha[x](b^*, z^{-\alpha})$$

[6] β_b corresponds to the row player, γ_b to the column player.

Since z is an equilibrium, $z^\alpha[b^*] = 0 \leq x[b^*]$, a contradiction. The uniqueness of equilibrium now follows from (4.1). $\square$

Corollary 4.3. *The identity function on* $[0,1]$ *is exactly representable. The* N *projection mappings* $(x_1, \ldots, x_N) \to x_k$ *are exactly representable* $[0,1]^N$.

Proof. Define for $x \in [0,1]$ and action profile z of the players on the same player set $(\alpha, \mathcal{P})$ as in $G_\aleph$ in Lemma 4.2, where α is now also a binary player,[7]

$$G_{id}[x](z) = G_\aleph[x, 1-x](z)$$

Then clearly G exactly represents the identity on $[0,1]$ via Player α; to represent the k-th projection mapping, take $G_k[x_1, \ldots, x_N](z) := G_{id}[x_k](z)$. $\square$

The next two easy corollaries get the difference between exactly representable and representable out of the way.

Corollary 4.4. *Suppose* $f : A \to \mathbb{R}$ *for* $A \subseteq \mathbb{R}^N$ *is exactly representable,* $a, b \in \mathbb{R}$ *such that* $0 \leq a \cdot f + b \leq 1$. *Then* $a \cdot f + b$ *is exactly representable. The same holds for correspondences.*

Proof. We prove the corollary for functions; for correspondences the result follows similarly. Let G_f exactly represent f on a set of binary players $(\alpha_f, \mathcal{P}_f)$. Denote $g = a \cdot f + b$. Let G_{id} exactly represent the identity function on $[0,1]$ on a set of binary players $(\alpha_{id}, \mathcal{P}_{id})$. Define on the set $(\alpha_f, \alpha_{id}, \mathcal{P}_f, \mathcal{P}_{id})$

$$G_g^{\alpha_f, \mathcal{P}_f}[x](z) = G_f[x](z^{\alpha_f, \mathcal{P}_f})$$
$$G_g^{\alpha_{id}, \mathcal{P}_{id}}[x](z) = G_{id}[a \cdot z^{\alpha_f} + b](z^{\alpha_{id}, \mathcal{P}_{id}})$$

Then G_g exactly represents g (via the Player α_{id}). $\square$

Corollary 4.5. *A representable function is exactly representable iff it takes values in* $[0,1]$. *A function* f *is representable iff for any* $a, b \in \mathbb{R}$, *with* $0 \leq a \cdot f + b \leq 1$, $a \cdot f + b$ *is exactly representable. The same facts hold for correspondences.*

The rest of this section is dedicated to building up the family of representable functions and correspondences, and to define the notion of a set being representable w.r.t. another set.

Proposition 4.6. *The identity function is representable on any bounded set. The projection mappings* $(x_1, \ldots, x_N) \to x_k$ *are representable on any bounded set.*

Proof. Let $G_{id,0}$ represent the identity on $[0,1]$. To represent it in $[a,b]$ (or any set contained in it), set

$$G_{id}[x](z) = G_{id,0}[\frac{x-a}{b-a}](z)$$

This game exactly represents $\frac{1}{b-a}(id - a)$. A similar argument for representing the projection mappings in a bounded set $[a,b]^N$ holds. $\square$

[7] We could have also used the column player from the $\mathbb{R}$-parametrized game $H[\cdot]$ above; the row player fails for $q = 0$ or $q = 1$.

For correspondences $F, H : A \Longrightarrow \mathbb{R}$, and an operation $*$ which is either $+$, $-$, or $\cdot$, define

$$(F * H)(x) = F(x) * H(x) = \{y * z \mid y \in F(x), z \in H(x)\}$$

and if $F_1, \ldots, F_K : \mathbb{R}^N \Longrightarrow \mathbb{R}$ and $H : \mathbb{R}^K \Longrightarrow \mathbb{R}$, then the composition $H \circ (F_1, \ldots, F_K)$ is defined as

$$(H \circ (F_1, \ldots, F_K))(x) = \{y \in \mathbb{R} \mid \exists z_1, \ldots, z_K \in \mathbb{R} \text{ s.t. } y \in H(z_1, \ldots, z_K) \text{ and } \forall j, z_j \in F_j(x)\}$$

Proposition 4.7. *The composition of representable functions is representable. The composition of representable correspondences is a representable correspondence.*

Proof. We prove the version for functions, the version of correspondences being similar: Let $f_1, \ldots, f_K, g$ be representable functions, such that for some $A, B \in \mathbb{R}$, $A \neq 0$, $A \cdot f_1 + B, \ldots, A \cdot f_K + B, A \cdot g + B$ are exactly represented by $\mathbb{R}^N$-parametrized games $G_1, \ldots, G_K$ and a $[0,1]^K$-parametrized game G_g, on player sets $(\alpha_j, \mathcal{P}_j)$, $j = 1, \ldots, K$, and $(\alpha_g, \mathcal{P}_g)$, respectively. Define G on the set of players $(\alpha_g, (\alpha_j)_j, \mathcal{P}_g, (\mathcal{P}_j)_j)$ by

$$G^{\alpha_j, \mathcal{P}_j}[x](z) = G_j[x](z^{\alpha_j, \mathcal{P}_j}),\ j = 1, \ldots, K$$

$$G^{\alpha_g, \mathcal{P}_g}[x](z) = G_g\big[\frac{1}{A}(z^{\alpha_1} - B), \ldots, \frac{1}{A}(z^{\alpha_K} - B)\big](z^{\alpha_g, \mathcal{P}_g})$$

Clearly G represents $g \circ (f_1, \ldots, f_K)$ (via the player α_g), since in any equilibrium z of $G[x]$, $\forall j, z^{\alpha_j} = A \cdot f_j(x) + B$. $\square$

Proposition 4.8. *The mappings on $\mathbb{R}^2 \to \mathbb{R}$ given by $(x_1, x_2) \to x_1 + x_2, x_1 - x_2, x_1 \cdot x_2$ are representable on any bounded subset of $\mathbb{R}^2$.*

Proof. Let $[a, b]^2$ contain the domain. Let $G_{id,0}, G_1, G_2$, be the $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^2$-parametrized games on player sets $(\alpha_{id}, \mathcal{P}_{id})$, $(\alpha_1, \mathcal{P}_1)$, $(\alpha_2, \mathcal{P}_2)$ which exactly represent the identity and the projections to the first and second coordinates in $[0,1], [0,1]^2, [0,1]^2$, respectively. Define the $\mathbb{R}^2$-parametrized game G on $(\alpha_{id}, \alpha_1, \alpha_2, \mathcal{P}_{id}, \mathcal{P}_1, \mathcal{P}_2)$,

$$G^{\alpha_j, \mathcal{P}_j}[x_1, x_2](z) = G_j[\frac{x_1 - a}{b - a}, \frac{x_2 - a}{b - a}](z^{\alpha_j, \mathcal{P}_j}),\ j = 1, 2$$

and

$$G^{\alpha_{id}, \mathcal{P}_{id}}[x](z) = G_{id}[\frac{1}{3}(z^{\alpha_1} * z^{\alpha_2} + 1)](z^{\alpha_{id}, \mathcal{P}_{id}})$$

where $* \in \{+, -, \cdot\}$. Note that regardless of which operation $*$ is, $\frac{1}{3}(x * y + 1) \in [0,1]$. Hence, if $* \in \{+, -\}$, G represents the $*$ operation, since for any equilibrium z of $G[x_1, x_2]$, we have

$$z^{\alpha_{id}} = \frac{1}{3}(z^{\alpha_1} + z^{\alpha_2} + 1) = \frac{1}{3}(\frac{1}{b-a}(x_1 + x_2 - 2a) + 1) \text{ (if } * = +)$$

or

$$z^{\alpha_{id}} = \frac{1}{3}(z^{\alpha_1} - z^{\alpha_2} + 1) = \frac{1}{3}(\frac{1}{b-a}(x_1 - x_2) + 1) \text{ (if } * = -)$$

and we have already shown the class of representable functions to be closed under addition or multiplication by a constant. In particular, if $L(x_1, x_2) = Ax_1 + Bx_2 + C$ for some $A, B, C \in \mathbb{R}$, L is representable. For multiplication,

$$z^{\alpha_{id}} = \frac{1}{3}(z^{\alpha_1} \cdot z^{\alpha_2} + 1) = \frac{1}{3}((\frac{1}{b-a})^2(x_1 - a)(x_2 - a) + 1) = \frac{1}{3(b-a)^2}x_1x_2 + L(x_1, x_2)$$

for some $L(x_1, x_2)$ as above, which is hence representable, and from what we've shown above, the representability of multiplication in $[a, b]^2$ follows. ☐

