

The Ethics of Intergenerational Risk

Paolo G. Piacquadio

August 7, 2015

University of Oslo, Moltke Moes vei 31, 0851, Oslo, Norway. Email:
p.g.piacquadio@econ.uio.no.

The paper reexamines the ethics of intergenerational risk. When risk resolves gradually, earlier decisions cannot depend on the realization of later shocks and, consequently, some inequalities across generations are inevitable. To account for these inequalities, risky intergenerational situations are assessed in relation to an endogenous reference. The reference is specific to each intergenerational resource distribution problem and captures information about the technology, the intensity of risk, and the way risk resolves over time. The characterized class of *reference-dependent utilitarian* criteria avoids serious drawbacks of existing alternatives, such as discounted expected utilitarianism. Specifically, the welfare criteria: (i) disentangle aversion to intergenerational inequality from aversion to risk; (ii) value an early resolution of risk; and (iii) discount the future based on the intensity and the time-resolution of risk.

JEL Classification: D63; D81; H43; Q54; Q56.

Keywords: Intergenerational justice; risk; social ordering; discounting.

1. Introduction

Human activities today impact the welfare of future generations. Some activities, like investing in new technologies, may improve future living conditions. Others, such as land grabs and deforestation, may worsen them. Still others, as those inducing climate change, have the potential to cause massive damages. To provide sensitive policy recommendations, different activities need to be compared by aggregating the conflicting interests of present and future generations. In economics, the ethical choices underlining these comparisons are generally expressed by a ranking of alternatives or a welfare criterion.

The workhorse of the economic literature to evaluate risky intergenerational social situations is (discounted) expected utilitarianism. This welfare criterion reinterprets in a dynamic framework Harsanyi's (1955) pioneering contribution: intergenerational situations ought to be ranked by the discounted sum of each generation's expected utility. Expected utilitarianism has been widely criticized for being insensitive to distributional issues (see Diamond (1967) and Broome (1984)).¹ While alternatives have been proposed (Epstein and Segal (1992), Adler and Sanchirico (2006), Grant et al. (2010), Fleurbaey (2010)), these all build on the "Harsanyi domain." The problem is to rank risky social situations. Each risky social situation is described by a utility lottery for each individual. When individuals are reinterpreted as generations, some specific aspects of intergenerational risk are disregarded in the Harsanyi domain. These aspects are ethically relevant and should be accounted for.

First, uncertainty resolves gradually over time. In the Harsanyi domain, risk is resolved in 'one shot'. The utility levels of present and future generations are revealed all together, as if society had a single and irreversible decision to make before risk is resolved. In contrast, intergenerational risk unfolds at different times. Moreover, society makes a succession of decisions. Thus, the information available when each decision is made might matter for the evaluation of social situations. Second, each such decision is taken without knowing their exact effect on future generations. Part (if not most) of the risk borne by future generations cannot be insured. Thus, unless society is willing to waste resources when more turn out to be available, risk makes intergenerational inequalities unavoidable. Third, generations are exposed to different amounts of risk. The well-being assigned to earlier generations cannot be contingent on events taking place after their lifetime; conversely, the well-being assigned to later generations can—and arguably should—depend on the events that will have occurred by that time.

The purpose of this paper is to study a set of principles of justice, which account for specific aspects of intergenerational risk. Each principle introduces a specific normative concern by comparing a restricted number of *prospects* (intergenerational risky social situations). Jointly, these principles single out a class of welfare criteria, named *reference-dependent utilitarian*. The reference-dependent utilitarian criteria assesses prospects in relation to an endogenous reference. This reference is specific to each problem faced by society as it accounts for the timing of resolution of risk, for the unavoidable inequalities across generations, and for their heterogeneity. These welfare criteria draw a parallel with reference-dependent preferences, developed to model individual behavior (see Kahneman and Tversky (1979), Koszegi and Rabin (2006), and Ok et al. (2015)). In contrast, both the criterion and the reference receive here a normative justification.

¹The utilitarian criterion is indifferent between permuting utilities across generations, even if this permutation worsens the welfare of the worst-off generation. Similarly, the utilitarian criterion is indifferent between permuting utilities across equally likely states of nature, even if this permutation increases inequality in each possible state of nature.

1.1. An overview

The contributions of the paper are developed in three steps. In Section 2, the reference is exogenous. I introduce a set of principles of justice and show how these lead to the class of reference-dependent utilitarian criteria. In Section 3, I endogenize the reference with an allocation rule, where a subset of feasible prospects is associated to each situation faced by society. Finally, in Section 4, I highlight the implications of the characterized class of welfare criteria and endogenous reference. All proofs are presented in the appendix. I briefly illustrate each step below.

Section 2 characterizes the class of reference-dependent utilitarian criteria. A prospect specifies a utility level for each generation at each state of nature. A welfare criterion is a ranking of prospects. The reference is a specific prospect that enters the ethical assessments through the principles of intergenerational equity and risk aversion.

The principle of intergenerational equity applies to the following situation. At each state of nature, one generation is assigned a utility that is larger than at the reference, while another generation is assigned a utility that is smaller than at the reference. Society considers a utility transfer from the first generation to the second. This transfer is equal across states of nature, adjusted by the probability of extinction, and such that the first generation's assignment remains larger than at the reference and the second generation's one smaller. Then, the transfer reduces inequality with respect to the reference in each state of nature. Society satisfies “reference-dependent intergenerational equity” if it considers the after-transfer prospect at least as desirable as the starting one.²

The principle of intergenerational equity applies, instead, to the following situation. In one state of nature, a generation is assigned a larger utility than at the reference; in a different state of nature, the same generation is assigned a lower utility than at the reference. Society considers a utility transfer from the first to the second. This transfer is mean-preserving and such that the utility in the first state of nature remains larger than at the reference, while the utility in the second one smaller. Then, the transfer reduces the gap between the prospect and the reference. Society satisfies “reference-dependent risk balancing” if it considers the after-transfer prospect at least as desirable as the starting one.³

For these transfer principles, the reference plays the role of a watershed: it separates the donors and the recipients of the utility transfer. I require the ranking to be invariant to proportional changes of the reference. This has two consequences. On the one side,

²Reference-dependent intergenerational equity is inspired by the Pigou-Dalton transfer principle, which assesses transfers of wealth or income among individuals. As the Pigou-Dalton transfer principle, it allows for any degree of inequality aversion, but rules out social preferences prone to intergenerational inequality.

³Reference-dependent risk balancing is inspired by the mean preserving spread (Rothschild and Stiglitz (1970)), introduced to assess individuals risk attitudes. It allows for any degree of risk aversion, but rules out social preferences prone to risk.

the information conveyed by the reference is limited to relative quantities. On the other side, society can assess the intergenerational inequality and risk for a larger number of comparisons between prospects, by opportunely scaling the reference. Consequently, this invariance condition strengthens the above transfer principles.

“Efficiency,” “continuity,” and specific “separability” conditions—typical of additive criteria—complete the characterization result and uniquely identify the class of reference-dependent utilitarian criteria.

Reference-dependent utilitarian criteria represent rather flexible social preferences. When society is indifferent to intergenerational inequality and risk, the reference becomes irrelevant and expected utilitarianism emerges as a special case. More generally, social welfare is measured in terms of the utility gain/loss of each generation at each state of nature with respect to the reference. At the limit for infinite aversion to intergenerational inequality and risk, society is egalitarian and maximizes the well-being of that generation and state of nature where the ratio between the assigned utility and the reference utility is smallest. It follows that the egalitarian society top-ranks the reference prospect, whenever this is efficient. The ethical interpretation of the reference then becomes clear: it is the most appealing way to distribute resources in the eyes of an egalitarian society.

Section 3 endogenizes the reference. An *intergenerational (resource distribution) problem* formalizes the decision problem faced by society; it specifies the set of feasible prospects, the timing of resolution of risk, and the extinction probabilities. For each intergenerational problem, I seek to identify the feasible prospects that an egalitarian society would select. This is equivalent to defining an allocation rule: a mapping that associates a subset of feasible prospects to each intergenerational problem.

I introduce two requirements for the allocation rule. The first is efficiency: a feasible prospect cannot be selected if another feasible prospect assigns at least as much utility to each generation at each state of nature and more to some. The second requirement is a recursive view of equity, similar to the one suggested by Asheim and Brekke (2002) for the intertemporal management of a risky capital. The utility assigned to a generation at a state of nature is equitable if: (i) it is as desirable as the utility lottery assigned to later generations at states of nature that can still occur; and (ii) later generations are assigned equitable utilities. These two requirements uniquely characterize the *recursive rule*. At the reference, the utility of a generation at a state of nature is the largest feasible among those that treat later generations alike and allow later generations to abide by the same principles of justice.

The recursive rule satisfies an “ex-ante” concern for equity. Before any risk is resolved, society considers each generation’s assignment equally desirable. The recursive rule also satisfies an “ex-post” concern for equity. At each point in time and based on the risk resolved by that time, society considers the assignment of the current and later generations

equally desirable.⁴ By means of examples, I show in Appendix B that the recursive rule selects a compelling reference prospect even when alternative rules, which are inspired by the ex-ante and ex-post approaches (Epstein and Segal (1992), Adler and Sanichirico (2006), Grant et al. (2010), Fleurbaey (2010)), do not.

Section 4 analyzes the ethical consequences of the reference-dependent utilitarian criterion when the reference is endogenously selected by the recursive rule.

The criterion disentangles risk aversion from intergenerational inequality aversion. This disentanglement is considered a major flaw of expected utilitarianism: the social attitude to risk cannot be differentiated from the attitude to intergenerational inequality. To account for such differences, a standard practice in dynamic welfare analysis is to assume social preferences which are inspired by the behavioral literature (Epstein and Zin (1989)).⁵ In contrast to “Epstein-Zin preferences,” the present criteria also distinguish between two types of risk: *intrinsic risk* is unavoidable even at the egalitarian reference and is specific to each intergenerational problem; *option risk*, instead, arouses only if the assigned prospect differs from the reference one.

A further feature of the criterion is to value an early resolution of risk. While the importance of the timing of resolution of risk has been addressed in the literature (see Arrow and Fisher (1974); Hammitt et al. (1992); Hanemann (1989); Henry (1974); Pindyck (2000)), it is generally disregarded when determining social preferences. Risk hinders an equitable treatment of generations: later generations might be better-off or worse-off than earlier ones, depending on the realization of risk. Thus, the earlier risk is resolved, the easier it is for society to treat generations equally and the higher is the social welfare that can be achieved.

Finally, the structure of social discounting is endogenous and depends on: extinction probabilities, the intensity of risk, its resolution over time, and society’s aversion to intrinsic risk. Assume extinction probabilities are constant over time. Then, exponential discounting arises when society is indifferent to intrinsic risk. Assume instead that the risk resolves all together after the first period. Then, the discount factors differ between the first period and later ones in a way that resembles quasi-hyperbolic discounting (see Laibson (1997)). It differs from quasi-hyperbolic, as discounting becomes exponential from the second period onward, after uncertainty is resolved.⁶ More in general, two

⁴In light of Fleurbaey (2010), the capacity of the rule to combine ex-ante and ex-post concerns might seem surprising. It is explained by the limited scope of a rule, selecting best alternatives, as compared to a welfare criterion, providing a fine-grained ranking of alternatives.

⁵This disentanglement has recently received some attention in the literature (Dasgupta (2008)) and is satisfied by the welfare criteria proposed by Traeger (2012) and Fleurbaey and Zuber (2015b).

