A solvable two-dimensional degenerate singular stochastic control problem with non convex costs*

Tiziano De Angelis† Giorgio Ferrari‡ John Moriarty§

January 14, 2015

Abstract. In this paper we provide a complete theoretical analysis of a two-dimensional degenerate non convex singular stochastic control problem. The optimisation is motivated by a storage-consumption model in an electricity market, and features a stochastic real-valued spot price modelled by Brownian motion. We find analytical expressions for the value function, the optimal control and the boundaries of the *action* and *inaction* regions. The optimal policy is characterised in terms of two monotone and discontinuous repelling free boundaries, although part of one boundary is constant and the smooth fit condition holds there.

Keywords: finite-fuel singular stochastic control; optimal stopping; free boundary; Hamilton-Jacobi-Bellmann equation; irreversible investment; electricity market.

MSC2010 subsject classification: 91B70, 93E20, 60G40, 49L20.

1 Introduction

In this paper we study a two-dimensional degenerate problem of singular stochastic control (SSC) with monotone, bounded controls and a non convex performance criterion that was introduced in [7] in the context of electricity markets. Here the first component of the state process is the electricity spot price, represented by a one-dimensional Brownian motion $B := (B_t)_{t\geq 0}$ carried by a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ and the optimisation problem detailed in [7] reads

$$U(x,c) = \inf_{\nu} \mathsf{E}\bigg[\int_0^\infty e^{-\lambda t}\lambda X_t^x \Phi(c+\nu_t)dt + \int_0^\infty e^{-\lambda t} X_t^x\, d\nu_t\bigg], \qquad (x,c) \in \mathbb{R}\times[0,1], \tag{1.1}$$

with $X_t^x := x + B_t$, $t \geq 0$, and where the infimum is taken over a suitable class of nondecreasing controls ν such that $c + \nu_t \leq 1$, P-a.s. for all $t \geq 0$. The constant λ denotes a positive discount factor and Φ is a strictly convex, twice continuously differentiable, decreasing function.

As discussed in Appendix A of [7], problem (1.1) is a *non convex* optimisation problem arising naturally from storage-consumption problems for electricity, when the spot price X is modelled by a continuous strong Markov process taking negative values with positive probability. In the storage-consumption problem $c + \nu_t$ represents the inventory level at time t of an electricity

*The first and the third authors were supported by EPSRC grant EP/K00557X/1; financial support by the German Research Foundation (DFG) via grant Ri–1128–4–1 is gratefully acknowledged by the second author.

†School of Mathematics, The University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom; tiziano.deangelis@manchester.ac.uk

‡Center for Mathematical Economics, Bielefeld University, Universitätsstrasse 25, D-33615 Bielefeld, Germany; giorgio.ferrari@uni-bielefeld.de

§School of Mathematics, The University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom; john.moriarty@manchester.ac.uk

storage facility such as a battery whose inventory level is c at time 0, so that ν_t is the cumulative amount of energy purchased up to time t. A *finite fuel* constraint $c + \nu_t \leq 1$, $c \in [0, 1]$, P-a.s. for all $t \geq 0$, reflects the fact that electricity storage has limited total capacity.

The study in [7], where the uncontrolled process X is of Ornstein-Uhlenbeck (OU) type, reveals how the non-convexity impacts in a complex way on the structure of the optimal control and on the connection between SSC problems and associated optimal stopping (OS) problems (as standard references on the subject see [15] and [16]). The analysis in [7] identifies three regimes, two of which are solved and the third of which is left as open problem under the OU dynamics. Here, with the aim of a full theoretical investigation, we take a more canonical example letting X be a Brownian motion. The complete solution that we provide also gives some insight in the open case of [7] since Brownian motion is a special case of OU with null rate of mean reversion. The methodology we employ here is different from that of [7], as we employ the characterisation via concavity of excessive functions for Brownian motion introduced in [8], Chapter 3 (later expanded in [5]) to study a parameterised family of OS problems in Section 3.3 below. This characterisation allows us to obtain the necessary monotonicity and regularity results for the optimal boundaries associated to (1.1). In contrast to the OU case, the Laplace transforms of the hitting times of Brownian motion are available in closed form and it is this feature which enables the method of the present paper.

From the mathematical point of view (1.1) falls into the class of finite fuel, singular stochastic control problems of monotone follower type (see, e.g., [3], [4], [9], [10], [17] and [18] as classical references on finite fuel monotone follower problems). As noted in [7] the total expected cost functional we aim at minimising in (1.1) is not convex in the control variable. In particular, by simply writing X as the difference of its positive and negative part, it is easy to see that the total expected cost functional in (1.1) can be written as a d.c. functional, i.e. as the difference of two convex functionals (see [13] or [14] for references on d.c. functions). As a consequence the differential connection between singular stochastic control and optimal stopping obtained, for example, in [9], [15] and [17] among others, does not directly address problem (1.1). Indeed it is shown in [7] that while connections to OS do hold for problem (1.1), they may or may not be of differential type depending on parameter values and the initial inventory level c. This suggests that the solutions of two-dimensional degenerate problems of this kind are complex and should be considered case by case.

In this paper we solve the third regime of [7] in the Brownian case, thus providing a full solution to the problem. While the action region is disconnected as expected from [7] the two free boundaries which we denote below by $c \mapsto \hat{\beta}(c)$ and $c \mapsto \hat{\gamma}(c)$ are *discontinuous*, the former being non-increasing everywhere but at a single jump and the latter being non-decreasing with a vertical asymptote (see Fig. 3). Through a verification argument we are able to show that the optimal control always acts by inducing discontinuities in the state process.

The free boundaries $\hat{\beta}$ and $\hat{\gamma}$ are therefore *repelling* (in the terminology of [6] or [19]). However, in contrast with most known examples of repelling boundaries, if the optimally controlled process hits $\hat{\beta}$ the controller does not immediately exercise all available control but, rather, causes the inventory level to jump to a *critical level* $\hat{c} \in (0, 1)$ (which coincides with the point of discontinuity of the upper boundary $c \mapsto \hat{\beta}(c)$). After this jump the optimally controlled process continues to evolve until hitting the lower boundary $\hat{\gamma}$ (the upper boundary is then formally infinite; for details see Sections 3 and 4).

This optimal process is unexpected in the light of [7], whose results might suggest the presence of a continuously reflecting boundary. However the present solution is not inconsistent with [7] and indeed does in part display a differential connection between SSC and OS. In particular when the initial inventory level c is strictly larger than the critical value $\hat{c}$ there is a single lower boundary $\hat{\gamma}$ which is constant and the optimal policy exercises all available control when the state process hits this boundary. However the so-called *smooth fit* condition holds at $\hat{\gamma}$ (for

$c > \hat{c}$), i.e. U_{xc} is continuous across it, and U_c coincides with the value function of an associated optimal stopping problem on $\mathbb{R} \times (\hat{c}, 1]$. This constant boundary can therefore be considered *discontinuously* reflecting, as a limiting case of the more canonical strictly decreasing reflecting boundaries. On the other hand, when the initial inventory level c is smaller than the critical value $\hat{c}$ we establish that the value function U of problem (1.1) coincides itself (and not through its derivative U_c) with the value function of an associated optimal stopping problem. This is done by solving the family of OS problems in Section 3.3 and examining their free boundaries. In this case it can easily be verified that U_{xc} is discontinuous across the optimal boundaries so that the smooth fit condition breaks down.

The rest of the paper is organised as follows. In Section 2 we set up the problem and in Section 3 we construct a candidate value function. A candidate optimal control for problem (1.1) and the candidate value function from Section 3 are then validated in Section 4 through a verification argument. Finally, proofs of some results needed in Section 3 are collected in Appendix A.

2 Setting and assumptions

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space carrying a one-dimensional standard Brownian motion $(B_t)_{t\geq 0}$ adapted to its natural filtration augmented by P-null sets $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$. We denote by X^x the Brownian motion starting from $x \in \mathbb{R}$ at time zero,

$$X_t^x = x + B_t, \quad t \geq 0. \tag{2.1}$$

It is well known that X^x is a (null) recurrent process with infinitesimal generator $\mathbb{L}_X := \frac{1}{2}\frac{d^2}{dx^2}$ and with fundamental decreasing and increasing solutions of the characteristic equation $(\mathbb{L}_X - \lambda)u = 0$ given by $\phi_\lambda(x) := e^{-\sqrt{2\lambda}x}$ and $\psi_\lambda(x) := e^{\sqrt{2\lambda}x}$, respectively.

Letting $c \in [0, 1]$ be constant, we denote by $C^{c,\nu}$ the purely controlled process evolving according to

$$C_t^{c,\nu} = c + \nu_t, \quad t \geq 0, \tag{2.2}$$

where ν is a control process belonging to the set

$$\mathcal{A}_c := \{\nu : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}_+, (\nu_t(\omega) := \nu(\omega, t))_{t\geq 0} \text{ is nondecreasing, left-continuous, adapted} \\ \text{with } c + \nu_t \leq 1 \ \ \forall t \geq 0, \ \nu_0 = 0 \ \ \mathsf{P}-\text{a.s.}\}.$$

From now on controls belonging to $\mathcal{A}_c$ will be called *admissible.*

Given a positive discount factor λ and a running cost function Φ, our problem (1.1) is to find

$$U(x, c) := \inf_{\nu \in \mathcal{A}_c} \mathcal{J}_{x,c}(\nu), \tag{2.3}$$

with

$$\mathcal{J}_{x,c}(\nu) := \mathsf{E}\bigg[\int_0^\infty e^{-\lambda s}\lambda X_s^x \Phi(C_s^{c,\nu})ds + \int_0^\infty e^{-\lambda s} X_s^x \, d\nu_s\bigg], \tag{2.4}$$

and to determine a minimising control policy ν^* if one exists.

Throughout this paper, for $t \geq 0$ and $\nu \in \mathcal{A}_c$ we will make use of the notation $\int_0^t e^{-\lambda s} X_s^x d\nu_s$ to indicate the Stieltjes integral $\int_{[0,t)} e^{-\lambda s} X_s^x d\nu_s$ with respect to ν. Moreover, from now on the following standing assumption on the running cost factor Φ will hold.

Assumption 2.1. $\Phi : \mathbb{R} \mapsto \mathbb{R}_+$ *lies in* $C^2(\mathbb{R})$ *and is decreasing and strictly convex with* $\Phi(1) = 0$.

As in [7] the sign of the function

$$k(c) := \lambda + \lambda\,\Phi'(c) \tag{2.5}$$

over $c \in [0,1]$ will also play a fundamental role in the solution of problem (2.3). Since $c \mapsto k(c)$ is strictly increasing by the strict convexity of Φ (cf. Assumption 2.1) we denote by $\hat{c} \in \mathbb{R}$ the unique solution of

$$\Phi'(c) = -1 \tag{2.6}$$

which we assume to lie in $(0,1)$. This corresponds to the third regime of [7] which is the subject of this paper (note that indeed $k(\hat{c}) = 0$).

3 Construction of a candidate value function

We begin by establishing that under our assumptions problem (2.3) is well posed with finite value function.

Proposition 3.1. *Let U be as in* (2.3)*. Then there exists $K > 0$ such that $|U(x,c)| \leq K(1+|x|)$ for any $(x,c) \in \mathbb{R} \times [0,1]$.*

Proof. Take $\nu \in \mathcal{A}_c$ and integrate by parts the cost term $\int_0^\infty e^{-\lambda s} X^x_s d\nu_s$ in (2.4) noting that $M_t := \int_0^t e^{-\lambda s}\nu_s dB_s$ is a uniformly integrable martingale. Then by well known estimates for the Brownian motion we obtain

$$|\mathcal{J}_{x,c}(\nu)| \leq \mathsf{E}\bigg[\int_0^\infty e^{-\lambda s}\lambda |X^x_s|\big[\Phi(C^{c,\nu}_s) + \nu_s\big]ds\bigg] \leq K(1+|x|), \tag{3.1}$$

for some suitable $K > 0$, since $\Phi(c) \leq \Phi(0)$, $c \in [0,1]$ by Assumption 2.1 and $\nu \in \mathcal{A}_c$ is bounded from above by 1. By (3.1) and the arbitrariness of $\nu \in \mathcal{A}_c$ the proposition is proved. $\square$

The aim of our study is to find analytical expressions for the value function U of problem (2.3) and the associated optimal control ν^*. We proceed in this section by constructing a solution W of the Hamilton-Jacobi-Bellman (HJB) equation naturally associated with U (cf. (3.2) below). The function W will be a candidate value function for (2.3) and in Section 4 we use a generalised version of Itô's formula to prove that $W = U$ using regularity results for W obtained in the present section. The optimal control will then be constructed by taking the free boundaries associated with W as the boundaries of *action* and *inaction* regions in the control problem, taking into account their geometry.

To be more precise, for $\mathcal{O} := \mathbb{R} \times (0,1)$ we aim at finding $W \in C^1(\mathcal{O}) \cap C(\overline{\mathcal{O}})$ such that $W_{xx} \in L^\infty_{loc}(\mathcal{O})$ and W solves the variational problem

$$\max\big\{(-\tfrac{1}{2}W_{xx} + \lambda W)(x,c) - \lambda x \Phi(c)\,,\, -W_c(x,c) - x\big\} = 0, \quad \text{for a.e. } (x,c) \in \mathcal{O} \tag{3.2}$$

with $W(x,1) = 0$, $x \in \mathbb{R}$. The candidate action and inaction regions associated to W are denoted $\mathcal{D}_W$ and $\mathcal{I}_W$, respectively and are defined by

$$\mathcal{D}_W := \big\{(x,c) \in \mathcal{O} : W_c(x,c) = -x\big\} \quad \text{and} \quad \mathcal{I}_W := \big\{(x,c) \in \mathcal{O} : W_c(x,c) > -x\big\}. \tag{3.3}$$

3.1 Heuristic study of the optimal policy

In order to derive a candidate solution to problem (2.3) we perform a preliminary heuristic analysis in the following three cases.

