Dynamic Debt Maturity*

Zhiguo He Konstantin Milbradt

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Abstract

We study a dynamic setting in which a firm chooses its debt maturity structure endogenously over time without commitment. In our model, the firm keeps its promised outstanding bond face-values constant, but can control the firm's maturity structure via the fraction of newly issued short-term bonds when refinancing its matured long-term and short-term bonds. As a baseline, we show that when the firm's cash-flows are constant then it is impossible to have the shortening equilibrium in which the firm keeps issuing short-term bonds and default consequently. Instead, when the cash-flows deteriorate over time so that the debt recovery value is affected by the endogenous default timing, then a shortening equilibrium with accelerated default can emerge. Self-enforcing shortening and lengthening equilibria exist, and the shortening equilibrium may be Pareto-dominated by the lengthening one.

Keywords: Maturity Structure, Dynamic Structural Models, Endogenous Default, No Commitment, Debt Rollover.

*He: University of Chicago, Booth School of Business; and NBER. Email: zhiguo.he@chicagobooth.edu. Milbradt: Northwestern University, Kellogg School of Management; and NBER. milbradt@gmail.com. We thank Guido Lorenzoni for helpful comments.

1 Introduction

The 2007/08 financial crisis has put debt maturity structure of financial institutions squarely in the focus of both policy discussions as well as the popular press. However, dynamic models of debt maturity structure are difficult to analyze, and hence academics are lagging behind in offering tractable frameworks in which the firm's debt maturity structure follows some endogenous dynamics. In fact, a widely used framework for analyzing debt maturity structure is based on Leland [1994b, 1998] and Leland and Toft [1996] who, for tractability's sake, take the frequency of refinancing/rollover as a fixed parameter. In that framework, equity holders are essentially able to commit to a policy of a constant debt maturity structure, which equals the inverse of the debt rollover frequency, until default. This stringent assumption is at odds with mounting empirical evidence that most firms have time-varying debt maturity structure; for instance, Chen et al. [2013] document that firms have pro-cyclical debt maturity structure; and Xu [2014] shows that speculative firms are actively managing their debt maturity structure via early refinancing.

This paper relaxes the assumption of a constant debt maturity structure by removing the equity holders' ability to commit to a future debt maturity structure. This results in a novel dynamic model that allows us to rigorously analyze how equity holders adjust the firm's debt maturity structure facing time-varying firm fundamentals and endogenous bond prices. In our model the firm has two kinds of debt, long and short term bonds, that mature with constant but different Poisson intensities. As the main innovation relative to the existing literature, we allow equity holders to control the firm's debt maturity structure endogenously by changing the maturity composition of current (rollover) debt issuances. When equity holders replace just-matured long-term debt by issuing short-term debt, the firm's debt maturity structure shortens. To focus on endogenous debt maturity dynamics only, we fix the firm's book leverage policy, by following the Leland-type model assumptions that the firm commits to maintaining a constant aggregate face-value of outstanding debt. This treatment is consistent with the fact that in practice, most of bond covenants have some restrictions regarding the firm's future leverage policies, but rarely on the firm's future maturity structures.

In refinancing their maturing bonds, equity holders are the claimants to the cash-flow gap between the face value of matured bonds and the proceeds from selling newly issued bonds at market price. When default is imminent, bond prices are low and equity holders are absorbing rollover losses. This so-called rollover risk may feed back to earlier default, an effect that emerged in a variant of the classic Leland model that involved finite maturity debt (Leland and Toft [1996] and Leland [1994b]). More importantly, as shown by He and Xiong [2012] and Diamond and He [2014], all else equal, equity holders are more likely to default if the firm has more a shorter debt maturity structure and thus needs to refinance more maturing bonds. The more debt has to be repriced, the heavier the rollover losses are for the firm when fundamentals deteriorate, thereby pushing the firm closer to default.

What is the equity holders' trade-off involved in shortening the maturity structure by issuing more short-term bonds today? The presence of default risk implies that "going short" offers higher issuance proceeds today. This is because short-term bonds fetch higher valuations relative to long-term bonds, as the former has a higher likelihood of maturing before the default event. Thus, the benefit of maturity shortening is to reduce the firm's rollover losses today.

However, as short-term debt comes due faster, shortening increases the future rollover frequency. Equity holders' exposure to future rollover increases, leading to earlier default and thus to lower equity value. This is the cost side of shortening maturity, and equity holders are cognizant of this negative long-term effect when deciding the optimal issuance policy. Combining both the benefit and cost gives rise to the equity holders' incentive compatibility condition for issuing short-term bonds, which plays a key role in our analysis. Our main research question is: Can situations arise in which this trade-off favors maturity shortening, so that, even though going short hastens default and thus hurts the social value of the firm, in equilibrium equity holders keep issuing short-term bonds due to an inability to commit?

As a benchmark, we first consider the case in which a firm produces constant cash flows but is waiting for an upside event (at which point the model ends). We show that there is never any slow drift towards inefficient default via shortening the firm's maturity structure, if there is a strictly positive loss-given-default for bond investors. Either the firm defaults immediately, or the firm lengthens its debt maturity structure by issuing long-term bonds and thus never defaults. This result of "no shortening equilibrium" is robust to various generalizations.

We establish the result of "no shortening equilibrium" by analyzing the equity holders' incentive compatibility condition in the vicinity of the default boundary. Interestingly, we show that the incentive compatibility condition is solely determined by the sign of the marginal impact of maturity shortening on the value of short-term bonds. More specifically, equity holders would like to issue more short-term bonds, if shortening the firm's debt maturity structure raises the market value of short-term bonds. Intuitively, right before default, the savings on today's rollover losses by issuing more short-term bonds just offset the increase of tomorrow's rollover losses; and the only effect at work is that maturity shortening edges the firm closer to default and hence affects the market value of bonds. However, given a positive loss-given-default, a lower distance-to-default drives down the market value of short-term bonds. As a result, the equity holders' incentive compatibility constraint is always violated in the vicinity of default, and the "no shortening equilibrium" result emerges.

This "no shortening equilibrium" is in sharp contrast to Brunnermeier and Oehmke [2013] who show that equity holders might want to privately renegotiate the bond maturity down (toward zero) with each individual bond investor. The key difference is on who bears the rollover losses when there is arrival of unfavorable news in an interim period. In Brunnermeier and Oehmke [2013], there is no covenants about the firm's aggregate face value of outstanding bonds, and after negative interim news the rollover losses of short-term bonds are absorbed by promising a sufficiently high new face-value to keep the short-term bond-holders in the firm. This increase in face-value dilutes the (non-renegotiating) existing long-term bond holders. In contrast, in our model equity holders are absorbing rollover losses through their own deep pockets (or through equity issuance), as increasing face value to dilute existing bond holders is prohibited by the assumption of a constant aggregate face value. By shutting down the interim dilution channel that drives the result in Brunnermeier and Oehmke [2013], we identify a new economic force that impacts maturity choice.

We then move on to show that for firms whose cash flows are deteriorating over time, it is possible to construct an equilibrium where equity holders shorten the firm's debt maturity structure and the firm drifts slowly towards inefficient early default. As in the constant cash-flow case, equity holders find it optimal to issue short-term bonds if maturity shortening increases the value of short-term bonds. However, there is a crucial difference between deteriorating cash flows and constant cash flows. For firms whose cash flows are deteriorating over time, all else equal debt values may be higher under an earlier default time. This is because bond holders will take over the firm earlier, at a higher fundamental level, resulting in higher debt recovery. This force, which is absent in the setting with constant cash flows, can entice equity holders to shorten the firm's debt maturity structure ex post, although committing to long debt maturity ex ante maximizes total welfare.

Indeed, in the case with deteriorating cash flows, starting at some initial state–i.e., current cash flows and maturity structure–that is sufficiently far away from bankruptcy, one can construct two equilibrium paths toward default, one with maturity shortening and the other with maturity lengthening. In the lengthening equilibrium, the firm's debt maturity structure grows longer and longer over time, as equity holders keep issuing long-term bonds to replace maturing short-term bonds. In our example, the firm in the lengthening equilibrium survives longer, resulting in higher overall welfare and even Pareto dominance over the shortening equilibrium.

A multiplicity of equilibria emerges in our model without much surprise. If bond investors expect equity holders to keep shortening the firm's maturity structure in the future, then bond investors price this expectation in the bond's market valuation, which can self-enforce the optimality of issuing short-term bonds only. Similarly, the belief of issuing long-term bonds always can be self-enforcing as well. However, we prove that when the firm is sufficiently close to default then the model has a unique equilibrium; intuitively, any future benign (malign) expectation of lengthening (shortening) maturity is "too late" to be self-enforcing.

There are two simplifying assumptions, however, that are crucial to the tractability of our model; they also may play some role in driving our main results. First, our analysis rules out Brownian cash-flow shocks, which are common in the existing Leland-type models. It is unclear how postponing default around the bankruptcy boundary due to Brownian uncertainty affects the clean relation between the equity holders' incentive compatibility of going short and its marginal impact on the value of short-term bonds. Allowing for Brownian shocks will necessarily involve a nontrivial two-dimensional analysis, and we await future research to consider this possibility. Second, in our model the firm cannot change the aggregate amount of face-value outstanding, which rules out diluting existing bond holders by promising higher face value to new incoming bond holders. Based on this dilution effect, Brunnermeier and Oehmke [2013] show that in a Merton-type model without endogenous default timing decisions, the firm might want to privately renegotiate the bond maturity down (toward zero) with each individual bond investor. To some extent, we rule out changes in face-value to purposefully isolate our effect from the effect of Brunnermeier and Oehmke [2013]. Having said that, it is interesting for the future research to study endogenous dynamic maturity structure and dynamic leverage simultaneously in the Leland-type model; see DeMarzo and He [2014] for some recent progress in modeling endogenous leverage dynamics without commitment.

Debt maturity is an active research area in corporate finance, and most of the early theoretical models were static models. Calomiris and Kahn [1991] and Diamond and Rajan [2001] emphasize the disciplinary role played by short-term debt, a force absent from our model. The repricing of short-term debt given news in Flannery [1986], Diamond [1991] and Flannery [1994] is related to the endogenous rollover losses of our paper. For dynamic corporate finance models with finite debt maturity, almost the entire existing literature is based on a Leland-type framework in which a firm commits to a constant debt maturity structure.[1] To the best of our knowledge, our model is the first that investigates the endogenous debt maturity dynamics. Our model nests the Leland framework (without Brownian shocks) if we assume that both long-term bonds and short-term bonds have the same maturity. In Leland [1994a] the firm is unable to commit not to default. Introducing a fixed rollover term in Leland [1994b] makes the outcome of this inability to commit worse as default occurs earlier the higher the rollover. We show that introducing a flexible maturity

[1] For more recent development, see He and Xiong [2012], Diamond and He [2014], Chen et al. [2014], He and Milbradt [2014], and McQuade [2013].

structure with an inability to commit might further worsen this default channel, even though a priori the added flexibility would seem work in equity holder's favor to move closer to the first-best welfare maximizing strategy.

Our paper is also related to the study of debt maturity and multiplicity of equilibria in the sovereign debt literature (e.g., Cole and Kehoe [2000]). Arellano and Ramanarayanan [2012] provide a quantitative model where the sovereign country can actively manage its debt maturity structure and leverage, and show that maturities shorten as the probability of default increases; a similar pattern emerges in Dovis [2012]. As typical in sovereign debt literature, one key motive for the risk-averse sovereign to borrow is for risk-sharing purposes in an incomplete market. Because debt maturity plays a role in how the available assets span shocks, the equilibrium risk-sharing outcomes are affected by debt maturity. This force is absent in most corporate finance models which are typically cast in a risk-neutral setting. A more related paper is Aguiar and Amador [2013] who, like us, provide a transparent and tractable framework for analyzing maturity choice in a dynamic framework without commitment. They study a drastically different economic question, however: there, a sovereign needs to reduce its debt and the debt maturity choices matter for the endogenous speed of deleveraging. In contrast, in our model the total face value of debt is fixed at a constant, and the maturity choice trades off rollover losses today versus higher rollover frequencies tomorrow.

We start by laying out our model generally in Section 2. We then solve the base model with constant cash flows in Section 3, and compare it with the setting where the firm's cash flows are decreasing over time in Section 4. We provide a numerical example in Section 4 to illustrate the nature of multiple equilibria in our model. Section 5 considers the possibility of interior equilibria, and Section 6 concludes. All proofs are in Appendix.

2 The Setting

2.1 Firm and Asset

All agents in the economy, that is equity and debt-holders, are risk-neutral with a constant discount rate r. The firm has assets-in-place generating cash flows at a rate of y_t, whose evolution will be specified later. There is a Poisson event arriving with a constant intensity $\zeta > 0$; at this event, assets-in-place pay off a sufficiently large constant X and the model ends. This event can also be interpreted as the realization of growth options, and throughout we call it the "upside event."

We allow the cash-flow rate y_t to be negative (e.g., operating losses). As y_t can take negative values, it might be optimal to abandon the asset at some finite time, denoted by T_a. We assume that abandonment is irreversible and costless. Given the cash-flow process y_t, the unlevered firm value (or asset value) is given by

$$A\left(y\right) = \mathbb{E}\left[\int_0^{\min\left(T_a, T_\zeta\right)} e^{-rt} y_t dt + \mathbf{1}_{\left\{T_\zeta < T_a\right\}} e^{-rT_\zeta} X\right], \tag{1}$$

The firm is financed by debt and equity. When equity holders default, debt holders take over the firm with some bankruptcy cost (to be specified later), so that the asset's recovery value from bankruptcy is $B\left(y\right) < A\left(y\right)$. We assume that $B'\left(y\right) > 0$, i.e., the firm's liquidation value is increasing in the current state of cash-flows.

2.2 Dynamic Maturity Structure and Debt Rollover

2.2.1 Assumptions

We study the dynamic maturity structure of the firm. To this end, we assume that the firm has two kinds of bonds outstanding: long-term bonds whose time-to-maturity follows an exponential distribution with mean $1/\delta_L$, and short-term bonds whose time-to-maturity follows an exponential distribution with mean $1/\delta_S$, where δ_i's are positive constants with $i \in \{S, L\}$ and $\delta_S > \delta_L$. Thus, bonds mature in an i.i.d. fashion with Poisson intensity δ_i. An equivalent interpretation is that of a sinking-fund bond as discussed in Leland [1994b, 1998].

Maturity is the only characteristic that differs across these two bonds. Both bonds have the same after-tax coupon rate c and the same principal normalized to 1. To avoid arbitrary valuation difference between two bonds, we set the before-tax coupon rate equal to the discount rate, i.e. $\rho c = r$ where $\rho \geq 1$ stands for a tax benefit per unit of coupon. This way, without default both bonds have a unit value, i.e., $D_L^{rf} = D_S^{rf} = 1$. We also assume both bonds have the same seniority to rule out trivial dilution motives. In bankruptcy, both bond holders receive, per unit of face-value, $B(y)$ as the asset's liquidation value. Throughout, we assume that

$$B(y) < D_i^{rf} = 1, \text{ for } i \in \{S, L\}. \tag{2}$$

This empirically relevant condition simply says that the loss-given-default for bond investors is strictly positive.

To focus on maturity structure only, throughout we assume that the firm commits to a constant "book leverage" policy. Specifically, following the canonical assumption in Leland [1998], the firm rolls over its bonds in such a way that the total promised face-value is kept at a constant normalized to 1 (hence, the total measure of these two bonds is 1).

We emphasize that this assumption can be motivated by bond covenants on future leverage policies taken by the firm. Essentially, this assumption rules out the "indirect" dilution effect caused by future net debt issuance in response to the firm's fundamental news, which is the economic force behind Brunnermeier and Oehmke [2013]. There, short-term bond holders have the advantage of repricing their individual bond face values given new information; since all bonds have the same seniority, a higher face value following negative news dilutes the existing long-term bond holders. Our constant face-value assumption explicitly rules out this indirect dilution effect, highlighting a complementary channel to Brunnermeier and Oehmke [2013], as discussed in Section 3.4.2.

Taking our assumptions together, we implicitly assume that debt covenants, while restricting the firm's future leverage policies, do not impose restrictions on a firm's future maturity. This assumption is realistic, as debt covenants often specify restrictions on firm leverage but rarely on debt maturity.

2.2.2 Maturity structure and its dynamics

The face value of short-term bonds at time t, denoted by $\phi_t \in [0,1]$, gives the *fraction of short-term bonds* outstanding. We call ϕ_t the current maturity structure of the firm. Given the current maturity structure ϕ_t, during $[t, t+dt]$ there are $m(\phi_t)\,dt$ dollars of bonds maturing, where

$$m(\phi_t) \equiv \phi_t \delta_S + (1-\phi_t)\,\delta_L. \tag{3}$$

The more short-term the current maturity structure is, the more the debt is rolled over each instant, as we have $m'(\phi) = \delta_S - \delta_L > 0$.

Recall that the constant book-leverage assumption implies that equity holders are issuing $m(\phi_t)\,dt$ units of new bonds to replace those maturing bonds. The main innovation of the paper is to allow equity holders to endogenously choose the proportion of newly issued short-term bonds, which we denote by $f_t \in [0,1]$.[2] Hence, the dynamics of maturity structure ϕ_t are given by

$$\frac{d\phi_t}{dt} = \underbrace{-\phi_t \cdot \delta_S}_{\text{Short-term maturing}} + \underbrace{m(\phi_t)\,f_t}_{\text{Newly issued short-term}}. \tag{4}$$

Most of our analysis focuses on constant issuance policies that take corner values 0 or 1, i.e. $f \in \{0,1\}$. Suppose that $f = 1$ always, so that the maturity structure is shortened; then $d\phi_t = \delta_L (1-\phi_t)\,dt > 0$, i.e., the maturity structure ϕ_t increases at the fraction of long-term debt multiplied by its maturing speed. Over time, the firm's maturity structure ϕ_t monotonically rises toward 100% of short-term debt. Similarly, if the firm keeps issuing long-term bonds so that $f = 0$, then $d\phi_t = -\phi_t \delta_S dt < 0$ and thus the maturity structure ϕ_t monotonically falls toward 0% of short-term debt.

[2] We assume that there is no debt buybacks, call provisions do not exist, and maturity of debt contracts cannot be changed once issued. We discuss the robustness of our result with respect to these assumptions in Section 3.5.3.

2.3 Rollover Losses and Default

2.3.1 Bond market prices

Given the equilibrium default time T_b (if $T_b = \infty$ then the firm never defaults), competitive bond investors price long-term and short-term bonds at $D_S(y_t, \phi_t)$ and $D_L(y_t, \phi_t)$ respectively. Even if T_b is deterministic, since we model bond maturity as a Poisson shock, bond holders are still exposed to the risk of default. Since we set $\rho c = r$, and the recovery value $B(\cdot)$ is below the face value 1, in general we have $D_L \leq D_S \leq 1$ (for the exact argument, see Section 3.2.1). This immediately implies the firm is incurring certain rollover losses, a topic we turn to now.

2.3.2 Rollover losses and default boundary

In Leland [1994b, 1998], equity holders commit to roll over (refinance) the firm's maturing bonds by re-issuing bonds of the same type. In our model, the firm can choose the fraction of short-term bonds f amongst the total of newly issued bonds. Per unit of face value, by issuing an f_t fraction of short-term bonds, the equity's net rollover cash-flows are

$$\underbrace{f_t D_S(y_t, \phi_t) + (1 - f_t) D_L(y_t, \phi_t)}_{\text{proceeds of newly issued bonds}} - \underbrace{1}_{\text{payment to maturing bonds}}.$$

We call this term "rollover losses."[3] Each instant there are $m(\phi_t)\,dt$ units of face value to be rolled over, hence the instantaneous expected cash flows to equity holders are

$$\underbrace{y_t}_{\text{operating CF}} - \underbrace{c}_{\text{coupon}} + \underbrace{\zeta E^{rf}}_{\text{upside event}} + \underbrace{m(\phi_t)\left[f_t D_S(y_t, \phi_t) + (1 - f_t) D_L(y_t, \phi_t) - 1\right]}_{\text{rollover losses}}. \tag{5}$$

Here, the third term "upside event" is the expected cash flows to equity of this event multiplied by its probability, where we define $E^{rf} \equiv X - D^{rf} = X - 1 > 0$.

[3] Equity holders are always facing rollover losses as long as $\rho c = r$ and $B(y_{T_b}) < 1$, which imply that $D_i < 1$. When $\rho c > r$, rollover gains occur for safe firms who are far from default. As emphasized in He and Xiong [2012], since rollover risk kicks in only when the firm is close to default, it is without loss of generality to focus on rollover losses only.

