Abstract

This paper extends the generalized expected utility model of the 2009 *Journal of Mathematical Economics* paper, [22], to the case of a Markov investment returns process. Using this generalization of the additively separable model, we derive the equity premium in a Lucas asset pricing equilibrium with Markov consumption growth and extend the Mehra and Prescott, [40] calibration to the non-additively separable case. Risk-free rates and levered equity premia near the historical averages can be obtained by combining the assumption of a relatively high elasticity of substitution with the assumption that the relative risk aversion measure is also high. In particluar we can, as is commonly done, assume an elasticity of substitution of one and combine this with the assumption of a high relative risk aversion measure. Surprisingly the calibration results for the generalized expected utility preferences are virtually the same as those obtained assuming Epstein-Zin, Weil (EZW) preferences with the same parameter values. The EZW generalization of the additively separable preferences does not assume expected utility maximization but is dynamically consistent while our generalization does retain the assumption of expected utility maximization but is not dynamically consistent. Throughout the paper we relate our approach to the standard development of the additively separable case and to the EZW approach. We also present a separate development of the case in which the elasticity of substitution is one, which in the additively separable case, is "the log case."

Consumption Based Asset Pricing in a General Model of Expected Utility Maximizing Investors

Richard Kihlstrom
Finance Department
The Wharton School

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1 Introduction

The foundations of modern asset pricing theory were laid during the late 1960's and early 1970's in a series of papers by Levhari and Srinavasan [31], Merton [37] and [38], Samuelson [45] and Hakansson [18]. These papers analyzed the consumption-savings and portfolio choice implications of a model, that I will refer to as the additively separable model, in which consumers are assumed to maximize the expected value of the discounted sum of utilities of per period consumption. Merton [39] used this framework to obtain a dynamic asset pricing model and Breeden [5] demonstrated how the equilibrium conditions obtained by Merton could be interpreted to yield what is now called the consumption CAPM. In his pioneering paper, [32], Lucas also used the additively separable model to construct a dynamic asset pricing model. The Lucas and Merton models provide the foundation for much of the subsequent work on dynamic asset pricing; see for example Grossman and Shiller, [15] and [16], Grossman, Melino and Shiller, [17], Hansen-Singleton, [20] and Mehra and Prescott [40]. In addition, many of the important developments of modern macroeconomics are obtained using the framework provided by the additively separable model.

The fact that the additively separable model is based on expected utility maximization and implies that the consumer's choices exhibit dynamic consistency makes it an attractive and widely accepted foundation for theory in macro and asset pricing. Because of the specific assumptions usually made about the utility function, it is also a quite tractable model and its implications have provided a useful starting point for empirical work in these broad fields.

In spite of its successes, there has been a growing recognition over the last thirty years that the restrictions imposed by the additively separable model may limit its ability to provide the basis for constructing an explanation for some important empirical phenomena. In particular, in [19] Hall emphasized that the model does not permit the measures of risk aversion and of the intertemporal elasticity of substitution to be chosen independently. This is not only restrictive, it also makes it difficult to interpret the implications of the model. For example, as Grossman and Shiller, [15], Grossman, Melino and Shiller, [17], and Mehra and Prescott [40] point out, the model in its simple form can only explain the equity premium if the risk aversion measure is quite high and the intertemporal elasticity of substitution is, consequently, low. If the risk aversion measure could be chosen independently of the intertemporal elasticity of substitution it might be possible to determine the independent effects of each of these parameters on the equity premium and on other aspects of investment and savings behavior.

Recent contributions by Epstein and Zin and Weil (EZW) have addressed the issue raised by Hall. In a series of papers, Epstein [11], Epstein and Zin [12], [13], [14], Weil [51] and [52] these authors developed a generalization of the additively separable model that allows the intertemporal elasticity of substitution and the risk aversion measure to be chosen independently. A general version of the preferences used by EZW had originally been proposed by Kreps and Porteus, [26], [27] and [28]. Selden, [46], [47], had also proposed the use of the same preferences but his discussion was restricted to the case of two periods. Since its introduction, the EZW generalization of the additively separable model has provided the basis for much important work in dynamic asset pricing; see for example Campbell, [7] and [8] and Bansal-Yaron [3].

The EZW generalization retains one of the attractive features of the additively separately model while sacrificing another. Specifically, the EZW generalization does not assume that consumers maximize expected utility but it does imply that their choices are dynamically consistent. Another attractive feature of the EZW generalization is its tractability.

In a 2009 *Journal of Mathematical Economics* paper, [22], the current author proposed an alternative generalization to the one proposed by EZW and, in the current paper, we continue to develop that approach. As noted in [22], when this alternative generalization of the additively separable model is used, the assumption that consumers maximize expected utility is retained but, in general, a dynamic inconsistency arises.

The generalized expected utility approach we propose is obtained by simply applying standard consumer theory to independently choose the intertemporal elasticity of substitution and the risk aversion measure. Consumer theory tells us that, when "ordinal preferences" for consumption streams can be represented by an additively separable utility function, that representation is just one of infinitely many "cardinal utility functions" that represent those preferences. Every monotonically increasing transformation of the additively separable utility function represents the same ordinal preferences and the intertemporal elasticity of substitution is unaffected by the monotonic transformation. That's true because the intertemporal elasticity of substitution is a property of the ordinal preferences and not of the particular cardinal utility function used to represent those ordinal preferences. When there is no uncertainty it doesn't matter which cardinal utility function we use to represent the ordinal preferences, and if it's simpler to use the additively separable representation that would seem to be the obvious choice. When there is uncertainty, and the consumer maximizes expected utility, different utility functions representing the same ordinal preferences will imply different attitudes toward risk. In the case of uncertainty, Kihlstrom-Mirman [23] and [24] describe how it is possible to compare the risk aversion of different cardinal representations of the same ordinal preferences. If we choose a non-additively separable representation of the ordinal preferences, we can compare the risk aversion of this alternative representation to the risk aversion of the additively separable representation. In fact, we can follow Kihlstrom and Mirman and compare the risk aversion of any pair of utility functions that represent the same ordinal preferences.

As is commonly done, we assume that the consumer's preferences are represented by a CES utility function of the consumption stream. We do not, however, restrict attention to the additively separable representation although that representation is one of the alternatives permitted by our approach. We also assume that, at each point in time, the consumer's current consumption-saving and portfolio choices maximize the expected value of his utility of current and future consumption. As noted, by taking this approach we are able retain one of the attractive features of the additively separable model, namely the assumption that the consumer maximizes expected utility. Because the consumer ignores past consumption, we refer to him as having "forward looking" von Neumann-Morgenstern preferences, but we sometimes simply refer to these preferences as the generalized expected utility model. Unfortunately, when the consumer maximizes the expected value of a non-additively separable utility function that is independent of past consumption, a dynamic inconsistency typically arises. Specifically, the consumer's current choices are not the ones he would have committed himself to make if such a commitment had been possible at an earlier point in time. Thus, while we are able to retain the assumption of expected utility maximization, the specific assumptions we make about the evolution of the consumer's preferences over time imply that it is impossible for us to retain the dynamic consistency of the additively separable model. We would emphasize, however, that the impact of this dynamic inconsistency is limited in an important way with our approach when it is applied to generalize the widely used additively separable case in which the log of consumption is the utility function of consumption. We refer to this special case of CES preferences as the "Cobb-Douglas" case. It is the case in which the elasticity of substitution is one. In the asset pricing and macro literature it is commonly referred to as the "log case."

The mention of dynamic inconsistency typically brings to mind the literature on hyperbolic discounting. As a consequence it is important to emphasize that the dynamic inconsistency we are, in general, forced to deal with is only superficially similar to but, in fact, quite distinct from that which arises in models of hyperbolic discounting. Hyperbolic discounting was, of course first considered by Strotz [49] and recently reconsidered in the work of Laibson [30], Harris and Laibson [21] and Luttmer and Marrioti [34] and [35]. In that literature, future utility is discounted differently as time passes and this gives rise to the dynamic inconsistency. In the case we consider of investors who have "forward looking" von Neumann-Morgenstern preferences, it is the fact that past consumption is ignored that gives rise to the dynamic inconsistently. As described in Proposition 1 of [22], this has the effect of changing the risk-aversion of the investor as time passes.

In analyzing the generalized expected utility maximizing model we propose, we do follow the "consistent planning" approach of Strotz [49] and assume that, when making his current choice, the consumer will "take account of future disobedience." This consistent planning approach was also used in some of the later literature cited above on hyperbolic discounting as well as by Pollak [43] and by Phelps and Pollak [42]. In the dynamic consumption-savings, portfolio model we consider, when the consumer makes his current choices he recognizes that his future choices will not be the ones he would currently like to commit himself to make in the future. Thus, the consumer chooses a consumption plan for the future that is, as Strotz asserted, "the best plan among those he will actually follow." Our approach can also be interpreted from the perspective of Peleg and Yaari [41] and is similar to that taken in the literature on durable goods monopoly and the Coase conjecture; see, for example, Coase [9], Stokey [48] and Bulow [6]. In following Peleg and Yaari and the durable goods monopoly literature, we view the current consumer as a leader in a leader-follower game in which the followers are the same consumer at future time periods. His current choices are a best response to the choices he knows he will want to make in the future. The result is a Nash equilibrium of a Stackleberg game in which each "player" is the consumer at a particular consumption period. We refer to our approach as one of "consistent planning" by a "forward looking" expected utility maximizing consumer.

As in our earlier paper, we investigate the asset pricing implications of this generalized expected utility model by deriving the equity premium in a Lucas asset pricing equilibrium. In the earlier paper, we limited attention to the case in which consumption growth was *i.i.d.* In the present paper we derive the asset pricing equilibrium for the case in which consumption growth is markov. This extension makes it possible for us to calibrate the model and examine the extent to which it might be able to improve on the additively separable model as calibrated by Mehra and Prescott, [40]. We chose to calibrate the model in this way to focus on the comparison of the results obtained from our generalization of the canonical additively separable model with those obtained from the alternative EZW generalization. Each of these generalizations allows the intertemporal elasticity of substitution and the risk aversion measure to be chosen independently. As mentioned above, the generalizations differ in that our approach retains the assumption of expected utility maximization but is not dynamically consistent while the EZW approach is dynamically consistent but does not retain expected utility maximization. In light of these differences it seems quite surprising that the calibration results obtained using the generalized expected utility preferences are virtually the same as those obtained assuming EZW preferences with the same parameter values.

We have remarked on the tractability of the EZW generalization and this feature makes it a useful general framework for analysis and starting point for empirical work. While the generalized expected utlity maximizing model we propose is less tractable than the EZW generalization, [22] did obtain, for the case of *i.i.d.* consumption growth, explicit solutions for the risk-free rate and the equity premium in the Lucas asset pricing equilibrium. In the present paper, we are also able to derive explicit conditions, for the case in which consumption growth is markov, that enable us to easily compute the steady state equity premium and risk-free rate in the Lucas asset pricing equilibrium.

As we have also mentioned [22] restricted attention to the case of *i.i.d.* consumption growth. This turns out to be a case in which the EZW generalization adds little in explanatory power to the additively separable model. Weil [51] for example noted that "the solution to the equity premium puzzle documented by Mehra and Prescott [40] cannot be found by simply separating risk aversion (from) intertemporal substitution. If the dividend growth process is *i.i.d.*, the risk-premium, when appropriately defined, is independent of the intertemporal elasticity of substitution, and thus is the same whether or not the time-additive, expected utility restriction is imposed. When the dividend growth process is non-*i.i.d.*, relaxing the parametric restriction adds, for plausible parameter values, a risk free rate puzzle to Mehra and Prescott's equity premium puzzle." Kocherlakota [25] demonstrates an even stronger result. In analyzing the Lucas asset pricing model he assumes EZW preferences and that "the growth rate of the aggregate endowment is *i.i.d."* Using this model he demonstrates that "an econometrician with data on asset prices and aggregate consumption cannot separately identify" the elasticity of substitution and the relative risk aversion. He concludes that the EZW preferences have "no more explanatory power than" the additively separable preferences. The inability of the EZW framework to add explanatory power beyond that of the additively separable model in the *i.i.d.* case is rooted in the observation made in a number of papers that, when the returns to the risky assets are *i.i.d.*, the EZW optimal portfolio depends only on the risk aversion measure and is independent of the elasticity of substitution. See for example, Swensson [50] and Bhamra and Uppal [4].

In our earlier paper we found that, in contrast to the EZW generalization, the risk premium obtained from our generalized expected utility maximization approach is affected by the elasticity of substitution as well as the risk aversion measure and this is true for the case of *i.i.d.* consumption growth. It thus does not suffer from the criticism of Kocherlakota and of Weil himself. In fact, one of the main results of our 2009 paper is that, in the case of *i.i.d.* consumption growth, our generalized expected utility model yields a higher risk premium than the standard additively separable model when and only when the elasticity of substitution in our model is exceeded by that of the additively separable model. This result had been pointed out by Kocherlakota.

1.1 Outline of the Paper and Preview of the Results

Throughout the paper we attempt to relate the formal development of our approach, which takes full advantage of the homotheticity of the preferences, to the standard development of the additively separable case and also to the EZW approach. Our development of the EZW approach, which will also make use of the homotheticity of the preferences, will differ from the usual development so that we can highlight the relationship of that approach to ours. Our theoretical development of the Lucas asset pricing equilibrium is, again somewhat different from the usual discussion because of the simplifications introduced by the homotheticity of the preferences. We have also chosen to present a separate development of what I refer to as the Cobb-Douglas case in which the elasticity of substitution is one. In the additively separable case, the Cobb-Douglas case is commonly refered to as the "the log case."

We begin Section 2 with a brief description of an infinite horizon consumption savings problem faced by a consumer for whom the return to saving each period is a stationary markov process. Section 2.1 describes how this consumption savings problem would be solved by a consumer who, in each period, maximizes the expected value of a CES utility function of current and future consumption. The utility function is not assumed to be additively separable. We refer to the consumer as having "forward looking" von Neumann Morgenstern preferences. Because of the non-additive-separability of the utility function and the fact that past consumption does not enter as an argument of the consumer's current utility function, this generalized expected utility model typically implies a dynamic inconsistency. We assume that the consumer takes this into account by choosing his current consumption to be a best response to the consumption choices he will make in the future. As we note, the task of obtaining a solution is simplified by the fact that the consumer faces the same problem each period. We refer to this solution to the consumption-savings problem as the "consistent planning" solution for a consumer with "forward looking" von Neumann Morgenstern preferences.

In Section 2.1.1 we interpret the consumer as a representative agent in an economy in which the return to savings is the return earned by real investments in a production economy. In this setting, we permit trading in a riskless asset that is in zero net supply, and derive an expression for the equilibrium riskless return. As usual, the riskless rate equals the "risk-neutral expectation" of the return earned by the real investment. Also, if we use the riskless rate to compute the present value of the "risk-neutral expectation" of the real investment's return we get the present value of these returns computed using Arrow-Debreu contingent claims prices which equal the ratio of the risk neutral density to the riskless rate. This expression is a generalization of the standard expression routinely obtained in the additively separable case.