1. We begin by comparing two strategies when $c \in (\hat{c}, 1]$, hence $k(c) > 0$. If control is never exercised, i.e. $\nu_t \equiv 0$, $t \geq 0$, one obtains from (2.4) an overall cost $\mathcal{J}_{x,c}(0) = x\Phi(c)$ by an

application of Fubini's theorem. If instead at time zero one increases the inventory by a small amount $\delta > 0$ and then does nothing for the remaining time, i.e. $\nu_t = \nu^\delta_t := \delta$ for $t > 0$ in (2.4), the total cost is $\mathcal{J}_{x,c}(\nu^\delta) = x(\delta + \Phi(c+\delta))$. Writing $\Phi(c+\delta) = \Phi(c) + \Phi'(c)\delta + o(\delta^2)$ we find that $\mathcal{J}_{x,c}(\nu^\delta) = \mathcal{J}_{x,c}(0) + \delta x(1 + \Phi'(c)) + o(\delta^2)$ so that exercising a small amount of control reduces future expected costs relative to a complete inaction strategy only if $xk(c)/\lambda < 0$, i.e. $x < 0$ since $k(c) > 0$. It is then natural to expect that for each $c \in (\hat{c}, 1]$ there should exist $\gamma(c) < 0$ such that it is optimal to exercise control only when $X^x_s < \gamma(c)$. We next want to understand whether a small control increment is more favourable than a large one and for that we consider a strategy where at time zero one exercises all available control, i.e. $\nu_t = \nu^f_t := 1 - c$ for $t > 0$. The latter produces a total expected cost equal to $\mathcal{J}_{x,c}(\nu^f) = x(1-c)$, so that for $x < 0$ and recalling that k is increasing one has

$$\mathcal{J}_{x,c}(\nu^f) - \mathcal{J}_{x,c}(\nu^\delta) = \frac{x}{\lambda}\Big(\int_c^1 k(y)dy - \delta k(c)\Big) + o(\delta^2) \leq \frac{x}{\lambda} k(c)(1 - c - \delta). \tag{3.4}$$

Since $k(c) > 0$ the last expression is negative whenever $1 - c > \delta$, so it is reasonable to expect that large control increments are more profitable than small ones. This suggests that the threshold γ introduced above should not be of the reflecting type (see for instance [9] or [11]) but rather of repelling type as observed in [1], [2] and [6] among others, and a corresponding free boundary problem is formulated and solved in Section 3.2.

2. Now consider the case $c \in [0, \hat{c})$, i.e. $k(c) < 0$ and argue similarly. If again we compare the cost associated with complete inaction to that associated with the strategy ν^δ we find that the latter is favourable if and only if $xk(c)/\lambda < 0$, i.e. $x > 0$ since now $k(c) < 0$. Hence we expect that for fixed $c \in [0, \hat{c})$ one should act when the process X exceeds a positive upper threshold $\beta(c)$. Then compare a small control increment with a large one, in particular consider a policy $\nu^{\hat{c}}$ that immediately exercises an amount $\hat{c} - c$ of control and then acts optimally for problem (2.3) with initial conditions $(x, \hat{c})$. The expected cost associated to $\nu^{\hat{c}}$ is $\mathcal{J}_{x,c}(\nu^{\hat{c}}) = x(\hat{c} - c) + U(x, \hat{c})$ and one has

$$\mathcal{J}_{x,c}(\nu^{\hat{c}}) - \mathcal{J}_{x,c}(\nu^\delta) \leq \frac{x}{\lambda}\Big(\int_c^{\hat{c}} k(y)dy - \delta k(c)\Big) + o(\delta^2) \tag{3.5}$$

where we have used that $U(x, \hat{c}) \leq x\Phi(\hat{c})$. If we fix $c \in [0, \hat{c})$ and $x > 0$, then for $\delta > 0$ sufficiently small the right-hand side of (3.5) becomes negative, which suggests that a reflection strategy at the upper boundary β would be less favourable than the strategy described by $\nu^{\hat{c}}$.

3. Taking $x < 0$ instead when $c \in [0, \hat{c})$, note that $U(x, \hat{c}) \leq x(1 - \hat{c})$ to obtain

$$\mathcal{J}_{x,c}(\nu^{\hat{c}}) - \mathcal{J}_{x,c}(0) \leq \frac{x}{\lambda}\int_c^1 k(y)dy = \frac{x}{\lambda}\Big(\int_c^{\hat{c}} k(y)dy + \int_{\hat{c}}^1 k(y)dy\Big). \tag{3.6}$$

The first integral on the right-hand side of (3.6) is negative but its absolute value can be made arbitrarily small by taking c close to $\hat{c}$. The second integral is positive and independent of c. Thus the overall expression becomes negative when c approaches $\hat{c}$ from the left. This suggests that when the inventory is a little below the critical value $\hat{c}$ a small investment sufficient to increase the inventory to the level $\hat{c}$ is preferable to inaction, after which the optimisation continues as discussed above for $c \in (\hat{c}, 1]$. The presence of both upper and lower repelling boundaries when c is a little below $\hat{c}$ is explored in Section 3.3.1, while the candidate solution for smaller values of c (when only an upper boundary is suggested) is constructed in Section 3.3.2.

3.2 Step 1: $c \in [\hat{c}, 1]$

We formulate the first heuristic of Section 3.1 mathematically by writing (3.2) as a free boundary problem, to find the couple of functions (u, γ) solving

$$\begin{cases} \frac{1}{2}u_{xx}(x,c) - \lambda u(x,c) = -\lambda x\Phi(c) & \text{for } x > \gamma(c),\ c \in [\hat{c}, 1) \\ \frac{1}{2}u_{xx}(x,c) - \lambda u(x,c) \geq -\lambda x\Phi(c) & \text{for a.e. } (x,c) \in \mathbb{R} \times [\hat{c}, 1) \\ u_c(x,c) \geq -x & \text{for } x \in \mathbb{R},\ c \in [\hat{c}, 1) \\ u(x,c) = x(1-c) & \text{for } x \leq \gamma(c),\ c \in [\hat{c}, 1] \\ u_x(x,c) = (1-c) & \text{for } x \leq \gamma(c),\ c \in [\hat{c}, 1) \\ u(x,1) = 0 & \text{for } x \in \mathbb{R}. \end{cases} \tag{3.7}$$

For future frequent use we also define

$$R(c) := 1 - c - \Phi(c), \qquad c \in [0, 1]. \tag{3.8}$$

Proposition 3.2. *The couple (W^o, γ^o) defined by $\gamma^o := -\frac{1}{\sqrt{2\lambda}}$ and*

$$W^o(x,c) := \begin{cases} -\frac{1}{\sqrt{2\lambda}}e^{-1}R(c)\phi_\lambda(x) + x\Phi(c), & x > \gamma^o, \\ x(1-c), & x \leq \gamma^o, \end{cases} \tag{3.9}$$

solves (3.7) *with $W^o \in C^1(\mathbb{R} \times [\hat{c}, 1])$ and $W^o_{xx} \in L^\infty_{loc}(\mathbb{R} \times (\hat{c}, 1))$.*

Proof. A general solution to the first equation in (3.7) is given by

$$u(x,c) = A^o(c)\psi_\lambda(x) + B^o(c)\phi_\lambda(x) + x\Phi(c), \qquad x > \gamma(c),$$

with A^o, B^o and γ to be determined. Since $\psi_\lambda(x)$ diverges with a superlinear trend as $x \to \infty$ and U has sublinear growth by Proposition 3.1, we set $A^o(c) \equiv 0$. Imposing the fourth and fifth conditions of (3.7) for $x = \gamma(c)$ and recalling the expression for R in (3.8) we have

$$B^o(c) := -\frac{1}{\sqrt{2\lambda}}e^{-1}R(c), \qquad \gamma(c) = \gamma^o = -\frac{1}{\sqrt{2\lambda}}. \tag{3.10}$$

This way the function W^o of (3.9) clearly satisfies $W^o(x,1) = 0$, W^o_x is continuous by construction and by some algebra it is not difficult to see that W^o_c is continuous on $\mathbb{R} \times [\hat{c}, 1]$ with $W^o_c(\gamma^o, c) = -\gamma^o$, $c \in [\hat{c}, 1]$. Moreover one also has

$$W^o_{cx}(x,c) + 1 = (1 + \Phi'(c))\big(1 - e^{-1}\phi_\lambda(x)\big) \geq 0, \qquad x > \gamma^o, c \in [\hat{c}, 1], \tag{3.11}$$

and hence $W^o_{cx}(\gamma^o, c) = -1$, for $c \in [\hat{c}, 1]$, i.e. the smooth fit condition holds, and $W^o_c(x,c) \geq -x$ on $\mathbb{R} \times [\hat{c}, 1]$ as required. It should be noted that W^o_{xx} fails to be continuous across the boundary although it remains bounded on any compact subset of $\mathbb{R} \times [\hat{c}, 1]$.

Finally we observe that

$$\frac{1}{2}W^o_{xx}(x,c) - \lambda W^o(x,c) = -\lambda x(1-c) \geq -\lambda x\Phi(c) \quad \text{for } x \leq \gamma^o,\ c \in [\hat{c}, 1], \tag{3.12}$$

since $\gamma^o < 0$ and $R(c) \geq 0$ on $c \in [\hat{c}, 1]$. $\square$

Remark 3.3. *In this case the differential connection between SSC and OS holds as in convex problems (see [15]). Differentiation of the first and third equations in* (3.7) *(or alternatively of* (3.9)*) shows that W^o_c solves a free boundary problem which is naturally associated to the following family of OS problems which, for fixed $x \in \mathbb{R}$, are parametric in $c \in [\hat{c}, 1)$:*

$$w(x,c) := \sup_{\tau \geq 0} \mathsf{E}\Big[\lambda\Phi'(c)\int_0^\tau e^{-\lambda t}X^x_t\,dt - e^{-\lambda\tau}X^x_\tau\Big] \tag{3.13}$$

Moreover $W^o_c(\,\cdot\,, c) \in C^1(\mathbb{R})$ for all $c \in [\hat{c}, 1)$, as proven above and hence from standard verification arguments it follows that $W^o_c = w$. We omit the details, which are standard.

3.3 Step 2: An auxiliary problem of optimal stopping for $c \in [0, \hat{c})$

To complete our construction of the candidate value function, we now use the second and third heuristics in Section 3.1 to determine a parametric family of optimal stopping problems. More precisely we aim at proving that when $c \in [0, \hat{c})$, the solution $U(\,\cdot\,, c)$ equals

$$W^1(x,c) := \inf_{\tau \geq 0} \mathsf{E}\bigg[\int_0^\tau e^{-\lambda t}\lambda X_t^x \Phi(c)dt + e^{-\lambda\tau}X_\tau^x(\hat{c}-c) + e^{-\lambda\tau}W^o(X_\tau^x, \hat{c})\bigg], \tag{3.14}$$

for $x \in \mathbb{R}$ and where the optimisation is taken over the set of $(\mathcal{F}_t)$-stopping times valued in $[0, \infty)$, P-a.s. We begin by using Itô's formula to express (3.14) as an OS problem in the form:

$$W^1(x,c) = x\Phi(c) + V(x,c), \tag{3.15}$$

where

$$V(x,c) := \inf_{\tau \geq 0} \mathsf{E}\big[e^{-\lambda\tau}G(X_\tau^x, c)\big], \tag{3.16}$$

$$G(x,c) := x(\hat{c} - c - \Phi(c)) + W^o(x, \hat{c}). \tag{3.17}$$

Note that $G \in C(\mathbb{R} \times [0, \hat{c}])$, $|G(x,c)| \leq C(1+|x|)$ for suitable $C > 0$ and $x \mapsto \mathsf{E}\big[e^{-\lambda\tau}G(X_\tau^x, c)\big]$ is continuous for any fixed τ and $c \in [0, \hat{c})$. Then from standard theory an optimal stopping time is $\tau_* := \inf\{t \geq 0 : X_t^x \in \mathcal{S}_c\}$ where

$$\mathcal{C}_c := \{x \in \mathbb{R} : V(x,c) < G(x,c)\} \quad \text{and} \quad \mathcal{S}_c := \{x \in \mathbb{R} : V(x,c) = G(x,c)\} \tag{3.18}$$

are continuation and stopping regions respectively, and V is finite valued and solves the variational problem

$$\max\big\{ -\tfrac{1}{2}u_{xx} + \lambda u\,,\, u - G\big\} = 0 \quad \text{for a.e. } (x,c) \in \mathbb{R} \times [0, \hat{c}). \tag{3.19}$$

At this point we depart from the approach of [7], instead using the geometric approach originally introduced in [8], Chapter 3, for Brownian motion (see also [5] for further extensions and details). We show that it is possible to use this approach to establish suitable properties of the optimal stopping boundaries as c varies, thus avoiding the difficulties encountered in the approach of [7]. As in [5], eq. (4.6), we define

$$F_\lambda(x) := \frac{\psi_\lambda(x)}{\phi_\lambda(x)} = e^{2\sqrt{2\lambda}x}, \qquad x \in \mathbb{R}, \tag{3.20}$$

together with its inverse

$$F_\lambda^{-1}(y) = \frac{1}{2\sqrt{2\lambda}}\ln(y), \qquad y > 0, \tag{3.21}$$

and the function

$$H(y,c) := \begin{cases} \frac{G(F_\lambda^{-1}(y),c)}{\phi_\lambda(F_\lambda^{-1}(y))}, & y > 0 \\ 0 & y = 0. \end{cases} \tag{3.22}$$

We can now restate part of Proposition 5.12 and Remark 5.13 of [5] as follows.