When the above cash flows in (5) are negative, these losses are covered by issuing additional equity, which dilutes the value of existing shares.[4] Equity holders are willing to buy more shares and bail out the maturing bond holders as long as the equity value is still positive (i.e. the option value of keeping the firm alive justifies absorbing these losses). When equity holders—protected by limited liability—declare default, equity value drops to zero, and bond holders receive the firm's liquidation value $B\left(y_{T_b}\right)$.

There are two distinct channels that expose equity holders to heavier losses, leading to default. The first, the cash-flow channel, has been studied extensively in the literature. When y_t deteriorates (say, y_t turns negative), equity holders are absorbing operating losses (the first term in (5)). Also, because a lower y_t leads to more imminent default (say, default occurs once y_t hits some lower boundary), bond prices D_S and D_L drop as well, leading to heavier rollover losses in the third term in (5) for any given $m\left(\phi\right)$.

The second channel, which is novel, is through the endogenous maturity structure ϕ_t. Fixing the issuance policy f, the greater ϕ_t, the higher the rollover frequency $m\left(\phi_t\right)$. Later we show that bond valuations D_i's are decreasing in ϕ as well, leading to heavier rollover losses. Both effects imply that given a shorter maturity structure ϕ, equity holders face worse rollover losses in (5) and are thus more prone to default, all else equal. Importantly, equity holders pick the path of the future maturity structure $\{\phi_s : s > t\}$ via equation (4) by choosing f_t endogenously subject to an incentive compatibility condition to be discussed shortly.

The above discussion suggests that there exists a default curve $\left(\Phi\left(y\right), y\right)$, where the increasing function $\Phi\left(\cdot\right)$ gives the threshold maturity structure given cash-flow y. In equilibrium, the firm defaults whenever the state lies in

$$\mathcal{B} = \left\{\left(\phi, y\right) \text{ such that } \phi \geq \Phi\left(y\right)\right\}.$$

[4] This assumption highlights the so-called "endogenous" default in that equity holders default when the are unwilling rather than unable to absorb the loss. The underlying assumption is that either equity holders have deep pockets or the firm faces a frictionless equity market.

Consistent with this observation, throughout we make the following assumption on off-equilibrium beliefs regarding default. When the firm stays alive at time t even though creditors expected it to be in default, new bond investors expect the firm to default as long as the $(\phi_s, y_s) \in \mathcal{B}$ for $s > t$. This implies that if in the next instant $(\phi_{t+dt}, y_{t+dt}) \in \mathcal{B}$, either because cash flow y_t is decreasing over time or the firm keeps issuing short-term debt so that $\phi_{t+dt} > \phi_t$, then bond investors apply the lowest possible bond value given by $D_L = D_S = B(y_{t+dt})$.

3 Baseline Model: Constant Cash-Flows

We first show a negative result for the constant cash-flow setting: There does not exist an equilibrium path in which equity holders keep shortening the firm's debt maturity structure and eventually default in the face of larger and larger rollover losses.

3.1 Setting

Consider the simplest setting with constant cash-flows, i.e., $y_t = y$. We denote by $D_S(\phi_\tau; y)$, $D_L(\phi_t; y)$, and $E(\phi_t; y)$ the short-term bond, long-term bond, and equity value, respectively. We explicitly write the cash-flow y into security valuations to emphasize their dependence on y.

Given maturity structure ϕ_t and issuance policy f_t, the expected cash-flows of equity is

$$y - c + \zeta E^{rf} + m(\phi_t)\left[f_t D_S(\phi_t; y) + (1 - f_t) D_L(\phi_t; y) - 1\right]. \tag{6}$$

The following Lemma characterizes two polar cases.

Lemma 1 *Default occurs immediately if $y - c + \zeta E^{rf} < 0$, and equity never defaults if $y - c + \zeta E^{rf} + \delta_S[B(y) - 1] \geq 0$.*

Intuitively, the rollover term in (6) at best is bounded above by zero, but at worst is $\delta_S[B(y) - 1]$ under the shortest maturity structure ($\phi = 1$) and the lowest debt price $B(y)$. Hence if $y - c + \zeta E^{rf} < 0$ then the equity's cash flows in (6) are always negative, leading to immediate default. On the other hand, if $y-c+\zeta E^{rf}+\delta_S\left[B\left(y\right)-1\right]>0$, then even under the most pessimistic beliefs equity holders never make losses and thus never default.

3.2 Shortening Equilibrium

When $0\leq y-c+\zeta E^{rf}<\delta_S\left[1-B\left(y\right)\right]$, there exist some nontrivial equilibria. We are interested in so-called "shortening" equilibria. Specifically, do there exist equilibria, in which equity holders setting $f=1$ (i.e., issuing short-term debt) from then on, so that ϕ increases over time and the firm eventually defaults in the face of larger and larger rollover losses?

3.2.1 Debt valuations

Bond holders are taking equity holders' policy $f=1$ as given. We treat the maturity structure ϕ as the state variable, which follows $d\phi_t=\left(1-\phi_\tau\right)\delta_L dt$ where we use (4) with $f=1$. Hence, the bond valuation equation with $i\in\{S,L\}$ is[5]

$$\underbrace{rD_i\left(\phi;y\right)}_{\text{required return}}=\underbrace{\rho c}_{\text{pre-tax coupon}=r}+\underbrace{\delta_i\left[1-D_i\left(\phi;y\right)\right]}_{\text{maturing}}+\underbrace{\zeta\left[1-D_i\left(\phi;y\right)\right]}_{\text{upside event}}+\underbrace{\left(1-\phi\right)\delta_L D_i'\left(\phi;y\right)}_{\text{state change}},\quad(7)$$

and by equal seniority we have the boundary condition

$$D_i\left(\Phi\left(y\right);y\right)=B\left(y\right).\quad(8)$$

Later analysis involves the price wedge between short-term and long-term bonds, which is defined as

$$\Delta\left(\phi;y\right)\equiv D_S\left(\phi;y\right)-D_L\left(\phi;y\right).$$

Applying δ_S and δ_L to (7) and taking differences, we obtain

$$\left(r+\delta_L+\zeta\right)\Delta\left(\phi\right)=\left(\delta_S-\delta_L\right)\left[1-D_S\left(\phi\right)\right]+\left(1-\phi\right)\delta_L\Delta'\left(\phi\right),\quad\text{and}\quad\Delta\left(\Phi\left(y\right);y\right)=0\quad(9)$$

[5]Bond holders get paid $D^{rf}=1$ in both the bond maturing event (occurring with intensity δ) and upside option event (occurring with intensity ζ).

As $1 - D_S(\phi; y) > 0$ if default is ever possible, we have

$$\Delta(\phi) > 0 \text{ for } \phi < \Phi(y), \tag{10}$$

i.e., short-term bonds have a higher price than long-term bonds. Intuitively, short-term bonds are paid back sooner and hence less likely to suffer default losses. Hence, short-term bonds are preferred if equity holders try to minimize the firm's current rollover losses.

3.2.2 Equity valuation and optimal issuance policy

Equity holders are not only minimizing the firm's current rollover losses; they also take into account any long-run effect brought on by issuing more short-term bonds. By issuing more short-term bonds today, it shortens the firm's future maturity structure going forward, aggravating future rollover losses and thus affecting possible default decisions.

Formally, equity holders are controlling the firm's dynamic maturity structure as in (4). The standard Hamilton-Jacobi-Bellman (HJB) equation for equity, with the choice variable f, can be written as

$$\begin{aligned} \underbrace{rE(\phi; y)}_{\text{required return}} &= y - c + \underbrace{\zeta \left[E^{rf} - E(\phi; y)\right]}_{\text{upside event}} + \\ &\max_{f \in [0,1]} \left\{ \begin{array}{c} \underbrace{m(\phi)\left[f D_S(\phi; y) + (1-f) D_L(\phi; y) - 1\right]}_{\text{rollover losses}} \\ + \underbrace{\left[-\phi\delta_S + m(\phi) f\right] E'(\phi; y)}_{\text{impact of maturity shortening}} \end{array} \right\}. \end{aligned} \tag{11}$$

Here, by choosing the fraction f of the newly issued short-term bonds, equity holders are balancing today's "rollover losses" against the "impact of maturity shortening" on the future equity value.

Due to linearity, the optimal incentive compatible issuance policy f is given by

$$f = \begin{cases} 1 & \text{if } \Delta(\phi; y) + E'(\phi; y) > 0, \\ 0 & \text{if } \Delta(\phi; y) + E'(\phi; y) < 0, \\ [0,1] & \text{if } \Delta(\phi; y) + E'(\phi; y) = 0. \end{cases} \tag{12}$$

We call $\Delta(\phi; y) + E'(\phi; y) > 0$ the *incentive compatibility* condition for equity issuing short-term debt, later IC for short. Issuing more short-term bonds lowers the firm's rollover losses today, as short-term bonds have higher prices ($\Delta(\phi; y) > 0$). However, issuing more short-term bonds today (higher f) makes the firm's future maturity structure more short-term (higher ϕ) and thus increase the rollover flow (higher $m(\phi)$). As we show next, this brings the firm closer to default and hurts equity holders' continuation value, leading to $E'(\phi; y) < 0$. The optimal issuance policy in (12) illustrates this trade-off faced by equity holders.

3.2.3 Endogenous default

Equity holders also choose when to default optimally. Since we are working with ϕ as the state variable, at the default boundary Φ we have these two standard value-matching and smooth-pasting conditions:

$$E(\Phi; y) = 0, \quad \text{and} \quad E'(\Phi; y) = 0. \tag{13}$$

The second smooth-pasting condition in (13) reflects the optimality of the default decision: The optimal default must occur when the change in equity value is zero.[6] Applying conditions in (13) to the equity equation (11), the equity's expected flow payoff at $\phi = \Phi$ equals to zero:

$$y - c + \zeta E^{rf} + \max_{f \in [0,1]} m(\Phi) \left[f D_S(\Phi; y) + (1-f) D_L(\Phi; y) - 1 \right] = 0. \tag{14}$$

[6] Rigorously, we should have the change of equity value with respect to time to be zero. Because ϕ and time have a one-to-one mapping given by $d\phi_t = (1-\phi_t)\,\delta_L dt$, the smooth-pasting condition in (13) follows.

In other words, in our model without diffusion terms, equity holders default exactly at the point when expected cash-flows turn negative.

Equation (14) pins down the default boundary $\Phi(y)$ as a function of the constant cash-flow y. At default, both bond values are given by $D_S(\Phi(y);y) = D_L(\Phi(y);y) = B(y)$, leading to a rollover term $m(\Phi)[B(y)-1]$ in (14) independent of the optimal issuance policy f. Plugging $m(\Phi)$ in (3), we have

$$\Phi(y) = \frac{1}{\delta_S - \delta_L}\left[\frac{y - c + \zeta E^{rf}}{1 - B(y)} - \delta_L\right]. \tag{15}$$

Because the recovery value $B(y)$ is increasing in y, one can verify that $\Phi(y)$ is increasing in y, as conjectured in Section 2.3.2.

3.3 Impossibility of Shortening Equilibria

We now give the formal definition for a shortening equilibrium.

Definition 1 *The equilibrium concept is that of subgame perfect equilibrium. Given an initial maturity structure $\phi_{t=0}$, a shortening equilibrium is a path of $\{\phi_{t=0} \to \Phi(y)\}$ with $f_t = 1$, so that (11) holds with boundary conditions (13); (7) holds with boundary conditions (8); and, the equity holders' incentive compatibility condition (12) holds with $f_t = 1$. Off equilibrium beliefs are assumed to treat any deviations by the equity holders as mistakes, and continue to believe in the closest equilibrium in terms of default time to the one before the deviation.*

Off-equilibrium beliefs here treat deviations as mistakes. For example, if everyone expected the firm to shorten the maturity structure, and then to default at a certain time, a deviation today of the shortening assumption does not alter the belief of investors that in the future the company will shorten the maturity structure and default. Another example would be that if the firm was supposed to default today, but did not, then investors assume it will default in the next instant. Essentially, sub-game perfection requires that after a deviation investor beliefs for future play have to be an equilibrium. To select amongst the possible multiple equilibria present after deviation, we impose the additional refinement that they "pick" the equilibrium that is closest to the one they had before in terms of ultimate default time. This is very much aking to a trembling-hand refinement.

To rule out any shortening equilibria, it is sufficient to analyze the equilibrium behavior immediately before default, i.e., $\phi = \Phi - \epsilon$ for a sufficiently small $\epsilon > 0$. In light of (12), we need to show that $\Delta(\Phi - \epsilon; y) + E'(\Phi - \epsilon; y) < 0$. Since at default we have $\Delta(\Phi; y) = 0$ in (9) and $E'(\Phi; y) = 0$ in (13), the IC condition $\Delta + E'$ is identically zero at Φ. The following lemma goes one order higher to sign the IC condition in the vicinity of the default boundary Φ.

Lemma 2 *It is never optimal to choose $f = 1$ right before default at $\Phi - \epsilon$ if*

$$\Delta'(\Phi; y) + E''(\Phi; y) > 0. \tag{16}$$

We first analyze the benefit of shortening $\Delta'(\Phi; y)$ in (16). From (9) we know that

$$\Delta'(\Phi; y) = -\frac{(\delta_S - \delta_L)}{(1 - \Phi)\,\delta_L}\,[1 - B(y)] < 0, \tag{17}$$

which says $\Delta(\Phi - \epsilon; y) > 0$. When the firm is a bit away from default, short-term bonds have the advantage of maturing before default, leading to a strictly higher price than long-term bonds. This is the benefit of issuing short-term bonds.

Equity holders have to balance this benefit with the cost of more imminent default; the latter is captured by the second term $E''(\Phi; y)$ in (16). This term is always positive, establishing the optimality of equity holders' endogenous default decision. The proof of Proposition 1 shows that

$$E''(\Phi) = \underbrace{\frac{(\delta_S - \delta_L)\,[1 - B(y)]}{(1 - \Phi)\,\delta_L}}_{= -\Delta'(\Phi; y)} - \underbrace{\frac{[\Phi\delta_S + (1 - \Phi)\,\delta_L]}{(1 - \Phi)\,\delta_L} D'_S(\Phi; y)}_{\text{impact on short-term bond}}. \tag{18}$$

Combining (17) and (18), we have

$$\Delta'(\Phi; y) + E''(\Phi; y) = -\frac{[\Phi\delta_S + (1 - \Phi)\,\delta_L]}{(1 - \Phi)\,\delta_L} D'_S(\Phi; y)\,.$$

Since $\Phi \in [0,1]$, the sign of IC condition $\Delta'(\Phi;y) + E''(\Phi;y)$ is the opposite of the sign of impact on short-term bond $D'_S(\Phi;y)$.

Proposition 1 *Consider the constant cash-flows setting. Right before default, given $f = 1$, the equity holders' incentive compatibility condition $\Delta'(\Phi;y) + E''(\Phi;y) \leq 0$ holds if and only if*

$$D'_S(\Phi;y) \geq 0. \tag{19}$$

Now we show that when y_t is constant at y, the sign of $D'_S(\Phi;y)$ is fully determined by the (opposite) sign of loss-given-default for bond investors. Recall that we assume that $B(y) < 1$, i.e., default leads to value losses for bond holders. From (7) with $\rho c = r$, we derive that[7]

$$D'_S(\Phi;y) = -\frac{(r + \delta_S + \zeta)[1 - B(y)]}{(1-\Phi)\delta_L} < 0.$$

In words, the shorter the firm's maturity structure, the closer the default, and hence the lower the bond value. The next corollary naturally follows from Lemma 2 and Proposition 1.

Corollary 1 *There do not exist shortening equilibria where equity holders keep issuing short-term bonds and then default at some finite future time in the constant cash-flow setting.*

3.4 Discussions

3.4.1 Intuitions

When choosing the fraction of newly issued short-term bonds, equity holders are weighing the benefit of reducing today's rollover losses against the cost of increasing future rollover losses. The negative result in Corollary 1 suggests that the cost of increasing future rollover losses always dominates the gain from today. What is the intuition behind this result?

We have shown that right before default, the future losses caused by maturity shortening, i.e., (18), equal the gain from reducing today's rollover loss, i.e., (17), plus the impact on the value of

[7] For the general case with $\rho c \neq r$, for default being losses to bond values we require $B(y) < D^{rf} = \frac{\rho c + \delta_S + \zeta}{r + \delta_S + \zeta}$.

short-term bonds. Why is this so? Suppose we are at $2dt$ before default; the reason that we need $2dt$ in this thought experiment is that we want to compare today's reduced rollover losses against tomorrow's heavier rollover losses, so we need at least one continuation period. More specifically, equity holders will roll over the maturing bonds at the end of dt, at which point bond holders have the chance of getting repaid fully. Between $[dt, 2dt]$ bond holders receives nothing as the firm defaults at the end of $2dt$.[8]

The short-term (long-term) bond will get a full payment of 1 with a probability of $\delta_S \cdot dt$ ($\delta_L \cdot dt$) over $[0, dt]$; otherwise both get the bankruptcy payout $B(y)$. This value difference $(\delta_S - \delta_L)[1 - B(y)]\, dt$ is reflected in the price wedge set by competitive bond investors. Hence, for equity holders who are refinancing a measure of $m(\phi)\, dt$ of maturing bonds, the relative benefit of issuing short-term bonds instead of long-term bonds (by setting $f = 1$ instead of $f = 0$) is

$$m(\phi)\, dt \cdot (\delta_S - \delta_L)[1 - B(y)]\, dt > 0. \tag{20}$$

However, given that short-term bonds have a higher intensity δ_S of coming due, equity holders realize that the next instant (at the end of dt) they are facing heavier rollover losses. Because at that time both bonds have the same price $B(y)$ which implies a financing short-fall of $B(y) - 1$, this effect equals

$$\begin{aligned} \frac{\partial}{\partial f}\left[(\phi\delta_S + (1-\phi)\,\delta_L)\, dt \cdot (B(y) - 1)\right] &= \frac{\partial \phi}{\partial f} \cdot \frac{\partial}{\partial \phi}\left[(\phi\delta_S + (1-\phi)\,\delta_L)(B(y) - 1)\, dt\right] \\ &= m(\phi)\, dt \cdot (\delta_S - \delta_L)(B(y) - 1)\, dt, \end{aligned} \tag{21}$$

where $\frac{\partial \phi}{\partial f} = m(\phi)\, dt$ from (4) captures how today's issuance policy f affects tomorrow's maturity structure ϕ. As a result, right before default so that only today and tomorrow count, the benefit from saving today's rollover losses in (20) exactly offsets the cost of having higher rollover losses (21) in the next instant!

[8] For illustration purpose, we can think of the coupon payment and upper side event occurs right after the equity holders' default decision.

In the above thought experiment we have kept bond prices unchanged, i.e. $D_S = D_L = B(y)$, so the rollover loss per unit of bond is always $B(y) - 1$. Because $\frac{\partial \phi}{\partial f} = m(\phi)\, dt > 0$, issuing short-term bonds pushes the maturity structure ϕ_t toward the default threshold Φ. This in turn pushes the firm closer to default, bringing about a first-order negative impact on bond prices and hence future rollover losses. Equity holders internalize this negative effect, which is captured by the second term in (18).[9] Consequently, Proposition 1 holds due to this additional negative effect on bond prices when shortening the firm's maturity structure.

3.4.2 Comparison to Brunnermeier and Oehmke (2013)

Our results highlight an economic mechanism that is different from Brunnermeier and Oehmke [2013]. In that paper, the firm with a long-term asset is borrowing from a continuum of identical creditors. Only standard debt contracts are considered with promised face value and maturity, and covenants are not allowed. News about the long-term asset arrives at interim periods, so that a debt contract maturing on that date will be repriced accordingly, as in Diamond [1991]. Under certain situations regarding interim news (e.g., whether it is about profitability or recovery value), Brunnermeier and Oehmke [2013] show that, given other creditors' debt contracts, equity holders find it optimal to deviate by offering any individual creditor a debt contract that matures one period earlier, so that it gets repriced sooner. In equilibrium, equity holders will offer the same deal to every creditor, and the firm's maturity will be "rat raced" to zero.

The repricing mechanism constitutes the key difference between Brunnermeier and Oehmke [2013] and our model. In their model, after negative interim news, a relative short-term bond gets repriced by adjusting up the promised face value to renegotiating bond holders. Because all bonds have the same seniority in sharing the positive recovery, including the repriced ones, repricing causes dilution of those relative long-term bonds without repricing opportunities. Put differently, the rollover losses are absorbed by the promised higher face values, which dilutes existing long-term

[9]The reason that only the short-term bond price D_S shows up is that equity is only issuing short-term bonds in the hypothetical shortening equilibrium. When we focus on lengthening equilibrium, only the long-term bond price D_L shows up; see Corollary 3.

bond holders, relieving equity from having to inject cash into the firm.

