# Bank Capital and Credit Cycles

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PRELIMINARY AND INCOMPLETE

**Abstract**

This paper proposes a simple theory of credit cycles that focuses on the role of bank capital and financing frictions. We build a continuous time general equilibrium model of an economy in which banks finance their loans by deposits and equity, while facing issuance costs when they raise new equity. The dynamics of the loan rate and the volume of lending in the economy are driven by the level of aggregate bank capitalization. The model has a unique Markov competitive equilibrium that can be solved in closed form. The explicit solutions facilitate the analysis of the full dynamics of the stochastic equilibrium. This dynamics is ergodic and typically exhibits quasi-cyclical patterns depending on the elasticities of credit demand, the fundamental volatility and the magnitude of issuance costs. We also perform a welfare analysis and show that lending decisions made by banks in a competitive equilibrium are socially inefficient.

**Keywords:** macro-model with a banking sector, credit cycles, bank capital

**JEL:** E21, E32, F44, G21, G28

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# 1 Introduction

Credit cycles, i.e., periodic variations of credit to GDP ratios, are a well documented phenomenon in many countries.[1] They differ from business cycles, and have in general a higher amplitude and a lower frequency. Credit booms (the upward sloping parts of the credit cycles) are sometimes followed by crises and recessions. Credit crunches (the downward sloping parts of the credit cycles) typically follow such recessions and crises. These credit cycles seem to be associated with an intertemporally inefficient allocation of capital to the productive sector.

Several theories have been put forward as potential explanations for credit cycles. The famous debt deflation mechanism identified by Fisher (1933) has been formalized by Bernanke et al. (1996) and Kiyotaki and Moore (1997). It attributes the origin of credit cycles to the fluctuations of the prices of the assets that are used as collateral by borrowers. Recent contributions by Lorenzoni (2008), Bianchi (2011), Jeanne and Korinek (2011) showed that collateral price fluctuations can be the source of welfare decreasing pecuniary externalities, which could justify countercyclical public policies. Such pecuniary externalities can also be generated by agency problems (see e.g. Gersbach and Rochet (2014)). Another strand of literature emphasizes the role of financial intermediaries, by pointing out that credit expansion is often accompanied by a loosening of lending standards and "systemic" risk-taking, whereas materialization of risk accumulated on the balance sheets of financial intermediaries leads to the contraction of credit (see e.g. Aitken et al. (2013), Dell'Ariccia and Marquez (2006), Jimenez and Saurina (2006)).

We propose an alternative but complementary explanation of the emergence of credit cycles that is rooted in external financial frictions faced by financial intermediaries. In our model, banks optimally adjust their lending volumes conditional on the level of equity capital. However, their capacity to adjust the level of equity is more limited, because

[1] See e.g. the recent contributions of Schularick and Taylor (2012), Jorda, Schularick and Taylor (2011a,b), Aikman, Haldane and Nelson (2013), Claessens, Kose and Terrones (2008, 2011a,b), Drehmann, Borio and Tsatsaronis (2012).

banks incur issuing costs when undertaking recapitalizations. As a result, temporary losses on lending activities may have a persistent impact on a bank's capital and, therefore, on credit supply.

Following Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013), we use a continuous time stochastic set-up. We model an economy where firms borrow from banks, and banks finance themselves by deposits and equity. Banks have diversified portfolios of loans, so that they are only affected by aggregate shocks that affect the average probability of default of the borrowers. These aggregate shocks are i.i.d. and represent the only source of risk in the economy. Bankers continuously adjust lending so as to maximize shareholder value. They also decide when to distribute dividends and when to issue new equity. The aggregate supply of bank credit is confronted with the firms' demand for credit, which decreases with the nominal loan rate. Credit supply and demand are equalized by the equilibrium loan rate.

In the absence of financial frictions (i.e., zero issuing cost of equity for banks) the equilibrium volume of lending and the nominal loan rate would be constant. The spread between the loan and deposit rates would equal the break-even rate reflecting the unconditional average of the firms' default probability. Aggregate risk would entirely be borne by the banks' shareholders: they would immediately distribute all profits as dividends and would inject new equity to offset losses. As a result, in this frictionless environment, there would be no aggregate fluctuations of credit and only small, i.i.d. fluctuations of output due to aggregate shocks.

In our model with financial frictions, the story is much more interesting. We are able to establish the existence of a unique competitive equilibrium, where the loan rate follows a Markov diffusion process reflected at two boundaries. At the lower boundary banks distribute dividends, whereas at the upper boundary they issue new equity. The equilibrium loan rate is a sufficient statistics for all relevant macro and financial variables: credit, output, bank leverage, bank equity (both book and market values) are deterministic functions of the equilibrium loan rate. We are able to obtain surprisingly simple explicit or quasi explicit expressions for all these functions, which substantially facilitates the analysis of the long-run behavior of the economy. For the most parameter values, the loan rate exhibits a quasi-cyclical behavior, which essentially comes from its reflection on both ends of its support. The frequency of the cycles depends on the banks' exposure to aggregate shocks and the magnitude of financing frictions. Moreover, the volatility of the loan rate (*endogenous volatility*) turns out to be the main determinant of the long run behavior of the economy. In particular, in the long run, the economy spends most of the time at the states with the lowest endogenous volatility, which under some parameter values gives rise to the persistent credit crunches.

The rest of the paper is structured as follows. Section 2 presents the model. In Section 3 we solve for the equilibrium. Section 4 illustrates the long run behavior of the economy and the role of bank capital in the propagation of aggregate shocks. In Section 5 we perform a welfare analysis. Section 6 concludes. All proofs are relegated to the Appendix.

# 2 The model

We consider a general equilibrium model in continuous time. There is only one physical good, taken as a numeraire, which can be consumed or invested. There are three types of agents: (i) depositors, who only play a passive role, (ii) investors, who own and manage the banks, and (iii) entrepreneurs, who manage the productive sector. Depositors are infinitely risk averse and discount the future at rate $r$. Investors and entrepreneurs are more impatient (their discount rate is $\rho > r$) but they are risk neutral.

## 2.1 Productive sector

The productive sector consists of a continuum of entrepreneurs controlling investment projects that are parametrized by a productivity parameter $x$. The productivity parameter $x$ is privately observed by each entrepreneur and is distributed according to a continuous distribution with density $g(x)$.

Although our model is in continuous time, we will start by presenting the production technology in discrete time set-up and then let the length $\Delta t$ of each period go to zero. Entrepreneurs' projects are short lived and each of them requires an investment of one unit of good. If successful, a project yields $(1 + x\Delta t)$ units of good in the next period and zero otherwise. Entrepreneurs have no own funds and thus must borrow from banks. They are protected by limited liability and default when their projects are not successful. Given a nominal loan rate $R\Delta t$ (for a loan of duration $\Delta t$), only the projects such that $x > R$ will be financed. Thus, the total volume of bank credit in the economy (that is also equal to the total volume of investment) will be

$$K(R) = \int_R^\infty g(x)dx.$$

The probability of default of a project of productivity $x$ is given by

$$\pi(x, \varepsilon_t) \equiv p(x)\Delta t + \Delta p(x)\sqrt{\Delta t}\varepsilon_t,$$

where $\varepsilon_t$ represents an aggregate shock faced simultaneously by all firms. For simplicity $\varepsilon_t$ is supposed to take only two values $+1$ (recession) and $-1$ (boom) with equal probabilities. Conditionally on the realization of $\varepsilon_t$, the net expected return per loan for a bank is

$$\mathbb{E}[R\Delta t(1 - \pi(x, \varepsilon_t)) - \pi(x, \varepsilon_t)|x > R] - r\Delta t =$$

$$(R - r)\Delta t - (1 + R\Delta t)(E[p(x)|x > R]\Delta t + E[\Delta p(x)|x > R]\sqrt{\Delta t}\varepsilon_t).$$

Taking the continuous time limit of the above expression, we obtain the net income per loan:

$$(R - r - E[(p(x)|x > R])dt - E[\Delta p(x)|x > R]dZ_t, \qquad (1)$$

where $\{Z_t, t \geq 0\}$ is a standard Brownian motion, the first term reflects the expected earnings per unit of time and the second term captures the exposure to aggregate shocks.