⁶Non-exponential discounting is known to lead to the problem of time inconsistency (see Koopmans (1960)). Nevertheless, time varying discounting seems necessary to combine reasonable short-term discount factors with sensitivity to the long-run effects of climate change (Karp (2005), Gerlagh and Liski (2012)). Furthermore, time inconsistency is proven to be unavoidable when aggregating heterogeneous opinions over the “correct” discount factors (Weitzman, 2001) or when aggregating individuals with different discount factors (Zuber (2011), Jackson and Yariv (2014)).

contrasting forces govern social discounting. The extinction probability makes saving less effective and reduces the social weight of future generations. The gradual resolution of risk makes it more difficult to contrast the inequalities faced by future generations and increases their social weight. As a result, the discount factor can be above or below 1, depending on which of these two effects prevails.

1.2. Related literature

The characterized class of reference-dependent utilitarian criteria contributes to the growing normative literature on intergenerational risk. Reinterpreting Harsanyi's (1955) setting in an intergenerational context, Fleurbaey and Zuber (2015a) generalize the class of criteria proposed by Fleurbaey (2010). Fleurbaey and Zuber (2015b), instead, compare several welfare criteria and highlight how risk, variable population, and inequalities affect social discounting. A different setting is recently proposed by Asheim and Zuber (2015). Building on recent advances in the utility-streams literature on intergenerational justice, and in particular on the rank-discounted utilitarian criterion (Zuber and Asheim (2012)), they study how to rank social situations in which each potential individual is characterized by a utility level and a probability of existence.

In contrast to this literature and the present paper, some authors have suggested that welfare criteria be based on individual decision-making. Reflecting a precautionary saving objective, the decision-maker's uncertainty about the growth rate of consumption leads to a declining schedule of social discounting (Gollier (2002)). More recently, Traeger (2012) proposes a criterion based on Kreps and Porteus (1978)'s decision tree representation of risk. A rational agent makes choices anticipating their implications for later choices. In a more recent extension, Traeger (2014) explores the effect of uncertainty, and not only risk. In a multi-agent framework, Weitzman (2001) and Heal and Millner (2014) assume disagreement in social discounting and propose a methodological framework to aggregate individual opinions.

This paper also makes a methodological contribution. Standard welfare criteria assess alternatives only based on the assigned well-being; as an example, expected utilitarianism requires information only about the assigned utility lotteries. These criteria are generally simple and analytically tractable; unfortunately, they do not take into account specific aspects of the decision problem faced by society, such as the timing of resolution of risk. A different branch of welfare economics, fair allocation theory, has addressed how assignments should depend on decision problems. The allocation rule approach studies the normative restrictions that appealing assignments should satisfy (see Thomson (2011) for a recent survey). This approach is flexible and provides policy recommendations tailored on the specific problems faced by society. Unfortunately, "optimal" decisions are often of little help in second best situations, where fine-grained welfare criteria are more

appropriate.

I argue for an approach that combines welfare criteria and allocation rules. The standard welfare criterion is modified so as to depend not only on the assigned well-being, but also on a reference. This reference summarizes the specificity of each problem and is endogenously determined by an allocation rule, as in fair allocation theory. This approach is related to Dhillon and Mertens (1999). As an alternative to expected utilitarianism, Dhillon and Mertens suggest additively aggregating normalized von Neumann-Morgenstern utility functions, where each individual's utility is set to have infimum 0 and supremum 1 on a set of admissible prospects. They suggest the admissible prospects to be "limited only by feasibility and justice" (1999, p. 476), but do not specify how. A distinguishing feature of the present contribution is to axiomatically formalize how the welfare criterion should depend on the decision problem.

2. The welfare criterion

2.1. The framework⁷

Time is discrete and the horizon finite: $T \equiv \{0, \dots, \bar{t}\}$, with $\bar{t} \geq 2$. States of the world are finite: $S \equiv \{0, \dots, \bar{s}\}$. The probability of each state is defined by the vector $\pi \equiv \{\pi^s\}_{s \in S} \gg 0$, with $\sum_{s \in S} \pi^s = 1$.⁸

In some states of nature, extinction is observed before the end-period \bar{t} . Let \overline{TS} be the set of period/state-of nature pairs $(t, s) \in T \times S$ of no extinction. Extinction is irreversible: $(t, s) \notin \overline{TS}$ implies that $(t+1, s) \notin \overline{TS}$. For each $t \in T$, let $S_t \subseteq S$ be the subset of states of nature with no extinction at t . Assume that at least 3 states of nature exist with no extinction at \bar{t} , that is $\#S_{\bar{t}} \geq 3$.

Each $(t, s) \in \overline{TS}$ identifies a **potential (representative) agent**.⁹ A potential agent (t, s) is alive at t only if state of nature s realizes; should s not realize, this potential agent remains unborn.¹⁰ For each $t \in T$, **generation t** is the set of potential agents at t . The existence probability of generation t is denoted by $\pi_t \equiv \sum_{s \in S_t} \pi^s$.

An **(intergenerational risky) prospect** $u \equiv \left\{ \{u(t, s)\}_{(t, s) \in \overline{TS}} \right\}$ assigns a utility

⁷Vector inequalities are defined as follows: $x \geq y \Leftrightarrow [x_i \geq y_i \forall i]$; $x > y \Leftrightarrow [x \geq y \text{ and } x \neq y]$; and $x \gg y \Leftrightarrow [x_i > y_i \forall i]$.

⁸The probability π captures the observer's belief about the likelihood of each state. The origin of such belief is assumed to be ethically irrelevant. The extension to cases in which the beliefs evolve over time can be analyzed on the lines of Dekel et al. (1998) and Karni and Vierø (2013).

⁹Population size can be thought of as constant over periods and states of nature without extinction. The result is unchanged when the framework is extended to exogenous population dynamics and quantities are interpreted in per-capita terms. The more interesting case of endogenous fertility requires a significantly different framework and is left to future research.

¹⁰This is a major difference with respect to Harsanyi's setting and rules out individual's preference for risk. Each potential agent is born after the realization of a specific state of nature. Thus, all the risk in the economy is borne by society.

level $u(t, s) > 0$ to each potential agent $(t, s) \in \overline{TS}$. The set of all intergenerational risky prospects is $U \equiv \mathbb{R}_{++}^{\# \overline{TS}}$.¹¹ A **reference (prospect)** is a prospect $x \equiv \left(\{x(t, s)\}_{(t, s) \in \overline{TS}} \right) \in U$. For the time being, this is exogenously given; it will be endogenized in Section 3.

The problem of society is to define a complete and transitive ranking of prospects for each given reference. For each $x \in U$, let the **social ordering for reference x** be denoted by R_x : $u R_x u'$ means that prospect u is socially at least as desirable as prospect u' for reference x . Strict preferences P_x and indifference I_x are the asymmetric and symmetric counterparts of R_x . Let a **social ordering function** be the mapping that associates to each reference $x \in U$ a social ordering for x .

2.2. The axioms

The first two axioms are standard. Among two different prospects, society prefers the one which assigns more utility to each potential agent.

Monotonicity: *Let $x \in U$. For each pair $u, \bar{u} \in U$, $u > \bar{u}$ implies that $u P_x \bar{u}$.*

The social ordering is required to be continuous. Small changes of the prospect determine small changes in social welfare.

Continuity: *Let $x \in U$. For each $u \in U$, the sets $\{\bar{u} \in U \mid \bar{u} R_x u\}$ and $\{\bar{u} \in U \mid u R_x \bar{u}\}$ are closed.*

The ethical concern for intergenerational equity is introduced as a multidimensional Pigou-Dalton transfer axiom (Pigou (1912); Dalton (1920)). The original version of the transfer principle considers a progressive transfer from a richer to a poorer agent. Provided the richer/poorer relation is not inverted, the transfer reduces inequality. Thus, the after-transfer allocation is at least as desirable as the starting one. I here introduce three differences.

First, the utility assigned to each potential agent at the reference determines which generation is to be considered rich, and which poor. One generation is rich (poor) if, at each state of nature, the utility assignment is larger (smaller) than at the reference. Second, the transfer from the rich to the poor is uniform across states of nature. Third, the transfer is discounted according to the existence probability: a transfer to future potential agents is less valuable when the probability that one of these agents will benefit from it is smaller.

(Reference-dependent) intergenerational equity: *Let $x \in U$. For each pair $u, \bar{u} \in U$, each pair $t, t' \in T$, and each $\delta \in \mathbb{R}_+$, if*

¹¹Similarly to Blackorby and Donaldson (1982), when the domain includes negative utilities, a discontinuity at 0 would emerge. This possibility is excluded by restricting the domain to strictly positive prospects.

- (i) $u(t, s) = \bar{u}(t, s) - \frac{\delta}{\pi_t} \geq x(t, s)$ for each $s \in S_t$;
- (ii) $u(t', s) = \bar{u}(t', s) + \frac{\delta}{\pi_{t'}} \leq x(t', s)$ for each $s \in S_{t'}$;
- (iii) $u(\tilde{t}, s) = \bar{u}(\tilde{t}, s)$ for each $(\tilde{t}, s) \in \overline{TS}$ with $\tilde{t} \neq t, t'$;

then $u R_x \bar{u}$.

The axiom reads as follows. Consider generations t and t' . At \bar{u} , t is assigned a larger utility than at the reference x in each possible state of nature (condition *i*); t' is assigned a smaller utility than at the reference x in each possible state of nature (condition *ii*). Define a transfer δ from t to t' which is: weighted by the respective extinction-probabilities; uniform across states of nature; and such that the first generation remains richer than the second even after the transfer. Then, *ceteris paribus* (condition *iii*), the after-transfer prospect u is at least as socially desirable as the initial one \bar{u} .

The ethical concern for risk is related to the mean preserving spread (Rothschild and Stiglitz (1970)), suggesting that, among equal-mean lotteries, society ought to prefer the one with lowest risk.¹²

The main difference with the mean-preserving spread is the introduction of the reference to determine which equal-mean transfers lead to ethically more appealing prospects. A prospect is judged socially at least as desirable if it is obtained through the following mean-preserving utility transfer: from a state of nature at which a generation is assigned a larger utility than at the reference, to another state of nature at which the same generation is assigned less than at the reference.

(Reference-depedent) risk balancing: Let $x \in U$. For each pair $u, \bar{u} \in U$, each $t \in T$, each pair $(t, s), (t, s') \in \overline{TS}$, and each $\delta \in \mathbb{R}_+$, if

- (i) $u(t, s) = \bar{u}(t, s) - \frac{\delta}{\pi_s} \geq x(t, s)$;
- (ii) $u(t, s') = \bar{u}(t, s') + \frac{\delta}{\pi_{s'}} \leq x(t, s')$;
- (iii) $u(\tilde{t}, \tilde{s}) = \bar{u}(\tilde{t}, \tilde{s})$ for each $(\tilde{t}, \tilde{s}) \in \overline{TS}$ with $(\tilde{t}, \tilde{s}) \neq (t, s), (t, s')$;

then $u R_x \bar{u}$.

The axiom reads as follows. Consider generation t . At \bar{u} , the potential agent (t, s) is assigned a larger utility than at the reference (condition *i*); the potential agent (t, s') is instead assigned a smaller utility than at the reference (condition *ii*). Define a transfer from (t, s) to (t, s') , weighted by the probability of each state of nature, such that the potential agent (t, s) remains richer than (t, s') even after the transfer. Then, *ceteris*

¹²The mean preserving spread is obtained by transferring probability mass to the tales of the distribution, but can be equivalently expressed as a regressive transfer across states of nature, weighted by the likelihood of each. See Atkinson (1970).

paribus (condition *iii*), the after-transfer prospect u is at least as socially desirable as the initial one \bar{u} .