The case in which the intertemporal elasticity of substitution is one, which we refer to as the Cobb-Douglas case, is an important one that must be dealt with separately and that is done in Section 2.1.2. In the extensive literature that has developed assuming additively separable preferences it is quite common to restrict attention to this case. This is often done because, in this case, the consumer's optimal consumption-savings decision is unaffected by the return to savings and hence by the risk of that return. This, of course, means that the optimal consumption-savings decision is the same whether the return to savings is risky or not. The same result holds when the consumer has "forward looking" von Neumann Morgenstern preferences and the elasticity of substitution is one. This implies that, in this important case, there is no dynamic inconsistency. Specifically, a consumer with "forward looking" von Neumann Morgenstern preferences and an elasticity of substitution of one makes dynamically consistent choices because his optimal consumption-savings decision is unaffected by the return to savings and hence by the risk of that return and by the consumer's risk aversion measure. In fact, in this case, the optimal consumption-savings decision is the same as in the additively separable case. Even in the Cobb-Douglas case, the consumer's portfolio choices are, of course, affected by his risk aversion measure as is the expression derived in Sections 2.1.2 for the equilibrium riskless return. In Section 4, which presents the simulation results, we discuss the extent to which the Cobb-Douglas case can provide an explanation for the historical equity premium and the average riskless rate.

Section 2.1.3 briefly describes two important special cases. The first is the case in which the returns to savings are *i.i.d.* Since this is the case studied in Kihlstrom, [22], the discussion is quite brief. The other special case is the one commonly analyzed in which the utility function is additively separable so that the elasticity of substitution is the inverse of the risk aversion measure. Since this latter case has been exhaustively studied, we focus on demonstrating that, in this special case, the optimal consumption choice for the problem described in Section 2.1 is a solution to the standard dynamic programming problem typically solved in the additively separable case. The "Bellman Value Function" of this dynamic programming problem is related to the concepts introduced in Section 2.1. For each of these special cases, the expression for the equilibrium riskless return simplifies dramatically and the simplified versions are described. We observe that when, in the *i.i.d.* case, utility is non-additively separable, the expression for the riskless rate differs from the one obtained in the additively separable case.

Section 2.2 describes how this consumption savings problem would be solved by a consumer with EZW preferences. Following Kreps and Porteus, Epstein-Zin and Weil demonstrated that the optimal consumption strategy could be obtained as the solution to a generalized dynamic programming problem. The Epstein-Zin value function is defined to be linear in wealth and, in general, the marginal value of wealth is a function of the return to savings. Following the discussion in Epstein-Zin, [12], we derive functional equations that must be satisfied by the Epstein-Zin marginal value of wealth function and by the optimal consumption strategy. In the process, a simple relationship between the marginal value of wealth and the optimal consumption strategy is obtained. In Section 2.2.1 we obtain an expression for the riskless rate that generalizes the expression obtained in the additively separable case. This expression once again equals a redefined "risk-neutral expectation" of the return earned by the real investment. The present value of the redefined "risk-neutral expectation" of the real investment's return again equals the present value of these returns computed using Arrow-Debreu contingent claims prices, which equal the ratio of the redefined risk neutral density to the riskless rate. The equation that yields the expression for the riskless rate is also shown to be equivalent to Epstein and Zin's "generalized Euler equation" which is equation (6.6) in Epstein and Zin, [12] and equation (16) in Epstein and Zin, [14].

With EZW preferences, the Cobb Douglas case must again be treated separately and that is done in Section 2.2.2. Again, in this case, the optimal consumption savings decision is unaffected by the return to savings. Hence, as is true with the generalized expected utility maximizing preferences treated in Section 2.1.2, the consumption savings decision is the same as that made by the consumers with additively separable preferences. The expression for the equilibrium riskless return obtained for EZW consumers does, however, differ from the expression obtained in Section 2.1.2 for the case of generalized expected utility maximizing preferences.

Section 2.2.3 describes the special case in which the return to savings are *i.i.d.* In that case, the expression for the equilibrium riskless return obtained for EZW consumers is, as Weil noted, the same as that for the case of additively separable preferences.

Section 3 begins with Section 3.1 that presents an introductory general discussion of the Lucas asset pricing equilibrium for the case in which the growth rates of the dividends paid by the Lucas tree are a stationary markov process. In Section 3.1 properties of the equilibrium are derived under general assumptions that are satisfied when the consumer has "forward looking" von Neumann Morgenstern preferences and "plans consistently" and when the consumer has EZW preferences. In particular, we derive a simple expression for the equilibrium price of the Lucas tree and for the price dividend ratio. The price dividend ratio is, in fact, easily seen to equal the ratio of the fraction of wealth saved to the fraction of wealth consumed. In the markov case, the fractions of wealth saved and consumed fluctuate with the consumption growth rate and, as a consequence, the price dividend ratio also fluctuates with this growth rate. These expressions can also be used to derive an expression that relates the equilibrium return on an investment in the Lucas tree to the consumption growth rate. Section 3.1.1 briefly describes the simplifications that emerge when the consumption growth rate process is *i.i.d.* In that case, which was analyzed in Kihlstrom, [22], the fractions of wealth saved and consumed are constant and as a result, the price dividend ratio is also constant. Also in this case, the price of the Lucas tree is simply the present value of a growing perpetuity that initially pays a dividend equal to the current dividend times the expected growth rate and then continues to grows at the expected growth rate. The discount rate used in computing the value of this perpetuity is the expected return on the Lucas tree.

Section 3.2 interprets the results of Sections 2.1 and 3.1 to describe the Lucas Asset Pricing Equilibrium in which the representative consumer has "forward looking" von Neumann Morgenstern preferences and "plans consistently." We substitute the expression for the return on the Lucas tree obtained in Section 3.1 in the first order conditions satisfied by the optimal fraction of wealth consumed by a consumer who maximizes expected utility and plans consistently; i.e., who solves the problem described in Section 2.1. The result is explicit expressions for the price dividend ratio, the fraction of wealth consumed, the fraction of wealth saved and for the return on the Lucas tree. In Section 3.2.1 we use the expression for the return on the Lucas tree obtained in Section 3.1 in the expression for the riskless rate obtained in Section 2.1.1. to obtain an expression for the riskless rate in the Lucas model. Using this expression and the expression for the return on the Lucas tree we calculate the equity premium. Section 3.2.2 briefly considers the Cobb Douglas case. Again the riskless rate and the equity premium are obtained for the Lucas Asset Pricing Equilibrium. Section 3.2.3 describes the special cases in which the consumption growth rates are *i.i.d.* and the case of additively separable preferences.

Section 3.3 interprets the results of Sections 2.2 and 3.1 to describe the Lucas Asset Pricing Equilibrium in which the representative consumer has EZW preferences. In Section 3.3.1 we use the expression for the return on the Lucas tree obtained in Section 3.1 in the expression for the riskless rate obtained in Section 2.2.1. to obtain an expression for the riskless rate in the Lucas model. Using this expression and the expression for the return on the Lucas tree we calculate the equity premium. Section 3.3.2 describes the special case in which the consumption growth rates are *i.i.d.*

Calibrations using our "generalized expected utility" preferences and using EZW preferences are described in Section 4. The calibrations are carried out using the binomial approach of Mehra and Prescott [40]. That is we assume that the only consumption growth rates are the two rates assumed by Mehra and Prescott. We also use the same transition probabilities that are used in their paper. Using the expressions derived in Section 3.2.1, we calculate the riskless rate and equity premium obtained when we assume that the representative consumer has forward looking von Neuman Morgenstern preferences and plans consistently. It is still true that it is necessary to assume a high measure of relative risk aversion to obtain a simulated equity premium that comes close to matching the 6.98% paid by Standard and Poor's 500 Composite Stock Index over the ninety years period considered by Mehra and Prescott. In the additively separable case when the risk aversion measure is high enough to come close to explaining the equity premium, the elaticity of substitution is of necessity low and the riskless rate is, as a result, much higher than the historical average of just under 1%. When the elasticity of subsitution can be chosen to be high at the same time that the risk aversion measure is high, it is possible to come close to matching both the riskless rate and the equity premium. Similar conclusions emerge if we use the expressions derived in Section 3.3.1, to calculate the riskless rate and equity premium obtained and assume that the representative consumer has EZW preferences. With these preferences, and only two possible growth rates, the riskless rate and the equity premium can, in fact, be computed quite simply using a calculation that is described in Section 4.1. The theoretical equity premia that we simulate are those paid by unlevered investments in the Lucas tree. In Section 4.2 we indicate how those premia can be adjusted for leverage. Using this leverage adjustment, it is possible to obtain realistic equity premia and riskless rates by assuming that the representative consumer has forward looking von Neuman Morgenstern preferences and plans consistently as well as by assuming EZW preferences. One case for which this is true is the one we have called the "Cobb-Douglas" case and which is commonly referred to as the "log case." We noted earlier that, in that case, our approach does not imply a dynamic inconsistency.

2 The Infinite Horizon Consumption Savings Problem

We suppose that the consumer looks forward to an infinite lifetime. He begins period t with initial wealth, W_t. His consumption in period t is C_t. The fraction of W_t consumed in period t is c_t. In each period, savings, which equal $(1-c_t)\,W_t$, can be invested only in a risky asset. The realized return on the risky asset is r_t. The random variable of which r_t is the realization is $\tilde{r}_t$. Period $t+1$ wealth is

$$W_{t+1} = W_t\,[1-c_t]\,\tilde{r}_t$$

and consumption in period t is

$$C_t = c_t W_t.$$

We assume that $\{\tilde{r}_t\}$ is a markov process and that the transition probabilities $\Pr\left(\tilde{r}_t > 0\,|r_{t-1}\right)$ are constant over time.

2.1 "Consistent Planning" with "Forward Looking" von Neumann Morgenstern Preferences

In discussing forward looking preferences in this infinite horizon setting, we are going to assume that, in each period, the preferences are CES and exhibit constant relative risk aversion CRRA in the Kihlstrom-Mirman [24] sense. Furthermore, the elasticity of substitution and the level of relative risk aversion are the same each period. Thus, in period t, the forward looking von Neumann-Morgenstern consumer maximizes

$$EU\left(C_t, \left\{\tilde{C}_{t+\tau}\right\}_{\tau=1}^{\infty}\right) = \frac{1}{1-\alpha}E\left[\left(C_t^{\rho} + \sum_{\tau=1}^{\infty}\beta^{\tau}\tilde{C}_{t+\tau}^{\rho}\right)^{\frac{1-\alpha}{\rho}}|r_{t-1}\right], \quad (1)$$

where $\beta < 1$, $\rho < 1$, $\rho \neq 0$, $\alpha \geq 0$ and $\alpha \neq 1$. The cases in which $\rho = 0$ and/or $\alpha = 1$ require special treatment and we briefly deal with them separately in Section 2.1.2 below. Following Kihlstrom-Mirman [24] we say that relative risk aversion of

$$U\left(C_t, \{C_{t+\tau}\}_{\tau=1}^{\infty}\right) = \frac{1}{1-\alpha}\left(C_t^{\rho} + \sum_{\tau=1}^{\infty}\beta^{\tau}C_{t+\tau}^{\rho}\right)^{\frac{1-\alpha}{\rho}} \quad (2)$$

is α. The "risk neutral" case is that in which $\alpha = 0$. Note when $\alpha = 0$, the utility function is homogeneous of degree one and it is also the least concave representation, in the sense defined by Debreu [10], of the implied ordinal preferences. For all α, the elasticity of substitution of (2) is

$$\sigma = \frac{1}{1-\rho}.$$

The additively separable case arises when $1-\alpha = \rho$ so that $\sigma = \frac{1}{\alpha}$. In this case, dynamic programming arguments imply that the consumer's choices exhibit dynamic consistency. Specifically, the plan for consumption at time $t+\tau$ that is optimal at time t is also optimal when time $t+\tau$ arrives.

When

$$1-\rho \neq \alpha,$$

the consumer's preferences are non-additively separable and his choices do not exhibit dynamic consistency. Specifically, the plan for consumption at time $t+\tau$ that maximizes (1) at time t is no longer optimal when time $t+\tau$ arrives. This is true because at time $t+\tau$ consumption at time t does not enter the utility function of the forward looking consumer. In the present discussion we will not elaborate further on this dynamic inconsistency. In Kihlstrom, [22], it is discussed further, although the discussion is limited to the case of three periods.

If, at time t, the consumer could commit himself to carry out the plan for consumption at time $t+\tau$ that maximizes (1), the dynamic inconsistency would not cause a problem. We assume, however, that such commitments are impossible. We thus follow Strotz [49] and assume that the consumer's plans for future consumption are "consistent;" viz, they are ones he will actually carry out. This means that the consumer is sophisticated and anticipates his future behavior. In the game theoretic view taken by Peleg and Yaari [41] of the situation considered by Strotz, the consumer at time t is a first mover in a game in which the other players are himself at later periods. In our setting, we implement Strotz's "consistent planning" approach by adopting the game theoretic view and assuming that the consumer's consumption choice at time t is a best response to the consumption choices he expects to make in the future. Thus, when the consumer chooses c_t, in period t he anticipates his future choices $\{c_{t+\tau}\}_{\tau=1}^{\infty}$. Because the consumer faces the same problem in every period, the equilibrium is one in which his choice startegy in every period is the same. Thus, for all t, $c_t, = \hat{c}(r_{t-1})$ where $\hat{c}(r_{t-1})$ is a best response to the fact that for all, $\tau > 0$, $c_{t+\tau}, = \hat{c}(r_{t+\tau-1})$. Specifically,

$$\begin{aligned} & \hat{c}(r_{t-1}) \\ = \; & \arg\max_{c} \left(E\left[\left(c^{\rho} + \beta [1-c]^{\rho} \tilde{r}_t^{\rho} \tilde{v} \right)^{\frac{1-\alpha}{\rho}} | r_{t-1} \right] \right)^{\frac{1}{1-\alpha}} \end{aligned} \tag{3}$$

where,

$$\begin{aligned} \tilde{v} &= v\left(\{\tilde{r}_{t+\tau}, \hat{c}(\tilde{r}_{t+\tau-1})\}_{\tau=1}^{\infty}\right) \\ &= [\hat{c}(\tilde{r}_t)]^{\rho} + \beta [1-\hat{c}(\tilde{r}_t)]^{\rho} \tilde{r}_{t+1}^{\rho} v\left(\{\tilde{r}_{t+1+\tau}, \hat{c}(\tilde{r}_{t+\tau})\}_{\tau=1}^{\infty}\right). \end{aligned} \tag{4}$$

It is easy to verify that

$$\begin{aligned} & v\left(\{r_{t+\tau},\hat{c}\left(r_{t+\tau-1}\right)\}_{\tau=1}^{\infty}\right) \\ = & \left[\left[\hat{c}\left(r_t\right)\right]^{\rho}+\sum_{\tau=1}^{\infty}\beta^{\tau}\left[\hat{c}\left(r_{t+\tau}\right)\right]^{\rho}\prod_{s=1}^{\tau}\left[1-\hat{c}\left(r_{t+s-1}\right)\right]^{\rho} r_{t+s}^{\rho}\right]. \end{aligned} \tag{5}$$

2.1.1 The Riskless Rate

We now interpret the consumer as a representative agent in an economy in which $\tilde{r}_t$ is the rate of return earned by real investments in a production economy. We also assume that a riskless asset is traded but in zero net supply. In this setting, when $\tilde{r}_{t-1}=r_{t-1}$, the equilibrium period t riskless return, $r_f\left(r_{t-1}\right)$, can be obtained from the consumer's first order condition.