Proposition 3.4. *Fix $c \in [0,\hat{c})$ and let $Q(\,\cdot\,, c)$ be the largest non-positive convex minorant of $H(\,\cdot\,, c)$ (cf. (3.22)), then $V(x,c) = \phi_\lambda(x)Q(F_\lambda(x), c)$ for all $x \in \mathbb{R}$. Moreover $\mathcal{S}_c = F_\lambda^{-1}(\mathcal{S}_c^Q)$, where $\mathcal{S}_c^Q := \{y > 0 : Q(y,c) = H(y,c)\}$ (cf. (3.18)).*

Fixing $c \in [0, \hat{c})$, we now determine H and Q as above and characterise the contact set $\mathcal{S}_c^Q$. We have (from (3.9) and (3.17))

$$G(x,c) = \begin{cases} xR(c), & x \le \gamma^o \\ -\frac{1}{\sqrt{2\lambda}} e^{-1} R(\hat{c})\phi_\lambda(x) + x(R(c) - R(\hat{c})), & x > \gamma^o. \end{cases} \tag{3.23}$$

Noting that $\phi_\lambda(F_\lambda^{-1}(y)) = y^{-\frac{1}{2}}$, $y > 0$, we obtain

$$H(y,c) = \begin{cases} 0, & y = 0 \\ \frac{1}{2\sqrt{2\lambda}} R(c) y^{\frac{1}{2}} \ln y, & 0 < y \le e^{-2} \\ -\frac{1}{\sqrt{2\lambda}} e^{-1} R(\hat{c}) + \frac{1}{2\sqrt{2\lambda}} (R(c) - R(\hat{c})) y^{\frac{1}{2}} \ln y, & y > e^{-2}. \end{cases} \tag{3.24}$$

Since the sign of $R(c)$ will play an important role in the geometry of the obstacle H, we denote by c_o the unique root of R (which was defined in (3.8)) should one exist. To simplify the following exposition we assume henceforth that c_o lies in $(0, 1)$, although the case where c_o does not exist in $(0, 1)$ is also covered by the same method presented in the next sections. The next result easily follows from properties of Φ.

Lemma 3.5. $R(1) = 0$ *and* R *is strictly concave, hence* R *is negative on* $[0, c_o)$ *and positive on* $(c_o, 1)$*; also* $\hat{c} > c_o$ *since* $k(c) = -\lambda R'(c)$ *so that at* $\hat{c}$ *the function* R *attains a positive maximum.*

We consider separately the cases $c \in [0, c_o)$ and $c \in (c_o, \hat{c})$. The intermediate case $c = c_o$ is obtained by pasting together the former two in the limits as $c \uparrow c_0$ and $c \downarrow c_0$ and noting that these limits coincide.

3.3.1 Step 2.1: $c \in (c_o, \hat{c})$

For $c \in (c_o, \hat{c})$, so that $R(c) > 0$ and $k(c) < 0$, we now apply the third heuristic of Section 3.1. Lemma 3.6 collects some elementary properties of H, while Proposition 3.7 establishes that in the present case, the minorant of Proposition 3.4 has the form illustrated in Figure 1.

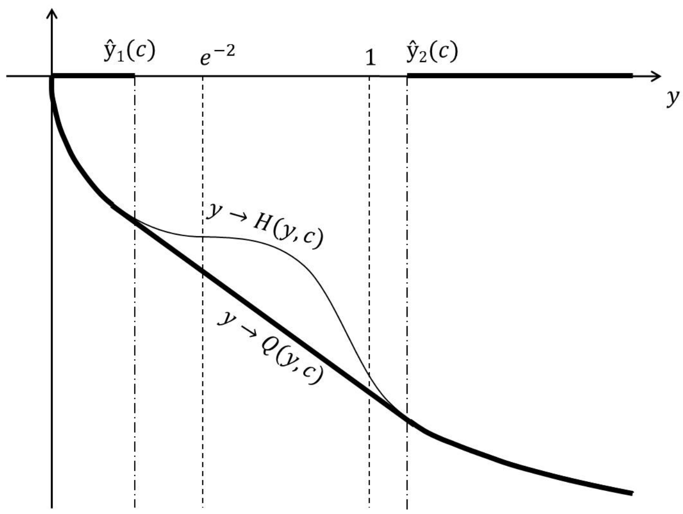

Figure 1: An illustrative plot of the functions $y \mapsto H(y,c)$ and $y \mapsto Q(y,c)$ (bold) of (3.24) and (3.31), respectively, for fixed $c \in (c_o, \hat{c})$. The bold region $[0, \hat{y}_1(c)] \cup [\hat{y}_2(c), \infty)$ on the y-axis is the stopping region $\mathcal{S}_c^Q$.

Lemma 3.6. *The function $H(\cdot, c)$ belongs to $C^1(0,\infty) \cap C([0,\infty))$. It is strictly decreasing, with $\lim_{y\downarrow 0} H_y(y,c) = -\infty$ and $\lim_{y\uparrow\infty} H_y(y,c) = 0$. Also $H_{yy}(\cdot, c) \in L^{\infty}_{loc}(\delta,\infty)$ for all $\delta > 0$ (with a single discontinuity at $y = e^{-2}$), with $H(\cdot, c)$ convex on $[0, e^{-2}) \cup (1,\infty)$ and concave in $[e^{-2}, 1]$.*

Proof. The function $H(\cdot, c)$ is continuous on $[0,\infty)$ by construction. We have

$$H_y(y,c) = \frac{1}{2\sqrt{2\lambda}} y^{-\frac{1}{2}}(1 + \frac{1}{2}\ln y) \times \begin{cases} R(c), & 0 < y \le e^{-2} \\ R(c) - R(\hat{c}), & y > e^{-2}, \end{cases} \tag{3.25}$$

$$H_{yy}(y;c) = -\frac{y^{-\frac{3}{2}}}{8\sqrt{2\lambda}} \ln(y) \times \begin{cases} R(c), & 0 < y \le e^{-2} \\ R(c) - R(\hat{c}), & y > e^{-2}, \end{cases} \tag{3.26}$$

so $H(\cdot, c) \in C^1(0,\infty)$. The remaining properties follow from the above and Lemma 3.5, since we assume $c \in (c_o, \hat{c})$. □

In the next Proposition we find a straight line tangent to $H(\cdot,c)$ at two points $\hat{y}_1(c) < e^{-2}$ and $\hat{y}_2(c) > 1$. The convexity/concavity of $H(\cdot,c)$ from Lemma 3.6 then guarantee that the largest non-positive convex minorant $Q(\cdot,c)$ of Proposition 3.4 is equal to this line on $(\hat{y}_1(c), \hat{y}_2(c))$ and equal to $H(\cdot,c)$ otherwise.

Proposition 3.7. *There exists a unique couple $(\hat{y}_1(c), \hat{y}_2(c))$ solving the system*

$$\begin{cases} H_y(y_1,c) = H_y(y_2,c) \\ H(y_1,c) - H_y(y_1,c)y_1 = H(y_2,c) - H_y(y_2,c)y_2 \end{cases} \tag{3.27}$$

for any $c \in (c_o, \hat{c})$ with $\hat{y}_1(c) \in (0, e^{-2})$ and $\hat{y}_2(c) > 1$.

Proof. Define

$$r_y(z) := H_y(y,c)(z - y) + H(y,c), \qquad y \ge 1, \tag{3.28}$$

$$g(y) := r_y(0) = -\tfrac{1}{\sqrt{2\lambda}} e^{-1} R(\hat{c}) + \tfrac{1}{2\sqrt{2\lambda}}(R(c) - R(\hat{c}))y^{\frac{1}{2}} \left(\frac{1}{2}\ln y - 1\right), \tag{3.29}$$

$$P_r(y,c) := \sup_{z\in[0,1]} h(z,y,c), \quad \text{where} \quad h(z,y,c) := r_y(z) - H(z,c), \tag{3.30}$$

so that $r_y(\cdot)$ is the straight line tangent to $H(\cdot,c)$ at y, with vertical intercept $g(y)$. The function $y \mapsto P_r(y,c)$ is decreasing and continuous and it is clear that $P_r(1,c) > 0$, since $H(\cdot,c)$ is concave on $[e^{-2}, 1]$. To establish the existence of a unique $\hat{y}_2(c) > 1$ such that $P_r(\hat{y}_2(c), c) = 0$, it is therefore sufficient to find $y > 1$ with $P_r(y,c) < 0$. Such a y exists since $g(y) \to -\infty$ as $y \to \infty$: it is clear from (3.30) that if $g(y) < H(1,c)$ then $P_r(y,c) < 0$ (both $r_y(\cdot)$ and $H(\cdot,c)$ are decreasing for $c \in (c_o, \hat{c})$).

Note that the map $z \mapsto h(z,y,c)$ is continuous, $h(1,y,c) < 0$ for $y > 1$ (cf. Figure 1) and $P_r(\hat{y}_2(c), c) = 0$; then when $y = \hat{y}_2(c)$, the supremum in (3.30) is attained on the convex portion of $H(\cdot,c)$ (i.e. in the interior of $[0,1]$) and thus is attained uniquely at a point $\hat{y}_1(c) \in (0, e^{-2})$. By construction $(\hat{y}_1(c), \hat{y}_2(c))$ uniquely solves system (3.27). □

The minorant Q is therefore

$$Q(y,c) = \begin{cases} H(y,c), & y \in [0, \hat{y}_1(c)], \\ H_y(\hat{y}_2(c),c)(y - \hat{y}_2(c)) + H(\hat{y}_2(c),c), & y \in (\hat{y}_1(c), \hat{y}_2(c)), \\ H(y,c), & y \in [\hat{y}_2(c), \infty). \end{cases} \tag{3.31}$$

The following proposition, which establishes properties of the optimal stopping boundaries which are necessary for construction of our candidate solution W, is proved in Appendix A.

Proposition 3.8. *The functions $\hat{y}_1$ and $\hat{y}_2$ of Proposition 3.7 belong to $C^1(c_o, \hat{c})$ with $c \mapsto \hat{y}_1(c)$ increasing and $c \mapsto \hat{y}_2(c)$ decreasing on $(c_o, \hat{c})$ and*

1. $\lim_{c \uparrow \hat{c}} \hat{y}_1(c) = e^{-2}$;
2. $\lim_{c \downarrow c_o} \hat{y}_1(c) = 0$;
3. $\hat{y}_2(c) < e^2$ *for all* $c \in (c_o, \hat{c})$;
4. $\lim_{c \uparrow \hat{c}} \hat{y}_1'(c) = 0$.

3.3.2 Step 2.2: $c \in [0, c_o)$.

The geometry indicated in Figure 1 does not hold in general. Indeed in the third heuristic of Section 3.1, a lower repelling boundary is suggested only for values of c close to $\hat{c}$. It turns out that 'close' in this sense means greater than or equal to c_o. We now take $c \in [0, c_o)$ and show that in this case the geometry of the auxiliary optimal stopping problems is as in Figure 2, so that each of these problems (which are parametrised by c) has just one boundary. The following Lemma has a proof very similar to that of Lemma 3.6 and it is therefore omitted.

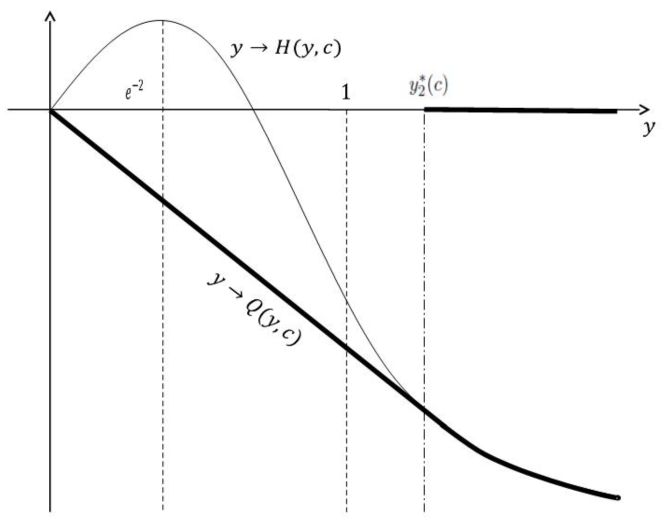

Figure 2: An illustrative plot of the functions $y \mapsto H(y, c)$ and $y \mapsto Q(y, c)$ (bold) of (3.24) and (3.37), respectively, for fixed $c \in [0, c_o)$. The bold interval $[y_2^*(c), \infty)$ on the y-axis is the stopping region $\mathcal{S}_c^Q$.

