A new class of costs for optimal transport problems

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In optimal mass transport theory, many problems can be written in the Monge-Kantorovich form

$$(MK) \qquad \inf\left\{\int_{X\times Y} c(x,y)\,d\gamma \;:\; \gamma\in\Pi(\mu,\nu)\right\}$$

where μ,ν are given probability measures on X,Y and $c : X\times Y \to [0,+\infty[$ is a cost function. Here the competitors are probability measures γ on $X\times Y$ with marginals μ and ν respectively (the set $\Pi(\mu,\nu)$ being the set of these transport plans). In the particular case where an optimal transport plan $\gamma\in\Pi(\mu,\nu)$ is carried by the graph of a map $T : X\to Y$ i.e. if

$$\langle\gamma,\varphi\rangle = \int_X \varphi(x,T(x))\,d\mu(x)$$

then T solves the original Monge problem : $\inf\left\{\int_X c(x,T(x))\,d\mu(x) \;:\; T^\sharp\mu=\nu\right\}$.

Here we are interested in a different case. Indeed in some applications to economy or in probability theory, it may be interesting to favour optimal plans which are not associated to a single valued transport map T. The idea is then to consider, instead of T, the family of conditional probabilities γ^x such that

$$\langle\gamma,\varphi\rangle = \int_X\int_Y \varphi(x,y)d\gamma^x(y)\,d\mu(x)\ ,$$

and to incorporate in problem (MK) an additional cost over γ^x as follows

$$(MK_2) \qquad \inf\left\{\int_{X\times Y} c(x,y)\,d\gamma(x,y) + \int_X G(x,\gamma^x)\,d\mu(x) \;:\; \gamma\in\Pi(\mu,\nu)\right\}\ ,$$

being $G : (x,p)\in X\times\mathcal{P}(X)\to[0,+\infty]$ a given non-linear function.

In this talk I will describe some recent results concerning problem (MK_2) (existence, duality principle, optimality conditions) and focus on specific examples where $X=Y$ and X is a convex compact subset of $\mathbb{R}^d$.

It is a joint work with Guy Bouchitté and Jean-Jacques Alibert (Université de Toulon).