If we let

$$\tilde{x}_t\left(r_{t-1}\right)=\left(\tilde{r}_t-r_f\left(r_{t-1}\right)\right)$$

and define $\hat{\gamma}$ by letting

$$\begin{aligned} & \hat{\gamma} \\ = & \arg\max_{\gamma}\left(E\left[\left(\hat{c}\left(r_{t-1}\right)^{\rho}+\beta\left[1-\hat{c}\left(r_{t-1}\right)\right]^{\rho}\left(r_f\left(r_{t-1}\right)+\gamma\tilde{x}_t\left(r_{t-1}\right)\right)^{\rho}\tilde{v}\right)^{\frac{1-\alpha}{\rho}}\left|r_{t-1}\right.\right]\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

then in equilibrium $\hat{\gamma}=1$.The first order condition satisfied at $\hat{\gamma}=1$ is

$$\begin{aligned} & E\left[\left(\hat{c}\left(r_{t-1}\right)^{\rho}+\beta\left[1-\hat{c}\left(r_{t-1}\right)\right]^{\rho}\tilde{r}_t^{\rho}\tilde{v}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{v}\tilde{r}_t^{\rho-1}\left[\tilde{r}_t-r_f\left(r_{t-1}\right)\right]\left|r_{t-1}\right.\right] \\ = & 0. \end{aligned} \tag{6}$$

So the riskless rate is

$$r_f\left(r_{t-1}\right)=E^*\left[\tilde{r}_t\left|r_{t-1}\right.\right] \tag{7}$$

where we define

$$\begin{aligned} & E^*\left[\tilde{r}_t\left|r_{t-1}\right.\right] \\ = & \frac{E\left[\left(\hat{c}\left(r_{t-1}\right)^{\rho}+\beta\left[1-\hat{c}\left(r_{t-1}\right)\right]^{\rho}\tilde{r}_t^{\rho}\tilde{v}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{v}\tilde{r}_t^{\rho}\left|r_{t-1}\right.\right]}{E\left[\left(\hat{c}\left(r_{t-1}\right)^{\rho}+\beta\left[1-\hat{c}\left(r_{t-1}\right)\right]^{\rho}\tilde{r}_t^{\rho}\tilde{v}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{v}\tilde{r}_t^{\rho-1}\left|r_{t-1}\right.\right]}. \end{aligned} \tag{8}$$

Remark 1 *If we let*

$$f\left(r_t\left|r_{t-1}\right.\right)$$

denote the conditional probability density of $\tilde{r}_t$ *when* $\tilde{r}_{t-1}=r_{t-1}$, *then we can define the "risk-neutral density" to be*

$$\begin{aligned} & f^*\left(r_t\left|r_{t-1}\right.\right) \\ = & \frac{E\left[\left(\hat{c}\left(r_{t-1}\right)^{\rho}+\beta\left[1-\hat{c}\left(r_{t-1}\right)\right]^{\rho}r_t^{\rho}\tilde{v}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{v}r_t^{\rho-1}\left|r_t\right.\right]f\left(r_t\left|r_{t-1}\right.\right)}{E\left[\left(\hat{c}\left(r_{t-1}\right)^{\rho}+\beta\left[1-\hat{c}\left(r_{t-1}\right)\right]^{\rho}\tilde{r}_t^{\rho}\tilde{v}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{v}\tilde{r}_t^{\rho-1}\left|r_{t-1}\right.\right]} \end{aligned} \tag{9}$$

where, it will be recalled that $\tilde{v}$ is defined in (4) and (5). Note that $f^*(r_t\,|r_{t-1})$ is, indeed, a density and

$$\frac{f^*(r_t\,|r_{t-1})}{r_f(r_{t-1})}$$

is the period $t-1$ price of an Arrow-Debreu claim to a dollar paid at time t contingent on $\tilde{r}_t = r_t$. These Arrow-Debreu contingent claims prices are normalized to sum to

$$\frac{1}{r_f(r_{t-1})},$$

the period $t-1$ value of a period t dollar, when the dollar is discounted at the riskless rate. The expectation $E^*[\tilde{r}_t\,|r_{t-1}]$ defined in (8) is the conditional "risk-neutral expectation" of $\tilde{r}_t$ and it is simply the expected value of $\tilde{r}_t$ computed using the distribution $f^*(r_t\,|r_{t-1})$. Note that,

$$\frac{E^*[\tilde{r}_t\,|r_{t-1}]}{r_f(r_{t-1})},$$

the present value of the "risk-neutral expectation" of $\tilde{r}_t$ is simply the period $t-1$ value of the possible returns $\tilde{r}_t$ earned by a dollar invested in the available real investment when the Arrow Debreu prices are used to compute the value of the investment. Since a dollar invested in the available real investment could also be invested in the riskless asset the period $t-1$ value, computed using the riskless rate $r_f(r_{t-1})$ as the discount rate, of the returns earned on the real investments must be one. That is we must have

$$1 = \frac{E^*[\tilde{r}_t\,|r_{t-1}]}{r_f(r_{t-1})},$$

which is true because of (7).

2.1.2 The Cobb Douglas Case

This is the case in which $\rho = 0$ and the intertemporal elasticity of substitution is one. In this case, when, $\alpha \neq 1$ and $\beta < 1$,

$$\begin{aligned}
&U_\alpha\left(C_t, \{C_{t+\tau}\}_{\tau=1}^{\infty}\right) \\
=\ &\left(\frac{1}{1-\alpha}\right) C_t^{(1-\alpha)(1-\beta)} \prod_{\tau=1}^{\infty} .C_{t+\tau}^{(1-\alpha)(1-\beta)\beta^\tau}.
\end{aligned}$$

The "risk neutral" case is again that in which $\alpha = 0$. Once again, when $\alpha = 0$, the utility function is homogeneous of degree one and it is also the least concave representation of the implied ordinal preferences. In this case, there is no dynamic inconsistency and, if we let

$$\begin{aligned}
&\Lambda(r_{t-1}) \\
=\ &E\left(\left[\hat{c}(\tilde{r}_t)^{(1-\beta)}\,\tilde{r}_t\right]^{(1-\alpha)\beta} \prod_{\tau=1}^{\infty}\left[\hat{c}(\tilde{r}_{t+\tau})\prod_{s=1}^{\tau}\left[1-\hat{c}(\tilde{r}_{t+s-1})\right]\tilde{r}_{t+s}\right]^{(1-\alpha)(1-\beta)\beta^{\tau+1}} |r_{t-1}\right),
\end{aligned}$$

then

$$
\begin{aligned}
& \hat{c}\left(r_{t-1}\right) \\
= \quad & \arg\max_{c} \left(c^{(1-\alpha)(1-\beta)}\left[1-c\right]^{(1-\alpha)\beta}\Lambda\left(r_{t-1}\right)\right)^{\frac{1}{1-\alpha}} \\
= \quad & \hat{c} = \arg\max_{c} c^{(1-\beta)}\left[1-c\right]^{\beta} \\
= \quad & (1-\beta)
\end{aligned}
$$

so that $\hat{c}\left(r_{t-1}\right)$ is independent of α and of r_{t-1}.This is also the solution in the additively separable case in which $1-\alpha=\rho=0$ and

$$
U\left(C_t,\left\{C_{t+\tau}\right\}_{\tau=1}^{\infty}\right)=\log\left(C_t\right)+\left[\sum_{\tau=1}^{\infty}\beta^{\tau}\log\left(C_{t+\tau}\right)\right].
$$

Note that when

$$
\hat{c}\left(r_{t-1}\right)=(1-\beta),
$$

$\Lambda\left(r_{t-1}\right)$ reduces to

$$
\begin{aligned}
& \Lambda\left(r_{t-1}\right) \\
= \quad & \left[(1-\beta)\beta^{\frac{\beta}{1-\beta}}\right]^{(1-\alpha)\beta} E\left(\tilde{r}_t^{(1-\alpha)\beta}\prod_{\tau=1}^{\infty}\left[\prod_{s=1}^{\tau}\tilde{r}_{t+s}\right]^{(1-\alpha)(1-\beta)\beta^{\tau+1}} |r_{t-1}\right) \\
= \quad & \left[(1-\beta)\beta^{\frac{\beta}{1-\beta}}\right]^{(1-\alpha)\beta} E\left(\tilde{r}_t^{(1-\alpha)\beta}\prod_{\tau=1}^{\infty}\tilde{r}_{t+\tau}^{(1-\alpha)\beta^{\tau+1}} |r_{t-1}\right)
\end{aligned}
$$

To obtain the riskless rate in the Cobb-Douglas case, we again let

$$
\tilde{x}_t\left(r_{t-1}\right)=\left(\tilde{r}_t-r_f\left(r_{t-1}\right)\right).
$$

and use

$$
\hat{c}\left(r_{t-1}\right)=(1-\beta).
$$

Now

$$
\begin{aligned}
& \hat{\gamma} \\
= \quad & \arg\max_{\gamma} E\left(\left(r_f+\gamma\tilde{x}_t\left(r_{t-1}\right)\right)^{(1-\alpha)\beta}\prod_{\tau=1}^{\infty}\tilde{r}_{t+\tau}^{(1-\alpha)\beta^{\tau+1}} |r_{t-1}\right)
\end{aligned}
$$

and the first order condition satisfied at $\hat{\gamma}=1$ is

$$
\begin{aligned}
& E\left(\tilde{r}_t^{(1-\alpha)\beta-1}\left[\tilde{r}_t-r_f\left(r_{t-1}\right)\right]\prod_{\tau=1}^{\infty}\tilde{r}_{t+\tau}^{(1-\alpha)\beta^{\tau+1}} |r_{t-1} \,|r_{t-1}\right) \\
= \quad & 0.
\end{aligned}
$$

So the riskless rate is

$$r_f(r_{t-1}) = E^*[\tilde{r}_t | r_{t-1}] \tag{10}$$

where now

$$E^*[\tilde{r}_t | r_{t-1}] = \frac{E\left(\prod_{\tau=1}^{\infty} \tilde{r}_{t+\tau-1}^{(1-\alpha)\beta^\tau} | r_{t-1}\right)}{E\left(\tilde{r}_t^{-1} \prod_{\tau=1}^{\infty} \tilde{r}_{t+\tau-1}^{(1-\alpha)\beta^\tau} | r_{t-1}\right)}.$$

Although the form of the expression for $E^*[\tilde{r}_t | r_{t-1}]$ changes when $\alpha = 0$, the discussion in Remark 1 continues to apply to this case if we also redefine $f^*(r_t | r_{t-1})$ to equal

$$\begin{aligned} & f^*(r_t | r_{t-1}) \\ = \; & \frac{E\left(\prod_{\tau=2}^{\infty} \tilde{r}_{t+\tau-1}^{(1-\alpha)\beta^\tau} | r_t\right) r_t^{(1-\alpha)\beta-1} f(r_t | r_{t-1})}{E\left(\tilde{r}_t^{-1} \prod_{\tau=1}^{\infty} \tilde{r}_{t+\tau-1}^{(1-\alpha)\beta^\tau} | r_{t-1}\right)}. \end{aligned}$$

2.1.3 Two Special Cases

In this section, we briefly discuss two important special cases. One is the case of *i.i.d.* returns considered in Kihlstrom. [22]. The other case is the additively separable case.

The Case of iid Returns In each period t, $c_t, = \hat{c}$ where $\hat{c}$ is a best response to the fact that for all, $\tau > 0$, $c_{t+\tau}, = \hat{c}$. Thus,

$$\begin{aligned} & \hat{c} \\ = \; & \arg\max_c \left(E\left[\left(c^\rho + \beta[1-c]^\rho \tilde{r}_t^\rho \tilde{v}\right)^{\frac{1-\alpha}{\rho}}\right]\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

where now

$$\begin{aligned} & \tilde{v} = v\left(\{\tilde{r}_{t+\tau}, \hat{c}\}_{\tau=1}^{\infty}\right) \\ = \; & \hat{c}^\rho \tilde{\varsigma} \end{aligned}$$

and

$$\tilde{\varsigma} = \left[1 + \sum_{\tau=1}^{\infty} \left(\beta[1-\hat{c}]^\rho\right)^\tau \prod_{s=1}^{\tau} \tilde{r}_{t+s}^\rho\right].$$

In this case, when $\rho \neq 0$, the riskless rate reduces to

$$r_f = \frac{E\left[\left(1 + \beta[1-\hat{c}]^\rho \tilde{r}_t^\rho \tilde{\varsigma}\right)^{\frac{1-\alpha}{\rho}-1} \tilde{\varsigma} \tilde{r}_t^\rho\right]}{E\left[\left(1 + \beta[1-\hat{c}]^\rho \tilde{r}_t^\rho \tilde{\varsigma}\right)^{\frac{1-\alpha}{\rho}-1} \tilde{\varsigma} \tilde{r}_t^{\rho-1}\right]}.$$

In the Cobb-Douglas case,

$$r_f = E^* \tilde{r}_t$$

where now

$$E^{*}\tilde{r}_t = \frac{E\tilde{r}^{(1-\alpha)\beta}}{E\tilde{r}^{(1-\alpha)\beta-1}}$$

The Additively Separable Case For this case, we demonstrate that problem (3) solved by the consumer of our model reduces to the one solved by a consumer who solves the standard "dynamic programming" problem. In the process we demonstrate how the "Bellman value function" is related to the function $v(\cdot)$ described in (5).

Recall that in this case, $1-\alpha=\rho$ so that (3) reduces to

$$\begin{aligned} & \hat{c}(r_{t-1}) \\ = \; & \arg\max_{c} \left(c^{1-\alpha} + \beta [1-c]^{1-\alpha} E\left[\tilde{r}_t^{1-\alpha}\tilde{v}\,|r_{t-1}\,\right]\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

where is as described in (4) and (5). In this case, we can define

$$\begin{aligned} & \omega(r_t) \\ = \; & \left(E\left[v\left(\{\tilde{r}_{t+\tau}, \hat{c}(\tilde{r}_{t+\tau-1})\}_{\tau=1}^{\infty}\right)|r_t\,\right]\right) \\ = \; & [\hat{c}(r_t)]^{1-\alpha} \\ & +\beta[1-\hat{c}(r_t)]^{1-\alpha} E\left[\tilde{r}_{t+1}^{1-\alpha} E\left[v\left(\{\tilde{r}_{t+1+\tau}, \hat{c}(\tilde{r}_{t+\tau})\}_{\tau=1}^{\infty}\right)|\tilde{r}_{t+1}\,\right]|r_t\,\right] \\ = \; & [\hat{c}(r_t)]^{1-\alpha} + \beta[1-\hat{c}(r_t)]^{1-\alpha} E\left[\tilde{r}_{t+1}^{1-\alpha}\omega(\tilde{r}_{t+1})\,|r_t\,\right] \end{aligned}$$

and let the "Bellman Value Function" equal

$$V(W_{t+1}, r_t) = \omega(r_t)\, W_{t+1}^{1-\alpha}. \tag{11}$$

With this notation

$$\begin{aligned} & \hat{c}(r_{t-1}) \\ = \; & \arg\max_{c} \left(c^{1-\alpha} + \beta [1-c]^{1-\alpha} E\left[\tilde{r}_t^{1-\alpha}\omega(\tilde{r}_t)\,|r_{t-1}\,\right]\right)^{\frac{1}{1-\alpha}} \end{aligned} \tag{12}$$

where the coefficient of the Bellman equation solves the functional equation

$$\begin{aligned} & \omega(r_t)^{\frac{1}{1-\alpha}} \\ = \; & \max_{c} \left(c^{1-\alpha} + \beta [1-c]^{1-\alpha} E\left[\tilde{r}_{t+1}^{1-\alpha}\omega(\tilde{r}_{t+1})\,|r_t\,\right]\right)^{\frac{1}{1-\alpha}}. \end{aligned} \tag{13}$$

Thus, it is indeed true that, in the additively separable case, the consumption choice that solves the problem (3) is a solution to the standard dynamic programming problem typically solved in that case where the "Bellman Value Function" is as defined in (11) and (13).