Lemma 3.9. *The function $H(\,\cdot\,, c)$ of (3.24) belongs to $C^1(0, \infty) \cap C([0, \infty))$. It is strictly increasing in $(0, e^{-2})$ and strictly decreasing in (e^{-2}, ∞). Moreover, $H_{yy}(\,\cdot\,, c) \in L^\infty_{loc}(\delta, \infty)$ for all $\delta > 0$ (with a single discontinuity at $y = e^{-2}$), $H_{yy}(1, c) = 0$ and $H(\,\cdot\,, c)$ is (strictly) concave in the interval $(0, 1)$ and it is (strictly) convex in $(1, \infty)$.*

The strict concavity of H in $(0, 1)$ suggests that there should exist a unique point $y_2^*(c) > 1$ solving

$$H_y(y, c)y = H(y, c). \tag{3.32}$$

The straight line $r_{y_2^*} : [0, \infty) \mapsto (-\infty, 0]$ given by

$$r_{y_2^*}(y) := H(y_2^*(c), c) + H_y(y_2^*(c), c)(y - y_2^*(c))$$

is then tangent to H at $y_2^*(c)$ and $r_{y_2^*}(0) = 0$. Notice that by (3.24) and (3.25), equation (3.32) may be rewritten in the equivalent form

$$F_3(y; c) = 0, \tag{3.33}$$

where we define the jointly continuous function $F_3 : (0, \infty) \times [0, 1] \mapsto \mathbb{R}$ as

$$F_3(y; c) := y^{\frac{1}{2}}\big(1 - \tfrac{1}{2}\ln y\big) - \frac{2e^{-1}R(\hat{c})}{R(\hat{c}) - R(c)}. \tag{3.34}$$

The proof of the next result may be found in Appendix A.

Proposition 3.10. *For each $c \in [0, c_o)$ there exists a unique point $y_2^*(c) \in (1, e^2)$ solving (3.33). The function $c \mapsto y_2^*(c)$ is decreasing and belongs to $C^1([0, c_o))$. Moreover, for $\hat{y}_2$ as in Proposition 3.7 one has*

$$y_2^*(c_o-) := \lim_{c\uparrow c_o} y_2^*(c) = \lim_{c\downarrow c_o} \hat{y}_2(c) =: \hat{y}_2(c_o+) \tag{3.35}$$

and

$$(y_2^*)'(c_o-) := \lim_{c\uparrow c_o} (y_2^*)'(c) = \lim_{c\downarrow c_o} (\hat{y}_2)'(c) =: (\hat{y}_2)'(c_o+). \tag{3.36}$$

The minorant Q is therefore

$$Q(y, c) = \begin{cases} H_y(y_2^*(c), c)y, & y \in [0, y_2^*(c)), \\ H(y, c), & y \in [y_2^*(c), \infty). \end{cases} \tag{3.37}$$

Note that (3.37) may be rewritten in the form (3.31) taking $\hat{y}_1(c) = 0$ and replacing $\hat{y}_2$ by y_2^*, since y_2^* solves (3.33).

3.3.3 The candidate value function W

We now combine the above cases, beginning with some definitions. The function $\hat{y}_1 \in C^1(c_o, \hat{c})$ of Section 3.3.1 may be extended to a function $\hat{y}_1 \in C^0([0, \hat{c}))$ by setting $\hat{y}_1(c) = 0$ for $c \in [0, c_o]$. The function $\hat{y}_2 \in C^1(c_o, \hat{c})$ may be extended to $\hat{y}_2 \in C^1([0, \hat{c}))$ (thanks to Proposition 3.10) by setting $\hat{y}_2(c) = y_2^*(c)$ for $c \in [0, c_o]$. With these definitions the expression (3.31), which we now recall, is valid for all $c \in [0, \hat{c})$:

$$Q(y, c) = \begin{cases} H(y, c), & y \in [0, \hat{y}_1(c)], \\ H_y(\hat{y}_2(c), c)(y - \hat{y}_2(c)) + H(\hat{y}_2(c), c), & y \in (\hat{y}_1(c), \hat{y}_2(c)), \\ H(y, c), & y \in [\hat{y}_2(c), \infty), \end{cases} \tag{3.38}$$

Note that by construction and thanks to the regularity of the boundaries (cf. Proposition 3.10) Q is well defined across c_o and is continuous on $(0, \infty) \times [0, \hat{c})$.

Proposition 3.11. *The function Q lies in $C^1((0, \infty) \times [0, \hat{c}))$.*

Proof. Denoting

$$A(y, c) := H_y(\hat{y}_2(c), c)(y - \hat{y}_2(c)) + H(\hat{y}_2(c), c),$$

the surface A is C^1 in the region $\{(y, c) : y \in [\hat{y}_1(c), \hat{y}_2(c)],\ c \in [0, \hat{c})\}$ (continuation set), since $\hat{y}_2 \in C^1([0, \hat{c}))$, whereas H is C^1 in the region $\{(y, c) : y \in (0, \hat{y}_1(c)] \cup [\hat{y}_2(c), \infty),\ c \in [0, \hat{c})\}$ (stopping set), where we adopt the convention $(0, \hat{y}_1(c)] = \emptyset$ for $c \in [0, c_o]$. As a consequence these properties hold for Q and it remains to verify whether the pasting across the boundaries is C^1 as well. At the two boundaries we clearly have (cf. Proposition 3.7)

$$H(\hat{y}_1(c),c) = A(\hat{y}_1(c),c), \quad c \in (c_o,\hat{c}) \ \text{ and } \ H(\hat{y}_2(c),c) = A(\hat{y}_2(c),c), \quad c \in [0,\hat{c}). \tag{3.39}$$

Recall that $\hat{y}_1 \in C^1(c_o,\hat{c})$, then an application of the chain rule to the left hand side of (3.39) gives

$$H_y(\hat{y}_1(c),c)\hat{y}_1'(c) + H_c(\hat{y}_1(c),c) = A_y(\hat{y}_1(c),c)\hat{y}_1'(c) + A_c(\hat{y}_1(c),c) \tag{3.40}$$

for $c \in (c_o,\hat{c})$. Hence $H_c(\hat{y}_1(c),c) = A_c(\hat{y}_1(c),c)$ for $c \in (c_o,\hat{c})$ since by our geometric construction of Q we know that $Q_y = A_y = H_y$ at the two boundaries. Similar arguments also provide $H_c(\hat{y}_2(c),c) = A_c(\hat{y}_2(c),c)$ for $c \in [0,\hat{c})$. □

Using Proposition 3.4 and setting

$$\hat{\beta}(c) := F_\lambda^{-1}(\hat{y}_2(c)),\ c \in [0,\hat{c}) \quad \text{and} \quad \hat{\gamma}(c) := \begin{cases} -\infty, & c \in [0,c_o], \\ F_\lambda^{-1}(\hat{y}_1(c)), & c \in (c_o,\hat{c}) \end{cases} \tag{3.41}$$

we obtain the following expression for V:

$$V(x,c) = \begin{cases} G(x,c), & x \in (-\infty,\hat{\gamma}(c)] \\ \phi_\lambda(x)\Big[H_y(F_\lambda(\hat{\beta}(c)),c)\Big(F_\lambda(x) - F_\lambda(\hat{\beta}(c))\Big) + H(F_\lambda(\hat{\beta}(c)),c)\Big], & x \in (\hat{\gamma}(c),\hat{\beta}(c)) \\ G(x,c), & x \in [\hat{\beta}(c),\infty). \end{cases} \tag{3.42}$$

Remark 3.12. *For $c \in (c_o,\hat{c})$ note that $\hat{y}_1$ and $\hat{y}_2$ solve (3.27) and the second expression in (3.38) may be equivalently rewritten in terms of $\hat{y}_1$, i.e. $Q(y,c) = H_y(\hat{y}_1(c),c)(y - \hat{y}_1(c)) + H(\hat{y}_1(c),c)$ for $y \in (\hat{y}_1(c),\hat{y}_2(c))$. Analogously (3.42) may be equivalently rewritten in terms of $\hat{\gamma}$, that is $V(x,c) = \phi_\lambda(x)\Big[H_y(F_\lambda(\hat{\gamma}(c)),c)\Big(F_\lambda(x) - F_\lambda(\hat{\gamma}(c))\Big) + H(F_\lambda(\hat{\gamma}(c)),c)\Big]$ for $x \in (\hat{\gamma}(c),\hat{\beta}(c))$.*

Corollary 3.13. *We have*

i) The boundary $\hat{\beta}$ lies in $C^1([0,\hat{c}))$ and it is strictly decreasing with $\hat{\beta}(c) \in (0, 1/\sqrt{2\lambda})$ for all $c \in [0,\hat{c})$;

ii) The boundary $\hat{\gamma}$ lies in $C^1((c_o,\hat{c}])$, it is strictly increasing with $\hat{\gamma}(c) \leq -1/\sqrt{2\lambda}$ for all $c \in [0,\hat{c})$.

Proof. This follows immediately from Propositions 3.8, 3.10 and (3.41). □

Proposition 3.14. *The value function V of (3.16) belongs to $C^1(\mathbb{R} \times [0,\hat{c}))$ with $V_{xx} \in L^\infty_{loc}(\mathbb{R} \times (0,\hat{c}))$. Moreover V satisfies*

$$\begin{cases} \frac{1}{2}V_{xx}(x,c) - \lambda V(x,c) = 0 & \text{for } \hat{\gamma}(c) < x < \hat{\beta}(c),\ c \in [0,\hat{c}) \\ \frac{1}{2}V_{xx}(x,c) - \lambda V(x,c) \geq 0 & \text{for a.e. } (x,c) \in \mathbb{R} \times [0,\hat{c}) \\ V(x,c) = G(x,c) & \text{for } x \leq \hat{\gamma}(c),\ x \geq \hat{\beta}(c),\ c \in [0,\hat{c}) \\ V_x(x,c) = G_x(x,c) & \text{for } x \leq \hat{\gamma}(c),\ x \geq \hat{\beta}(c),\ c \in [0,\hat{c}) \\ V_c(x,c) = G_c(x,c) & \text{for } x \leq \hat{\gamma}(c),\ x \geq \hat{\beta}(c),\ c \in [0,\hat{c}). \end{cases} \tag{3.43}$$

Proof. Step 1. From Proposition 3.4 we have that $Q \in C^1((0,\infty)\times[0,\hat{c}))$ implies $V \in C^1(\mathbb{R}\times[0,\hat{c}))$. Analogously to prove that V_{xx} is locally bounded it suffices to show it for Q_{yy}. Since $Q_{yy} = H_{yy}$ for $x \le \hat{\gamma}(c)$, $x \ge \hat{\beta}(c)$, $c \in [0,\hat{c})$ and Q is linear in y elsewhere the claim follows.

Step 2. The fact that V solves (3.43) is a consequence of its regularity and derives from standard Markovian arguments which are well known in optimal stopping theory (see for example [20], Sec. 7). $\square$

Thanks to Proposition 3.14 we establish some properties of $W^1(x,c) = x\Phi(c) + V(x,c)$ (Eq. (3.15)) required in order to verify that W^1 and W^o, when pasted together at $c = \hat{c}$, provide a function W that solves the HJB equation (3.2).

Corollary 3.15. *$W^1 \in C^1(\mathbb{R}\times[0,\hat{c}))$, with $W^1_{xx} \in L^\infty_{loc}(\mathbb{R}\times(0,\hat{c}))$ and in particular we have*

$$W^1_c(x,c) = -x \quad \text{and} \quad W^1_x(x,c) = \hat{c} - c + W^o_x(x,\hat{c}) \tag{3.44}$$

for $x \in (-\infty, \hat{\gamma}(c)] \cup [\hat{\beta}(c), +\infty)$ and $c \in [0,\hat{c})$.

The next two propositions follow from results collected above and their detailed proofs are given in Appendix A.

Proposition 3.16. *$W^1_c(x;c) \ge -x$ for all $(x,c) \in \mathbb{R}\times[0,\hat{c})$.*

Proposition 3.17. *Let*

$$W(x,c) := \begin{cases} W^1(x,c), & \text{for } (x,c) \in \mathbb{R}\times[0,\hat{c}) \\ W^o(x,c), & \text{for } (x,c) \in \mathbb{R}\times[\hat{c},1], \end{cases} \tag{3.45}$$

then $W \in C^1(\mathbb{R}\times[0,1])$ and $W_{xx} \in L^\infty_{loc}(\mathbb{R}\times[0,1])$.

Remark 3.18. *The case $c = \hat{c}$ deserves special mention. From the point of view of the auxiliary optimal stopping problem, the interval (γ^o,∞) is then a region of "indifference": continuation inside this region produces the same performance as stopping at once (by an application of Itô's formula). Conventionally this would be resolved by taking the smallest possible optimal stopping time (which, in this case, is $\tau = 0$ almost surely). From the point of view of our SSC problem, we have the heuristic from Section 3.2 that exercising control is strictly suboptimal when $(x,c) \in (\gamma^o,\infty)\times\{\hat{c}\}$ and our heuristics are confirmed as a result of the verification argument in the next section. We conclude that when $(x,c) \in (\gamma^o,\infty)\times\{\hat{c}\}$, it would be contradictory to exercise any nonzero amount of control, however small, before the first hitting time of $(\gamma^o,\hat{c})$ by $(X_t,\hat{c})$. When $c = \hat{c}$ there is therefore no indifference in the SSC problem: we are forced to choose the* largest *possible optimal stopping time in the auxiliary optimal stopping problem, that is $\tau_o := \inf\{t \ge 0 : X_t \le \gamma^o\}$.*

4 Verification theorem and the optimal control

The function $\hat{\gamma}$ can be extended to $[\hat{c},1]$ by putting $\hat{\gamma}(c) = \gamma^o$, $c \in [\hat{c},1]$ (by (1) and (2) of Proposition 3.8 and (3.41)). This extension is C^1 on $(c_o,1]$ and will be assumed in the rest of the paper. We also set $\hat{\beta}(c) = +\infty$ for $c \in [\hat{c},1]$.

