Collateralizing Liquidity

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Abstract

I study a dynamic model of optimal funding to understand why liquid financial assets are used as collateral. Firms need to borrow to invest in risky projects with non-verifiable returns. Since projects are profitable and assets allow firms to invest in them, firms with investment opportunities value the asset more than those without them. When investment opportunities are persistent, current borrowers also value the asset more in the future and financial assets are optimally used as collateral. Assets carry liquidity and collateral premia. As liquidity increases, the liquidity premium increases while the collateral premium decreases. The asset's debt capacity increases with liquidity.

1 Introduction

Collateralized debt is a widely used form of financing. Trillions of dollars are traded daily in debt collateralized by diverse financial assets, such as sale and repurchase agreements (repos) and collateralized over-the-counter derivative trades.[1] Many financial institutions use collateralized debt to raise funds that allow them to provide intermediation services. Some of these institutions are private depository institutions, credit unions, mortgage real estate investment trusts, and security

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[1] See Gorton and Metrick (2012a), Gorton and Metrick (2012b), and Copeland et al. (2011 (revised 2012).

brokers and dealer. Most of these institutions are highly levered and they use the repo market as a source of financing.

The assets that are "sold" using repos or offered as collateral in collateralized derivative trades are financial assets which could also be sold in financial markets. These financial assets are not productive assets and, in principle, their intrinsic value is the same independently of the assets' holder. Then, why do so many financial institutions choose to use these financial assets as collateral instead of selling them to raise funds?

In this paper I develop a model in which borrowers and lenders value the asset equally in autarky, but they assign different values to it in equilibrium. In an environment with incomplete contracts, this endogenous difference in valuations implies that collateralized debt contracts implement the optimal funding contract. Since the contract is an equilibrium outcome, I can also characterize the amount that can be borrowed against an asset (i.e., its debt capacity) and its determinants.

The model is a discrete-time, infinite-horizon model. There are two types of risk neutral agents, borrowers and lenders, and one durable asset which pays dividends each period. Borrowers can invest in risky projects but they need external funds to do so. The return of the projects is private information of the agent who invested in them and, thus, it is not contractible. In order to raise funds, borrowers enter into a state contingent contract with lenders. In equilibrium, firms value the asset more than lenders, and, therefore, choose to use collateral contracts.

In my model, borrowers have the investment opportunity before the asset's dividends are paid and, thus, cannot invest without external financing. This timing implies a maturity mismatch for the borrower between the need of funds to invest and the availability of the dividends. Lenders do not have access to investment opportunities. Therefore, if left in autarky, both risk neutral borrowers and lenders would value the asset by the expected discounted sum of dividends. However, once the agents are allowed to trade, the equilibrium features endogenous differences in valuations and, thus, collateral contracts are optimal. The main assumptions that lead to this result is the persistence in the role as borrowers and lenders, and the asymmetric information about the borrower's ability to repay.

Suppose that an agent in this economy will have an investment opportunity tomorrow. Holding the asset tomorrow will allow the agent to invest in the project either by selling the asset or by pledging it as collateral. Being able to raise funds against the asset in the funding market will solve the maturity mismatch problem between the timing of the dividend realization and the investment opportunity. In turn, this implies that agents with investment opportunities tomorrow will value the asset more than those without them. If the role as borrowers and lenders is persistent, as it is the case in many collateralized debt markets, it follows that agents who are borrowers today will value the asset tomorrow more than lenders will.[2] This difference in valuations, together with the asymmetric information about the borrower's ability to repay, implies that collateral contracts arise optimally in equilibrium.

The extra value borrowers assign to the asset on top of the expected discounted value of dividends can be decomposed in two premia for the borrower: a liquidity premium and a collateral premium. The liquidity premium for the borrowers arises from them being able to sell the asset and use the funds to invest in the risky projects (from solving the maturity mismatch mentioned above). The collateral premium for the borrowers is the additional value they can obtain by using the asset as collateral instead of selling it.

The asset's debt capacity depends on the insurance the asset provides to the lender if there is default and on the incentives it provides borrowers to repay. The larger the liquidity in the asset market, the higher the asset's debt capacity.

I extend the baseline model in several dimensions. First, I make the asset risky and allow for the dividends to be correlated with the return of the borrower's investment opportunities. An increase in this correlation increases the asset's debt capacity since it makes it costlier for the borrower to default when he can avoid it. It makes the asset better collateral. Second, I make the access to investment opportunities stochastic. As long as the access to investment opportunities is persistent, collateralized debt remains optimal. Third, I allow for savings within periods and show that borrowers would rather buy assets and be able to pledge them as collateral than saving the consumption goods and having them available to invest. Finally, I introduce heterogeneity among the borrowers' investment opportunities. Depending on the liquidity in the asset market two regimes may arise in equilibrium. When the liquidity is low, all borrowers use the asset as collateral. When the liquidity is high, some borrowers use the asset as collateral while others choose to sell the asset to raise funds.

My paper is related to the literature on optimal contracting in the presence of agency problems. In particular, it is closely related to papers that deal with incomplete contracts either due to private information or moral hazard. In Bolton and Scharfstein (1990) the optimal contract is a long-term

[2] This persistence is consistent with the observation that, in many collateralized debt markets, different types of institutions specialize in borrowing or lending. For example, in the repo market, money market funds are usually lenders whereas hedge funds and specialty lenders are usually borrowers.

contract in which the lender commits to liquidate the borrower's firm if default occurs. In Lacker (2001) non-pecuniary costs of default are assumed while in Rampini (2005) default penalties are modeled as transfers of goods only valued by the borrower. Gale and Hellwig (1985), Bernanke and Gertler (1989) and Bernanke et al. (1998) also get optimal contracts that resemble collateralized debt. They follow Townsend (1979) and allow for costly state verification to partially resolve the agency problems. In all of this literature, including in my paper, the optimal contract involves some credible punishment for the borrower if the bad state is realized which incentivizes him to behave, and report the state truthfully. The main contribution of my paper is that the cost of default for the borrower, which determines the optimality of collateralized debt, arises endogenously and is not assumed.

In my model, contracts are incomplete but perfectly enforceable.[3] However, as Barro (1976) shows, enforcement frictions can also give rise to collateral contracts in equilibrium. Along these lines, Stiglitz and Weiss (1981), Chan and Kanatas (1985), and Kocherlakota (2001) analyze collateral as a mechanism to enforce contracts and deter default. Kiyotaki and Moore (1997) and Rampini and Viswanathan (2013) explore the effect of collateral constraints on the business cycles and the capital structure of the firm, respectively. In most of these papers, the assets that are used as collateral are either illiquid or they are inputs of production so selling them in order to raise funds is not an option. In contrast, in my paper the asset that is used as collateral in equilibrium is a liquid financial asset.

One of the main advantages of deriving collateralized debt as part of the optimal contract is that the amount that can be borrowed against the asset, its debt capacity, is an equilibrium outcome. In this regard, my paper is also related to the endogenous leverage literature based on the collateral equilibrium models developed Araujo et al. (1994), Geanakoplos (2003a, b), Geanakoplos and Zame (1997), and Fostel and Geanakoplos (2008) In these models the collateral requirements are also equilibrium outcomes. However, in contrast to my paper, these papers restrict the contract space to collateralized debt.

Finally, in a search model with bilateral trading, Monnet and Narajabad (2012) show that agents prefer to conduct repurchase agreements than asset sales when they face substantial uncertainty about the private value of holding the asset in the future.

The rest of the paper is organized as follows. Section 2 presents the baseline model. Section 3

[3]Hart and Moore (1985) study the optimal contract in the presence of incomplete contracts that can be renegotiated.

defines and characterizes equilibrium. Section 4 analyzes liquidity as a determinant of collateralized debt. Section 5 presents several extensions to the baseline model. Section 6 concludes.

2 Model

In this section I present the baseline model and derive the main result that collateralized debt arises in equilibrium as part of the optimal funding contract.

Time is discrete, starts at $t = 0$, and goes on forever. Each period t is divided in two subperiods, morning and afternoon.[4] There are two different types of non-storable, consumption goods: a morning specific good and an afternoon specific good. A unit of the morning (afternoon) specific good at time t that is not consumed within the morning (afternoon) at time t, perishes and disappears. There is a one infinitely lived asset in the economy which is in fixed supply $\bar{k}$. Holding k units of the asset yield $d_t k$ units of (afternoon) consumption good as dividend at the end of each afternoon t. In this section, I will assume that the dividend is constant, $d_t = \bar{d}$, and, thus, the asset is riskless.

There is a large number N of each of the two types of agents in the economy, borrowers (B) and lenders (L). All agents are risk-neutral, live forever and share the same discount factor $\beta \in (0, 1)$. Lenders are endowed with $k_{L,0}$ units of the asset at $t = 0$ and each period t they receive an endowment of e_L^m and e_L^a units of the morning and afternoon consumption goods respectively. Borrowers start their life with $k_{B,0}$ units of the asset and, every period t, they receive e_B^m units of the morning consumption good. Borrowers receive no good endowment in the afternoon. Instead, each afternoon, each borrower has access to his own investment opportunity.

Each afternoon t a borrower j can invest in his own risky, constant return, short-term project: one unit of (afternoon) consumption good invested in his project at the beginning of afternoon t yields a random payoff at the end of the period which is only observable by the agent who invested in the project. This payoff is given by $\theta_t^j \in \{\theta_L, \theta_H\}$ where $\theta_L < \theta_H$. Each period, a fraction p_L of borrowers get a low return θ_L whereas a fraction p_H gets a high return θ_H. Therefore, $p_i = \Pr\left(\theta_t^j = \theta_i\right)$ for $i = L, H$. I assume that the project is profitable in expectation but it incurs losses if the low state is realized, i.e., $\theta_L < 1 < \mathbb{E}\left(\theta_t^j\right)$.

Assumption The discount factor β satisfies $\beta \frac{\mathbb{E}(\theta) - \theta_L}{1 - \theta_L} < 1$.

[4] I follow Lagos and Wright (2005) in this regard.

Figure 1: **Timing**

The assumption above ensures that borrowers want to use their assets to invest today rather than waiting and use them to produce tomorrow.

There are two bilateral markets in the economy in which each borrower is randomly matched with a lender: an asset market and a funding market. The asset market opens every morning and the bilateral terms of trade are determined through Nash bargaining. The funding market opens every afternoon. The bilateral funding contract between a borrower and a lender in the funding market is chosen optimally by the borrower who has all the bargaining power in this market. Since matches are random both in the asset market and in the funding market, the distribution of assets in the hands of lenders and borrowers in each subperiod is a relevant state variable. Let $F_i^s(k)$ be the fraction of agents of type $i = L, B$ in subperiod $s = m, a$ that holds less than k units of the asset. For example, F_L^m (F_B^a) is the cumulative distribution of assets in the hands of lenders (borrowers) in the morning (afternoon). Given these definitions, the aggregate state of the economy is given by $\phi = (d, F_L^m, F_B^m, F_L^a, F_B^a)$.

Each morning, borrowers and lenders meet in the asset market and adjust their asset holdings. Each afternoon, borrowers go to the funding market to fund their investment opportunities, they invest in them and, finally, after the projects and assets pay off, they settle their funding contracts. This timing is depicted in figure 1.

Remark 1 ***Timing.*** *Since the asset's dividend is paid after the investment in the risky projects needs to be made, the dividends cannot be invested in the risky projects by the borrower. This timing assumption implies that borrowers need to borrow in order to take advantage of their profitable investment opportunities. Moreover, it implies that, if agents were in autarky, they would all value the asset at the discounted sum of dividends,* $\frac{d}{1-\beta}$.

Remark 2 ***Non-contractibility of returns.*** *The key friction of the model is the non-contractibility* *of the return of the risky projects. Since the return of the risky projects is only observed by the borrower and is not verifiable by lenders, the funding contract cannot be contingent on the realized value of the return of the risky projects. This non-contractibility is the only source of inefficiency in the model. Without it, the first-best outcome will be attained: lenders would transfer all their endowment to the borrowers who would invest it in the risky projects and then pay lenders back an expected payoff of* 1*. In the first best, the optimal funding contract is indeterminate. In particular, the borrowers would not need to hold the asset to invest since they would be able to pledge the return of the risky projects credibly.*

However, once one introduces the non-contractibility of the return of the risky projects and given that the projects may incur losses, the borrowers will only be able to invest in their investment opportunities if they hold some assets. A borrower without assets would never be able to raise funds since he would always claim that the low state was realized giving the lender an expected gross return of $\theta_L < 1$*. Holding assets solves this problem: it allows the borrower to raise funds by selling the asset, by pledging the asset's dividends as collateral, or by pledging the asset itself as collateral.*

2.1 Asset Market

At the beginning of each morning, each borrower is randomly matched with a lender in the asset market. The terms of trade, quantity and price, are determined through Nash bargaining. The bargaining power of borrowers is equal to $\gamma \in (0,1]$.[5] As I discuss in section 4, γ can be interpreted as a measure of liquidity in the asset market.

Each pair matched in the asset market is characterized by the asset holdings of the agents who are being matched. If a borrower with k_B assets is matched with a lender with k_L assets the match will be characterized by the pair (k_B, k_L). A lender in a match indexed by (k_B, k_L) will transfer $k^T(k_B, k_L; \phi)$ units of the asset to the borrower in exchange for $P(k_B, k_L; \phi)$ units of the afternoon consumption good. If $k^T(k_B, k_L; \phi) > 0$ the lender is selling assets to the borrower whereas if $k^T(k_B, k_L; \phi) < 0$ the lender is buying assets from the borrower. The pair $\left(k^T(k_B, k_L; \phi), P(k_B, k_L; \phi)\right)$ determines the bilateral terms of trade in the asset market.

The value of a borrower in the morning, before being matched with a lender, is

$$\bar{V}_B^m(k_B; \phi) = \int \left[\mathbb{E}_{k_L}\left(V_B^a\left(k_B + k^T(k_B, k_L; \phi), k_L; \phi\right)\right) - P(k_B, k_L; \phi)\right] dF_L^m(k_L) \qquad (1)$$

where $V_B^a(k_B, k_L; \phi)$ is the value of a borrower with assets k_B who enters a loan contract with a

[5] If $\gamma = 0$ both agents value the asset the same in equilibrium and the optimal funding contract is indeterminate.

lender with assets k_L in the afternoon and $\mathbb{E}_x$ is the expectation operator over the random variable x.

Similarly, the value of a lender in the morning, before being matched with a borrower, is

$$\bar{V}_L^m(k_L;\phi) = \int \left[\mathbb{E}_{k_B}\left(V_L^a\left(k_L - k^T(k_B,k_L;\phi),k_B;\phi\right)\right) + P(k_B,k_L;\phi)\right] dF_B^m(k_B) \tag{2}$$

where $V_L^a(k_B,k_L;\phi)$ is the value of a lender in a match (k_B,k_L) in the afternoon.

I assume that borrowers and lenders have enough consumption good each morning to buy all the assets from their counterpart on the asset market. By making this assumption, the model abstracts from borrowing constraints in the asset market. Since the main focus of the paper is to understand the use of financial assets as collateral, considering such borrowing constraints, though interesting and realistic, makes it harder to disentangle the forces at work without adding much to the analysis.