Throughout the paper, we will focus on the simple case where $p(x) \equiv p$ and $\Delta p(x) \equiv \sigma_0$, so that the unconditional probability of default and the exposure to aggregate shocks are the same for all firms. The net total output per period is then

$$dY_t = F(K(R))dt - \sigma_0 K(R) dZ_t, \tag{2}$$

where the aggregate production function $F(K)$ is defined implicitly by

$$F(K(R)) = \int_R^\infty x g(x) dx - pK(R).$$

Notice that $F'(K(R)) = R - p$ so that total surplus $F(K(R)) - rK(R)$ is maximized for $R_{FB} = r + p$. Thus, in the first best allocation of credit, the cost of funding for firms has two components: the riskless rate and the unconditional probability of default. Consequently, banks make zero expected profit and the total volume of credit in the economy is given by $K(R_{FB})$.

## 2.2 Banking sector

Banks behave competitively and finance loans to businesses by a combination of deposits and equity. Since we focus on credit, we do not introduce explicit liquidity provision activities associated with bank deposits. These deposits are modeled in a parsimonious fashion: depositors are infinitely risk averse and have a constant discount factor $r$. This implies two things: first, deposits must be absolutely riskless (all the risks will thus be borne by bank shareholders); second, depositors are indifferent to the level of deposits and timing of withdrawal provided that they receive an interest rate $r$. In sum, banks can collect any amount of deposits (i.e., deposits represent an infinitely inelastic source of funding) provided they pay the interest rate $r$ and fully guarantee their value.

The main financial friction in the model is that banks face a proportional issuance cost

$\gamma$ when they want to issue new equity.[2] Because of this deadweight issuance cost, banks will be reluctant to issue new equity too often and will mostly rely on retained earnings as a way to accumulate capital. For simplicity, we will neglect other external frictions such as adjustment costs for loans or fixed costs of issuing equity.[3] This implies that our economy exhibit a homotheticity property: all banks' decisions (lending, dividends, recapitalization) will be proportional to their equity levels. In other words, all banks will make the same decisions at the same moment, up to a scaling factor equal to their equity level. This entails an important simplification: only the aggregate size of the banking sector reflected by the aggregate bank capitalization will matter for our analysis, whereas the number of banks and their individual sizes will not play any role.

In this context, it is legitimate to anticipate the existence of a Markovian competitive equilibrium, where all aggregate variables depend on a single state variable, which itself follows a Markov diffusion process. In such an equilibrium, aggregate bank credit $K_t$, aggregate bank equity $E_t$ and the loan rate $R_t$ are perfectly correlated. It turns out that it is convenient to use the loan rate $R_t$ as the state variable, and look for the deterministic function $E_t = E(R_t)$ and a Markovian dynamics:

$$dR_t = \mu(R_t)dt + \sigma(R_t)dZ_t, \tag{3}$$

that are compatible with the equilibrium conditions.

# 3 The competitive equilibrium

Our objective in this section is to characterize the competitive equilibrium by mapping the history of fundamental shocks $\{dZ_s, s \leq t\}$ into the dynamics of the risk-adjusted spread $R_t$ and aggregate bank capital $E_t$. We start by characterizing the optimal policies

[2]For the empirical estimates of the proportional equity issuance costs see e.g. Hennessy and Whited (2007).

[3]We also disregard any frictions caused by governance problems inside the banks or government explicit/implicit guarantees.

of an individual bank, while taking the loan rate $R_t$ as given. Then, we proceed with aggregation and determine the equilibrium dynamics of the loan rate and its mapping into the value of aggregate bank capital. For the rest of the paper we assume that $r = 0$ and treat the case $r > 0$ in Appendix B.

### 3.1 Profit-maximization problem of an individual bank

Consider first the optimal decision problem of a bank as a function of its level of equity $e_t$ and loan rate $R_t$. In a competitive equilibrium, bank shareholders take the loan rate $R_t$ as given and choose lending $k_t \geq 0$, dividend $d\delta_t \geq 0$ and recapitalization $di_t \geq 0$ policies so as to maximize the market value of equity:[4]

$$v(e, R) = \max_{k_t, d\delta_t, di_t} \quad \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\left(d\delta_t - (1+\gamma)di_t\right)\right], \tag{4}$$

where $R_t$ evolves according to (3) and the equity value follows

$$de_t = k_t[(R_t - p)dt \quad - \sigma_0 dZ_t] - d\delta_t + di_t. \tag{5}$$

The following proposition formalizes the conditions under which the above problem has a non-degenerate solution.

**Proposition 1** *Consider the Markov process $R_t$, defined implicitly by the diffusion equation*

$$dR_t = \mu(R_t)dt + \sigma(R_t)dZ_t,$$

*on the interval $[R_{min}, R_{max}]$, where both boundaries are reflecting, $\mu(.)$ and $\sigma(.)$ are continuously differentiable and $\sigma(.) > 0$.*

*The maximization problem of an individual bank has a non-degenerate solution if and*

[4]Throughout the paper, we use lower case letters for individual variables and upper case letters for aggregate variables.

*only if:*

$$\int_{R_{min}}^{R_{max}} \frac{R-p}{\sigma_0 \sigma(R)} dR = \log(1+\gamma) \tag{6}$$

*and*

$$\mu(R) = \frac{\sigma(R)}{2} \left( \frac{2\rho\sigma_0 - \sigma(R)}{R-p} - \frac{R-p}{\sigma_0} + \sigma'(R) \right). \tag{7}$$

A fundamental property of the individual decision problem of a bank is that the feasible set, in terms of trajectories of $(k_t, d\delta_t, di_t)$, and the objective function are homogenous of degree one in $e_t$. Therefore, the value function itself must satisfy:

$$v(e, R) = eu(R),$$

where $u(R)$ can be thought of as the market-to-book value of equity for banks.

Using the above property and applying standard dynamic programming methods (see Appendix), it can be shown that the optimal lending policy of the bank is indeterminate, i.e., bank shareholders are indifferent with respect to the volume of lending. The latter is entirely determined by the firms' demand for credit. This situation if analogous to the case of an economy with constant returns to scale, where the equilibrium price of any output is only determined by technology (constant marginal cost) and the volume of activity is then determined by the demand side.

The optimal dividend and recapitalization policies are of the so-called "barrier type" and depend uniquely on the market-to-book value $u(R)$, which reflects the marginal value of equity capital. In particular, dividends are distributed only when $R_t = R_{min}$, where $R_{min}$ is such that $u(R_{min}) = 1$. In other words, distribution of dividends only takes place when the market-to-book ratio equals one. Recapitalizations occur only when $R_t = R_{max}$, where $R_{max}$ satisfies $u(R_{max}) = 1 + \gamma$, i.e., when the marginal value of equity equals the marginal cost of equity issuance. As long as the loan rate $R$ lies strictly in between $R_{min}$ and $R_{max}$, bank equity only changes due to retained earnings/absorbed losses.