The next axiom is an informational parsimony requirement. The ranking should not vary when the reference expands or contracts proportionally. This is related to the ratio-scale property discussed in Blackorby and Donaldson (1982), but crucially differs because it is imposed on the reference instead of the utilities. This axiom implies that the information conveyed by the reference is about relative quantities.

Proportionality: *Let $x \in U$. For each pair $u, \bar{u} \in U$ and each $\alpha > 0$, $u R_x \bar{u}$ if and only if $u R_{\alpha x} \bar{u}$.*

This axiom immediately complements and strengthens the previous two. *Intergenerational equity* and *risk balancing* require the potential agents involved in the transfers to have more (for the donors) and less (for the recipients) utility than at the reference. By *proportionality*, it is possible to identify donors and recipients in a larger number of cases. Thus, more comparisons can be assessed.

The next two axioms introduce separability in the evaluation. The first separability condition is across time. If the utilities of a generation are unaffected, the assigned level is irrelevant for the ethical assessment. This is standard in the literature and closely related to “Independence of the Utility of the Dead” (Blackorby et al. (2005)).

Intergenerational separability: *Let $x \in U$. For each $u, \bar{u}, \tilde{u}, \hat{u} \in U$ and each $t \in T$ such that:*

- (i) $u(t, s) = \bar{u}(t, s)$ and $\tilde{u}(t, s) = \hat{u}(t, s)$ for each $(t, s) \in \overline{TS}$;
- (ii) $u(t', s) = \tilde{u}(t', s)$ and $\bar{u}(t', s) = \hat{u}(t', s)$ for each $(t', s) \in \overline{TS}$ with $t' \neq t$;

then $u R_x \bar{u}$ if and only if $\tilde{u} R_x \hat{u}$.

The second separability condition is across states of nature, but within a period of time. Consider two prospects u and \bar{u} that assign the same utilities to each potential agent except those belonging to generation t . If furthermore a potential agent at t is unaffected by the choice, her level of utility is irrelevant for the ethical assessment.

Intragenerational separability: *Let $x \in U$. For each $u, \bar{u}, \tilde{u}, \hat{u} \in U$, and each $(t, s) \in \overline{TS}$ such that:*

- (i) $u(t, s) = \bar{u}(t, s)$ and $\tilde{u}(t, s) = \hat{u}(t, s)$;
- (ii) $u(t, s') = \tilde{u}(t, s')$ and $\bar{u}(t, s') = \hat{u}(t, s')$ for each $(t, s') \in \overline{TS}$ with $s' \neq s$;
- (iii) $u(\tilde{t}, \tilde{s}) = \bar{u}(\tilde{t}, \tilde{s}) = \tilde{u}(\tilde{t}, \tilde{s}) = \hat{u}(\tilde{t}, \tilde{s})$ for each $(\tilde{t}, \tilde{s}) \in \overline{TS}$ with $\tilde{t} \neq t$;

then $u R_x \bar{u}$ if and only if $\tilde{u} R_x \hat{u}$.

Although separability conditions are strong assumptions to impose, these nevertheless have valuable implications and are common in the literature. First, they introduce additivity in the social welfare functions and, thus, significantly simplify their application in optimization problems. Second, they provide informational parsimony: the comparison of prospects requires information only about the generations/potential agents affected by the choice. Finally and most importantly, they ensure tractability of the representation result and help disentangle the effects of the reference on the social evaluation.

2.3. The reference-dependent utilitarian criterion

I first define the social ordering. Let $x \in U$ and define the **(expected) reference utility of generation t** as $x_t \equiv \sum_{s \in S_t} \pi^s x(t, s)$. For each $t \in T$, let $r_t \in (-\infty, 1]$ and define $\mathbf{r} \equiv (\{r_t\}_{t \in T})$. The **welfare of generation t** at $u \in U$ is given by:

$$w_t(u; x, r_t) \equiv \left[\frac{1}{x_t} \sum_{s \in S_t} \pi^s x(t, s) \left(\frac{u(t, s)}{x(t, s)} \right)^{r_t} \right]^{\frac{1}{r_t}} \text{ if } r_t \neq 0; \quad (1)$$

$$w_t(u; x, r_t) \equiv \exp \left[\frac{1}{x_t} \sum_{s \in S_t} \pi^s x(t, s) \ln \left(\frac{u(t, s)}{x(t, s)} \right) \right] \text{ if } r_t = 0. \quad (2)$$

What matters for welfare is not the absolute level of utility assigned to each potential agent, but the ratio between the assigned utility and the reference one. The parameter r_t measures the aversion to risk in terms of the deviations of the assigned utilities from the reference. Set $\rho \in (-\infty, 1]$. **Intergenerational social welfare** at $u \in U$ is given by:

$$W(u; x, \mathbf{r}, \rho) \equiv \frac{1}{\rho} \sum_{t \in T} x_t w_t(u; x, r)^{\rho} \text{ if } \rho \neq 0; \quad (3)$$

$$W(u; x, \mathbf{r}, \rho) \equiv \sum_{t \in T} x_t \ln w_t(u; x, r) \text{ if } \rho = 0. \quad (4)$$

The welfare of each generation is first transformed by a concave power function and then additively aggregated. The parameter ρ measures the aversion to inequalities across generations. The weight of each generation t is given by the reference utility x_t . It follows that the **discount factor** between any two generations $t, t' \in T$ with $t' > t$ is $\beta(t', t) \equiv \frac{x_{t'}}{x_t}$.

Definition. *The social ordering function is **reference-dependent (generalized) utilitarian** if there exist $(\mathbf{r}, \rho) \in (-\infty, 1]^{\bar{t}+1}$ such that, for each reference $x \in U$, the social ordering for x can be represented by the intergenerational social welfare function $W(\cdot; x, \mathbf{r}, \rho)$; that is, for each pair of prospects $u, \bar{u} \in U$:*

$$u R_x \bar{u} \Leftrightarrow W(u; x, \mathbf{r}, \rho) \geq W(\bar{u}; x, \mathbf{r}, \rho).$$

The first result establishes the equivalence between the above-introduced axioms and the reference-dependent utilitarian criterion.

Theorem 1. *The following statements are equivalent:*

- (i) *the social ordering function satisfies monotonicity, continuity, intergenerational equity, risk balancing, proportionality, intergenerational separability, and intragenerational separability;*
- (ii) *the social ordering function is reference-dependent utilitarian.*

The intuition of the result goes as follows. *Monotonicity, continuity, intergenerational separability, and intragenerational separability* require welfare to be increasing with respect to the utility of each potential agent, continuous, additive over time, and, for each period, additive over states of nature. *Risk balancing* requires the criterion to be defined in terms of the ratio between the assigned utility and the reference utility of each potential agent. Together with *proportionality*, this forces the welfare of each generation t to be measured as a mean of order $r_t \leq 1$. *Intergenerational equity* and *proportionality* determine a similar power representation (with $\rho \leq 1$) for comparisons across generations and introduce social discounting.

2.4. Special cases

In a static framework, social welfare is ordinally equivalent to the welfare of a generation in (1) or (2) and would be uniquely characterized by *monotonicity, continuity, risk balancing, proportionality, and intragenerational separability*. This welfare measure allows society to introduce individual heterogeneity with respect to needs, circumstances, or merits. This incorporates equivalent scales (Ebert and Moyes (2003)) in a complete ranking of alternatives.

In the present dynamic setting, reference-dependent utilitarianism simplifies as follows, for specific choices of parameters and reference.

Case 1. Discounted expected utilitarianism. When $\rho = r_t = 1$ for each $t \in T$:

$$W(u; x, \mathbf{1}, 1) = \sum_{(t,s) \in \overline{TS}} \pi^s u(t, s).$$

Intergenerational social welfare is given by the expected utility allocated to each potential agent. The independence from the reference follows from the ethical indifference to risk and intergenerational inequalities, measured in contrast to the reference.

Case 2. Reference-dependent power utilitarianism. When $r_t = \rho = \alpha \leq 1$ for each $t \in T$:

$$W(u; x, \alpha, \alpha) = \frac{1}{\alpha} \sum_{(t,s) \in \overline{TS}} \pi^s x(t, s) \left(\frac{u(t, s)}{x(t, s)} \right)^\alpha \text{ if } \alpha \neq 0;$$

$$W(u; x, \mathbf{0}, 0) = \sum_{(t,s) \in \overline{TS}} \pi^s x(t, s) \ln \left(\frac{u(t, s)}{x(t, s)} \right) \text{ if } \alpha = 0.$$

Intergenerational social welfare is a weighted sum of a power transformation of utilities. Each utility is first divided by the reference and then concavely transformed. These transformed relative utilities are then weighted by the reference-adjusted probability $\pi^s x(t, s)$ and added up. The separability across generations and states of nature is a result of the aversion to risk, measured by \mathbf{r} , being equal to the aversion to intergenerational inequality, measured by ρ .

Case 3. Nested power utilitarianism. Let the reference prospect be constant across time and states of nature. By *proportionality*, the criterion is unchanged when the reference prospect is multiplied by a positive constant. Thus, let $x = 1_{\overline{TS}}$ be the reference that assigns a utility of 1 to each potential agent. It follows that $x_t = \pi_t$ for each $t \in T$ and the welfare representation simplifies (assume $r, \rho \neq 0$ for simplicity) as:

$$W(u; 1_{\overline{TS}}, \mathbf{r}, \rho) = \frac{1}{\rho} \sum_{t \in T} \pi_t \left[\sum_{s \in S_t} \frac{\pi^s}{\pi_t} u(t, s)^{r_t} \right]^{\frac{\rho}{r_t}}. \quad (5)$$

Case 4. Reference-dependent maximin. At the limit for $\rho \rightarrow -\infty$ and $r_t \rightarrow -\infty$ for each $t \in T$, any two prospects are compared by the maximin ranking of the assigned utility with respect to the reference:¹³

$$u R_x \bar{u} \Rightarrow \min_{(t,s) \in \overline{TS}} \frac{u(t, s)}{x(t, s)} \geq \min_{(t,s) \in \overline{TS}} \frac{\bar{u}(t, s)}{x(t, s)}.$$

A society that is infinitely averse to intergenerational inequality and risk ought to select the prospect that maximizes the relative utility of the worst-off potential agent. Should the reference assign all available resources, it is the first ranked alternative according to the reference-dependent maximin criterion. This property guides the choice of the reference, as discussed in the next section.