In this case, the riskless rate reduces to

$$r_f(r_{t-1}) = E^{*}\left[\tilde{r}_t\,|r_{t-1}\,\right]$$

where

$$E^* \left[\tilde{r}_t \left| r_{t-1} \right.\right] = \frac{E\left[\omega\left(\tilde{r}_t\right)\tilde{r}_t^{1-\alpha}\left| r_{t-1} \right.\right]}{E\left[\omega\left(\tilde{r}_t\right)\tilde{r}_t^{-\alpha}\left| r_{t-1} \right.\right]}.$$

When the returns are iid,

$$\begin{aligned} & \omega \\ = \; & \hat{c}^{1-\alpha} + \beta\omega\left[1-\hat{c}\right]^{1-\alpha} E\left[\tilde{r}^{1-\alpha}\right] \\ = \; & \hat{c}^{-\alpha} \end{aligned}$$

where

$$\hat{c} = 1 - \left(\beta E\left[\tilde{r}_{t+1}^{1-\alpha}\right]\right)^{\frac{1}{\alpha}}.$$

In this case, the risk free rate reduces to

$$r_f = E^* \tilde{r}_t$$

where

$$E^* \tilde{r}_t = \frac{E\left[\tilde{r}^{1-\alpha}\right]}{E\left[\tilde{r}^{-\alpha}\right]}.$$

2.2 Epstein-Zin Weil Preferences

The formal descriptions of these preferences by Epstein-Zin and by Weil differ slightly. The current exposition follows that in Section 5 of Epstein and Zin, [12]. We begin by obtaining the Epstein-Zin value function as the solution to a functional equation that is the analog of (13). The value function is

$$V\left(W, r_{t-1}\right) = \xi\left(r_{t-1}\right) W$$

where $\xi(\cdot)$ is a solution to the functional equation

$$\begin{aligned} & \xi\left(r_{t-1}\right) \\ = \; & \max_c \left[\left(c^\rho + \beta\left[1-c\right]^\rho \left[E\left(\left[\xi\left(\tilde{r}_t\right)\tilde{r}_t\right]^{1-\alpha}\left| r_{t-1} \right.\right)\right]^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{\rho}}\right]. \end{aligned} \tag{14}$$

We let

$$\begin{aligned} & \hat{c}\left(r_{t-1}\right) \\ = \; & \arg\max_c \left[\left(c^\rho + \beta\left[1-c\right]^\rho \left[E\left(\left[\xi\left(\tilde{r}_t\right)\tilde{r}_t\right]^{1-\alpha}\left| r_{t-1} \right.\right)\right]^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{\rho}}\right]. \end{aligned}$$

Clearly, when $1-\alpha = \rho$,(14) reduces to (13) and

$$\xi\left(r_{t-1}\right) = \omega\left(r_{t-1}\right)^{\frac{1}{1-\alpha}}.$$

The first order condition satisfied at $\hat{c}\left(r_{t-1}\right)$ is

$$\left[\hat{c}\left(r_{t-1}\right)\right]^{\rho-1} - \beta\left[1-\hat{c}\left(r_{t-1}\right)\right]^{\rho-1}\left[E\left(\left[\xi\left(\tilde{r}_t\right)\tilde{r}_t\right]^{1-\alpha}\left| r_{t-1} \right.\right)\right]^{\frac{\rho}{1-\alpha}} = 0$$

and the solution is

$$\hat{c}\left(r_{t-1}\right)=\frac{1}{1+\left(\beta\left[E\left(\left[\xi\left(\tilde{r}_{t}\right)\tilde{r}_{t}\right]^{1-\alpha}\left|r_{t-1}\right.\right)\right]^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{1-\rho}}}. \tag{15}$$

This implies that

$$\frac{1-\hat{c}\left(r_{t-1}\right)}{\hat{c}\left(r_{t-1}\right)}=\frac{1}{\beta\left(\left[E\left(\left[\xi\left(\tilde{r}_{t}\right)\tilde{r}_{t}\right]^{1-\alpha}\left|r_{t-1}\right.\right)\right]^{\frac{1}{1-\alpha}}\right)^{\frac{\rho}{\rho-1}}}. \tag{16}$$

Substituting the expression (15) for $\hat{c}\left(r_{t-1}\right)$ in

$$\begin{aligned}&\xi\left(r_{t-1}\right)\\ =\;&\left(\left[\hat{c}\left(r_{t-1}\right)\right]^{\rho}+\beta\left[1-\hat{c}\left(r_{t-1}\right)\right]^{\rho}\left[E\left(\left[\left(\xi\left(\tilde{r}_{t}\right)\tilde{r}_{t}\right)\right]^{1-\alpha}\left|r_{t-1}\right.\right)\right]^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{\rho}}\end{aligned}$$

allows us to replace the functional equation (14) by the functional equation

$$\xi\left(r_{t-1}\right)=\left(1+\left[\left(\beta\left[E\left(\left[\xi\left(\tilde{r}_{t}\right)\tilde{r}_{t}\right]^{1-\alpha}\left|r_{t-1}\right.\right)\right]^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{1-\rho}}\right]\right)^{\frac{1-\rho}{\rho}}. \tag{17}$$

Clearly, (15) and (17) combine to imply

$$\xi\left(r_{t-1}\right)=\left[\hat{c}\left(r_{t-1}\right)\right]^{\frac{\rho-1}{\rho}}. \tag{18}$$

Using (18), the expression (15) for $\hat{c}\left(r_{t-1}\right)$ becomes

$$\hat{c}\left(r_{t-1}\right)=\frac{1}{1+\left(\beta\left[E\left(\left[\left[\hat{c}\left(r_{t}\right)\right]^{\frac{\rho-1}{\rho}}\tilde{r}_{t}\right]^{1-\alpha}\left|r_{t-1}\right.\right)\right]^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{1-\rho}}}, \tag{19}$$

a functional equation that can be used to obtain $\hat{c}\left(\cdot\right)$. Alternatively the functional equation (17) can be used to obtain $\xi\left(\cdot\right)$.

2.2.1 The Riskless Rate

To obtain the riskless rate with Epstein-Zin preferences, we once again let

$$\tilde{x}_{t}\left(r_{t-1}\right)=\left(\tilde{r}_{t}-r_{f}\left(r_{t-1}\right)\right)$$

and derive the first order condition satisfied at $\hat{\gamma}=1$,where now

$$\hat{\gamma}=\arg\max_{\gamma}\left[E\left(\left[\xi\left(\tilde{r}_{t}\right)\left(r_{f}+\gamma\tilde{x}_{t}\left(r_{t-1}\right)\right)\right]^{1-\alpha}\left|r_{t-1}\right.\right)\right]^{\frac{1}{1-\alpha}}.$$

The first order condition is

$$
\begin{aligned}
& E\left(\xi\left(\tilde{r}_t\right)^{1-\alpha} \tilde{r}_t^{-\alpha}\left(\tilde{r}_t-r_f\left(r_{t-1}\right)\right)\left|r_{t-1}\right.\right) \\
= \quad & 0.
\end{aligned}
$$

So the riskless rate is

$$
r_f\left(r_{t-1}\right)=E^*\left(\tilde{r}_t\left|r_{t-1}\right.\right) \tag{20}
$$

where now

$$
E^*\left(\tilde{r}_t\left|r_{t-1}\right.\right)=\frac{E\left(\left[\xi\left(\tilde{r}_t\right)\tilde{r}_t\right]^{1-\alpha}\left|r_{t-1}\right.\right)}{E\left(\xi\left(\tilde{r}_t\right)^{1-\alpha} \tilde{r}_t^{-\alpha}\left|r_{t-1}\right.\right)}. \tag{21}
$$

It will be noted that, because the coefficient $\xi\left(r_t\right)$ is a solution to the functional equation (17), it depends on the parameter ρ that determines the intertemporal elasticity of substitution. Thus, the equity premium is affected by ρ as well as the risk aversion measure α. The dependence on ρ, becomes more explicit if we use (18) and (21) to rewrite (20) as

$$
r_f\left(r_{t-1}\right)=\frac{E\left(\left[\hat{c}\left(\tilde{r}_t\right)\right]^{(1-\alpha)\frac{\rho-1}{\rho}} \tilde{r}_t^{1-\alpha}\left|r_{t-1}\right.\right)}{E\left(\left[\hat{c}\left(\tilde{r}_t\right)\right]^{(1-\alpha)\frac{\rho-1}{\rho}} \tilde{r}_t^{-\alpha}\left|r_{t-1}\right.\right)}.
$$

Remark 2 *If we let*

$$
f\left(r_t\left|r_{t-1}\right.\right)
$$

denote the conditional probability density of $\tilde{r}_t$ *when* $\tilde{r}_{t-1}=r_{t-1}$*, then we can redefine the "risk-neutral density" to be*

$$
\begin{aligned}
& f^*\left(r_t\left|r_{t-1}\right.\right) \\
= \quad & \frac{\xi\left(r_t\right)^{1-\alpha} r_t^{-\alpha}\left|r_{t-1}\right. f\left(r_t\left|r_{t-1}\right.\right)}{E\left(\xi\left(\tilde{r}_t\right)^{1-\alpha} \tilde{r}_t^{-\alpha}\left|r_{t-1}\right.\right)}
\end{aligned}
$$

We again note that $f^*\left(r_t\left|r_{t-1}\right.\right)$ *is, indeed, a density and*

$$
\frac{f^*\left(r_t\left|r_{t-1}\right.\right)}{r_f\left(r_{t-1}\right)}
$$

is again the period $t-1$ *price of an Arrow-Debreu claim to a dollar paid at time* t *contingent on* $\tilde{r}_t=r_t$*. As before, these Arrow-Debreu contingent claims prices are normalized to sum to*

$$
\frac{1}{r_f\left(r_{t-1}\right)},
$$

the period $t-1$ *value of a period* t *dollar, when the dollar is discounted at the riskless rate. The expectation* $E^*\left[\tilde{r}_t\left|r_{t-1}\right.\right]$ *as redefined in (21) is again the conditional "risk-neutral expectation" of* $\tilde{r}_t$ *and it is simply the expected value* *of $\tilde{r}_t$ computed using the redefined distribution $f^*(r_t|r_{t-1})$. The interpretation of*

$$\frac{E^*[\tilde{r}_t|r_{t-1}]}{r_f(r_{t-1})},$$

is as in Remark 1. Thus, we must again have

$$1=\frac{E^*[\tilde{r}_t|r_{t-1}]}{r_f(r_{t-1})},$$

and now this is true because of (20).

Remark 3 *This expression (20) for the riskless rate can be shown to be equivalent to equation (6.6) in Epstein and Zin, [12]. To see this note that we can rewrite (20) as*

$$1=\left[\frac{E\left(\xi(\tilde{r}_t)^{1-\alpha}\tilde{r}_t^{-\alpha}r_f(r_{t-1})|r_{t-1}\right)}{E\left([\xi(\tilde{r}_t)\tilde{r}_t]^{1-\alpha}|r_{t-1}\right)}\right]^{\frac{1}{(1-\alpha)}}.$$

Also using (16) we get

$$\beta^{\frac{1}{\rho}}\left(\frac{1-\hat{c}(r_{t-1})}{\hat{c}(r_{t-1})}\right)^{\frac{\rho-1}{\rho}}=\frac{1}{\left[E\left([\xi(\tilde{r}_t)\tilde{r}_t]^{1-\alpha}|r_{t-1}\right)\right]^{\frac{1}{(1-\alpha)}}}.$$

Combining these expressions we get

$$1=\beta^{\frac{1}{\rho}}\left(\frac{1-\hat{c}(r_{t-1})}{\hat{c}(r_{t-1})}\right)^{\frac{\rho-1}{\rho}}\left[E\left(\xi(\tilde{r}_t)^{1-\alpha}\tilde{r}_t^{-\alpha}r_f(r_{t-1})|r_{t-1}\right)\right]^{\frac{1}{(1-\alpha)}}$$

We can once again use (18) and rewrite this expression to get

$$1=\beta\left[E\left(\left(\frac{[1-\hat{c}(r_{t-1})]W_{t-1}\hat{c}(\tilde{r}_t)\tilde{r}_t}{\hat{c}(r_{t-1})W_{t-1}}\right)^{(1-\alpha)\frac{\rho-1}{\rho}}\tilde{r}_t^{\frac{1-\alpha-\rho}{\rho}}r_f(r_{t-1})|r_{t-1}\right)\right]^{\frac{\rho}{1-\alpha}}$$

which is the same as equation (6.6) in Epstein and Zin, [12]. This expression is also that same as equation (1) in Bansal Yaron [3]

2.2.2 The Cobb Douglas Case

In this case, the Epstein-Zin value function is

$$V(W,r_{t-1})=\xi(r_{t-1})W$$

where now $\xi(\cdot)$ is a solution to the functional equation

$$\begin{aligned}&\xi(r_{t-1})\\ =\;&\max_c c^{(1-\beta)}[1-c]^{\beta}\left[E\left([(\xi(\tilde{r}_t)\tilde{r}_t)]^{1-\alpha}|r_{t-1}\right)\right]^{\frac{\beta}{1-\alpha}}\end{aligned}$$

The solution for $\hat{c}\left(r_{t-1}\right)$ is again the same as in the additively separable case. Specifically,

$$\begin{aligned}
& \hat{c}\left(r_{t-1}\right) \\
= \; & \arg\max_{c} c^{(1-\beta)} \left[1-c\right]^{\beta} \left[E\left(\left[\left(\xi\left(\tilde{r}_t\right)\tilde{r}_t\right)\right]^{1-\alpha} | r_{t-1}\right)\right]^{\frac{\beta}{1-\alpha}} \\
= \; & \hat{c} = \arg\max_{c} c^{(1-\beta)} \left[1-c\right]^{\beta} \\
= \; & (1-\beta)
\end{aligned}$$

The expression for the riskless rate is again given by (20) where now $\xi\left(\cdot\right)$ in equation (21) is the solution to

$$\begin{aligned}
& \xi\left(r_{t-1}\right) \\
= \; & (1-\beta)^{(1-\beta)} \beta^{\beta} \left[E\left(\left[\left(\xi\left(\tilde{r}_t\right)\tilde{r}_t\right)\right]^{1-\alpha} | r_{t-1}\right)\right]^{\frac{\beta}{1-\alpha}}.
\end{aligned} \tag{22}$$

2.2.3 The Case of iid Consumption Growth

In this case, the solution to the functional equation simplifies to

$$V\left(W\right) = \xi W$$

where

$$\begin{aligned}
& \xi \\
= \; & \max_{c} \left[\left(c^{\rho} + \beta\xi^{\rho}\left[1-c\right]^{\rho}\left[E\left(\left[\tilde{r}_t\right]^{1-\alpha}\right)\right]^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{\rho}}\right]
\end{aligned}$$

and

$$\begin{aligned}
& \hat{c} \\
= \; & \arg\max_{c} \left[\left(c^{\rho} + \beta\xi^{\rho}\left[1-c\right]^{\rho}\left[E\left(\left[\tilde{r}_t\right]^{1-\alpha}\right)\right]^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{\rho}}\right].
\end{aligned}$$

The solutions simplify to

$$\hat{c} = 1 - \left(\beta\left[E\left(\tilde{r}_t^{1-\alpha}\right)\right]^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{1-\rho}}.$$

and

$$\xi = \left[1 - \left(\beta\left[E\left(\tilde{r}_t^{1-\alpha}\right)\right]^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{1-\rho}}\right]^{\frac{\rho-1}{\rho}}.$$

Because the coefficient ξ is a constant, the expression for the riskless rate reduces to

$$r_f = E^{*}\tilde{r}_t$$

$$E^{*}\tilde{r}_{t}=\frac{E\left[\tilde{r}^{1-\alpha}\right]}{E\left[\tilde{r}^{-\alpha}\right]} \tag{23}$$

which is not only independent of ρ, it is the same as that obtained in the additively separable case. This result is the basis for the observation in Kocherlakota [25] that the EZW preferences have "no more explanatory power than" the additively separable preferences. It also underlies the similar point made by Weil [51].