Note that the loan risk-adjusted spread[5] $R - p$, is strictly positive in the region $(R_{min}, R_{max})$. To see the intuition for this property, it is instructive to consider the marginal impact of lending on expected shareholder value:

$$\mathbb{E}\left[\frac{d(e_t u(R_t))}{dk_t}\right] = \mathbb{E}\left[\frac{u(R_t)de_t + e_t u'(R_t)dR_t + u'(R_t)dR_t de_t}{dk_t}\right], \tag{8}$$

which, after easy computations, reduces to

$$[(R_t - p)u(R_t) - \sigma_0 \sigma(R_t) u'(R_t)]dt. \tag{9}$$

The first-order condition with respect to $k_t$ thus implies that

$$R_t - p = \sigma_0 \sigma(R_t) \frac{u'(R_t)}{u(R_t)}. \tag{10}$$

The left-hand side of expression (10) captures the expected profit from lending, whereas its right-hand side can be interpreted as the marginal cost of lending. In fact, even though shareholders are risk-neutral, they recognize that aggregate shocks have an impact on $R_t$ and thus on the value-to-book ratio. As can be easily seen from the expressions describing the dynamics of $e_t$ and $R_t$, a loss ($dZ_t > 0$) depletes bank equity $e_t$, while pushing up the loan rate $R_t$ and thus the value of $u(R_t)$. As a result, the adverse impact of a loss on shareholder value is magnified. Symmetrically, a gain ($dZ_t < 0$) translates into a reduction of $u(R_t)$, which reduces the impact of positive profits on shareholder value. This mechanism gives rise to a kind of *induced risk aversion*, which actually explains why bankers require a positive spread for accepting to lend, even though they are risk-neutral.

The market-to-book ratio function can be easily computed by integrating relation (10)

[5] In the general case where $r > 0$, the loan risk-adjusted spread is given by $R - p - r$.

between $R_{min}$ and $R$, while using the boundary condition $u(R_{min}) = 1$. This yields:

$$u(R) = exp\Big(\int_{R_{min}}^{R} \frac{s-p}{\sigma_0\sigma(s)}ds\Big). \tag{11}$$

## 3.2 Equilibrium

Having determined the optimal policies of an individual bank, we are now ready to proceed with aggregation. In the region $R \in (R_{min}, R_{max})$, aggregate bank capital evolves according to

$$dE_t = K(R_t)[(R_t - p)dt - \sigma_0 dZ_t], \tag{12}$$

where $K(R_t)$ is the aggregate demand for bank loans.

An equilibrium is characterized by two stochastic processes: a loan rate $R_t$ and aggregate bank capital $E_t$ such that the lending market clears. We focus on equilibria that are Markovian with a single state variable $R_t$. If such an equilibrium exists, one must have $E_t = E(R_t)$. From Itô's lemma, it follows that:

$$dE_t = \left(\mu(R_t)E'(R_t) + \frac{\sigma^2(R_t)}{2}E''(R_t)\right)dt + \sigma(R_t)E'(R_t)dZ_t. \tag{13}$$

Matching the drift and volatility terms of $E_t$ defined by (12) and $E(R_t)$ defined by (13) yields a system of two equations:

$$(R-p)K(R) = \mu(R)E'(R) + \frac{\sigma^2(R)}{2}E''(R), \tag{14}$$

$$-\sigma_0 K(R) = \sigma(R)E'(R). \tag{15}$$

Moreover, recall that, from Proposition 1, we know the expression of $\mu(R)$ as a function of $\sigma(R)$. Solving the system of equations (14)-(15), while taking into account expression (7), enables us to obtain the explicit characterization of the equilibrium (see Appendix for the details).

**Proposition 2** *For any given loan demand function $K(.)$, there exists a unique Markov* *equilibrium characterized by a volatility function*

$$\sigma(R) = \frac{[2\rho\sigma_0^2 + (R-p)^2]K(R)}{\sigma_0[K(R) - (R-p)K'(R)]}. \tag{16}$$

*The drift function is defined by*

$$\mu(R) = \frac{\sigma(R)}{2}\left(\frac{\sigma(p) - \sigma(R)}{R-p} - \frac{R-p}{\sigma_0} + \sigma'(R)\right). \tag{17}$$

*Reflecting boundaries are given by $R_{min} = p$ and $R_{max}$ such that*

$$\int_p^{R_{max}} \frac{R-p}{\sigma_0\sigma(R)}dR = \log(1+\gamma). \tag{18}$$

*The aggregate bank capital function $E(R)$ is given by*[6]

$$E(R) = \int_R^{R_{max}} \frac{\sigma_0 K(s)}{\sigma(s)}ds. \tag{19}$$

Thus, the dynamics of $R_t$ and $E_t$ depend on the credit demand function $K(R)$ and four parameters: exposure to aggregate shocks (or fundamental volatility) $\sigma_0$, the unconditional probability of default $p$, discount factor $\rho$ and financing frictions $\gamma$. In equilibrium, the loan rate fluctuates in between its first-best level $p$ and $R_{max}$ that is increasing with the magnitude of financing frictions, $\gamma$.[7] Observe that bank capital is also increasing with $R_{max}$. Thus, stronger financing frictions will induce banks to hold more capital for any given level of $R$.[8] In the absence of financing frictions, i.e., when $\gamma = 0$, the recapitalization barrier would coincide with the payout barrier, so that the loan rate will be permanently fixed at $R_t \equiv p$ implying the first-best allocation of credit.

[6]Note that $E(R_{max}) = 0$.

[7]Interestingly, financing frictions only affect $R_{max}$, without having any impact on $\mu(R)$ and $\sigma(R)$.

[8]It also follows that market capitalization of the banking sector, $V(R) = u(R)E(R)$, is increasing with the magnitude of financing frictions.

# 4 Credit cycles and financial (in)stability

In this section we discuss the properties of the equilibrium and the behavior of the economy in the long run. We start by applying the impulse response methodology to study the stability of the deterministic steady state. Then, we discuss the ergodic properties of the system, showing that the system behavior in a stochastic environment can be in sharp contrast to the behavior predicted by the analysis conducted in a deterministic setting.

## 4.1 Impulse response analysis

The usual methodology to analyze the long-term behavior of macro-variables in a DSGE model is to linearize around the deterministic steady-state and perturb the system by a single unanticipated shock. The equivalent here would be to look at the case where $dZ_t \equiv 0$ for $t > 0$. The system then becomes deterministic:

$$dR_t = \mu(R_t)dt,$$

and the initial shock determines $R_0 > p$.

For the purpose of our analysis, it is useful to rewrite the general expression of $\mu(R)$ stated in Proposition 2 in the following way:

$$\mu(R) \equiv \sigma(R)H(R), \tag{20}$$

where

$$H(R) = \frac{(R-p)^2 K'(R)}{\sigma_0[K(R)-(R-p)K'(R)]} + \frac{(R-p)[(R-p)^2 + 2\rho\sigma_0^2]K''(R)}{2\sigma_0[K(R)-(R-p)K'(R)]^2}. \tag{21}$$

It is easy to see from expression (21) that $\mu(p) = 0$. Hence, the frictionless loan rate ($R_t \equiv p$) is an equilibrium of the deterministic system that is further referred to as the deterministic steady-state (DSS). It is *locally* stable when $\mu'(p) < 0$ and is *globally* stable when $\mu(R) < 0$ for all $R$. In the neighborhood of the DSS, $\mu'(R)$ can be approximated by

$$\mu'(R) \approx \mu(p) + \mu'(p)(R-p) = 2\rho^2\sigma_0^2 \frac{K''(R)}{K(R)}(R-p).$$

Hence, the DSS is locally stable when $K''(R) < 0$. Moreover, it also follows from (21), that condition $K''(R) < 0$ ensures global stability.

**Proposition 3** *a) When $K''(R) < 0$, the DSS is globally stable. b) When $K''(R) > 0$, the DSS is locally unstable and the trend $\mu(R)$ has at least one change of sign over the interval $[p, R_{max}]$.*

To illustrate the properties of the equilibrium, consider the following specification of the loan demand function:

$$K(R) = (\overline{R} - R)^{\beta}, \tag{22}$$

where $\beta > 0$ and $p < \overline{R}$.