Since the terms of trade are determined by Nash bargaining, $P(k_B,k_L;\phi)$ and $k^T(k_B,k_L;\phi)$ solve

$$\begin{aligned}\max_{\substack{P_0\in\left[-e_L^m,e_B^m\right]\\ k_0\in\left[-k_B,k_L\right]}} &\left(\mathbb{E}_{k_L}\left(V_B^a(k_B+k_0,k_L;\phi)\right) - P_0 - \mathbb{E}_{k_L}\left(V_B^a(k_B,k_L;\phi)\right)\right)^{\gamma} \\ &\times\left(\mathbb{E}_{k_B}\left(V_L^a(k_L-k_0,k_B;\phi)\right) + P_0 - \mathbb{E}_{k_B}\left(V_L^a(k_L,k_B;\phi)\right)\right)^{1-\gamma}\end{aligned} \tag{3}$$

where γ is the bargaining power of the borrower. The first term in the objective function is the differential utility a borrowers gets from participating in the asset market. The second term is the differential utility a lender gets from participating in the asset market.

2.2 Funding Market

Every afternoon, a bilateral funding market opens. Each borrower is matched randomly with a lender and the terms of the loan contract are determined by the borrower. Loan contracts are one-subperiod contracts and they consist of a loan amount in terms of (afternoon) consumption good, q, and contingent repayments in terms of (afternoon) consumption, r_i, and in terms of asset transfers, t_i, $i = L, H$.

Definition 1 *A contract* (q, r_L, r_H, t_L, t_H) *is feasible at time* t *if*

$$\begin{aligned}&0 \le q \le e_L^a \\ &0 \le r_i \le dk_{Bt} + \theta_i q \quad & i = L, H \\ &0 \le t_i \le k_{Bt} & i = L, H\end{aligned}$$

where k_{Bt} is the amount of assets held by the borrower in afternoon t.

The first constraint in the definition above states that the size of the loan q has to be non-negative and that it cannot be larger than the endowment of consumption good of the lender in the afternoon. The second set of constraints imply that the state contingent repayments in terms of consumption good cannot be negative nor can they be more than the amount of the good the borrower has at the end of the afternoon in each state. Similarly, the third set of constraints imply that the borrower cannot transfer more assets than the amount he holds.

As I mentioned above, since the return of the risky projects is only observed by the borrower who invested in them and is not verifiable by the lenders, the loan contract cannot be contingent on the realization of the return θ_t. However, the contracts can be made contingent on the reported return of the risky projects as long as the contracts are incentive compatible.

Definition 2 *A contract (q, r_L, r_H, t_L, t_H) is incentive compatible if*

$$-r_L + \beta \mathbb{E}_{\phi'}\left(\bar{V}_B^m\left(k_B - t_L; \phi'\right)|\theta_H\right) \leq -r_H + \beta \mathbb{E}_{\phi'}\left(\bar{V}_B^m\left(k_B - t_H; \phi'\right)|\theta_H\right) \tag{IC_H}$$

and whenever $r_H \leq \theta_L(dk_B + q)$

$$-r_L + \beta \mathbb{E}_{\phi'}\left(\bar{V}_B^m\left(k_B - t_L; \phi'\right)|\theta_L\right) \geq -r_H + \beta \mathbb{E}_{\phi'}\left(\bar{V}_B^m\left(k_B - t_H; \phi'\right)|\theta_L\right) \tag{IC_L}$$

In an incentive compatible allocation it is always at least as good for the borrower to tell the truth and report state that has been realized than to lie. This is captured by constraint IC_H if the realized return is high and by IC_L if the realized return is low. However, the constraint IC_L is only active when lying in the low state is feasible, i.e., when there are enough resources in the low state to match the contingent repayment in terms of goods in the high state, r_H.

Let $V_B^a(k_B, k_L; \phi)$ be the value of a borrower with assets k_B who is matched with a lender with assets k_L in the loan market. Then,

$$\begin{aligned} V_B^a(k_B, k_L; \phi) = & \sup_{q, r_L, r_H, t_H, t_L} \mathbb{E}(\theta) q - p_H r_H - p_L r_L + dk_B \\ & + \beta p_H \mathbb{E}_{\phi'}\left(\bar{V}_B^m\left(k_B - t_H; \phi'\right)|\theta_H\right) + \beta p_L \mathbb{E}_{\phi'}\left(\bar{V}_B^m\left(k_B - t_L; \phi'\right)|\theta_L\right) \end{aligned} \tag{4}$$

s.t.

$$(q, r_L, r_H, t_L, t_H) \text{ is feasible and IC}$$

$$\begin{aligned} \beta \mathbb{E}_{\phi'}\left(\bar{V}_L^m\left(k_L; \phi'\right)\right) \leq & -q + p_H r_H + \beta p_H \mathbb{E}_{\phi'}\left(\bar{V}_L^m\left(k_L + t_H; \phi'\right)|\theta_H\right) \\ & + p_L r_L + \beta p_L \mathbb{E}_{\phi'}\left(\bar{V}_L^m\left(k_L + t_L; \phi'\right)|\theta_L\right) \end{aligned} \tag{PC}$$

$$\phi' = H(\phi; \theta) \tag{LOM}$$

where $H(\cdot;\theta)$ is the law of motion of the aggregate state ϕ given the realized return θ.

The borrower chooses a feasible and incentive compatible contract to maximize his expected utility subject to the lender's participation constraint PC and given the perceived law of motion for the aggregate state in LOM. Since $\mathbb{E}(\theta) > 1$, the borrower invests all the loan amount, q, in the risky technology and gets an expected return $\mathbb{E}(\theta) q$. He expects to repay $p_H r_H + p_L r_L$ in terms of consumption good and he gets dividends dk_B from his asset holdings at the beginning of the afternoon. Finally, his expected continuation value (his value the following morning) depends on the contingent transfers of assets t_L and t_H that are part of the contract.

PC states that the lender has to be at least as good participating in the contract as he would be if he didn't participate in it. If the lender participates in the contract he gives up q units of consumption good at the beginning of the afternoon in exchange for contingent repayments in terms of consumption good and asset. With probability p_i state $i = L, H$ is reported and the lender receives r_i units of the consumption good and a t_i additional units of the asset with which to enter the asset market the following morning.

Assumption The lender's endowment in the afternoon e_L^a satisfies $\frac{d\bar{k}}{1-\theta_L} < e_L^a$.

The assumption above implies that the equilibrium loan amount is never restricted by the amount of the consumption good owned by the lenders, i.e., $q < e_L^a$. If this was not the case, a borrower with a sufficiently high amount of assets could issue risk free debt by pledging the dividends paid by the asset and the incentive compatibility constraints would not bind.

By inspecting the constraints in the problem above, one can considerably simplify the borrower's problem. First, in any equilibrium, the participation constraint for lenders PC will hold with equality. If it did not, the borrower could increase the loan amount and increase his expected utility without violating any of the additional constraints. Similarly, as is usual in this kind of problems, the incentive compatibility constraint will bind in the high state and IC_H will hold with equality. Finally, in order to maximize the size of the loan, the repayment in terms of goods in the low state, r_L, will be the maximum possible, i.e., $r_L = \theta_L q + dk_B$. The following proposition, which is proved in the appendix, formalizes these arguments.

Proposition 1 *In the optimal lending contract the participation constraint for the lender binds*

$$\begin{aligned}
\beta \mathbb{E}_{\phi'}\left(\bar{V}_L^m\left(k_L;\phi'\right)\right) &= -q + p_H r_H + \beta p_H \mathbb{E}_{\phi'}\left(\bar{V}_L^m\left(k_L + t_H;\phi'\right)|\theta_H\right) \\
&\quad + p_L r_L + \beta p_L \mathbb{E}_{\phi'}\left(\bar{V}_L^m\left(k_L + t_L;\phi'\right)|\theta_L\right),
\end{aligned}$$

the incentive compatibility constraint for the borrower in the high state binds

$$-r_L + \beta \mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_L; \phi' \right) | \theta_H \right) = -r_H + \beta \mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_H; \phi' \right) | \theta_H \right),$$

and the repayment in terms of consumption good in the low state is maximal

$$r_L = dk_B + \theta_L q.$$

From the proposition above it follows that the borrower's problem in the funding market reduces to choosing only the two asset transfers.

Corollary 1 *The borrower's problem can be rewritten as*

$$\begin{aligned}
&V_B^a \left(k_B, k_L; \phi \right) = \sup_{(t_H, t_L) \in [0, k_B]^2} \frac{\left(\mathbb{E} \left(\theta \right) - \theta_L \right)}{1 - \theta_L} dk_B \\
&+ \frac{\left(\mathbb{E} \left(\theta \right) - \theta_L \right)}{1 - \theta_L} \beta \left(p_H V_L^m \left(k_L + t_H; \phi' \right) + p_L V_L^m \left(k_L + t_L; \phi' \right) - \mathbb{E}_{\phi'} \left(V_L^m \left(k_L; \phi' \right) \right) \right) \\
&+ \frac{\left(\mathbb{E} \left(\theta \right) - 1 \right)}{1 - \theta_L} \beta p_H \left(\mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_H; \phi' \right) | \theta_H \right) - \mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_L; \phi' \right) | \theta_H \right) \right) \\
&+ \beta \left(p_L \mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_L; \phi' \right) | \theta_L \right) + p_H \mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_H; \phi' \right) | \theta_H \right) \right)
\end{aligned}$$

subject to

$$\begin{aligned}
q^* &= \frac{dk_B + \beta \left(p_H \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_H; \phi' \right) | \theta_H \right) + p_L \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_L; \phi' \right) | \theta_L \right) \right)}{1 - \theta_L} \\
&+ \frac{\beta p_H \left(\mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_H; \phi' \right) - \bar{V}_B^m \left(k_B - t_L; \phi' \right) | \theta_H \right) \right) - \beta \mathbb{E}_{\phi'} \left(V_L^m \left(k_L; \phi' \right) \right)}{1 - \theta_L} \\
q^* &\geq \max\{0, \beta p_H \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_H; \phi' \right) | \theta_H \right) + \beta p_L \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_L; \phi' \right) | \theta_L \right) \\
&\quad - p_L \beta \mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_H; \phi' \right) - \bar{V}_B^m \left(k_B - t_L; \phi' \right) | \theta_H \right) - \beta \mathbb{E}_{\phi'} \left(V_L^m \left(k_L; \phi' \right) \right) \} \\
\phi' &= H \left(\phi; \theta \right)
\end{aligned}$$

3 Equilibrium

In this section I define a recursive equilibrium of the model and characterize the unique affine equilibrium which features an optimal lending contract that can be implemented with a combination of riskless and collateralized debt.

Definition 3 *A recursive equilibrium in this economy is a pair of value functions for borrowers, in the morning and in the afternoon, $\bar{V}_B^m \left(k_B; \phi \right)$ and $V_B^a \left(k_B, k_L; \phi \right)$, a value function for lenders,* *in the morning and the afternoon,* $\bar{V}_L^m(k_B;\phi)$ *and* $V_L^a(k_L,k_B;\phi)$, *price and quantity functions in the bilateral asset market,* $P(k_B,k_L;\phi)$ *and* $k^T(k_B,k_L;\phi)$, *a loan contract*

$$(q(k_B,k_L;\phi), r_L(k_B,k_L;\phi), r_H(k_B,k_L;\phi), t_L(k_B,k_L;\phi), t_H(k_B,k_L;\phi))$$

and a law of motion for ϕ, $H(\phi;\theta)$, *such that* (1), (2), (3), *and* (4) *are satisfied and the law of motion for* ϕ *satisfies:*

$$\begin{aligned}
F_L^{m\prime}(k) &= p_H\left(\int F_L^a(k - t_H(k_B,k;d,\phi))\,dF_B^a(k_B)\right) \\
&\quad + p_L\left(\int F_L^a(k - t_L(k_B,k;d,\phi))\,dF_B^a(k_B)\right) \\
F_B^{a\prime}(k) &= \int F_B^m\left(k - k^T(k,k_L;d,\phi)\right)dF_L^m(k_L) \\
F_B^{m\prime}(k) &= p_H\left(\int F_B^a(k + t_H(k,k_L;d,\phi))\,dF_L^a(k_L)\right) \\
&\quad + p_L\left(\int F_B^a(k + t_L(k,k_L;d,\phi))\,dF_L^a(k_L)\right) \\
F_L^{a\prime}(k) &= \int F_L^{m\prime}\left(k + k^T(k_B,k;d,\phi)\right)dF_B^{m\prime}(k_B).
\end{aligned}$$

Since the borrower has all the bargaining power in the funding market and lenders are left indifferent between participating in the funding market or not, in equilibrium

$$\bar{V}_L^a(k_L;\phi) = dk_L + \beta\mathbb{E}_{\phi'}\left(\bar{V}_L^m(k_L,\phi')\right)$$

where $\bar{V}_L^a(k,\phi') = \mathbb{E}_{k_B}\left(V_L^a(k,k_B;\phi')\right)$. Moreover, the equilibria in the asset and funding markets depend only on the expected value of the borrowers and lenders in the following subperiod. Using (1) one gets

$$\bar{V}_B^m(k_B;\phi) = \int\left(\bar{V}_B^a(k_B;\phi) - P(k_B,k_L;\phi)\right)dF_L^m(k_L)$$

where $\bar{V}_B^a(k,\phi') = \mathbb{E}_{k_L}\left(V_B^a(k,k_L;\phi')\right)$

Remark 3 *An equilibrium can be fully characterized by* $\bar{V}_B^a(k_B;\phi)$ *and* $\bar{V}_L^m(k_L;\phi)$.

3.1 Affine Equilibrium

For the remainder of the paper, I will focus on recursive affine equilibria, i.e., equilibria in which the value functions are affine in asset holdings. Within this class of equilibria, the equilibrium is unique.[6] All proofs are in the appendix.

[6] The mapping that characterizes an equilibrium is not a contraction mapping. Therefore I cannot show that the equilibrium is unique within a more general class of value functions.

From remark 3 an affine equilibrium is fully characterized by

$$\begin{aligned} \bar{V}_B^a(k_B;\phi) &= c_B(\phi)\,k_B + a_B(\phi) \\ \bar{V}_L^m(k_L;\phi) &= c_L(\phi)\,k_L + a_L(\phi) \end{aligned}$$

Using this characterization in (3) to determine the terms of trade in the asset market gives

$$\max_{P_0,k_0}\left(P_0 - dk_0 - \beta c_L(\phi)\,k_0\right)^{1-\gamma} \times \left(c_B(\phi)\,k_0 - P_0\right)^{\gamma}$$

which implies that

$$P_0 = (1-\gamma)\,c_B(\phi)\,k_0 + \gamma\left(dk_0 + \beta c_L(\phi)\,k_0\right)$$

and

$$k^T(k_B,k_L;\phi) = \arg\max_{k_0\in[-k_B,k_L]}\left(c_B(\phi) - d - \beta c_L(\phi)\right)k_0.$$

As it is usually the case with linear value functions when the terms of trade are determined through Nash bargaining, the price is an average of marginal valuations of the counterparts where the weight on an agent's marginal valuation is his counterpart's Nash bargaining weight. Moreover, the quantity traded maximizes aggregate surplus and, thus, trades in the asset market are efficient.