In this section we establish the optimality of the candidate value function W and construct the optimal control policy for problem (2.3), which is purely discontinuous since all free boundaries are repelling. Several results obtained in Section 3 are summarised in the following Proposition.

Proposition 4.1. *The function W of* (3.45) *solves the variational problem* (3.2)*. Moreover, $|W(x,c)| \le K(1+|x|)$ or some $K>0$ and $W(x,1)=U(x,1)=0$.*

Proof. The functions W^o and W^1 solve the HJB equation separately on $\mathbb{R}\times[\hat c,1)$ and $\mathbb{R}\times(0,\hat c)$ (see Proposition 3.2 for the claim regarding W^o and Propositions 3.14, 3.16 and Corollary 3.15 for the claim regarding W^1). Then Proposition 3.17 guarantees that W solves the HJB on $\mathbb{R}\times(0,1)$ as required. From the definitions of W, W^o and W^1 (see (3.45), (3.9) and (3.14)) one also obtains the sublinear growth property and $W(x,1)=0$. □

Let $(x,c)\in\mathbb{R}\times[0,1]$ be arbitrary but fixed and consider the two dimensional dynamics $(X^x, C^{c,\nu})$ for an arbitrary admissible control ν. Define the stopping times

$$\tau_{\hat\beta} := \inf\{t\ge 0 : X^x_t \ge \hat\beta(C^{c,\nu}_t)\}, \qquad \tau_{\hat\gamma} := \inf\{t\ge 0 : X^x_t \le \hat\gamma(C^{c,\nu}_t)\}, \tag{4.1}$$

and

$$\tau^* := \tau_{\hat\beta}\wedge\tau_{\hat\gamma}, \qquad \sigma^* := \inf\{t\ge \tau_{\hat\beta} : X^x_t \le \hat\gamma(C^{c,\nu}_t)\}, \tag{4.2}$$

with the convention $\inf\emptyset = +\infty$. Here we note that such stopping times could be P-a.s. infinite depending on the particular choice of ν but in general $\sigma^*\ge\tau_{\hat\beta}\ge\tau^*$ P-a.s. Then, recalling that $\hat c$ is the unique solution in $(0,1)$ of (2.6) we introduce the admissible purely discontinuous control

$$\nu^*_t := (1-c)\mathbb{1}_{\{t>\tau^*\}}\mathbb{1}_{\{\tau^*=\tau_{\hat\gamma}\}} + \big[(\hat c - c)\mathbb{1}_{\{t\le\sigma^*\}} + (1-\hat c)\mathbb{1}_{\{t>\sigma^*\}}\big]\mathbb{1}_{\{t>\tau^*\}}\mathbb{1}_{\{\tau^*=\tau_{\hat\beta}\}}. \tag{4.3}$$

The control ν^* prescribes to do nothing until the uncontrolled process X^x leaves the interval $(\hat\gamma(c),\hat\beta(c))$, where $c\in[0,1)$ is the initial inventory level. Then, if $\tau_{\hat\gamma}<\tau_{\hat\beta}$ one should immediately exert all the available control after hitting the lower boundary $\hat\gamma(c)$, if instead $\tau_{\hat\gamma}>\tau_{\hat\beta}$ one should initially increase the inventory to $\hat c$ after hitting the upper boundary $\hat\beta(c)$ and then wait until X hits the new value $\hat\gamma(\hat c)$ of the lower boundary before exerting all available control. We remark that under the strategy ν^* the stopping time $\tau_{\hat\gamma}$ is P-a.s. finite. In Figure 3 below we provide an illustrative diagram of the boundaries and of the behaviour of control ν^* of (4.3). Optimality of ν^* for problem (2.3) is proved in the next theorem.

Theorem 4.2. *The admissible control ν^* of* (4.3) *is optimal for problem* (2.3) *and $W\equiv U$.*

Proof. The proof is based on a verification argument and, as usual, it splits into two steps.

Step 1. Fix $(x,c)\in\mathbb{R}\times[0,1]$ and take $R>0$. Set $\tau_R := \inf\big\{t\ge0 : X^x_t\notin(-R,R)\big\}$, take an admissible control ν, and recall the regularity results for W in Proposition 3.17. Then we can use Itô's formula in the weak version of [12], Chapter 8, Section VIII.4, Theorem 4.1, up to the stopping time $\tau_R\wedge T$, for some $T>0$, to obtain

$$\begin{aligned}
W(x;c) =& \mathsf{E}\Big[e^{-\lambda(\tau_R\wedge T)}W(X^x_{\tau_R\wedge T}, C^{c,\nu}_{\tau_R\wedge T})\Big] - \mathsf{E}\Big[\int_0^{\tau_R\wedge T} e^{-\lambda s}(\mathbb{L}_X-\lambda)W(X^x_s,C^{c,\nu}_s)ds\Big]\\
&- \mathsf{E}\Big[\int_0^{\tau_R\wedge T} e^{-\lambda s}W_c(X^x_s,C^{c,\nu}_s)d\nu_s\Big]\\
&- \mathsf{E}\Big[\sum_{0\le s<\tau_R\wedge T} e^{-\lambda s}\big(W(X^x_s,C^{c,\nu}_{s+}) - W(X^x_s,C^{c,\nu}_s) - W_c(X^x_s,C^{c,\nu}_s)\Delta\nu_s\big)\Big]
\end{aligned}$$

where $\Delta\nu_s := \nu_{s+}-\nu_s$ and the expectation of the stochastic integral vanishes since W_x is bounded on $(x,c)\in[-R,R]\times[0,1]$.

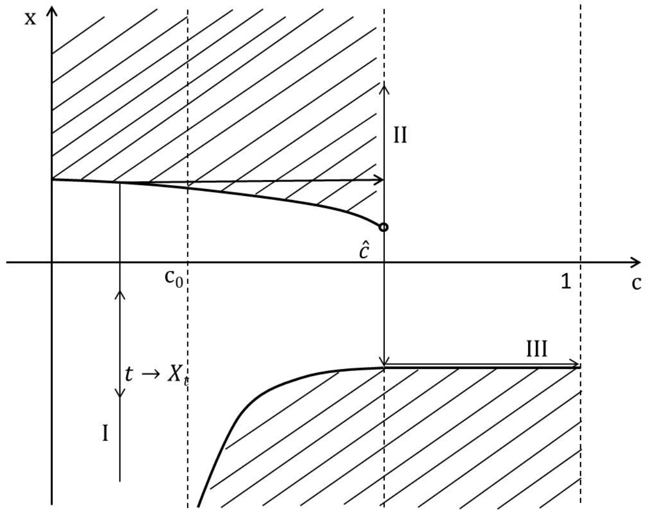

Figure 3: An illustrative diagram of the the free boundaries and of the optimal control ν^* of (4.3). The upper boundary $\hat{\beta}$ and the lower boundary $\hat{\gamma}$ split the state space into the inaction region (white) and action region (hatched). When the initial state is (x, c) with $c \in [0, c_o)$ and $x < \hat{\beta}(c)$ one observes the following three regimes: in regime (I) the process X diffuses until hitting $\hat{\beta}(c)$, then an amount $\Delta\nu = \hat{c} - c$ of control is exerted, horizontally pushing the process (X, C) to the regime (II); there X continues to diffuse until it hits γ^o and at that point all remaining control is exercised and (X, C) is pushed horizontally until the inventory reaches its maximum (III).

Now, recalling that any $\nu \in \mathcal{A}_c$ can be decomposed into the sum of its continuous part and its pure jump part, i.e. $d\nu = d\nu^{cont} + \Delta\nu$, one has (see [12], Chapter 8, Section VIII.4, Theorem 4.1 at pp. 301-302)

$$\begin{aligned} W(x,c) =& \mathsf{E}\left[e^{-\lambda(\tau_R\wedge T)}W(X^x_{\tau_R\wedge T}, C^{c,\nu}_{\tau_R\wedge T})\right] - \mathsf{E}\left[\int_0^{\tau_R\wedge T} e^{-\lambda s}(\mathbb{L}_X - \lambda)W(X^x_s, C^{c,\nu}_s)ds\right] \\ &- \mathsf{E}\left[\int_0^{\tau_R\wedge T} e^{-\lambda s}W_c(X^x_s, C^{c,\nu}_s)d\nu^{cont}_s + \sum_{0\le s<\tau_R\wedge T} e^{-\lambda s}\left(W(X^x_s, C^{c,\nu}_{s+}) - W(X^x_s, C^{c,\nu}_s)\right)\right]. \end{aligned}$$

Since W satisfies the Hamilton-Jacobi-Bellman equation (3.2) (cf. Proposition 4.1) and by writing

$$W(X^x_s, C^{c,\nu}_{s+}) - W(X^x_s, C^{c,\nu}_s) = \int_0^{\Delta\nu_s} W_c(X^x_s, C^{c,\nu}_s + u)du, \tag{4.4}$$

we obtain

$$\begin{aligned} W(x,c) \le& \mathsf{E}\left[e^{-\lambda(\tau_R\wedge T)}W(X^x_{\tau_R\wedge T}, C^{c,\nu}_{\tau_R\wedge T})\right] + \mathsf{E}\left[\int_0^{\tau_R\wedge T} e^{-\lambda s}\lambda X^x_s\Phi(C^{c,\nu}_s)ds\right] \\ &+ \mathsf{E}\left[\int_0^{\tau_R\wedge T} e^{-\lambda s}X^x_s d\nu^{cont}_s\right] + \mathsf{E}\left[\sum_{0\le s<\tau_R\wedge T} e^{-\lambda s}X^x_s\Delta\nu_s\right] \\ =& \mathsf{E}\left[e^{-\lambda(\tau_R\wedge T)}W(X^x_{\tau_R\wedge T}, C^{c,\nu}_{\tau_R\wedge T}) + \int_0^{\tau_R\wedge T} e^{-\lambda s}\lambda X^x_s\Phi(C^{c,\nu}_s)ds + \int_0^{\tau_R\wedge T} e^{-\lambda s}X^x_s d\nu_s\right]. \end{aligned} \tag{4.5}$$

When taking limits as $R \to \infty$ we have $\tau_R \wedge T \to T$, P-a.s. By standard properties of Brownian motion it is easy to prove that the integral terms in the last expression on the right hand side of (4.5) are uniformly bounded in $L^2(\Omega, \mathsf{P})$, hence uniformly integrable. Moreover, W

has sub-linear growth by Proposition 4.1. Then we also take limits as $T \uparrow \infty$ and it follows that

$$W(x,c) \leq \mathsf{E}\bigg[\int_0^\infty e^{-\lambda s}\lambda X^x_s \Phi(C^{c,\nu}_s)ds + \int_0^\infty e^{-\lambda s}X^x_s d\nu_s\bigg], \tag{4.6}$$

due to the fact that $\lim_{T\to\infty}\mathsf{E}[e^{-\lambda T}W(X^x_T, C^{c,\nu}_T)] = 0$. Since the latter holds for all admissible ν we have $W(x,c) \leq U(x;c)$.

Step 2.

If $c = 1$ then $W(x,1) = 0 = U(x,1)$. Then take $c \in [0,1)$ and define $C^*_t := C^{c,\nu^*}_t = c + \nu^*_t$, with ν^* as in (4.3). Applying Itô's formula again (possibly using localisation arguments as in Step 1.) up to time $\tau_{\hat\gamma}$ (cf. (4.1)) we find

$$\begin{aligned} W(x,c) =&\mathsf{E}\bigg[e^{-\lambda\tau_{\hat\gamma}}W(X^x_{\tau_{\hat\gamma}}, C^*_{\tau_{\hat\gamma}}) - \int_0^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds\bigg] \\ &- \mathsf{E}\bigg[\sum_{0\leq s<\tau_{\hat\gamma}} e^{-\lambda s}\big(W(X^x_s, C^*_{s+}) - W(X^x_s, C^*_s)\big)\bigg], \end{aligned} \tag{4.7}$$

where we have used that ν^* does not have a continuous part. We also recall, as already observed, that $\tau_{\hat\gamma} < +\infty$, P-a.s. under the control policy of ν^*.