Under the above specifications, the volatility of the loan rate is

$$\sigma(R) = \frac{[2\rho\sigma_0^2 + (R-p)^2]\,(\overline{R} - R)}{\sigma_0[\overline{R} + (\beta - 1)R - \beta p]}. \tag{23}$$

The drift of the loan rate is given by

$$\mu(R) = \sigma(R)\frac{\beta(R-p)Q(R)}{2\sigma_0[\overline{R} + (\beta-1)R - \beta p]^2}, \tag{24}$$

where $Q(R)$ is a quadratic polynomial:

$$Q(R) = (1-\beta)((R-p)^2 - 2\rho\sigma_0^2) - 2(R-p)(\overline{R} - p). \tag{25}$$

With the above specification, it is easy to see that, in the neighborhood of $p$, $\mu'(R)$ has the same sign as the polynomial $Q(R)$. It can be shown that $Q(R)$ is increasing with $p$ for any level of $R \in [p, \overline{R}]$ and $p < \overline{R}$. For $R = p$ and $p \to \overline{R}$, we have $\lim Q(p) \to$

$-2\rho\sigma^2(1-\beta)$. Hence, it follows that, when $\beta < 1$ (which is equivalent to $K''(R) < 0$), we have $\mu'(p) < 0$ and thus the DSS is *locally* stable. It is also easy to see that, when $R = \overline{R}$ and $p \to \overline{R}$, we have $\lim Q(\overline{R}) = Q(p) < 0$, which guarantees that $\mu(R) < 0$ in the entire interval $[p, \overline{R}]$.[9] Thus, the DSS is also *globally* stable when $\beta < 1$, which corresponds to a low elasticity of credit demand.

Consider now the case in which $\beta > 1$ (which is equivalent to $K''(R) > 0$). In this case $\mu'(p) > 0$ (i.e., the DSS is locally unstable), and there exists a unique $R^* \in (p, \overline{R})$ such that $\mu(R)$ is positive in the region $(0, R^*)$ and negative in the region $(R^*, \overline{R})$.

## 4.2 Long run behavior in the stochastic set-up

After studying the properties of the deterministic equilibrium, we consider the full dynamics of the stochastic equilibrium. It turns out that the system is ergodic and thus the long run behavior of the economy can be described by the ergodic density function. The ergodic density measures the average time spent by the economy in the neighborhood of each possible loan rate $R$: the states with lower $R$ can be interpreted as "boom" states and the states with higher $R$ can be thought of as "bust" states.

**Proposition 4** *If $\sigma(R) > 0$ for $\forall R \in [p, R_{max}]$, there exists a unique ergodic distribution of $R$ characterized by the density function*

$$f(R) = \frac{C_0}{\sigma^2(R)} exp\Big(\int_p^R \frac{2\mu(s)}{\sigma^2(s)} ds\Big), \tag{26}$$

*where the constant $C_0$ is such that $\int_p^{R_{max}} f(R)dR = 1$.*

It can easily be seen from the expression of $\sigma(R)$ provided in Proposition 2 that, for any loan demand specifications such that $K'(R) < 0$ and $K(R) > 0$ in the region $[p, R_{max}]$, the volatility of the loan rate remains strictly positive. Thus there exists an ergodic distribution of $R$. By differentiating the logarithm of the ergodic density defined

[9]Note that the interval $[p, \overline{R}]$ comprises the interval $[p, R_{max}]$.

in (26), we obtain:

$$\frac{f'(R)}{f(R)} = \frac{2}{\sigma(R)} \left( \frac{\mu(R)}{\sigma(R)} - \sigma'(R) \right). \tag{27}$$

By using the general formulas for $\sigma(R)$ and $\mu(R)$ defined in Proposition 2, it can be shown that $\sigma(p) = 2\rho\sigma_0$, $\sigma'(p) = 2\rho\sigma_0 \frac{K'(R)}{K(R)} < 0$ and $\mu(p) = 0$. Hence, $f'(p) > 0$, which means that the state $R = p$ corresponding to the DSS is definitely *not* the one at which the economy spends most of the time in the stochastic set up. To get a deeper understanding of the determinants of the system behavior in the long run, we resort to the numerical example. Figure 1 reports the typical patterns of the endogenous volatility $\sigma(R)$ (left panel) and the ergodic density $f(R)$ (right panel) for the loan demand specification defined in (22).

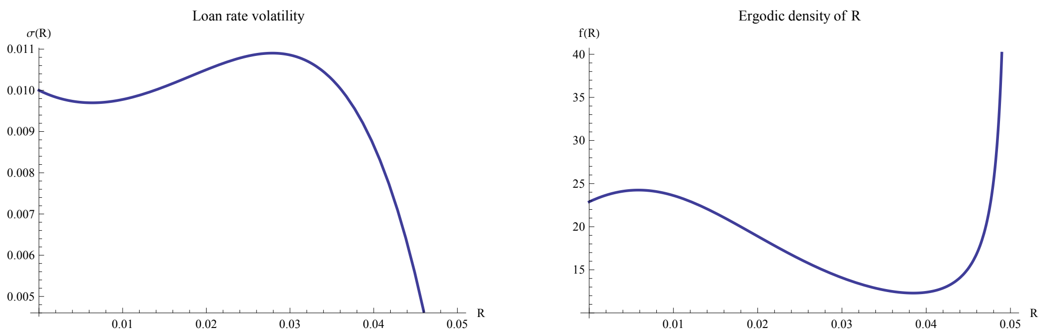


Figure 1: Endogenous volatility and ergodic density functions

Figure 1 shows that the extrema of the ergodic density almost coincide with those of the endogenous volatility function, i.e., the economy spends most of the time at the states with the lowest endogenous volatility. Intuitively, the economy can get "trapped" in the states with low endogenous volatility because the endogenous drift is generally too small to move it away from these states. In fact, $\sigma(R)$ turns out to be several times larger than $\mu(R)$ for any level of $R$.[10]

Note that functions $\sigma(.)$ and $f(.)$ must be truncated (and, in the case of the ergodic density, rescaled) on $[p, R_{max}]$, where $R_{max}$ depends on the magnitude of issuing costs $\gamma$. For the chosen specification of the loan demand function, $K(R) = (\overline{R} - R)^{\beta}$, we always

[10] Formally, this can be observed from the expression (24): in fact, $\mu(R) \equiv \sigma(R)H(R)$, where $H(R)$ is typically very small.

have $R_{max} < \overline{R}$. However, $R_{max}$ can be arbitrary close to $\overline{R}$. In that case the economy will spend quite some time in the region where the loan rate is close to $R_{max}$. We interpret this situation as a persistent "credit crunch": it manifests itself via scarce bank equity capital, high loan rates and low volumes of lending.

This "credit crunch" scenario is reminiscent to the "net worth trap" documented by Brunnermeier and Sannikov (2014) and Isohätälä, Milne and Roberston (2014). However, the existence of the slow-recovery region in our setting has a different "raison d' être" than in the above-mentioned papers. Specifically, when adverse shocks deplete bank capital, banks reduce their risk exposure by cutting down credit supply. In fact, scaling down operations enables banks to manage their risk so as to reduce the probability of costly recapitalizations. However, in the environment with high aggregate risk $\sigma_0$ and low elasticity of loan demand, banks will struggle for a long time to rebuild equity, so that the economy may be locked in a long phase of recession.

To sum up, the analysis conducted in this section suggests that the long run behavior of the economy in the stochastic environment are determined by the endogenous volatility, rather than the endogenous drift. Thus, relying on the results of the impulse response analysis in order to draw the insights about the long-run behavior of the economy in the stochastic environment might be misleading.