Corollary 4.9. *The sum, difference, and product of representable functions or correspondences is representable.*

Corollary 4.10. *Every polynomial is representable on any bounded set.*

For semi-algebraic sets $A, B \subseteq \mathbb{R}^N$, we let $\Xi\{A/B\} : B \Longrightarrow [0, 1]$ denote the correspondence defined by

$$\Xi\{A/B\}(x) = \begin{cases} \{0\} & \text{if } x \notin \overline{A} \\ [0,1] & \text{if } x \in \partial^B A \\ \{1\} & \text{if } x \in A \backslash \partial^B A \end{cases}$$

where $\partial^B A$ denotes the relative boundary of A in B; i.e., those $x \in B$ such that for each neighbourhood $U \subseteq \mathbb{R}^N$ of x, U intersects both $A \cap B$ and $B \backslash A$.[8]

Definition 4.11. *We will say that A is representable w.r.t. B if the correspondence $\Xi\{A/B\} : B \Longrightarrow [0, 1]$ is representable.*[9]

It is immediate from our above results that:

Proposition 4.12. *Let $V : \mathbb{R}^N \to \mathbb{R}^N$ be a linear automorphism.*[10] *Then A is representable w.r.t. B iff $V(A)$ is representable w.r.t. $V(B)$.*

Proposition 4.13. *The set $\leq = \{(x, z) \in \mathbb{R}^2 \mid x \leq z\}$ is representable in $\mathbb{R}^2$.*

We denote $\leq$ instead of $\Xi\{\leq, \mathbb{R}^2\}$.

Proof. Define the following $\mathbb{R}^2$-parametrized game G with a single player α:

$$G^\alpha(x, z)(a) = \boxed{z \mid x}$$

Clearly this game represents $\leq$. ☐

[8] The proof that the relative boundary of a semi-algebraic set w.r.t. a semi-algebraic set is also semi-algebraic a standard application of the Tarski-Seidenberg theorem (Theorem 2.1) and we do not repeat it; see, e.g., the proof of [3, Prop. 2.2.2].

[9] Since $0 \leq \Xi\{A/B\} \leq 1$, this is the same as requiring it to be exactly representable.

[10] I.e., a bijective linear mapping.

Corollary 4.14. *The absolute-value function is representable on any bounded subset of* $\mathbb{R}$.

Proof. Clearly,[11] $|x| = (2 \cdot \leq (0, x) - 1) \cdot x$. $\square$

Hence, it follows by Corollary 4.9, Corollary 4.14, and Propositions 4.7:

Corollary 4.15. *The functions* $\mathbb{R}^N \to [0,1]$ *given by* $(x_1, \ldots, x_N) \to \min[x_1, \ldots, x_N]$ *and* $\to \max[x_1, \ldots, x_N]$ *are representable on any bounded subset of* $\mathbb{R}^N$.

4.3 From Representable Sets to Functions

Lemma 4.16. *Let* $B \subseteq \mathbb{R}^N$ *be open and semi-algebraic, and let* $f : B \to \mathbb{R}$ *be continuous such that the sets*

$$B_+ = \{(x, y) \in B \times [0, 1] \mid y > f(x)\}$$

$$B_- = \{(x, y) \in B \times [0, 1] \mid y < f(x)\}$$

are representable w.r.t. $B \times (-M, M)$ *for some* $M > \max_{x \in B} |f(x)|$. *Then* f *is representable. If* $0 \leq f \leq 1$, *then* f *is exactly representable.*

Proof. By Proposition 4.12 and Corollary 4.5, and by observing the normalisation $\frac{1}{2M} f + \frac{1}{2}$ for some such M, we may assume $0 < f < 1$. Define $F : B \times (0, 1) \Longrightarrow \mathbb{R}$ by

$$F = \Xi\{B_- / B \times (0, 1)\} - \Xi\{B_+ / B \times (0, 1)\}$$

By assumption, and by Propositions 4.8 and 4.7, there is an $\mathbb{R}^{N+1}$-parametrized game G_F on a set of binary players $(\alpha_F, \mathcal{P}_F)$ which exactly represents $\frac{1}{2}(F+1)$. Define a $\mathbb{R}^N$-parametrized game G on a set of binary players $(\alpha_f, \alpha_F, \mathcal{P}_F)$ by

$$G^{\alpha_F, \mathcal{P}_F}[x_1, \ldots, x_n](z) = G_F[x_1, \ldots, x_n, z^{\alpha_f}](z^{\alpha_F, \mathcal{P}_F})$$

$$G^{\alpha_f}[x_1, \ldots, x_n](z) = \boxed{2 \cdot z^{\alpha_F} - 1 \mid 0}$$

G then represents f. (See Figure 2.) Indeed, let z be an equilibrium. If $z^{\alpha_f} < f(x) (< 1)$, then (x, z^{α_f}) is in the interior of B_-,[12], and hence also $x \notin \overline{B_+}$, so $F(x, z^{\alpha_f}) = 1$. Therefore, $z^{\alpha_F} = 1$, and therefore $z^{\alpha_f} = 1$, a contradiction. A similar contradiction is reached if $z^{\alpha_f} > f(x) (> 0)$.

$\square$

[11] Although the right-hand side is technically a correspondence, it is single valued.

[12] It is here we use the fact that B is open.

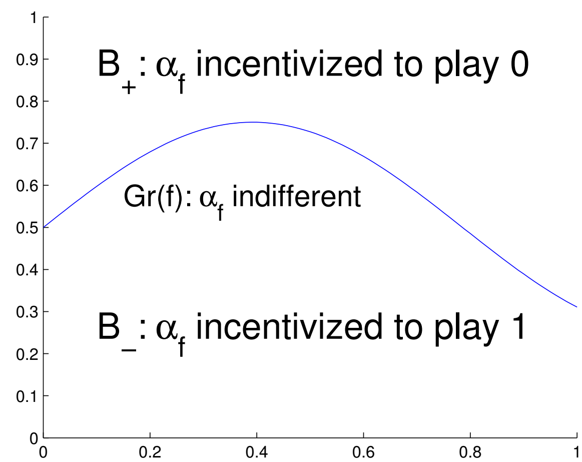

Figure 2: The sets B_+, B_-.

4.4 Cylindrical Algebraic Decompositions & Lojasiewicz's Inequality

A Cylindrical Algebraic Decomposition (C.A.D.) of $\mathbb{R}^N$ which is adapted to a finite set of polynomials $\mathcal{R}$ in $\mathbb{R}[x_1, \dots, x_N]$ is a sequence $\mathcal{C}_1, \dots, \mathcal{C}_N$ where for $1 \leq K \leq N$, $\mathcal{C}_K$ is a finite partition of $\mathbb{R}^K$ into semialgebraic subsets (called *cells*), satisfying:

- Each cell in $\mathcal{C}_1$ is either a point or an open interval.
- For every K, $1 \leq K < N$, and every $C \in \mathcal{C}_K$, there are finitely many continuous semi-algebraic functions $\xi_{C,1} < \xi_{C,2} < \cdots < \xi_{C,\ell_C} : C \to \mathbb{R}$, such that the cylinder $C \times \mathbb{R} \subseteq \mathbb{R}^{K+1}$ is the disjoint unions of cells of $\mathcal{C}_{K+1}$ which are each either:
  - a graph of one of the functions $\xi_{C,j}$, i.e., of the form
  $$A_{C,j} = \{(x', x_{K+1}) \in C \times \mathbb{R} \mid x_{K+1} = \xi_{C,j}(x')\} \tag{4.2}$$
  - or a band of cylinder bounded from below and from above by the graphs of the function $\xi_{C,j}$ and $\xi_{C,j+1}$, where we take $\xi_{C,0} = -\infty$ and $\xi_{C,\ell_C+1} = \infty$:
  $$B_{C,j} = \{(x', x_{K+1}) \in C \times \mathbb{R} \mid \xi_{C,j}(x') < x_{K+1} < \xi_{C,j+1}(x')\} \tag{4.3}$$
- The sign of each polynomial in $\mathcal{R}$ is constant in each cell in $\mathcal{C}_N$.[13] In particular, any $\mathcal{R}$-semi-algebraic set is the union of cells of $\mathcal{C}_N$.