From (4.2) one has $\tau^* \leq \tau_{\hat\gamma}$, P-a.s. and therefore we can always write

$$\begin{aligned} &\int_0^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds \\ &= \int_0^{\tau^*} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds + \int_{\tau^*}^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds \\ &= -\int_0^{\tau^*} e^{-\lambda s}\lambda X^x_s\Phi(C^*_s)ds + \int_{\tau^*}^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds \end{aligned} \tag{4.8}$$

where the last equality follows by recalling that $(\mathbb{L}_X - \lambda)W(x,c) = -\lambda x\Phi(c)$ for $\hat\gamma(c) < x < \hat\beta(c)$ and hence it holds in the first integral for $s \leq \tau^*$. To evaluate the last term of (4.8) we study separately the events $\{\tau^* = \tau_{\hat\beta}\}$ and $\{\tau^* = \tau_{\hat\gamma}\}$. We start by observing that under the control strategy ν^* one has $\{\tau^* = \tau_{\hat\beta}\} = \{\tau_{\hat\gamma} = \sigma^*\}$ and we get

$$\mathbb{1}_{\{\tau^*=\tau_{\hat\beta}\}}\int_{\tau^*}^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds = -\mathbb{1}_{\{\tau^*=\tau_{\hat\beta}\}}\int_{\tau^*}^{\tau_{\hat\gamma}} e^{-\lambda s}\lambda X^x_s\Phi(C^*_s)ds \tag{4.9}$$

by Proposition 3.2 since $(X^x_s, C^*_s) = (X^x_s, \hat c)$ for any $\tau^* < s \leq \tau_{\hat\gamma} = \sigma^*$ on $\{\tau^* = \tau_{\hat\beta}\}$. On the other hand

$$\mathbb{1}_{\{\tau^*=\tau_{\hat\gamma}\}}\int_{\tau^*}^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds = 0 = \mathbb{1}_{\{\tau^*=\tau_{\hat\gamma}\}}\int_{\tau^*}^{\tau_{\hat\gamma}} e^{-\lambda s}\lambda X^x_s\Phi(C^*_s)ds. \tag{4.10}$$

Then it follows from (4.8), (4.9) and (4.10) that

$$\int_0^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds = -\int_0^{\tau_{\hat\gamma}} \lambda X^x_s\Phi(C^*_s)ds. \tag{4.11}$$

Moreover $\Phi(C^*_s) = 0$ for any $s > \tau_{\hat\gamma}$ because $C^*_s = 1$ for any such s and thus we finally get from (4.11)

$$\int_0^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds = -\int_0^{\infty} \lambda X^x_s\Phi(C^*_s)ds. \tag{4.12}$$

Note that under the control strategy ν^* we also have $\{\tau_{\hat\gamma} < \tau_{\hat\beta}\} = \{\tau_{\hat\gamma} < \sigma^*\}$ and $\{\tau_{\hat\gamma} > \tau_{\hat\beta}\} = \{\tau_{\hat\gamma} = \sigma^*\}$, then from (4.3) we have

$$\begin{aligned}
&\mathsf{E}\left[e^{-\lambda\tau_{\hat\gamma}}W(X^x_{\tau_{\hat\gamma}}, C^*_{\tau_{\hat\gamma}})\right] \\
=&\mathsf{E}\left[\mathbb{1}_{\{\tau_{\hat\gamma}>\tau_{\hat\beta}\}}e^{-\lambda\tau_{\hat\gamma}}W(\hat\gamma(\hat c),\hat c)\right] + \mathsf{E}\left[\mathbb{1}_{\{\tau_{\hat\gamma}<\tau_{\hat\beta}\}}e^{-\lambda\tau_{\hat\gamma}}W(\hat\gamma(c),c)\right] \\
=&\mathsf{E}\left[\mathbb{1}_{\{\tau_{\hat\gamma}>\tau_{\hat\beta}\}}e^{-\lambda\tau_{\hat\gamma}}\hat\gamma(\hat c)(1-\hat c)\right] + \mathsf{E}\left[\mathbb{1}_{\{\tau_{\hat\gamma}<\tau_{\hat\beta}\}}e^{-\lambda\tau_{\hat\gamma}}\hat\gamma(c)(1-c)\right] \\
=&\mathsf{E}\left[\int_{\tau_{\hat\gamma}}^{\infty} e^{-\lambda s}X^x_s d\nu^*_s\right]
\end{aligned}\tag{4.13}$$

by using that $W(\hat\gamma(c),c) = \hat\gamma(c)(1-c)$ for all $c\in[0,1)$ as proved in Section 3 (see also Figure 3).

For the jump part of the control, i.e. for the last term in (4.7), again we argue in a similar way as above and use that on the event $\{\tau^* = \tau_{\hat\gamma}\}$ there is no jump strictly prior to $\tau_{\hat\gamma}$ and the sum in (4.7) is zero, whereas on the event $\{\tau^* = \tau_{\hat\beta}\}$ a single jump occurs prior to $\tau_{\hat\gamma}$, precisely at $\tau_{\hat\beta}$. This gives

$$\begin{aligned}
&\mathsf{E}\Big[\sum_{0\le s<\tau_{\hat\gamma}} e^{-\lambda s}\big(W(X^x_s, C^*_{s+}) - W(X^x_s, C^*_s)\big)\Big] \\
=&\Big[\mathbb{1}_{\{\tau^*=\tau_{\hat\gamma}\}}\cdot 0 + \mathbb{1}_{\{\tau^*=\tau_{\hat\beta}\}}e^{-\lambda\tau_{\hat\beta}}X^x_{\tau_{\hat\beta}}(\hat c - c)\Big] = \mathsf{E}\Big[\int_0^{\tau_{\hat\gamma}} e^{-\lambda s}X^x_s d\,\nu^*_s\Big].
\end{aligned}\tag{4.14}$$

Combining (4.12), (4.13) and (4.14) it follows from (4.7) that

$$W(x,c) = \mathsf{E}\Big[\int_0^\infty e^{-\lambda s}\lambda\, X^x_s\,\Phi(C^*_s)ds + \int_0^\infty e^{-\lambda s}X^x_s d\nu^*_s\Big] \ge U(x,c), \tag{4.15}$$

which together with Step 1. implies $W(x,c) = U(x,c)$, $(x,c)\in\mathbb{R}\times[0,1]$ and ν^* of (4.3) is optimal. $\square$

Remark 4.3. *Note that, unusually, the regions* (3.3) *actually differ from the shaded region in Figure 3 along the line $c = \hat c$, since from* (3.9) *we see that $W^o_c(x,\hat c) = -x$ for all $x\in\mathbb{R}$. This is firstly possible because $(\mathbb{L}_X - \lambda)W(x,\hat c) = -\lambda\Phi(\hat c)x$ for $x > \gamma^o$ and therefore, as long as X stays above γ^o, an inaction strategy does not increase the overall costs; secondly, we have pointed out in Remark 3.18 that acting while X is above γ^o is strictly suboptimal. The action and inaction regions depicted in Figure 3, and* not *those guessed in* (3.3)*, therefore give the unique optimal policy.*

A Some proofs needed in Section 3

Proof. **[Proposition 3.8]**

Rewrite (3.27) as

$$F_1(\hat y_1(c),\hat y_2(c);c) = 0 \quad \text{and} \quad F_2(\hat y_1(c),\hat y_2(c);c) = 0, \tag{A-1}$$

with the two functions $F_i : (0,\infty)\times(0,\infty)\times[0,1]$, $i = 1,2$, defined by

$$F_1(x,y;c) := x^{-\frac12}(1+\tfrac12\ln x)R(c) - y^{-\frac12}(1+\tfrac12\ln y)(R(c)-R(\hat c)), \tag{A-2}$$

$$F_2(x,y;c) := x^{\frac12}(1-\tfrac12\ln x)R(c) - y^{\frac12}(1-\tfrac12\ln y)(R(c)-R(\hat c)) - 2e^{-1}R(\hat c). \tag{A-3}$$

The Jacobian matrix

$$J(x,y,c)=\begin{bmatrix}\frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y}\\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y}\end{bmatrix}(x,y,c)=\frac{1}{4}\begin{bmatrix}-R(c)x^{-\frac{3}{2}}\ln x & (R(c)-R(\hat c))y^{-\frac{3}{2}}\ln y\\ -R(c)x^{-\frac{1}{2}}\ln x & (R(c)-R(\hat c))y^{-\frac{1}{2}}\ln y\end{bmatrix}$$

has determinant

$$D(x,y,c):=\frac{1}{16}\big[R(c)-R(\hat c)\big]R(c)\frac{1}{\sqrt{xy}}\Big(\frac{1}{y}-\frac{1}{x}\Big)\ln x\ln y$$

which is strictly negative when $x<e^{-2}$, $y>1$ and $c\in(c_0,\hat c)$. To simplify notation we suppress the dependency of $\hat y_1$ and $\hat y_2$ on c. Total differentiation of (A-1) with respect to c and the Implicit Function Theorem imply that $\hat y_1$ and $\hat y_2$ lie in $C^1(c_o,\hat c)$, with

$$\begin{bmatrix}\hat y_1'(c)\\ \hat y_2'(c)\end{bmatrix}=J^{-1}\begin{bmatrix}-\frac{\partial F_1}{\partial c}\\ -\frac{\partial F_1}{\partial c}\end{bmatrix}(\hat y_1,\hat y_2,c)=R'(c)J^{-1}(\hat y_1,\hat y_2,c)\begin{bmatrix}\hat y_1^{-\frac{1}{2}}(1+\frac{1}{2}\ln\hat y_1)-\hat y_2^{-\frac{1}{2}}(1+\frac{1}{2}\ln\hat y_2)\\ \hat y_1^{\frac{1}{2}}(1-\frac{1}{2}\ln\hat y_1)-\hat y_2^{\frac{1}{2}}(1-\frac{1}{2}\ln\hat y_2)\end{bmatrix}$$

so that

$$\begin{aligned}\hat y_1'(c)=&\frac{1}{4}\frac{R'(c)\big[R(c)-R(\hat c)\big]}{D(\hat y_1,\hat y_2,c)}\Big[\sqrt{\frac{\hat y_1}{\hat y_2}}(1-\frac{1}{2}\ln\hat y_1)+\ln\hat y_2-\sqrt{\frac{\hat y_2}{\hat y_1}}(1+\frac{1}{2}\ln\hat y_1)\Big]\hat y_2^{-1}\ln\hat y_2\\ =:&D_1(\hat y_1,\hat y_2,c)/D(\hat y_1,\hat y_2,c)\end{aligned}\tag{A-4}$$

and

$$\begin{aligned}\hat y_2'(c)=&-\frac{1}{4}\frac{R(c)R'(c)}{D(\hat y_1,\hat y_2,c)}\Big[\hat y_1^{\frac{1}{2}}\ln\hat y_1-\hat y_1\hat y_2^{-\frac{1}{2}}(1+\frac{1}{2}\ln\hat y_2)+\hat y_2^{\frac{1}{2}}(1-\frac{1}{2}\ln\hat y_2)\Big]\hat y_1^{-\frac{3}{2}}\ln\hat y_1\\ =:&D_2(\hat y_1,\hat y_2,c)/D(\hat y_1,\hat y_2,c)\end{aligned}\tag{A-5}$$

It is not hard to verify that $\hat y_1'(c)>0$ by using $0<\hat y_1<e^{-2}$, $\hat y_2>1$ and $R(c)<R(\hat c)$. The sign of the right hand side of (A-5) is the opposite of the one of

$$\hat D:=\hat y_1^{\frac{1}{2}}\ln\hat y_1-\hat y_1\hat y_2^{-\frac{1}{2}}(1+\frac{1}{2}\ln\hat y_2)+\hat y_2^{\frac{1}{2}}(1-\frac{1}{2}\ln\hat y_2)\tag{A-6}$$

since $\ln\hat y_1<0$, $R'(c)>0$ and $D(\hat y_1,\hat y_2,c)<0$, $c\in(c_0,\hat c)$. Recalling now (A-1), (A-2) and (A-3), we obtain

$$\hat y_2^{\frac{1}{2}}\big(1-\frac{1}{2}\ln\hat y_2\big)=\frac{R(c)}{R(c)-R(\hat c)}\hat y_1^{\frac{1}{2}}\big(1-\frac{1}{2}\ln\hat y_1\big)-\frac{2e^{-1}}{R(c)-R(\hat c)}R(\hat c),$$

and

$$\hat y_2^{-\frac{1}{2}}\big(1+\frac{1}{2}\ln\hat y_2\big)=\frac{R(c)}{R(c)-R(\hat c)}\hat y_1^{-\frac{1}{2}}\big(1+\frac{1}{2}\ln\hat y_1\big),$$

which plugged into (A-6) give

$$\hat D=-\frac{R(\hat c)}{(R(c)-R(\hat c))}(\sqrt{\hat y_1}\ln\hat y_1+2e^{-1})=:-\frac{R(\hat c)}{(R(c)-R(\hat c))}q(\hat y_1).$$

It is now easy to see that $x\mapsto q(x)$ is strictly decreasing on $(0,e^{-2})$ and such that $q(e^{-2})=0$ and $\lim_{x\downarrow 0}q(x)=2e^{-1}>0$. Hence $q(\hat y_1)>0$ implies that $\hat D>0$ and $\hat y_2'(c)<0$.

To complete the proof we need to show properties (1)-(4). We observe that due to monotonicity of $\hat y_i(\cdot)$, $i=1,2$, on $(c_o,\hat c)$ their limits exist at all points of such interval.

(1) Taking limits as $c \uparrow \hat{c}$ in the second of (A-1), using (A-3) and defining $\hat{y}_1(\hat{c}-) := \lim_{c\uparrow\hat{c}} \hat{y}_1(c)$ we get

$$\hat{y}_1^{\frac{1}{2}}(\hat{c}-)\big(1 - \frac{1}{2}\ln \hat{y}_1(\hat{c}-)\big) = 2e^{-1},$$

which is uniquely solved by $\hat{y}_1(\hat{c}-) = e^{-2}$.

(2) We take limits as $c \uparrow \hat{c}$ in (A-4) and note that $\hat{y}_1'(c) \to 0$ since $R'(c) \to 0$.

(3) We argue by contradiction and assume that $\lim_{c\downarrow c_o} \hat{y}_1(c) = \overline{y}_1 > 0$. Then taking limits as $c \downarrow c_o$ in the first of (A-1) and recalling that $R(c_o) = 0$ we find

$$R(\hat{c})\sqrt{\hat{y}_2(c_o+)}\big[1 + \frac{1}{2}\ln \hat{y}_2(c_o+)\big] = 0,$$

which is clearly impossible since $\hat{y}_2(c_o+) \geq 1$ due to the fact that $\hat{y}_2(c) > 1$ for any $c \in (c_o, \hat{c})$.