### 4.3 Cycles

Besides the ergodic distribution, which describes the long term average behavior of the economy, it is also important to look at the spectral distribution of $R_t$, which provides information about the cyclical behaviors of the economy. To get an intuition about the determinants of these cyclical behaviors, we use the observation made in the previous section that, for most parameter values, the ratio $\frac{\mu(R)}{\sigma(R)}$ is close to zero and $\sigma(R)$ is almost constant. Thus, the cyclical behaviors of $R_t$ can be approximated by the diffusion process with a constant volatility $\overline{\sigma} = E[\sigma(R)]$, which is reflected at the both ends of the interval $[p, R_{max}]$. It turns out that the spectral behavior of this reflected Brownian motion is entirely determined by the ratio $\frac{\overline{\sigma}}{R_{max}-p}$. Indeed, the eigenvalues of the associated Sturm-Liouville operator (see Linetsky (2005)) are given by

$$\lambda_n = \frac{1}{2}\Big(\frac{\pi n \overline{\sigma}}{R_{max} - p}\Big)^2, \qquad n = 1, 2, .... \tag{28}$$

When the ratio $\frac{\overline{\sigma}}{R_{max}-p}$ is small, these eigenvalues are clustered around zero, and the process $R_t$ resembles a pure white noise. However, when $\frac{\overline{\sigma}}{R_{max}-p}$ is large, the eigenvalues are more spread out and the system exhibits quasi-cycles of frequency

$$\frac{\sqrt{\lambda}}{2\pi} = \frac{1}{2\sqrt{2}} \frac{\overline{\sigma}}{R_{max} - p}.$$

# 5 Welfare analysis

In our simple set-up where deposit taking does not generate any surplus, social welfare can easily be computed as the sum of the market value of the firms (i.e., the expected discounted profit of the productive sector) and the market value of the banks' equity. In this section, we show that the competitive allocation of credit does not maximize social welfare, under the constraint that the government is subject to the same frictions as the private investors. This means that the government cannot directly transfer wealth (through taxes and subsidies) between the productive and the banking sectors. In this set up, it is natural to take as a state variable the total capitalization $E_t$ of the banking sector, rather than the loan rate $R_t$. We begin by reformulating our characterization of the competitive equilibrium by using $E_t$ as a state variable. We then show how social welfare can be increased by modifying the competitive allocation of credit $K_c(E_t)$.

## 5.1 Competitive equilibrium with aggregate equity as the state variable

Suppose we want to characterize directly the competitive allocation of credit $K_c(E)$ and the loan rate $R_c(E)$ as functions of aggregate equity $E$, which of course implies $K_c(E) = K[R_c(E)]$. Let $E_{min} = E(R_{max}) \equiv 0$ and $E_{max} = E(p)$ denote respectively the minimum and the maximum leves of aggregate bank equity in the economy. In the region $[E_{min}, E_{max}]$, the dynamics of aggregate equity $E_t$ satisfies

$$dE_t = K_c(E_t)[(R_c(E_t) - p)dt - \sigma_0 dZ_t], \tag{29}$$

whereas the equity of a particular bank $e_t$ follows

$$de_t = k_t[(R_c(E_t) - p)dt - \sigma_0 dZ_t], \tag{30}$$

as a function of its lending volume $k_t$.

Using $e_t$ and $E_t$ as the state variables in the maximization problem of an individual bank and applying the same arguments as before, yields the condition

$$\frac{\hat{u}'(E)}{\hat{u}(E)} = -\frac{R_c(E) - p}{\sigma_0^2 K_c(E)}, \tag{31}$$

where $\hat{u}(E) = u(R_c(E))$.

Observe that $\hat{u}'(E) = u'(R_c(E))R_c'(E)$ and from (10) we have $u'(R) = u(R)\frac{R-p}{\sigma_0\sigma(R)}$. Hence, it follows that

$$R_c'(E) = -\frac{\sigma(R)}{\sigma_0 K_c(E)} = -\frac{(R_c(E) - p)^2 + 2\rho\sigma_0^2}{\sigma_0^2[K(R_c) - (R_c(E) - p)K'(R_c)]}. \tag{32}$$

Note that, when $E$ is used as the state variable, the equilibrium loan rate $R_c(E)$ cannot be determined explicitly. This is precisely the reason why we have chosen to use $R$ as the state variable in our core analysis. However, for the welfare analysis, the former approach is more natural. Moreover, it also allows us to express the marginal (private) cost of lending as a function of the aggregate capitalization of the banking sector. Indeed, condition (31) can be rewritten as

$$R_c(E) - p = \sigma_0^2 K_c(E)\Big[ -\frac{\hat{u}'(E)}{\hat{u}(E)}\Big], \tag{33}$$

where the left-hand and the right-hand sides capture the marginal benefit and the marginal cost of lending for the bank respectively.[11] Notice that $\hat{u}'(E) = u'(R_c(E))R_c'(E) < 0$, so that the right-hand side of (33) is positive. As will be shown in the sequel, the private marginal cost of credit in a competitive equilibrium diverges from the social marginal cost of credit.

## 5.2 Computing social welfare

For $E \in [E_{min}, E_{max}]$, the social welfare function at the competitive equilibrium, $W(E)$, satisfies the following differential equation

$$\rho W(E) = \pi_F\Big[K_c(E)\Big] + \pi_B\Big[K_c(E)\Big]W'(E) + \frac{\sigma_0^2}{2}K_c^2(E)W''(E), \tag{34}$$

where $\pi_F(K_c) = F(K_c) - K_c F'(K_c)$ is the expected profit of the firms per unit of time and $\pi_B(K_c) = K_c F'(K_c)$ is the expected profit of the banks per unit of time.

Note that dividend distribution and bank recapitalizations have no immediate impact on the firms' profit. This consideration yields us two boundary conditions, $W'(E_{min}) = V'(E(R_{max})) = 1 + \gamma$ and $W'(E_{max}) = V'(E(R_{min})) = 1$. Thus, the welfare function can be computed numerically. To illustrate the relation between the level of the aggregate bank capitalization and welfare, we compute $W(E)$ in the simple case where $\beta = 1$ (see Appendix C for the details). The knowledge of $W(E)$ enables us to easily compute the

[11] This condition is analogous to condition (10) obtained in Section 3.1.

expected social welfare loss as a function of the bank capital loss, i.e.,

$$\frac{W(E_0) - W(E_0 - \Delta E)}{W(E_0)} 100\%,$$

where $\Delta E > 0$ is the loss of bank capital and $E_0$ is the level of bank capital before the loss.

The left panel of Figure 2 shows that social welfare is an increasing function of aggregate bank capitalization.[12] The right panel of Figure 2 reports the expected welfare loss as the function of bank capital loss, where $E_0 = E_{max}$ and $\Delta E \in [0, E_{max} - E_{min}]$.

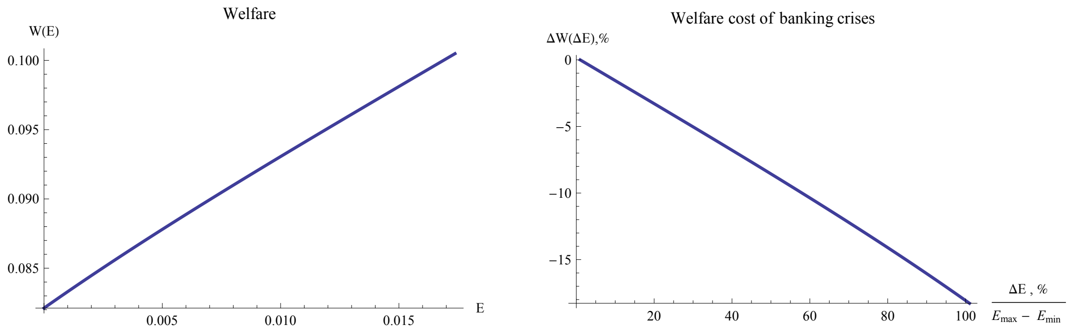


Figure 2: Social welfare and the welfare cost of banking crises ($\rho = 0.05$, $\beta = 1$, $p = 0.02$, $\overline{R} = 0.12$, $\sigma_0 = 0.1$, $\gamma = 0.2$)

## 5.3 Market failure

Consider now a small variation of lending around $K_c(E)$, given the aggregate bank equity level $E$. Its impact on social welfare is given by the first derivative of the right-hand side of equation (34) with respect to $K_c(E)$:

$$\begin{aligned}\mathcal{L}(E) = \pi_F'\Big[K_c(E)\Big] + \pi_B'\Big[K_c(E)\Big]W'(E) + \sigma_0^2 K_c(E)W''(E) = \\ = -K_c(E)F''(K_c(E)) + [K_c(E)F''(K_c(E)) + R_c(E) - p]W'(E) + \sigma_0^2 K_c(E)W''(E),\end{aligned} \tag{35}$$

[12]Note that the maximum level of capitalization is attained for $R = p$ and the maximum variation of welfare depends on the magnitude of financing costs.

where $F''(K_c(E)) = R'_c(K_c(E))$. The right-hand side of the above expression vanishes when the loan spread is equal to the social costs of lending, i.e., when

$$R_c(E) - p = \sigma_0^2 K_c(E)\Big[-\frac{W''(E)}{W'(E)}\Big] + R'_c(K_c(E))K_c(E)\Big[1 - \frac{1}{W'(E)}\Big]. \tag{36}$$

Comparing expressions (36) and (33) suggests that the allocation of credit in the competitive equilibrium is distorted in two ways: first, because of the difference in the private and social induced risk aversion, i.e., $IRA_p(E) = -\frac{\hat{u}'(E)}{\hat{u}(E)}$ and $IRA_s(E) = -\frac{W''(E)}{W'(E)}$; second, because an increase in lending decreases the loan spread (recall that $R'_c(K_c) < 0$) and thus reduces the marginal earnings of banks. Indeed, since $W'(E) > 0$, the second term at the right-hand side of expression (36) is negative.