The linearity of the value function also implies that $k^T(k_B,k_L;\phi) \in \{-k_B,k_L\}$. Since the borrower can wait until the afternoon and sell the asset in the loan market, by setting $t_L = t_H = k_B$, he will never choose to sell in the morning. Selling in the afternoon, allows the borrower to invest the proceeds of the sale in the risky projects which has a higher expected return than consuming the goods in the morning. Therefore, the terms of trade in the asset market are independent of the asset stock with which the borrower enters the market and $k^T(k_B,k_L;\phi) = k_L$, for all k_B, k_L, and ϕ.

Given the affine specification of the value functions in an affine equilibrium, the solution to the borrower's problem in the funding market will be a corner solution. There are four possible solutions: $t_L = t_H = 0$, $t_L = 0$ and $t_H = k_B$, $t_L = k_B$ and $t_H = 0$, and $t_L = t_H = k_B$.

Lemma 1 *In an affine equilibrium, a borrower will choose*

$$\begin{aligned} t_L &= k_B \\ t_H &= \begin{cases} k_B & \text{if } c_B(\phi) < c_L(\phi) \\ 0 & \text{if } c_L(\phi) < c_B(\phi) \end{cases} \end{aligned}$$

If the borrower values the asset as much as the lender, he will set $t_L = t_H = k_B$ and "sell" the asset to the lender. In exchange, the lender will lend the borrower $c_L(\phi) = c_B(\phi)$ which is a fair compensation from the borrower's perspective.

However, if the borrower values the asset more than the lender, he would be getting less than his valuation if he chose to sell the asset to the lender: the lender would still pay a price $c_L(\phi)$ per unit where now $c_L(\phi) < c_B(\phi)$. In this case, the borrower will only transfer the asset if he cannot avoid it, i.e., if the low return is realized and he does not have enough resources to compensate the lender in consumption goods. Transferring the asset is costly for the borrower since he gets $c_L(\phi)$ units for it while he values it $c_B(\phi) > c_L(\phi)$. This difference in valuation can be interpreted as the punishment for lying which induces truth telling and makes the state contingent contract contract incentive compatible.

Proposition 2 *In the unique affine equilibrium,*

$$c_B(\phi) = c_B(d) > c_L(d) = c_L(\phi)$$

where $c_B(d)$ and $c_L(d)$ are affine functions of the dividend d.

In equilibrium, a borrower values the asset more than the lender. Holding assets allows the borrower to take advantage of his profitable investment opportunities either by selling the assets, by pledging its dividends or by pledging the assets themselves. Lenders, in contrast, can only eat the dividends or sell the asset in the asset market. Since $\mathbb{E}(\theta) > 1$, borrowers value the asset more than lenders. Since investment opportunities are persistent, borrowers will value the asset more than lenders in the future as well.

Corollary 2 *In an affine equilibrium, the optimal loan contract $(q^*, r_L^*, r_H^*, t_L^*, t_H^*)$ is given by*

$$q^*(d) = \frac{(d + \beta p_L c_L(d) + \beta p_H c_B(d))}{1 - \theta_L} k_B$$

$$\begin{aligned} r_L^*(d) &= \frac{(\theta_L q^*(d) + d)}{1 - \theta_L} k_B \\ r_H^*(d) &= r_L(d) + \beta c_B(d) k_B \\ t_L^* &= k_B,\ t_H^* = 0 \end{aligned}$$

The optimal loan contract only depends on the aggregate state through the dividend level d. As one can see from corollary 2, borrowers are collateral constrained in the optimal loan contract: the maximum amount that the borrowers are able to borrow depends linearly on the amount of assets they have. The amount that can be borrowed per unit of asset held is an equilibrium outcome and it depends on how much the expected holder values the asset. With probability p_L the return of the projects is low and the borrower transfers all his asset holdings to the lender whose expected discounted value of one unit of asset is $\beta c_L(d)$. With probability p_H the return of the projects is high and the borrower keeps all his assets which he values $\beta c_B(d)$ per unit.

3.2 Implementation of the Optimal Loan Contract

The optimal loan contract can be implemented using two different debt contracts: riskless debt and collateralized debt. This implementation, however, is not unique. I'll be looking at the implementation with the maximal amount of risk-free debt.

Lemma 2 *In any implementation the risk free rate is* 0*.*

In any implementation, lenders have to be indifferent between lending at the risk-free rate or by taking on some risk. Since the expected return of the optimal loan contract for lenders is 0, the risk free rate has to be 0.

The maximum amount that can be repaid independently of the realized state, risklessly, is $r_L^*(d)$. Therefore, since the risk-free rate is 0, the maximum amount of riskless debt is $r_L^*(d)$. The remaining part of the loan amount $q^*(d)$ is repaid in consumption goods only if the return of the project is high, whereas if the return is low, the lender receives an asset transfer from the borrower. I will refer to this fraction of the loan amount as collateralized debt.

In this implementation collateralized debt will be characterizes by two quantities: a loan amount

$$q_c := q(d) - r_L^*(d) = \beta\left(p_L c_L(d) + p_H c_B(d)\right) k_B$$

and an interest rate i_c

$$i_c = \frac{r_H(d) - r_L(d)}{q_c k_b} - 1 = \frac{p_L\left(c_B(d) - c_L(d)\right)}{\left(p_L c_L(d) + p_H\left(c_B(d)\right)\right)}.$$

The loan amount q_c is equal to the expected repayment the lender gets from the borrower: with probability p_L the lender gets paid in assets which he values $\beta c_L(d)$ per unit and with probability p_H he gets paid $\beta c_B(d) k_B$ in afternoon consumption good, which is the maximum amount the borrower is willing to give up in order to keep his assets. The higher the difference between the borrower's and lender's valuations of the asset, the higher the repayment in the high state to which the borrower can credibly promise and, therefore, the higher the interest rate on the collateralized debt.

Since the implementation above is the one with the maximum amount of collateralized debt, the maximum amount that can be borrowed against one unit of the asset, its debt capacity, is given by

$$D = \beta \left(c_L \left(d \right) + p_H \left(c_B \left(d \right) - c_L \left(d \right) \right) \right).$$

The asset's debt capacity depends on the value of collateral for lenders when there is default (insurance) and on the extra value borrowers attach to collateral when there isn't default (incentives). Ceteris paribus, a higher value of collateral for lenders increases the loan amount since they can recover more when there is default, while a higher (extra) value of collateral for borrowers decreases the borrowers' incentives to lie and, therefore, allows them to borrow more.

3.3 Premia

As I mentioned above, if borrowers and lenders were in autarky and could not trade with each other, they would all value the asset at its fundamental value $\frac{d}{1-\beta}$. However, when borrowers and lenders are allowed to trade, their marginal valuation for the asset differs from the fundamental value and between borrowers and lenders.

This difference between the agents' valuation of the asset and the fundamental value of the asset can be decomposed in several premia. A borrower values the asset more than the expected discounted dividend stream for two reasons. The first reason is that the asset serves as a liquidity transformation device, it allows the borrower to invest when he has the investment opportunity. The extra value due to this function is captured by the private liquidity premium, which I define as the difference between how much a borrower would value the asset if he chose to sell it to get funds and the fundamental value of the asset. If a borrower chose to sell the asset in the afternoon, he would get $d + \beta c_L \left(d \right)$ per unit of asset, which is the maximum amount the lender would be willing to pay for it. With this funds, the borrower would be able to invest in his risky projects and he would get the return on equity $\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}$ on them. Therefore, the borrower would value each unit of asset $\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L} \left(d + \beta c_L \left(d \right) \right)$, and the private liquidity premium is defined as

$$\frac{\mathbb{E} \left(\theta \right) - \theta_L}{1 - \theta_L} \left(d + \beta c_L \left(d \right) \right) - \left(\frac{d}{1 - \beta} \right)$$

The second reason why a borrower values the asset more than the fundamental value is that he expects to use it as collateral in the following period. I define the private collateral premium as the extra value a borrower gets from using the asset as collateral instead of selling it to raise funds. In an affine equilibrium, a borrower who chooses to use the asset as collateral values it $c_B(d)$ per unit. Then, the private collateral premium is

$$c_B(d) - \frac{\mathbb{E}(\theta) - \theta_L}{1 - \theta_L}(d + \beta c_L(d)) = \frac{\mathbb{E}(\theta) - \theta_L}{1 - \theta_L}\beta p_H (c_B(d) - c_L(d)).$$

This premium depends on the difference in valuations of the borrower and the lender. If both agents value the asset the same, the private collateral premium is 0. In this case, the borrower would be indifferent between selling the asset and pledging it as collateral. When the borrower values the asset more than the lender, the private collateral premium is positive and the borrower chooses to use the asset as collateral in the optimal funding contract.

Finally, a lender may value the asset more than the expected discounted sum of its dividends if he has some bargaining power in the asset market. By being able to sell the asset to agents that value the asset more than themselves, lenders can extract some of this surplus whenever their bargaining power is positive, i.e., $\gamma < 1$. This extra value is what I call a liquidity premium and it is defined as

$$c_L(d) - \left(\frac{d}{1-\beta}\right) = (1-\gamma)\frac{\mathbb{E}(\theta) - 1}{1 - \theta_L}\left(\frac{\frac{d}{1-\beta} + \beta p_H (c_B(d) - c_L(d))}{1 - \beta\left(\gamma + (1-\gamma)\frac{\mathbb{E}(\theta) - \theta_L}{1-\theta_L}\right)}\right).$$

When $\gamma = 1$ this premium is 0 since borrowers have all the bargaining power in the asset market and keep all the surplus from the transaction. In this sense, γ is a measure of the assets liquidity. When γ is high, liquidity is low and the price at which the asset is sold in the asset market is closer to the fundamental value $\frac{d}{1-\beta}$.

4 Market liquidity

In the baseline model I have assumed that meetings in the asset market are random and that trading is bilateral. However, all results can be generalized to the case in which the asset market is competitive. In the next subsection I characterize the competitive asset market equilibrium in an affine equilibrium and show that assuming a competitive asset market or a bilateral one is equivalent as long as the bargaining power of the borrowers is chosen appropriately. This result implies that the parameter γ can be interpreted as a measure of the market liquidity.

4.1 Competitive asset market

Assume that the asset market in the morning is competitive and that we are in an affine equilibrium in which $c_B(d)$ and $d + \beta c_L(d)$ are the borrowers' and lenders' marginal valuation of the asset, respectively. Then, the equilibrium price in a competitive asset market is be given by the amount of consumption good in the market relative to the stock of assets being supplied - there is *cash-in-the-market* pricing. Borrowers spend all the consumption good they have in the asset as long as the price does not exceed their marginal valuation $c_B(d)$. For a given asset price p, the demand for the asset is be given by

$$D(p) = \min\left\{c_B(d), \frac{e_B^m}{p}\right\}$$

Similarly, lenders are willing to sell their asset holdings k_L if the price is at least as high as their marginal valuation $d + c_L(d)$. For a given asset price p, the supply of the asset will be given by

$$S(p) = \begin{cases} 0 & \text{if } p < d + \beta c_L(d) \\ s \in [0, k_L] & p = d + \beta c_L(d) \\ k_L & p > d + \beta c_L(d) \end{cases}$$

Therefore, equilibrium price in the competitive asset market will be given by

$$p^* = \min\left\{\max\left\{d + \beta c_L(d), \frac{e_B^m}{k_L}\right\}, c_B(d)\right\}$$

The equilibrium in the competitive asset market is shown in figure 4.1.

Proposition 3 *For any competitive equilibrium in the asset market there exists γ such that the equilibrium prices in the bilateral asset market are the same as those in the competitive market.*

Since borrowers will never pay more than their marginal valuation for the asset and lenders will not sell for less the asset for less than their marginal valuation, the equilibrium price in the competitive asset market p^* can always be expressed as

$$p^* = \gamma(d + \beta c_L(d)) + (1 - \gamma) c_B(d)$$

for some $\gamma \in [0, 1]$. Note that the expression above for p^* is exactly the same per unit price one would get in a bilateral market in which the bargaining power of borrowers is γ.

There is a one-to-one relationship between γ and the relative amount of liquidity in the market. When the market liquidity is high, i.e., when the amount of consumption good relative to the asset supplied is high, the market is a "sellers' market" and γ is low (buyers have low bargaining power).

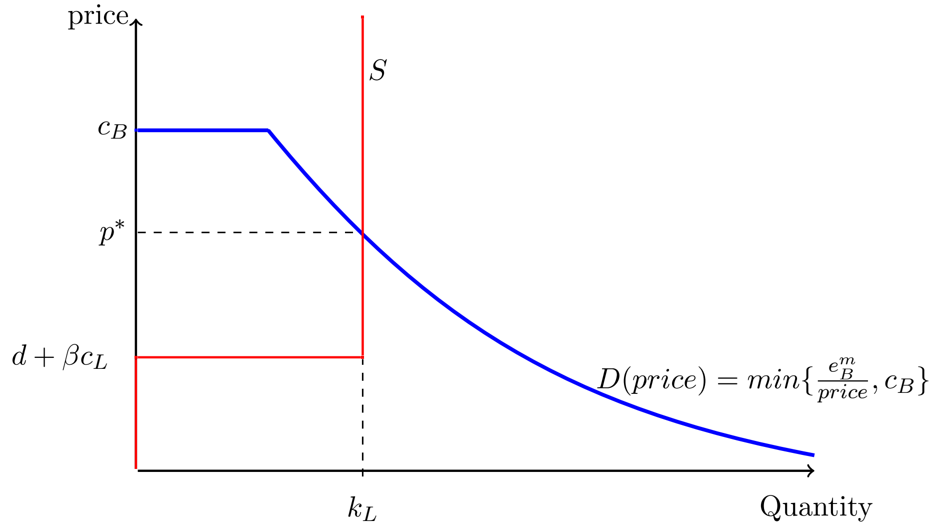

Figure 2: **Competitive asset market**

Alternatively, when the market liquidity is low, i.e., when there are few consumption goods per unit of asset supplied, the market is a "buyers' market" and γ is high (buyers have high bargaining power). In this sense, γ can be interpreted as a measure of the liquidity in the asset market.

Proposition 4 *The debt capacity of the asset increases with the liquidity in the asset market, .i.e.,*

$$\frac{\partial D}{\partial \gamma} < 0$$

A more liquid asset market provides better insurance to the lender and worse incentives to the borrower. On one hand, a lower γ increases the surplus lenders can extract from the borrowers, i.e.,

$$\frac{\partial c_L(d)}{\partial \gamma} < 0$$

On the other hand, a lower γ closes the gap between the borrower's and lender's valuation and, thus, decreases the cost of defaulting, i.e.,

$$\frac{\partial (c_B(d) - c_L(d))}{\partial \gamma} > 0$$

As the proposition above shows, the effect on the asset's quality as insurance for the lenders dominates and more liquid asset have a higher debt capacity.