[13] The sign is either > 0, $= 0$, or < 0; if $P(x) > 0$ (resp. $P(x) = 0$, resp. $P(x) < 0$), then $sign(P(x)) = 1$ (resp. 0, resp. -1).

The following can be found, e.g., as [2, Thm. 5.6]:

Theorem 4.1. *For any finite collection $\mathcal{R}$ of polynomials in $\mathbb{R}[x_1, \ldots, x_N]$, there exist a C.A.D. adapted to $\mathcal{R}$.*

Inductively, we can define the *dimension* of cells: In $\mathcal{C}_1$, the dimension of a point is 0, while the dimension of an open interval is 1. For $1 < K \leq N$, if C is a cell in $\mathcal{C}_K$ of dimension d, then $A_{C,j}$ given by (4.2) is also of dimension d, while $B_{C,j}$ given by (4.3) is of dimension $d + 1$.

For a semi-algebraic set A, a C.A.D. is *A-adapted* if A is the union of cells in the C.A.D.. (See Figure 3.[14]) Clearly, if A is $\mathcal{R}$-semi-algebraic, then A is the union of cells of any C.A.D. which is $\mathcal{R}$-adapted. The *dimension of a semi-algebraic set A* is defined as the largest dimension of any cell contained in A of a C.A.D. which is A-adapted; it can be shown that this definition is independent of the choice of the C.A.D..

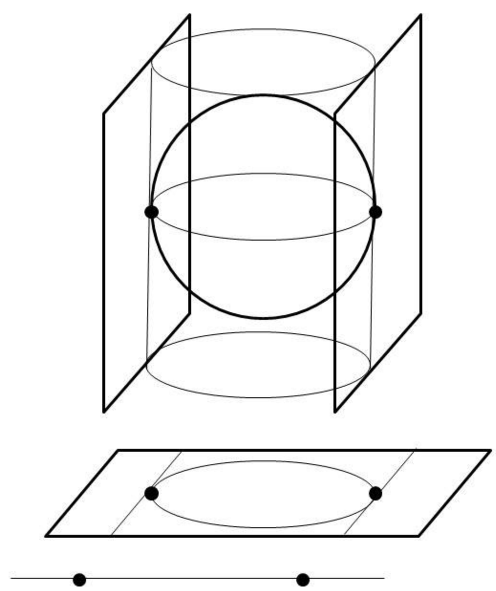

Figure 3: C.A.D. adapted to the sphere.

[14]The author is grateful to Bary Pradelski for creating this image, which was inspired by Figure 5.1 of [2].

We remark at this point that it is generally *NOT* true for polynomials $Q_1, \ldots, Q_m$ that

$$\overline{\{x \mid \forall j, Q_j(x) > 0\}} = \{x \mid \forall j, Q_j(x) \geq 0\}$$

As such, we will need a more refined type of C.A.D., and we will need to take care throughout.

A C.A.D. is called *stratified* if every cell is a smooth semi-algebraic manifold,[15] and the closure of each cell is the union of itself and a finite union of (strictly) lower-dimensional cells. By Propositions 5.39, 5.40, 5.41 of [2]:

Proposition 4.17. *Let $\mathcal{R} \subseteq \mathbb{R}[x_1, \ldots, x_N]$ be finite. Then there exists a linear automorphism $V : \mathbb{R}^N \to \mathbb{R}^N$ and collections of polynomials $\mathcal{Q}_1, \ldots, \mathcal{Q}_N$, with $\mathcal{Q}_K \subseteq \mathbb{R}[x_1, \ldots, x_K]$, such that $P \circ V \in \mathcal{Q} := \cup_j \mathcal{Q}_j$ for all $P \in \mathcal{R}$, and there is a C.A.D. $\mathcal{C}_1, \ldots, \mathcal{C}_N$ of $\mathbb{R}^N$ which is $\mathcal{Q}$-adapted such that:*

- *The C.A.D. is stratified.*
- *The cells in $\mathcal{C}_K$ are precisely all the (non-empty) sets of the form*

$$\{x \in \mathbb{R}^K \mid \forall Q \in \mathcal{Q}_K, sign(Q(x)) = \sigma(Q)\} \tag{4.4}$$

  for some $\sigma \in \{-1, 0, 1\}^{\mathcal{Q}_K}$ for which the above set is non-empty.

- *For a (non-empty) cell $C \in \mathcal{C}_K$ of the form (4.4),*

$$\overline{C} = \{x \in \mathbb{R}^K \mid \forall Q \in \mathcal{Q}_K, sign(Q(x)) \in \overline{\sigma(Q)}\} \tag{4.5}$$

  where $\overline{-1} = \{0, -1\}$, $\overline{1} = \{0, 1\}$, $\overline{0} = \{0\}$,

- *For a (non-empty) cell $C \in \mathcal{C}_K$, the functions $(\xi_{C,j})_j$ in (4.2) and (4.3) extend to continuous semi-algebra functions $(\xi_{\overline{C},j})_j$ on $\overline{C}$ and furtheremore*

$$\overline{A_{C,j}} = \{(x', x_{K+1}) \in \overline{C} \times \mathbb{R} \mid x_{K+1} = \xi_{\overline{C},j}(x')\} \tag{4.6}$$

$$\overline{B_{C,j}} = \{(x', x_{K+1}) \in \overline{C} \times \mathbb{R} \mid \xi_{\overline{C},j}(x') \leq x_{K+1} \leq \xi_{\overline{C},j+1}(x')\} \tag{4.7}$$

We recall Lojasiewicz's inequality (e.g., [3, Thm 2.6.6]):

Proposition 4.18. *Let A be a locally closed*[16] *and bounded semi-algebraic set and let $f, g : A \to \mathbb{R}$ be semi-algebraic and continuous, such that for $x \in A$, $f(x) = 0$ implies $g(x) = 0$. Then there is $K \in \mathbb{N}$ and $c > 0$ such that $g^K \leq c \cdot f$ on A.*

[15] That is, for each cell C and each $x \in C$, there is a neighbourhood U of x in C and a semi-algebraic infinitely continuously differentiable diffeomorphism of U with an open subset of $\mathbb{R}^{dim(C)}$.

[16] I.e., the intersection of a closed and open set.