(4) From the second equality in (A-1) and (A-3) one finds

$$\sqrt{\hat{y}_2(c)}\big[1 - \frac{1}{2}\ln \hat{y}_2(c)\big] = \frac{2e^{-1}R(\hat{c}) - R(c)\sqrt{\hat{y}_1(c)}\big[1 - \frac{1}{2}\ln \hat{y}_1(c)\big]}{R(\hat{c}) - R(c)} \geq 2e^{-1} > 0 \qquad \text{(A-7)}$$

where the first lower bound follows by the fact that $x \mapsto \sqrt{x}[1 - \frac{1}{2}\ln(x)]$ is strictly increasing and positive on $[0, e^{-2}]$, with maximum value $2e^{-1}$. Since also $\hat{y}_2(c) > 0$, from (A-7) we conclude that $\big[1 - \frac{1}{2}\ln \hat{y}_2(c)\big] > 0$, thus implying $\hat{y}_2(c) < e^2$.

□

Proof. **[Proposition 3.10]**

The proof is carried out in three steps and for simplicity we omit the dependency of y_2^* on c.

Step 1. The function $f(y) := \sqrt{y}(1 - \frac{1}{2}\ln(y))$ is strictly decreasing on $(1, e^2)$ with $f(1) = 1$ and $f(e^2) = 0$ so, since the absolute value of the second term of (3.34) is smaller than one, there exists a unique $y_2^*(c) \in (1, e^2)$ solving (3.33). Moreover since

$$\frac{\partial F_3}{\partial y}(y, c) = -\tfrac{1}{4}y^{-\frac{1}{2}}\ln y < 0 \quad \text{for } (y, c) \in (1, e^2) \times [0, c_o) \qquad \text{(A-8)}$$

we can use the implicit function theorem to conclude that $y_2^* \in C^1([0, c_o))$ and

$$(y_2^*)'(c) = -\Big(\frac{\partial F_3}{\partial y}\Big/\frac{\partial F_3}{\partial c}\Big)(y_2^*, c) = -\frac{8e^{-1}R(\hat{c})R'(c)}{\big(R(\hat{c}) - R(c)\big)^2}\frac{\sqrt{y_2^*}}{\ln y_2^*} < 0 \quad \text{for } c \in [0, c_o). \qquad \text{(A-9)}$$

Step 2. The limit $y_2^*(c_o-) := \lim_{c\uparrow c_o} y_2^*(c)$ exists by monotonicity and so by continuity we have $F_3(y_2^*(c_o-); c_o) = 0$, i.e.

$$\sqrt{y_2^*(c_o-)}\big(1 - \tfrac{1}{2}\ln y_2^*(c_o-)\big) = 2e^{-1}. \qquad \text{(A-10)}$$

We now take limits as $c \downarrow c_o$ in (A-3) and use part 2 of Proposition 3.8 to conclude that

$$\sqrt{\hat{y}_2(c_o+)}\big(1 - \tfrac{1}{2}\ln \hat{y}_2(c_o+)\big) = 2e^{-1}, \qquad \text{(A-11)}$$

where $\hat{y}_2(c_o+) := \lim_{c\downarrow c_o} \hat{y}_2(c)$ exists by monotonicity of $\hat{y}_2$ (cf. Proposition 3.8). Hence from (A-10), (A-11) and uniqueness of the solution to $F_3(y;c_o)=0$ we obtain (3.35).

Step 3. Setting $\hat{y}_2(c_o) := y_2^*(c_o-) = \hat{y}_2(c_o+)$ and taking limits as $c \uparrow c_o$ in (A-9) we obtain

$$(y_2^*)'(c_o-) = -\frac{8e^{-1}R'(c_o)\sqrt{\hat{y}_2(c_o)}}{R(\hat{c})\ln \hat{y}_2(c_o)}. \tag{A-12}$$

We now turn to study the limit of $\hat{y}_2'(c)$ when $c \downarrow c_o$. Recall (A-4) and (A-5). We have $\hat{y}_1(c) \downarrow 0$ and $R(c) \downarrow 0$, however from point (3) in the proof of Proposition 3.8 and by taking limits in the first equation of (A-1) it turns out that it must be

$$\lim_{c\downarrow c_o} R(c)\hat{y}_1^{-\frac{1}{2}}(c)\ln \hat{y}_1(c) = -\ell \quad \text{for some } \ell > 0. \tag{A-13}$$

Therefore as c approaches c_o from above we have the following asymptotic behaviours

$$D(\hat{y}_1(c), \hat{y}_2(c), c) \approx \frac{1}{16}R(\hat{c})R(c)\hat{y}_2^{-\frac{1}{2}}(c)\ln \hat{y}_2(c)\hat{y}_1^{-\frac{3}{2}}(c)\ln \hat{y}_1(c),$$

and

$$D_2(\hat{y}_1(c), \hat{y}_2(c), c) \approx -\tfrac{1}{4}R'(c)R(c)\hat{y}_2^{\frac{1}{2}}(c)\big(1-\tfrac{1}{2}\ln \hat{y}_2(c)\big)\hat{y}_1^{-\frac{3}{2}}(c)\ln \hat{y}_1(c).$$

Hence

$$\hat{y}_2'(c) \approx -4\frac{R'(c)}{R(\hat{c})}\left[\frac{\hat{y}_2^{\frac{1}{2}}(c)\Big(1-\frac{1}{2}\ln \hat{y}_2(c)\Big)}{\hat{y}_2^{-\frac{1}{2}}(c)\ln \hat{y}_2(c)}\right]. \tag{A-14}$$

and (3.36) now follows from (A-11) in the limit as $c \downarrow c_o$. □

Proof. **[Proposition 3.16]**

Recalling (3.15), (3.16), (3.17) and Proposition 3.14 we see that it suffices to show that $V_c(x,c) \geq G_c(x,c)$ for any $x \in (\hat{\gamma}(c), \hat{\beta}(c))$ and $c \in [0,\hat{c})$. The proof is performed in two steps.

Step 1. Fix $c \in [0, c_o]$ and recall that (cf. Section 3.3.2 and (3.18)) for any such c the continuation set is of the form $(-\infty, \hat{\beta}(c))$. Define $u := V_c - G_c$, then it is not hard to see by (3.17), Proposition 3.14 and (3.43) that $u \in C(\mathbb{R}\times[0,c_o])$ and it is the unique classical solution of

$$(\mathbb{L}_X - \lambda)u(x,c) = -\lambda x(1+\Phi'(c)), \quad \text{for } x < \hat{\beta}(c) \text{ with } u(\hat{\beta}(c),c)=0. \tag{A-15}$$

Therefore, setting $\tau_\beta := \inf\{t \geq 0 : X_t^x \geq \hat{\beta}(c)\}$ and using the Feynmann-Kac representation formula (possibly up to a standard localisation argument), we get

$$u(x,c) = \mathsf{E}\left[e^{-\lambda\tau_\beta}u(X^x_{\tau_\beta},c) + \lambda(1+\Phi'(c))\int_0^{\tau_\beta} e^{-\lambda t}X_t^x dt\right] = (1+\Phi'(c))\mathsf{E}\left[\int_0^{\tau_\beta}\lambda e^{-\lambda t}X_t^x dt\right], \tag{A-16}$$

where we have used that $u(X^x_{\tau_\beta},c)=0$ P-a.s. since $\tau_\beta < \infty$ P-a.s. by the recurrence property of Brownian motion. Recalling that $X_t^x = x + B_t$ (cf. (2.1)) we can write

$$x = \mathsf{E}\left[\int_0^\infty \lambda e^{-\lambda t}X_t^x dt\right] \tag{A-17}$$

and then by strong Markov property and standard formulae for the Laplace transform of τ_β it easily follows from (A-16)

$$u(x,c) = (1+\Phi'(c))\Big(x - \mathsf{E}\big[e^{-\lambda\tau_\beta}X^x_{\tau_\beta}\big]\Big) = (1+\Phi'(c))\left[x - \hat{\beta}(c)\frac{\psi_\lambda(x)}{\psi_\lambda(\hat{\beta}(c))}\right]. \tag{A-18}$$

Since $(1+\Phi'(c)) \le 0$ for $c \in [0, c_o]$, we have $u(x,c) = V_c(x,c) - G_c(x,c) \ge 0$ if and only if

$$\theta(x,c) := x - \hat{\beta}(c)\frac{\psi_\lambda(x)}{\psi_\lambda(\hat{\beta}(c))} \le 0 \quad \text{for } x < \hat{\beta}(c). \tag{A-19}$$

From Proposition 3.7 we obtain $1 < \hat{y}_2(c) < e^2$ and hence $0 < \hat{\beta}(c) < 1/\sqrt{2\lambda}$. Therefore, also recalling that $\psi_\lambda(x) = e^{\sqrt{2\lambda}x}$, one has for any $x < \hat{\beta}(c)$

$$\theta_x(x,c) = 1 - \hat{\beta}(c)\sqrt{2\lambda}e^{\sqrt{2\lambda}(x-\hat{\beta}(c))} \ge 1 - \hat{\beta}(c)\sqrt{2\lambda} \ge 0.$$

We can now conclude that (A-19) is fulfilled by noting that $\theta(\hat{\beta}(c), c) = 0$, hence $u \ge 0$ in $(-\infty, \hat{\beta}(c)) \times [0, c_o]$.

Step 2. Fix now $c \in (c_o, \hat{c})$, take $x \in (\hat{\gamma}(c), \hat{\beta}(c))$ and denote again $u := V_c - G_c$. As in *Step 1* it is not hard to see that

$$(\mathbb{L}_X - \lambda)u(x,c) = -\lambda x(1+\Phi'(c)) \quad \text{for } x \in (\hat{\gamma}(c), \hat{\beta}(c)) \text{ and } u(\hat{\gamma}(c), c) = u(\hat{\beta}(c), c) = 0. \tag{A-20}$$

Set $\tau_{\gamma,\beta} := \tau_\gamma \wedge \tau_\beta$ with τ_β as in *Step 1* above and $\tau_\gamma := \inf\{t \ge 0 : X^x_t \le \hat{\gamma}(c)\}$. Then u is continuous and it admits the Feynmann-Kac representation

$$u(x,c) = (1+\Phi'(c))\mathsf{E}\bigg[\int_0^{\tau_{\gamma,\beta}} \lambda e^{-\lambda t} X^x_t dt\bigg] \tag{A-21}$$

where we have used that $u(X^x_{\tau_{\gamma,\beta}}, c) = 0$ P-a.s. due to (A-20) and to the fact that $\tau_{\gamma,\beta} < \infty$ P-a.s. by the recurrence property of Brownian motion. Since $(1+\Phi'(c)) < 0$ on $[c_o, \hat{c})$ then $u(x,c) \ge 0$ on $(\hat{\gamma}(c), \hat{\beta}(c))$ (i.e. $V_c \ge G_c$) if and only if $\mathsf{E}[\int_0^{\tau_{\gamma,\beta}} \lambda e^{-\lambda t} X^x_t dt] \le 0$ for $x \in (\hat{\gamma}(c), \hat{\beta}(c))$. Thanks to (A-17), strong Markov property and Green's formula (cf. also [5], eq. (4.3))

$$\begin{aligned}
\mathsf{E}\bigg[\int_0^{\tau_{\gamma,\beta}} \lambda e^{-\lambda t} X^x_t dt\bigg] &= x - \mathsf{E}\big[e^{-\lambda \tau_{\gamma,\beta}} X^x_{\tau_{\gamma,\beta}}\big] \\
&= x - \Big\{\hat{\gamma}(c)\mathsf{E}\big[e^{-\lambda\tau_\gamma}\mathbb{1}_{\{\tau_\gamma < \tau_\beta\}}\big] + \hat{\beta}(c)\mathsf{E}\big[e^{-\lambda\tau_\beta}\mathbb{1}_{\{\tau_\beta < \tau_\gamma\}}\big]\Big\} \\
&= x - \left\{\hat{\gamma}(c)\frac{\sinh\big(\sqrt{2\lambda}(\hat{\beta}(c) - x)\big)}{\sinh\big(\sqrt{2\lambda}(\hat{\beta}(c) - \hat{\gamma}(c))\big)} + \hat{\beta}(c)\frac{\sinh\big(\sqrt{2\lambda}(x - \hat{\gamma}(c))\big)}{\sinh\big(\sqrt{2\lambda}(\hat{\beta}(c) - \hat{\gamma}(c))\big)}\right\} \\
&= \frac{1}{\sinh\big(\sqrt{2\lambda}(\hat{\beta}(c) - \hat{\gamma}(c))\big)}\Theta(x, c; \hat{\beta}(c), \hat{\gamma}(c)),
\end{aligned} \tag{A-22}$$

where we define

$$\begin{aligned}
&\Theta(x, c; \hat{\gamma}(c), \hat{\beta}(c)) \\
&:= \Big[x \sinh\big(\sqrt{2\lambda}(\hat{\beta}(c) - \hat{\gamma}(c))\big) - \hat{\gamma}(c)\sinh\big(\sqrt{2\lambda}(\hat{\beta}(c) - x)\big) - \hat{\beta}(c)\sinh\big(\sqrt{2\lambda}(x - \hat{\gamma}(c))\big)\Big].
\end{aligned} \tag{A-23}$$