¹³In the literature, it is standard to introduce its leximin extension, although it violates *continuity*. More

specifically, for each $x, u \in U$ define the vector $\left(\frac{u}{x} \right) \equiv \left(\left\{ \frac{u(t, s)}{x(t, s)} \right\}_{(t,s) \in \overline{TS}} \right)$ where potential agents

$(t, s) \in \overline{TS}$ are rearranged in increasing order of relative utility $\frac{u(t, s)}{x(t, s)}$. Let \geq_{lex} denote the ordinary leximin criterion, which evaluates two vectors by first comparing the smallest component; if they are equal, it compares the second smallest component, and so on. Let $x \in U$. Then, for each pair $u, \bar{u} \in U$, the reference-dependent leximin criterion is defined by:

$$u R_x \bar{u} \Leftrightarrow \left(\frac{u}{x} \right) \geq_{\text{lex}} \left(\frac{\bar{u}}{x} \right).$$

3. Selection of the reference

3.1. An intergenerational problem

At each period/state of nature $(t, s) \in \overline{TS}$, a capital stock $k(t, s) \geq 0$ is available. Production takes place. The output can be partly allocated to consumption of the potential agent (t, s) and, for the remaining part, to the next period capital $k(t+1, s)$ for the benefit of later potential agents. The feasible utilities of potential agent (t, s) can be compactly described as:

$$0 \leq u(t, s) \leq f_{(t,s)}(k(t, s), k(t+1, s)), \quad (6)$$

where the function $f_{(t,s)}$ is continuous, increasing in $k(t, s)$, decreasing in $k(t+1, s)$, and satisfies no free lunch, i.e. $f_{(t,s)}(0, 0) = 0$. Function $f_{(t,s)}$ summarizes both technology and preferences: it could be written as the composition of a utility transformation of the consumption that is available for given capital stock and savings. Assuming that utilities are constant across potential agents, I refer to each $f_{(t,s)}$ as the technology at period t and state s .¹⁴ Let the technology $F \equiv \left(\{f_{(t,s)}\}_{(t,s) \in \overline{TS}}\right)$ be a set of functions satisfying the above assumptions; their domain is denoted by \mathcal{F} . Assume that initial capital is positive: $\bar{k}_0 \equiv \left(\{\bar{k}(0, s)\}_{s \in S_0}\right) \gg 0$.

Society is uncertain about future technology. Let each state of nature define a sequence of technologies. Risk and its resolution over time are formalized by an event tree. For each $(t, s) \in \overline{TS}$, denote by $P(t, s) \subseteq S$ the subset of the states of the world that can still realize from (t, s) ; then, for each pair $(t, s), (t', s') \in \overline{TS}$ with $t' > t$, it must hold that either the later partition is finer than the previous one, i.e. $P(t, s) \supseteq P(t', s')$, or that the partitions are disjoint, i.e. $P(t, s) \cap P(t', s') = \emptyset$. At the last period \bar{t} , all risk is resolved: $P(\bar{t}, s) = \{s\}$ for each $s \in S_{\bar{t}}$. A sequence of such partitions $P \equiv \left(\{P(t, s)\}_{(t,s) \in \overline{TS}}\right)$ uniquely identifies an event tree. Let \mathcal{P} be the domain of such partitions.¹⁵ The event tree structure implies that, at each period, the production functions are identical across states of nature belonging to the same partition: for each pair $(t, s), (t, s') \in \overline{TS}$, if $s, s' \in P(t, s)$ then $f_{(t,s)} = f_{(t,s')}$.

An **intergenerational problem** $I \equiv \{\bar{k}_0, F, P, \}$ is defined by an initial capital $\bar{k}_0 \gg 0$, a technology $F \in \mathcal{F}$, and an event tree $P \in \mathcal{P}$. Let \mathcal{I} be the domain of intergenerational problems satisfying the above assumptions.

For each $I \in \mathcal{I}$, let $U^I \subset \mathbb{R}_+^{\overline{TS}}$ denote the **set of feasible prospects for I**. Each $u \in U^I$ is such that: (6) holds for each $(t, s) \in \overline{TS}$; $k(0, s) = \bar{k}(0, s)$ for each $s \in S_0$; and

¹⁴The results are not affected when consumption and/or a multidimensional commodity space are explicitly introduced.

¹⁵The setting is slightly more general than in the literature: I do not assume that later partitions are strictly finer, nor that the event tree has a unique initial node. This allows the case of one-shot resolution of uncertainty to be included as a special case.

$u(t, s) = u(t, s')$ and $k(t, s) = k(t, s')$ for pair $(t, s), (t, s') \in \overline{TS}$ such that $s, s' \in P(t, s)$.

An **(allocation) rule** ϕ associates to each intergenerational problem $I \in \mathcal{I}$ a non-empty subset of its feasible prospects $\phi(I) \subseteq U^I$. The properties imposed on the rule express the ethical concerns that the selected reference embodies and are discussed next.

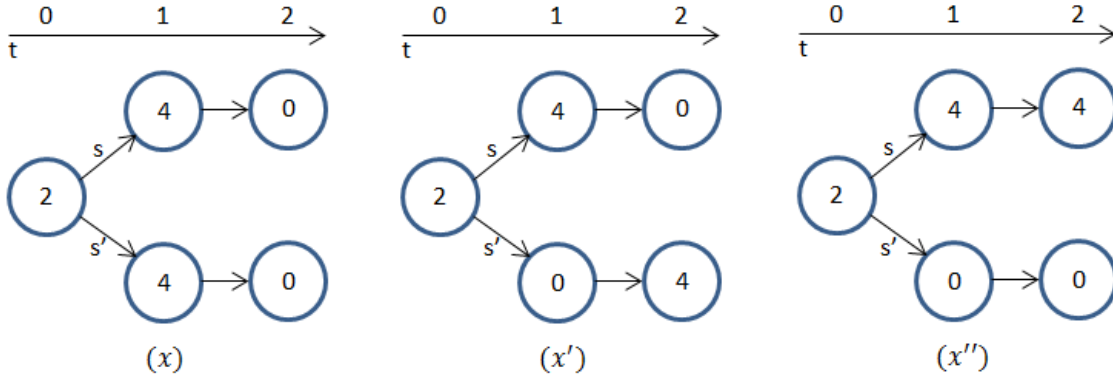
3.2. The recursive rule

The first axiom guarantees that the selected prospect assigns all the resources available in the economy: any larger prospect is not feasible.

Maximality: For each $I \in \mathcal{I}$, $x \in \phi(I)$ and $x' > x$ imply that $x' \notin U^I$.

The second axiom ensures equity across potential agents. It consists of two conditions. First, society should consider each generation's utility lottery equally desirable. Second, the utility assigned to each potential agent should be as desirable as the lottery of later generations, restricted to the states of nature that can still occur.

To illustrate this axiom consider the following prospects. The intergenerational problem consists of three periods, $T \equiv \{0, 1, 2\}$, and two equally likely states of nature, $S \equiv \{s, s'\}$; states of nature are revealed at $t = 1$. Prospects x , x' , and x'' are equally desirable for the expected utilitarian society: these are obtained by permuting utilities across time and equally likely states of nature.



The first condition is similar to an “ex-ante” concern for equity introduced by Diamond (1967) and ensures that x' is socially preferred to x . Prospect x does not guarantee “a fair shake” to generations 1 and 2. Conversely, at x' and x'' both generations are given an equal chance of achieving the high and low utilities.

The second condition is similar to an “ex-post” concern for equity and ensures that x'' is socially preferred to x' . Broome (1991) suggests that although x'' and x' are ex-ante identical for all individuals, x' is preferable as it leads to lower inequality at each possible state of nature. In the present setting, an argument similar to Asheim and Brekke (2002) seems more appropriate. At x' , potential agent $(1, s)$ is assigned a high utility even though society knows at the time the choice is made that this will lead to a lower utility for the later generation. At x'' , instead, the utility of potential agent $(1, s)$ is as desirable as the

utility lottery assigned to the later generation at states of nature that can still occur, as $x''(1, s) = x''(2, s)$ and $x''(1, s') = x''(2, s')$.

Although this axiom combines ethical concerns for both ex-ante and ex-post equity, these concerns assume here a different meaning than in Harsanyi's setting. By assumption, potential agents do not face any risk and all risk in the economy is borne by society. Thus, instead of adopting individual's preferences as in Harsanyi, each generation's lottery is evaluated by society uniquely based on ethical arguments. Let $\mu : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous, strictly increasing, and concave function. Society evaluates each lottery by the generalized weighted average with transformation μ of the assigned utilities. When μ is linear, the evaluation follows the classical weighted average; as μ becomes more and more concave, risk aversion increases and society gives more and more weight to low utilities.

For each $P \in \mathcal{P}$ and $(t, s), (t, s') \in \overline{TS}$, with $s' \in P(t, s)$, let $\pi_{P(t,s)}^{s'} \equiv \frac{\pi^{s'}}{\sum_{\bar{s} \in P(t,s)} \pi^{\bar{s}}}$ be the conditional probability that state s' realizes, given that partition $P(t, s)$ is reached.

Recursive (μ -expected) equity: For each $I \in \mathcal{I}$, $x \in \phi(I)$ implies that:

- (i)
$$\sum_{s \in S_t} \frac{\pi^s}{\pi_t} \mu(x(t, s)) = \sum_{s \in S_{t'}} \frac{\pi^s}{\pi_{t'}} \mu(x(t', s)) \text{ for each } t, t' \in T;$$
- (ii)
$$\mu(x(t, s)) = \sum_{s' \in P(t,s)} \pi_{P(t,s)}^{s'} \mu(x(t', s')) \text{ for each } (t, s) \in \overline{TS} \text{ and each } t' > t.$$

Since $\pi_t \equiv \sum_{s \in S_t} \pi^s$, $\frac{\pi^s}{\pi_t}$ is the probability of state s to occur at t , conditional on the existence of generation t . Thus, Requirement (i) forces all generations to be treated alike before any risk is resolved. Requirement (ii) introduces the recursive evaluation of lotteries.

Let the **recursive (μ -expected) rule** be the rule $\phi^{R\mu}$ that satisfies *maximality* and *recursive equity*.

Proposition 1. *On the domain \mathcal{I} , the recursive rule $\phi^{R\mu}$ is well-defined.*

Importantly, the selection of the recursive rule is unique and belongs to the set of strictly positive prospects U , as required by Theorem 1.

Corollary 1. *On the domain \mathcal{I} , the recursive rule $\phi^{R\mu}$ selects a unique and strictly positive reference.*

Alternative rules to select the reference exist. I discuss their axiomatic characterization and welfare implications in Appendix B. By ways of examples, I argue that the recursive rule is the only one that selects a compelling reference for each intergenerational problem.

4. The recursive-rule utilitarian criterion

In this section, I highlight the consequences of the reference-dependent generalized utilitarian criterion, when the reference prospect is endogenously determined by the recursive rule $\phi^{R\mu}$. I refer to such criterion as the **recursive-rule utilitarian** criterion.

Given the representation result of Theorem 1, the weight attributed to a generation's welfare (1) or (2) is given by the expected reference utility of that generation. As a reminder, this is defined as the weighted average over the utilities assigned to potential agent at the reference: $x_t \equiv \sum_{s \in S_t} \pi^s x(t, s)$ for each $t \in T$. The reference is selected by the recursive rule $\phi^{R\mu}$ based on: (i) the intensity of risk faced in the allocation problem, (ii) the timing of its resolution, and (iii) the aversion of society to the inequalities risk generates. Consequently, the factor $\beta(t', t) = \frac{x_{t'}}{x_t}$ should be interpreted as a risk-adjusted time discount factor. Moreover, depending on the problem faced by society and its ethical principles, a wide spectrum of discounting formulas emerge. The following discussion sheds light on the determinants of discounting and is organized in a number of propositions. As the results are straightforward, I only provide a sketch of the proofs.

The first proposition tells that with a larger concavity of the function μ , measuring the observer's aversion to risk in the selection of the reference, more weight is assigned to future generations. Future generations face more risk. Thus, more aversion to risk leads to a reference in which on average higher utility is assigned to later generations. As a consequence, society places more weight on future generations.

Proposition 2. *On the domain \mathcal{I} , let the social ordering function be recursive-rule utilitarian. Then, as μ becomes more concave, the discount factors increase.*

Let $I \in \mathcal{I}$ and consider generations $t, t' \in T$ with $t' > t$. By Condition (ii) of recursive equity, $x \in \phi^{R\mu}(I)$ is such that for each $(t, s) \in \overline{TS}$, $\mu(x(t, s)) = \sum_{s' \in P(t, s)} \pi_{P(t, s)}^{s'} \mu(x(t', s'))$.