Corollary 4.19. *Let $\mathcal{C}_1, \dots, \mathcal{C}_N$ be a stratified C.A.D. of $\mathbb{R}^N$, let $1 \le K < N$, and let $C \in \mathcal{C}_N$ be a K-dimensional cell. Let $\mathcal{D}_K$ denote the union of all cells of dimension $\le K$.*[17] *Let $\zeta : C \to \mathbb{R}$ be semi-algebraic and continuous with $\zeta > 0$. Then for each $M > 0$, there is a polynomial Q satisfying $Q(x) = 0$ for $x \in \partial^{\mathcal{D}_K} C$, and $0 < Q(x) < \zeta(x)$ for $x \in C \cap B_M^N$.*

Proof. Let $M > 0$, and let $P_1, \dots, P_l, Q_1, \dots, Q_m$ be polynomials such that $C = (\cap_{i=1}^l P_i = 0) \cap (\cap_{j=1}^m Q_j > 0)$ and such that $\overline{C} = (\cap_{i=1}^l P_i = 0) \cap (\cap_{j=1}^m Q_j \ge 0)$; such exist because the C.A.D. is stratified. C is clearly locally closed, and since $dim(C) = K$ and $\mathcal{D}_K$ is closed, $\partial^{\mathcal{D}_K} C = \overline{C} \backslash C$. Define $P = \prod_{j=1}^m Q_j$. By assumption, $P(x) = 0$ for $x \in \partial^{\mathcal{D}_K} C = \overline{C} \backslash C$ and $P(x) > 0$ for $x \in C$. By Lojasiewicz's inequality, there is $K \in \mathbb{N}$ and $c > 0$ such that $P^K \le c \cdot \zeta$ on $\overline{C} \cap B_M^N$. Hence, take $Q = \frac{1}{2c} P^K$. $\square$

4.5 Proof of Theorem 3.2

Recall $B_M^N = (-M, M)^N$ for $N \in \mathbb{N}$, $M > 0$.

Proposition 4.20. *Let $0 \le K \le N$, let $\mathcal{D}_K$ be the union of the cells in a stratified C.A.D. $\mathcal{C}_1, \dots, \mathcal{C}_N$ of $\mathbb{R}^N$ of dimension $\le K$; let C be a particular K-dimension cell in $\mathcal{C}_N$. Then C is representable w.r.t. $\mathcal{D}_K \cap B_M^N$ for any $M > 0$.*

Proposition 4.21. *Let $M > 0$, and let $X \subseteq \mathbb{R}^N$ be semi-algebraic. Then X is representable w.r.t. B_M^N for any $M > 0$.*

We prove Propositions 4.20 and 4.21 and Theorem 3.2 now together inductively. Suppose we have proven Propositions 4.20 and 4.21 for dimension $\le N$ and all $K \le N$ (note that $\mathbb{R}^0$ is a singleton) and Theorem 3.2 for dimension $< N$. We prove Propositions 4.20 and 4.21 for dimension $N + 1$, and Theorem 3.2 for dimension N.

Throughout the proofs, for brevity, if $A, B \subseteq \mathbb{R}^N$ and $M > 0$, we write $\Xi_M\{A/B\}$ instead of $\Xi\{A/B \cap B_M^N\}$.

Remark 4.22. In the proofs, given $F : B \Longrightarrow [0, 1]$, we may write $F(y)$ for $y \notin B$, which is clearly not defined. When doing so, we mean to use any extension of F to all of $\mathbb{R}^N$, $\overline{F} : \mathbb{R}^N \Longrightarrow [0, 1]$. Hence, when writing an equality involving F taking inputs on a domain larger than B, we mean to *first* to extend F to $\mathbb{R}^N$, and the meaning of the equality is precisely that it does *not* matter which extension is used (as long as it still takes values in $[0, 1]$). For example, suppose for some $M > 0$, $G : \mathbb{R}^N \Longrightarrow \mathbb{R}$ satisfies $G(x) = \{0\}$ for $x \notin B_M^N$, and $F : B_{2M}^N \Longrightarrow [0, 1]$ satisfies $F(x) = \{0\}$ for $x \in B_M^N$. Then we can write

$$G(x) \times F(x) = \{0\}, \forall x \in \mathbb{R}^N$$

even though the second term in the product is not defined for $x \notin B_{2M}^N$. However, it is clear that no matter how it is extended to $\mathbb{R}^N$ the equality would continue to hold.

[17] Observe that $\mathcal{D}_K$ is closed.

Remark 4.23. In continuation to Remark 4.22, if furthermore F is exactly representable, F clearly extends to an exactly representable $\overline{F} : \mathbb{R}^N \Longrightarrow [0, 1]$.

4.5.1 Induction Step for Proposition 4.20 (from N to $N+1$)

Fix $M > 0$. First take the case $K = N + 1$. In this case, $\mathcal{D}_{N+1} = \mathbb{R}^{N+1}$. Let $C \in \mathcal{C}_{N+1}$ with $dim(C) = N + 1$, and let $Q_1, \ldots, Q_m \in \mathbb{R}[x_1, \ldots, x_{N+1}]$ such that $C = \cap_{j=1}^m (Q_j > 0)$ and $\overline{C} = \cap_{j=1}^m (Q_j \geq 0)$. Since the C.A.D. is stratified and C is open, such exist. Hence for all $x \in B_M^{N+1}$,[18]

$$\Xi_M\{C, \mathbb{R}^{N+1}\}(x) =\leq (0, \min[Q_1(x), \ldots, Q_m(x)])$$

Therefore C is representable w.r.t. B_M^{N+1}.

Now, we work backwards: take $K < N+1$; assume we have proven the claim of representability of the cells of dimension $K+1$ w.r.t. $\mathcal{D}_{K+1} \cap B_{M+1}^{N+1}$.[19] We deal with two cases:

First let's deal with the case that C is of the form $A_{C',j}$ as in (4.2) for some $C' \in \mathcal{C}_{N-1}$ with $dim(C') = K$ and some j. Call such cells *horizontal*. Let $\mathcal{D}'_K \subseteq \mathbb{R}^N$ be the union of cells of dimensional $\leq K$ in $\mathcal{C}_N$. By the induction hypothesis, C' is representable w.r.t. $\mathcal{D}'_K \cap B_M^N$. By Corollary 4.19, there is a polynomial $Q \in \mathbb{R}[x_1, \ldots, x_N]$ such that $Q = 0$ on $\partial^{\mathcal{D}'_K} C' \cap B_M^N$ and such that on $C' \cap B_M^N$,

$$0 < Q < \min\left[\min_i [\xi_{C',i+1} - \xi_{C',i}], 1\right]$$

Let C_+, C_- be the cells $B_{C',j}, B_{C',j-1}$ as in (4.3) (those are the cells of dimension $K+1$ whose boundaries include C and which lie just 'above' and just 'below' C). We contend that by our assumptions on C' and Q, $\Xi_M\{C/\mathcal{D}_K\}$ satisfies for all $x = (y, x_{N+1}) \in \mathcal{D}_K \cap B_M^{N+1}$,[20]

$$\begin{aligned}\Xi_M\{C/\mathcal{D}_K\}(y, x_{N+1}) = \Xi_M&\{C'/\mathcal{D}'_K\}(y) \\ &\times \Xi_{M+1}\{C_+/\mathcal{D}_{K+1}\}(y, x_{N+1} + Q(y)) \\ &\times \Xi_{M+1}\{C_-/\mathcal{D}_{K+1}\}(y, x_{N+1} - Q(y))\end{aligned}$$

If we prove this equality, C is representable w.r.t. $\mathcal{D}_K \cap B_M^{N+1}$, since we have shown the representability for cells of dimension $K+1$ w.r.t. $\mathcal{D}_{K+1} \cap B_{M+1}^{N+1}$ and by the induction hypothesis Proposition 4.20 holds in dimension $\leq N$, and using Proposition 4.7. Observe that[21] $\partial^{\mathcal{D}_K} C = \xi_{\overline{C'},j}(\partial^{\mathcal{D}'_K} C')$ and $\overline{C} = \xi_{\overline{C'},j}(\overline{C'})$. We check all options for $x = (y, x_{N+1}) \in \mathcal{D}_K \cap B_M^{N+1}$. (See Figure 4.)

[18]Recall the notation $\leq (\cdot, \cdot)$ from Proposition 4.13.

[19]Note that we have written B_{M+1}^{N+1} instead of B_M^{N+1}; the reason will become apparent. Since M was arbitrary, this assumption is permissible.