5 Extensions

In this section I introduce four different extensions of the baseline model. I first consider stochastic dividends which are correlated with the returns of the risky projects. Second, I introduce stochastic investment opportunities. Third, I allow for savings between the morning and the afternoon. And fourth, I allow for heterogeneity in the borrowers' investment opportunities. All omitted proofs are in the appendix.

5.1 Correlated dividends and returns

The model is the same as the one presented in the previous section with the only difference that the dividend paid by the asset is stochastic and it is potentially correlated with the return of the risky projects of the borrowers. Formally, I assume that there is an underlying unobservable i.i.d. aggregate state $\omega_t \in (\omega_1, \omega_2)$ that determines the probability of success of the risky projects and the dividend level. The fraction of risky projects with high and low returns depends on the aggregate state: if the aggregate state is ω_n a fraction p_i^n of the borrowers gets a return θ_i, $i = L, H$. As before, the unconditional probability of a borrower getting a return $\theta_t^j = \theta_i$ in the afternoon of period t is given by p_i, $i = L, H$ and the returns of the risky projects are unconditionally i.i.d across time and borrowers.

I assume that the returns of the risky projects in period t and dividends in period $t+1$ are correlated and that their joint distribution is stationary. Therefore, the expected dividend given an individual realization θ_i is given by

$$\mathbb{E}\left(d_{t+1}|\theta_t^j = \theta_i\right) = \sum_{n=1,2} \Pr\left(\omega_t = \omega_n|\theta_t^j = \theta_i\right) E\left(d_{t+1}|\omega_t = \omega_n\right) := d_i \text{ for } i = L, H, \forall j, \forall t \geq 0$$

Finally, since the aggregate state is i.i.d. across time, $\mathbb{E}\left(d_{t+1}\right) = \bar{d}$ and $\mathbb{E}\left(\theta_t^j\right) = \mathbb{E}\left(\theta\right)$ for all t for all j.

Given that the coefficients c_B and c_L in the affine equilibrium above are affine in the dividend level, it is easy to see that all the results in the previous section hold. In particular, the debt capacity of the asset when returns and dividends are correlated is given by

$$D = \beta\left(c_L\left(d_L\right) + p_H\left(c_B\left(d_H\right) - c_L\left(d_L\right)\right)\right)$$

Proposition 5 *Assets that have dividends that are more highly positively correlated with the risky projects have a higher debt capacity.*

$$\frac{\partial D}{\partial d_H|_{\bar{d}}} > 0$$

In the model, the asset partly resolves the non-contractibility of the return of the risky projects and allows the borrower raise funds. Since in equilibrium the borrower values the asset more than the lender does, $c_B(d_H) - c_L(d_L)$ is the endogenous cost of defaulting on the promised amount r_H and it allows the borrower to credibly commit to reporting the projects' return truthfully. Assets that have dividends that are more highly positively correlated with the return of the risky projects have a higher d_H and a lower d_L which imply a higher cost of default for borrowers. This higher cost of default allows the borrower to commit a larger amount of goods in the high state and, thus, increase the debt capacity of the asset and make it better collateral.

Proposition 6 *If $d_H \geq d_{\min}$, where $d_{\min} < \bar{d}$, a mean preserving spread of the returns of the projects decreases the debt capacity of the asset, i.e.*

$$\frac{\partial D}{\partial \theta_H|_{\mathbb{E}(\theta)}} < 0$$

When the return of the projects and the future dividend level are sufficiently positively correlated, an increase in the riskiness of the projects decreases the debt capacity of the asset. A mean preserving spread of the projects' return increases θ_H and decreases p_H. On top of increasing the probability of default directly, the shift in the structure of the risky projects' return affects both the insurance and incentive components of the debt capacity. First, since default is more likely, the value of the asset for lenders when default happens, $c_L(d_L)$, decreases and so does the lender's willingness to lend to the borrower. On the other hand, this decrease in value for the lender increases the borrower's cost of misreporting which would increase the asset's debt capacity. However, since p_H decreases after a mean preserving state, the debt capacity of the asset puts more weight on the quality of the asset to provides insurance than to provide incentives. When the dividend is sufficiently positively correlated with the return of the risky projects the decrease in the asset quality to provide insurance to lenders dominates and the debt capacity of the asset decreases.

5.2 Stochastic investment opportunities

In the baseline model, borrowers always have access to the investment opportunities while lenders can never access them. This assumption can be relaxed to allow for probabilistic access to the risky projects.

The model is the same as the baseline model with the exception that now borrowers remain borrowers (and lenders remain lenders) with probability ρ. An agent who has a borrowing opportunity at time t keeps his investment opportunity at time $t+1$ with probability ρ while an agent without a lending opportunity at time t acquires one with probability $(1-\rho)$. Whether an agent has access to an investment opportunity or not at time t is known at the beginning of the morning of time t before the asset market opens. To keep things more tractable, I assume that the agents that get an investment opportunity don't get any endowment of afternoon good.

The only difference in the borrowers' and lenders' problems in this setup and in the baseline model is in the continuation values. Whereas in the baseline model agents know in which side of the market they will participate in the future, in the current setup their role in the economy is random. Therefore, everything derived in the baseline model at the beginning of the paper holds by substituting $\bar{V}_B^m\left(\cdot;\phi'\right)$ by

$$\rho\bar{V}_B^m\left(\cdot;\phi'\right)+(1-\rho)\,\bar{V}_L^m\left(\cdot;\phi'\right)$$

and $\bar{V}_L^m\left(\cdot;\phi'\right)$ by

$$\rho\bar{V}_L^m\left(\cdot;\phi'\right)+(1-\rho)\,\bar{V}_B^m\left(\cdot;\phi'\right)$$

Analogously to the baseline case borrowers will only choose to transfer the asset in the high state if the lender values the asset at least as much as he does.

Lemma 3 *In an affine equilibrium, a borrower will choose*

$$\begin{aligned} t_L &= k_B \\ t_H &= \begin{cases} k_B & \text{if } (1-2\rho)\left(c_L(d)-c_B(d)\right)<0 \\ 0 & \text{if } (1-2\rho)\left(c_L(d)-c_B(d)\right)\geq 0 \end{cases} \end{aligned}$$

As before, what determines whether assets are sold or used as collateral is the difference in future valuations for the asset between borrowers and lenders. When investment opportunities are random, this difference in valuation will depend not only on the valuation for the asset of borrowers and lenders, $c_B(d)$ and $c_L(d)$, but also on the persistence of the investment opportunities, ρ.

Lemma 4 *Current borrowers always value the asset more than current lenders*

$$c_B(d)>c_L(d)$$

At the beginning of time t, before accessing the asset market, borrowers value the asset more than lenders do. Having the asset allows borrowers to take advantage of their investment opportunities in the afternoon whereas lenders have to wait at least a period to have access to the same investment opportunities.

From the lemmas above we have the following proposition.

Proposition 7 *The asset will be used as collateral if and only if $\rho > 0.5$.*

Since having the asset is more valuable while having access to the investment opportunity, borrowers will value the asset more than lenders only if their likelihood of being borrowers is higher than the lenders'. Therefore, as long as the investment opportunities are persistent, borrowers will always choose to use it as collateral rather than selling it.

Suppose that $\rho > 0.5$. Then, the debt capacity of the asset is given by

$$D = \beta\left(\rho c_L(d) + (1-\rho) c_B(d) + p_H(2\rho - 1)(c_B(d) - c_L(d))\right)$$

As in the baseline case, the debt capacity has two components: one that provides insurance to lenders and one that provides incentives to borrowers. Lenders are always willing to lend their own valuation $\beta\left(\rho c_L(d) + (1-\rho) c_B(d)\right)$ if the loan is collateralized by the asset. Since lenders know that defaulting is costly for borrowers, they know that borrowers will have incentives to pay more in the high state in order to avoid losing the asset. The extra amount that borrowers will be willing to pay in the high state is $\beta(2\rho - 1)(c_B(d) - c_L(d))$.

5.3 Savings between morning and afternoon

The baseline model can also be extended to allow for savings of consumption good between the morning and afternoon. If borrowers are allowed to bring consumption goods into the afternoon, this would allow them to get leverage and they would get a return of $\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}$ per unit saved. Then, the surplus for a borrower who acquires k_0 units of the asset in the asset market and pays P_0 for them is

$$V_B^a(k_B + k_0; \phi) - \frac{\mathbb{E}(\theta) - \theta_L}{1 - \theta_L} P_0 - P_0 - V_B^a(k_B; \phi)$$

where $V_B^a(k_B; \phi)$ is the value of a borrower who enters the afternoon with assets k_B. Similarly, the surplus of a lender who sells k_0 units of the asset at a price P_0 is

$$V_L^a(k_L - k_0, k_B; \phi) + P_0 - V_L^a(k_L, k_B; \phi)$$

In this case, the price for k_0 units of the asset in the morning asset market will be given by

$$\bar{P}_0 = \frac{(1-\gamma) c_B(d) + \gamma(d + \beta c_L(d))}{1 + (1-\gamma)\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}} k_0$$

Proposition 8 *If savings are allowed, borrowers will choose to buy as many assets as they can in the asset market, i.e., $k_0 = k_L$.*

Proof. The price for k_0 units of the asset in the morning asset market will be given by

$$\bar{P}_0 = \frac{(1-\gamma) c_B(d) + \gamma(d + \beta c_L(d))}{1 + (1-\gamma)\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}} k_0$$

A borrower will strictly prefer to buy an additional unit of the asset rather than saving if

$$\begin{aligned} \frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}\bar{P}_0 &< c_B(d) k_0 \\ \frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}\gamma(d+\beta c_L(d)) &< c_B(d) \end{aligned}$$

This condition holds for all $\gamma > 0$ since the borrower could always sell the asset in the funding market and get $\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}(d + \beta c_L(d))$. ■

5.4 Multiple Project Types

In this section I extend the model in the previous section to allow for heterogeneity among borrowers. Each borrower is characterized by the returns of the projects in which he is able to invest. I assume that the projects available to different borrowers differ in their correlation with the dividends paid by the asset but they share the same success probability and unconditional expected return. There are J types of projects and a fraction μ_j of borrowers can invest in projects of type j, $j = 1, ..., J$, where $\sum_j \mu_j = 1$

In this case, in an affine equilibrium, the marginal value of assets for a lenders is

$$c_L(d) = (1-\gamma)\sum_{i \in I} \mu_i c_B^i(d) + \gamma\left(d + \beta c_L\left(\bar{d}\right)\right).$$

The borrowers' problem remains unchanged, though now the asset value for lenders depends on the average valuation among borrowers.

Proposition 9 *The borrowers with the highest marginal valuation of the asset will always use it as collateral.*

Borrowers whose projects have the highest positive correlation with the dividend paid by the asset value the asset the most. The higher this correlation, the larger the amount that can be borrowed against the asset since it is better at providing incentives to solve the asymmetric information problem.

Proposition 10 *There exists $\bar{\gamma}$ such that all borrowers use the asset as collateral in equilibrium if and only if $\gamma > \bar{\gamma}$.*

Depending on the parameters of the model, two different kinds of regimes might arise. In one, all borrowers choose to use the asset as collateral. This is clearly the case when $\gamma = 1$ since the multiple project type model and the benchmark model give the same contract for each agent. The existence of other types of borrowers only matters through the resale value of the asset in the asset market. If the sellers don't get any surplus from this sale, then the price of the asset in the asset market will be equal to the expected discounted value of dividend and, thus, it would be independent of the distribution of borrower types in the economy.

If $\gamma < 1$ there might be borrowers who choose to sell the asset at the beginning of the afternoon and invest those funds rather than pledging the asset as collateral. Since the expected price at which lenders can sell the asset in the asset market depends on the borrowers' average valuation of the asset, it may be the case that this average valuation is high enough to motivate some of the borrowers who value the asset the least to sell the asset to raise funds. Whether there are some borrowers that don't use the asset as collateral or not depends on the value of γ. For high values of γ everybody uses the asset as collateral in the only symmetric equilibrium. For lower values of γ some borrowers may choose to sell the asset instead of using it as collateral.

6 Conclusion

In this paper I showed that when the roles as borrowers and lenders are persistent, and the return on the risky projects is non-contractible, debt collateralized by liquid financial assets arises as an optimal way for borrowers to raise funds. In equilibrium, borrowers value the assets more than lenders and, therefore, borrowers would rather offer their assets as collateral than sell them. If borrowers sold their assets, they would get at most the valuation of lenders, whereas by offering them as collateral borrowers keep their assets when there is no default.

This difference in marginal valuations of the asset between borrowers and lenders is an equilibrium outcome. In autarky, both borrowers and lenders value the asset as the expected discounted sum of the dividend stream. When the agents are able to trade, the borrower values the asset more then the lender and they both value the asset (weakly) more than its fundamental value. The borrowers' excess valuation can be divided in two premia: a private liquidity premium and a private collateral premium. The first comes from the asset solving a maturity mismatch for the borrower: the asset pays dividends in the future but the borrower has an investment opportunity today. Being able to sell the asset provides the borrower with funds in the moment he needs them.

The private collateral premium is the extra value the borrower assigns to the asset as collateral which is an instrument to solve the non-contractibility of the projects' returns.

Since collateralized debt is optimal in this setup, and the marginal valuations of both borrowers and lenders are endogenous, the maximum amount that can be borrowed against the asset, its debt capacity, is also an equilibrium outcome. Some of the main determinants of the asset's debt capacity are the liquidity in the market in which they are traded, the correlation of their dividends with the return of the investments made by the borrowers and the riskiness of those investments. Assets traded in more liquid markets have more insurance value for lenders and are better collateral than those traded in less liquid markets. Similarly, assets that have dividends that are more highly correlated with the ability of the borrowers to repay are more costly for borrowers to lose and, thus, have a higher debt capacity. Finally, borrower with riskier projects are able to borrow less against the same asset. When one allows for heterogeneity in the types of investments available to borrowers, there will always be at least one type of borrower using the asset as collateral. Whether all borrowers use collateralized debt will depend on the market's liquidity.

We know from previous literature that changes in margins and haircuts played an important role in recent crises.[7] Having a model that characterizes these objects as equilibrium outcomes is important both from a positive and a normative point of view. In positive terms, it is interesting to see where the financial shocks come from and how they interact with the fundamentals of the economy. From the normative side, policies that aim at stabilizing the cycle and preventing financial crises should take into account what drives changes in the financing conditions faced by financial intermediaries, firms, and households. This paper delivers some of the insights needed to understand collateralized debt markets better.

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[7] See Jermann and Quadrini (forthcoming), Perri and Quadrini (2012).

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7 Appendix

7.1 Borrower's Problem

Lemma 5 *Without loss of generality, PC can be replaced by*

$$\begin{aligned} \beta \mathbb{E}_{\phi'}\left(V_L^m\left(k_L;\phi'\right)\right) &= -q + p_H r_H + \beta p_H \mathbb{E}_{\phi'}\left(V_L^m\left(k_L + t_H;\phi'\right)|\theta_H\right) \\ &\quad + p_L r_L + \beta p_L \mathbb{E}_{\phi'}\left(V_L^m\left(k_s + t_L;\phi'\right)|\theta_L\right). \end{aligned} \tag{5}$$

in the borrower's problem.