Thus, as μ becomes more concave, the ratio between $\sum_{s' \in P_t^s} \pi_{P(t, s)}^{s'} x(t', s')$ and $x(t, s)$ increases. As this holds for each $s \in S_t$, $\frac{x_{t'}}{x_t}$ increases as does, by definition, the discount factor $\beta(t', t)$.

Consider the limit case of a linear μ : society is indifferent to risk in the selection of the reference. For such an ethical viewpoint, the only reason to discount future generations is the risk of extinction and the discount factor simplifies as $\beta(t', t) = \frac{\pi_{t'}}{\pi_t}$. Furthermore, if the risk of extinction is constant over time, discounting is exponential.

Proposition 3. *On the domain \mathcal{I} , let the social ordering function be recursive-rule utilitarian. Then, if μ is linear, the discount factors are determined by the extinction probabilities. If, furthermore, the extinction probability is constant, discounting is exponential.*

The same type of discounting arises in one additional cases: when risk is resolved at 0, before any decision is taken. The distribution problem at each state of nature can be solved separately. Thus, Requirement (ii) of *recursive equity* ensures that, at each state of nature, generations are all assigned the same utility, implying that $x(t, s) = x(t', s)$ for each pair $(t, s), (t', s) \in \overline{TS}$.

Proposition 4. *On the domain \mathcal{I} , let the social ordering function be recursive-rule utilitarian. Then, the discount factors are determined by the extinction probabilities if $I \in \mathcal{I}$ is such that all risk is resolved at 0, i.e. $P(0, s) = \{s\}$ for each $s \in S$. If, furthermore, the extinction probability is constant, discounting is exponential.*

Another interesting case emerges when no information is available at 0, but risk resolves entirely in period 1. Let $\mathcal{I}^1 \subset \mathcal{I}$ be the sub-domain of intergenerational problems $I \in \mathcal{I}$ such that: $P(0, s) = S$ for each $s \in S$; and $P(t, s) = \{s\}$ for each $(t, s) \in \overline{TS}$ with $t \neq 0$. The full evolution of technology becomes completely known in period 1, independently of the state of nature. Applying the conditions of *recursive equity*, this implies that $x(t, s) = x(t', s)$ for each pair $(t, s), (t', s) \in \overline{TS}$ with $t, t' \geq 1$. If, furthermore, the extinction probability is constant over time, discounting is similar to the “quasi-hyperbolic discounting” described by Laibson (1997). As in quasi-hyperbolic discounting, the discount factors are $\beta(0, 0) = 1$ and $\beta(t, 0) = \beta\delta^t$ for each $t > 0$; in contrast to quasi-hyperbolic discounting, the discount factors between later periods are constant, that is $\beta(\tau, t) = \delta^{\tau-t}$ for each $\tau \geq t > 0$ as in exponential discounting.¹⁶

Proposition 5. *On the domain \mathcal{I}^1 , let the social ordering function be recursive rule utilitarian. Then, if the extinction probability is constant, discount factors are $\beta(0, 0) = 1$ and $\beta(t, 0) = \beta\delta^t$ for each $t > 0$.*

For each $t, t' \geq 1$, $x(t, s) = x(t', s)$ for each pair $(t, s), (t', s) \in \overline{TS}$ implies that $\beta(t', t) = \frac{\pi_{t'}}{\pi_t} = \delta^{t'-t}$ for some $0 < \delta \leq 1$. Let $\beta \equiv \frac{\beta(1, 0)}{\delta} = \frac{x_1}{x_0} \cdot \frac{1}{\pi_1}$. Then, $\beta(0, 0) = 1$ and $\beta(t, 0) = \beta\delta^t$ for each $t > 0$.

To summarize, two opposite forces characterize the risk-adjusted discount factor. The extinction probability leads society to assign less weight to future generations: the higher the extinction probability, the more likely it is that resources are lost. The gradual resolution of risk leads society to assign more weight to future generations: the slower the resolution of risk, the larger the inequalities that future generations might face. The balance between these two effects is ambiguous. If the extinction effect prevails, the discount factor is lower than 1, in accordance with the literature on discounting. Conversely, if the risk-resolution effect prevails, the discount factor is larger than 1. In this case, society is more concerned about the riskier welfare of later generations than the less risky welfare of earlier ones.

¹⁶This result provides a rationale for using time varying discounting in the evaluation of climate change effects, as recently done by Karp (2005) and Gerlagh and Liski (2012).

5. Conclusions

In the literature, welfare issues involving intergenerational risk are generally addressed by analogy with Harsanyi's (1955) pioneering contribution to risky social situations. Agents are simply reinterpreted as generations and time discounting is added. I claim that such approaches disregard essential aspects of intergenerational risk:

- risk resolves gradually over time;
- it exposes generations to different types and quantity of risk;
- it is, to a large extent, uninsurable; and, consequently,
- it naturally generates inequalities across generations, independent of the state of nature that eventually occurs.

In this paper, I propose and characterize a class of welfare criteria that use this information for the normative assessment of intergenerational risk. Each prospect, which defines a utility level for each generation at each state of nature, is assessed in contrast to an endogenous reference. This reference captures the specificity of each intergenerational problem faced by society: it is the most equitable and efficient prospect that is feasible. It thus accounts for the time resolution of risk, the heterogeneous risk faced by the generations, and the unavoidable inequalities among generations.

The axiomatic analysis produces a class of welfare criteria, called *reference-dependent utilitarian*, that avoid some serious drawbacks of alternative criteria, such as discounted expected utilitarianism. Four aspects distinguish the reference-dependent utilitarian criteria. First, these criteria disentangle aversion to intergenerational inequality from aversion to risk. Second, they discern two types of risk: intrinsic risk is unavoidable and specific to each intergenerational decision problem; option risk is the residual risk which is incurred in when deviating from the egalitarian reference. Third, they rationalize social preferences for an early resolution of uncertainty. Fourth, they ethically justify a variety of discounting structures, depending on the timing of resolution of risk, the intensity of risk, and society's aversion to risk.

Each member of the characterized class of welfare criteria is identified by three ethical choices: social aversion to intergenerational inequality, social aversion to intrinsic risk, and social aversion to option risk. These ethical choices certainly play a crucial role in defining optimal policies and may be subject to an ethical debate similar to the discount factor. However, the methodology adopted indicates a solution which might avoid such controversies. While the endogenous reference ensures that the welfare criterion is tailored to each intergenerational problem, the ethical principles embodied by these ethical choices remain unchanged over the entire domain of intergenerational problems. The social aversion to intergenerational inequality, intrinsic risk, and option risk can then

be justified through Rawls' (1971) method of reflective equilibrium to "revise or criticize ethical norms in the light of their implications" (Dasgupta and Heal (1979); p. 311). The ethical choices should be revised until the consequent policy recommendations are socially desirable for each intergenerational problem. When a reflective equilibrium is reached, the criterion is "normatively robust" and can be applied to the real problem faced by society.

Several important features of intergenerational risk require further investigation. The ethical treatment of intergenerational issues falls short of an endogenous dimension such as population size (see Blackorby et al. (2005)), which might substantially aggravate or alleviate future resource scarcity. The "event tree" structure of information disclosure does not allow society to address unawareness about future events (see Dekel et al. (1998)). Finally, the restriction to a single dimension of well-being, with neither overlapping generations nor multiple commodities, rules out the ethical difficulties of confronting conflicting views about what constitutes a good life (see Piacquadio (2014)) and might lead to underestimating the effects of environmental damages (see Sterner and Persson (2008)).

Acknowledgments. The author thanks Geir Asheim, Claude d'Aspremont, Marc Fleurbaey, Reyer Gerlach, Peter Hammond, Bård Harstad, Francois Maniquet, Kalle Moene, William Thomson, Yves Sprumont, Kjetil Storesletten, and Christian Traeger. Thanks also to the audiences at BEEER (Bergen), FEEM (Milano, Venice), SSCW (Boston), SDU (Odense), CESifo (Munich), NMBU (Ås), CORE (Louvain-la-Neuve), and EAERE (Helsinki). The paper subsumes an earlier manuscript entitled "Fair intergenerational utilitarianism." This paper is part of the research activities at the Centre for the Study of Equality, Social Organization, and Performance (ESOP), supported by the Research Council of Norway, project number 179552. The research leading to these results has received funding from the ERC grant agreement n. 283236 (FP7/2007-2013).

A. Proofs

A.1. Theorem 1, Part 1: (ii) implies (i)

Since the welfare criterion is increasing in the assigned utilities, it satisfies *monotonicity*. Since it is continuous, it satisfies *continuity*. Since it is homogeneous with respect to the reference $x \in U$, it satisfies *proportionality*. Since it is additive over each generation's welfare, it satisfies *intergenerational separability*. Since for each $t \in T$, the assigned utilities enter additively in $w_t(u; x, r_t)$, the welfare criterion also satisfies *intragenerational sep-*

arability. The implications for *risk balancing* and *intergenerational equity* are presented as lemmas.

Lemma 1. *If a social ordering is reference-dependent utilitarian, then it satisfies risk balancing.*

Proof. Let $x \in U$. Let a pair $u, \bar{u} \in U$ be such that, for some $t \in T$, a pair $s, s' \in S_t$, and $\delta \in \mathbb{R}_+$, the following conditions hold: (i) $u(t, s) = \bar{u}(t, s) - \frac{\delta}{\pi^s} \geq x(t, s)$; (ii) $u(t, s') = \bar{u}(t, s') + \frac{\delta}{\pi^{s'}} \leq x(t, s')$; (iii) $u(\tilde{t}, \tilde{s}) = \bar{u}(\tilde{t}, \tilde{s})$ for each $(\tilde{t}, \tilde{s}) \neq (t, s), (t, s')$. I need to prove that $u R_x \bar{u}$.

Define $a \equiv \frac{u(t, s)}{x(t, s)}$, $\bar{a} \equiv \frac{\bar{u}(t, s)}{x(t, s)}$, $b \equiv \frac{u(t, s')}{x(t, s')}$, and $\bar{b} \equiv \frac{\bar{u}(t, s')}{x(t, s')}$; by (i) and (ii) it follows that $\bar{a} > a \geq b > \bar{b}$. Condition (iii) implies that:

$$W(u; x, \mathbf{r}, \rho) - W(\bar{u}; x, \mathbf{r}, \rho) \geq 0 \Leftrightarrow w_t(u; x, r_t) - w_t(\bar{u}; x, r_t) \geq 0.$$

Case $r_t \neq 0$. By condition (iii), $w_t(u; x, r_t) - w_t(\bar{u}; x, r_t) \geq 0$ if only if:

$$\frac{1}{r_t} \left[\pi^s x(t, s) (a^{r_t} - \bar{a}^{r_t}) + \pi^{s'} x(t, s') (b^{r_t} - \bar{b}^{r_t}) \right] \geq 0.$$

Define $\Delta \equiv \frac{\bar{a}^{r_t} - b^{r_t}}{\bar{a} - b}$. It follows that $\Delta > 0$ if $r_t > 0$ and $\Delta < 0$ if $r_t < 0$.