In the particular case where $\beta = 1$, we are able to compute $IRA_p(E)$ and $IRA_s(E)$ and numerically compare the private and social cost of lending. The left panel of Figure 3 depicts the difference between the coefficients of the private and the social *induced* risk aversion, i.e., $IRA_p(E) - IRA_s(E)$. It turns out that the private induced risk aversion is lower than the social induced risk aversion when the aggregate bank equity is low and vise versa when the aggregate level of equity is high. However, the negative effect of the second term in (36) dominates, so that the private cost of lending is always higher than the social cost of lending (see the right panel of Figure 3).

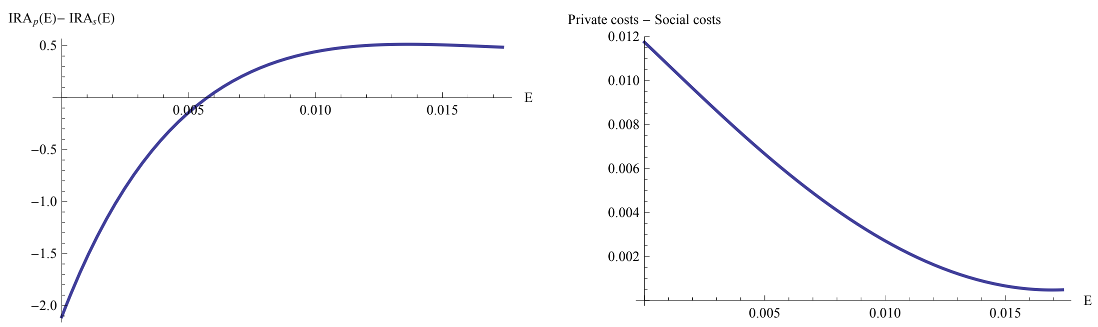


Figure 3: Private *vs* social costs of lending for $\beta = 1$

We are also able to estimate the sign of $\mathcal{L}(E)$ and thus to have a precise picture of the relation between the levels of bank capitalization and welfare distortions. It turns out that

$\mathcal{L}(E)$ is positively signed for the lower levels of bank capitalization and becomes negative for the higher level of capitalization (see Figure 4). This suggests that, for the lower levels of bank capital, welfare can be improved by increasing credit to the productive sector, whereas for the higher level of bank capitalization welfare can be improved by reducing credit. Put differently, competitive banks lend too much when things go well (high equity), and too little when things go badly (low equity).

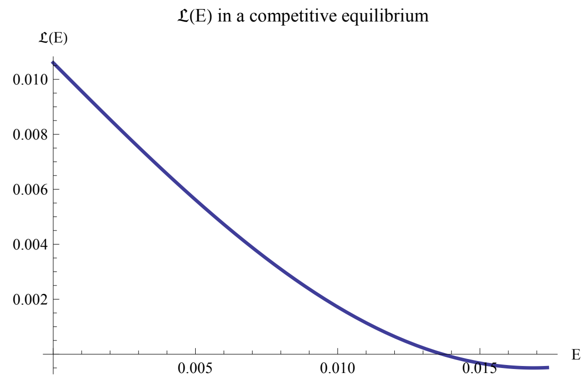


Figure 4: Market failure

# 6 Conclusion [To be completed]

# Appendix A. Proofs

**Proof of Proposition 1.** By the standard dynamic programming arguments, $v(e,R)$ must satisfy the Bellman equation:[13]

$$\begin{aligned}\rho v = \max_{k,d\delta,di} \Big\{ & d\delta(1-v_e) - di(1+\gamma-v_e)+ \\ & + k[(R-p)v_e - \sigma_0\sigma(R)v_{eR}] + \frac{k^2\sigma_0^2}{2}v_{ee} \\ & + \mu(R)v_R + \frac{\sigma^2(R)}{2}v_{RR}\Big\}.\end{aligned} \tag{A1}$$

Using the fact that $v(e,R) = eu(R)$, one can rewrite the Bellman equation (A1) as follows:

$$\begin{aligned}\rho u(R) = \max_{k,d\delta,di} \Big\{ & \frac{d\delta}{e}[1-u(R)] - \frac{di}{e}[1+\gamma-u(R)] + \frac{k}{e}[(R-p)u(R) - \sigma_0\sigma(R)u'(R)]+ \\ & + \mu(R)u'(R) + \frac{\sigma^2(R)}{2}u''(R)\Big\}\end{aligned} \tag{A2}$$

A solution to the maximization problem in $k$ only exists when

$$\frac{u'(R)}{u(R)} = \frac{R-p}{\sigma_0\sigma(R)}. \tag{A3}$$

It follows from the above expression that $u(R)$ is increasing with $R$. Then, the optimal payout policy maximizing the right-hand side of (A2) is characterized by a critical barrier $R_{min}$ satisfying

$$u(R_{min}) = 1, \tag{A4}$$

and the optimal recapitalization policy is characterized by a barrier $R_{max}$ such that

$$u(R_{max}) = 1+\gamma. \tag{A5}$$

In other words, dividends are only distributed when $R_t$ reaches $R_{min}$, whereas recapitalization occurs only when $R_t$ reaches $R_{max}$.

Conditions (A3), (A4) and (A5) can be summarized in a single condition:

$$\int_{R_{min}}^{R_{max}} \frac{R-p}{\sigma_0\sigma(R)}dR = \log(1+\gamma). \tag{A6}$$

To obtain the second condition stated in the Proposition 1, notice that in the region $R \in (R_{min}, R_{max})$, market-to-book value $u(R)$ satisfies:

$$\rho u(R) = \mu(R)u'(R) + \frac{\sigma^2(R)}{2}u''(R). \tag{A7}$$

[13] For the sake of space, we omit the arguments of function $v(e,R)$.

Taking the first derivative of (A3), we can compute $u''(R)$. Plugging $u''(R)$ and $u'(R)$ into (A7) and rearranging terms yields:

$$\mu(R) = \frac{\sigma(R)}{2}\left(\frac{2\rho\sigma_0 - \sigma(R)}{R-p} - \frac{R-p}{\sigma_0} + \sigma'(R)\right). \tag{A8}$$

**Proof of Proposition 2.** First, we derive the expression of $\sigma(R)$. From (15), we have

$$E'(R) = -\frac{\sigma_0(R)}{\sigma(R)}K(R). \tag{A9}$$

By differentiating the above equation, we obtain $E''(R)$. Plugging $E'(R)$ and $E''(R)$ into (14) and solving it with respect to $\mu(R)$ yields:

$$\mu(R) = \frac{\sigma(R)}{2}\left(-\frac{2(R-p)}{\sigma_0} - \sigma(R)\frac{K'(R)}{K(R)} + \sigma'(R)\right) \tag{A10}$$

Recall that we have another expression for $\mu(R)$ resulting from the individual bank's maximization problem (see (A8)). Equilibrium implies the existence of a unique $\mu(R)$. Equating the right-hand sides of (A10) and (A8) yields the expression for $\sigma(R)$. Note that $\sigma(p) = 2\rho\sigma_0$.