3 The Infinite Period Lucas Asset Pricing Equilibrium

3.1 Some Simple Introductory Results

We denote the Lucas tree dividend at time t by s_t. It is the realization of a random variable $\tilde{s}_t$. Between periods t and $t+1$ the dividend grows at rate g_t which is the realization of the random variable $\tilde{g}_t$.We thus, have

$$\tilde{s}_{t+1}=\tilde{g}_t s_t.$$

We assume that the dividend growth rate process is markov and that the transition probabilities $\Pr\left(\tilde{g}_t>0\,|g_{t-1}\right)$ are constant over time. We also assume that

$$\Pr\left(\tilde{g}_t>0\,|g_{t-1}\right)=1.$$

Let $P\left(s_t\right)$ be the period t price of the Lucas tree. Then the return on savings invested at time t is

$$r\left(s_{t+1},s_t\right)=\frac{s_{t+1}+P\left(s_{t+1}\right)}{P\left(s_t\right)}. \tag{24}$$

In equilibrium, wealth in period t is

$$W_t\left(s_t\right)=s_t+P\left(s_t\right),$$

consumption in period t is s_t and savings in period t is $P\left(s_t\right)$.

When, for all t, $c_t=\hat{c}\left(g_{t-1}\right)$, then

$$\hat{c}\left(g_{t-1}\right)=\frac{s_t}{s_t+P\left(s_t\right)}$$

and

$$1-\hat{c}\left(g_{t-1}\right)=\frac{P\left(s_t\right)}{s_t+P\left(s_t\right)}.$$

This implies that the price dividend ratio is

$$\frac{P\left(s_t\right)}{s_t}=\frac{1-\hat{c}\left(g_{t-1}\right)}{\hat{c}\left(g_{t-1}\right)}$$

and that

$$P\left(s_t\right)=\frac{1-\hat{c}\left(g_{t-1}\right)}{\hat{c}\left(g_{t-1}\right)}s_t. \tag{25}$$

Substituting (25) in (24) we get that

$$\begin{aligned} r_t &= r\left(s_{t+1},s_t\right) \\ &= \frac{s_{t+1}+P\left(s_{t+1}\right)}{P\left(s_t\right)} \\ &= \frac{s_{t+1}+\frac{1-\hat{c}(g_t)}{\hat{c}(g_t)}s_{t+1}}{\frac{1-\hat{c}(g_{t-1})}{\hat{c}(g_{t-1})}s_t} \\ &= \frac{\hat{c}\left(g_{t-1}\right)}{\hat{c}\left(g_t\right)\left[1-\hat{c}\left(g_{t-1}\right)\right]}\frac{s_{t+1}}{s_t} \\ &= \frac{\hat{c}\left(g_{t-1}\right)}{\left[1-\hat{c}\left(g_{t-1}\right)\right]}\frac{g_t}{\hat{c}\left(g_t\right)} \end{aligned} \tag{26}$$

Note that (26) implies that the expected return on the Lucas tree each period is

$$E\left[\tilde{r}_t \left| g_{t-1}\right.\right]=\left[\frac{\hat{c}\left(g_{t-1}\right)}{\left[1-\hat{c}\left(g_{t-1}\right)\right]}\right]E\left[\frac{\tilde{g}_t}{\hat{c}\left(\tilde{g}_t\right)}\left|g_{t-1}\right.\right]. \tag{27}$$

It should be pointed out that the equilibrium results just described are all a consequence of the assumptions made about the growth rate process and the fact that, for all t, $c_t=\hat{c}\left(g_{t-1}\right)$. They do not depend on any other assumption about preferences. This will imply that these results apply when we assume consistent planning by a forward looking expected utility maximizing consumer as well as in the case of Epstein-Zin and Weil preferences.

3.1.1 The Case of i.i.d. Returns

As noted, this is the case dealt with in Kihlstrom [22]. In each period t, $c_t, =\hat{c}$ where $\hat{c}$ is a best response to the fact that for all, $\tau>0$, $c_{t+\tau},=\hat{c}$.Thus, for all t,

$$\hat{c}=\frac{s_t}{s_t+P\left(s_t\right)}$$

and

$$1-\hat{c}=\frac{P\left(s_t\right)}{s_t+P\left(s_t\right)}.$$

This implies that the price dividend ratio is constant and is given by

$$\frac{P\left(s_t\right)}{s_t}=\frac{1-\hat{c}}{\hat{c}}.$$

As a result (25) becomes

$$P\left(s_t\right)=\frac{1-\hat{c}}{\hat{c}}s_t \tag{28}$$

and (26) reduces to

$$r_t = r\left(s_{t+1}, s_t\right) = \frac{g_t}{[1-\hat{c}]} \tag{29}$$

Thus, in this case, the expected return on the Lucas tree each period is simply

$$E\tilde{r}_t = \frac{E\tilde{g}_t}{[1-\hat{c}]}. \tag{30}$$

Also (28) can be rewritten as

$$P\left(s_t\right) = \frac{1}{\frac{1}{1-\hat{c}} - 1} s_t. \tag{31}$$

Using (29), (31) reduces to

$$P\left(s_t\right) = \frac{E\tilde{g}}{E\tilde{r} - E\tilde{g}} s_t. \tag{32}$$

Thus, in this case, the equilibrium value of the Lucas tree at time t is the present value of a dividend stream that starts at s_t and grows at the expected rate $E\tilde{g}$. The discount rate used to compute the present value is the expected return on the Lucas tree.

3.2 The "Consistent Planning" Approach with "Forward Looking" von Neumann Morgenstern Preferences

Substituting

$$r_t = \frac{\hat{c}\left(g_{t-1}\right)}{\hat{c}\left(g_t\right)\left[1-\hat{c}\left(g_{t-1}\right)\right]} g_t \tag{33}$$

in the expression (5), we get

$$v\left(\left\{r_{t+\tau}, \hat{c}\left(r_{t+\tau-1}\right)\right\}_{\tau=1}^{\infty}\right) = \left[\hat{c}\left(g_t\right)\right]^{\rho} \theta\left(\left\{g_{t+\tau}\right\}_{\tau=1}^{\infty}\right) \tag{34}$$

where

$$\theta\left(\left\{g_{t+\tau}\right\}_{\tau=1}^{\infty}\right) = \left[1 + \sum_{\tau=1}^{\infty} \beta^{\tau} \prod_{s=1}^{\tau} g_{t+s}^{\rho}\right]. \tag{35}$$

Using (34), the maximand in (3) becomes

$$\begin{aligned} & E\left[\left(c^{\rho} + \beta\left[1-c\right]^{\rho} \tilde{r}_t^{\rho} \tilde{v}\right)^{\frac{1-\alpha}{\rho}} | r_{t-1}\right] && (36) \\ = \; & E\left[\left(c^{\rho} + \beta\left[1-c\right]^{\rho}\left[\frac{\hat{c}\left(g_{t-1}\right)}{1-\hat{c}\left(g_{t-1}\right)}\right]^{\rho} \tilde{g}_t^{\rho} \tilde{\theta}\right)^{\frac{1-\alpha}{\rho}} | g_{t-1}\right] \end{aligned}$$

where

$$\tilde{\theta} = \theta\left(\left\{\tilde{g}_{t+\tau}\right\}_{\tau=1}^{\infty}\right). \tag{37}$$

Remark 4 *To interpret (35), note that*

$$\left[\theta\left(\{g_{t+\tau}\}_{\tau=1}^{\infty}\right)\right]^{\frac{1}{\rho}} = \left[1+\sum_{\tau=1}^{\infty}\beta^{\tau}\prod_{s=1}^{\tau}g_{t+s}^{\rho}\right]^{\frac{1}{\rho}}$$

is the utility of the consumption stream

$$\{C_{t+1+\tau}\}_{\tau=0}^{\infty} = \left\{1, \left\{\prod_{s=1}^{\tau}g_{t+s}\right\}_{\tau=1}^{\infty}\right\}.$$

This is the consumption stream obtained when consumption at time $t+1$ is one and the subsequent consumption growth path is $\{g_{t+\tau}\}_{\tau=1}^{\infty}$. The utility is computed using the homogeneous of the degree one representation.

Also note that

$$\theta\left(\{g_{t-1+\tau}\}_{\tau=1}^{\infty}\right) = 1 + g_t^{\rho}\theta\left(\{g_{t+\tau}\}_{\tau=1}^{\infty}\right).$$

The first order condition satisfied at

$$\begin{aligned} & \hat{c}\left(g_{t-1}\right) \\ = \ & \arg\max_{c}\left(E\left[\left(c^{\rho}+\beta\left[1-c\right]^{\rho}\left[\frac{\hat{c}\left(g_{t-1}\right)}{1-\hat{c}\left(g_{t-1}\right)}\right]^{\rho}\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}}|g_{t-1}\right]\right)^{\frac{1}{1-\alpha}}. \end{aligned}$$

is

$$E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\left(1-\beta\left[\frac{\hat{c}\left(g_{t-1}\right)}{1-\hat{c}\left(g_{t-1}\right)}\right]\tilde{g}_t^{\rho}\tilde{\theta}\right)|g_{t-1}\right] = 0.$$

Solving this first order condition, we get the following expression for the price dividend ratio

$$\begin{aligned} \frac{P\left(s_t\right)}{s_t} &= \left[\frac{1-\hat{c}\left(g_{t-1}\right)}{\hat{c}\left(g_{t-1}\right)}\right] \qquad (38) \\ &= \frac{\beta E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho}\tilde{\theta}\,|g_{t-1}\right]}{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}|g_{t-1}\right]}. \end{aligned}$$

Thus,

$$\hat{c}\left(g_{t-1}\right) = \frac{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}|g_{t-1}\right]}{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}}|g_{t-1}\right]} \qquad (39)$$

and

$$1-\hat{c}\left(g_{t-1}\right) = \frac{\beta E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho}\tilde{\theta}\,|g_{t-1}\right]}{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}}|g_{t-1}\right]}. \qquad (40)$$

3.2.1 The Riskless Rate and The Equity Premium

The equilibrium return on a riskless asset in zero net supply is $r_f(g_{t-1})$. It's obtained as the solution to the first order condition

$$\begin{aligned} & E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho-1}\tilde{\theta}\left[\frac{\hat{c}(g_{t-1})}{[1-\hat{c}(g_{t-1})]}\tilde{g}_t-r_f(g_{t-1})\right]|g_{t-1}\right] \\ = \; & 0, \end{aligned}$$

which is the same as (6) when we use (33), (34), (35) and (37). Solving the first order condition for $r_f(g_{t-1})$, we get

$$r_f(g_{t-1})=\frac{\hat{c}(g_{t-1})}{[1-\hat{c}(g_{t-1})]}\frac{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho}\tilde{\theta}\,|g_{t-1}\right]}{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho-1}\hat{c}(\tilde{g}_t)\,\tilde{\theta}\,|g_{t-1}\right]}. \tag{41}$$

The expression (41) is the same as (7) when we use(8), (33), (34), (35) and (37).

If we let

$$E^*\left[\frac{\tilde{g}_t}{\hat{c}(\tilde{g}_t)}\,|g_{t-1}\right]=\frac{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\frac{\tilde{g}_t^{\rho}}{\hat{c}(\tilde{g}_t)}\hat{c}(\tilde{g}_t)\,\tilde{\theta}\,|g_{t-1}\right]}{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho-1}\hat{c}(\tilde{g}_t)\,\tilde{\theta}\,|g_{t-1}\right]}, \tag{42}$$

we can rewrite the expression for the riskless rate as

$$r_f(g_{t-1})=\frac{\hat{c}(g_{t-1})}{[1-\hat{c}(g_{t-1})]}E^*\left[\frac{\tilde{g}_t}{\hat{c}(\tilde{g}_t)}\,|g_{t-1}\right]. \tag{43}$$

Since

$$E\left[r(\tilde{g}_t)\,|g_{t-1}\right]=\frac{\hat{c}(g_{t-1})}{[1-\hat{c}(g_{t-1})]}E\left[\frac{\tilde{g}_t}{\hat{c}(\tilde{g}_t)}\,|g_{t-1}\right] \tag{44}$$

we have

$$\frac{E\left[r(\tilde{g}_t)\,|g_{t-1}\right]}{r_f(g_{t-1})}=\frac{E\left[\frac{\tilde{g}_t}{\hat{c}(\tilde{g}_t)}\,|g_{t-1}\right]}{E^*\left[\frac{\tilde{g}_t}{\hat{c}(\tilde{g}_t)}\,|g_{t-1}\right]}. \tag{45}$$

We can also rewrite the expression (41) for $r_f(g_{t-1})$, as

$$r_f(g_{t-1})=\frac{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}|g_{t-1}\right]}{\beta E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho-1}\hat{c}(\tilde{g}_t)\,\tilde{\theta}\,|g_{t-1}\right]} \tag{46}$$

Then the expression for the equity premium becomes

$$
\begin{aligned}
& E\left[r\left(\tilde{g}_t\right)|g_{t-1}\right]-r_f\left(g_{t-1}\right) && (47)\\
= & \frac{\hat{c}\left(g_{t-1}\right)}{\left[1-\hat{c}\left(g_{t-1}\right)\right]}\left(E\left[\frac{\tilde{g}_t}{\hat{c}\left(\tilde{g}_t\right)}|g_{t-1}\right]-E^*\left[\frac{\tilde{g}_t}{\hat{c}\left(\tilde{g}_t\right)}|g_{t-1}\right]\right)\\
= & \left(\frac{\hat{c}\left(g_{t-1}\right)}{\left[1-\hat{c}\left(g_{t-1}\right)\right]}\right)E\left[\frac{\tilde{g}_t}{\hat{c}\left(\tilde{g}_t\right)}|g_{t-1}\right]\\
& -\frac{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}|g_{t-1}\right]}{\beta E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho-1}\hat{c}\left(\tilde{g}_t\right)\tilde{\theta}\,|g_{t-1}\right]}.
\end{aligned}
$$