Subcase $r_t \in (0, 1]$. Thus, $\Delta > 0$. By first order linear approximation:

$$\begin{aligned} a^{r_t} &= \left(\bar{a} - \frac{\delta}{\pi^s x(t, s)} \right)^{r_t} \geq \bar{a}^{r_t} - \frac{\delta}{\pi^s x(t, s)} \Delta \text{ and} \\ b^{r_t} &= \left(\bar{b} + \frac{\delta}{\pi^{s'} x(t, s')} \right)^{r_t} \geq \bar{b}^{r_t} + \frac{\delta}{\pi^{s'} x(t, s')} \Delta. \end{aligned}$$

Premultiply the first by $\pi^s x(t, s)$ and the second by $\pi^{s'} x(t, s')$. Adding up and simplifying, gives:

$$\pi^s x(t, s) (a^{r_t} - \bar{a}^{r_t}) + \pi^{s'} x(t, s') (b^{r_t} - \bar{b}^{r_t}) \geq 0.$$

Since $r_t > 0$, this proves that $W(u; x, \mathbf{r}, \rho) - W(\bar{u}; x, \mathbf{r}, \rho) \geq 0$ and $u R_x \bar{u}$.

Subcase $r_t < 0$. Thus, $\Delta < 0$. By first order linear approximation:

$$\begin{aligned} a^{r_t} &= \left(\bar{a} - \frac{\delta}{\pi^s x(t, s)} \right)^{r_t} \leq \bar{a}^{r_t} - \frac{\delta}{\pi^s x(t, s)} \Delta \text{ and} \\ b^{r_t} &= \left(\bar{b} + \frac{\delta}{\pi^{s'} x(t, s')} \right)^{r_t} \leq \bar{b}^{r_t} + \frac{\delta}{\pi^{s'} x(t, s')} \Delta. \end{aligned}$$

Premultiply the first by $\pi^s x(t, s)$ and the second by $\pi^{s'} x(t, s')$. Add up and simplifying, gives:

$$\pi^s x(t, s) (a^{r_t} - \bar{a}^{r_t}) + \pi^{s'} x(t, s') (b^{r_t} - \bar{b}^{r_t}) \leq 0.$$

Since $r_t < 0$, this proves that $W(u; x, \mathbf{r}, \rho) - W(\bar{u}; x, \mathbf{r}, \rho) \geq 0$ and $u R_x \bar{u}$.

Case $r_t = 0$. By condition (iii), $w_t(u; x, 0) - w_t(\bar{u}; x, 0) \geq 0$ if only if:

$$\pi^s x(t, s) (\ln a - \ln \bar{a}) + \pi^{s'} x(t, s') (\ln b - \ln \bar{b}) \geq 0.$$

Define $\Delta \equiv \frac{\ln \bar{a} - \ln \bar{b}}{\bar{a} - \bar{b}}$; since $\bar{a} > \bar{b}$, $\Delta > 0$. By first order linear approximation:

$$\ln a = \ln \left(\bar{a} - \frac{\delta}{\pi^s x(t, s)} \right) \geq \ln \bar{a} - \frac{\delta}{\pi^s x(t, s)} \Delta \text{ and}$$

$$\ln b = \ln \left(\bar{b} + \frac{\delta}{\pi^{s'} x(t, s')} \right) \geq \ln \bar{b} + \frac{\delta}{\pi^{s'} x(t, s')} \Delta.$$

Premultiply the first by $\pi^s x(t, s)$ and the second by $\pi^{s'} x(t, s')$. Adding up and simplifying gives the required inequality. This proves that $W(u; x, \mathbf{r}, \rho) - W(\bar{u}; x, \mathbf{r}, \rho) \geq 0$ and $u R_x \bar{u}$. \square

Lemma 2. *If a social ordering is reference-dependent utilitarian, then it satisfies inter-generational equity.*

Proof. Let $x \in U$. Let a pair $u, \bar{u} \in U$ be such that for some $t, t' \in T$, with $t' > t$, and $a \in \mathbb{R}_+$ the following conditions hold: (i) $\frac{u(t, s)}{x(t, s)} = \frac{\bar{u}(t, s)}{x(t, s)} - \frac{a}{\pi_t x(t, s)} \geq 1$ for each $s \in S_t$; (ii) $\frac{u(t', s)}{x(t', s)} = \frac{\bar{u}(t', s)}{x(t', s)} + \frac{a}{\pi_{t'} x(t', s)} \leq 1$ for each $s \in S_{t'}$; (iii) $u(\tilde{t}, s) = \bar{u}(\tilde{t}, s)$ for each $\tilde{t} \neq t, t'$ and each $s \in S_{\tilde{t}}$. I need to prove that $u R_x \bar{u}$.

Define $\delta \equiv \frac{a}{\bar{k}}$ for $\bar{k} \in \mathbb{N}_+$. Let $\left(\{u^k\}_{k \in [1, \bar{k}]} \right) \in U^{\bar{k}}$ be such that: (I) $u^1 = u$ and $u^{\bar{k}} = \bar{u}$; (II) for each $k \in [1, \bar{k} - 1]$, $\frac{u^k(t, s)}{x(t, s)} = \frac{u^{k+1}(t, s)}{x(t, s)} - \frac{\delta}{\pi_t x(t, s)}$ for each $s \in S_t$ and $\frac{u^k(t', s)}{x(t', s)} = \frac{u^{k+1}(t', s)}{x(t', s)} + \frac{\delta}{\pi_{t'} x(t', s)}$ for each $s \in S_{t'}$; (III) $u^k(\tilde{t}, s) = u(\tilde{t}, s)$ for each $\tilde{t} \neq t, t'$, each $s \in S_{\tilde{t}}$, and each $k \in [1, \bar{k}]$.

I show next that at the limit for $\bar{k} \rightarrow \infty$ (and thus for $\delta \rightarrow 0$), $W(u^k; x, \mathbf{r}, \rho) - W(u^{k+1}; x, \mathbf{r}, \rho) \geq 0$. By transitivity the result follows.

Case $\rho \neq 0$. By condition (III),

$$\begin{aligned} W(u^k; x, \mathbf{r}, \rho) - W(u^{k+1}; x, \mathbf{r}, \rho) &= \frac{1}{\rho} x_t [w_t(u^k; x, r_t)^\rho - w_t(u^{k+1}; x, r_t)^\rho] + \\ &\quad \frac{1}{\rho} x_{t'} [w_{t'}(u^k; x, r_{t'})^\rho - w_{t'}(u^{k+1}; x, r_{t'})^\rho]. \end{aligned} \quad (7)$$

By condition (II), u^{k+1} can be written as a function of u^k and δ . Define the “equally distributed equivalent” for t and t' at c^{k+1} as:

$$e_t(\delta) = w_t(u^{k+1}; x, r_t),$$

$$e_{t'}(\delta) = w_{t'}(u^{k+1}; x, r_{t'}).$$

It follows that $e_t(0) = w_t(u^k; x, r_t)$ and $e_{t'}(0) = w_{t'}(u^k; x, r_{t'})$. Thus (7) can be written

as:

$$W(u^k; x, \mathbf{r}, \rho) - W(u^{k+1}; x, \mathbf{r}, \rho) = \frac{1}{\rho} x_t [e_t(0)^\rho - e_t(\delta)^\rho] + \frac{1}{\rho} x_{t'} [e_{t'}(\delta)^\rho - e_{t'}(0)^\rho].$$

Divide by δ , and take the limit for $\bar{k} \rightarrow \infty$. As $\delta \rightarrow 0$, $\frac{1}{\rho} x_t \frac{e_t(0)^\rho - e_t(\delta)^\rho}{\delta}$ tends to:

$$\frac{1}{\rho} x_t \frac{\partial}{\partial \delta} e_t(\delta)^\rho \Big|_{\delta=0} = x_t e_t(0)^{\rho-1} \frac{\partial e_t(\delta)}{\partial \delta} \Big|_{\delta=0}, \quad (8)$$

while $\frac{1}{\rho} x_{t'} \frac{e_{t'}(0)^\rho - e_{t'}(\delta)^\rho}{\delta}$ tends to:

$$\frac{1}{\rho} x_{t'} \frac{\partial}{\partial \delta} e_{t'}(\delta)^\rho \Big|_{\delta=0} = x_{t'} e_{t'}(0)^{\rho-1} \frac{\partial e_{t'}(\delta)}{\partial \delta} \Big|_{\delta=0}. \quad (9)$$

Computing the derivatives of e_t and $e_{t'}$, yields, for all $r_t, r_{t'} \in (-\infty, 1)$:

$$\frac{\partial e_t(\delta)}{\partial \delta} \Big|_{\delta=0} = -\frac{1}{\pi_t x_t} e_t(0)^{1-r_t} \sum_{s \in S_t} \pi^s \left(\frac{u^k(t, s)}{x(t, s)} \right)^{r_t-1}; \quad (10)$$

$$\frac{\partial e_{t'}(\delta)}{\partial \delta} \Big|_{\delta=0} = \frac{1}{\pi_{t'} x_{t'}} e_{t'}(0)^{1-r_{t'}} \sum_{s \in S_{t'}} \pi^s \left(\frac{u^k(t', s)}{x(t', s)} \right)^{r_{t'}-1}. \quad (11)$$

Substituting (10) in (8), leads to:

$$\frac{1}{\rho} x_t \frac{\partial}{\partial \delta} e_t(\delta)^\rho \Big|_{\delta=0} = -e_t(0)^{\rho-1} \frac{\sum_{s \in S_t} \pi^s \left(\frac{u(t, s)}{x(t, s)} \right)^{r_t-1}}{\sum_{s \in S_t} \pi^s (e_t(0))^{r_t-1}}.$$

Since $\frac{u^k(t, s)}{x(t, s)} \geq 1$ for each $s \in S_t$, $e_t(0) \geq 1$; moreover $\rho \leq 1$; thus, $e_t(0)^{\rho-1} \leq 1$.

Similarly, since $\frac{u^k(t, s)}{x(t, s)} \geq 1$ for each $s \in S_t$, $e_t(0) \geq 1$, and $r_t < 1$, it follows that

$$\sum_{s \in S_t} \pi^s (e_t(0))^{r_t-1} \geq \sum_{s \in S_t} \pi^s \left(\frac{u(t, s)}{x(t, s)} \right)^{r_t-1}. \text{ These imply that } \frac{1}{\rho} x_t \frac{\partial}{\partial \delta} e_t(\delta)^\rho \Big|_{\delta=0} \geq -1.$$

Similarly, substitute (11) in (9) to get:

$$\frac{1}{\rho} x_{t'} \frac{\partial}{\partial \delta} e_{t'}(\delta)^\rho \Big|_{\delta=0} = e_{t'}(0)^{\rho-1} \frac{\sum_{s \in S_{t'}} \pi^s \left(\frac{u(t', s)}{x(t', s)} \right)^{r_{t'}-1}}{\sum_{s \in S_{t'}} \pi^s (e_{t'}(0))^{r_{t'}-1}}.$$

Since $\frac{u^k(t', s)}{x(t', s)} \leq 1$ for each $s \in S_{t'}$, $e_{t'}(0) \leq 1$; moreover $\rho \leq 1$; thus, $e_{t'}(0)^{\rho-1} \geq 1$.

Similarly, since $\frac{u^k(t', s)}{x(t', s)} \leq 1$ for each $s \in S_{t'}$, $e_{t'}(0) \leq 1$, and $r_t < 1$, it follows that

$$\sum_{s \in S_t} \pi^s (e_t(0))^{r_t-1} \leq \sum_{s \in S_t} \pi^s \left(\frac{u(t,s)}{x(t,s)} \right)^{r_t-1}. \text{ These imply that } \left. \frac{1}{\rho} x_{t'} \frac{\partial}{\partial \delta} e_{t'}(\delta)^\rho \right|_{\delta=0} \geq 1.$$

Substituting in (7), shows that:

$$\lim_{\bar{k} \rightarrow \infty} \frac{(W(u^k; x, \mathbf{r}, \rho) - W(u^{k+1}; x, \mathbf{r}, \rho))}{\delta} \geq 0.$$

Since this inequality is true for each $k \in [1, \bar{k}]$, transitivity implies that $W(u^1; x, \mathbf{r}, \rho) \geq W(u^{\bar{k}}; x, \mathbf{r}, \rho)$ or, equivalently, $W(u; x, \mathbf{r}, \rho) \geq W(\bar{u}; x, \mathbf{r}, \rho)$ and $u R_x \bar{u}$.