Second, from the maximization problem of the individual bank, we know that the maximum value of loan rate, $R_{max}$, is the solution of equation $u(R_{max}) = 1 + \gamma$, which can be rewritten as follows:

$$\int_{R_{min}}^{R_{max}} \frac{R-p}{\sigma_0\sigma(R)}dR = \log(1+\gamma). \tag{A11}$$

To finalize the characterization of the equilibrium, it remains to determine $R_{min}$ and $E(R_{max})$. To this purpose, consider first derivative of the market value of the entire banking sector, $V(R) \equiv E(R)u(R)$. At $R_{min}$ and $R_{max}$, we must have $V'(R_{min}) = E'(R_{min})$ and $V'(R_{max}) = (1+\gamma)E'(R_{max})$. This implies respectively that $u'(R_{max})E(R_{max}) = 0$ and $u'(R_{min})E(R_{min}) = 0$. Notice that $u'(R_{max}) > 0$ and $E(R_{min}) > 0$ (the latter must hold because the value equity is decreasing with $R$). These considerations yield us two conditions: $E(R_{max}) = 0$ and $u'(R_{min}) = 0$.

From equation (A3) it immediately follows that $u'(R_{min}) = 0$ if and only if $R_{min} = p$. Solving (A9) under the boundary condition $E(R_{max}) = 0$ yields the equity value function stated in (19).

# Appendix B. Solving for the equilibrium when $r > 0$

In this subsection, we solve for the equilibrium in the set up where $r > 0$. In this case, the dynamics of equity value of an individual bank follows:

$$de_t = d(k_t - D_t) = re_t + k_t[(R-p-r)dt \quad - \sigma_0 dZ_t] - d\delta_t + di_t. \tag{A12}$$

Solving the shareholders' maximization problem in the same way as we did in the proof of Proposition 1 yields two equations:

$$\frac{u'(R)}{u(R)} = \frac{R-p-r}{\sigma_0\sigma(R)}, \tag{A13}$$

$$(\rho - r)u(R) = \mu(R)u'(R) + \frac{\sigma^2(R)}{2}u''(R). \tag{A14}$$

Substituting $u'(R)$ and $u''(R)$ in (A14) enables us to express $\mu(R)$ as a function of $\sigma(R)$:

$$\mu(R) = \frac{\sigma(R)}{2}\left(\frac{2\sigma_0(\rho-r)}{R-p-r} - \frac{\sigma(R)}{R-p-r} - \frac{(R-p-r)}{\sigma_0} + \sigma'(R)\right). \tag{A15}$$

In the region $R \in (R_{min}, R_{max})$, the aggregate equity of the banking sector follows:

$$dE_t = [K(R_t)(R_t - p - r) + rE_t]dt - \sigma_0 K(R_t)dZ_t. \tag{A16}$$

Applying Itô's lemma to $E(R_t)$ and matching the drift and volatility terms with those from expression (A16), we get the system of equations:

$$K(R)(R-p-r) + rE = \mu(R)E'(R) + \frac{\sigma^2(R)}{2}E''(R), \tag{A17}$$

$$-\sigma_0 K(R) = \sigma(R)E'(R). \tag{A18}$$

Proceeding in the same way as in the proof of Proposition 2, we obtain the expression for $\sigma(R)$:

$$\sigma(R) = \frac{K(R)[(R-p-r)^2 + 2\sigma_0^2(\rho-r)] + 2r(R-p-r)E(R)}{\sigma_0\left[K(R) + (R-p-r)K'(R)\right]}. \tag{A19}$$

Substituting $\sigma(R)$ into (A18) yields the first-order differential equation:

$$E'(R) = -\frac{\sigma_0^2\left[K(R) + (R-p-r)K'(R)\right]}{K(R)[(R-p-r)^2 + 2\sigma_0^2(\rho-r)] + 2r(R-p-r)E(R)}, \tag{A20}$$

that can be solved numerically under the boundary condition $E(R_{max}) = 0$.

The recapitalization barrier $R_{max}$ must be computed numerically by solving equation

$$\int_{R_{min}}^{R_{max}} E'(s)\frac{(s-p-r)}{\sigma_0^2 K(s)}ds = \log(1+\gamma). \tag{A21}$$

Note that the left-hand side of the above expression is increasing in $R_{max}$. Hence, there exists a unique solution to (A21). The minimum loan rate is given by $R_{min} = r + p$.

# Appendix C. Computing social welfare

Consider the simple case where $\beta = 1$. The credit demand is then $K(R) = \overline{R} - R$ and the equity value function can be computed explicitely:

$$E(R) = -\frac{(\overline{R} - p)\sigma_0}{\sqrt{2\rho}} \arctan\left(\frac{R - p}{\sqrt{2\rho}\sigma_0}\right) + E_0, \tag{A22}$$

where the constant $E_0$ is given by

$$E_0 = \frac{(\overline{R} - p)\sigma_0}{\sqrt{2\rho}} \arctan\left(\frac{R_{max} - p}{\sqrt{2\rho}\sigma_0}\right). \tag{A23}$$

The maximum level of equity is given by $E_{max} = E(p)$.
Rewriting the loan rate $R$ as a function of $E$ yields

$$R_c(E) = p + \sqrt{2\rho}\sigma_0 \tan\left(\frac{\sqrt{2\rho}}{\sigma_0(\overline{R} - p)}(E_0 - E)\right), \tag{A24}$$

and thus

$$K_c(E) = \overline{R} - p - \sqrt{2\rho}\sigma_0 \tan\left(\frac{\sqrt{2\rho}}{\sigma_0(\overline{R} - p)}(E_0 - E)\right). \tag{A25}$$

To recover the production function, $F(K)$, recall that $F'(K) = R - p$. Using the fact that $R = \overline{R} - K$, we obtain $F'(K) = (\overline{R} - p - K)$ and, thereby,

$$F(K) = (\overline{R} - p)K - \frac{K^2}{2}. \tag{A26}$$

The expected profit of firms is then

$$\pi_F(K_c(E)) = F(K_c(E)) - K_c(E)F'(K_c(E)) = \frac{[K_c(E)]^2}{2},$$

and the expected profit of banks is

$$\pi_B(K_c(E)) = K_c(E)F'(K_c(E)) = \overline{R} - p - K_c(E).$$

Then, for $\beta = 1$, social welfare follows ODE:

$$\rho W(E) = \frac{K_c^2(E)}{2} + K_c(E)(\overline{R} - p - K_c(E))W'(E) + \frac{\sigma_0^2}{2}[K_c(E)]^2 W''(E), \tag{A27}$$

given that $W'(0) = 1 + \gamma$ and $W'(E_{max}) = 1$.
Differentiating the above expression with respect to $E$ and solving the obtained equation numerically with respect to $W'(E)$ enables us uncover $W(E)$.

# Appendix D. Leverage regulation

In the core of the paper we were focusing on the "laissez-faire" environment in which banks face no regulation. Our objective here is to understand how does *leverage regulation* affect the optimal banks' policies and the equilibrium behavior of the economy. Assume that, under leverage regulation, each bank must maintain equity capital above a certain fraction of loans, i.e.,

$$e_t \geq \Lambda k_t,$$

where $\Lambda$ is a leverage ratio.

Notice that banks have two options to comply with leverage ratio. The first option involves costly recapitalization. The second option consists in cutting on lending and paying back debt. Anecdotal evidence suggests that, in practice, bank shareholders prefer to use the latter option, rather than to undertake recapitalizations. As will become apparent below, consistent with anecdotal evidence, in our model, bank shareholders will also prefer to control their leverage by cutting on lending and will use recapitalizations as a last resort.