[20]Note that this implies $y \in \mathcal{D}'_K \cap B_M^N$ and $(x, y_{N+1} \pm Q(y)) \in \mathcal{D}_{K+1} \cap B_{M+1}^{N+1}$, and hence the terms on the right-hand side are well-defined. This is why we required to assume the representation of cells of dimension $K+1$ w.r.t. $\mathcal{D}_{K+1} \cap B_{M+1}^{N+1}$, not $\mathcal{D}_{K+1} \cap B_M^{N+1}$, since it may be that $(y, x_{N+1} \pm Q(y))$ are in B_{M+1}^{N+1} but not in B_M^{N+1}.

[21]Recall the extension $\xi_{\overline{C'},j}$ of $\xi_{C',j}$ from (4.6).

- (C1) If $y \notin \overline{C'}$, then $x \notin \overline{C}$ and $\Xi_M\{C'/\mathcal{D}'_K\}(y) = \Xi_M\{C/\mathcal{D}_K\}(x) = \{0\}$, so both sides are $\{0\}$.
- (C2) If $y \in \partial^{\mathcal{D}'_K} C'$ and $x_{N+1} = \xi_{\overline{C'},j}(y)$, then $Q(y) = 0$ and $x \in \partial^{\mathcal{D}_K} C$, $x \in \partial^{\mathcal{D}_{K+1}} C_+$, $x \in \partial^{\mathcal{D}_{K+1}} C_-$, so all terms on both sides are $[0,1]$.
- (C3) If $y \in \partial^{\mathcal{D}'_K} C'$ and $x_{N+1} \neq \xi_{\overline{C'},j}(y)$, then $x \notin \overline{C}$, $Q(y) = 0$, and either $\Xi_{M+1}\{C_+/\mathcal{D}_{K+1}\}(x) = \{0\}$ (if $x_{N+1} < \xi_{\overline{C'},j}(y)$ and hence $x \notin \overline{C_+}$) or $\Xi_{M+1}\{C_-/\mathcal{D}_{K+1}\}(x) = \{0\}$ (if $x_{N+1} > \xi_{\overline{C'},j}(y)$ and hence $x \notin \overline{C_-}$), so both sides are $\{0\}$.

Options (C2) and (C3) are vacuous if $K = 0$, since $\mathcal{D}'_0$ is finite so $\partial^{\mathcal{D}'_0} C' = \emptyset$.

- (C4) If $y \in C'$, and $x_{N+1} = \xi_{\overline{C'},j}(y)$, then $x \in C$ and since $\forall i,\ Q < \xi_{C',i+1} - \xi_{C',i}$, we have $(y, x_{N+1} \pm Q(y)) \in C_\pm$; hence all terms on both sides are $\{1\}$.
- (C5) If $y \in C'$, and $x_{N+1} \neq \xi_{\overline{C'},j}(y)$, $\Xi_{M+1}\{C/\mathcal{D}_K\}(x) = \{0\}$ and at least one of the inclusions $(y, x_{N+1} \pm Q(y)) \in \overline{C_\pm}$ does not hold (as $\forall i, Q < \xi_{C',i+1} - \xi_{C',i}$ in C'), so both sides are $\{0\}$.

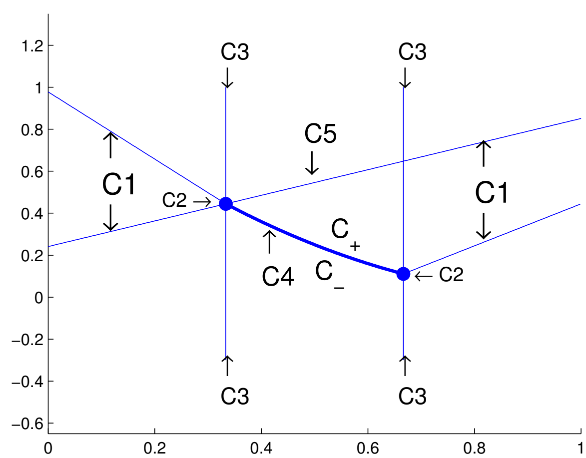

Figure 4: The Cases (C1)-(C5) for $K = 1, N = 2$, $C' = [\frac{1}{3}, \frac{2}{3}]$, $\xi_{C',1}(x) = (x-1)^2$

Now we deal with the case that C is of the form $B_{C',j}$ as in (4.3) for some $C' \in \mathcal{C}_N$ with $dim(C') = K - 1$; let $\xi_{\overline{C'},j}, \xi_{\overline{C'},j+1}$ be as in (4.7). Call such cells *vertical*. Let $\mathcal{D}'_{K-1} \subseteq \mathbb{R}^N$ be the union of cells of dimensional $\leq K - 1$ in $\mathcal{C}_N$.

By the induction hypothesis, C' is representable w.r.t. $\mathcal{D}'_{K-1} \cap B^N_M$. We claim that $\Xi_M\{C/\mathcal{D}_K\}$ is given, for $(y, x_{N+1}) \in \mathcal{D}_K \cap B^{N+1}_M$ by[22]

$$\begin{aligned}\Xi_M\{C/\mathcal{D}_K\}(y, x_{N+1}) = \Xi_M\{C'/\mathcal{D}'_{K-1}\}(y) \times \leq (\xi_{\overline{C'},j}(y), x_{N+1}) \cdot \\ \times \leq (x_{N+1}, \xi_{\overline{C'},j+1}(y)) \times \Big(1 - \max_{C^* \in T^*} \Xi_M\{C^*/\mathcal{D}_K\}(y, x_{N+1})\Big)\end{aligned}$$

where T^* denotes the collection of K-dimensional horizontal cells in $\mathcal{C}_{N+1}$. By the induction hypotheses for Theorem 3.2 holding in dimension $< N$ and for Proposition 4.20 holding in dimension $\leq N$, and by Corollary 4.15, Proposition 4.7, and Remark 4.23, the right-hand side is representable. Again, to prove the equality we need to verify the various cases for $x = (y, x_{N+1}) \in \mathcal{D}_K \cap B^{N+1}_M$ (see Figure 5).

- (D1) If $y \notin \overline{C'}$, then $x \notin \overline{C}$, and either $\Xi_M\{C'/\mathcal{D}'_{K-1}\}(y) = \{0\}$ (if $y \in \mathcal{D}'_{K-1}$) or there is $C^* \in T^*$ with $\Xi_M\{C^*/\mathcal{D}_K\}(y, x_{N+1}) = \{1\}$ (if $y \in \mathcal{D}'_K \backslash \mathcal{D}'_{K-1}$) - so both sides are $\{0\}$.

For the cases (D2) and (D4), observe that $x \in \overline{C}$ implies $\max_{C^* \in T^*} \Xi_M\{C^*/\mathcal{D}_K\}(y, x_{N+1}) = [0,1]$ or $= \{0\}$.

- (D2) If $y \in \partial^{\mathcal{D}'_{K-1}} C' (\subseteq \overline{C'})$ and $\xi_{\overline{C'},j}(y) \leq x_{N+1} \leq \xi_{\overline{C'},j+1}(y)$, then the left-hand side and the first term on the right-hand side are $[0,1]$, $x \in \overline{C}$, and the two middle terms in the product are either $\{1\}$ or $[0,1]$ depending on whether $x_{N+1} = \xi_{\overline{C'},j}(y)$, $x_{N+1} = \xi_{\overline{C'},j+1}(y)$, or $\xi_{\overline{C'},j}(y) < x_{N+1} < \xi_{\overline{C'},j+1}(y)$. (When $K = 1$, this case is vacuous, as $\partial^{\mathcal{D}'_0} \mathcal{C}' = \emptyset$ for any 0-dimensional C'.)
- (D3) If $y \in \overline{C'}$ and $x_{N+1} < \xi_{\overline{C'},j}(y)$ or $\xi_{\overline{C'},j+1}(y) < x_{N+1}$, then the left-hand side is $\{0\}$ (since $x \notin \overline{C}$) and so is at least one of the terms $\leq (\xi_{\overline{C'},j}(y), x_{N+1})$ or $\leq (x_{N+1}, \xi_{\overline{C'},j+1}(y))$.
- (D4) If $y \in C'$ and $x_{N+1} = \xi_{\overline{C'},j}(y)$ or $x_{N+1} = \xi_{\overline{C'},j+1}(y)$, then $x \in \partial^{\mathcal{D}_K} C$ so $\Xi_M\{C/\mathcal{D}_K\}(y, x_{N+1}) = [0,1]$, and each term on the right-hand side is either $\{1\}$ or $[0,1]$, with at least one of the two middle terms being $[0,1]$.
- (D5) If $y \in C'$ and $\xi_{\overline{C'},j}(y) < x_{N+1} < \xi_{\overline{C'},j+1}(y)$, then all terms on both sides are $\{1\}$ (in particular, $\Xi_M\{C^*/\mathcal{D}_K\}(y, x_{N+1}) = \{0\}$ for all $C^* \in T^*$, since the boundary of a cell of dimension K must be the union of cells of strictly lower dimension.)