As emphasized in Section 2.2.1 when we lay out the assumptions, in our model the firm commits to maintain a constant total outstanding face value when refinancing its maturing bonds. This amounts to a bond covenant about the firm's "book leverage," so that equity holders cannot simply issue more bonds to cover the firm's rollover losses. Instead, equity holders in our model are absorbing these losses through their own deep pockets (or through equity issuance), and existing long-term bonds remain undiluted. Interestingly, once we shut down the interim dilution channel that drives the result in Brunnermeier and Oehmke [2013], we identify a new economic force not present in their paper.

We make the constant face-value assumption for two reasons. First, as it is a standard assumption in the dynamic structural corporate finance models starting from Leland and Toft [1996], our analysis represents the minimum departure from the literature. More importantly, the full commitment on the firm's book leverage policies isolates the standard dilution issues (via promised face values) from the firm's endogenous maturity decisions, which is the focus of our paper. Besides, in practice, most of bond covenants have some restrictions regarding the firm's future leverage policies, but rarely on the firm's future maturity structures. This empirical observation lends support to our premise of a full commitment on the firm's book leverage policy but no commitment on its debt maturity structure policy.

3.5 Robustness of Corollary 1

Before we move on to the next section, we demonstrate that Corollary 1 is robust to several natural extensions, including exogenous default. Readers may skip this section without loss of understanding of the rest of the paper.

3.5.1 Exogenous default boundary

We have so far followed the Leland tradition by assuming that either equity holders have deep pockets or can issue equity in a frictionless fashion. Hence, the default boundary is determined endogenously when the equity's option value of keeping the firm alive is zero, leading to the smooth-pasting condition $E'(\Phi) = 0$. This condition implies a zero IC condition $\Delta(\Phi; y) + E'(\Phi; y) = 0$ at default, and we need the help of Lemma 2 by going one order of derivative higher.

Suppose instead that equity holders are forced to default before they are willing to; this can happen for liquidity reasons if equity holders do not have deep pockets, or financial markets become illiquid due to information-driven problems. Say that the default boundary is $\hat{\Phi}$ with $E\left(\hat{\Phi}\right) = 0$. Then, we must have $E'\left(\hat{\Phi}\right) < 0$ as equity holders always have the option to default earlier than $\hat{\Phi}$; the fact that they hang on during the process $\phi \uparrow \hat{\Phi}$ and strictly prefer to hang on at $\hat{\Phi}$ implies that $E(\phi) > E\left(\hat{\Phi}\right) = 0$ for $\phi < \hat{\Phi}$. In other words, $\hat{\Phi}$ matters only when $\hat{\Phi} < \Phi(y)$. On the other hand, equal seniority implies a zero debt price wedge $\Delta\left(\hat{\Phi}\right) = 0$. As a result, $\Delta\left(\hat{\Phi}\right) + E'\left(\hat{\Phi}\right) < 0$ right before default, and equity holders always want to issue long-term bonds ($f = 0$). This rules out the possibility of shortening equilibria.

3.5.2 Exogenous Poisson default event

In the baseline model the only way to generate a positive price wedge Δ is by the endogenous default decision of the equity holders. However, a positive bond price wedge exists if the firm experiences some exogenous default events. Suppose that the firm is forced to liquidate exogenously after some independent Poisson shock with intensity $\xi > 0$, with the same liquidation value $B(y)$ as endogenous default. Appendix A.5.1 shows that shortening equilibrium cannot exist either in the setting with exogenous Poisson default events.

Moreover, one might think our result in Corollary 1 is partly driven by the particular "no-news-is-bad-news" information setting in the baseline model. The introduction of downward negative liquidation shock with interim bad news rules out this concern.

3.5.3 Relaxed reissuing strategy space

As suggested in (12), the key IC condition compares the pricing wedge to the long-run impact of maturity shortening to equity. It turns out that only the valuation of short-term bonds D_S matters in Corollary 1, although intuitively this condition should involve the valuation of long-term bonds as well. As explained in footnote 9, this is because in shortening equilibria the firm is issuing short-term bonds only, i.e., f is cornered to $f = 1$ given the allowable set of $[0, 1]$.

The assumption of $f \in [0, 1]$ might be violated, as firms can repurchase bonds, or may face certain covenants restricting the firm to reissue certain long-term bonds at minimum. We hence modify the allowable set for the fraction of newly short-term bonds to be $f \in [f_l, f_h]$. Under this assumption, in shortening equilibria the firm takes the highest fraction f_h, which can be either below 1 so that the firm is issuing some mixture of short-term and long-term bonds, or above 1 to accommodate repurchases. In Appendix A.5.1 we show that our result in Corollary 1 holds in this relaxed stetting.

4 Maturity Shortening with Time-Decreasing Cash-Flows

In contrast to Corollary 1, shortening equilibria exist when the firm's cash-flows are deteriorating slowly over time. We show that the general intuition discussed in Section 3.4.1 yields a similar necessary condition for shortening equilibria as in (19); time-varying cash-flows, however, have profound implications which may overturn the negative result in Corollary 1. And, even though lengthening the firm's debt maturity structure can be the more efficient equilibrium, equilibria involving maturity shortening and inefficient early endogenous default can exist.

4.1 Deterministic and Cornered Equilibria

In this section we focus on equilibria where equity holders are taking "deterministic" and "cornered" issuance strategies. Section 5 considers deterministic equilibria with "deterministic" interior issuance policies.

Definition 2 *Equilibria are considered "deterministic" if the firm's issuance policy f_τ is a deterministic function of time-to-default. Equilibria are "deterministic" and "cornered" if the firm's deterministic issuance policy takes a corner solution $f_\tau \in \{0, 1\}$.*

As an example, suppose that we are in the constant cash-flows case studied in Section 3. Proposition 1 and Lemma 1 together imply that there are two possible deterministic and cornered equilibria: either the firm defaults immediately, or the firm keeps issuing long-term bonds and never defaults. In contrast, we will show the equilibrium structure is much richer in the time-varying cash-flow case.

Because cash-flows depend on time-to-default deterministically and there are no other payoff-relevant shocks in the model (other than the upside event shock), focusing on "deterministic" issuance policies essentially rules out sun-spot type equilibria. Cornered strategies are in general optimal for risk-neutral equity holders who are solving a linear problem, and note that the class of "deterministic" and "cornered" equilibria have not ruled out time-varying issuance polices.[10] However, cornered strategies indeed impose restrictions on the set of equilibria. Section 5 considers all possible equilibria, including $f_\tau \in (0,1)$ for some τ.[11]

4.2 Setting and Valuations

In this section, illustration is more straightforward in terms of the dynamics of the firm's time-to-default $\tau \equiv T_b - t$; recall T_b is the firm's endogenous default time. Naturally, $d\tau = -dt$, and y_τ and ϕ_τ are the cash-flow and the maturity structure with τ periods left until default. We call the cash-flow when the firm defaults, i.e., $y_b = y_{\tau=0}$, defaulting or *ultimate* cash-flow; it plays an important role in later analysis.

Let us introduce a time-dependent cash-flow y_τ with drift

$$dy_\tau = \mu_y (y_\tau) \, d\tau, \tag{22}$$

with $\mu_y (y) > 0$. Here, y_τ is increasing with time-to-maturity or y_t is decreasing over time.

[10] For instance, we could have some issuance policy that jumps from $f_\tau = 0$ to $f_{\tau+} = 1$ at certain pre-specified time-to-default τ. However, Lemma 5 in the Appendix shows that this never holds on equilibrium paths.

[11] For instance, an interior issuance policy say $f \in (0,1)$ which affects bond valuations can make equity holders indifferent between shortening ($f = 1$) or lengthening ($f = 0$), which in turn implies the optimality of an interior policy f.

4.2.1 Incentive compatibility and endogenous default

We now have both current cash-flow y and debt maturity ϕ as state variables. Bond values solve the following Partial Differential Equation (PDE) where $i \in \{S, L\}$:

$$\begin{aligned}\underbrace{rD_i(\phi,y)}_{\text{req return}} &= \underbrace{\rho c}_{\text{pre-tax coupon}} + \underbrace{\delta_i\left[1-D_i(\phi,y)\right]}_{\text{maturing}} + \underbrace{\zeta\left[1-D_i(\phi,y)\right]}_{\text{upside option}} \\ &\quad + \underbrace{\left[-\phi\delta_S + m(\phi) f\right]\frac{\partial}{\partial\phi}D_i(\phi,y)}_{\text{maturity structure change}} + \underbrace{\mu_y(y)\frac{\partial}{\partial y}D_i(\phi,y)}_{y\text{change}},\end{aligned} \tag{23}$$

and equity value solves the following PDE

$$\begin{aligned}\underbrace{rE(\phi,y)}_{\text{req return}} &= \underbrace{y-c}_{\text{CF net coupon}} + \underbrace{\zeta\left[E^{rf}-E(\phi,y)\right]}_{\text{upside event}} + \underbrace{\mu_y(y)\frac{\partial}{\partial y}E(\phi,y)}_{y\text{change}} \\ &\quad + \max_{f\in[0,1]}\left\{\begin{array}{c}\underbrace{m(\phi)\left[fD_S(\phi,y)+(1-f)D_L(\phi,y)-1\right]}_{\text{rollover losses}} \\ +\underbrace{\left[-\phi\delta_S+m(\phi)f\right]\frac{\partial}{\partial\phi}E(\phi,y)}_{\text{maturity shortening}}\end{array}\right\}.\end{aligned} \tag{24}$$

The same argument as Section 3.2.2 leads to the same IC condition (12) for equity holders, with a necessary modification to a partial derivative with respect to ϕ due to two-dimensional state space:

$$f = \begin{cases} 1 & \text{if } E_\phi(\phi,y)+\Delta(\phi,y)>0 \\ [0,1] & \text{if } E_\phi(\phi,y)+\Delta(\phi,y)=0 \\ 0 & \text{if } E_\phi(\phi,y)+\Delta(\phi,y)<0 \end{cases}, \tag{25}$$

where throughout we use the subscript notation $E_\phi(\phi,y) \equiv \frac{\partial}{\partial\phi}E(\phi,y)$ to indicate partial derivatives. Let us define $IC(\phi,y) \equiv \Delta(\phi,y) + E_\phi(\phi,y)$, so $IC(\phi,y) > 0$ implies $f = 1$.

Similar to the discussion in Section 3.2.3, at the optimal default boundary equity holders' instantaneous expected flow payoff equals zero. This implies the same default boundary given in (15), which is reproduced here (recall $y_b = y_{\tau=0}$ denotes the defaulting or *ultimate* cash-flow)

$$\Phi\left(y_b\right) = \frac{1}{\delta_S - \delta_L}\left[\frac{y_b - c + \zeta E^{rf}}{1 - B\left(y_b\right)} - \delta_L\right], \text{ with } \Phi'\left(y_b\right) > 0.$$

This gives the endogenous default boundary in the (y, ϕ) space. Lemma 3 gives the smooth pasting property of $E(\cdot, \cdot)$ at the default boundary on the state space of (ϕ, y).

Lemma 3 *At the endogenous default boundary we have value matching condition $E\left(\Phi\left(y_b\right), y_b\right) = 0$, and two smooth-pasting conditions on each dimension $E_\phi\left(\Phi\left(y_b\right), y_b\right) = 0$ and $E_y\left(\Phi\left(y_b\right), y_b\right) = 0$.*

4.2.2 Time-to-default and valuations

In our deterministic model with only Poisson jumps, given the ultimate bankruptcy state ,

$$\left(\phi_{\tau=0} = \Phi\left(y_b\right), y_{\tau=0} = y_b\right)$$

the equilibrium path (ϕ_τ, y_τ) is essentially one-dimensional indexed by time-to-default τ, working our way back from the boundary. Hence given any equilibrium path we can rewrite the bond and equity values by $D_i\left(\tau, y_b\right)$ and $E\left(\tau, y_b\right)$ respectively as a function of τ only, while treating the defaulting cash-flow state y_b as a parameter. Thus, we can rewrite the above two PDEs in their ODE forms:

$$\begin{aligned} rD_i\left(\tau, y_b\right) &= \rho c + \delta_i\left[1 - D_i\left(\tau, y_b\right)\right] + \zeta\left[1 - D_i\left(\tau, y_b\right)\right] - \frac{\partial}{\partial\tau}D_i\left(\tau, y_b\right), \text{ for } i \in \{S, L\}, \quad (26) \\ rE\left(\tau, y_b\right) &= y\left(\tau, y_b\right) - c + \zeta\left[E^{rf} - E\left(\tau, y_b\right)\right] \\ &\quad + m\left(\phi\left(\tau, \phi_b\right)\right)\left[f_\tau D_S\left(\tau, y_b\right) + \left(1 - f_\tau\right)D_L\left(\tau, y_b\right) - 1\right] - \frac{\partial}{\partial\tau}E\left(\tau, y_b\right). \quad (27) \end{aligned}$$

where $y\left(\tau, y_b\right)$ is the cash-flow y_τ given ultimate (defaulting) cash-flow y_b and $\phi\left(\tau, \phi_b\right)$ is the maturity structure ϕ_τ given ultimate (defaulting) maturity structure ϕ_b, and f_τ is the optimal issuance strategy at time-to-default τ. The closed-form solutions for bond values are (recall $\rho c = r$)

$$D_i(\tau, y_b) = 1 - e^{-(r+\delta_i+\zeta)\tau}\left[1 - B(y_b)\right], \text{ for } i \in \{S, L\}. \tag{28}$$

For the solution to equity $E(\tau, y_b)$, see Appendix A.1.3.

Effectively, we are working with the state space of (τ, y_b) instead of the state space of (ϕ, y). Given any (deterministic) equilibrium issuance policy $\{f_\tau\}$, there is a deterministic mapping between these two state spaces.[12] Consequently, via changing coordinates, one can translate $D_i(\tau, y_b)$ and $E(\tau, y_b)$ back to the form of $D_i(\phi, y)$ and $E(\phi, y)$ by solving for τ as a function of (ϕ, y).

4.3 Can Shortening Equilibria Exist?

We revisit the possibility of shortening equilibria in this section, and show a positive result: shortening equilibria can occur in the setting with deteriorating cash flows.

4.3.1 Incentive compatibility condition right before default

As before, we postulate a shortening equilibrium, and evaluate the IC condition (25) right before the default boundary $\Phi(y_b)$. Again, we have a zero IC condition at default: $E_\phi(\Phi(y_b), y_b) = 0$ (Lemma 3) due to the equity's optimal default decision, and equal seniority implies a zero short-long price wedge $\Delta(\Phi(y_b), y_b) = 0$. Hence we analyze the sign of $E_\phi(\phi, y) + \Delta(\phi, y)$ slightly away from $\tau = 0$ along the path of (ϕ_τ, y_τ), i.e., the path which originates at the default state $(\phi_{\tau=0} = \phi_b = \Phi(y_b), y_b)$. Differentiating the IC condition respect to τ, we need to evaluate the sign of

$$IC_\tau(\tau, y_b)\big|_{\tau=0} = \frac{\partial}{\partial\tau}\left[E_\phi(\tau, y_b) + \Delta(\tau, y_b)\right]\Big|_{\tau=0}. \tag{29}$$

If (29) is strictly positive, then $E_\phi(\phi, y) + \Delta(\phi, y) > 0$ for $\tau > 0$ right before default, implying issuing short-term bonds right before default is incentive compatible. Similar to Proposition 1 with constant cash-flows, the next proposition shows that a necessary condition for shortening equilibria

[12] For the technical details on this change of variables, see Appendix A.1.1.

to exist is that shortening debt maturity has a strictly positive partial impact on the value of short-term debt around the vicinity of default.

Proposition 2 *The unique cornered shortening equilibrium, $f = 1$, occurs in the vicinity of $\tau = 0$ if and only if*

$$\frac{\partial}{\partial \phi} D_S \left(\Phi \left(y_b \right), y_b \right) \geq 0. \tag{30}$$

Recall that Corollary 1 states that in the constant cash-flow case, we have $D_S' \left(\Phi; y \right) < 0$ given a positive loss-given-default ($B \left(y \right) < 1$), which rules out the possibility of any shortening equilibria. However, for deteriorating cash-flows the shortening equilibrium exists even with a positive loss-given-default. The next section explains the economic intuition behind this difference.

4.3.2 How condition (30) differs from condition (19)

The condition (30) in Proposition 2 and the condition (19) in Corollary 1 are similar; but they differ in one crucial aspect. Although both involve taking derivative with respect to ϕ, $D_S' \left(\Phi; y \right) > 0$ in (19) has a "total" derivative while $\frac{\partial}{\partial \phi} D_S \left(\Phi \left(y_b \right), y_b \right) > 0$ in (30) has a "partial" derivative. This difference is highlighted when cash-flows are deteriorating over time. In short, when y_τ is time-varying, the cash-flows at the time of default, $y_{\tau=0} = y_b$, and hence the bond recovery value $B \left(y_b \right)$, become endogenous. The partial derivative in (30) exactly reflects this important effect.

We investigate the marginal impact of maturity shortening on bond values around the default boundary. Taking the partial derivative of $D_S \left(\phi, y \right)$ at $\left(\Phi \left(y_b \right), y_b \right)$ with respect to ϕ, i.e., $\frac{\partial D_S \left(\Phi \left(y_b \right), y_b \right)}{\partial \phi}$, and translating everything into the $\left(\tau, y_b \right)$ space, we have:

$$\left. \frac{\partial D_S \left(\tau \left(\phi, y \right), y_b \left(\phi, y \right) \right)}{\partial \phi} \right|_{\tau=0} = \underbrace{\left. \frac{\partial D_S \left(\tau, y_b \right)}{\partial \tau} \frac{\partial \tau}{\partial \phi} \right|_{\tau=0}}_{\text{time-to-default, } (-)} + \underbrace{\left. \frac{\partial D_S \left(\tau, y_b \right)}{\partial y_b} \frac{\partial y_b}{\partial \phi} \right|_{\tau=0}}_{\text{default CF level, } (+)}. \tag{31}$$

The first term captures how maturity shortening affects the firm's time-to-default, which is present in the constant cash-flow case. The novel second term captures the resulting change of default cash-flow level y_b, which directly affects the recovery value $B \left(y_b \right)$ received by bond investors. In

Appendix A.1.1 we show the following intuitive results:

$$\left.\frac{\partial \tau}{\partial \phi}\right|_{\tau=0} < 0, \quad \text{and} \quad \left.\frac{\partial y_b}{\partial \phi}\right|_{\tau=0} > 0. \tag{32}$$

The first sign says that fixing the current cash-flow state, shortening maturity worsens rollover losses and hence reduces the time-to-default τ. For the second sign, because cash-flows are decreasing over time, the reduction of time-to-maturity increases the cash-flows at default.

We analyze the first term in (31). Using (28), we derive the impact of time-to-default on the bond value as (recall $B(y_b) < 1$ and $\rho c = r$)

$$\left.\frac{\partial D_S(\tau, y_b)}{\partial \tau}\right|_{\tau=0} = (r + \delta_S + \zeta)\left[1 - B(y_b)\right] > 0.$$

Together with $\frac{\partial \tau}{\partial \phi} < 0$ in (32), we see that the first term in (31) is negative. Intuitively, shortening the maturity structure edges the firm closer to default, hurting bond values. The same negative force is present in the constant cash-flow case, which goes against condition (30).

In contrast to the constant cash-flow case, there is a second term present when cash-flows are time varying. We derive $\frac{\partial D_S(\tau, y_b)}{\partial y_b}$ using (28):

$$\left.\frac{\partial D_S(\tau, y_b)}{\partial y_b}\right|_{\tau=0} = B'(y_b) > 0.$$

Here, the last inequality holds as the firm's liquidation value is increasing in its profitability. Because $\frac{\partial y}{\partial \phi} > 0$ in (32), the second term in (31) is positive. Intuitively, by bringing the firm closer to default, shortening the maturity structure allows the bond holders to take over the firm earlier with a better fundamental y_b, raising bond values. When the positive second term dominates the negative first term, condition (30) holds and hence shortening equilibria may exist. Section 4.6 gives a numerical example in which the firm follows the path of a shortening equilibrium.