Case $\rho = 0$. Similar steps lead to:

$$\begin{aligned} x_t \frac{\partial}{\partial \delta} \ln e_t(\delta) \Big|_{\delta=0} &= -e_t(0)^{-1} \frac{\sum_{s \in S_t} \pi^s \left(\frac{u(t,s)}{x(t,s)} \right)^{r_t-1}}{\sum_{s \in S_t} \pi^s (e_t(0))^{r_t-1}} \geq -1, \\ x_{t'} \frac{\partial}{\partial \delta} \ln e_{t'}(\delta) \Big|_{\delta=0} &= e_{t'}(0)^{-1} \frac{\sum_{s \in S_{t'}} \pi^s \left(\frac{u(t',s)}{x(t',s)} \right)^{r_{t'}-1}}{\sum_{s \in S_{t'}} \pi^s (e_{t'}(0))^{r_{t'}-1}} \geq 1. \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{\bar{k} \rightarrow \infty} \frac{(W(u^k; x, \mathbf{r}, \rho) - W(u^{k+1}; x, \mathbf{r}, \rho))}{\delta} &= \\ x_t \frac{\partial}{\partial \delta} \ln e_t(\delta) \Big|_{\delta=0} + x_{t'} \frac{\partial}{\partial \delta} \ln e_{t'}(\delta) \Big|_{\delta=0} &\geq 0, \end{aligned}$$

and, by transitivity, $u R_x \bar{u}$ follows. \square

A.2. Theorem 1, Part 2: (i) implies (ii)

The (mathematically involved) proof is divided in 10 steps. Assume *monotonicity*, *continuity*, *intergenerational equity*, *risk balancing*, *proportionality*, *intergenerational separability*, and *intragenerational separability* hold and let $x \in U$.

The first step shows that the social ordering for x , R_x , has a specific functional representation: it is continuous, additive across time and for each period additive across states, and increasing in the utility assigned to each potential agent.

Step 1. For each $t \in T$ and each $s \in S_t$, there exist continuous and strictly increasing functions q_t and $\bar{v}_{(t,s)}$ such that R_x is represented by:

$$V(u; x) = \sum_{t \in T} q_t \left(\sum_{s \in S_t} \bar{v}_{(t,s)}(u(t,s)) \right). \quad (12)$$

Proof. By Gorman (1968)'s theorem on overlapping separable sets, *continuity*, *intergenerational separability*, and *intragenerational separability* imply that there exist continuous

functions q_t (one for each $t \in T$) and $\bar{v}_{(t,s)}$ (one for each $(t,s) \in \overline{TS}$) such that R_x is represented by (12). By *monotonicity*, it must be true that, for each $t \in T$ and each $s \in S_t$, either q_t and $\bar{v}_{(t,s)}$ are all strictly increasing or these are all strictly decreasing. Either choices lead to ordinally equivalent representations of R_x . \square

The next step shows that each $\bar{v}_{(t,s)}$ can be rewritten in terms of the “relative utility” $u(t,s)/x(t,s)$. Moreover, such function is concave in relative utility and is equal across potential agents belonging to the same generation, up to an additive constant.

Step 2. For each $t \in T$, there exist strictly increasing and concave function $v_t : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for each $a \in \mathbb{R}_+$ and each $s \in S_t$:

$$v_t(a) = \frac{\bar{v}_{(t,s)}(ax(t,s))}{\pi^s x(t,s)} + \chi(t,s) \text{ for some } \chi(t,s) \in \mathbb{R}.$$

Proof. For each $t \in T$, each $s \in S_t$, and each $u(t,s) \in \mathbb{R}_+$ define:

$$v_{(t,s)}\left(\frac{u(t,s)}{x(t,s)}\right) \equiv (\pi^s x(t,s))^{-1} \bar{v}_{(t,s)}(u(t,s)).$$

Since $\bar{v}_{(t,s)}$ is strictly increasing (by Step 1), also $v_{(t,s)}$ is.

Let a pair $u, \bar{u} \in U$ be such that for some $t \in T$, a pair $s, s' \in S_t$, and a $\delta \in \mathbb{R}_+$ the following conditions hold:

- (i) $u(t,s) = \bar{u}(t,s) - \frac{\delta}{\pi^s} \geq x(t,s)$;
- (ii) $u(t,s') = \bar{u}(t,s') + \frac{\delta}{\pi^{s'}} \leq x(t,s')$;
- (iii) $u(\tilde{t}, \tilde{s}) = \bar{u}(\tilde{t}, \tilde{s})$ for each $(\tilde{t}, \tilde{s}) \in \overline{TS}$ with $(\tilde{t}, \tilde{s}) \neq (t,s), (t,s')$.

By *risk balancing*, $u R_x \bar{u}$. By Step 1, this requires that $V(u;x) - V(\bar{u};x) \geq 0$ or, using (iii), that:

$$\begin{aligned} & \bar{v}_{(t,s)}(u(t,s)) - \bar{v}_{(t,s)}\left(u(t,s) + \frac{\delta}{\pi^s}\right) + \\ & \bar{v}_{(t,s')}(u(t,s')) - \bar{v}_{(t,s')}\left(u(t,s') - \frac{\delta}{\pi^{s'}}\right) \geq 0 \end{aligned} \tag{13}$$

Substituting the utility functions $v_{(t,s)}$ and $v_{(t,s')}$ in (13), gives:

$$\begin{aligned} & \pi^s x(t,s) \left[v_{(t,s)}\left(\frac{u(t,s)}{x(t,s)}\right) - v_{(t,s)}\left(\frac{u(t,s)}{x(t,s)} + \frac{\delta}{\pi^s x(t,s)}\right) \right] + \\ & \pi^{s'} x(t,s') \left[v_{(t,s')}\left(\frac{u(t,s')}{x(t,s')}\right) - v_{(t,s')}\left(\frac{u(t,s')}{x(t,s')} - \frac{\delta}{\pi^{s'} x(t,s')}\right) \right] \geq 0. \end{aligned}$$

If $v_{(t,s)}$ and $v_{(t,s')}$ are differentiable at $\left(\frac{u(t,s)}{x(t,s)}\right)$ and $\left(\frac{u(t,s')}{x(t,s')}\right)$ respectively, dividing by

δ and taking the limit for $\delta \rightarrow 0$, yields:

$$v'_{(t,s')} \left(\frac{u(t,s)}{x(t,s)} \right) \leq v'_{(t,s')} \left(\frac{u(t,s')}{x(t,s')} \right). \quad (14)$$

Since $v_{(t,s)}$ and $v_{(t,s')}$ are strictly increasing, these are differentiable almost everywhere. Thus, equation (14) holds for almost all $\left(\frac{u(t,s)}{x(t,s)} \right) > 1 > \left(\frac{u(t,s')}{x(t,s')} \right)$ and, symmetrically, the reverse inequality holds for almost all $\left(\frac{u(t,s)}{x(t,s)} \right) < 1 < \left(\frac{u(t,s')}{x(t,s')} \right)$. Thus, if the functions are differentiable at 1, $v'_{(t,s)}(1) = v'_{(t,s')}(1)$.

Let $a > 0$. By *proportionality* and Step 1, $V(u; x) \geq V(\bar{u}; x)$ if and only if $V(u; ax) \geq V(\bar{u}; ax)$. Since this equivalence holds for each $a > 0$, equation (14) holds almost everywhere for each $\left(\frac{u(t,s)}{x(t,s)} \right) > a > \left(\frac{u(t,s')}{x(t,s')} \right)$ and each $a > 0$. Thus $v'_{(t,s)}(a) = v'_{(t,s')}(a)$ almost everywhere for each $a > 0$ and $v_{(t,s)}$ and $v_{(t,s')}$ are concave. This also implies that for each $t \in T$, there exists a strictly increasing and concave function $v_t : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a constant $\chi(t, s) \in \mathbb{R}$ for each $s \in S_t$ such that for for each $b \in \mathbb{R}_+$, $v_t(b) = v_{(t,s)}(b) + \chi(t, s)$. \square

Step 3. For each $t \in T$, v_t is differentiable.

Proof. Let $t \in T$. By contradiction, assume v_t is not differentiable at $a \in \mathbb{R}_+$. Then, left and right derivative at a are such that $v'_t(a^-) \neq v'_t(a^+)$. By continuity and almost everywhere differentiability of v_t , there exist a pair $u, \bar{u} \in C$ such that: (i) $u(t, s) > \bar{u}(t, s) = a = \bar{u}(t, s') > u(t, s')$ for some $s, s' \in S_t$; (ii) $u(\bar{t}, \bar{s}) = \bar{u}(\bar{t}, \bar{s})$ for each $(\bar{t}, \bar{s}) \neq (t, s), (t, s')$; (iii) $V(u; x) = V(\bar{u}; x)$; and (iv) v_t is differentiable at $u(t, s)$ and $u(t, s')$. Define $\Delta V \equiv V(u; x) - V(\bar{u}; x)$; by the previous steps and (iii):

$$\begin{aligned} \Delta V &= \left[\pi^s x(t, s) v_t \left(\frac{u(t, s)}{x(t, s)} \right) + \pi^{s'} x(t, s') v_t \left(\frac{u(t, s')}{x(t, s')} \right) \right] \\ &\quad - \left[\pi^s x(t, s) v_t \left(\frac{a}{x(t, s)} \right) + \pi^{s'} x(t, s') v_t \left(\frac{a}{x(t, s')} \right) \right] = 0. \end{aligned}$$

For each $b > 0$, define $\Delta V(b) \equiv V(u; bx) - V(\bar{u}; bx)$ or, substituting:

$$\begin{aligned} \Delta V(b) &= b \left[\pi^s x(t, s) v_t \left(\frac{u(t, s)}{bx(t, s)} \right) + \pi^{s'} x(t, s') v_t \left(\frac{u(t, s')}{bx(t, s')} \right) \right] \\ &\quad - b \left[\pi^s x(t, s) v_t \left(\frac{a}{bx(t, s)} \right) + \pi^{s'} x(t, s') v_t \left(\frac{a}{bx(t, s')} \right) \right]. \end{aligned}$$

By *proportionality*, $\Delta V(b) = 0$ for each $b > 0$. Thus, by differentiating $\Delta V(b)$ with respect to b , it follows that $\left. \frac{\partial \Delta V(b)}{\partial b} \right|_{b=b^*} = 0$ for each $b^* > 0$. Let $\bar{b} \equiv (x(t, s))^{-1}$. Using the

fact that $v'_t(a^-) \neq v'_t(a^+)$ and that $a = \frac{a}{bx(t, s)}$, it follows that $\left. \frac{\partial \Delta V(b)}{\partial b} \right|_{b=\bar{b}^-} \neq \left. \frac{\partial \Delta V(b)}{\partial b} \right|_{b=\bar{b}^+}$. A contradiction. \square

The next step proves that v_t is a mean of order r_t : it either has a power functional form or the logarithmic form.