Proof. Let V^* be the solution to the borrower's problem. Let $\{V_j\}_{j\geq 0}$ be such that $\lim_{j\to\infty} V_j = V^*$, where

$$\begin{aligned} &V_j = \mathbb{E}\left(\theta\right) q_j - p_H r_{Hj} - p_L r_{Lj} + dk_B \\ &+\beta p_H \mathbb{E}_{\phi'}\left(V_B^m\left(k_B - t_{Hj};\phi'\right)|\theta_t^i = \theta_H\right) + \beta p_L \mathbb{E}_{\phi'}\left(V_B^m\left(k_B - t_{Lj};\phi'\right)|\theta_t^i = \theta_L\right) \end{aligned} \tag{6}$$

for some feasible and incentive compatible $\{q_j, r_{Lj}, r_{Hj}, t_{Lj}, t_{Hj}\}$ that satisfies the participation constraint PC. Suppose that for some $j \geq 0$, $(q_j, r_{Lj}, r_{Hj}, t_{Lj}, t_{Hj})$ is such that PC is slack. Then, one could increase q_j and increase V_j to V_j^0 still satisfying all the other constraints. Let $\left\{V_j'\right\}_{j\geq 0}$ be a sequence identical to $\{V_j\}_{j\geq 0}$ if at $(q_j, r_{Lj}, r_{Hj}, t_{Lj}, t_{Hj})$ PC holds with equality and V_j^0 otherwise. Then, by construction, $V_j^{'} \geq V_j$ and therefore,

$$\lim_{j\to\infty} V_j' \geq \lim_{j\to\infty} V_j = V^*.$$

Therefore, we can replace PC by (5) in the borrower's problem. ■

Lemma 6 *Without loss of generality, the incentive compatibility constraints can be replaced by*

$$-r_L + \beta \mathbb{E}_{\phi'}\left(\bar{V}_B^m\left(k_B - t_L;\phi'\right)|\theta_t^i = \theta_H\right) = -r_H + \beta \mathbb{E}_{\phi'}\left(\bar{V}_B^m\left(k_B - t_H;\phi'\right)|\theta_t^i = \theta_H\right).$$

in the borrower's problem.

Proof. Let V^* be the solution to the borrower's problem. Let $\{V_j\}_{j\geq 0}$ be such that $\lim_{j\to\infty} V_j = V^*$, where

$$\begin{aligned} &V_j = \mathbb{E}\left(\theta\right) q_j - p_H r_{Hj} - p_L r_{Lj} + dk_B \\ &+\beta p_H \mathbb{E}_{\phi'}\left(V_B^m\left(k_B - t_{Hj};\phi'\right)|\theta_t^i = \theta_H\right) + \beta p_L \mathbb{E}_{\phi'}\left(V_B^m\left(k_B - t_{Lj};\phi'\right)|\theta_t^i = \theta_L\right) \end{aligned} \tag{7}$$

for some feasible and incentive compatible $\{q_j, r_{Lj}, r_{Hj}, t_{Lj}, t_{Hj}\}$ that satisfies the participation constraint (5). Suppose that for some $s \geq 0$, no incentive compatibility constraint binds. Then, there exists $\varepsilon_s > 0$ such that

$$\begin{aligned} \beta \mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_{Hs}; \phi' \right) - \bar{V}_B^m \left(k_B - t_{Ls}; \phi' \right) | \theta_L \right) &\leq r_{Hs} - r_{Ls} + \left(\theta_H - \theta_L \right) \varepsilon_s \\ r_{Hs} - r_{Ls} + \left(\theta_H - \theta_L \right) \varepsilon_s &\leq \beta \mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_{Hs}; \phi' \right) - \bar{V}_B^m \left(k_B - t_{Ls}; \phi' \right) | \theta_H \right). \end{aligned}$$

Replace $\{q_s, r_{Ls}, r_{Hs}, t_{Ls}, t_{Hs}\}$ by $\{q_s + \varepsilon_s + \varepsilon_0, r_{Ls} + \theta_L \varepsilon_s, r_{Hs} + \theta_H \varepsilon_s, t_{Lj}, t_{Hj}\}$ where $\varepsilon_0 > 0$ is such that the participation constraint binds. This contract still satisfies all the constraints, but it attains a value $V_s^0 > V_s$.

If $r_{Hs} > r_{Ls}$, IC_H is the only relevant incentive compatibility constraint. For all s such that $r_{Hs} > r_{Ls}$ and IC_H is not binding, the previous argument applies and a value $V_s^0 > V_s$ can be attained.

Now consider those $s \geq 0$ such that $r_H s \leq r_L s < \theta_L q_s + dk_B$. If IC_L binds, one could keep $r_{Hs} - r_{Ls_s}$ constant by increasing both r_{Ls} and r_{Hs} and by increasing q_s to keep the participation constraint binding which would result in an increase in the objective function. Let this new value be V_s^0. If for $s \geq 0$, $r_{Hs} \leq r_{Ls} = \theta_L q_s + dk_B$, IC_L doesn't bind unless IC_H binds. Suppose IC_L binds and IC_H doesn't. Then, one could increase r_{Hs} still satisfying incentive compatibility and relaxing the participation constraint. Therefore, one could increase q_s which would increase the objective function and give a value $V_s^0 \geq 0$. Therefore, once can construct a new sequence $\left\{V_j'\right\}_{j \geq 0}$, $V_j' \geq V_j$ such that $V_j' = V_j$ is the incentive compatibility constraint in the high state binds and $\mathrm{V}_j' = V_j^0$ if it doesn't. By construction,

$$\lim_{j \to \infty} V_j^{'} \geq \lim_{j \to \infty} V_j = V^*.$$

Therefore, without loss of generality one can concentrate on those sequences that are feasible in which IC_H holds with equality, i.e.,

$$r_{H_j} - r_{L_j} = \beta \mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_{Hj}; \phi' \right) - \bar{V}_B^m \left(k_B - t_{Lj}; \phi' \right) | \theta_H \right) \text{ for all } j. \tag{8}$$

■

Lemma 7 *Without loss of generality, the feasibility constraints on contingent transfers in consumption good can be replaced by*

$$r_L = \theta_L q + dk_B \text{ and } r_H \geq 0$$

in the borrower's problem.

Proof. By assumption $q \leq \mathbb{E}_L^a$ will not bind in a solution to the borrower's problem. Using lemma 5, the participation constraint can be assumed to hold with equality, and using this in the objective function one can see that the objective function is always increasing in the amount of the loan q. Using lemma 6, the incentive compatibility constraint holds with equality which implies that the upper bound for q is given by the maximum amount that can be repaid in the low state, i.e., by $r_L = \theta_L q + dk_B$.

Let V^* be a solution to the borrower's problem. Let $\{V_j\}$ be a sequence such that $lim_{j\to\infty} V_j = V^*$ and where

$$V_j = (\mathbb{E}(\theta) - 1) q_j + \beta p_H \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_{Hj}; \phi' \right) | \theta_t^i = \theta_H \right) + \beta p_L \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_{Lj}; \phi' \right) | \theta_t^i = \theta_L \right) + dk_B$$
$$+ \beta p_H \mathbb{E}_{\phi'} \left(V_B^m \left(k_B - t_{Hj}; \phi' \right) | \theta_t^i = \theta_H \right) + \beta p_L \mathbb{E}_{\phi'} \left(V_B^m \left(k_B - t_{Lj}; \phi' \right) | \theta_t^i = \theta_L \right) - \beta \mathbb{E}_{\phi'} \left(V_L^m \left(k_L; \phi' \right) \right) \quad (9)$$

for some feasible and incentive compatible that satisfies (8), and PC with equality, that is that,

$$\begin{aligned} r_{Lj} &= q^* - \beta \left(p_H \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_{Hj}; \phi' \right) | \theta_H \right) + p_L \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_{Lj}; \phi' \right) | \theta_L \right) \right) \\ &\quad - \beta p_H \left(\mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_{Hj}; \phi' \right) - \bar{V}_B^m \left(k_B - t_{Lj}; \phi' \right) | \theta_H \right) \right) + \beta \mathbb{E}_{\phi'} \left(V_L^m \left(k_L; \phi' \right) \right) \quad (10) \\ r_{Hj} &= q^* - \beta \left(p_H \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_{Hj}; \phi' \right) | \theta_H \right) + p_L \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_{Lj}; \phi' \right) | \theta_L \right) \right) \\ &\quad - \beta p_L \left(\mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_{Lj}; \phi' \right) - \bar{V}_B^m \left(k_B - t_{Hj}; \phi' \right) | \theta_H \right) \right) + \beta \mathbb{E}_{\phi'} \left(V_L^m \left(k_L; \phi' \right) \right) \quad (11) \end{aligned}$$

Then, for all j, the contract can be summarized by $\{q_j, t_{Lj}, t_{Hj}\}$. The feasibility constraints for r_L and r_H imply the following constraints

$$\begin{aligned} q_j &\leq \frac{dk_B + \beta \left(p_H \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_{Hj}; \phi' \right) | \theta_H \right) + p_L \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_{Lj}; \phi' \right) | \theta_L \right) \right)}{1 - \theta_L} \\ &\quad + \frac{\beta p_H \left(\mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_{Hj}; \phi' \right) - \bar{V}_B^m \left(k_B - t_{Lj}; \phi' \right) | \theta_H \right) \right) - \beta \mathbb{E}_{\phi'} \left(V_L^m \left(k_L; \phi' \right) \right)}{1 - \theta_L}, \quad (12) \\ q_j &\geq \beta p_H \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_{Hj}; \phi' \right) | \theta_H \right) + \beta p_L \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_{Lj}; \phi' \right) | \theta_L \right) \\ &\quad + p_H \beta \mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_{Hj}; \phi' \right) - \bar{V}_B^m \left(k_B - t_{Lj}; \phi' \right) | \theta_H \right) - \beta \mathbb{E}_{\phi'} \left(V_L^m \left(k_L; \phi' \right) \right), \quad (13) \\ q_j &\geq \frac{dk_B + \beta p_H \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_{Hj}; \phi' \right) | \theta_H \right) + \beta p_L \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_{Lj}; \phi' \right) | \theta_L \right)}{1 - \theta_H} \\ &\quad + \frac{p_L \beta \mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_{Lj}; \phi' \right) - \bar{V}_B^m \left(k_B - t_{Hj}; \phi' \right) | \theta_H \right) - \beta \mathbb{E}_{\phi'} \left(V_L^m \left(k_L; \phi' \right) \right)}{1 - \theta_H} \quad (14) \\ q_j &\geq \beta p_H \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_{Hj}; \phi' \right) | \theta_H \right) + \beta p_L \mathbb{E}_{\phi'} \left(V_L^m \left(k_L + t_{Lj}; \phi' \right) | \theta_L \right) \\ &\quad - p_L \beta \mathbb{E}_{\phi'} \left(\bar{V}_B^m \left(k_B - t_{Hj}; \phi' \right) - \bar{V}_B^m \left(k_B - t_{Lj}; \phi' \right) | \theta_H \right) - \beta \mathbb{E}_{\phi'} \left(V_L^m \left(k_L; \phi' \right) \right). \quad (15) \end{aligned}$$

Construct the following sequence $\left\{ V_j' \right\}$: if $\{q_j, t_{Lj}, t_{Hj}\}$ is such that (12) holds with equality, set $V_j' = V_j$. If $\{q_j, t_{Lj}, t_{Hj}\}$ is such that (12) is slack let V_j' be the value attained by the contract that satisfies (12) with equality. Since the transfers in terms of consumption good are defined by (10) and (11), this contract is still incentive compatible and feasible. Moreover, $q_j' > q_j$ and $V_j' > V_j$. Therefore,

$$\lim_{j\to\infty} V_j' \geq \lim_{j\to\infty} V_j = V^*$$

and without loss of generality one can concentrate on the sequences $\{V_j\}$ as defined above in (9), such that the loan quantities $\{q_j\}$ satisfy (12) with equality. Having this constraint hold with equality implies $r_{Lj} = \theta_L q_j + dk_B$. Since $q_j \geq 0$ always, this implies that all contracts along this sequence satisfy $r_L > 0$ which is the same as satisfying (13) with strict inequality.

Suppose that for some j (14) holds with equality. This implies $r_{Hj} = \theta_H q_j + dk_B$ and since $r_{Lj} = \theta_L q_j + dk_B$ this would imply that participation constraint is slack, and that the producer is giving the non-producer all the gains from the project. From lemma 5, there exists a feasible and incentive compatible contract that attains a higher value that contract j and therefore, without loss of generality we can ignore sequences in which for some elements j, (14) holds with equality. ■

7.2 Uniqueness

7.2.1 Asset Market

Given the affine structure of the equilibrium, using the results for prices and quantities in the asset market, one gets that the value functions for lenders and borrowers in the morning before entering the asset market are, respectively,

$$\bar{V}_L^m(k_L;\phi) = P(k_B, k_L;\phi) = \left[(1-\gamma)\, c_B(\phi) + \gamma\left(d + \beta\mathbb{E}_{\phi'} c_L(\phi')\right)\right] k_L$$

and

$$\bar{V}_B^m(k_B;\phi) = \gamma\left[c_B(\phi) - \left(d + \beta\mathbb{E}_{\phi'|\phi} c_L(\phi')\right)\right]\int k dF_L^m(k) + c_B(\phi)\, k_B.$$

Using the guessed functional form for the value functions gives

$$c_L(\phi) = (1-\gamma)\, c_B(\phi) + \gamma\left(d + \beta\mathbb{E}_{\phi'} c_L(\phi')\right)$$

and

$$\mathbb{E}_{\phi'} c_L(\phi') = \frac{\left[(1-\gamma)\,\mathbb{E}_{\phi'} c_B(\phi') + \gamma d\right]}{1-\gamma\beta}.$$

Therefore,

$$c_L(\phi) = \left[(1-\gamma)\, c_B(\phi) + \gamma\left(d + \beta\frac{(1-\gamma)\,\mathbb{E}_{\phi'} c_B(\phi') + \gamma d}{1-\gamma\beta}\right)\right].$$

7.2.2 Funding Market

Applying the results in proposition 1 implies that, in an affine equilibrium, the value of a borrower who enters the loan market with k_B units of the asset and is matched with a lender with k_L units of the asset can be written as

$$\begin{aligned}
V_B^a(k_B, k_L; \phi) &= \max_{t_H, t_L \in [0, k_B]^2} (\mathbb{E}(\theta) - 1) q^* + dk_B + \beta (p_H c_L(\phi) t_H + p_H c_L(\phi) t_L) \\
&\quad + \beta p_H \mathbb{E}_{\phi'} (c_B(\phi)(k_B - t_H) | \theta_L) + \beta p_L \mathbb{E}_{\phi'} (c_B(\phi)(k_B - t_L) | \theta_H) \\
&\quad + \beta \mathbb{E}_{\phi'} \left(\gamma \left[c_B(\phi') - (d + \beta c_L(\phi')) \right] \int k dF_L^{m'}(k) \right)
\end{aligned}$$

s.t.