[22] The first term, $\Xi_M\{C'/\mathcal{D}'_{K-1}\}(y)$, is only defined on $\mathcal{D}'_{K-1}$, while the terms $\leq (\xi_{\overline{C'},j}(y), x_{N+1}), \leq (x_{N+1}, \xi_{\overline{C'},j+1}(y))$ are only defined when $y \in \overline{C'}$; however, recall Remark 4.22. Also recall the notation $\leq (\cdot,\cdot)$ from Proposition 4.13; recall that it is possible that $\xi_{\overline{C'},j} \equiv -\infty$ or $\xi_{\overline{C'},j+1} \equiv \infty$, in which case the corresponding $\leq (\xi_{\overline{C'},j}, x_{N+1})$ or $\leq (x_{N+1}, \xi_{\overline{C'},j+1})$ is constant $\{1\}$ and can be removed from the product.

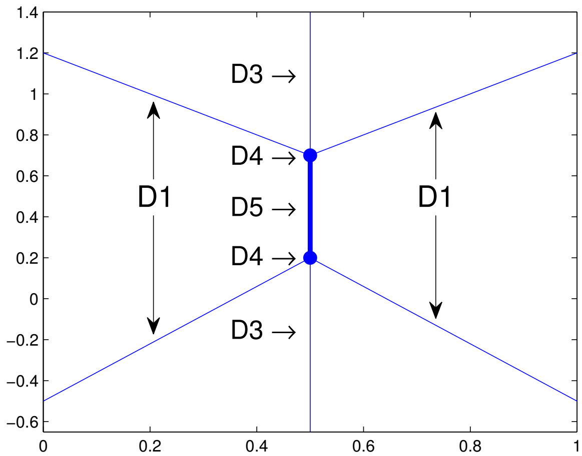

Figure 5: The Cases (D1), (D3)-(D5) for $K = 1, N = 2$, $C' = \{\frac{1}{2}\}$

4.5.2 Induction Step for Proposition 4.21 (from N to $N + 1$)

Let $M > 0$, let X be semi-algebraic, and let $\mathcal{C}_1, \ldots, \mathcal{C}_{N+1}$ be a stratified C.A.D. which is X-adapted; by applying a linear automorphism to the space we can assume such exists, and by Proposition 4.12 it suffices to proof the claim after application a linear automorphism.[23]

Inductively, for $K = 0, \ldots, N + 1$, we show that X is representable w.r.t. $\mathcal{D}_K \cap B_M^{N+1}$.[24] For $K = 0$, $B_M^{N+1} \cap \mathcal{D}_0$ is a finite set, hence this case follows from the following elementary fact: For every every pair of finite collections $x^1, \ldots, x^m \in \mathbb{R}^N$ and $y^1, \ldots, y^m \in \mathbb{R}$, there is a polynomial Q with $Q(x^j) = y^j$ for $j = 1, \ldots, m$.

Suppose we have shown X is representable w.r.t. $\mathcal{D}_{K-1} \cap B_M^N$. Let $T = \{C \in \mathcal{C}_N \mid dim(C) = K, C \subseteq X\}$, $T^o = \{C \in \mathcal{C}_N \mid dim(C) = K, C \cap X = \emptyset\}$ be the collection of K-dimensional cells that are contained in and not contained in X, respectively. Then we claim that $\Xi_M\{X/\mathcal{D}_K\}$ satisfies for each $x \in$

[23] Also, observe that for each $M > 0$ and each linear automorphism V, there is $M' > 0$ such that $V(B_M^{N+1}) \subseteq B_{M'}^{N+1}$.

[24] Note that we do not directly rely on the induction hypothesis of representability of subsets of $\mathbb{R}^N$ w.r.t. B_M^N, but we rely on the hypothesis indirectly, since it was used to prove Theorem 3.2 for domains of dimension $< N$, which in turn was used to prove Proposition 4.20 for dimension $N + 1$, which will be used in this induction step.

$\mathcal{D}_K \cap B_M^{N+1}$,[25]

$$\Xi_M\{X/\mathcal{D}_K\}(x) = \min\Big[\max\big[\max_{C\in T}\Xi_M\{C/\mathcal{D}_K\}(x), \Xi_M\{X/\mathcal{D}_{K-1}\}(x)\big], \\ \big(1 - \max_{C\in T^o}\Xi_M\{C/\mathcal{D}_K\}(x)\big)\Big] \tag{4.8}$$

Proving (4.8) suffices since the right-hand side is representable by Corollary 4.15 and Proposition 4.7, and Remark 4.23. Indeed, we check various cases (see Figure 6 and Figure 7):

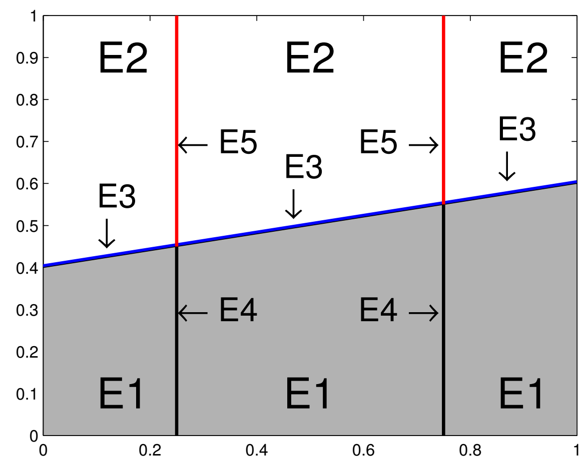

Figure 6: The Cases (E1)-(E5) for $K = N = 2$, $X = \{(x, y) \in \mathbb{R}^2 \mid y < \frac{1}{5}(x+2)\}$

- (E1) If $x \in C^* \in T$, then $\max_{C\in T}\Xi_M\{C/\mathcal{D}_K\}(x) = \{1\}$ and (since $x \notin \overline{C}$ for all $C \in T^o$) $\max_{C\in T^o}\Xi_M\{C/\mathcal{D}_K\}(x) = \{0\}$, and also $\Xi_M\{X/\mathcal{D}_K\}(x) = \{1\}$.
- (E2) If $x \in C^* \in T^o$, we see symmetrically that both sides are $\{0\}$.
- (E3) If $x \in \partial^{\mathcal{D}_K} X$, then $\Xi_M\{X/\mathcal{D}_K\}(x) = [0, 1]$, and also $x \in \partial^{\mathcal{D}_K} C_1 \cap \partial^{\mathcal{D}_K} C_2$ for some $C_1 \in T, C_2 \in T^o$, and hence

$$\max_{C\in T}\Xi_M\{C/\mathcal{D}_K\}(x) = 1 - \max_{C\in T^o}\Xi_M\{C/\mathcal{D}_K\}(x) = [0, 1]$$

so both sides are $[0, 1]$.

[25]The term $\Xi_M\{X/\mathcal{D}_{K-1}\}(x)$ is only defined when $x \in \mathcal{D}_{K-1}$; however, recall Remark 4.22.