For better illustration, Figure 1 schematically depicts potential paths of a shortening equilibrium for both the case of constant cash-flows and that of time-decreasing cash-flows. In the left panel with

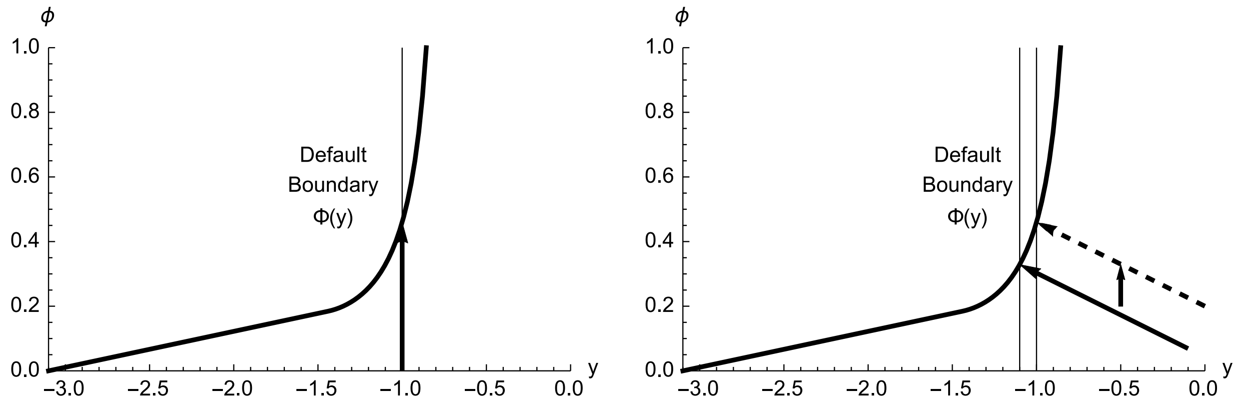

Figure 1: **Schematic graph of shortening equilibrium path. Left panel:** constant cash-flows. **Right panel:** time-decreasing cash-flows. The default boundary $\Phi(y)$ is the same for both settings. The key difference is the impact of shortening maturity on cash-flows at default. In the left panel, the cash-flows at default are fixed at $y_{\tau=0} = y$ irrespective of issuance policy. In contrast, in the right panel, shortening the maturity structure leads to higher cash-flows $y_{\tau=0}$ at default.

constant cash-flows, when the firm issues more short-term bonds, the firm moves closer to default; however, the equilibrium path, as well as the bond recovery value, are unchanged. In contrast, in the right panel with time-decreasing cash-flows, issuing more short-term bonds shortens the firm's survival time, but the firm lands on a path sitting above the equilibrium one. As the second term in (31) captures, this deviating path features greater cash-flows at default and hence a higher bond recovery value.

4.4 Lengthening Equilibria

We now study the equilibria in which equity holders are lengthening the firm's maturity structure. Because of deteriorating cash-flows, equity will default eventually even if the firm keeps lengthening its debt maturity, i.e., $f_\tau = 0$. Almost exactly the same analysis as in Proposition 2 applies in this case. In words, we have a lengthening equilibrium if, at the default boundary, the value of long-term bond gets hurt by maturity shortening.

Proposition 3 *The unique cornered lengthening equilibrium, $f = 0$, occurs in the vicinity of $\tau = 0$*

if and only if

$$\frac{\partial}{\partial \phi} D_L\left(\Phi\left(y_b\right), y_b\right) \leq 0. \tag{33}$$

4.5 Multiple Equilibria and Uniqueness towards Bankruptcy

We have seen that either a unique shortening equilibrium or a unique lengthening equilibrium might exist given the right *ultimate* bankruptcy state. But, can a cornered shortening equilibrium switch to a lengthening equilibrium along the equilibrium path? Lemma 5 in the Appendix gives a negative answer. It shows that, within the class of deterministic equilibria, there cannot be any jumps in equilibrium issuance strategies–either from shortening to lengthening or vice versa. This property helps greatly reduce the dimensionality of the multiplicity of equilibria. An immediate implication is that given any initial state (ϕ, y) (different from the *ultimate* bankruptcy state), within the class of deterministic cornered equilibria, there are at most two unique cornered paths leading to default, either always shortening with $f = 1$ or always lengthening with $f = 0$.

This multiplicity of either shortening equilibrium or lengthening equilibrium emerges without too much surprise, as the intuition is similar to the notion of self-enforcing default in the literature of sovereign debt (e.g., Cole and Kehoe [2000]). More specifically, if bond investors expect equity holders to keep shortening the firm's maturity structure in the future, then bond investors price this expectation in the bond's market valuation, which can self-enforce the optimality of issuing short-term bonds only. Similarly, the belief of issuing long-term bonds always can be self-enforcing as well.

The next proposition shows the optimality of a cornered issuance strategies along the whole path, if indeed such a strategy is optimal at the time of default. In other words, working backwards from the boundary, if $f_0 \in \{0, 1\}$ then equity holders find it optimal to set $f_s = f_0$ for the whole path traced out by $s \in [0, \tau]$, i.e., cornered paths always stay cornered (i.e., never become interior) away from the boundary. Combined with Lemma 5 in the Appendix, we also establish that there exist at most two deterministic cornered equilibria.

Proposition 4 *Given the initial starting value (ϕ, y), there exist (at most) two deterministic cornered equilibria: one with shortening always $f_s = 1$ for $s \in \left[0, \tau^S\right]$ such that $y\left(\tau^S, y_b^S\right) = y$ and $\phi\left(\tau^S, y_b^S\right) = \phi$, and the other with lengthening always $f_s = 0$ for $s \in \left[0, \tau^L\right]$ with $y\left(\tau^L, y_b^L\right) = y$ and $\phi\left(\tau^L, y_b^L\right) = \phi$. Moreover, for the continuous IC condition of either $f_s = 1$ or $f_s = 0$ along the whole path $s \in \left[0, \tau^i\right]$, it is sufficient to check the IC condition on the default boundary given by either (30) or (33), respectively.*

The dynamics embedded in our model allow us to say more. The existence of multiple equilibria is not guaranteed, and for some initial state, either the shortening equilibrium or the lengthening one becomes the unique equilibrium. Intuitively, if the firm starts off extremely close to the default boundary satisfying (30) in Proposition 2, then the only equilibrium path is indeed the shortening equilibrium, as a benign expectation of lengthening maturity in the future is "too late" to save the firm. This intuition can also be expressed in a geometric way, because the respective regions on the boundary for lengthening and shortening equilibria are non-overlapping. Hence, for points close to the boundary, even if we change the issuance strategy arbitrarily, we cannot change the path fast enough by $\left|\frac{d\phi}{dt}\right| < \infty$ to avoid hitting the specific region, due to the bounded issuance strategy space (here, $f \in [0, 1]$).[13] The following proposition summarizes this observation:

Proposition 5 *There exists a no-return region with positive measure, in which starting from there either shortening equilibrium or lengthening equilibrium is the unique equilibrium.*

4.6 An Example with Constant Negative Drift

We now consider the case in which the cash-flow drift is a negative constant, i.e., $dy_t = -\mu dt$ where $\mu > 0$ is a positive constant.

[13]If the issuance strategy space is unbounded, then the firm can change its maturity structure instantaneously so that $\left|\frac{d\phi}{dt}\right| = \infty$, and hence this argument fails.

4.6.1 Liquidation value $B(y)$

We now derive the firm's liquidation value $B(y)$. Motivated by bankruptcy cost, we assume that debt holders are less efficient in running the liquidated firm, relative to equity holders. Specifically, we assume that, post-default, the upside payoff X becomes $\alpha_X X > 0$ with $\alpha_X \in (0,1)$; and, given the current cash-flow y_τ, we assume that the cash-flow post-default becomes $\alpha_y y_\tau$. Since in our numerical examples the defaulting cash-flows $y_b < 0$, to capture the inefficiency we set $\alpha_y > 1$. This specification is similar to Mella-Barral and Perraudin [1997].

For simplicity the liquidated firm is assumed to be unlevered. Also, debt holders will optimally terminate the firm when the expected flow payoff $\alpha_y y_t + \zeta\alpha_X X$ hits zero from above, which implies $B(y) = 0$ at $y = -\frac{\alpha_X}{\alpha_y}\zeta X$. Given this boundary condition, the liquidation value $B(y)$ which satisfies $rB(y) = \alpha_y y + \zeta\left[\alpha_X X - B(y)\right] - \mu_y B'(y)$ can be solved as:

$$B(y) = \begin{cases} \frac{\zeta\alpha_X X + \alpha_y y}{r+\zeta} + \frac{\left(\exp\left[-\frac{(r+\zeta)}{\mu_y}(\zeta\alpha_X X + \alpha_y y)\right] - 1\right)\mu_y}{(r+\zeta)^2} & \text{for } y > -\frac{\alpha_X}{\alpha_y}\zeta X. \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

By setting $\alpha_X = \alpha_y = 1$ we recover the unlevered asset value $A(y)$ defined in (1). The difference $A(y) - B(y) > 0$ is due to the inefficient management of debt holders and thus can be interpreted as a bankruptcy cost.

4.6.2 Shortening and lengthening equilibria

Figure 2 shows the two unique cornered equilibrium paths starting from the same $(\phi, y) = (0, .99)$, one a shortening equilibrium and the other a lengthening equilibrium, together with the default boundary $\Phi(y)$. In the shortening equilibrium, the firm keeps issuing short-term bonds and defaults at $\left(\phi_b^S = \Phi\left(y_b^S\right), y_b^S\right)$ if the upside event fails to realize along the path. Since the defaulting cash-flow y_b^S is negative, $\alpha_y > 1$ says that the firm is experiencing even worse (negative) cash-flows under the debt holders' management. The higher the α_y, the greater the sensitivity of the recovery value to cash-flows, i.e., $B'(y_b)$. From (31) and the discussion afterward, we know that $B'(y_b)$ contributes to the second positive term in (31), which is crucial to guarantee the equity's incentive compatibility condition in the shortening equilibrium.

As shown in Figure 2, there is another lengthening equilibrium given the same initial state, in which equity holders find it optimal to keep issuing long-term bonds and default at $\left(\phi_b^L = \Phi\left(y_b^L\right), y_b^L\right)$. The times of default, T_b, differ greatly across these two equilibria: $T_b^S = 0.43$ for the shortening equilibrium while $T_b^L = 1.55$ for the lengthening equilibrium. In the next section we analyze the welfare of these two equilibria in detail.

Suppose that we are in the shortening equilibrium, i.e., bond investors believe that equity holders will keep shortening the firm's maturity structure. As we mentioned in Section 4.5, if the belief of bond investors switches to "equity holders will keep issuing long-term bonds" in an unanticipated way, then we can switch to a lengthening equilibrium, provided that we are sufficiently far away from default. Once we are too close to the default boundary, however, there cannot be such a switch of belief any more, because the lengthening path would hit $\Phi\left(y_b\right)$ in a shortening region. In other words, lengthening beliefs are inconsistent by backward induction. In some sense, there is a no-return region or "black hole" in the state space: there, the firm is absorbed into the shortening equilibrium, without any hope of returning.

4.7 Welfare Analysis

We study the welfare question in this section. In the setting with time-deteriorating cash-flows, there is a natural optimal stopping time even for unlevered firms. The welfare analysis becomes interesting when we layer this optimal stopping problem on top of a standard equity-debt agency frictions, in which equity is choosing the optimal debt maturity structure to maximize equity value only. We base our analysis on the example in Section 4.6, but we will comment on the generality of our results.

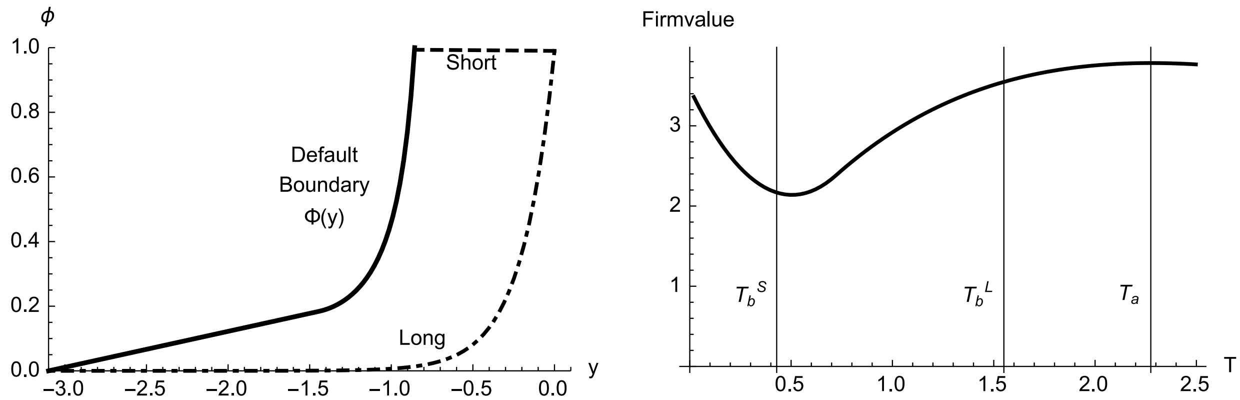

Figure 2: Example with $\rho = 1$, $c = r = 10\%$, $D^{rf} = 1$, $E^{rf} = 12$, $\mu = 2$, $\zeta = .35$, $\delta_S = 10$, $\delta_L = 1$, $\alpha_y = 3$, $\alpha_X = .95$. The initial state is $(y, \phi) = (0, .99)$. **Left panel:** Default boundary (solid line), shortening equilibrium path (dashed line), lengthening equilibrium path (dot-dashed line). **Right panel:** Total firm value as a function of default time T. Here, the default time is $T_b^S = 0.43$ ($T_b^L = 1.55$) for the shortening (lengthening) equilibrium, and the first-best $T_a = T_{FB} = 2.28$ (without tax benefit so $\rho = 1$).

4.7.1 Time of default and firm value

Take any arbitrary, not necessarily equilibrium, time of default denoted by T; we investigate the levered firm value as a function of T in general.

Each instant, in expectation the firm generates cash-flows $[y_t + \zeta X + (\rho - 1)\, c]\, dt$. In default, the firm recovers $B(y_T)$. The cash-flows are discounted at a rate $r + \zeta > r$ due to the upside event. Hence, the levered firm value, given the default time T (we omit the dependence on the initial cash-flow y), is

$$V(T) \equiv \int_0^T e^{-(r+\zeta)t} [y_t + \zeta X + (\rho - 1)\, c]\, dt + e^{-(r+\zeta)T} B(y_T). \tag{35}$$

There are two differences when we compare the levered firm value (35) to the asset's unlevered value $A(y)$ in (1). First, the levered firm receives a tax subsidy $(\rho - 1)\, c$. Second, the levered firm defaults to induce bankruptcy costs, i.e, $B(y) < A(y)$. Note that $B(\cdot)$ has taken into account the potential optimal abandonment time after default in the setting of time-decreasing cash-flows.

Our discussion focuses on the inefficient default timing caused by the equilibrium debt maturity dynamics. For clarity of illustration, in the following analysis we eliminate the debt tax subsidy by setting $\rho = 1$. This way, the only difference between (35) and (1) is the bankruptcy cost embedded in the difference between $B(y)$ and $A(y)$.

Without tax subsidy $\rho = 1$, the optimal stopping time which maximizes (35), call it $T_{FB} = \arg\max_T V(T)$, is simple. Basically, we should set T_{FB} to be T_a, i.e., the optimal abandonment time of the unlevered firm (recall Section 2.1):

$$T_{FB} = T_a = \inf\{t : y_t < -\zeta X\}.$$

This way, we maximize the firm value and minimize the bankruptcy cost to zero.[14]

Although the global optimum of $V(T)$ is trivial, the local behavior of $V(T)$ can be more intriguing for the relevant region $T < T_a = \arg\max_T V(T)$ in which T_b^S and T_b^L lie. The right panel of Figure 2 plots $V(T)$, which is non-monotone in T for T far less than T_a. This implies that there is a region in which faster default, potentially due to shortening of debt maturity, can be welfare enhancing locally!

To better understand the mechanism, we investigate $V'(T)$ which is the marginal impact of delaying default on firm value (multiplying both sides by $e^{(r+\zeta)T}$):

$$\begin{aligned} e^{(r+\zeta)T}V'(T) &= y_T + \zeta X - (r+\zeta)B(y_T) + B'(y_T)\frac{dy_T}{dT} \\ &= \underbrace{y_T + \zeta X - (r+\zeta)A(y_T) + A'(y_T)\frac{dy_T}{dT}}_{\text{first-best stopping problem, (0)}} \\ &\quad + \underbrace{(r+\zeta)\left[A(y_T) - B(y_T)\right]}_{\text{inefficient def. (+)}} + \underbrace{\left[B'(y_T) - A'(y_T)\right]\frac{dy_T}{dT}}_{\text{impact on liq. value, (?)}}. \end{aligned} \tag{36}$$

As standard in any frictionless optimal stopping problem, the first term is zero.[15] The second term captures the positive bankruptcy cost. The third term captures the impact of delaying default on

[14] At $y = -\zeta X$, both $B(y) = A(y) = 0$ because both equity and debt holders will terminate the firm immediately. This implies a zero bankruptcy cost.

[15] This is because the unlevered firm value $A(y)$ satisfies the differential equation $rA(y) = y + \zeta[X - A(y)] + A'(y)\frac{dy}{dt}$.

the firm's liquidation value. In our example with $\alpha_y > 1$, it is possible that $B'(y_T) - A'(y_T) > 0$ because of worsening cash-flows in default. Together with deteriorating cash-flows $\frac{dy_T}{dT} < 0$, this force can make the third term negative. As a result, $V(T)$ may not be always increasing in T, and the right panel of Figure 2 shows that indeed at the shortening equilibrium we have $V'\left(T_b^S\right) < 0$.

Here is another intuition. Recall that $A(y) - B(y)$ captures the (endogenous) bankruptcy cost in our model. When $A'(y_T) - B'(y_T) < 0$, we have an endogenous bankruptcy cost that is decreasing in the defaulting cash-flows. The earlier the default, the higher the defaulting cash-flows, and the smaller the bankruptcy cost. This force contributes to the non-monotonicity of the levered firm value as a function of the default time.

4.7.2 Inefficiency of shortening equilibrium: global versus local

We use the welfare function $V(T)$ to evaluate the welfare across two equilibria. For the example considered in Section 4.6.2, we highlight the two equilibrium default times, T_b^S for shortening equilibrium and T_b^L for lengthening equilibrium. By $\Phi'(y_b) > 0$, shortening equilibria always have a smaller default time than lengthening equilibria, i.e., $T_b^S < T_b^L$. This is indicated by the vertical lines in the right panel of Figure 2. Further, at the initial point the shortening equilibrium is inferior to the lengthening equilibrium, i.e., $V\left(T_b^S\right) < V\left(T_b^L\right)$.

The fact that $V(T)$ is downward sloping at T_b^S is intriguing, which indicates that equity holders are maximizing the whole firm value by shortening the maturity structure if only local deviations were allowed. It turns out that this is not a coincidence. To see this, let us take the derivative of the firm value $V = E + \phi D_S + (1-\phi) D_L$ with respect to the maturity structure ϕ:

$$V_\phi(\phi, y) = \underbrace{\Delta(\phi, y) + E_\phi(\phi, y)}_{\text{Incentive compatibility}} + \underbrace{\phi \frac{\partial}{\partial \phi} D_S(\phi, y)}_{\text{Impact on ST bonds}} + \underbrace{(1-\phi) \frac{\partial}{\partial \phi} D_L(\phi, y)}_{\text{Impact on LT bonds}}. \tag{37}$$

Suppose that we can show that $V_\phi(\phi, y) > 0$ in any shortening equilibrium around $(\phi = \Phi(y), y)$. Then, since maturity shortening leads to earlier default in an equilibrium with cornered strategies, i.e. $\frac{\partial T_b}{\partial \phi} < 0$, we have $V'\left(T_b^S\right) = V_\phi / \frac{\partial T_b^S}{\partial \phi} < 0$ in the vicinity of the default boundary for shortening

Initial $(y,\phi) = (0, .99)$	T_b	$V(T_b)$	$E(\phi, y)$	$D_S(\phi, y)$	$D_L(\phi, y)$
Lengthening equilibrium	$T_b^L = 1.55$	3.55	2.55	0.99	0.89
Shortening equilibrium	$T_b^S = 0.43$	2.17	1.17	0.99	0.82

Table 1: Firm, equity, long-term bond, and short-term bond values for $\rho = 1$, $c = r = 10\%$, $D^{rf} = 1$, $E^{rf} = 12$, $\mu = 2$, $\zeta = .35$, $\delta_S = 10$, $\delta_L = 1$, $\alpha_y = 3$, $\alpha_X = .95$, and initial point $(y,\phi) = (0, .99)$.

equilibria, i.e., the firm value is higher by defaulting earlier locally.[16]

In (37), the first part is equity's IC condition is non-negative in any shortening equilibrium; and the second term is positive due to condition (30) near the bankruptcy boundary. For the third term which is the impact on long-term bonds, under $\rho c = r$ one can show that $\frac{\partial}{\partial\phi} D_L(\phi, y) > \frac{\partial}{\partial\phi} D_S(\phi, y)$ at the default boundary.[17] Thus, in the vicinity of the boundary we have $\frac{\partial}{\partial\phi} D_L(\phi, y) > \frac{\partial}{\partial\phi} D_S(\phi, y) > 0$ in any shortening equilibrium, i.e., long-term bond holders also gains from the earlier default caused by maturity shortening.