$$\begin{aligned}
q^* &= \frac{dk_B + \beta (p_H c_L(\phi) t_H + p_L c_L(\phi) t_L) - p_H \beta c_B(\phi)(t_H - t_L)}{1 - \theta_L} && (16) \\
q^* &\geq \max\left\{ \beta \left(p_H \mathbb{E}_{\phi'} \left(c_L(\phi') | \theta_H \right) t_H + p_L \mathbb{E}_{\phi'} \left(c_L(\phi') | \theta_L \right) t_L \right) + p_L \beta \mathbb{E}_{\phi'} \left(c_B(\phi') | \theta_L \right)(t_H - t_L), 0 \right\} && (17) \\
r_H &= q^* - \beta p_H \mathbb{E}_{\phi'} \left(c_L(\phi') | \theta_H \right) t_H - \beta p_L \mathbb{E}_{\phi'} \left(c_L(\phi') | \theta_L \right) t_L + p_L \beta \mathbb{E}_{\phi'} \left(c_B(\phi') | \theta_L \right)(t_H - t_L) \\
r_L &= q^* - \beta p_H \mathbb{E}_{\phi'} \left(c_L(\phi') | \theta_H \right) t_H - \beta p_L \mathbb{E}_{\phi'} \left(c_L(\phi') | \theta_L \right) t_L + p_H \beta \mathbb{E}_{\phi'} \left(c_B(\phi') | \theta_H \right)(t_H - t_L)
\end{aligned}$$

Given the affine specification of the utility functions, the solution to the borrower's problem in the afternoon will be in corner solution. If the constraint (17) is ignored, there are four possible solutions: $t_L = 0 = t_H$, $t_L = 0$ and $t_H = k_B$, $t_L = k_B$ and $t_H = 0$, and $t_L = k_B = t_H$. If $t_L \geq t_H$ then the constraints on q are satisfied. If $t_L < t_H$, (17) might bind. In the appendix I show that (17) can't bind in equilibrium. Therefore, ignoring the constraints on q^*, (17) ,and using the guessed functional form for $V_P^a(k_B, k_L; \phi)$, one can match coefficients and get

$$\begin{aligned}
c_B(d) &= \frac{(\mathbb{E}(\theta) - \theta_L)}{1 - \theta_L} \left(d + \beta \left(p_H \mathbb{E}_{\phi'} \left(c_L(\phi') | \theta_H \right) \frac{\partial t_H}{\partial k_B} + p_L \mathbb{E}_{\phi'} \left(c_L(\phi') | \theta_L \right) \frac{\partial t_L}{\partial k_B} \right) \right) \\
&\quad - \frac{(\mathbb{E}(\theta) - 1)}{1 - \theta_L} p_H \beta \mathbb{E}_{\phi'} \left(c_B(\phi') | \theta_H \right) \left(\frac{\partial t_H}{\partial k_B} - \frac{\partial t_L}{\partial k_B} \right) \\
&\quad + \beta p_H \mathbb{E}_{\phi'} \left(c_B(\phi') | \theta_H \right) \left(1 - \frac{\partial t_H}{\partial k_B} \right) + \beta p_L \mathbb{E}_{\phi'} \left(c_L(\phi') | \theta_L \right) \left(1 - \frac{\partial t_L}{\partial k_B} \right) && (18) \\
a_B(\phi) &= \beta \mathbb{E}_{\phi'} \left(\gamma \left[c_B(\phi') - \left(d' + \beta \mathbb{E}_{\phi''} \left(c_B(\phi'') \right) \right) \right] \int k dF_L^{m'}(k) \right).
\end{aligned}$$

The following four lemma's help characterize the optimal funding contract.

Lemma 8 (17) *can't bind in equilibrium.*

Proof. Suppose (17) binds, then,

$$\frac{dk_B + \theta_L \beta \left(p_H \mathbb{E}_{\phi'} \left(c_L \left(\phi'\right) |\theta_H\right) t_H + p_L \mathbb{E}_{\phi'} \left(c_L \left(\phi'\right) |\theta_L\right) t_L\right)}{\beta c_B \left(d_H\right)} = t_H - t_L. \tag{19}$$

If $r_H = 0$ binds,

$$q^* = \left(\beta p_H \mathbb{E}_{\phi'} \left(c_L \left(\phi'\right) |\theta_H\right) t_H - \beta p_L \mathbb{E}_{\phi'} \left(c_L \left(\phi'\right) |\theta_L\right) t_L\right) (1 - \theta_L) + p_L dk_B$$

and using the definition of q^* together with (11) this implies

$$q^* = \frac{p_L dk_B + \beta \left(p_H \mathbb{E}_{\phi'} \left(c_L \left(\phi'\right) |\theta_H\right) t_H + p_L \mathbb{E}_{\phi'} \left(c_L \left(\phi'\right) |\theta_L\right) t_L\right) (1 - p_H \theta_L)}{1 - \theta_L}.$$

Putting these last two equations together gives

$$\beta \left(p_H \mathbb{E}_{\phi'} \left(c_L \left(\phi'\right) |\theta_H\right) t_H + p_L \mathbb{E}_{\phi'} \left(c_L \left(\phi'\right) |\theta_L\right) t_L\right) (2 - p_H - \theta_L) \theta_L = -\theta_L p_L dk_B$$

which implies $p_H \mathbb{E}_{\phi'} \left(c_L \left(\phi'\right) |\theta_H\right) t_H + p_L \mathbb{E}_{\phi'} \left(c_L \left(\phi'\right) |\theta_L\right) t_L < 0$, a contradiction. ■

Proposition 11 *The distributions of assets converges to a degenerate distribution*

$$\lim_{t\to\infty} \left(F^m_{B,t}(k), F^m_{L,t}(k), F^a_{B,t}(k), F^a_{L,t}(k)\right) = (1,)$$

Proof. If asset transfers are not contingent, $H(\phi) = \phi$. Suppose that the borrowers transfer the asset only if the low state is realizes. Then, the probability that a borrower has 0 assets in the afternoon of time $t+1$ is

$$\Pr\left(k^a_{B,t+1} = 0\right) = \left(\Pr\left(k^a_{B,t} = 0\right) + \sum_{s=1}^{\infty} p_L \Pr\left(k^a_{B,t} = s\right)\right) \left(\Pr\left(k^a_{B,t} = 0\right) + p_H \sum_{n=1}^{\infty} \Pr\left(k^a_{B,t} = n\right)\right)$$

where the first term represents the fraction of agents who transferred s units of the asset in the afternoon of time t and who had 0 assets at the beginning of the afternoon. The second term represents the probability meeting a lender in the asset market who did not hold any assets: the lenders who were matched with borrowers with no assets the previous afternoon or those who were matched with borrowers who did not transfer assets.

Then,

$$\begin{aligned}\Pr\left(k^a_{B,t+1} = 0\right) &= \left(p_H \Pr\left(k^a_{B,t} = 0\right) + p_L\right)\left(p_L \Pr\left(k^a_{B,t} = 0\right) + p_H\right) \\ &= p_H p_L \Pr\left(k^a_{B,t} = 0\right)^2 + \left(p_L^2 + p_H^2\right) \Pr\left(k^a_{B,t} = 0\right) + p_H p_L\end{aligned}$$

for all $t > 0$ and $\Pr\left(k^a_{B,0} = 0\right) = 0$. Let $p_H = p$. Define the operator T as follows,

$$Tx = x + p(1-p)(1+x)^2$$

Note that if $x \in [0,1]$, then $Tx \in [0,1]$. Moreover,

$$\Pr\left(k^a_{B,t+1} = 0\right) = T \Pr\left(k^a_{B,t} = 0\right)$$

and

$$\Pr\left(k^a_{B,t+1} = 0\right) = T^{t+1} \Pr\left(k^a_{B,0} = 0\right) = T^{t+1}0$$

Note that

$$\begin{aligned} |x - Tx| &= p(1-p)(1+x)^2 \\ \left|T^t 0 - T^{t+1}0\right| &= p(1-p)\left(1 + T^t 0\right)^2 < \frac{1}{4}\left(1 + T^t 0\right)^2 \end{aligned}$$

Taking limits as $t \to \infty$, one gets

$$\lim_{t\to\infty} \left|T^t 0 - T^{t+1}0\right| < \lim_{t\to\infty} \frac{1}{4}\left(1 + T^t 0\right)^2 = 0$$

and thus

$$\lim_{t\to\infty} \Pr\left(k^a_{B,t+1} = 0\right) = \Pr\left(k_{B,\lim} = 0\right)$$

Moreover,

$$\begin{aligned} \Pr\left(k_{B,\lim} = 0\right) &= p_H p_L \Pr\left(k_{B,\lim} = 0\right)^2 + \left(p_L^2 + p_H^2\right) \Pr\left(k_{B,\lim} = 0\right) + p_H p_L \\ 0 &= p_H p_L \Pr\left(k_{B,\lim} = 0\right)^2 + \left(\left(p_L^2 + p_H^2\right) - 1\right) \Pr\left(k_{B,\lim} = 0\right) + p_H p_L \end{aligned}$$

The expression above is always positive for $\Pr\left(k_{B,\lim} = 0\right) \in [0,1]$ and, in this interval, the only solution of the equation above is $\Pr\left(k_{B,\lim} = 0\right) = 1$.

The analogous follows for the case in which the borrowers transfers the assets when the high state is realized.

Then, in the limit, $\phi_t \to \bar{\phi}$. ■

Lemma 9 *In equilibrium,* $t_L \neq 0$

Proof. Suppose $t_L = 0$ in equilibrium. If the objective function is decreasing in t_L, then it is also decreasing in t_H. Therefore, $t_L = 0$ implies $t_H = 0$. The coefficients of the value functions in an affine equilibrium would then become

$$c_B(\phi) = \frac{(\mathbb{E}(\theta) - \theta_L)}{1 - \theta_L} d + \beta \mathbb{E}_{\phi'}\left(c_B(\phi')\right)$$

and

$$c_L(\phi) = (1-\gamma)\, c_B(d) + \gamma\left(d + \beta \mathbb{E}_{\phi'} c_L(\phi')\right).$$

Since in this case the asset transfers are not state contingent, $\phi = \phi^{'}$ and

$$c_B(\phi) = \frac{(\mathbb{E}(\theta) - \theta_L)}{1 - \theta_L} \frac{d}{1 - \beta}$$

$$c_L(\phi) = \frac{(1 - \gamma) \frac{(\mathbb{E}(\theta) - \theta_L)}{1 - \theta_L} \frac{d}{1 - \beta} + \gamma d}{1 - \gamma\beta}$$

where the marginal valuation of the asset depends on the state only through the dividend level d (and are linear in it).

The derivative of the objective function with respect to t_L is

$$\frac{(\mathbb{E}(\theta) - \theta_L)}{1 - \theta_L} \beta p_L \mathbb{E}(c_L(d) | \theta_L) + \frac{(\mathbb{E}(\theta) - 1)}{1 - \theta_L} p_H \beta \mathbb{E}(c_B(d) | \theta_H) - \beta p_L \mathbb{E}(c_B(d) | \theta_L).$$

If $t_L = 0$ and $t_H = 0$ this becomes

$$\frac{(\mathbb{E}(\theta) - \theta_L)}{1 - \theta_L} \frac{\beta p_L}{1 - \gamma\beta} \left((1 - \gamma\beta)(1 - \gamma) \frac{(\mathbb{E}(\theta) - 1)}{1 - \theta_L} d + \left(\frac{(\mathbb{E}(\theta) - \theta_L)}{1 - \theta_L} + \left(\gamma^2 - 1 + \gamma\beta(1 - \gamma)\right) \beta \right) \frac{d}{1 - \beta} \right)$$
$$+ \frac{(\mathbb{E}(\theta) - \theta_L)}{1 - \theta_L} \frac{(\mathbb{E}(\theta) - 1)}{1 - \theta_L} p_H \beta \left(\frac{d}{1 - \beta} \right)$$

which is > 0 and therefore contradicts $t_L = 0$. ■

Lemma 1 In an affine equilibrium, a borrower will choose

$$\begin{aligned} t_L &= k_B \\ t_H &= \left\{ \begin{array}{ll} k_B & \text{if } c_B(\phi) \le c_L(\phi) \\ 0 & \text{if } c_L(\phi) \le c_B(\phi) \end{array} \right| \end{aligned}$$

Proof. The proof of this proposition follows from the FOC of the borrower's problem in the funding market using the two lemmas above. ■

Lemma 10 $t_L = k_B = t_H$ *is not an equilibrium.*

Proof. Suppose $t_L = k_B = t_H$ in equilibrium. In this case, $\phi = \phi'$ and

$$\begin{aligned} c_B(\phi) &= \frac{(\mathbb{E}(\theta) - \theta_L)}{1 - \theta_L} (d + \beta c_L(\phi)) \\ c_L(\phi) &= \left[(1 - \gamma) c_B(d) + \gamma \left(d + \beta \frac{(1 - \gamma) c_B(\phi) + \gamma d}{1 - \gamma\beta} \right) \right] \end{aligned}$$

Therefore,

$$c_L(\phi) = \frac{(1 - \gamma) c_B(\phi) + \gamma \bar{d}}{1 - \gamma\beta},$$

and

$$c_B(\phi) = \frac{\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L} d}{\left(1-\beta\left(\gamma+\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)\right)}$$

which implies

$$c_L(\phi) = \frac{\left(\gamma+\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)}{\left(1-\beta\left(\gamma+\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)\right)} d$$

and

$$c_B(\phi) = \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left(d+\beta\frac{\left(\gamma+\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right) d}{\left(1-\beta\left(\gamma+\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)\right)}\right).$$

Thus,

$$c_L(\phi) - c_B(\phi) = \gamma\left(-\frac{(\mathbb{E}(\theta)-1)}{1-\theta_L}\left(d+\beta\frac{\left(\gamma+\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)}{\left(1-\beta\left(\gamma+\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)\right)} d\right)\right) < 0$$

what would imply that the derivative of the objective function is decreasing in t_H and, thus, $t_H = 0$, which is a contradiction. ■

Proposition 2 In the only affine equilibrium, $t_L^* = k_B$ and $t_H^* = 0$.