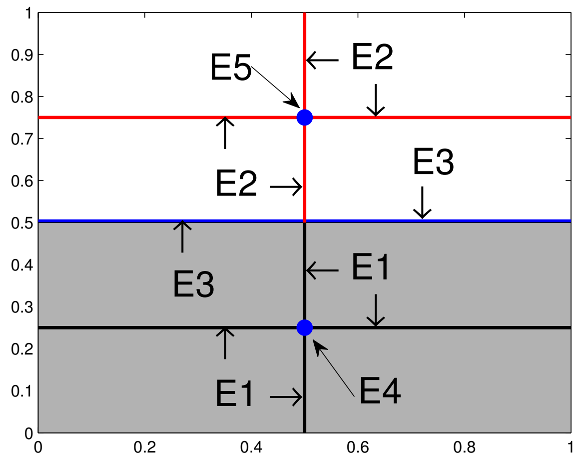

Figure 7: The Cases (E1)-(E5) for $K = 1, N = 2$, $X = \{(x, y) \in \mathbb{R}^2 \mid y < \frac{1}{2}\}$

If none of these cases hold, then $x \in \mathcal{D}_{K-1} \backslash \partial^{\mathcal{D}_K} X$. Observe that[26] $\partial^{\mathcal{D}_K} X \supseteq \partial^{\mathcal{D}_{K-1}} X$. We deal with two sub-cases, $x \in X$ or $x \notin X$:

- (E4) If $x \in (X \cap \mathcal{D}_{K-1}) \backslash \partial^{\mathcal{D}_K} X \subseteq X \backslash \partial^{\mathcal{D}_{K-1}} X$, then $\Xi_M\{X/\mathcal{D}_{K-1}\}(x) = \{1\}$ and $\Xi_M\{C/\mathcal{D}_K\}(x) = \{0\}$ for all $C \in T^o$ (since otherwise, $x \in \overline{C}$ for some $C \in T^o$, which together with $x \in X$ implies $x \in \partial^{\mathcal{D}_K} X$, a contradiction), so both sides are $\{1\}$.

- (E5) If $x \in (\mathcal{D}_{K-1} \backslash X) \backslash \partial^{\mathcal{D}_K} X \subseteq (\mathcal{D}_{K-1} \backslash X) \backslash \partial^{\mathcal{D}_{K-1}} X$, then $\Xi_M\{X/\mathcal{D}_{K-1}\}(x) = \{0\}$ and (similar to E4) $\Xi_M\{C/\mathcal{D}_K\}(x) = \{0\}$ for all $C \in T$, so both sides are $\{0\}$.

4.5.3 Induction Step for Theorem 3.2 (from $N - 1$ to N)

First take the case $K = 1$, i.e., $f : A \to [0, 1]$. Fix $M > 1$ such that $A \subseteq B_M^N$, and extend f to a continuous semi-algebraic function $\hat{f}$ on B_M^N such that $0 \leq \hat{f} \leq 1$; this can be done by the Teitze extension theorem for semi-algebraic function (e.g., [3, Prop. 2.6.9]).[27] Because Proposition 4.21 holds for dimensions $\leq N+1$ and since $M > \sup_{x \in B_M^N} |\hat{f}|$, one can then apply Lemma 4.16 to represent $\hat{f}$ in B_M^N.

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[26] If every neighbourhood of a point intersects $\mathcal{D}_{K-1}$ both in A and not in A, then every neighbourhood also intersects $\mathcal{D}_K$ in A and not in A.

[27] This is the sole reason we required in Theorem 3.2 for the domain to be closed - so we could apply the Teitze extension theorem.

Now suppose $f = (f_1, \dots, f_K)$. Informally, for each $x \in B_M^N$, we represent each of the f_j on their own and then play them 'independently'. Formally, for each $1 \leq j \leq K$, there is an $\mathbb{R}^N$-parametrized game G_j on the set of binary players $(\alpha_j, \mathcal{P}_j)$ which exactly represents G_j. Then define the $\mathbb{R}^N$-parametrized game G on the set of players $(\alpha_1, \dots, \alpha_K, \mathcal{P}_1, \dots, \mathcal{P}_K)$ by

$$G^{\alpha_j, \mathcal{P}_j}[x](z) = G_j[x](z^{\alpha_j, \mathcal{P}_j})$$

G is clearly the desired game.

4.6 Proof of Theorem 3.1

In this section, let I be a finite set of players with finite sets $(A^i)_{i \in I}$ of actions,.

Proposition 4.24. *Let $X \subseteq \mathbb{R}^N$ be semi-algebraic and compact, and $f : X \to \prod_i \Delta(A^i)$ be a continuous semi-algebraic function. Then there exists an $\mathbb{R}^N$-parametrized game G on a set of players $(I, \mathcal{P})$, where the $\mathcal{P}$ are binary players, such that for each $x \in X$, in any equilibrium z of $G[x]$, we have $z^I = f(x)$.*

Proof. By Theorem 3.2, there is an $\mathbb{R}^N$-parametrized game on a set of binary players $\mathcal{P}_f := \cup\{(\alpha_{i,b}, \mathcal{P}_{i,b}) \mid i \in I, b \in A^i\}$ such that for each $x \in X$ and each equilibrium z of $G_f[x]$, $z^{\alpha_{i,b}} = f^{i,b}(x)$, where $f^{i,b}$ is the coordinate of f for the action $b \in A^i$.

Also, by Lemma 4.2, for each $i \in I$ there is a $\mathbb{R}^{A^i}$-parametrized game G_i on a set of players $(i, \mathcal{P}_i)$ where $\mathcal{P}_i$ consists of binary players, such that for each $y \in \Delta(A^i)$ and any equilibrium z of $G_i[y]$, $z^i = x$.

Hence, on the set of players $(I, (\mathcal{P}_i)_{i \in I}, \mathcal{P}_f)$, define the $\mathbb{R}^N$-parametrized game G by

- For $i \in I$, $b \in A^i$, $G^{\mathcal{P}_f}[x](u) = G_f[x](u^{\mathcal{P}_f})$.
- For each $i \in I$,

$$G^{i, \mathcal{P}_i}[x](u) = G_i((u^{\alpha_{i,b}})_{b \in A^i})(u^{i, \mathcal{P}_f})$$

  G is then the desired game.

□

Corollary 4.25. *Let $\psi : [0,1]^N \to \prod_i \Delta(A^i)$ be semi-algebraic and continuous. Then there exists a game G on a set of players $(I, \mathcal{P})$, where the $\mathcal{P}$ are binary, such that for the image of ψ, denoted $E := \psi([0,1]^N)$, we have*

$$E = \{z^I \mid z \textit{ is an equilibrium of } G\}$$

Proof. By Proposition 4.24, there is an $\mathbb{R}^N$-parametrized game G_ψ on a set of players $(I, \mathcal{P}_\psi)$ such that for each $x \in [0,1]^N$ and any equilibrium z of $G_\psi[x]$, $z^I = f(x)$. Define G on a set of players $(I, \alpha_1, \dots, \alpha_N, \mathcal{P}_\psi)$ by

$$G^{\alpha_j} \equiv 0, \ \forall j = 1, \dots, N$$

and

$$G^{I,\mathcal{P}_\psi}(z^{I,\mathcal{P}}, z^{\alpha_1}, \ldots, z^{\alpha_N}) = G_\psi[z^{\alpha_1}, \ldots, z^{\alpha_N}](z^{I,\mathcal{P}})$$

The result follows, since the equilibrium profiles for $\alpha_1, \ldots, \alpha_N$ range over all of $[0,1]^N$. $\square$

Now we prove Theorem 3.1: Using the C.A.D. (or, alternatively, the well-known triangulation of semi-algebraic sets, e.g., [3, Ch. 9]) it follows that for $N = dim(X)$, there is a finite collection Ψ of continuous semi-algebraic functions $\psi : [0,1]^N \to \prod_{i \in I} \Delta(A^i)$ such that $X = \cup_{\psi \in \Psi} D_\psi$, where $D_\psi = \psi([0,1]^N)$. By Proposition 4.24, for each $\psi \in \Psi$ there is a game H_ψ on the set of players $(I, \mathcal{P}_\psi)$, where $\mathcal{P}_\psi$ consists of binary players, such that if E_ψ is the set of equilibria of H_ψ, then $D_\psi = \{z^I \mid z \in E_\psi\}$. Denote $M = |\Psi|$. We can replace all the $\mathcal{P}_\psi$ with the largest of the sets $(\mathcal{P}_\psi)_{\psi \in \Psi}$, which we denote simply $\mathcal{P}_\Psi$, and henceforth all the games G_ψ are the on the set $(I, \mathcal{P}_\Psi)$.