The above discussion implies that, in the shortening equilibrium, when the firm is close to default, maturity shortening taken by equity holders improves the firm value locally. In other words, all parties in the firm–right before default–will vote against lengthening the firm's maturity structure marginally! This holds despite the fact that, when the firm is far away from the default boundary, all parties should be better off by taking the globally more efficient lengthening equilibrium. Indeed, in our example, the lengthening equilibrium Pareto dominates the shortening equilibrium, as Table 1 reveals.[18]

We would like to point out that the local-efficiency property of the shortening equilibrium, while intriguing, is less general. For instance, the result that $\frac{\partial}{\partial\phi} D_L(\phi, y) > \frac{\partial}{\partial\phi} D_S(\phi, y)$ in the vicinity of the bankruptcy boundary might change if we do not have $\rho c = r$. Perhaps more empirically relevant situations are that there are other stakeholders in the firm who may suffer from earlier default. In

[16]The opposite holds for lengthening equilibria, where we have $V'\left(T_b^L\right) > 0$ in the vicinity of the bankruptcy boundary. Because we know the first-best $T_{FB} = T_a$ has the longest survival, it is not that surprising to see that lengthening improves the firm value.

[17]It is easy to show that $\left[\frac{\partial D_L(\Phi(y_b), y_b)}{\partial\phi} - \frac{\partial D_S(\Phi(y_b), y_b)}{\partial\phi}\right]_{\tau=0} = (\delta_L - \delta_S)\left[1 - B(y_b)\right]\frac{\partial\tau}{\partial\phi} > 0$ because $\left.\frac{\partial\tau}{\partial\phi}\right|_{\tau=0} < 0$ and $\delta_S > \delta_L$. Intuitively, since short-term bonds are more likely to be paid in full, they depend less on the bankruptcy recovery relative to their long-term counterpart.

[18]This property of Pareto dominance may not holds generally, and we find other numerical examples in which relative to the shortening equilibrium, equity and short-term bond holders gain in the lengthening equilibrium while long-term bond holders lose strictly.

Appendix A.5.2 we consider the situation where the firm has another group of debt holders holding consol bonds whose valuation does not enter the equity holders' rollover decisions at all. Because the value of consol bonds suffer losses due to earlier default, the maturity-shortening equilibrium may become locally inefficient, in the sense that right before default the firm value is improved by marginally lengthening the firm's maturity structure.

5 Equilibria with Interior Issuance Policies

Our analysis has so far focused on deterministic cornered issuance policies, i.e., $f \in \{0, 1\}$. This section extends our analysis to the class of all deterministic equilibria by allowing for the possibility of interior issuance policies so that $f \in [0, 1]$ and shows uniqueness of the equilibrium on the default boundary.

5.1 Unique Equilibrium around Default Boundary

We work on the state space of (τ, y_b), and denote the equilibrium issuance strategy by $f(\tau, y_b) \in [0, 1]$. Recall the equity's IC condition $IC(\tau, y_b) \equiv \Delta(\tau, y_b) + E_\phi(\tau, y_b)$ with $IC(0, y_b) = 0$. We investigate $IC_\tau(\tau, y_b)$, which is the partial derivative of IC condition along the direction of time-to-default. Evaluating on the default boundary, i.e., $\tau = 0$, we show in the Appendix that

$$IC_\tau(0, y_b) = m(\phi_b)\left[f(0, y_b)\frac{\partial}{\partial\phi}\Delta(0, y_b) + \frac{\partial}{\partial\phi}D_L(0, y_b)\right]. \tag{38}$$

Denote by $f_{\tau=0}$ the equilibrium issuance policy $f(0, y_b)$ at $\tau = 0$. We have shown that, if $f_{\tau=0}$ takes cornered values, we must have $IC_\tau(0, y_b)|_{f_{\tau=0}=1} > 0 \Leftrightarrow \frac{\partial D_S(\Phi(y_b), y_b)}{\partial\phi} > 0$ or $IC_\tau(0, y_b)|_{f_{\tau=0}=0} < 0 \Leftrightarrow \frac{\partial D_L(\Phi(y_b), y_b)}{\partial\phi} < 0$ so that equity finds it optimal to issue short-term or long-term debt right before default, respectively. If $f \in (0, 1)$ which is an interior value, then we must have $IC_\tau(0, y_b) = 0$ in (38), i.e.,[19]

$$f = -\frac{\frac{\partial}{\partial\phi}D_L(0, y_b)}{\frac{\partial}{\partial\phi}\Delta(0, y_b)}. \tag{39}$$

[19]We show that the right hand side of (39) is independent of f in the proof of Proposition 6 in the Appendix.

We now show generally that on the default boundary the equilibrium is unique, either interior or cornered. Thus, there is no-return regions in which the multiplicity of equilibria vanishes to yield a unique one when sufficiently close to the default boundary, as shown by Proposition 5.

Proposition 6 *Focus on the class of deterministic equilibria. Then, if admissible as given by (A.16), there exists a unique equilibrium $f_{\tau=0}$ on the default boundary. Further, if $f_{\tau=0} \in \{0,1\}$ and we are at the point where the constraint $f \in [0,1]$ is strictly binding, we have a unique equilibrium also in the vicinity of the default boundary $(\Phi(y_b), y_b)$.*

The intuition is similar to the discussion at the end of Section 4.5. The source of multiplicity comes from the self-enforcing expectations of future (issuance) policies. However, if the firm is close to the default boundary, then there will be not enough room for future expectations to be self-enforcing, and a unique equilibrium arises. Once we move further away from the boundary, there might be enough time for self-enforcing expectations to introduce multiplicity. Geometrically, this implies that sufficiently far away from the default boundary, paths can cross each other, and hence multiple equilibria emerge (see Figure 2).

5.2 An Example of Equilibrium with Interior Issuance Policy

When we are away from default, the analysis becomes more complicated when allowing for interior issuance polices. We show in the proof of Lemma 7 in the Appendix that along any path that has $IC(\tau, y_b) = 0$ for $\tau \in [0, s]$ a unique $f_\tau \in (0,1)$ exists for every $\tau \in [0, s]$. Further, we can derive the unique interior issuance policy f_τ at τ explicitly given the forward-looking endogenous equilibrium objects.[20] These endogenous equilibrium objects are essentially functions of the equilibrium path $\{f_s\}_{s=0}^{\tau}$, and together with $f_{\tau=0}$ given in (39) we can solve for f_τ by backward induction. Thus, in conjunction with Proposition 6 this implies that there exists a unique path to any admissible bankruptcy point $(\Phi(y_b), y_b)$.[21] The idea is that if the equilibrium issuance policy $f_\tau \in (0,1)$ for

[20] That is, incorporating all times from today until the default time.

[21] Essentially, the bankruptcy point is *admissible* if the path is pointing away from the bankruptcy set $\mathcal{B}$, not into $\mathcal{B}$, with respect to τ.

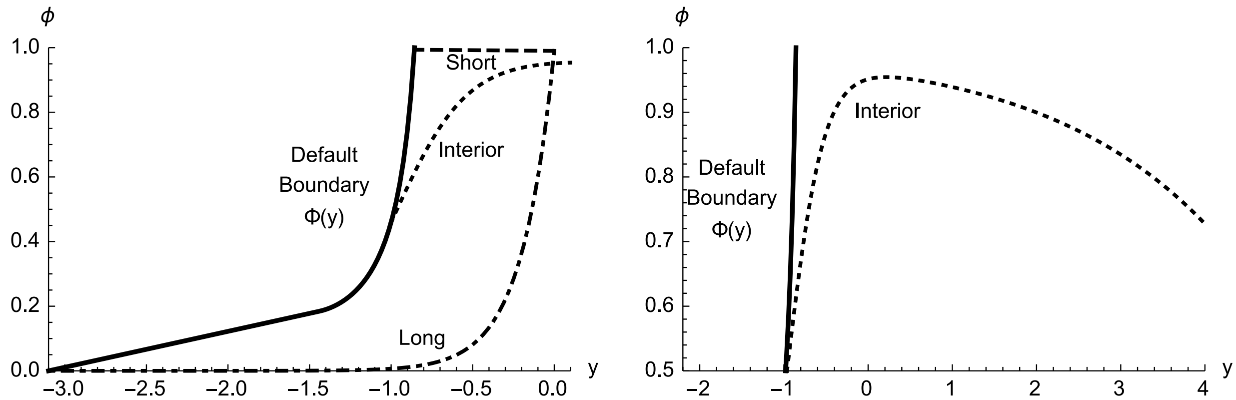

Figure 3: **Equilibrium with interior issuance policy. Left panel:** Interior issuance policies while also showing cornered strategy equilibria. **Right panel:** extended graph of interior issuance equilibrium path with non-monotone path (first shortening, then lengthening when close to default).

$\tau > 0$, then $IC(\tau, y_b)$ has to remain at zero; and $IC(\tau, y_b)$ becomes strictly positive (negative) if $f_\tau = 1$ ($f_\tau = 0$).

Let us pick an ultimate point on the boundary that lies in the region of interior equilibria, and work our way backward to trace out the path. Figure 3 maps one such path, first in the left panel in relation to the previous analyzed corner equilibrium paths, and then in a zoomed-out fashion in the right panel. In the left panel, sufficiently far away from the default boundary our interior path crosses with our previous lengthening equilibrium path, leading to multiple equilibria at that intersection point: one lengthening, and (at least) one interior equilibrium. The right panel reveals that the firm's maturity structure is no longer monotone in time (as implied by the cornered equilibria) in this interior issuance equilibrium, i.e., $\frac{d\phi(t)}{dt}$ switches signs. For large y's the firm is shortening its maturity structure (high f, although not cornered), leading to a slow rise in ϕ. However, once the firm is getting close enough to $\Phi(y)$, it will reverse course and start lengthening its maturity structure (low f, although not cornered).

6 Conclusion

We study a dynamic setting in which a firm can commit to keeping the overall face-value of debt outstanding constant, but cannot commit to its future maturity structure. Instead, the firm chooses its debt maturity structure optimally over time in response to observable firm fundamentals. It controls its maturity structure via choosing the fraction of newly issued short-term bonds when refinancing its matured bonds. As a baseline, we show that when the firm's cash-flows are constant then it is impossible to have the "shortening-to-death" equilibrium where the firm keeps issuing short-term bonds and default consequently. This is because the recovery in default is constant, and the maturity structure just imposes faster default which hurts bond-holders.

In contrast, when cash-flows deteriorate over time so that the debt recovery value is affected by the endogenous default timing, then a shortening equilibrium can emerge. For a shortening equilibrium to arise, from the perspective of bond holders, the benefit of a more favorable recovery value by taking the firm over earlier must outweigh the increased expected default risk due to earlier default. The shortening equilibrium can be locally efficient while being globally inefficient, relative to the lengthening equilibrium. We further show that the model has a unique cornered equilibrium when a firm is sufficiently close to default.

References

Mark Aguiar and Manuel Amador. Take the short route: How to repay and restructure sovereign debt with multiple maturities. *Working Paper*, 2013.

Cristina Arellano and Ananth Ramanarayanan. Default and the maturity structure in sovereign bonds. *Journal of Political Economy*, 120:187–232, 2012.

Markus K Brunnermeier and Martin Oehmke. The Maturity Rat Race. *Journal of Finance*, 68(2): 483–521, 2013.

Charles W Calomiris and Charles M Kahn. The Role of Demandable Debt in Structuring Optimal Banking Arrangements. *American Economic Review*, 81(3):497–513, June 1991. URL `http://www.jstor.org/stable/2006515`.

Hui Chen, Yu Xu, and Jun Yang. Systematic Risk, Debt Maturity, and the Term Structure of Credit Spreads. *Working Paper*, 2013.

Hui Chen, Rui Cui, Zhiguo He, and Konstantin Milbradt. Liquidity and default of corporate bonds over the business cycle. *Working Paper*, 2014.

Harold L Cole and T J Kehoe. Self-fulfilling debt crises. *Review of Economic Studies*, 67:91–116, 2000.

Peter DeMarzo and Zhiguo He. Leverage dynamics without commitment. *Working Paper*, 2014.

D W Diamond. Debt Maturity and Liquidity Risk. *Quarterly Journal of Economics*, 106(3):709–737, August 1991. URL `http://www.jstor.org/stable/2937924`.

D W Diamond and Zhiguo He. A theory of debt maturity: the long and short of debt overhang. *Journal of Finance*, 69(2):719–762, 2014.

Douglas W Diamond and Raghuram G Rajan. Liquidity Risk, Liquidity Creation, and Financial

Fragility: A Theory of Banking. *Journal of Political Economy*, 109(2):287–327, April 2001. URL `http://www.jstor.org/stable/3078420`.

Alessandro Dovis. Efficient sovereign default. *Working Paper*, 2012.

Mark J Flannery. Asymmetric information and risky debt maturity choice. *Journal of Finance*, 41 (1):19–37, March 1986. URL `http://www.jstor.org/stable/2328342`.

Mark J Flannery. Debt Maturity and the Deadweight Cost of Leverage: Optimally Financing Banking Firms. *American Economic Review*, 84(1):320–331, March 1994. URL `http://www.jstor.org/stable/2117987`.

Zhiguo He and Konstantin Milbradt. Endogenous liquidity and defaultable bonds. *Econometrica*, 82(4):1443–1508, 2014.

Zhiguo He and Wei Xiong. Rollover Risk and Credit Risk. *Journal of Finance*, 67(2):391–429, 2012. URL `http://www.jstor.org/stable/41419701`.

Hayne Leland. Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance*, 49(4):1213–1252, September 1994a. URL `http://www.jstor.org/stable/2329184`.

Hayne Leland. Bond prices, yield spreads, and optimal capital structure with default risk. *Working Paper*, 1994b.

Hayne Leland. Agency costs, risk management, and capital structure. *Journal of Finance*, 53(4): 1213–1243, August 1998. URL `http://www.jstor.org/stable/117400`.

Hayne Leland and Klaus Bjerre Toft. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *Journal of Finance*, 51(3):987–1019, July 1996. URL `http://www.jstor.org/stable/2329229`.

Timothy J. McQuade. Stochastic Volatility and Asset Pricing Puzzles. *Working Paper*, 2013.

Pierre Mella-Barral and William Perraudin. Strategic debt service. *Journal of Finance*, 52(2): 531–556, June 1997. URL `http://www.jstor.org/stable/2329489`.

Qiping Xu. Kicking the Maturity Down the Road: Early Refinancing and Maturity Management in the Corporate Bond Market. *Working Paper*, 2014.

A Appendix

A.1 Change of variables, value functions, recovery value and admissibility

A.1.1 Change of variables

We will solve the model in terms of (τ, y_b), that is time to maturity and cash-flow at time of bankruptcy. This makes the equations all ODEs that we have to consider on the equilibrium path. We then calculate separately the IC conditions via the derivatives of E_ϕ under different assumptions of the issuance strategies. For the moment, fix the issuance strategy $f(\tau)$.

The proportion of short-term debt $\phi(\tau, y_b)$. Recall that we have $\phi \equiv \frac{S}{P}$ is the proportion of short-term debt. Consider an arbitrary path $f(\tau) \in [0,1]$ for the issuance strategy. Then, we have

$$\phi'(\tau) = \phi(\tau)\left[\delta_S(1 - f(\tau)) + f(\tau)\delta_L\right] - \delta_L f(\tau)$$

Integrating up, imposing $\phi(0) = \phi_b = \Phi(y_b)$, we have

$$\phi(\tau, y_b) = e^{\int_0^\tau [\delta_S(1-f_s)+f_s\delta_L]ds}\left[\Phi(y_b) - \delta_L \int_0^\tau e^{-\int_0^s [\delta_S(1-f_u)+f_u\delta_L]du} f_s ds\right] \tag{A.1}$$

Taking derivatives, while keeping $f(\tau)$ fixed, we have

$$\frac{\partial h_1}{\partial \tau} = \frac{\partial \phi(\tau, y_b)}{\partial \tau} = \phi(\tau, y_b)\left[\delta_S(1 - f(\tau)) + f(\tau)\delta_L\right] - \delta_L f(\tau) \tag{A.2}$$

$$\frac{\partial h_1}{\partial y_b} = \frac{\partial \phi(\tau, y_b)}{\partial y_b} = \Phi'(y_b)\, e^{\int_0^\tau [\delta_S(1-f_s)+f_s\delta_L]ds} \tag{A.3}$$

The current cash-flow state $y(\tau, y_b)$. Next, let us assume there exists h_0 so that $h_0(y_\tau) = h_0(y_b) + \mu\tau$. In the linear growth specification, we have $h_0(x) = x$, whereas in the exponential growth specification we have $h_0(x) = \log(x)$.

Derivatives w.r.t. ϕ. The ODEs are solved in terms of

$$\mathbf{z} = (\tau, y_b) \tag{A.4}$$

However, the incentives of the equity holders are derived from the Markov system

$$\mathbf{x} = (\phi, y), \tag{A.5}$$

as the optimal f requires the derivative E_ϕ. We are looking for points $\mathbf{z} = \mathbf{g}(\mathbf{x})$ such that $\mathbf{h}(\mathbf{x}, \mathbf{z}) = \mathbf{h}(\mathbf{x}, \mathbf{g}(\mathbf{x})) = \mathbf{0}$ where

$$\mathbf{h}(\mathbf{x}, \mathbf{z}) = \begin{bmatrix} h_1(\mathbf{x}, \mathbf{z}) \\ h_2(\mathbf{x}, \mathbf{z}) \end{bmatrix} = \begin{bmatrix} -\phi + \phi(\tau, y_b) \\ -h_0(y) + h_0(y(\tau, y_b)) \end{bmatrix} = \mathbf{0} \tag{A.6}$$

and where

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} \tau(\phi, y) \\ y_b(\phi, y) \end{bmatrix} \tag{A.7}$$

To calculate the derivative of of for example$E(\tau, y_b) = E(\mathbf{z})$ w.r.t. ϕ, we have to use

$$\frac{\partial}{\partial \phi} E(\tau, y_b) = E_\tau(\tau, y_b)\frac{\partial \tau}{\partial \phi} + E_{y_b}(\tau, y_b)\frac{\partial y_b}{\partial \phi} = \left[\frac{\partial}{\partial \mathbf{z}} E(\mathbf{z})\right] \cdot \left[\frac{\partial \mathbf{z}}{\partial \phi}\right] \tag{A.8}$$

The Jacobian matrix is given by

$$\mathbf{J} = \frac{\partial \mathbf{h}(\mathbf{x}, \mathbf{z})}{\partial \mathbf{z}} = \begin{bmatrix} \frac{\partial h_1}{\partial \tau} & \frac{\partial h_1}{\partial y_b} \\ \frac{\partial h_2}{\partial \tau} & \frac{\partial h_2}{\partial y_b} \end{bmatrix} \tag{A.9}$$

Then, applying the chain rule when taking the derivative w.r.t. x_i, $\frac{\partial \mathbf{h}}{\partial x_i} + \frac{\partial \mathbf{h}}{\partial \mathbf{z}}\frac{\partial \mathbf{z}}{\partial x_i} = \mathbf{0}$, we have for $x_i = \phi$,

$$\frac{\partial \mathbf{z}}{\partial \phi} = \frac{\partial \mathbf{g}(\mathbf{x})}{\partial \phi} = \frac{\partial}{\partial \phi}\begin{bmatrix} \tau(\phi, y) \\ y_b(\phi, y) \end{bmatrix} = -\mathbf{J}^{-1}\frac{\partial}{\partial \phi}\mathbf{h}(\mathbf{x}, \mathbf{z}) \tag{A.10}$$

Let us calculate the different derivatives. First, we have

$$\frac{\partial h_1}{\partial \phi} = -1 \tag{A.11}$$
$$\frac{\partial h_2}{\partial \phi} = 0 \tag{A.12}$$

so that $\frac{\partial}{\partial \phi}\mathbf{h}(\mathbf{x},\mathbf{z}) = -[1,0]^\top$. Then, we have

$$\begin{aligned}
\frac{\partial \mathbf{z}}{\partial \phi} = \begin{bmatrix} \frac{\partial \tau(\phi,y)}{\partial \phi} \\ \frac{\partial y_b(\phi,y)}{\partial \phi} \end{bmatrix} &= -\begin{bmatrix} \frac{\partial h_1}{\partial \tau} & \frac{\partial h_1}{\partial y_b} \\ \frac{\partial h_2}{\partial \tau} & \frac{\partial h_2}{\partial y_b} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial h_1}{\partial \phi} \\ \frac{\partial h_2}{\partial \phi} \end{bmatrix} \\
&= \frac{1}{\frac{\partial h_1}{\partial \tau}\frac{\partial h_2}{\partial y_b} - \frac{\partial h_1}{\partial y_b}\frac{\partial h_2}{\partial \tau}} \begin{bmatrix} \frac{\partial h_2}{\partial y_b} & -\frac{\partial h_1}{\partial y_b} \\ -\frac{\partial h_2}{\partial \tau} & \frac{\partial h_1}{\partial \tau} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{\frac{\partial h_1}{\partial \tau}\frac{\partial h_2}{\partial y_b} - \frac{\partial h_1}{\partial y_b}\frac{\partial h_2}{\partial \tau}} \begin{bmatrix} \frac{\partial h_2}{\partial y_b} \\ -\frac{\partial h_2}{\partial \tau} \end{bmatrix} \\
&= \frac{1}{h_0'(y_b)\frac{\partial \phi(\tau,y_b)}{\partial \tau} - \mu \frac{\partial \phi(\tau,y_b)}{\partial y_b}} \begin{bmatrix} h_0'(y_b) \\ -\mu \end{bmatrix}
\end{aligned} \tag{A.13}$$

Thus, we ultimately have

$$\begin{bmatrix} \frac{\partial \tau(\phi,y)}{\partial \phi} \\ \frac{\partial y_b(\phi,y)}{\partial \phi} \end{bmatrix} = \frac{1}{h_0'(y_b)\left\{\phi(\tau,y_b)\left[\delta_S(1-f(\tau)) + f(\tau)\delta_L\right] - \delta_L f(\tau)\right\} - \mu \Phi'(y_b)\, e^{\int_0^\tau [\delta_S(1-f_s)+f_s\delta_L]ds}} \begin{bmatrix} h_0'(y_b) \\ -\mu \end{bmatrix} \tag{A.14}$$

A.1.2 Admissible paths

The bankruptcy boundary and the change of variables interact in a specific way. Essentially, we cannot allow such f's that will point inside the bankruptcy region $\mathcal{B}$ when increasing τ. To this end, we need to impose

$$\Phi'(y_b) > \frac{\frac{d}{d\tau}\phi(\tau,y_b)\big|_{\tau=0}}{\frac{d}{d\tau}y(\tau,y_b)\big|_{\tau=0}} = \frac{\Phi(y_b)\left[\delta_S(1-f) + f\delta_L\right] - \delta_L f}{\mu / h_0'(y_b)} \tag{A.15}$$

at $\tau = 0$. Multiplying through by $\mu/h_0'(y_b) > 0$, and rearranging, we have the following inequality that defines admissible f:

$$0 > h_0'(y_b)\left\{\Phi(y_b)\left[\delta_S(1-f) + f\delta_L\right] - \delta_L f\right\} - \mu \Phi'(y_b) \tag{A.16}$$

A.1.3 Debt and Equity solutions for Section 4.2

Next, let us derive debt and equity values for given paths of f for (τ, y_b).