3.2.2 The Cobb Douglas Case

In this case, we have noted that

$$
\begin{aligned}
& \hat{c}\left(r_{t-1}\right)\\
= & \left(1-\beta\right).
\end{aligned}
$$

Thus, the expression (38) for the price dividend ratio reduces to

$$
\begin{aligned}
& \frac{P\left(s_t\right)}{s_t}=\left[\frac{1-\hat{c}}{\hat{c}}\right]\\
= & \frac{\beta}{\left(1-\beta\right)}.
\end{aligned}
$$

Also (26) reduces to

$$
E\tilde{r}_t=\frac{E\left[\tilde{g}_t|g_{t-1}\right]}{\beta}.
$$

The expression (10) for the riskless rate becomes

$$
r_f\left(g_{t-1}\right)=\frac{E^*\left[\tilde{g}_t|g_{t-1}\right]}{\beta}
$$

where

$$
E^*\left[\tilde{g}_t|g_{t-1}\right]=\frac{E\left(\prod_{\tau=1}^{\infty}\tilde{g}_{t+\tau-1}^{(1-\alpha)\beta^{\tau}}|g_{t-1}\right)}{E\left(\tilde{g}_t^{-1}\prod_{\tau=1}^{\infty}\tilde{g}_{t+\tau-1}^{(1-\alpha)\beta^{\tau}}|g_{t-1}\right)}.
$$

Thus,

$$
r_f\left(g_{t-1}\right)=\frac{E\left(\prod_{\tau=1}^{\infty}\tilde{g}_{t+\tau-1}^{(1-\alpha)\beta^{\tau}}|g_{t-1}\right)}{\beta E\left(\tilde{g}_t^{-1}\prod_{\tau=1}^{\infty}\tilde{g}_{t+\tau-1}^{(1-\alpha)\beta^{\tau}}|g_{t-1}\right)}.
$$

3.2.3 Some Special Cases

In this section, we briefly return to the special cases of *i.i.d.* returns and of additively separable preferences. Here when the returns are *i.i.d.* it is because consumption growth is *i.i.d.*

The Case of iid Consumption Growth Now (38) reduces to

$$\frac{P(s_t)}{s_t} = \left[\frac{1-\hat{c}}{\hat{c}}\right]$$

$$= \frac{\beta E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho}\tilde{\theta}\right]}{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\right]}.$$

Thus, (39) and (40) reduce to

$$\hat{c} = \frac{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\right]}{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}}\right]}$$

and

$$1-\hat{c} = \frac{\beta E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho}\tilde{\theta}\right]}{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}}\right]}.$$

In Kihlstrom [22] the following proposition was proved. This generalizes a result obtained in Kihlstrom and Mirman [23].

Proposition 5 $1-\hat{c}$ *is a decreasing (increasing) function of* α *if* $\varepsilon = \frac{1}{1-\rho} > (<)\, 1$.

We then have the obvious corollary.

Corollary 6

$$E\tilde{r}_t = \frac{E\tilde{g}_t}{[1-\hat{c}]}$$

is an increasing (decreasing) function of α *if* $\varepsilon = \frac{1}{1-\rho} > (<)\, 1$.

In this case, the riskfree rate is

$$r_f = \frac{E^*\tilde{g}_t}{[1-\hat{c}]}$$

where

$$E^*\tilde{g}_t = \frac{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho}\tilde{\theta}\right]}{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho-1}\tilde{\theta}\right]}. \tag{48}$$

Thus, (45) simplifies to

$$\frac{E\tilde{r}_t}{r_f} = \frac{E\tilde{g}_t}{E^*\tilde{g}_t}.$$

Also (46) reduces to

$$r_f = \frac{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}}\right]}{\beta E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho-1}\tilde{\theta}\right]}. \tag{49}$$

So the expression (47) for the equity premium becomes

$$\begin{aligned}
& E\tilde{r}_t - r_f \\
= \;& \left(\frac{1}{[1-\hat{c}]}\right)(E\tilde{g}_t - E^*\tilde{g}_t) \\
= \;& \left(\frac{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}}\right]}{\beta E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho}\tilde{\theta}\right]}\right)E\tilde{g}_t \\
& -\frac{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}}\right]}{\beta E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho-1}\tilde{\theta}\right]}
\end{aligned}$$

In the Cobb-Douglas case, the expression for the riskless rate becomes

$$r_f = \frac{E\left(\tilde{g}_t^{(1-\alpha)\beta}\right)}{\beta E\left(\tilde{g}_t^{(1-\alpha)\beta-1}\right)}$$

So the equity premium becomes

$$E\tilde{r}_t - r_f = \frac{E\tilde{g}_t}{\beta} - \frac{E\left(\tilde{g}_t^{(1-\alpha)\beta}\right)}{\beta E\left(\tilde{g}_t^{(1-\alpha)\beta-1}\right)}$$

The Additively Separable Case Using the expression (34), and the condition

$$1-\alpha = \rho$$

that holds in the additively separable case, we observe that (36) becomes

$$\begin{aligned}
&\hat{c}(g_t) \quad (50)\\
= \quad & \arg\max_c \left(c^{1-\alpha} + \beta [1-c]^{1-\alpha} \left[\frac{\hat{c}(g_t)}{1-\hat{c}(g_t)} \right]^{1-\alpha} E\left[\tilde{g}_{t+1}^{1-\alpha} \tilde{\theta} \,|g_t \right] \right)^{\frac{1}{1-\alpha}}.
\end{aligned}$$

where $\tilde{\theta}$ is defined by (37) and (35) also holds.

Remark 7 *If we define*

$$\omega(g_t) = [\hat{c}(g_t)]^{1-\alpha} \varphi(g_t) \tag{51}$$

where

$$\begin{aligned}
\varphi(g_t) &= E\left[\theta\left(\{\tilde{g}_{t+\tau}\}_{\tau=1}^{\infty}\right)|g_t\right] \qquad (52)\\
&= 1 + E\left[\left(\sum_{\tau=1}^{\infty} \beta^{\tau} \prod_{s=1}^{\tau} \tilde{g}_{t+s}^{1-\alpha}\right)|g_t\right]\\
&= 1 + \beta E\left[\tilde{g}_{t+1}^{1-\alpha} \varphi(\tilde{g}_{t+1})\,|g_t\right].
\end{aligned}$$

Then, the maximand in (50) becomes

$$\begin{aligned}
& c^{1-\alpha} + \beta [1-c]^{1-\alpha} \left[\frac{\hat{c}(g_t)}{1-\hat{c}(g_t)} \right]^{1-\alpha} E\left[\tilde{g}_{t+1}^{1-\alpha} \tilde{\theta} \,|g_t \right]\\
= \quad & c^{1-\alpha} + \beta [1-c]^{1-\alpha} \left[\frac{\hat{c}(g_t)}{1-\hat{c}(g_t)} \right]^{1-\alpha} E\left[\tilde{g}_{t+1}^{1-\alpha} \varphi(\tilde{g}_{t+1}) \,|g_t \right]\\
= \quad & c^{1-\alpha} + \beta [1-c]^{1-\alpha} E\left[\tilde{r}_t^{1-\alpha} \omega(\tilde{g}_{t+1}) \,|g_t \right].
\end{aligned}$$

where $\tilde{r}_t$ is as in (26). Thus, when we use the notation introduced in (26), (51) and (52), the maximand in (50) reduces to the maximand in (13) as it must.

In this case, the expression (38) for the price dividend ratio reduces to

$$\begin{aligned}
\frac{P(s_t)}{s_t} &= \left[\frac{1-\hat{c}(g_t)}{\hat{c}(g_t)} \right] \qquad (53)\\
&= \beta E\left[\tilde{g}_{t+1}^{1-\alpha} \varphi(\tilde{g}_{t+1})\,|g_t\right].
\end{aligned}$$

As a consequence, (39) and (40) reduce to

$$\begin{aligned}
\hat{c}(g_t) &= \frac{1}{1+\beta E\left[\tilde{g}_{t+1}^{1-\alpha} \varphi(\tilde{g}_{t+1})\,|g_t\right]} \qquad (54)\\
&= \frac{1}{\varphi(g_t)}
\end{aligned}$$

and

$$1-\hat{c}(g_t) = \frac{\beta E\left[\tilde{g}_{t+1}^{1-\alpha} \varphi(\tilde{g}_{t+1})\,|g_t\right]}{1+\beta E\left[\tilde{g}_{t+1}^{1-\alpha} \varphi(\tilde{g}_{t+1})\,|g_t\right]}. \tag{55}$$

Also (42)

$$E^*\left[\frac{\tilde{g}_{t+1}}{\hat{c}\left(\tilde{g}_{t+1}\right)}\left|g_t\right.\right] = \frac{E\left[\frac{\tilde{g}_{t+1}}{\hat{c}(\tilde{g}_{t+1})}\tilde{g}_{t+1}^{-\alpha}\hat{c}\left(g_{t+1}\right)\varphi\left(\tilde{g}_{t+1}\right)\left|g_t\right.\right]}{E\left[\tilde{g}_t^{-\alpha}\hat{c}\left(\tilde{g}_{t+1}\right)\varphi\left(\tilde{g}_{t+1}\right)\left|g_t\right.\right]} = \frac{E\left[\tilde{g}_{t+1}^{1-\alpha}\varphi\left(\tilde{g}_{t+1}\right)\left|g_t\right.\right]}{E\left[\tilde{g}_t^{-\alpha}\left|g_t\right.\right]}. \tag{56}$$

Using (53) and (56) in (43) we get

$$r_f\left(g_t\right) = \frac{1}{\beta E\left[\tilde{g}_{t+1}^{-\alpha}\left|g_t\right.\right]}. \tag{57}$$

We also have

$$E\left[\frac{\tilde{g}_{t+1}}{\hat{c}\left(\tilde{g}_{t+1}\right)}\left|g_t\right.\right] = E\left[\tilde{g}_{t+1}\varphi\left(\tilde{g}_{t+1}\right)\left|g_t\right.\right]$$

which together with (53) implies that (44) reduces to

$$E\left[\tilde{r}\left(g_t,\tilde{g}_{t+1}\right)\left|g_t\right.\right] = \frac{E\left[\tilde{g}_{t+1}\varphi\left(\tilde{g}_{t+1}\right)\left|g_t\right.\right]}{\beta E\left[\tilde{g}_{t+1}^{1-\alpha}\varphi\left(\tilde{g}_{t+1}\right)\left|g_t\right.\right]}.$$

We then have expressions for the equity premium. Specifically,

$$\frac{E\left[\tilde{r}\left(g_t,\tilde{g}_{t+1}\right)\left|g_t\right.\right]}{r_f\left(g_t\right)} = \frac{E\left[\frac{\tilde{g}_{t+1}}{\hat{c}(\tilde{g}_{t+1})}\left|g_t\right.\right]}{E^*\left[\frac{\tilde{g}_{t+1}}{\hat{c}(\tilde{g}_{t+1})}\left|g_t\right.\right]} = \frac{E\left[\tilde{g}_{t+1}\varphi\left(\tilde{g}_{t+1}\right)\left|g_t\right.\right]E\left[\tilde{g}_{t+1}^{-\alpha}\left|g_t\right.\right]}{E\left[\tilde{g}_{t+1}^{1-\alpha}\varphi\left(\tilde{g}_{t+1}\right)\left|g_t\right.\right]} \tag{58}$$

and

$$E\left[\tilde{r}\left(g_t,\tilde{g}_{t+1}\right)\left|g_t\right.\right] - r_f\left(g_t\right) = \left[\frac{E\left[\tilde{g}_{t+1}\varphi\left(\tilde{g}_{t+1}\right)\left|g_t\right.\right]}{\beta E\left[\tilde{g}_{t+1}^{1-\alpha}\varphi\left(\tilde{g}_{t+1}\right)\left|g_t\right.\right]}\right] - \frac{1}{\beta E\left[\tilde{g}_{t+1}^{-\alpha}\left|g_t\right.\right]}$$

The Case of iid Consumption Growth and Additively Separable Preferences In this case (53) becomes simply

$$\frac{P\left(s_t\right)}{s_t} = \left[\frac{1-\hat{c}}{\hat{c}}\right] = \beta E\left[\tilde{g}_t^{1-\alpha}\tilde{\theta}\right].$$

where

$$\begin{aligned}
\beta E\left[\tilde{g}_t^{1-\alpha}\tilde{\theta}\right] &= \sum_{\tau=1}^{\infty}\beta^{\tau}\prod_{s=1}^{\tau}E\tilde{g}_{t+s}^{1-\alpha} \\
&= \cdot\sum_{\tau=1}^{\infty}\left[\beta E\tilde{g}^{1-\alpha}\right]^{\tau} \\
&= \frac{\beta E\tilde{g}^{1-\alpha}}{1-\beta E\tilde{g}^{1-\alpha}}.
\end{aligned}$$

Combining these observations (53) becomes

$$\frac{P\left(s_t\right)}{s_t} = \left[\frac{1-\hat{c}}{\hat{c}}\right] = \frac{\beta E\tilde{g}^{1-\alpha}}{1-\beta E\tilde{g}^{1-\alpha}}.$$

Also (54) and (55) reduce to

$$\hat{c} = 1-\beta E\tilde{g}^{1-\alpha}$$

and

$$1-\hat{c} = \beta E\tilde{g}^{1-\alpha}.$$

Also (48) and (56) reduce to

$$E^{*}\tilde{g} = \frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}$$

So the expressions (49) and (57) for the riskless rate simplify to

$$\begin{aligned}
r_f &= \frac{1}{[1-\hat{c}]}E^{*}\tilde{g} \\
&= \frac{1}{\beta E\tilde{g}^{-\alpha}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{E\tilde{r}}{r_f} &= \frac{E\tilde{g}}{E^{*}\tilde{g}} \\
&= \frac{E\tilde{g}E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}
\end{aligned}$$

and the equity premium is

$$\begin{aligned}
& E\tilde{r}-r_f \\
= \; & \frac{E\tilde{g}}{\beta E\tilde{g}^{1-\alpha}} - \frac{1}{\beta E\tilde{g}^{-\alpha}}.
\end{aligned}$$

Kocherlakota has noted that the following result holds. The result and its proof appear in Kihlstrom [22].

Proposition 8

$$E^*\tilde{g} = \frac{E\left[\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}^{\rho}\tilde{\theta}\right]}{E\left[\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}^{\rho-1}\tilde{\theta}\right]} < \frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}$$

Iff

$$\varepsilon = \frac{1}{1-\rho} < \frac{1}{\alpha}$$

This implies that

$$\frac{E\tilde{r}}{r_f} = \frac{E\tilde{g}}{E^*\tilde{g}}$$

is larger in the non-additively separable case than in the additively separable case when and only when the elasticity of substitution is larger in the non-additively separable case than in the additively separable case with the same α.