**Maximization problem of a representative bank.** Under leverage regulation, the maximization problem of a representative bank is similar to the one considered in the unregulated environment, except the fact that now lending decisions are subject to the leverage constraint:

$$v_\Lambda(e, R) = \max_{k_t \geq \frac{e}{\Lambda}, d\delta_t, di_t} \quad \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\left(d\delta_t - (1+\gamma)di_t\right)\right]. \tag{A28}$$

Exploiting the homogeneity property of the value function, we can rewrite the maximization problem as follows:

$$\rho u(R) = \max_{k \geq \frac{e}{\Lambda}, d\delta, di} \quad \left\{\frac{d\delta}{e}[1 - u(R)] - \frac{di}{e}[1 + \gamma - u(R)] + \frac{k}{e}[(R-p)u(R) - \sigma_0\sigma(R)u'(R)] + \right.$$
$$\left. + \mu(R)u'(R) + \frac{\sigma^2(R)}{2}u''(R)\right\}. \tag{A29}$$

Recall that bank equity is decreasing in $R$. Then, there exists a critical level of the loan rate $R_\Lambda$ such that constraint $k \geq \frac{e}{\Lambda}$ binds for any $R \in [R_\Lambda, R_{max}]$. Then, we can split the above maximization problem into the unconstrained problem, whose solution is described in Section 2.1, and the constrained problem. Following the same arguments as in the unregulated set up, the optimal recapitalization and payout policies will be of the barrier type. Specifically, the optimal payout barrier $R_{min}$ is such that $u(R_{min}) = 1$ and recapitalization barrier $R_{max}$ satisfies $u(R_{max}) = 1 + \gamma$. To determine the market-to-book value function $u(R)$, notice that, in the unconstrained region $[R_{min}, R_\Lambda]$, it can be computed according to:

$$u(R) = \exp\left(\int_{R_{min}}^{R} \frac{s-p}{\sigma_0\sigma(s)}ds\right).$$

In the constrained region $[R_\Lambda, R_{max}]$, function $u(R)$ satisfies the differential equation

$$\rho u(R) = \frac{1}{\Lambda}[(R-p)u(R) - \sigma_0\sigma(R)u'(R)] + \mu(R)u'(R) + \frac{\sigma^2(R)}{2}u''(R),$$

subject to the matching and smooth-pasting conditions $u_-(R_\Lambda) = u_+(R_\Lambda)$ and $u'_-(R_\Lambda) = u'_+(R_\Lambda)$.

In order to solve the above equation (numerically), we must define the law of motion of $R$ and determine the threshold $R_\Lambda$.

**Constrained equilibrium.** We will further refer to the equilibrium emerging under leverage regulation as to the *constrained equilibrium.* To solve for the constrained equilibrium, consider first the region $[R_\Lambda, R_{max}]$. In this region, the leverage constraint is binding, which immediately yields the value of equity:

$$E(R) = \Lambda K(R).$$

Thus, plugging $E'(R) = \Lambda K'(R)$ and $E''(R) = \Lambda K''(R)$ into the system (14)-(15), one can easily uncover the law of motion of $R$:

$$\sigma_\Lambda(R) = -\frac{\sigma_0}{\Lambda}\frac{K(R)}{K'(R)}, \tag{A30}$$

$$\mu_\Lambda(R) = \sigma(R)\left(-\frac{(R-p)}{\sigma_0} - \frac{\sigma(R)}{2}\frac{K''(R)}{K'(R)}\right). \tag{A31}$$

In the region $[R_{min}, R_\Lambda)$, the law of motion of the loan rate is defined by the same expressions as in the unregulated setting (see Proposition 2). However, the value of the aggregate bank capital changes for:

$$E(R) = -\int_{R_{min}}^{R} \frac{\sigma_0 K(s)}{\sigma(s)} ds + E_0, \tag{A32}$$

where constant $E_0$ is chosen so as to ensure the value-matching condition $E(R_\Lambda) = \Lambda K(R_\Lambda)$ and $\sigma(R)$ is defined in (16).

The last ingredient we need to complete the characterization of the constrained equilibrium is the threshold $R_\Lambda$. Its value can be inferred from the smooth-pasting condition $E'_-(R_\Lambda) = E'_+(R_\Lambda)$, which can be rewritten as follows:

$$-\frac{\sigma_0}{\sigma(R)}\frac{K(R)}{K'(R)} = \Lambda,$$

where $\sigma(R)$ is defined in (16).

**The impact of leverage regulation: numerical example.** To illustrate the impact of leverage regulation on bank policies, we resort to the numerical analysis. Table 1 reports the values of $R_\Lambda$ and $R_{max}$ computed for the different levels of leverage ratio $\Lambda$.

The numerical analysis suggests that there exists a critical leverage ratio $\underline{\Lambda}$, below which leverage regulation does not affect the optimal bank policies, so that $R_{max}$ is the

Table 1: Impact of leverage regulation

| | $\beta = 2, \sigma = 0.1$ | | $\beta = 4, \sigma = 0.1$ | | $\beta = 2, \sigma = 0.05$ | |
|---|---|---|---|---|---|---|
| | $R_\Lambda$ | $R_{max}$ | $R_\Lambda$ | $R_{max}$ | $R_\Lambda$ | $R_{max}$ |
| $\Lambda = 0$ | – | **0.1105** | – | **0.0859** | – | **0.0792** |
| $\Lambda = 0.05$ | – | 0.1105 | – | 0.0859 | – | 0.0792 |
| $\Lambda = 0.10$ | – | 0.1105 | – | 0.0859 | 0.0541 | 0.0723 |
| $\Lambda = 0.15$ | 0.0938 | 0.1085 | 0.0794 | 0.0855 | 0.0420 | 0.0643 |
| $\Lambda = 0.20$ | 0.0769 | 0.1023 | 0.0618 | 0.0812 | 0.0349 | 0.0582 |
| $\Lambda = 0.25$ | 0.0657 | 0.0961 | 0.0500 | 0.0756 | 0.0300 | 0.0534 |

Table 1 reports the values of $R_\Lambda$ and $R_{max}$ for different levels of $\Lambda$. Parameter values common to all scenarios are: $\rho = 0.05$, $\overline{R} = 0.2$, $\gamma = 10$, $p = 0$.

same as in the unregulated environment. However, for any $\Lambda > \underline{\Lambda}$, both $R_\Lambda$ and $R_{max}$ are decreasing in $\Lambda$. Thus, a tighter leverage ratio would induce banks to recapitalize at a lower $R_{max}$, thereby, reducing the maximum amplitude of the loan rate. Moreover, when faced with leverage regulation, banks would maintain more equity capital even in the region in which the leverage constraint is not binding (see Figure 5). It is also easy to see that $E(R_{max}) > 0$, i.e., in contrast to the unregulated setting, banks undertake recapitalizations, while holding a strictly positive level of capital.

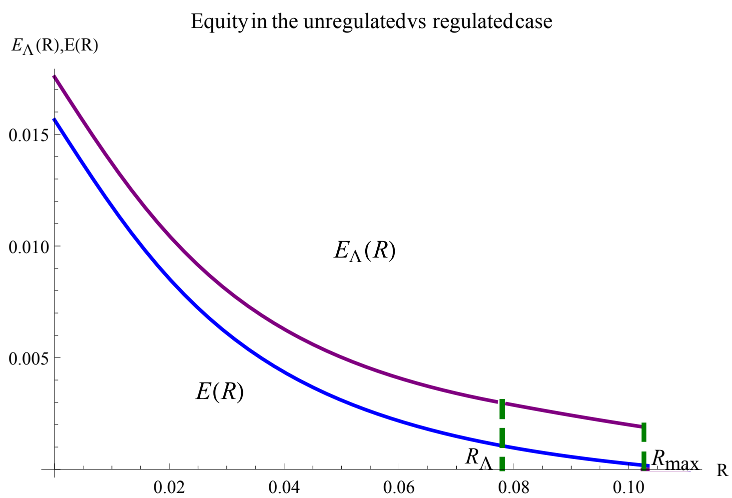


Figure 5: Impact of leverage regulation on bank capital

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