Debt. Debt has an ODE

$$(r + \delta_i + \zeta) D_i(\tau, y_b) = (\rho c + \delta_i + \zeta) - \frac{\partial}{\partial \tau} D_i(\tau, y_b)$$

that is solved by

$$D_S(\tau, y_b) = \frac{\rho c + \delta_S + \zeta}{r + \delta_S + \zeta} + e^{-(r+\delta_S+\zeta)\tau}\left[B(y_b) - \frac{\rho c + \delta_S + \zeta}{r + \delta_S + \zeta}\right] \tag{A.17}$$
$$D_L(\tau, y_b) = \frac{\rho c + \delta_L + \zeta}{r + \delta_L + \zeta} + e^{-(r+\delta_L+\zeta)\tau}\left[B(y_b) - \frac{\rho c + \delta_L + \zeta}{r + \delta_L + \zeta}\right] \tag{A.18}$$

Importantly, for a given (τ, y_b) debt values are independent of the path of f. Imposing $\rho c = 1$ we get the result in the main text.

Equity. Equity solves the ODE where $y = y(\tau, y_b)$ and $\phi = \phi(\tau, y_b)$

$$(r + \zeta) E(\tau, y_b) = y + \zeta E^{rf} - c + m(\phi)\left[f D_S(\tau, y_b) + (1-f) D_L(\tau, y_b) - 1\right] - \frac{\partial}{\partial \tau} E(\tau, y_b) \tag{A.19}$$

with boundary condition $\frac{\partial}{\partial \tau} E(\tau, y_b)\big|_{\tau=0} = 0$. For future reference, we will differentiate w.r.t. ϕ to get

$$E_{\tau\phi}(\tau, y_b) = m'(\phi)\left[f D_S(\tau, y_b) + (1-f) D_L(\tau, y_b) - 1\right] + m(\phi)\left[f \frac{\partial D_S(\tau, y_b)}{\partial \phi} + (1-f)\frac{\partial D_L(\tau, y_b)}{\partial \phi}\right] - (r+\zeta) E_\phi(\tau, y_b) \tag{A.20}$$

where we abused notation for $E_{\tau\phi}$. Differentiating w.r.t. y_b, we have

$$\begin{aligned}(r+\zeta)\frac{\partial E(\tau, y_b)}{\partial y_b} &= \frac{\partial y(\tau, y_b)}{\partial y_b} + (\delta_S - \delta_L)\left[f D_S(\tau, y_b) + (1-f) D_L(\tau, y_b) - 1\right]\frac{\partial \phi(\tau, y_b)}{\partial y_b} \\ &\quad + m(\phi)\left[f\frac{\partial D_S(\tau, y_b)}{\partial y_b} + (1-f)\frac{\partial D_L(\tau, y_b)}{\partial y_b}\right] - \frac{\partial}{\partial \tau}\left(\frac{\partial E(\tau, y_b)}{\partial y_b}\right)\end{aligned} \tag{A.21}$$

with boundary condition $\frac{\partial}{\partial \tau}\left(\frac{\partial E(\tau, y_b)}{\partial y_b}\right)\Big|_{\tau=0} = 0$ and where we used $m'(\phi) = \delta_S - \delta_L$. Integrating up for a given path of f, we have

$$\begin{aligned}E(\tau, y_b) &= \int_0^\tau e^{(r+\zeta)(u-\tau)}\Big\{y(u, y_b) + \zeta E^{rf} - c \\ &\quad + m(\phi(u, y_b))\left[f_u D_S(u, y_b) + (1-f_u) D_L(u, y_b) - 1\right]\Big\} du\end{aligned} \tag{A.22}$$

Here, we can see how f affects the value of equity even for a given (τ, y_b). Integrating up $\frac{\partial E(\tau, y_b)}{\partial y_b}$, we have

$$\begin{aligned}\frac{\partial E(\tau, y_b)}{\partial y_b} &= \int_0^\tau e^{(r+\zeta)(u-\tau)}\Bigg\{\frac{\partial y(u, y_b)}{\partial y_b} \\ &\quad + (\delta_S - \delta_L)\left[f_u D_S(u, y_b) + (1-f_u) D_L(u, y_b) - 1\right]\frac{\partial \phi(u, y_b)}{\partial y_b} \\ &\quad + m(\phi(u, y_b))\left[f_u\frac{\partial D_S(u, y_b)}{\partial y_b} + (1-f_u)\frac{\partial D_L(u, y_b)}{\partial y_b}\right]\Bigg\} du\end{aligned} \tag{A.23}$$

A.2 Proofs of Section 3

Proof of Lemma 1. We use the fact that $B(y) \le D_S \le D^{rf} = 1$ and $B(y) \le D_L \le D^{rf} = 1$ to bound the rollover term in (6):

$$\begin{aligned}0 \ge\, & m(\phi_t)\left[f_t D_S(\phi_t; y) + (1-f_t) D_L(\phi_t; y) - 1\right] \ge m(\phi_t)\left[f_t B(y) + (1-f_t) B(y) - 1\right] \\ =\, & m(\phi_t)\left[B(y) - 1\right] \ge \delta_S\left[B(y) - 1\right],\end{aligned}$$

where we used $\delta_S = \max_{\phi\in[0,1]} m(\phi)$. Hence if $y - c + \zeta E^{rf} < 0$ then the cash flows to equity are always negative, leading to immediate default. On the other hand, if $y - c + \zeta E^{rf} + \delta_S[B(y) - 1] > 0$, then even under the most pessimistic beliefs equity holders never make losses and thus never default. ■

Proof of Lemma 2. We have shown that at default point Φ the incentive compatibility condition (12) just holds with equality. If (16) holds, then $\Delta(\Phi - \epsilon; y) + E'(\Phi - \epsilon; y)$ is strictly negative. According to (12), it is never optimal to choose $f = 1$ right before default at $\phi = \Phi - \epsilon$. ■

Proof of Proposition 1. The equation of (11) for $f = 1$ is:

$$rE(\phi; y) = y - c + \zeta\left[E^{rf} - E(\phi; y)\right] + \left[\phi\delta_S + (1-\phi)\delta_L\right]\left[D_S(\phi; y) - 1\right] + (1-\phi)\delta_L E'(\phi; y) \tag{A.24}$$

We then take the derivative with respect to ϕ of (A.24):

$$(r+\zeta) E'(\phi; y) = (\delta_S - \delta_L)\left[D_S(\phi; y) - 1\right] + \left[\phi\delta_S + (1-\phi)\delta_L\right] D_S'(\phi; y) - \delta_L E'(\phi; y) + (1-\phi)\delta_L E''(\phi; y).$$

Evaluating this equation at the default boundary Φ, together with $E'(\Phi; y) = 0$ and $D_S(\Phi; y) = B(y)$, we have equation (18) in the main text. ■

A.3 Proofs of Section 4

Proof of Lemma 3. $E(\Phi(y_b), y_b) = 0$ at default is obvious as equity defaults when their cash-flows turn exactly zero in our deterministic setting. Plugging in $E(\Phi(y_b), y_b) = 0$ into the ODE for equity valuation, we see that

$E_\tau(\tau, y_b)|_{\tau=0} = 0$. Change the coordinates of the state space to (ϕ, y), we have

$$E_\phi(\Phi(y_b), y_b)|_{\tau=0} = E_\tau(\tau, y_b) \left.\frac{\partial \tau}{\partial \phi}\right|_{\tau=0} + E_{y_b}(\tau, y_b) \left.\frac{\partial y_b}{\partial \phi}\right|_{\tau=0} = E_{y_b}(\tau, y_b) \left.\frac{\partial y_b}{\partial \phi}\right|_{\tau=0}, \tag{A.25}$$

where we use $E_\tau(\tau, y_b)|_{\tau=0} = 0$. However, we have $E_{y_b}(\tau, y_b)|_{\tau=0}$, because equity defaults at $\tau = 0$ and y_b only affects the recovery value that bond holders receives and equity holders get nothing. (Note that $E_{y_b}(\tau, y_b)$ fixes τ while changes y_b; it differs from $E_y(\phi, y)$). Similarly we can show $E_y(\Phi(y_b), y_b) = 0$. ■

Proof of Proposition 2. See proof of Proposition 6 ■

Proof of Proposition 3. See proof of Proposition 6 ■

Proof of Proposition 4. Combine proof of Proposition 6 with Lemma 4 below. ■

Lemma 4 *For cornered equilibria, the IC condition on the default boundary is sufficient for the IC along the whole path.*

Proof of Lemma 4. We start with the following observation. Suppose the current state of the system is given by (ϕ, y). Firm-value is then given by

$$V(\phi, y) = E(\phi, y) + \phi D_S(\phi, y) + (1 - \phi) D_L(\phi, y)$$

Suppose we consider an arbitrary equilibrium path $(\phi, y) \to (\Phi(y_b), y_b)$ where default occurs at the point $(\Phi(y_b), y_b)$. We know that the default time is

$$\tau = \frac{y - y_b}{\mu}$$

by the linear growth specification of y. That is, we fix the starting point and the end-point of the path, and thereby the time to default, but leave the actual issuance strategy $\{f_\tau\}$ and thus the actual path taken by ϕ undefined. Let us sum up all the cash-flows to get an alternate expression for firm value,

$$V(\tau, y_b) = \int_0^\tau e^{(r+\zeta)(s-\tau)} [y_b + \mu s + \zeta X + (\rho - 1) c]\, ds + e^{-(r+\zeta)\tau} B(y_b)$$

and we can thus define

$$E(\tau, y_b) = V(\tau, y_b) - \phi(\tau, y_b) D_S(\tau, y_b) - [1 - \phi(\tau, y_b)] D_L(\tau, y_b)$$

Importantly, we see that equity value is invariant to the specific path of ϕ taken as long as y_b and thus τ is held fixed. However, incentives are not invariant to the path taken, as we will show below. Consider now

$$\begin{aligned} E_\phi &= \frac{\partial}{\partial \phi}[V(\phi, y) - \phi D_S(\phi, y) - (1 - \phi) D_L(\phi, y)] \\ &= \left\{\frac{\partial}{\partial \tau} V(\tau, y_b) - \phi \frac{\partial}{\partial \tau} D_S(\tau, y_b) - (1 - \phi) \frac{\partial}{\partial \tau} D_L(\tau, y_b)\right\} \frac{\partial \tau}{\partial \phi} \\ &\quad + \left\{\frac{\partial}{\partial y_b} V(\tau, y_b) - \phi \frac{\partial}{\partial y_b} D_S(\tau, y_b) - (1 - \phi) \frac{\partial}{\partial y_b} D_L(\tau, y_b)\right\} \frac{\partial y_b}{\partial \phi} \\ &\quad - D_S(\phi, y) + D_L(\phi, y) \end{aligned}$$

so that we have, after rearranging

$$\begin{aligned} IC(\tau, y_b) &= \Delta(\tau, y_b) + E_\phi(\tau, y_b) \\ &= \left\{\frac{\partial}{\partial \tau} V(\tau, y_b) - \phi(\tau, y_b) \frac{\partial}{\partial \tau} D_S(\tau, y_b) - [1 - \phi(\tau, y_b)] \frac{\partial}{\partial \tau} D_L(\tau, y_b)\right\} \frac{\partial \tau}{\partial \phi} \\ &\quad + \left\{\frac{\partial}{\partial y_b} V(\tau, y_b) - \phi(\tau, y_b) \frac{\partial}{\partial y_b} D_S(\tau, y_b) - [1 - \phi(\tau, y_b)] \frac{\partial}{\partial y_b} D_L(\tau, y_b)\right\} \frac{\partial y_b}{\partial \phi} \end{aligned}$$

Thus, importantly, we see that the path of f for a given $\phi(\tau, y_b)$ only is reflected in $e^{\int_0^\tau [\delta_S(1-f_s) + \delta_L f_s] ds}$ that enters through the change-of-variables. Note that

$$
\begin{aligned}
V(\tau, y_b) &= \left[y_b + \zeta X + (\rho - 1)c\right]\frac{1 - e^{-(r+\zeta)\tau}}{r+\zeta} + \mu\frac{e^{-(r+\zeta)\tau} - 1 + (r+\zeta)\tau}{(r+\zeta)^2} + e^{-(r+\zeta)\tau}B(y_b) \\
\frac{\partial}{\partial\tau}V(\tau, y_b) &= e^{-(r+\zeta)\tau}\left[y_b + \zeta X + (\rho-1)c - (r+\zeta)B(y_b)\right] + \mu\frac{1 - e^{-(r+\zeta)\tau}}{r+\zeta} \\
\frac{\partial}{\partial y_b}V(\tau, y_b) &= \frac{1 - e^{-(r+\zeta)\tau}}{r+\zeta} + e^{-(r+\zeta)\tau}B'(y_b)
\end{aligned}
$$

And thus, plugging in, we have

$$
\begin{aligned}
\left\{\frac{\partial}{\partial\tau}(\cdot)\right\} &= e^{-(r+\zeta)\tau}\left[y_b + \zeta X + (\rho-1)c + B(y_b)\right] + \mu\frac{1 - e^{-(r+\zeta)\tau}}{r+\zeta} \\
&\quad + \phi(\tau, y_b)(r + \delta_S + \zeta)e^{-(r+\delta_S+\zeta)\tau}\left[B(y_b) - \frac{\rho c + \delta_S + \zeta}{r + \delta_S + \zeta}\right] \\
&\quad + \left[1 - \phi(\tau, y_b)\right](r + \delta_L + \zeta)e^{-(r+\delta_L+\zeta)\tau}\left[B(y_b) - \frac{\rho c + \delta_L + \zeta}{r + \delta_L + \zeta}\right] \\
\left\{\frac{\partial}{\partial y_b}(\cdot)\right\} &= \frac{1 - e^{-(r+\zeta)\tau}}{r+\zeta} + B'(y_b)\left\{e^{-(r+\zeta)\tau} - \phi(\tau, y_b)e^{-(r+\delta_S+\zeta)\tau} - \left[1 - \phi(\tau, y_b)\right]e^{-(r+\delta_L+\zeta)\tau}\right\}
\end{aligned}
$$

We note that $\left\{\frac{\partial}{\partial\tau}(\cdot)\right\}_{\tau=0} = 0$ and $\left\{\frac{\partial}{\partial y_b}(\cdot)\right\}_{\tau=0} = 0$, so that indeed we have $IC(0, y_b) = 0$.

In the linear specification, we note that

$$
\begin{aligned}
\frac{\partial y_b}{\partial\phi} = -\mu\frac{\partial\tau}{\partial\phi} &= -\frac{\mu}{\phi_\tau(\tau, y_b) - \mu\phi_{y_b}(\tau, y_b)} \\
&= -\frac{\mu}{\left\{\phi(\tau, y_b)\left[\delta_S(1-f) + f\delta_L\right] - f\delta_L\right\} - \mu\Phi'(y_b)e^{\int_0^\tau[\delta_S(1-f)+f\delta_L]ds}} > 0
\end{aligned}
$$

as for $f \in \{0, 1\}$ paths do not cross and we must have $\frac{\partial\tau}{\partial\phi} < 0$. As the $IC(\tau, y_b)$ condition is not monotone in τ, we use a scaled up version $e^{k\tau}IC(\tau, y_b)$ for a specific k. We can show that for shortening equilibria (i.e. $f = 1$)

$$
\begin{aligned}
\frac{\partial}{\partial\tau}\left[e^{(r+\zeta+\delta_L)\tau}IC\left(\tau, y_b^S\right)\right] &= \frac{e^{-\delta_S\tau}\left\{\delta_S - (\delta_S - \delta_L)\left[1 - \Phi\left(y_b^S\right)\right]e^{\delta_L\tau}\right\}\left\{(\delta_S + r + \zeta)\left[1 - B\left(y_b^S\right)\right] - \mu B'\left(y_b^S\right)\right\}}{\left[\delta_L\Phi\left(y_b^S\right) - \delta_L\right] - \mu\Phi'\left(y_b^S\right)} \\
&= \frac{e^{-\delta_S\tau}\left\{\delta_S - (\delta_S - \delta_L)\left[1 - \phi\left(\tau, y_b^S\right)\right]\right\}\left\{(\delta_S + r + \zeta)\left[1 - B\left(y_b^S\right)\right] - \mu B'\left(y_b^S\right)\right\}}{\left[\delta_L\Phi\left(y_b^S\right) - \delta_L\right] - \mu\Phi'\left(y_b^S\right)} \\
&= e^{-\delta_S\tau}m\left(\phi\left(\tau, y_b^S\right)\right)\frac{(\delta_S + r + \zeta)\left[1 - B\left(y_b^S\right)\right] - \mu B'\left(y_b^S\right)}{\left[\delta_L\Phi\left(y_b^S\right) - \delta_L\right] - \mu\Phi'\left(y_b^S\right)}
\end{aligned}
$$

and for lengthening equilibria (i.e. $f = 0$) we have

$$
\begin{aligned}
\frac{\partial}{\partial\tau}\left[e^{(r+\zeta+\delta_S)\tau}IC\left(\tau, y_b^L\right)\right] &= \frac{e^{-\delta_L\tau}\left[\delta_L + (\delta_S - \delta_L)\Phi\left(y_b^L\right)e^{\delta_S\tau}\right]\left\{(\delta_L + r + \zeta)\left[1 - B\left(y_b^L\right)\right] - \mu B'\left(y_b^L\right)\right\}}{\delta_S\Phi\left(y_b^L\right) - \mu\Phi'\left(y_b^L\right)} \\
&= \frac{e^{-\delta_L\tau}\left[\delta_L + (\delta_S - \delta_L)\phi\left(\tau, y_b^L\right)\right]\left\{(\delta_L + r + \zeta)\left[1 - B\left(y_b^L\right)\right] - \mu B'\left(y_b^L\right)\right\}}{\delta_S\Phi\left(y_b^L\right) - \mu\Phi'\left(y_b^L\right)} \\
&= e^{-\delta_L\tau}m\left(\phi\left(\tau, y_b^L\right)\right)\frac{(\delta_L + r + \zeta)\left[1 - B\left(y_b^L\right)\right] - \mu B'\left(y_b^L\right)}{\delta_S\Phi\left(y_b^L\right) - \mu\Phi'\left(y_b^L\right)}
\end{aligned}
$$

Here, by $\phi_\tau(\tau, y_b) = \phi(\tau, y_b) - m(\phi(\tau, y_b))f$ in the the shortening equilibrium we have $\left[1 - \phi\left(\tau, y_b^S\right)\right] = \left[1 - \Phi\left(y_b^S\right)\right]e^{\delta_L\tau}$ and in the lengthening we have $\phi\left(\tau, y_b^L\right) = \Phi\left(y_b^L\right)e^{\delta_S\tau}$. We see that at $\tau = 0$ these collapse to our boundary conditions for shortening and lengthening equilibria, respectively. Note that $m(\phi) \in [\delta_L, \delta_S]$ as ϕ is bounded, we know there is a maximal τ corresponding to any ultimate bankruptcy cash-flow y_b (beyond this τ we would have $\phi(\tau, y_b) \notin [0, 1]$; recall we are reversing time and there is divergence!). Thus, the IC condition at 0 is sufficient for all paths. ■

Proof of Lemma 5. Let us first discuss admissibility: on the boundary, the path has to point away from the default region $\mathcal{B}$ in terms of τ. Suppose then that we consider a point in the vicinity of $\tau = 0$, say $\tau = 0 + \varepsilon$. We know that there is a maximum adjustment speed of $\frac{d\phi}{d\tau}$ and that ϕ is continuous in τ. This implies that a point (ϕ, y_b) that is very close to the boundary where only a shortening equilibrium exists cannot change ϕ quickly enough to reach either the interior equilibrium or lengthening equilibrium. Similar reasoning applies for lengthening equilibria. ■

A.4 Proofs of Section 5

We will proof the main result, Proposition 6, in a sequence of lemmas. Lemma 5 establishes that there is continuity of the issuance strategy f w.r.t. time-to-maturity τ. Lemma 6 establishes the uniqueness of $f_{\tau=0}$ if it is admissible on the boundary. Lemma 7 provides the uniqueness of a path leading away from the boundary.