To simplify notation we set $\vartheta(x,c) := \Theta(x, c; \hat{\gamma}(c), \hat{\beta}(c))$. The right-hand side of (A-22) is negative for any $x \in (\hat{\gamma}(c), \hat{\beta}(c))$ if and only if $\vartheta(x,c) \le 0$ therein. To study the sign of ϑ we first note that $\vartheta(\hat{\gamma}(c), c) = 0 = \vartheta(\hat{\beta}(c), c)$ and

$$\begin{cases}
\vartheta_x(x,c) = \sinh\big(\sqrt{2\lambda}(\hat{\beta} - \hat{\gamma})(c)\big) \\
\qquad\qquad + \sqrt{2\lambda}\Big[\hat{\gamma}(c)\cosh\big(\sqrt{2\lambda}(\hat{\beta}(c) - x)\big) - \hat{\beta}(c)\cosh\big(\sqrt{2\lambda}(x - \hat{\gamma}(c))\big)\Big] \\
\vartheta_{xx}(x,c) = -2\lambda\hat{\gamma}(c)\sinh\big(\sqrt{2\lambda}(\hat{\beta}(c) - x)\big) - 2\lambda\hat{\beta}(c)\sinh\big(\sqrt{2\lambda}(x - \hat{\gamma}(c))\big) \\
\vartheta_{xxx}(x,c) = 2\lambda\sqrt{2\lambda}\Big[\hat{\gamma}(c)\cosh\big(\sqrt{2\lambda}(\hat{\beta}(c) - x)\big) - \hat{\beta}(c)\cosh\big(\sqrt{2\lambda}(x - \hat{\gamma}(c))\big)\Big].
\end{cases} \tag{A-24}$$

From (A-24) it is easy to see that *i)* $\vartheta_x(\hat{\gamma}(c),c)<0$, since $\hat{\gamma}(c)\le -1/\sqrt{2\lambda}$, *ii)* $\vartheta_{xx}(\hat{\gamma}(c),c)>0$, $\vartheta_{xx}(\hat{\beta}(c),c)<0$ and *iii)* $\vartheta_{xxx}(x,c)<0$. Hence $x\mapsto\vartheta_{xx}(x,c)$ is strictly decreasing and there exists a unique point $x_*:=x_*(c)$ such that $\vartheta_{xx}(x_*,c)=0$. Clearly x_* is a maximum of $x\mapsto\vartheta_x(x,c)$ in $(\hat{\gamma}(c),\hat{\beta}(c))$. We claim now and we will prove it later that $\vartheta_x(\hat{\beta}(c),c)>0$. Then $\vartheta_x(x,c)>0$ for $x\in(x_*,\hat{\beta}(c))$. Moreover since $\vartheta_x(\hat{\gamma}(c),c)<0$, then there exists a unique point $x'_*:=x'_*(c)<x_*$ such that $\vartheta_x(x'_*,c)=0$. Such x'_* is the unique stationary point of $\vartheta(\,\cdot\,,c)$ in $(\hat{\gamma}(c),\hat{\beta}(c))$ and it is a negative minimum due to the fact that $\vartheta_{xx}(x,c)>0$ for any $x<x_*$. Therefore, recalling also $\vartheta(\hat{\gamma}(c),c)=0=\vartheta(\hat{\beta}(c),c)$, we conclude that $\vartheta(x,c)<0$ for any $x\in(\hat{\gamma}(c),\hat{\beta}(c))$. From (A-21) and (A-22) we thus get $u(x,c)\ge 0$ for any $x\in(\hat{\gamma}(c),\hat{\beta}(c))$.

To complete the proof it remains to show that $\vartheta_x(\hat{\beta}(c),c)>0$. For that it is convenient to rewrite the first of (A-24) in terms of $\hat{y}_1(c)$ and $\hat{y}_2(c)$ (cf. (3.41)) so to have

$$\begin{aligned}\vartheta_x(\hat{\beta}(c),c)&=\Theta_x(F_\lambda^{-1}(\hat{y}_2(c)),c;F_\lambda^{-1}(\hat{y}_1(c)),F_\lambda^{-1}(\hat{y}_2(c)))\\&=\hat{y}_2^{\frac12}(c)\hat{y}_1^{-\frac12}(c)\big(1-\tfrac12\ln\hat{y}_2(c)\big)-\hat{y}_2^{-\frac12}(c)\hat{y}_1^{\frac12}(c)\big(1+\tfrac12\ln\hat{y}_2(c)\big).\end{aligned}\tag{A-25}$$

From system (3.27) (see also (A-1), (A-2) and (A-3)) we obtain

$$\begin{cases}\hat{y}_2^{\frac12}(c)\big(1-\frac12\ln\hat{y}_2(c)\big)=\dfrac{-2e^{-1}R(\hat{c})+R(c)\hat{y}_1^{\frac12}(c)\big(1-\frac12\ln\hat{y}_1(c)\big)}{R(c)-R(\hat{c})}\\ \hat{y}_2^{-\frac12}(c)\big(1+\frac12\ln\hat{y}_2(c)\big)=\dfrac{R(c)}{R(c)-R(\hat{c})}\hat{y}_1^{-\frac12}(c)\big(1+\frac12\ln\hat{y}_1(c)\big),\end{cases}\tag{A-26}$$

which plugged into (A-25) give

$$\begin{aligned}2\vartheta_x(\hat{\beta}(c),c)&=2\Theta_x(F_\lambda^{-1}(\hat{y}_2(c)),c;F_\lambda^{-1}(\hat{y}_1(c)),F_\lambda^{-1}(\hat{y}_2(c)))\\&=\frac{\hat{y}_1^{-\frac12}(c)}{R(\hat{c})-R(c)}\Big[2e^{-1}R(\hat{c})+R(c)\sqrt{\hat{y}_1(c)}\ln\hat{y}_1(c)\Big].\end{aligned}\tag{A-27}$$

Recalling now that $0<\hat{y}_1(c)<e^{-2}$, $R(\hat{c})>R(c)>0$ and noting that the function $\sqrt{x}\ln(x)$ is nonnegative on $[0,e^{-2}]$, we conclude by (A-27) that $\vartheta_x(\hat{\beta}(c),c)>0$ for all $c\in(c_o,\hat{c})$ as claimed. □

Proof. **[Proposition 3.17]**

From (3.15), (3.31) and (3.42), Corollaries 3.13 and 3.15 and by using Remark 3.12 and (1) – (2) of Proposition 3.8 we observe that for all $x\in\mathbb{R}$

$$W^1(x,\hat{c}-):=\lim_{c\uparrow\hat{c}}W^1(x,c),\quad W^1_c(x,\hat{c}-):=\lim_{c\uparrow\hat{c}}W^1_c(x,c)\quad\text{and}\quad W^1_x(x,\hat{c}-):=\lim_{c\uparrow\hat{c}}W^1_x(x,c)$$

exist and that $W^1_c(\,\cdot\,,\hat{c}-)$ and $W^1_x(\,\cdot\,,\hat{c}-)$ are continuous. Therefore W^1 has a C^1 extension to $\mathbb{R}\times[0,\hat{c}]$ which we denote again by W^1.

For $x\in(-\infty,\gamma^o]\cup[\hat{\beta}(\hat{c}-),+\infty)$ we have $W^1(x,\hat{c})=W^o(x,\hat{c})$, $W^1_c(x,\hat{c})=W^o_c(x,\hat{c})$ and $W^1_x(x,\hat{c})=W^o_x(x,\hat{c})$ since $V=G$, $V_c=G_c$ and $V_x=G_x$ in that set (cf. (3.16), (3.17) and (3.9)). For $x\in(\gamma^o,\hat{\beta}(\hat{c}-))$ we have

$$W^1(x,\hat{c})=x\Phi(\hat{c})+\phi_\lambda(x)Q(F_\lambda(x),\hat{c}-)\tag{A-28}$$

$$W^1_c(x,\hat{c})=x\Phi'(\hat{c})+\phi_\lambda(x)Q_c(F_\lambda(x),\hat{c}-)\tag{A-29}$$

$$W^1_x(x,\hat{c})=\Phi(\hat{c})+\phi_\lambda(x)\Big[Q_x(F_\lambda(x),\hat{c}-)F'_\lambda(x)-\sqrt{2\lambda}Q(F_\lambda(x),\hat{c}-)\Big]\tag{A-30}$$

by (3.15) and Proposition 3.4. To find an explicit expression of (A-28) we study $Q(y, \hat{c}-)$ for $y \in (e^{-2}, \hat{y}_2(\hat{c}-))$ (see (1) of Proposition 3.8). In particular from (3.31), Remark 3.12 and Proposition 3.8 (noting that $\hat{y}_1(c) < e^{-2}$ for $c < \hat{c}$) we find

$$Q(y, \hat{c}-) = H_y(e^{-2}-, \hat{c})(y - e^{-2}) + H(e^{-2}-, \hat{c}) = -\tfrac{1}{\sqrt{2\lambda}} R(\hat{c}) e^{-1}. \tag{A-31}$$

It then follows $W^1(x, \hat{c}) = W^o(x, \hat{c})$ by simple calculations, (3.9) and (3.20).

For (A-29) we consider $Q_c(y, \hat{c}-)$ for $y \in (e^{-2}, \hat{y}_2(\hat{c}-))$ and arguing as above we obtain

$$Q_c(y, \hat{c}-) = \Big[H_{yc}(e^{-2}-, \hat{c}) + H_{yy}(e^{-2}-, \hat{c})\hat{y}_1'(\hat{c}-)\Big](y - e^{-2}) + H_c(e^{-2}-, \hat{c}) = 0 \tag{A-32}$$

by (3.24), (3.25) and (3.26), hence $V_c(x, \hat{c}-) = 0$ and $W^1_c(x, \hat{c}) = W^o_c(x, \hat{c}) = -x$ by recalling that $\Phi'(\hat{c}) = -1$ (cf. (2.6)).

To conclude the proof we observe that $Q_y(y, \hat{c}-) = H_y(\hat{y}_1(\hat{c}-), \hat{c}) = 0$ for $y \in (e^{-2}, \hat{y}_2(\hat{c}-))$, hence (A-31) and (A-30) give $W^1_x(x, c) = W^o_x(x, \hat{c}) = \Phi(\hat{c}) + \phi_\lambda(x) R(\hat{c}) e^{-1}$. $\square$

References

[1] Alvarez, L.H.R. (1999). *A Class of Solvable Singular Stochastic Control Problems*, Stoch. Stoch. Rep. 67, pp. 83–122.

[2] Alvarez, L.H.R. (1999). *Singular Stochastic Control, Linear Diffusions, and Optimal Stopping: a Class of Solvable Problems*, SIAM J. Control Optim. 39(6), pp. 1697–1710.

[3] Beneš, V.E., Shepp, L.A., Witsenhausen, H.S. (1980). *Some Solvable Stochastic Control Problems*, Stochastics 4, pp. 39–83.

[4] Chow, P.L., Menaldi, J.L., Robin, M. (1985). *Additive Control of Stochastic Linear Systems with Finite Horizon*, SIAM J. Control Optim. 23, pp. 858–899.

[5] Dayanik, S., Karatzas, I. (2003). *On the Optimal Stopping Problem for One-Dimensional Diffusions*, Stochastic Process. Appl. 107(2), pp. 173–212.

[6] Davis, M., Zervos, M. (1994). *A Problem of Singular Stochastic Control with Discretionary Stopping*, Ann. Appl. Probab. 4(1), pp. 226–240.

[7] De Angelis, T., Ferrari, G., and Moriarty, J. (2014). *A Non Convex Singular Stochastic Control Problem and its Related Optimal Stopping Boundaries*, preprint. **arXiv**: 1405.2442.

[8] Dynkin, E.B., Yushkevich, A.A. (1969). *Markov Processes: Theorems and Problems.* Plenum Press, New York.

[9] El Karoui, N., Karatzas, I. (1988). *Probabilistic Aspects of Finite-fuel, Reflected Follower Problems.* Acta Applicandae Math. 11, pp. 223–258.

[10] El Karoui, N., Karatzas, I. (1991). *A New Approach to the Skorokhod Problem and its Applications.* Stoch. Stoch. Rep. 34, pp. 57–82.

[11] Federico, S., Pham, H. (2013). *Characterization of the Optimal Boundaries in Reversible Investment Problems*, SIAM J. Control Optim. 52(4), pp. 2180–2223.

[12] Fleming, W.H., Soner, H.M. (2005). *Controlled Markov Processes and Viscosity Solutions*, 2nd Edition. Springer.

[13] Horst, R., Pardalos, P.M., Thoai, N.V. (editors) (1995). *Introduction to Global Optimization*, Vol. 3 of Nonconvex Optimization and its Applications, Kluwer Academic Publishers, Dordrecht.

[14] Horst, R., Pardalos, P.M., Thoai, N.V. (1999). *DC Programming: Overview*, J. Optim. Theory Appl. 103(1), pp. 1-43.

[15] Karatzas, I., Shreve, S.E. (1984). *Connections between Optimal Stopping and Singular Stochastic Control I. Monotone Follower Problems*, SIAM J. Control Optim. 22, pp. 856–877.

[16] Karatzas, I., Shreve, S.E. (1985). *Connections between Optimal Stopping and Singular Stochastic Control II. Reflected Follower Problems*, SIAM J. Control Optim. 23, pp. 433–451.

[17] Karatzas, I. (1985). *Probabilistic Aspects of Finite-fuel Stochastic Control*, Proc. Nat'l. Acad. Sci. U.S.A., 82(17), pp. 5579–5581.

[18] Karatzas, I., Shreve, S.E. (1986). *Equivalent Models for Finite-fuel Stochastic Control*, Stochastics 18, pp. 245–276.

[19] Karatzas, I., Ocone, D., Wang, H., Zervos, M. (2000). *Finite-Fuel Singular Control with Discretionary Stopping*, Stoch. Stoch. Rep. 71, pp. 1–50.

[20] Peskir, G., Shiryaev, A. (2006). *Optimal Stopping and Free-Boundary Problems*, Lectures in Mathematics, ETH Zurich.