3.3 Epstein-Zin Weil Preferences

Substituting (26) in the functional equation (19) yields

$$\hat{c}\left(g_{t-1}\right) = \frac{1}{1+\beta\left[E\left(\left[\left(\frac{\tilde{g}_t^{1-\alpha}}{\left[\hat{c}\left(\tilde{g}_t\right)\right]^{\frac{1-\alpha}{\rho}}}\right)\right]|g_{t-1}\right)\right]^{\frac{\rho}{1-\alpha}}} \tag{59}$$

and

$$\frac{\hat{c}\left(g_{t-1}\right)}{\left[1-\hat{c}\left(g_{t-1}\right)\right]} = \frac{1}{\beta\left[E\left(\left[\left(\frac{\tilde{g}_t^{1-\alpha}}{\left[\hat{c}\left(\tilde{g}_t\right)\right]^{\frac{1-\alpha}{\rho}}}\right)\right]|g_{t-1}\right)\right]^{\frac{\rho}{1-\alpha}}} \tag{60}$$

Using (18), (59) becomes

$$\xi\left(g_{t-1}\right) = \frac{1}{\left[1+\beta\left[E\left(\left[\xi\left(\tilde{g}_t\right)^{\frac{1-\alpha}{1-\rho}}\tilde{g}_t^{1-\alpha}\right]|g_{t-1}\right)\right]^{\frac{\rho}{1-\alpha}}\right]^{\frac{\rho-1}{\rho}}}$$

which is the form taken by the functional equation (17) in the Lucas asset pricing equilibrium.

3.3.1 The Riskless Rate and The Equity Premium

Again using (26) in the expression (20) for the riskless rate, the expression becomes

$$r_f\left(g_t\right) = \frac{\hat{c}\left(g_t\right)}{\left[1-\hat{c}\left(g_t\right)\right]}E^*\left[\frac{\tilde{g}_{t+1}}{\hat{c}\left(\tilde{g}_{t+1}\right)}|g_t\right] \tag{61}$$

where

$$E^*\left[\frac{\tilde{g}_{t+1}}{\hat{c}\left(\tilde{g}_{t+1}\right)}\,|g_t\right]=\frac{E\left(\left[\xi\left(\tilde{g}_t\right)\frac{\tilde{g}_t}{\hat{c}(\tilde{g}_t)}\right]^{1-\alpha}|g_{t-1}\right)}{E\left(\left[\xi\left(\tilde{g}_t\right)\right]^{1-\alpha}\left[\frac{\tilde{g}_t}{\hat{c}(\tilde{g}_t)}\right]^{-\alpha}|g_{t-1}\right)} \tag{62}$$

Using (18), (62) and (60) the expression (61) for the riskless rate becomes

$$r_f\left(g_{t-1}\right)=\frac{E\left(\left[\hat{c}\left(\tilde{g}_t\right)\right]^{-\frac{1-\alpha}{\rho}}\tilde{g}_t^{1-\alpha}\,|g_{t-1}\right)^{1-\frac{\rho}{1-\alpha}}}{\beta E\left(\left[\hat{c}\left(\tilde{g}_t\right)\right]^{-\frac{1-\rho-\alpha}{\rho}}\tilde{g}_t^{-\alpha}\,|g_{t-1}\right)} \tag{63}$$

Also (60) implies that

$$\begin{aligned} E\tilde{r}_t\left(g_{t-1}\right) &= \frac{\hat{c}\left(g_t\right)}{\left[1-\hat{c}\left(g_t\right)\right]}E\left[\frac{\tilde{g}_t}{\hat{c}\left(\tilde{g}_t\right)}\,|g_{t-1}\right] \\ &= \frac{E\left[\frac{\tilde{g}_t}{\hat{c}(\tilde{g}_t)}\,|g_{t-1}\right]}{\beta\left[E\left(\left[\hat{c}\left(\tilde{g}_t\right)\right]^{-\frac{1-\alpha}{\rho}}\tilde{g}_t^{1-\alpha}\,|g_{t-1}\right)\right]^{\frac{\rho}{1-\alpha}}}. \end{aligned} \tag{64}$$

We, therefore, have

$$\begin{aligned} \frac{E\tilde{r}_t\left(g_{t-1}\right)}{r_f\left(g_{t-1}\right)} &= \frac{E\left[\frac{\tilde{g}_t}{\hat{c}(\tilde{g}_t)}\,|g_{t-1}\right]E\left(\left[\xi\left(\tilde{g}_t\right)\right]^{1-\alpha}\left[\frac{\tilde{g}_t}{\hat{c}(\tilde{g}_t)}\right]^{-\alpha}|g_{t-1}\right)}{E\left(\left[\xi\left(\tilde{g}_t\right)\frac{\tilde{g}_t}{\hat{c}(\tilde{g}_t)}\right]^{1-\alpha}|g_{t-1}\right)} \\ &= \frac{E\left[\frac{\tilde{g}_t}{\hat{c}(\tilde{g}_t)}\,|g_{t-1}\right]E\left(\left[\hat{c}\left(\tilde{g}_t\right)\right]^{-\frac{1-\rho-\alpha}{\rho}}\tilde{g}_t^{-\alpha}\,|g_{t-1}\right)}{E\left(\left[\hat{c}\left(\tilde{g}_t\right)\right]^{-\frac{1-\alpha}{\rho}}\tilde{g}_t^{1-\alpha}\,|g_{t-1}\right)}. \end{aligned}$$

The equity premium is

$$\begin{aligned} & E\tilde{r}_t\left(g_{t-1}\right)-r_f\left(g_{t-1}\right) \\ =\; & \frac{E\left[\frac{\tilde{g}_t}{\hat{c}(\tilde{g}_t)}\,|g_{t-1}\right]}{\beta\left[E\left(\left[\hat{c}\left(\tilde{g}_t\right)\right]^{-\frac{1-\alpha}{\rho}}\tilde{g}_t^{1-\alpha}\,|g_{t-1}\right)\right]^{\frac{\rho}{1-\alpha}}} \\ & -\frac{E\left(\left[\hat{c}\left(\tilde{g}_t\right)\right]^{-\frac{1-\alpha}{\rho}}\tilde{g}_t^{1-\alpha}\,|g_{t-1}\right)^{1-\frac{\rho}{1-\alpha}}}{\beta E\left(\left[\hat{c}\left(\tilde{g}_t\right)\right]^{-\frac{1-\rho-\alpha}{\rho}}\tilde{g}_t^{-\alpha}\,|g_{t-1}\right)} \end{aligned} \tag{65}$$

3.3.2 The Cobb-Douglas Case

Since

$$\hat{c}\left(g_{t-1}\right)=\left(1-\beta\right),$$

$$E\tilde{r}_t = \frac{E\left[\tilde{g}_t \left| g_{t-1} \right.\right]}{\beta}$$

and (22) becomes

$$\begin{aligned} & \xi\left(g_{t-1}\right) \\ = \quad & (1-\beta)^{(1-\beta)} \left[E\left(\left[\left(\xi\left(\tilde{g}_t\right)\tilde{g}_t\right)\right]^{1-\alpha} \left| g_{t-1}\right.\right)\right]^{\frac{\beta}{1-\alpha}}. \end{aligned}$$

Also (21) reduces to

$$E^*\left(\tilde{r}_t \left| g_{t-1} \right.\right) = \frac{E\left(\left[\xi\left(\tilde{g}_t\right)\tilde{g}_t\right]^{1-\alpha} \left| g_{t-1}\right.\right)}{\beta E\left(\xi\left(\tilde{g}_t\right)^{1-\alpha} \tilde{g}_t^{-\alpha} \left| g_{t-1}\right.\right)}$$

and the expression for the riskless rate becomes

$$r_f\left(g_{t-1}\right) = \frac{E^*\left(\tilde{g}_t \left| g_{t-1} \right.\right)}{\beta}$$

where

$$E^*\left(\tilde{g}_t \left| g_{t-1} \right.\right) = \frac{E\left(\left[\xi\left(\tilde{g}_t\right)\tilde{g}_t\right]^{1-\alpha} \left| g_{t-1}\right.\right)}{E\left(\xi\left(\tilde{g}_t\right)^{1-\alpha} \tilde{g}_t^{-\alpha} \left| g_{t-1}\right.\right)}.$$

3.3.3 The Case of iid Consumption Growth

In this case, (59) becomes

$$\hat{c} = \frac{1}{1 + \beta\hat{c}^{-1}\left[E\left(\tilde{g}^{1-\alpha}\right)\right]^{\frac{\rho}{1-\alpha}}}$$

and the solution is

$$\hat{c} = 1 - \beta\left[E\left(\tilde{g}^{1-\alpha}\right)\right]^{\frac{\rho}{1-\alpha}}. \tag{66}$$

Using (29), the expression (23) for the riskless rate reduces to

$$r_f = \frac{1}{\left[1-\hat{c}\right]} E^*\tilde{g} \tag{67}$$

where

$$E^*\tilde{g} = \frac{E\left(\tilde{g}^{1-\alpha}\right)}{\beta E\left(\tilde{g}^{-\alpha}\right)} \tag{68}$$

Taken together, (66), (67) and (68) imply that the riskless rate is

$$r_f = \frac{\left[E\left(\tilde{g}^{1-\alpha}\right)\right]^{1-\frac{\rho}{1-\alpha}}}{\beta E\left(\tilde{g}^{-\alpha}\right)}.$$

Also, (30) and (67) imply that

$$\begin{aligned}\frac{E\tilde{r}}{r_f} &= \frac{E\tilde{g}}{E^*\tilde{g}} \\ &= \frac{E\tilde{g}E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}\end{aligned}$$

and this expression is therefore the same in the case of EZW preferences as in the case of additively separable preferences.

4 Calibrations

In the calibrations, we initially follow exactly the approach taken by Mehra-Prescott [40] in Section 4 of their paper. That is we assume that two rates of real consumption growth are possible. The high growth rate is

$$g_H = 1.054 \tag{69}$$

and the low growth rate is

$$g_L = .982. \tag{70}$$

The transition probabilities are

$$\begin{aligned} & p_{HH} \\ = {} & \Pr\left(\tilde{g}_t = g_H \left| g_{t-1} = g_H\right.\right) \\ = {} & \Pr\left(\tilde{g}_t = g_L \left| g_{t-1} = g_L\right.\right) \\ = {} & p_{LL} \\ = {} & .43 \end{aligned} \tag{71}$$

and

$$\begin{aligned} & p_{LH} \\ = {} & \Pr\left(\tilde{g}_t = g_L \left| g_{t-1} = g_H\right.\right) \\ = {} & \Pr\left(\tilde{g}_t = g_H \left| g_{t-1} = g_L\right.\right) \\ = {} & p_{HL} \\ = {} & .57 \end{aligned} \tag{72}$$

Thus, in the steady state stationary distribution

$$Pr\left(\tilde{g} = g_H\right) = .5,$$

$$E\tilde{g} = 1.018,$$

$$\sigma_{\tilde{g}} = .036$$

and the serial correlation of the growth rates is $-.14$. Mehra and Prescott note that "The parameters were selected so that the average growth rate of per capita consumption, the standard deviation of the growth rate of per capita consumption and the first-order serial correlation of this growth rate, all with respect to the model's stationary distribution, matched the sample values for the U.S. economy between 1889-1978. The sample values for the U.S. economy were 0.018, 0.036 and -0.14, respectively."[1] The data used to obtain these sample values is described in Section 3 of their paper. Mehra and Prescott assume that, in the Lucas asset pricing model, the representative consumer's preferences are, in our terminology, additively separable. In the notation of their paper, as in ours when the utility function is additively separable, α measures the relative risk aversion as well as the inverse of the intertemporal elasticity of substitution and β is the discount rate. Mehra and Prescott then assume that in the Lucas model consumption growth is as specified in (69), (70) , (71) and (72) and they "search for parameters α and β for which the model's averaged risk-free rate and equity risk premium match those observed for the U.S. economy over this ninety-year period."[2] They note that "The average real return on relatively riskless, short-term securities over the 1889-1978 period was 0.80 percent."[3] They also note that "The average real return on the Standard and Poor's 500 Composite Stock Index over the ninety years considered was 6.98 percent per annum. This leads to an average equity premium of 6.18 percent (standard error 1.76 percent)."[4]

The simulation results for the riskless rate, equation (46), and the equity premia, equation (47), obtained by assuming consistent planning by consumers with forward looking von Neuman Morgenstern preferences are as described in Tables 1 and 2. Note that in these tables the diagonals correspond to the additively separable cases in which $\alpha = 1 - \rho$. When $\alpha = 15 = 1 - \rho$, $\sigma = 1/15 = .067$ and the simulated equity premium equals 4.9% . For this case, the simulated riskless rate is, however, 17%. If we drop the assumption that $\alpha = 1 - \rho$, we can raise the risk aversion measure to $\alpha = 25$ while keeping the elasticity of substitution at a higher level of $\sigma = .5$ by choosing $\rho = -1$. In that case we get a lower simulated equity premium of 2.6% but the simulated riskless rate is 3%. In the Cobb-Douglas case, for which $\rho = 0$ and $\sigma = 1$, the equity premium falls to 2.5% and the riskless rate is also 2.5%.

Table1
Riskfree Rates:
Computed Using Equation (46) and $\beta = .97$
Consistent Planning with Forward Looking von Neuman Morgenstern Preferences

[1] Mehra-Prescott [40], p. 154.
[2] Mehra-Prescott [40], p. 154.
[3] Mehra-Prescott [40], p. 155.
[4] Mehra-Prescott [40], p. 155-6.

ρ	1/3	0	−1	−4	−9	−14	−19	−24
σ	1.5	1	.5	.2	.1	.067	.05	.04
α								
2	1.0414	1.0471	1.0645	1.1178	1.2083	1.2969	1.3806	1.4583
5	1.0386	1.0439	1.0593	1.1069	1.1876	1.2674	1.3426	1.4587
10	1.0347	1.0386	1.0506	1.0889	1.1544	1.2182	1.278	1.3314
15	1.0296	1.0341	1.0438	1.0735	1.1223	1.1719	1.2179	1.2599
20	1.027	1.0296	1.0368	1.0574	1.0939	1.1296	1.162	1.1912
25	1.024	1.0247	1.0304	1.0454	1.0671	1.0899	1.1116	1.1297

Table 2
Equity Premia:
Computed Using Equation (47) and $\beta = .97$
Consistent Planning with Forward Looking von Neuman Morgenstern Preferences

ρ	1/3	0	−1	−4	−9	−14	−19	−24
σ	1.5	1	.5	.2	.1	.067	.05	.04
α								
2	.0022	.0024	.0029	0.005	.0112	.0217	.0347	.0458
5	.0055	.0056	0.0066	0.0098	0.0177	.0277	0.0398	0.0582
10	.0101	.011	0.0128	0.0173	0.0269	0.0395	0.0543	0.0718
15	.015	.0154	0.0194	.0233	0.0362	0.0492	0.0653	0.0804
20	.019	.02	0.0219	0.0320	0.0424	0.0556	0.072	0.0913
25	.023	.0249	0.026	.0325	0.0474	0.0611	0.08	0.0976

For purposes of comparison we have also computed (using a calculation described in the next section) the riskless rates and the equity premia assuming EZW preferences. With EZW preferences the riskless rate and the equity premia can be easily computed when there are only two possible consumption growth rates. The results of these computations appear in Tables 3 and 4. For the case, in which $\alpha = 25$, $\rho = -1$ and $\sigma = .5$, EZW preferences yield an equity premium of 2.7% and a riskless rate of 2.9%. For the Cobb Douglas case in which $\alpha = 25$, $\rho = 0$ and $\sigma = 1$, the EZW equity premium is 2.4% and the riskless rate is 2.5%. For these cases the EZW riskless rates and equity premia are virtually the same as those obtained with consistent planning and forward looking von Neumann Morgenstern preferences. In fact, it is somewhat striking that, for all of the parameter pairs reported, similar results are obtained from the EZW preferences and from the assumption of consistent planning with forward looking von Neumann Morgenstern preferences.