Lemma 5 *There is no discontinuities in f on any equilibrium path, i.e.* $\left|\frac{f_{t+dt}-f_t}{dt}\right| < \infty$ *everywhere.*

Proof of Lemma 5. Note that $E_\phi(\phi, y)$ can be calculated as

$$\frac{\partial}{\partial \phi} E(\phi, y) = \frac{\partial}{\partial \phi} E(\tau, y_b) = E_\tau(\tau, y_b)\frac{\partial \tau}{\partial \phi} + E_{y_b}(\tau, y_b)\frac{\partial y_b}{\partial \phi}. \tag{A.26}$$

As before, the first term captures the effect of time-to-default τ, while the second term captures the effect of defaulting cash-flows y_b. Suppose now there exists a time-to-default $\hat{\tau}$ at which there is a jump in f, i.e., $f_{\hat{\tau}-} \neq f_{\hat{\tau}+}$. Equity values and debt values (and thus the bond value wedge Δ) are continuous across $\hat{\tau}$ along the path (ϕ_τ, y_τ) by inspection of (A.17), (A.18) and (A.22). However, equity's derivative with respect to τ, i.e., E_τ, displays a discontinuity at the policy switching point $\hat{\tau}$. Plugging into (A.19), we have

$$E_{\hat{\tau}-} - E_{\hat{\tau}+} = m(\phi)\Delta \cdot (f_{\hat{\tau}-} - f_{\hat{\tau}+}) = m(\phi)\Delta. \tag{A.27}$$

Since $m(\phi)\Delta > 0$, it implies that when equity switches to issuing more short-term bonds at $\hat{\tau}$, i.e., $f_{\hat{\tau}-} - f_{\hat{\tau}+}$, the equity value's derivative with respect to τ jumps up, i.e., the benefit of surviving longer goes up.

In the original (ϕ, y) state space, denote the corresponding switching points to be $\left(\hat{\phi}_-, \hat{y}_-\right)$ and $\left(\hat{\phi}_+, \hat{y}_+\right)$. Equity's incentive compatibility condition depends on $\frac{\partial}{\partial \phi}E_\phi(\phi, y)$ at these two points. By writing out the terms in integral form, and noting that any f are bounded, we can show that in (A.26), both the $\frac{\partial \tau}{\partial \phi}$ in the first term, and the entire second term related to y_b, i.e., $E_{y_b}(\tau, y_b)\frac{\partial y_b}{\partial \phi}$, are continuous at the switching point. Hence, equation (A.27) implies that

$$E_\phi\left(\hat{\phi}_-, \hat{y}_-\right) - E_\phi\left(\hat{\phi}_+, \hat{y}_+\right) = (E_{\tau-} - E_{\tau+})\frac{\partial \tau}{\partial \phi} = m(\phi)\Delta(f_{\hat{\tau}-} - f_{\hat{\tau}+}) \cdot \frac{\partial \tau}{\partial \phi}.$$

Next, note that $\frac{\partial \tau}{\partial \phi} < 0$, i.e., shortening maturity gives rise to a shorter time-to-default. Following the intuition right after (A.27), when equity switches to issuing short-term bonds, the benefit of surviving longer going up implies that marginal negative impact of shortening maturity is more severe. To make the general point, let us write

$$\begin{aligned}
IC\left(\hat{\phi}_+, \hat{y}_+\right) &= \Delta\left(\hat{\phi}, \hat{y}\right) + E_\phi\left(\hat{\phi}_+, \hat{y}_+\right) \\
&= \Delta\left(\hat{\phi}, \hat{y}\right) + E_\phi\left(\hat{\phi}_-, \hat{y}_-\right) + \left[-m(\phi)\Delta(f_{\hat{\tau}-} - f_{\hat{\tau}+})\frac{\partial \tau}{\partial \phi}\right] \\
&= IC\left(\hat{\phi}_-, \hat{y}_-\right) + \left[m(\phi)\Delta(f_{\hat{\tau}-} - f_{\hat{\tau}+})\left(-\frac{\partial \tau}{\partial \phi}\right)\right]
\end{aligned}$$

Consider first the case when $f_{\hat{\tau}-} = 1$ and $f_{\hat{\tau}+} < 1$. This implies that $\left[m(\phi)\Delta(f_{\hat{\tau}-} - f_{\hat{\tau}+})\left(-\frac{\partial \tau}{\partial \phi}\right)\right] > 0$ and we immediately have a violation: if $f_{\hat{\tau}-} = 1$was optimal, then $IC\left(\hat{\phi}_+, \hat{y}_+\right) > IC\left(\hat{\phi}_-, \hat{y}_-\right) \geq 0$ and thus $f_{\hat{\tau}+} < 1$ violates the IC condition. Next, consider the case when $f_{\hat{\tau}-} = 0$ and $f_{\hat{\tau}+} > 0$. This implies that $\left[m(\phi)\Delta(f_{\hat{\tau}-} - f_{\hat{\tau}+})\left(-\frac{\partial \tau}{\partial \phi}\right)\right] < 0$, which implies $IC\left(\hat{\phi}_+, \hat{y}_+\right) < IC\left(\hat{\phi}_-, \hat{y}_-\right) \leq 0$ and thus invalidates $f_{\hat{\tau}+} > 0$. Lastly, consider the case when $f_{\hat{\tau}-} \in [0, 1]$ such that $IC\left(\hat{\phi}_-, \hat{y}_-\right) = 0$. Then we immediately see that any $f_{\hat{\tau}+} \neq f_{\hat{\tau}-}$ violates IC: (i) if $f_{\hat{\tau}-} \in (0, 1)$, then we must have $IC\left(\hat{\phi}_+, \hat{y}_+\right) = 0$ as well, which is violated by $\left[m(\phi)\Delta(f_{\hat{\tau}-} - f_{\hat{\tau}+})\left(-\frac{\partial \tau}{\partial \phi}\right)\right] \neq 0$. (ii) if $f_{\hat{\tau}-} \in \{0, 1\}$, then we are in the above proofs, and see that the violation exactly runs counter to the IC condition. ■

Lemma 6 *If an equilibrium $f_{\tau=0}$ exists on the boundary, it is unique. It might not exists due to the admissibility condition.*

Proof of Lemma 6. First, let us concentrate on f on the boundary. Taking derivatives of (A.17) and (A.18) w.r.t. ϕ via (A.14), and evaluating at $\tau = 0$, we have

$$\frac{\partial D_i}{\partial \phi} = \frac{h_0'(y_b)\{(\rho c_i - r) + (r + \zeta + \delta_i)[1 - B(y_b)]\} - \mu B'(y_b)}{h_0'(y_b)\{\Phi(y_b)[\delta_S(1-f) + f\delta_L] - \delta_L f\} - \mu \Phi'(y_b)} \tag{A.28}$$

Differentiating IC w.r.t. τ, we have $IC_\tau = \Delta_\tau(\tau, y_b) + E_{\phi\tau}(\tau, y_b)$. Plugging in for $E_{\phi\tau}(\tau, y_b)$ from (A.20), evaluating at $\tau = 0$ so that $IC = \Delta = E_\phi = 0$, and noting that $\Delta_\tau = (\delta_S - \delta_L)[1 - B(y_b)]$, we have

$$\begin{aligned} \left.\frac{\partial IC}{\partial \tau}\right|_{\tau=0} &= m(\Phi(y_b)) \left[f \frac{\partial D_S(\tau, y_b)}{\partial \phi} + (1-f) \frac{\partial D_L(\tau, y_b)}{\partial \phi} \right]_{\tau=0}. & \text{(A.29)} \\ &= m(\Phi(y_b)) \frac{h_0'(y_b)\{[\rho c - r] + [r + \zeta + f\delta_S + (1-f)\delta_L][1 - B(y_b)]\} - \mu B'(y_b)}{h_0'(y_b)\{\Phi(y_b)[\delta_S(1-f) + f\delta_L] - \delta_L f\} - \mu\Phi'(y_b)} & \text{(A.30)} \end{aligned}$$

As $m(\phi) > 0$, we can ignore this term for determining the sign. Next, let us collect all terms in the numerator multiplying f, which are given by $\{h_0'(y_b)(\delta_S - \delta_L)[1 - B(y_b)]\}$. Further, we know from condition (A.16) that for all admissible f the denominator has to be negative. Thus, we can concentrate on the numerator to determine the optimal f. We have

$$\begin{aligned} & h_0'(y_b)\{[\rho c - r] + [r + \zeta + f\delta_S + (1-f)\delta_L][1 - B(y_b)]\} - \mu B'(y_b) \\ = \; & \left[h_0'(y_b)\{(\rho c_L - r) + (r + \zeta + \delta_L)[1 - B(y_b)]\} - \mu B'(y_b)\right] \\ & + f \cdot h_0'(y_b)(\delta_S - \delta_L)[1 - B(y_b)] \end{aligned} \tag{A.31}$$

and we see that we have a linear function in f, which is increasing by

$$(\delta_S - \delta_L)[1 - B(y_b)] > 0 \tag{A.32}$$

Thus, we have at most one unique root in (A.31). Importantly, we also know that (A.31) crosses 0 from above if at all. As the numerator is monotone, this implies a unique equilibrium. If the numerator is everywhere negative for $f \in [0, 1]$, then $f = 1$. If the numerator is everywhere positive for $f \in [0, 1]$, then $f = 0$ if this is admissible. Lastly, if there exits an admissible

$$\hat{f} = \frac{\mu B'(y_b) - h_0'(y_b)\{(\rho c_L - r) + (r + \zeta + \delta_L)[1 - B(y_b)]\}}{h_0'(y_b)(\delta_S - \delta_L)[1 - B(y_b)]} \in (0, 1) \tag{A.33}$$

then this is the unique equilibrium. We thus have

$$f_{\tau=0} = \min\left[1, \max\left[\hat{f}, 0\right]\right] \tag{A.34}$$

as the unique equilibrium subject to admissibility (). ■

Lemma 7 *For a given point $(\Phi(y_b), y_b)$ there exists a unique equilibrium path τ leading away from the boundary.*

Proof of Lemma 7. Writing out $IC(\tau, y_b)$, we have

$$\begin{aligned} IC(\tau, y_b) &= \Delta(\tau, y_b) + E_\phi(\tau, y_b) \\ &= D_S(\tau, y_b) - D_L(\tau, y_b) + \frac{\partial y_b}{\partial \phi}\left\{\frac{\partial}{\partial y_b} E(\tau, y_b)\right\} \\ &\quad + \frac{\partial \tau}{\partial \phi}\left\{\begin{array}{c} y(\tau, y_b) - c + \zeta E^{rf} - (r + \zeta) E(\tau, y_b) \\ + m(\phi(\tau, y_b))[f D_S(\tau, y_b) + (1-f) D_L(\tau, y_b) - 1] \end{array}\right\} \end{aligned} \tag{A.35}$$

Let us move things under the common denominator $h_0'(y_b)\{\phi(\tau, y_b)[\delta_S(1-f) + f\delta_L] - \delta_L f\} - \mu \frac{\partial}{\partial y_b}\phi(\tau, y_b)$ that comes from $\frac{\partial y_b}{\partial \phi}$ and $\frac{\partial \tau}{\partial \phi}$. Plugging in for $\frac{\partial y_b}{\partial \phi}$ and $\frac{\partial \tau}{\partial \phi}$, we have

$$\begin{aligned} IC(\tau, y_b) = \; & \frac{1}{h_0'(y_b)\{\phi(\tau, y_b)[\delta_S(1-f) + f\delta_L] - \delta_L f\} - \mu \frac{\partial}{\partial y_b}\phi(\tau, y_b)} \\ & \times \left\{\begin{array}{c} \Delta(\tau, y_b)\left[h_0'(y_b)\{\phi(\tau, y_b)[\delta_S(1-f) + f\delta_L] - \delta_L f\} - \mu \frac{\partial}{\partial y_b}\phi(\tau, y_b)\right] - \mu \frac{\partial}{\partial y_b} E(\tau, y_b) \\ + h_0'(y_b)\left[\begin{array}{c} y(\tau, y_b) - c + \zeta E^{rf} - (r + \zeta) E(\tau, y_b) \\ + m(\phi(\tau, y_b))[f D_S(\tau, y_b) + (1-f) D_L(\tau, y_b) - 1] \end{array}\right] \end{array}\right\} \end{aligned} \tag{A.36}$$

Suppose we have an interior equilibrium. For interior equilibria we have $IC(\tau, y_b) = 0$, so that for non-zero denominators, we must have

$$\begin{aligned} 0 &= \Delta(\tau, y_b)\left[h_0'(y_b)\{\phi(\tau, y_b)[\delta_S - f(\delta_S - \delta_L)] - \delta_L f\} - \mu\frac{\partial}{\partial y_b}\phi_0(\tau, y_b)\right] - \mu\frac{\partial}{\partial y_b}E(\tau, y_b) \\ &\quad + h_0'(y_b)\left\{\begin{array}{c} y(\tau, y_b) - c + \zeta E^{rf} - (r+\zeta)E(\tau, y_b) \\ +m(\phi(\tau, y_b))[f\Delta(\tau, y_b) + D_L(\tau, y_b) - 1] \end{array}\right\} \end{aligned} \tag{A.37}$$

Plugging in, we see that f cancels out:

$$\begin{aligned} & \overbrace{\{h_0'(y_b)[\phi(\tau, y_b)(\delta_S - \delta_L) + \delta_L] - h_0'(y_b)\cdot m(\phi(\tau, y_b))\}}^{=h_0'(y_b)[\phi(\delta_S-\delta_L)+\delta_L - m(\phi)]=0}\Delta(\tau, y_b) f \\ = \; & \Delta(\tau, y_b)\left[h_0'(y_b)\delta_S\phi(\tau, y_b) - \mu\frac{\partial}{\partial y_b}\phi(\tau, y_b)\right] - \mu\frac{\partial}{\partial y_b}E(\tau, y_b) \\ & + h_0'(y_b)\left\{\begin{array}{c} y(\tau, y_b) - c + \zeta E^{rf} - (r+\zeta)E(\tau, y_b) \\ +m(\phi(\tau, y_b))[D_L(\tau, y_b) - 1] \end{array}\right\} \end{aligned} \tag{A.38}$$

Let us take the derivative with respect to τ of the RHS only, noting that the LHS is identically 0 across τ as long as we have an interior equilibrium. We then have

$$\begin{aligned} 0 &= \left[h_0'(y_b)\delta_S\phi(\tau, y_b) - \mu\frac{\partial\phi(\tau, y_b)}{\partial y_b}\right]\frac{\partial\Delta(\tau, y_b)}{\partial\tau} \\ &\quad + \Delta(\tau, y_b)\left[h_0'(y_b)\delta_S\boldsymbol{\frac{\partial\phi(\tau, y_b)}{\partial\tau}} - \mu\boldsymbol{\frac{\partial^2\phi(\tau, y_b)}{\partial y_b\partial\tau}}\right] - \mu\boldsymbol{\frac{\partial^2 E(\tau, y_b)}{\partial y_b\partial\tau}} \\ &\quad + h_0'(y_b)\left\{\begin{array}{c} \frac{\partial y(\tau, y_b)}{\partial\tau} + m'(\phi(\tau, y_b))[D_L(\tau, y_b) - 1]\boldsymbol{\frac{\partial\phi(\tau, y_b)}{\partial\tau}} \\ +m(\phi(\tau, y_b))\frac{\partial D_L(\tau, y_b)}{\partial\tau} - (r+\zeta)\boldsymbol{\frac{\partial E(\tau, y_b)}{\partial\tau}} \end{array}\right\} \end{aligned} \tag{A.39}$$

where bold-face functions indicate (linear) functions of **contemporaneous** f. Plugging in for the bold-face functions, dropping (τ, y_b) for brevity, we have

$$\begin{aligned} 0 &= \left[h_0'(y_b)\delta_S\phi - \mu\frac{\partial\phi}{\partial y_b}\right]\frac{\partial\Delta}{\partial\tau} \\ &\quad + \Delta\left[h_0'(y_b)\delta_S[f\boldsymbol{\{-m(\phi)\}} + \delta_S\phi] - \mu\left[f\left\{\boldsymbol{(\delta_L - \delta_S)\frac{\partial\phi}{\partial y_b}}\right\} + \delta_S\frac{\partial}{\partial y_b}\phi\right]\right] \\ &\quad - \mu\left[\begin{array}{c} \frac{\partial y}{\partial y_b} + (\delta_S - \delta_L)[D_L - 1]\frac{\partial\phi}{\partial y_b} \\ +f\left\{\boldsymbol{(\delta_S - \delta_L)\Delta\frac{\partial\phi}{\partial y_b} + m(\phi)\frac{\partial\Delta}{\partial y_b}}\right\} \\ +m(\phi)\frac{\partial D_L}{\partial y_b} - (r+\zeta)\frac{\partial E}{\partial y_b} \end{array}\right] \\ &\quad + h_0'(y_b)\left\{\begin{array}{c} \frac{\partial y}{\partial\tau} + m'(\phi)[D_L - 1][f\boldsymbol{\{-m(\phi)\}} + \delta_S\phi] \\ +m(\phi)\frac{\partial D_L}{\partial\tau} - (r+\zeta)\left[\begin{array}{c} y + \zeta E^{rf} - c + m(\phi)[D_L - 1] \\ +f\cdot\boldsymbol{\{m(\phi)\Delta\}} - (r+\zeta)E \end{array}\right] \end{array}\right\} \end{aligned} \tag{A.40}$$

where we left terms multiplying f bold-face. Gathering terms as

$$0 = (numerator) - (denominator)f \iff f = \frac{(numerator)}{(denominator)} \tag{A.41}$$

we have

$$\begin{aligned} denominator &= m(\phi)\left[h_0'(y_b)\{\Delta[\delta_S + (r+\zeta)] - (\delta_S - \delta_L)(1 - D_L)\} + \mu\frac{\partial\Delta}{\partial y_b}\right] \\ numerator &= \left[h_0'(y_b)\delta_S\phi - \mu\frac{\partial\phi}{\partial y_b}\right]\frac{\partial\Delta}{\partial\tau} + \Delta\delta_S\left[h_0'(y_b)\delta_S\phi - \mu\frac{\partial\phi}{\partial y_b}\right] \\ &\quad - \mu\left[\begin{array}{c} \frac{\partial y}{\partial y_b} + (\delta_S - \delta_L)[D_L - 1]\frac{\partial\phi}{\partial y_b} \\ +m(\phi)\frac{\partial D_L}{\partial y_b} - (r+\zeta)\frac{\partial E}{\partial y_b} \end{array}\right] \\ &\quad + h_0'(y_b)\left\{\begin{array}{c} \frac{\partial y}{\partial\tau} + m'(\phi)[D_L - 1]\delta_S\phi + m(\phi)\frac{\partial D_L}{\partial\tau} \\ -(r+\zeta)\left[y + \zeta E^{rf} - c + m(\phi)[D_L - 1] - (r+\zeta)E\right] \end{array}\right\} \end{aligned} \tag{A.42}$$