Let G_M be a game on a set of binary players $\mathcal{P}_M$ with finitely many equilibria, at least M of them, such that they are linearly independent in the space of mixed strategies (viewed as a subspace of $\mathbb{R}^{2 \times \mathcal{P}_M}$). For example: We may assume (since we can allow repeats in Ψ) that $M = 3^K$ for some K. Then, let H be a 2×2 coordination game with 3 equilibria (2 pure and one mixed), $E = \{e_1, e_2, e_3\} \subseteq \Delta(\{left, right\})^2 \subseteq \mathbb{R}^4$, and let G_M be defined with $2K$ players $\mathcal{P}_M$: that is, K 'pairs', each pair of which is playing H. The set of equilibrium of G_M is $E_M := E^K$. Let $\iota : \psi \to E_M$ be a bijection.

Since the set $E_M = (\iota(\psi))_{\psi \in \Psi}$ is linearly independent in $R^{2 \times \mathcal{P}_M}$, there is a $\mathbb{R}^{2 \times \mathcal{P}_M}$-parametrized game $G_\sqsupset$ on the set of players (I, P_Ψ) such that for each $\psi \in \Psi$, $x \in [0,1]^N$, and action profile z for the players $I \cup \mathcal{P}$,

$$H_\psi(z) = G_\sqsupset[\iota(\psi)](z)$$

Finally, define G on the set of players $(I, \mathcal{P}_\Psi, \mathcal{P}_M)$ by:

$$G^{\mathcal{P}_M}(z) = G_M(z^{\mathcal{P}_M})$$

$$G^{I,\mathcal{P}_\Psi}(z) = G_\sqsupset[z^{\mathcal{P}_M}](z^{I,\mathcal{P}_\Psi})$$

Clearly the equilibria of G are all of the form $(z^{I,\Psi}, z^M)$ with $z^M \in E_M$ and $z^{I,\mathcal{P}_\Psi}$ being an equilibrium of $H_{\iota^{-1}(z^M)}$, and hence by our assumptions on the family of functions Ψ and the games $(H_\psi)_{\psi \in \Psi}$, G is the required game.

4.7 Proof of Theorem 3.3 (The Case $K = 1$)

First, we prove Theorem 3.3 for the case $K = 1$ (i.e., the case that the range is one-dimensional). Fix $M > 1$ large enough such that $\overline{A} \subseteq B := B^N_M$. Define a correspondence $\Phi : B \to \mathbb{R}$ by

$$\Phi(x) = \{y \in \mathbb{R} \mid \exists z \in \overline{A}, ||z - x|| = ||x - A||, (z, y) \in \overline{Gr(\Psi)}\}$$

where $|| \cdot ||$ is any fixed semi-algebraic norm (e.g., the Euclidean norm) and $||x - A|| = \inf_{z' \in A} ||x - z'||$.

Lemma 4.26. Φ *is u.s.c. and semi-algebraic with non-empty compact values contained in* $[0,1]$*. Furtheremore,* $\Phi(x) = \Psi(x)$ *for all* $x \in A$*.*

Proof. The mapping $x \to \inf_{z' \in A} ||z' - x||$ is continuous and semi-algebraic (e.g., [3, Prop. 2.2.8]), so the semi-algebracity of $Gr(\Phi)$ follows by the Tarski-Seidenberg theorem. The non-emptiness of values is trivial, as is the fact that they are contained in $[0,1]$.

If $x \in A$, the only z with $||z - x|| = ||x - A||$ is $z = x$; since Ψ is u.s.c. on A, $\overline{Gr(\Psi)} \cap A \times \mathbb{R} = Gr(\Psi)$, and hence $\Phi(x) = \Psi(x)$.

Now we show that $Gr(\Phi)$ is closed. For each $n \in \mathbb{N}$, suppose $(x_n, y_n) \in Gr(\Phi)$ and let $z_n \in \overline{A}$ be such that $||x_n - z_n|| = ||x_n - A||$ and $(z_n, y_n) \in \overline{Gr(\Psi)}$. Suppose $(x_n, y_n) \to (x, y)$. Since $\overline{A}$ is compact, we may assume $z_n \to z \in \overline{A}$. Since the mapping $x \to ||x - A||$ is continuous, we have $||z - x|| = ||x - A||$ and $(z, y) \in \overline{Gr(\Psi)}$ so $(x, y) \in Gr(\Phi)$. □

Hence, we show that Φ is representable. Define the sets

$$B_+ = \{(x, y) \in B \times (-M, M) \mid \forall t \in \Phi(x), y > t)\}$$

$$B_- = \{(x, y) \in B \times (-M, M) \mid \forall t \in \Phi(x), y < t\}$$

By the upper-semicontinuity of Φ, the sets B_+, B_- are open. By Proposition 4.21, these sets are representable w.r.t. $B_M^{N+1} = B \times (-M, M)$. The proof then follows precisely as the proof of Lemma 4.16.

4.8 Proof of Theorem 3.3 (General Case)

Now we prove the general version of Theorem 3.3 - i.e., for range of arbitrary dimension K. Let F be as in Theorem 3.3. For each $j = 0, \ldots, K-1$, define[28]

$$\Pi_j = \{(x, y_1, \ldots, y_j) \mid \exists (y_{j+1}, \ldots, y_K) \text{ s.t. } (x, y_1, \ldots, y_j, y_{j+1}, \ldots, y_k) \in Gr(F)\}$$

By the Tarski-Seidenberg theorem (Theorem 2.1), each Π_j is semi-algebraic; each is also easily seen (since F takes values in the compact set $[0,1]^K$) to be bounded. Then for each $j = 1, \ldots, K$, let $H_j : \Pi_{j-1} \to [0,1]$ be such that $Gr(H_j) = \Pi_j$. H_j is then semi-algebraic, and seen to have non-empty convex values contained in $[0,1]$. Then clearly,[29]

$$F(x) = \{(y_1, \ldots, y_K) \mid \forall j = 1, \ldots, K, y_j \in H_j(x, y_1, \ldots, y_{j-1}) \tag{4.9}$$

We can now prove Theorem 3.3: By Section 4.7, there are games $G_1, \ldots, G_K$, where G_j is a $\mathbb{R}^{N+j-1}$-parametrized game on a binary player set $(\alpha_j, \mathcal{P}_j)$ such that for each $(x, y_1, \ldots, y_{j-1}) \in \Pi_{j-1}$,

$$H_j(x, y_1, \ldots, y_{j-1}) = \{z^{\alpha_j} \mid z \text{ is an equilibrium of } G_j[x, y_1, \ldots, y_{j-1}]\}$$

[28] Since F has non-empty values, $\Pi_0 = A$.

[29] Note that if $y_i \in H_i(x, y_1, \ldots, y_{i-1})$ for $i < j$, then $(x, y_1, \ldots, y_{j-1}) \in \Pi_{j-1}$, and the right-hand side of 4.9 has a well-defined truth value.

Now, define the $\mathbb{R}^N$-parametrized game G on the set of players $\cup_j(\alpha_j, \mathcal{P}_j)$ by

$$G^{\alpha_j,\mathcal{P}_j}[x](z) = G_j[x, z^{\alpha_1}, \cdots, z^{\alpha_{j-1}}](z^{\alpha_j,\mathcal{P}_j})$$

Then G is the required game; inductively, one shows that for each $x \in A$ and each profile of actions z, z is an equilibrium of $G[x]$ iff $\forall j, z^{\alpha_j} \in H_j(x, z^{\alpha_1}, \ldots, z^{\alpha_{j-1}})$; hence the result follows by (4.9).

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