Table 3
Riskfree Rates:

Computed Using Equation (63) and $\beta = .97$
EZW Preferences

ρ	1/3	0	-1	-4	-9	-14	-19	-24
σ	1.5	1	.5	.2	.1	.067	.05	.04
α								
2	1.0413	1.0471	1.0645	1.1186	1.2153	1.3210	1.4369	1.5639
5	1.0386	1.0437	1.0591	1.1069	1.1917	1.2836	1.3834	1.4918
10	1.0343	1.0383	1.0506	1.0882	1.1544	1.2250	1.3005	1.3815
15	1.0303	1.0334	1.0426	1.0709	1.1201	1.1719	1.2266	1.2845
20	1.0268	1.0289	1.0355	1.0553	1.0894	1.1250	1.1620	1.2008
25	1.0238	1.0252	1.0292	1.0415	1.0624	1.0841	1.1065	1.1297

Table 4
Equity Premia:
Computed Using Equation (43) and $\beta = .97$
EZW Preferences

ρ	1/3	0	-1	-4	-9	-14	-19	-24
σ	1.5	1	.5	.2	.1	.067	.05	.04
α								
2	.0023	.0024	.0029	0.0047	.0082	.0125	.0174	.0227
5	.0055	.0058	0.0067	0.0098	0.0158	.0225	0.0299	0.0379
10	0.0106	.0011	0.0128	0.0179	0.0272	0.0373	0.0482	0.0598
15	0.0154	.0161	0.0183	0.0250	0.0368	0.0495	0.0628	0.0769
20	0.0197	.0205	0.0231	0.0309	0.0444	0.0587	0.0735	0.0890
25	0.0234	.0243	0.0271	0.0355	0.0500	0.0650	0.0805	0.0965

4.1 Computing the Riskless Rate and the Equity Premium with Epstein-Zin, Weil Preferences

When the consumer has EZW preferences and only two rates of real consumption growth are possible, the riskless rate and the equity premium can be computed quite simply . In particular we let

$$\hat{c}\left(g_{t-1}\right) = \begin{cases} \hat{c}\left(g_H\right), \text{ if } g_{t-1} = g_H \\ \hat{c}\left(g_L\right), \text{ if } g_{t-1} = g_L \end{cases}.$$

Then (59) becomes

$$\hat{c}\left(g_H\right) = \frac{1}{1 + \beta\left[p_{HH}\left[\left(\frac{g_H}{\left[\hat{c}\left(g_H\right)\right]^{\frac{1}{\rho}}}\right)\right]^{1-\alpha} + p_{LH}\left[\left(\frac{g_L}{\left[\hat{c}\left(g_L\right)\right]^{\frac{1}{\rho}}}\right)\right]^{1-\alpha}\right]^{\frac{\rho}{1-\alpha}}} \tag{73}$$

when $g_{t-1} = g_H$ and

$$\hat{c}(g_L) = \frac{1}{1 + \beta\left[p_{HL}\left[\left(\frac{g_H}{[\hat{c}(g_H)]^{\frac{1}{\rho}}}\right)\right]^{1-\alpha} + p_{LL}\left[\left(\frac{g_L}{[\hat{c}(g_L)]^{\frac{1}{\rho}}}\right)\right]^{1-\alpha}\right]^{\frac{\rho}{1-\alpha}}} \tag{74}$$

when $g_{t-1} = g_L$.

If we let

$$x = \frac{\hat{c}(g_H)}{\hat{c}(g_L)}$$

then (73) becomes

$$\hat{c}(g_H) = 1 - \beta\left[p_{HH} g_H^{1-\alpha} + p_{LH} x^{\frac{1-\alpha}{\rho}} g_L^{1-\alpha}\right]^{\frac{\rho}{1-\alpha}} \tag{75}$$

and (74) becomes

$$\hat{c}(g_L) = 1 - \beta\left[p_{LL} g_L^{1-\alpha} + p_{HL} x^{\frac{\alpha-1}{\rho}} g_H^{1-\alpha}\right]^{\frac{\rho}{1-\alpha}}. \tag{76}$$

Then (75) and (76) combine to imply that

$$x = \frac{1 - \beta\left[p_{HH} g_H^{1-\alpha} + p_{LH} x^{\frac{1-\alpha}{\rho}} g_l^{1-\alpha}\right]^{\frac{\rho}{1-\alpha}}}{1 - \beta\left[p_{LL} g_L^{1-\alpha} + p_{HL} x^{\frac{\alpha-1}{\rho}} g_H^{1-\alpha}\right]^{\frac{\rho}{1-\alpha}}}.$$

This can be solved for x and the result can be substituted in (75)and (76) to get $\hat{c}(g_H)$ and $\hat{c}(g_L)$. We can then use $\hat{c}(g_H)$ and $\hat{c}(g_L)$ and (63) to compute $r_f(g_H)$ and $r_f(g_L)$ where for $\alpha = H$ and $\alpha = L$,

$$r_f(g_\alpha) = \frac{E\left([\hat{c}(\tilde{g}_t)]^{-\frac{1-\alpha}{\rho}} \tilde{g}_t^{1-\alpha} \,|g_{t-1} = g_\alpha\right)^{1-\frac{\rho}{1-\alpha}}}{\beta E\left([\hat{c}(\tilde{g}_t)]^{-\frac{1-\rho-\alpha}{\rho}} \tilde{g}_t^{-\alpha} \,|g_{t-1} = g_\alpha\right)}.$$

Similarly we can use $\hat{c}(g_H)$, $\hat{c}(g_l)$ and (65) to compute $E\tilde{r}_t(g_H)$ and $E\tilde{r}_t(g_L)$ where for $\alpha = H$ and $\alpha = L$,

$$E\tilde{r}_t(g_\alpha) = \frac{E\left[\frac{\tilde{g}_t}{\hat{c}(\tilde{g}_t)} \,|g_{t-1} = g_\alpha\right]}{\beta\left[E\left([\hat{c}(\tilde{g}_t)]^{-\frac{1-\alpha}{\rho}} \tilde{g}_t^{1-\alpha} \,|g_{t-1} = g_\alpha\right)\right]^{\frac{\rho}{1-\alpha}}}.$$

In the steady state,

$$r_f = .5\left(r_f(g_H) + r_f(g_L)\right)$$

and

$$E\tilde{r}_t = .5\left(E\tilde{r}_t(g_H) + E\tilde{r}_t(g_L)\right).$$

4.1.1 The Cobb-Douglas Case

In this case, we let

$$\xi\left(g_{t-1}\right)=\begin{cases}\xi\left(g_H\right), & \text{if } g_{t-1}=g_H\\ \xi\left(g_L\right), & \text{if } g_{t-1}=g_L\end{cases}.$$

substituting $g_{t-1}=g_H$ and $g_{t-1}=g_L$ in

$$\begin{aligned}&\xi\left(g_{t-1}\right)\\ =\;& (1-\beta)^{(1-\beta)}\left[E\left(\left[\left(\xi\left(\tilde{g}_t\right)\tilde{g}_t\right)\right]^{1-\alpha}\left|g_{t-1}\right.\right)\right]^{\frac{\beta}{1-\alpha}}\end{aligned}$$

we get

$$\xi\left(g_H\right)^{\frac{1-\alpha}{\beta}}=\left[1-\beta\right]^{\frac{(1-\beta)(1-\alpha)}{\beta}}\left[\left[\xi\left(g_H\right)g_H\right]^{1-\alpha}p_{HH}+\left[\xi\left(g_L\right)g_L\right]^{1-\alpha}p_{LH}\right] \quad (77)$$

and

$$\xi\left(g_L\right)^{\frac{1-\alpha}{\beta}}=\left[1-\beta\right]^{\frac{(1-\beta)(1-\alpha)}{\beta}}\left[\left[\xi\left(g_H\right)g_H\right]^{1-\alpha}p_{HL}+\left[\xi\left(g_L\right)g_L\right]^{1-\alpha}p_{LL}\right]. \quad (78)$$

Letting

$$x=\left(\frac{\xi\left(g_H\right)}{\xi\left(g_L\right)}\right)^{1-\alpha}$$

(77) and (78) combine to imply that

$$x^{\frac{1}{\beta}}=\frac{xg_H^{1-\alpha}p_{HH}+g_L^{1-\alpha}p_{LH}}{xg_H^{1-\alpha}p_{HL}+g_L^{1-\alpha}p_{LL}}.$$

Then (77) implies that

$$\xi\left(g_H\right)^{1-\alpha}=\left[1-\beta\right]^{1-\alpha}\left[g_H^{1-\alpha}p_{HH}+x^{-1}g_L^{1-\alpha}p_{LH}\right]^{\frac{\beta}{1-\beta}}$$

and (78) implies that

$$\xi\left(g_L\right)^{1-\alpha}=\left[1-\beta\right]^{1-\alpha}\left[xg_H^{1-\alpha}p_{HL}+g_L^{1-\alpha}p_{LL}\right]^{\frac{\beta}{1-\beta}}.$$

Using these expressions, the risk free rate

$$r_f\left(g_{t-1}\right)=\frac{E^*\left(\tilde{g}_t\left|g_{t-1}\right.\right)}{\beta}$$

can be obtained for $g_{t-1}=g_H$ and $g_{t-1}=g_L$ by computing

$$E^*\left(\tilde{g}_t\left|g_{t-1}\right.\right)=\frac{E\left(\left[\xi\left(\tilde{g}_t\right)\tilde{g}_t\right]^{1-\alpha}\left|g_{t-1}\right.\right)}{\beta E\left(\xi\left(\tilde{g}_t\right)^{1-\alpha}\tilde{g}_t^{-\alpha}\left|g_{t-1}\right.\right)}$$

for each of these two possible g_{t-1} values.

4.2 Adjusting for Leverage

In the section of their paper [40] entitled "The effects of firm leverage", Mehra and Prescott point out that

> "The security priced in our model does not correspond to the common stocks traded in the U.S. economy. In our model there is only one type of capital, while in an actual economy there is virtually a continuum of capital types with widely varying risk characteristics. The stock of a typical firm traded in the stock market entitles its owner to the residual claim on output after all other claims including wages have been paid. The share of output accruing to stockholders is much more variable than that accruing to holders of other claims against the firm. Labor contracts, for instance, may incorporate an insurance feature, as labor claims on output are in part fixed, having been negotiated prior to the realization of output. Hence, a disproportionate part of the uncertainty in output is probably borne by equity owners.
>
> The firm in our model corresponds to one producing the entire output of the economy. Clearly, the riskiness of the stock of this firm is not the same as that of the Standard and Poor's 500 Composite Stock Price Index."[5]

We can adjust for the fact that the firms whose shares enter the S & P 500 Index are levered by computing the equity premium of a levered investment in the Lucas tree whose return we have calculated. In doing this we make an adjustment for leverage that is consistent with one made in Abel, [2]. In his paper, Abel calibrated a version of the continuous time Lucas asset pricing model in which consumers preferences have a "catching up with the Jones feature" which had been introduced in an earlier paper Abel, [1]. In the simple two state discrete time model calibrated here the adjustment for leverage is done by letting the return on the levered firm's equity be given by r_L where

$$r_L = \frac{D+E}{E} r_U - \frac{D}{E} r_D,$$

r_U is the return on the unlevered firm, r_D is the return on the firm's debt, D is the market value of the firm's outstanding debt and E is the market value of the firm's outstanding equity. If we assume that the risk of default on a levered investment in the Lucas tree is zero we can let

$$r_D = r_F.$$

Thus, the return on a levered investment in the Lucas tree is

$$r_L = r_U + \frac{D}{E}\left(r_U - r_F\right)$$

[5] Mehra-Prescott [40], p157.

while the equity premium paid by this levered investment is

$$r_L - r_F = \left(\frac{D+E}{E}\right)(r_U - r_F)\,.$$

Abel [2] noted that Masulis [36] had reported market debt to value ratios for US corporations in the range of .13 to .44 during the period from 1929 to 1986. We noted earlier that when we assume consistent planning by consumers with forward looking von Neuman Morgenstern preferences and let $\alpha = 25$ and $\sigma = .5$, we get a simulated unlevered equity premium of 2.6% and a simulated riskless rate of 3%. With a debt to value ratio of .4, the levered equity premium is

$$\frac{2.6\%}{.6} = 4.33\%.$$

When $\alpha = 25$ and $\sigma = 1$, the unlevered equity premium was 2.5% and the riskless rate was also 2.5%. With a debt to value ratio of .4 the levered equity premium is

$$\frac{2.5\%}{.6} = 4.2\%.$$

As noted EZW preferences yield virtually the same results in both of these cases.

5 Summary

This paper extends the generalized expected utility model of the 2009 *Journal of Mathematical Economics* paper, [22], to the case in which the returns to investment follow a Markov process. Using this generalization of the additively separable model, we derive the equity premium in a Lucas asset pricing equilibrium for the case in which consumption growth is Markov. The derivation of the equity premium in our earlier paper restricted attention to the case in which consumption growth was *i.i.d.* The extension to markov consumption growth makes it possible to extend the Mehra and Prescott, [40] calibration to the non-additively separable case. With additively separable preferences an equity premium close to the historical premium of just over 6% can be obtained with a relative risk aversion measure of 15. But in the additively separable case, the elasticity of substitution is the inverse of the relative risk aversion. So in that case, when the relative risk aversion is high, the elasticity of subsitution is, of necessity, low. As a consequence with additively separable preferences and a relative risk aversion measure of 15, the calibrated riskfree rate is 17% which is well above the historical average. When we calibrate the generalized expected utility model, riskfree rates and equity premia near the historical averages can be obtained by combining the assumption of a relatively high elasticity of substitution with the assumption that the relative risk aversion measure is also high. For example, with a relative risk aversion measure of 25 and an elasticity of substitution of .5 the simulated unlevered equity premium is 2.6% and the simulated riskless rate is 3%. With a debt to value ratio of .4 (which is in the range of the historical average), the unlevered equity premium of 2.6% becomes a levered equity premium of 4.33%. If, as is commonly done, we assume an elasticity of substitution equal to one, a relative risk aversion measure of 25 implies that the unlevered equity premium is 2.5%, the levered equity premium (again with a debt to value ratio of .4) is 4.2% and the simulated riskless rate is 2.5%. The case in which the elasticity of substitution is one is the case in which our generalized expected utility approach It is, of course, also true that, with EZW preferences, the assumption of a relatively high elasticity of substitution can be combined with the assumption that the relative risk aversion measure is also high and the same calibration can be done with those preferences. What is striking is that the calibration results just described for the generalized expected utility preferences are virtually the same as those obtained assuming EZW preferences with the same parameter values. As we have repeatedly noted, the EZW generalization of the additively separable preferences does not assume expected utility maximization but is dynamically consistent while our generalization does retain the assumption of expected utility maximization but is not dynamically consistent.

Throughout the paper we have attempted to relate the formal development of our approach, which relies heavily on the homotheticity of the preferences, to the standard development of the additively separable case and also to the EZW approach. Our development of the EZW approach differs from the usual development so that we can highlight the relationship of that approach to ours. It has also been necessary to present a separate development of what I refer to as the Cobb-Douglas case in which the elasticity of substitution is one. In the additively separable case, the Cobb-Douglas case is commonly called the "the log case."

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