Thus, by linearity we have a unique candidate f_τ. The bold terms feature **contemporaneous** f that is linear in all cases:

$$m(\phi) = \delta_L + \phi(\delta_S - \delta_L) \tag{A.43}$$

$$y(\tau, y_b) = \begin{cases} y_b + \mu\tau & \text{linear} \\ y_b e^{\mu\tau} & \text{exponential} \end{cases} \tag{A.44}$$

$$\frac{\partial}{\partial y_b} y(\tau, y_b) = \begin{cases} 1 & \text{linear} \\ e^{\mu\tau} & \text{exponential} \end{cases} \tag{A.45}$$

$$\frac{\partial}{\partial \tau} y(\tau, y_b) = \begin{cases} \mu & \text{linear} \\ \mu y_b e^{\mu\tau} & \text{exponential} \end{cases} \tag{A.46}$$

$$\phi(\tau, y_b) = e^{\int_0^\tau [\delta_S(1-f_s)+f_s\delta_L]ds} \left[\Phi(y_b) - \delta_L \int_0^\tau e^{-\int_0^s [\delta_S(1-f_u)+f_u\delta_L]du} f_s ds\right] \tag{A.47}$$

$$\frac{\partial}{\partial y_b}\phi(\tau, y_b) = e^{\int_0^\tau [\delta_S(1-f_s)+f_s\delta_L]ds} \Phi'(y_b) \tag{A.48}$$

$$\boldsymbol{\frac{\partial}{\partial \tau}\phi(\tau, y_b)} = f\boldsymbol{\{-m(\phi(\tau, y_b))\}} + \delta_S \phi(\tau, y_b) \tag{A.49}$$

$$\boldsymbol{\frac{\partial}{\partial \tau}\frac{\partial}{\partial y_b}\phi(\tau, y_b)} = f\left\{\boldsymbol{(\delta_L - \delta_S)\frac{\partial}{\partial y_b}\phi(\tau, y_b)}\right\} + \delta_S \frac{\partial}{\partial y_b}\phi(\tau, y_b) \tag{A.50}$$

$$D_S(\tau, y_b) = \frac{\rho c + \delta_S + \zeta}{r + \delta_S + \zeta} + e^{-(r+\delta_S+\zeta)\tau}\left[B(y_b) - \frac{\rho c + \delta_S + \zeta}{r + \delta_S + \zeta}\right] \tag{A.51}$$

$$\frac{\partial}{\partial \tau} D_S(\tau, y_b) = -(r + \delta_S + \zeta) e^{-(r+\delta_S+\zeta)\tau}\left[B(y_b) - \frac{\rho c + \delta_S + \zeta}{r + \delta_S + \zeta}\right] \tag{A.52}$$

$$\frac{\partial}{\partial y_b} D_S(\tau, y_b) = e^{-(r+\delta_S+\zeta)\tau} B'(y_b) \tag{A.53}$$

$$E(\tau, y_b) = \int_0^\tau e^{(r+\zeta)(u-\tau)} \left\{ y(u, y_b) + \zeta E^{rf} - c \right. \tag{A.54}$$

$$\left. + m(\phi(u, y_b))\left[f_u D_S(u, y_b) + (1 - f_u) D_L(u, y_b) - 1\right]\right\} du \tag{A.55}$$

$$\boldsymbol{\frac{\partial}{\partial \tau}E(\tau, y_b)} = y + \zeta E^{rf} - c + m(\phi(\tau, y_b))\left[D_L(\tau, y_b) - 1\right] \tag{A.56}$$

$$+ f \cdot \boldsymbol{\{m(\phi(\tau, y_b))\left[D_S(\tau, y_b) - D_L(\tau, y_b)\right]\}} - (r + \zeta) E(\tau, y_b) \tag{A.57}$$

$$\frac{\partial}{\partial y_b} E(\tau, y_b) = \int_0^\tau e^{(r+\zeta)(u-\tau)} \left\{ \frac{\partial}{\partial y_b} y(u, y_b) \right. \tag{A.58}$$

$$+ (\delta_S - \delta_L)\left[f_u D_S(u, y_b) + (1 - f_u) D_L(u, y_b) - 1\right] \frac{\partial}{\partial y_b}\phi(u, y_b) \tag{A.59}$$

$$\left. + m(\phi(u, y_b))\left[f_u \frac{\partial}{\partial y_b} D_S(u, y_b) + (1 - f_u) \frac{\partial}{\partial y_b} D_L(u, y_b)\right]\right\} du \tag{A.60}$$

$$\begin{aligned} \boldsymbol{\frac{\partial}{\partial \tau}\frac{\partial}{\partial y_b}E(\tau, y_b)} = {} & \frac{\partial y(\tau, y_b)}{\partial y_b} + (\delta_S - \delta_L)\left[D_L(\tau, y_b) - 1\right]\frac{\partial \phi(\tau, y_b)}{\partial y_b} \\ & + f\left\{ \begin{array}{c} \boldsymbol{(\delta_S - \delta_L)\left[D_S(\tau, y_b) - D_L(\tau, y_b)\right] \frac{\partial \phi(\tau, y_b)}{\partial y_b}} \\ \boldsymbol{+ m(\phi(\tau, y_b))\left[\frac{\partial D_S(\tau, y_b)}{\partial y_b} - \frac{\partial D_L(\tau, y_b)}{\partial y_b}\right]} \end{array} \right\} \\ & + m(\phi(\tau, y_b))\frac{\partial D_L(\tau, y_b)}{\partial y_b} - (r + \zeta)\frac{\partial E(\tau, y_b)}{\partial y_b} \end{aligned} \tag{A.61}$$

Note that the interior equilibrium path is unique for any *ultimate* bankruptcy state $(\Phi(y_b), y_b)$ as it is stems from a linear equation. Thus, suppose that $f_{\tau=0} \in \{0, 1\}$. Then we know that $IC_\tau(0, y_b) \gtrless 0$ and $f_{\tau=0}$ stays cornered until a time τ at which $IC(\tau, y_b) = 0$. Suppose $f_{\tau=0} \in (0, 1)$. Then immediately we have, by Lemma 5, as f is continuous that the above determines the path of f uniquely as it is a linear equation, until a time τ at which f becomes cornered. In this case, then, IC starts diverging from 0 and again f is uniquely determined by the sign of IC. They key step here is to note that IC is continuous by the functions involved and by the continuity of f. ■

Proof of Proposition 6. Uniqueness of $f_{\tau=0}$ on the default boundary follows from Lemma 6. Continuity of ϕ follows from Lemma 5. The existence of unique paths leading away from any admissible boundary point is established by

Lemma 7. Finally, for cornered equilibria, the fact that they stay cornered for some distance away from the boundary implies that paths cannot cross, and additionally we know that ϕ has a bounded rate of change. Thus, as Proposition 5 showed, the equilibrium stays unique for some distance away from the boundary for $f_{\tau=0} \in \{0,1\}$ with the restriction on f strictly binding. ■

A.4.1 Welfare

The total value of the firm is given by both (35) and

$$V = E + \phi D_S + (1-\phi) D_L \tag{A.62}$$

as there is no claimants to the cash-flow stream here besides debt and equity. Suppose that for the equilibrium we are investigating, we have $\tau = T_b(\phi, y)$ as the time-to-default. It is easy to show that for any cornered strategy we have $\frac{\partial T_b}{\partial \phi} < 0$ by $\Phi'(y_b) > 0$ (for non-cornered strategies, this does not have to hold as f is free to adjust). We thus have, by value equivalence,

$$V(\phi, y) = E(\phi, y) + \phi D_S(\phi, y) + (1-\phi) D_L(\phi, y) = V(T_b(\phi, y), y)$$

Whether f is socially optimal depends on if V_ϕ has the appropriate sign. Taking derivatives w.r.t. ϕ, we have

$$V_\phi(\phi, y) = \underbrace{E_\phi + \Delta}_{IC} + \left[\phi \frac{\partial D_S}{\partial \phi} + (1-\phi)\frac{\partial D_L}{\partial \phi}\right] = V_T(T_b(\phi, y), y) \left.\frac{\partial \tau}{\partial \phi}\right|_{\tau = T_b} \tag{A.63}$$

There is some caution warranted here – away from $\tau = 0$, the derivatives of the debt valuations w.r.t. ϕ has to include changes in the policy functions $\frac{\partial f(s)}{\partial \phi}$ for $s \leq \tau$ unless we are looking at corner paths only. Suppose we are looking at a cornered equilibrium with $f = 1$. Then we know we must have $IC \geq 0$. Next, when evaluating at $\tau = 0$, we have $IC = 0$ and thus the sign of V_ϕ and social optimality of f is determined by the sign of

$$\left[\phi \frac{\partial D_S}{\partial \phi} + (1-\phi)\frac{\partial D_L}{\partial \phi}\right] \tag{A.64}$$

Suppose that on the boundary $\frac{\partial D_L}{\partial \phi} > \frac{\partial D_S}{\partial \phi} > 0$ so $f_{\tau=0} = 1$ is both an equilibrium and locally socially optimal. By f being cornered, we also know that $f = 1$ even for slight changes to ϕ, and by continuity of the value functions $\left[\phi \frac{\partial D_S}{\partial \phi} + (1-\phi)\frac{\partial D_L}{\partial \phi}\right] > 0$ for some time even away from the boundary. This of course implies that

$$V_T(T_b(\phi, y), y) = \frac{IC(\phi, y) + \phi \frac{\partial D_S}{\partial \phi} + (1-\phi)\frac{\partial D_L}{\partial \phi}}{\left.\frac{\partial \tau}{\partial \phi}\right|_{\tau = T_b}} < 0$$

Thus, any shortening-only equilibrium has, at least in the vicinity of the default boundary, a local maximum to the left. Note here the divergence of private incentives (which feature the full choice f of the proportions of short versus long in the infinitesimal period) and the social incentives (which have to deal with aggregate proportions ϕ of short versus long debt when f is changed).

A.5 Extensions

A.5.1 Extensions of Section 3

Exogenous default Poisson event. Most of derivation changes slightly; for instance, for bond valuations, in contrast to (7) we have

$$\underbrace{rD_i(\phi; y)}_{\text{required return}} = \underbrace{\rho c}_{\text{coupon}} + \underbrace{\delta_i [1 - D_i(\phi; y)]}_{\text{maturing}} + \underbrace{\zeta [1 - D_i(\phi; y)]}_{\text{upside option}} + \underbrace{\xi [B(y) - D_i(\phi; y)]}_{\text{exogenous default}} + \underbrace{(1-\phi)\delta_L D_i'(\phi; y)}_{\text{state change}} \tag{A.65}$$

Corollary 2 *Even with exogenous liquidation shocks $\xi > 0$, there does not exist equilibria in which equity holders keep issuing short-term bonds and then default endogenously at some finite future time.*

Relaxed issuance space. We require the firm's maturity structure is shortening at the hypothetical default point Φ, i.e., we must have ϕ increasing

$$\left.\frac{d\phi}{dt}\right|_{\phi=\Phi} = -\Phi\delta_S + m(\Phi) f_h > 0.$$

Equity solves

$$\begin{aligned} rE(\phi; y) &= y - c + \zeta\left[E^{rf} - E(\phi; y)\right] \\ &\quad + \max_{f\in[f_l, f_h]} \left\{m(\phi)\left[f D_S(\phi; y) + (1-f) D_L(\phi; y) - 1\right] + \left[-\phi\delta_S + m(\phi) f\right] E'(\phi; y)\right\}. \end{aligned}$$

Assuming $f = f_h$ is optimal, we have

$$\begin{aligned} rE(\phi; y) &= y - c + \zeta\left[E^{rf} - E(\phi; y)\right] \\ &\quad + \left[\phi\delta_S + (1-\phi)\delta_L\right]\left[f_h D_S(\phi; y) + (1-f_h) D_L(\phi; y) - 1\right] \\ &\quad + \left[-\phi\delta_S + (\phi\delta_S + (1-\phi)\delta_L) f_h\right] E'(\phi; y) \end{aligned} \tag{A.66}$$

Taking derivatives w.r.t. ϕ, we have

$$\begin{aligned} (r+\zeta) E'(\phi; y) &= (\delta_S - \delta_L)\left[f_h D_S(\phi; y) + (1-f_h) D_L(\phi; y) - 1\right] \\ &\quad + \left[\phi\delta_S + (1-\phi)\delta_L\right]\left[f_h D_S'(\phi; y) + (1-f_h) D_L'(\phi; y)\right] \\ &\quad + \left[-\delta_S + (\delta_S - \delta_L) f_h\right] E'(\phi; y) + \left[-\phi\delta_S + (\phi\delta_S + (1-\phi)\delta_L) f_h\right] E''(\phi; y) \end{aligned}$$

Since $E = 0, E' = 0$ at Φ, and equal seniority $D_S = D_L = B(y)$, we have

$$0 = (\delta_S - \delta_L)\left[B(y) - 1\right] + m(\Phi)\left[f_h D_S'(\phi; y) + (1-f_h) D_L'(\phi; y)\right] + \left[-\phi\delta_S + m(\Phi) f_h\right] E''(\phi; y).$$

Rearranging, we have

$$E''(\phi, y) = \frac{(\delta_S - \delta_L)\left[1 - B(y)\right]}{-\phi\delta_S + m(\Phi) f_h} - \frac{m(\Phi)}{-\phi\delta_S + m(\Phi) f_h}\left[f_h D_S'(\phi; y) + (1-f_h) D_L'(\phi; y)\right]$$

Recall that for shortening we require $\left[-\phi\delta_S + m(\Phi) f_h\right] > 0$, which rules out $f_h < 0$. Next, note that under this assumption we have

$$\Delta'(\Phi; y) = -\frac{(\delta_S - \delta_L)\left[1 - B(y)\right]}{-\Phi\delta_S + m(\Phi) f_h} < 0$$

Plugging in, we have

$$\Delta'(\Phi; y) + E''(\phi; y) = -\frac{m(\Phi)}{-\phi\delta_S + m(\Phi) f_h}\left[f_h D_S'(\phi; y) + (1-f_h) D_L'(\phi; y)\right]$$

From the bond ODE we have

$$\begin{aligned} D_L'(\Phi; y) &= \frac{-(r + \delta_L + \zeta)\left[1 - B(y)\right]}{-\Phi\delta_S + m(\Phi) f_h} \\ f_h\Delta'(\Phi; y) + D_L'(\Phi; y) &= -\frac{\left[f_h(\delta_S - \delta_L) + r + \delta_L + \zeta\right]\left[1 - B(y)\right]}{-\Phi\delta_S + m(\Phi) f_h} \\ &= -\frac{\left[(f_h\delta_S + (1-f_h)\delta_L + r + \zeta)\right]\left[1 - B(y)\right]}{-\Phi\delta_S + m(\Phi) f_h} \end{aligned}$$

so that finally

$$IC_\phi(\Phi) = \Delta'(\Phi; y) + E''(\phi; y) = \frac{m(\Phi)}{-\phi\delta_S + m(\Phi) f_h}\frac{\left[(f_h\delta_S + (1-f_h)\delta_L + r + \zeta)\right]\left[1 - B(y)\right]}{-\Phi\delta_S + m(\Phi) f_h}$$

We need $IC_{\phi}(\Phi) < 0$ as we want $IC(\Phi - \varepsilon) > 0$ as this implies $f = 1$ at $\Phi - \varepsilon$. Using the approximation $IC(\Phi - \varepsilon) \approx IC(\Phi) - IC_{\Phi}\varepsilon > 0$, we see that we need so only when f_h is so negative

$$[(f_h \delta_S + (1 - f_h)\delta_L + r + \zeta)] < 0 \iff f_h < -\frac{r + \delta_L + \zeta}{\delta_S - \delta_L}$$

the firm might choose the highest f_h and default slowly. But when $f_h < 0$, the firm is repurchasing back short-term debt!

A.5.2 Extensions of Section 4

Suppose that the firm borrows from another group of debt holders holding consol bonds with coupon c_{consol} a la Leland (1994), which is absent from rollover concerns. To make the analysis stark and simple, we assume that these consol bonds get zero payment in both the upper and the default events.[22] As a result, the valuation formula for the long-term and short-term bonds remain identical. The equity holder's problem remains almost the same, with the only adjustment of an additional coupon outflow of c_{consol}. The default boundary becomes

$$\Phi(y_b) = \frac{1}{\delta_S - \delta_L}\left[\frac{y_b - c - c_{consol} + \zeta E^{rf}}{1 - B(y_b)} - \delta_L\right],$$

which affects the endogenous time-to-default τ. The value of consol bonds, denoted by D_{consol}, is given by

$$D_{consol}(\tau, y_b) = \frac{\rho c_{consol}}{r + \zeta}\left[1 - e^{-(r+\zeta)\tau}\right],$$

with $\frac{\partial}{\partial \phi} D_{consol}(\phi, y)\Big|_{\tau=0} = \rho c_{consol} \frac{\partial \tau}{\partial \phi} < 0$. Intuitively, shortening maturity structure leads to an earlier default and hence a lower value of consol bonds.

Now the firm value includes the value of consol bonds. As before, we can decompose the local effect of maturity shortening on the firm value, i.e., $V_{\phi}(\phi, y)$, into

$$V_{\phi}(\phi, y) = \underbrace{E_{\phi}(\phi, y) + \Delta(\phi, y)}_{\text{Incentive compatibility}} + \underbrace{\phi \frac{\partial}{\partial \phi} D_S(\phi, y) + (1 - \phi)\frac{\partial}{\partial \phi} D_L(\phi, y)}_{\text{Impact on ST \& LT bonds}} + \underbrace{\frac{\partial}{\partial \phi} D_{consol}(\phi, y)}_{\text{Impact on consol bonds}}. \tag{A.67}$$

The last negative term is increasing in c_{consol} and may dominate the second positive term in a maturity-shortening equilibrium, leading to $V_{\phi}(\Phi(y_b), y_b) < 0$.

[22] Zero recovery in the default event can be justified by the assumption that the consol bonds are junior to the term bonds we analyzed so far.

A.6 Additional Graphs

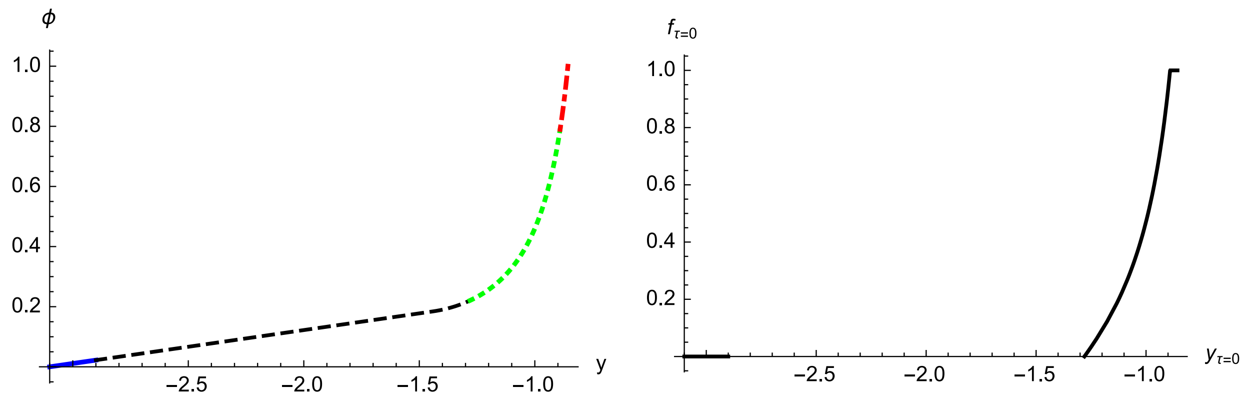

Figure 4: **Default regions and default issuance policies. Left panel:** Equilibrium default regions. Blue solid is long-only equilibria, red dot-dashed is short-only equilibria, green dotted is interior equilibria, and dashed are inaccessible parts of the boundary (no path starting from outside the bankruptcy region can end here). **Right panel:** Issuance policies at default, $f_{\tau=0}$, as a function of default cash-flow $y_{\tau=0}$. The blue solid segment of the left panel corresponds the issuance line flat at 0. The green dotted segment of the left panel corresponds to the strictly increasing part of the issuance line. The red dot-dashed segment of the left panel corresponds to the issuance line flat at 1. The dashed segment of the left panel corresponds to the discontinuity in the issuance line.