Proof. From the previous lemmas in this section, the only candidate left for equilibrium is $t_L^* = k_B$ and $t_H^* = 0$. Assume $t_L^* = k_B$ and $t_H^* = 0$. Then, (18) becomes

$$\begin{aligned} c_B(\phi) &= \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left(d+\beta\left(p_L\mathbb{E}_{\phi'}\left(c_B(\phi')|\theta_L\right)+p_H\mathbb{E}_{\phi'}\left(c_B(\phi')|\theta_H\right)\right)\right), \text{ and} \\ \mathbb{E}_{\phi'}\left(c_B(\phi')|\theta_H\right) &= \frac{\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left(\mathbb{E}(d|\theta_H)+\beta p_L\mathbb{E}_{\phi'}\left(c_B(\phi')|\theta_L\right)\right)}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}. \end{aligned}$$

$$c_B(\phi) = \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left(d+\frac{\beta\left(p_L\mathbb{E}_{\phi'}\left(c_B(\phi')|\theta_L\right)+p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}d_H\right)}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\right). \tag{20}$$

To have $t_L^* = k_B$ and $t_H^* = 0$ be chosen by the producer, the objective function must be increasing in t_L and decreasing in t_H, i.e.,

$$\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}\beta p_L\mathbb{E}_{\phi'}\left(c_L(\phi')|\theta_L\right) - \beta p_L\mathbb{E}_{\phi'}\left(c_B(\phi')|\theta_L\right) + \frac{\mathbb{E}(\theta)-1}{1-\theta_L}\beta p_H\mathbb{E}_{\phi'}\left(c_B(\phi')|\theta_H\right) > 0 \tag{21}$$

and

$$\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}\beta\left(\mathbb{E}_{\phi'}\left(c_L(\phi')|\theta_H\right) - \mathbb{E}_{\phi'}\left(c_B(\phi')|\theta_H\right)\right) < 0. \tag{22}$$

Let $d_i = \mathbb{E}(d|\theta_i)$ and $\bar{d} = \mathbb{E}(d)$. Using (20) evaluated at $d = d_L$ and $d = d_H$ the derivative of the objective function with respect to t_L becomes

$$\frac{\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}\beta(1-\beta)}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\left(\begin{array}{c} p_L\left(\mathbb{E}_{\phi'}\left(c_L(\phi')|\theta_L\right)-\left(d_L+\beta\frac{\bar{d}}{(1-\beta)}\right)\right)-p_H\left(d_H+\beta\frac{\bar{d}}{(1-\beta)}\right) \\ +\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}p_H\frac{(1-\beta p_L)d_H+\beta p_L d_L}{(1-\beta)} \end{array}\right)$$

Using that

$$(1-\beta p_L)\,d_H+\beta p_L d_L=(1-\beta+\beta p_H)\,d_H+\beta p_L d_L=(1-\beta)\,d_H+\beta\bar{d}$$

the expression above becomes

$$\frac{\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}\beta\,(1-\beta)}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\left(p_L\left(\mathbb{E}_{\phi'}\left(c_L\left(\phi'\right)|\theta_L\right)-\left(d_L+\beta\frac{\bar{d}}{(1-\beta)}\right)\right)+\frac{(\mathbb{E}\,(\theta)-1)}{1-\theta_L}p_H\left(d_H^i+\frac{\beta\bar{d}}{(1-\beta)}\right)\right)>0$$

since $\mathbb{E}_{\phi'}\left(c_L\left(\phi'\right)|\theta_L\right)-\left(d_L+\beta\frac{\bar{d}}{(1-\beta)}\right)\geq 0$. Therefore, the derivative of the objective function is positive under this guess which in turn implies $t_L>0$.

The sign of the derivative of the objective function with respect to t_H depends on the sign of the difference in marginal valuations between the borrower (borrower) and the lender. Using the definition of $c_L\,(\phi)$ and $c_B\,(\phi)$ together with the law of iterated expectations and that ϕ' converges, one gets that $c_L\,(\phi)=c_L\,(d)$ and $c_B\,(\phi)=c_B\,(d)$. Using this in the equation above, this difference can be rewritten as

$$\begin{aligned}
c_L\,(d)-c_B\,(d) &= \left[(1-\gamma)\,c_B\,(d)+\gamma\left(d+\beta\frac{(1-\gamma)\,c_B\left(\bar{d}\right)+\gamma\bar{d}}{1-\gamma\beta}\right)\right]-c_B\,(d)\\
&= \gamma\left(d+\beta\frac{(1-\gamma)\,c_B\left(\bar{d}\right)+\gamma\bar{d}}{1-\gamma\beta}-c_B\,(d)\right)\\
&= \gamma\left(-\frac{(\mathbb{E}\,(\theta)-1)}{1-\theta_L}d+\frac{(1-\beta)}{1-\gamma\beta}\left(\left((1-\gamma)\,\frac{(\mathbb{E}\,(\theta)-\theta_L)}{1-\theta_L}+\gamma\right)\frac{\beta\bar{d}}{1-\beta}-c_B\left(\bar{d}\right)+\frac{(\mathbb{E}\,(\theta)-\theta_L)}{1-\theta_L}\bar{d}\right)\right)\\
&\leq \gamma\left(-\frac{(\mathbb{E}\,(\theta)-1)}{1-\theta_L}d+\frac{(1-\beta)}{1-\gamma\beta}\left(\frac{(\mathbb{E}\,(\theta)-\theta_L)}{1-\theta_L}\frac{\bar{d}}{1-\beta}-c_B\left(\bar{d}\right)\right)\right)
\end{aligned}$$

But

$$\begin{aligned}
c_B\left(\bar{d}\right) &> \frac{(\mathbb{E}\,(\theta)-\theta_L)}{1-\theta_L}\frac{\bar{d}}{1-\beta}\\
p_Lc_L\,(d_L)+p_H\frac{(\mathbb{E}\,(\theta)-\theta_L)}{1-\theta_L}d_H &> \frac{\bar{d}}{1-\beta}\left(1-\beta p_H\frac{(\mathbb{E}\,(\theta)-\theta_L)}{1-\theta_L}\right)\\
p_H\frac{(\mathbb{E}\,(\theta)-\theta_L)}{1-\theta_L}\left(d_H+\frac{\beta\bar{d}}{1-\beta}\right) &> \frac{\bar{d}}{1-\beta}-p_Lc_L\,(d_L)\\
p_H\frac{(\mathbb{E}\,(\theta)-\theta_L)}{1-\theta_L}\left(d_H+\frac{\beta\bar{d}}{1-\beta}\right) &> p_H\left(d_H+\frac{\beta\bar{d}}{1-\beta}\right)\\
&= \frac{\bar{d}}{1-\beta}-p_L\left(d_L+\beta\frac{\bar{d}}{1-\beta}\right)\geq\frac{\bar{d}}{1-\beta}-p_Lc_L\,(d_L)
\end{aligned}$$

since $\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}>1$.

Then,

$$c_L\,(d_H)-c_B\,(d_H)\leq\gamma\left(-\frac{(\mathbb{E}\,(\theta)-1)}{1-\theta_L}d_H+\frac{(1-\beta)}{1-\gamma\beta}\left(\frac{(\mathbb{E}\,(\theta)-\theta_L)}{1-\theta_L}\frac{\bar{d}}{1-\beta}-c_B\left(\bar{d}\right)\right)\right)<0,$$

and this implies $t_H = 0$. Therefore, if the contract implied by $t_L^* = k_B$ and $t_H^* = 0$ is a solution to the borrower's problem in the project market. ■

7.2.3 Value function coefficients with one project

When $d_t = d$ for all t, the coefficients for the marginal valuations are given by

$$\begin{aligned} c_L(d) &= \frac{\left((1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L} + \gamma\left(1-\beta p_H \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)\right)}{\left((1-\gamma\beta)\left(1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right) - (1-\gamma)\beta p_L \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)} d \\ c_B(d) &= \left(\frac{\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}(1-\gamma\beta p_H)}{\left((1-\gamma\beta)\left(1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right) - (1-\gamma)\beta p_L\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)}\right) d \end{aligned}$$

Proposition 4 The debt capacity of the asset increases with the assets liquidity, .i.e.,

$$\frac{\partial D}{\partial \gamma} < 0$$

Problem 1 *Using the closed form for the coefficients of the value function above it is easy to see that*

$$\begin{aligned} \frac{\partial D}{\partial \gamma} &= \beta\left(\frac{\partial c_L(d)}{\partial \gamma} + p_H \frac{\partial\left(c_B(d) - c_L(d)\right)}{\partial \gamma}\right) \\ &= -\beta \frac{\left(1 - \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta\right)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_L + \frac{(\mathbb{E}(\theta)-1)}{1-\theta_L}\left(1 - p_H\left(1-\beta\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)\right)\left(1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)}{\left((1-\gamma\beta)\left(1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right) - (1-\gamma)\beta p_L\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)^2} d < 0 \end{aligned}$$

7.3 Correlated dividends and returns

7.3.1 Value function coefficients

If one allows for returns to be correlated with the dividend level, i.e., $d_i = \mathbb{E}(d|\theta_i)$, the coefficients for the marginal valuations are given by the following system

$$\begin{aligned} c_L(d) &= \left[(1-\gamma)c_B(d) + \gamma\left(d + \beta\frac{(1-\gamma)c_B(\bar{d}) + \gamma\bar{d}}{1-\gamma\beta}\right)\right] \\ c_B(d) &= \frac{(E(\theta)-\theta_L)}{1-\theta_L}\left(d + \frac{\beta\left(p_L c_L(d_L) + p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}d_H\right)}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}}\right) \end{aligned}$$

which in turn are all functions of $c_L(d_L)$ and $c_B(\bar{d})$ which are given by

$$\begin{aligned} c_L(d_L) &= \frac{\left(\left((1-\gamma)\frac{E(\theta)-\theta_L}{1-\theta_L}+\gamma\right)\left(1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)\gamma+(1-\gamma)\left(\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)^2\right)}{\left((1-\gamma\beta)\left(1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)-(1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_L\right)}\beta p_H d_H \\ &+\frac{\left((1-\gamma)\frac{E(\theta)-\theta_L}{1-\theta_L}+\gamma\right)\left(1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)(1-\gamma\beta p_H)}{\left((1-\gamma\beta)\left(1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}\right)-(1-\gamma)\frac{(E(\theta)-\theta_L)}{1-\theta_L}\beta p_L\right)}d_L \end{aligned}$$

and

$$c_B(\bar{d})=\frac{(E(\theta)-\theta_L)}{1-\theta_L}\left(\bar{d}+\frac{\beta\left(p_L c_L(d_L)+p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}d_H\right)}{1-\beta p_H\frac{(E(\theta)-\theta_L)}{1-\theta_L}}\right).$$

7.3.2 Results

Proposition 5 Assets that have dividends that are more highly correlated with the risky projects have a higher debt capacity.

$$\frac{\partial D}{\partial d_H|_{\bar{d}}}>0$$

Proof. From the definition of debt capacity it follows that

$$\begin{aligned} \frac{\partial D}{\partial d_L|_{\bar{d}}} &= \beta\frac{\partial(p_H c_B(d_H)+p_L c_L(d_L))}{\partial d_L|_{\bar{d}}} \\ &\propto -\beta p_L\left(\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}-\frac{\partial c_L(d_L)}{\partial d_L}\right) \\ &\propto -\beta p_L\frac{(1-\gamma\beta)\gamma\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left(1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)}{1-\gamma\beta-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta(1-\gamma(\beta p_H+p_L))} \end{aligned}$$

Therefore,

$$sign\left(\frac{\partial D}{\partial d_L|_{\bar{d}}}\right)=-sign\left(1-\gamma\beta-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta(1-\gamma(\beta p_H+p_L))\right).$$

Let $f(\gamma):=1-\gamma\beta-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta(1-\gamma(\beta p_H+p_L))$. Since by assumption $\left(1-\beta\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}\right)>0$, $f(\gamma)>0\ \forall\gamma\in[0,1]$ and

$$\frac{\partial D}{\partial d_L|_{\bar{d}}}<0\Longleftrightarrow\frac{\partial D}{\partial d_H|_{\bar{d}}}>0$$

■

Without loss of generality, θ_L can be set to 0. In this case, the variance of the project is given by

$$V(\theta)=\mathbb{E}(\theta)\theta_H-\mathbb{E}(\theta)^2$$

and one can get a mean preserving spread by increasing θ_H and setting

$$p_H = \frac{\mathbb{E}(\theta)}{\theta_H}$$

Proposition 6 If $d_H \geq d_{\min}$, where $d_{\min} < \bar{d}$, a mean preserving spread in the returns of the projects decreases the debt capacity of the asset, i.e.

$$\frac{\partial D}{\partial p_H|_{\mathbb{E}(\theta)}} > 0.$$

Proof. From the definition of debt capacity

$$\begin{aligned}
\frac{\partial D}{\partial p_H|_{\mathbb{E}(\theta)}} &= \beta \frac{\partial \left(p_H c_B\left(d_H\right) + p_L c_L\left(d_L\right)\right)}{\partial p_H|_{\mathbb{E}(\theta)}} \\
&= \beta \frac{\left(c_B\left(d_H\right) - c_L\left(d_L\right) + p_L \frac{\partial c_L(d_L)}{\partial p_H|_{\mathbb{E}(\theta)}}\right)}{1 - \beta p_H \frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}} \\
&= \beta \frac{\left(\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}\left(d_H - d_L\right) + \gamma\left(c_B\left(d_L\right) - \left(d_L + \beta c_L\left(\bar{d}\right)\right)\right) + p_L \frac{\partial c_L(d_L)}{\partial p_H|_{\mathbb{E}(\theta)}}\right)}{1 - \beta p_H \frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}}
\end{aligned}$$

Differentiating the expression for $c_L(d_L)$ found in the appendix with respect to p_H keeping $\mathbb{E}(\theta)$ fixed, one can see that

$$\frac{\partial c_L\left(d_L\right)}{\partial p_H|_{\mathbb{E}(\theta)}} > 0.$$

Therefore, if $d_H \geq d_{\min}$

$$\frac{\partial D}{\partial p_H|_{\mathbb{E}(\theta)}} > 0 \Longleftrightarrow \frac{\partial D}{\partial \theta_H|_{\mathbb{E}(\theta)}} < 0$$

where $d_{\min} < \bar{d}$ and $d_{\min}$ solves

$$\frac{\mathbb{E}(\theta) - \theta_L}{1 - \theta_L}\frac{-\bar{d} + d_{\min}}{p_L} + \gamma\left(c_B\left(\frac{\bar{d} - p_H d_{\min}}{p_L}\right) - \left(\frac{\bar{d} - p_H d_{\min}}{p_L} + \beta c_L\left(\bar{d}\right)\right)\right) + p_L \frac{\partial c_L\left(\frac{\bar{d}-p_H d_{\min}}{p_L}\right)}{\partial p_H|_{\mathbb{E}(\theta)}} = 0$$

■

7.4 Stochastic investment opportunities

Lemma 4 Current borrowers always value the asset more than current lenders

$$c_B(d) > c_L(d)$$

Proof. The solution to the bargaining problem in the asset market implies

$$\begin{aligned}
c_L(d) &= \left[(1-\gamma) c_B(d) + \gamma\left(d + \beta\left(\rho c_L(d) + (1-\rho) c_B(d)\right)\right)\right] \\
c_L(d) &= \frac{\left[\left((1-\gamma) + \gamma\beta(1-\rho)\right) c_B(d) + \gamma d\right]}{(1-\gamma\beta\rho)} \qquad (23)
\end{aligned}$$

Moreover, from 23 in the derivation of the equilibrium in the baseline model adjusting the continuation values to allow for random investment opportunities we have

$$\begin{aligned} c_B(d) &= \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left(d+\beta\left(p_L\left(\rho c_L(d)+(1-\rho)c_B(d)\right)\right)+p_H\beta\left(\rho c_B(d)+(1-\rho)c_L(d)\right)\right) \\ c_B(d) &= \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left(d+\beta\left(\left(p_L\rho+p_H(1-\rho)\right)c_L(d)+\left(p_L(1-\rho)+p_H\rho\right)c_B(d)\right)\right) \end{aligned}$$

Let

$$\hat{p}_L \equiv \left(p_L\rho+p_H(1-\rho)\right)$$

and

$$\hat{p}_H \equiv \left(p_L(1-\rho)+p_H\rho\right)$$

Then,

$$c_B(d)=\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left(d+\beta\left(\hat{p}_L c_L(d)+\hat{p}_H c_B(d)\right)\right)$$

Using the expression for $c_L(d)$ in (23) in the expression above we get

$$c_B(d)=\frac{\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left(1+\beta\frac{\hat{p}_L}{(1-\gamma\beta\rho)}\gamma\right)}{\left(1-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta\left(1-(1-\beta)\frac{\hat{p}_L}{(1-\gamma\beta\rho)}\gamma\right)\right)}d$$

and

$$c_L(d)=\frac{\left[\left(\beta\gamma p_H(1-2\rho)+(1-\gamma)\right)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}+\gamma\right]d}{(1-\gamma\beta\rho)\left(1-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta\left(1-(1-\beta)\frac{\hat{p}_L}{(1-\gamma\beta\rho)}\gamma\right)\right)}$$

Moreover,

$$c_L(d)-c_B(d)=\frac{-(1-\beta)\gamma c_B(d)+\gamma d}{(1-\gamma\beta\rho)}$$

Note that

$$(1-\beta)\gamma c_B(d)>\gamma d$$

since $\mathbb{E}(\theta)>1$. Then, the equilibrium coefficients are exactly the same as in the baseline model substituting p_i by $\hat{p}_i$, $i=L,H$. Therefore, $c_B(d)>c_L(d)$.

$$\begin{aligned} c_L(d) &= \frac{\left[\left(\beta\gamma p_H(1-2\rho)+(1-\gamma)\right)X+\gamma\right]d}{(1-\gamma\beta\rho)\left(1-X\beta\left(1-\frac{(1-\beta)(p_L\rho+p_H(1-\rho))\gamma}{(1-\gamma\beta\rho)}\right)\right)} \\ c_L(d) &= \frac{\left[\left(\beta\gamma p_H(1-2\rho)+(1-\gamma)\right)X+\gamma\right]d}{\left((1-\gamma\beta\rho)(1-X\beta)+(1-\beta)\gamma+(1-2\rho)(1-\beta)\gamma p_H\right)} \end{aligned}$$

■

7.5 Equilibrium with multiple project types.

Proposition 12 *The borrowers with the highest marginal valuation of the asset always use it as collateral.*

Proof. Let $c_B^{\max}$ be the marginal valuation of the borrower who values the asset the most. Then,

$$c_L(d) - c_B^{\max}(d) = (1-\gamma)\sum_{j\in J}\mu_j\left(c_B^j(d) - c_B^{\max}(d)\right) + \gamma\left(d + \beta c_L\left(\bar{d}\right) - c_B^{\max}(d)\right) \text{ for all } d.$$

By definition of $c_B^{\max}$ the first term is (weakly) negative. Moreover, we know that $c_B^{\max} \geq \frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}\left(d + \beta c_L\left(\bar{d}\right)\right) > d + \beta c_L\left(\bar{d}\right)$ since the borrower can always choose to sell the asset in the afternoon and invest the proceeds in the risky projects. Therefore, the borrower with the maximum marginal valuation of the asset chooses not to set transfers of asset to 0 when the realization of the return of the projects is high.

$$\begin{aligned}
&\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}\beta\left(\begin{array}{c} p_L c_L\left(d_L^j\right) - p_L\left(d_L^i + \beta\frac{p_L c_L\left(d_L^j\right) + p_H\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}d_H^j}{1-\beta p_H\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}}\right) \\ +\frac{\mathbb{E}(\theta)-1}{1-\theta_L}p_H\left(d_H^j + \beta\frac{p_L c_L\left(d_L^j\right) + p_H\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}d_H^j}{1-\beta p_H\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}}\right)\end{array}\right) \\
=\ &\frac{\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta}{1-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_H}\left(p_L(1-\beta)c_L\left(d_L^j\right) - \bar{d} + (1-\beta p_L)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}p_H d_H^j + \beta p_H\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L}p_L d_L^j\right) \\
=\ &\frac{\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}{1-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_H}\frac{\beta}{1-\beta}\left(p_L\left(c_L\left(d_L^j\right) - \left(d_L^j + \beta\frac{\bar{d}}{1-\beta}\right)\right) + p_H\frac{\mathbb{E}(\theta)-1}{1-\theta_L}\left(d_H^j + \beta\frac{\bar{d}}{1-\beta}\right)\right) > 0
\end{aligned}$$

■

7.5.1 Value function coefficients with multiple project types

All borrowers use the asset as collateral From sections 3 and 5, the marginal value of the asset for a producer of type j is given .

$$c_B^j(d) = \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left(d + \beta\left(p_L c_L\left(d_L^j\right) + p_H c_B^j\left(d_H^j\right)\right)\right)$$

which gives

$$c_B^j\left(d_H^j\right) = \frac{\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left(d_H^j + \beta p_L c_L\left(d_L^j\right)\right)}{1-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_H}$$

and

$$c_B^j(d) = \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left[d + \beta\frac{p_L c_L\left(d_L^j\right) + p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}d_H^j}{1-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_H}\right]$$

Moreover, the marginal value of the asset for lenders is

$$c_L(d) = (1-\gamma)\sum_{j\in J}\mu_j c_B^j(d) + \gamma\left(d + \beta c_L\left(\bar{d}\right)\right).$$

Then,

$$c_L\left(\bar{d}\right) = \frac{(1-\gamma)\sum_{j\in J}\mu_j c_B^j\left(\bar{d}\right) + \gamma\bar{d}}{1-\gamma\beta}$$

and

$$c_L(d) = \frac{(1-\gamma\beta)(1-\gamma)\sum_{j\in J}\mu_j c_B^j(d) + (1-\gamma\beta)\gamma d + \gamma\beta(1-\gamma)\sum_{j\in J}\mu_j c_B^j\left(\bar{d}\right) + \gamma\beta\gamma\bar{d}}{1-\gamma\beta}.$$

Using the definition of c_B^j gives

$$\begin{aligned}(1-\gamma\beta)\,c_L(d) &= (1-\gamma)\,\beta\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\frac{p_L c_L\left(\sum_{j\in J}\mu_j d_L^j\right) + p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J}\mu_j d_H^j}{1-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_H} \\ &\quad + \left((1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L} + \gamma\right)\left((1-\gamma\beta)\,d + \gamma\beta\bar{d}\right).\end{aligned}$$

Rearranging terms and setting $d = \sum_{j\in J}\mu_j d_L^j$,

$$\begin{aligned}&\left((1-\gamma\beta)\left(1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right) - (1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_L\right) c_L\left(\sum_{j\in J}\mu_j d_L^j\right) \\ =\ &\left(\left(1-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_H\right)\left((1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L} + \gamma\right)\gamma + (1-\gamma)\left(\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)^2\right)\beta\bar{d} \\ &+ \left(\begin{array}{c}\left(1-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_H\right)\left((1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L} + \gamma\right)(1-\gamma\beta) \\ -(1-\gamma)\,p_L\left(\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)^2\beta\end{array}\right)\sum_{j\in J}\mu_j d_L^j.\end{aligned}$$

The guess is correct, the coefficients are affine in the dividend level.

Some borrowers don't use the asset as collateral It can be shown that the distribution of assets converges over time. Therefore, analogously with the case in which there is just one project, $c_B^j(\phi) = c_B^j(d)$ for all j.

From previous sections the marginal value of a borrower type $j\in J^C$ that uses the asset as collateral is given by

$$c_B^j(d) = \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left[d + \beta\frac{p_L c_L\left(d_L^j\right) + p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}d_H^j}{1-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_H}\right].$$

The marginal value of a borrower type $j \in J^{NC}$ who chooses not to use the asset as collateral is

$$c_B^j(d) = \frac{(\mathbb{E}(\theta) - \theta_L)}{1-\theta_L}\left[d + \beta c_L\left(\bar{d}\right)\right] = c_B^{NC}(d).$$

Moreover, the marginal value of the asset for lenders is

$$c_L(d) = \frac{(1-\gamma\beta)(1-\gamma)\sum_{j\in J}\mu_j c_B^j(d) + (1-\gamma\beta)\gamma d + \gamma\beta(1-\gamma)\sum_{j\in J}\mu_j c_B^j\left(\bar{d}\right) + \gamma\beta\gamma\bar{d}}{1-\gamma\beta}.$$

Using the definition of c_B^j gives

$$\begin{aligned}(1-\gamma\beta)\, c_L(d) &= (1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta\left(\frac{\sum_{j\in J^C}\mu_j p_L c_L\left(d_L^j\right) + p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J^C}\mu_j d_H^j}{1-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_H} + \sum_{j\in J^{NC}}\mu_j c_L\left(\bar{d}\right)\right)\\ &\quad + \left(\gamma + \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)\left(\gamma\beta\bar{d} + (1-\gamma\beta)\, d\right).\end{aligned}$$

Using that

$$c_L\left(\bar{d}\right) = \frac{(1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta\left(\frac{p_L\sum_{j\in J^C}\mu c_L\left({}_j d_L^j\right)+p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J^C}\mu_j d_H^j}{1-\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_H}\right) + \left(\gamma + \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}(1-\gamma)\right)\bar{d}}{\left((1-\gamma\beta) - (1-\gamma)\sum_{j\in J^{NC}}\mu_j\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta\right)},$$

the coefficients can be recovered using the solution to

$$\begin{bmatrix} 1 + \frac{\sum_{j\in J^{NC}}\mu_j p_L(1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}} & -\sum_{j\in J^C}\mu_j\beta\left(\gamma + (1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\frac{\left(1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J^{NC}}\mu_j\right)}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\right) \\ \frac{(1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_L}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}} & 1-\beta\left(\gamma + (1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\frac{\left(1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J^{NC}}\mu_j\right)}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\right)\end{bmatrix}\times$$

$$\begin{bmatrix}\sum_{j\in J^C}\mu_j c_L\left(d_L^j\right) \\ c_L\left(\bar{d}\right)\end{bmatrix}$$

$$= \begin{bmatrix}\left((1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}+\gamma\right)\sum_{j\in J^C}\mu_j d_L^j + \frac{\sum_{j\in J^C}\mu_j(1-\gamma)\left(\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)^2\beta p_H\sum_{j\in J^C}\mu_j d_H^j}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}} \\ \left((1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}+\gamma\right)\bar{d} + \frac{(1-\gamma)\left(\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)^2\beta p_H\sum_{j\in J^C}\mu_j d_H^j}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\end{bmatrix}$$

The guess is correct. Moreover, the coefficients are affine in the dividend level.

In order to have some borrowers not using the asset as collateral it must be that

$$\begin{aligned} c_L(d) &\geq c_B^{NC}(d) \\ c_L(d) &\geq \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left[d + \beta c_L\left(\bar{d}\right)\right] \\ c_L(d) &\geq \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left[d + \beta c_L\left(\bar{d}\right)\right]\end{aligned}$$

$$
\begin{aligned}
D\mathbb{E}T \;=\; & \left(1-\beta\left(\gamma+(1-\gamma)\frac{\left(\mathbb{E}\left(\theta\right)-\theta_L\right)}{1-\theta_L}\frac{\left(1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J^{NC}}\mu_j\right)}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\right)\right)\left(1+\frac{\sum_{j\in J^{NC}}\mu_j p_L\left(1-\gamma\right)\frac{(\mathbb{E}(\theta)-\theta_L}{1-\theta_L}}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\right. \\
& +\frac{(1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_L}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\sum_{j\in J^C}\mu_j\beta\left(\gamma+(1-\gamma)\frac{\left(\mathbb{E}\left(\theta\right)-\theta_L\right)}{1-\theta_L}\frac{\left(1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J^{NC}}\mu_j\right)}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\right)
\end{aligned}
$$

$$
\begin{bmatrix} \sum_{j\in J^C}\mu_j c_L\left(d_L^j\right) \\ c_L\left(\bar{d}\right) \end{bmatrix} = \frac{1}{D\mathbb{E}T}\begin{bmatrix} A \\ B \end{bmatrix}
$$

$$
\begin{aligned}
A \;=\; & \left(1-\beta\left(\gamma+(1-\gamma)\frac{\left(\mathbb{E}\left(\theta\right)-\theta_L\right)}{1-\theta_L}\frac{\left(1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J^{NC}}\mu_j\right)}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\right)\right)\times \\
& \left(\left((1-\gamma)\frac{\left(\mathbb{E}\left(\theta\right)-\theta_L\right)}{1-\theta_L}+\gamma\right)\sum_{j\in J^C}\mu_j d_L^j+\frac{\sum_{j\in J^C}\mu_j\left(1-\gamma\right)\left(\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)^2\beta p_H\sum_{j\in J^C}\mu_j d_H^j}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\right) \\
& +\sum_{j\in J^C}\mu_j\beta\left(\gamma+(1-\gamma)\frac{\left(\mathbb{E}\left(\theta\right)-\theta_L\right)}{1-\theta_L}\frac{\left(1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\sum_{j\in J^{NC}}\mu_j\right)}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\right)\times \\
& \left(\left((1-\gamma)\frac{\left(\mathbb{E}\left(\theta\right)-\theta_L\right)}{1-\theta_L}+\gamma\right)\bar{d}+\frac{(1-\gamma)\left(\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)^2\beta p_H\sum_{j\in J^C}\mu_j d_H^j}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\right)
\end{aligned}
$$

$$
\begin{aligned}
B \;=\; & -\frac{(1-\gamma)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta p_L}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\times \\
& \left(\left((1-\gamma)\frac{\left(\mathbb{E}\left(\theta\right)-\theta_L\right)}{1-\theta_L}+\gamma\right)\sum_{j\in J^C}\mu_j d_L^j+\frac{\sum_{j\in J^C}\mu_j\left(1-\gamma\right)\left(\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)^2\beta p_H\sum_{j\in J^C}\mu_j d_H^j}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\right) \\
& +\left(1+\frac{\sum_{j\in J^{NC}}\mu_j p_L\left(1-\gamma\right)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\beta}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\right)\times \\
& \left(\left((1-\gamma)\frac{\left(\mathbb{E}\left(\theta\right)-\theta_L\right)}{1-\theta_L}+\gamma\right)\bar{d}+\frac{(1-\gamma)\left(\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\right)^2\beta p_H\sum_{j\in J^C}\mu_j d_H^j}{1-\beta p_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}}\right)
\end{aligned}
$$