Step 4. For each $t \in T$, there exist constants $\bar{\eta}_t, \eta_t, r_t \in \mathbb{R}$ such that for each $y > 0$:

$$v_t(y) = \bar{\eta}_t + \frac{\eta_t}{r_t} y^{r_t} \text{ if } r_t \neq 0 \text{ and}$$

$$v_t(y) = \bar{\eta}_t + \eta_t \ln y \text{ if } r_t = 0.$$

Proof. Let a pair $u, \bar{u} \in U$ be such that for some $t \in T$ and a pair $s, s' \in S_t$:

- (i) $u(\tilde{t}, \tilde{s}) = \bar{u}(\tilde{t}, \tilde{s})$ for each $(\tilde{t}, \tilde{s}) \neq (t, s), (t, s')$;
- (ii) $\sum_{s \in S_{t'}} \pi^s x(t', s) v_{t'} \left(\frac{u(t', s)}{x(t', s)} \right) = \sum_{s \in S_{t'}} \pi^s x(t', s) v_{t'} \left(\frac{\bar{u}(t', s)}{x(t', s)} \right)$ for each $t' \in T$; and
- (iii) $V(u; x) = V(\bar{u}; x)$.

This yields:

$$\begin{aligned} \pi^s x(t, s) v_t \left(\frac{u(t, s)}{x(t, s)} \right) + \pi^{s'} x(t, s') v_t \left(\frac{u(t, s')}{x(t, s')} \right) = \\ \pi^s x(t, s) v_t \left(\frac{\bar{u}(t, s)}{x(t, s)} \right) + \pi^{s'} x(t, s') v_t \left(\frac{\bar{u}(t, s')}{x(t, s')} \right), \end{aligned} \quad (15)$$

and, by *proportionality*, for each $a > 0$ also:

$$\begin{aligned} \pi^s x(t, s) v_t \left(\frac{u(t, s)}{ax(t, s)} \right) + \pi^{s'} x(t, s') v_t \left(\frac{u(t, s')}{ax(t, s')} \right) = \\ \pi^s x(t, s) v_t \left(\frac{\bar{u}(t, s)}{ax(t, s)} \right) + \pi^{s'} x(t, s') v_t \left(\frac{\bar{u}(t, s')}{ax(t, s')} \right) \end{aligned} \quad (16)$$

For $b \in \mathbb{R}$, let u_b be a smooth path through U , which satisfies (i)-(iii) for all $b \neq 0$ and such that $u_0 = u$. Thus equations (15) and (16) are satisfied when u is replaced by u_b for each $b \in \mathbb{R}$. Differentiate with respect to b , evaluate at $b = 0$, and simplify to get:

$$\begin{aligned} \pi^s v'_t \left(\frac{u(t, s)}{x(t, s)} \right) \frac{\partial u_b(t, s)}{\partial b} \Big|_{b=0} + \pi^{s'} v'_t \left(\frac{u(t, s')}{x(t, s')} \right) \frac{\partial u_b(t, s')}{\partial b} \Big|_{b=0} = 0, \\ \pi^s v'_t \left(\frac{u(t, s)}{ax(t, s)} \right) \frac{\partial u_b(t, s)}{\partial b} \Big|_{b=0} + \pi^{s'} v'_t \left(\frac{u(t, s')}{ax(t, s')} \right) \frac{\partial u_b(t, s')}{\partial b} \Big|_{b=0} = 0. \end{aligned}$$

Combining these equations leads to:

$$\frac{v'_t \left(\frac{u(t, s)}{ax(t, s)} \right)}{v'_t \left(\frac{u(t, s)}{x(t, s)} \right)} = \frac{v'_t \left(\frac{u(t, s')}{ax(t, s')} \right)}{v'_t \left(\frac{u(t, s')}{x(t, s')} \right)}. \quad (17)$$

Define the function $\lambda_t(a) \equiv \frac{v'_t\left(\frac{u(t,s')}{ax(t,s')}\right)}{v'_t\left(\frac{u(t,s')}{x(t,s')}\right)}$; by the properties of v_t , $\lambda_t(a)$ is continuous and such that $\lambda_t(a) > 0$ for each $a > 0$. Substituting in (17) and taking the log transformation gives:

$$\ln v'_t\left(\frac{u(t,s)}{ax(t,s)}\right) - \ln v'_t\left(\frac{u(t,s)}{x(t,s)}\right) = \ln \lambda_t(a). \quad (18)$$

Equation (18) holds for each $u(t,s) > 0$. Define $y \equiv \frac{u(t,s)}{x(t,s)}$ and the transformation $g(\ln(y)) \equiv v'_t(y)$ for each $y > 0$. Substituting and rearranging gives $\ln g(\ln y) - \ln g(\ln y - \ln a) = -\ln \lambda_t(a)$ for each $y > 0$. Divide by $\ln a$ and take the limit for $a \rightarrow 1$ to obtain:

$$\frac{d \ln g(\ln y)}{d \ln y} = -\lim_{a \rightarrow 1} \frac{\ln \lambda_t(a)}{\ln a}.$$

By differentiability of v_t (see Step 3), the limit exists and the RHS of this equation is finite. Let $r_t \equiv 1 - \lim_{a \rightarrow 1} \frac{\ln \lambda_t(a)}{\ln a}$. Integrating with respect to y gives:

$$g(\ln y) = v'_t(y) = \eta_t y^{r_t-1}, \quad (19)$$

for some integrating constant $\ln \eta_t$. Further integrating, with integrating constant $\bar{\eta}_t$, gives $v_t(y) = \bar{\eta}_t + \frac{\eta_t}{r_t} y^{r_t}$ if $r_t \neq 0$ and $v_t(y) = \bar{\eta}_t + \eta_t \ln y$ otherwise. \square

The function q_t needs to be homothetic with respect to x and, thus, assumes a specific form, as highlighted next.

Step 5. For each $t \in T$ and each $u \in U$, $q_t = \psi_t(\bar{q}_t(u; x))$ where:

$$\bar{q}_t(u; x) = \bar{q}_t(u; x, r_t) \equiv x_t \left[\frac{1}{x_t} \sum_{s \in S_t} \pi^s x(t, s) \left(\frac{u(t, s)}{x(t, s)} \right)^{r_t} \right]^{\frac{1}{r_t}} \text{ if } r_t \neq 0 \text{ and}$$

$$\bar{q}_t(u; x) = \bar{q}_t(u; x, 0) \equiv x_t \exp \left[\frac{1}{x_t} \sum_{s \in S_t} \pi^s x(t, s) \ln \left(\frac{u(t, s)}{x(t, s)} \right) \right] \text{ if } r_t = 0.$$

Proof. Substitute $\bar{v}_{(t,s)}$ from Step 2 in (12), and rewrite as $V(u; x) = \sum_{t \in T} q_t(u(t); x(t))$, where:

$$q_t(u(t); x(t)) \equiv q_t \left(\chi_t + \sum_{s \in S_t} \pi^s x(t, s) v_t \left(\frac{u(t, s)}{x(t, s)} \right) \right),$$

and where $u(t) \equiv (\{u_s\}_{s \in S_t})$, $x(t) \equiv (\{x_s\}_{s \in S_t})$, and $\chi_t \equiv -\sum_{s \in S_t} \pi^s x(t, s) \chi(t, s)$ for each $t \in T$.

Let $u, \bar{u} \in U$ and $t \in T$ be such that $u'_t = \bar{u}'_t$ for each $t' \neq t$. By *proportionality*, for each $a > 0$, $V(u; x) \geq V(\bar{u}; x)$ if and only if $V(u; ax) \geq V(\bar{u}; ax)$. Since V is additive over time, this statement is equivalent to $q_t(u(t); x(t)) \geq q_t(\bar{u}(t); x(t))$ if and only if

$q_t(u(t); ax(t)) \geq q_t(\bar{u}(t); ax(t))$. Thus q_t is homothetic with respect to $x(t)$. It follows that it can be written as $q_t(u(t); x(t)) = \tilde{\psi}_t(\tilde{q}_t(u(t); x(t)))$ where \tilde{q}_t is positively linearly homogeneous and such that:

$$\tilde{q}_t(u(t); x(t)) = \tilde{q}_t^* \left(\chi_t + \sum_{s \in S_t} \pi^s x(t, s) v_t \left(\frac{u(t, s)}{x(t, s)} \right) \right),$$

with \tilde{q}_t^* continuous and strictly increasing.

Case 1. Assume $r_t \neq 0$. Substitute $v_t \left(\frac{u(t, s)}{x(t, s)} \right) = \bar{\eta}_t + \frac{\eta_t}{r_t} \left(\frac{u(t, s)}{x(t, s)} \right)^{r_t}$ from Step 4:

$$\tilde{q}_t(u(t); x(t)) = \tilde{q}_t^* \left(\chi_t + x_t \bar{\eta}_t + \frac{\eta_t}{r_t} \sum_{s \in S_t} \pi^s x(t, s) \left(\frac{u(t, s)}{x(t, s)} \right)^{r_t} \right).$$

Since $\tilde{q}_t(u(t); x(t))$ is positively linearly homogeneous, $\tilde{q}_t(u(t); ax(t)) = a \tilde{q}_t(u(t); x(t))$ for each $a > 0$. Thus:

$$\begin{aligned} \tilde{q}_t^* \left(a \chi_t + a x_t \bar{\eta}_t + a^{1-r_t} \frac{\eta_t}{r_t} \sum_{s \in S_t} \pi^s x(t, s) \left(\frac{u(t, s)}{x(t, s)} \right)^{r_t} \right) = \\ a \tilde{q}_t^* \left(\chi_t + x_t \bar{\eta}_t + \frac{\eta_t}{r_t} \sum_{s \in S_t} \pi^s x(t, s) \left(\frac{u(t, s)}{x(t, s)} \right)^{r_t} \right). \end{aligned}$$

Since this needs to hold for each $u \in U$, it follows that:

$$\tilde{q}_t^*(y) = x_t \left(r_t \frac{y - \chi_t - x_t \bar{\eta}_t}{x_t} \right)^{\frac{1}{r_t}} \text{ for each } y \in \mathbb{R},$$

and, substituting:

$$\tilde{q}_t(u(t); x(t)) = \eta_t^{\frac{1}{r_t}} x_t \left(\frac{1}{x_t} \sum_{s \in S_t} \pi^s x(t, s) \left(\frac{u(t, s)}{x(t, s)} \right)^{r_t} \right)^{\frac{1}{r_t}}.$$

Define $\psi_t \equiv \eta_t^{\frac{1}{r_t}} \tilde{\psi}_t$ and $\bar{q}_t(u; x) = \eta_t^{-\frac{1}{r_t}} \tilde{q}_t(u(t); x(t))$ to obtain the result.

Case 2. Assume $r_t = 0$. Substitute $v_t \left(\frac{u(t, s)}{x(t, s)} \right) = \bar{\eta}_t + \eta_t \ln \left(\frac{u(t, s)}{x(t, s)} \right)$ from Step 4:

$$\tilde{q}_t(u(t); x(t)) = \tilde{q}_t^* \left(\chi_t + x_t \bar{\eta}_t + \eta_t \sum_{s \in S_t} \pi^s x(t, s) \ln \left(\frac{u(t, s)}{x(t, s)} \right) \right).$$

Since $\tilde{q}_t(u(t); x(t))$ is positively linearly homogeneous, $\tilde{q}_t(u(t); ax(t)) = a \tilde{q}_t(u(t); x(t))$

for each $a > 0$. Thus:

$$\begin{aligned} q_t^* \left(a\chi_t + ax_t\bar{\eta}_t + a\eta_t \sum_{s \in S_t} \pi^s x(t, s) \left[\ln \left(\frac{u(t, s)}{x(t, s)} \right) - \ln a \right] \right) = \\ a q_t^* \left(\chi_t + x_t\bar{\eta}_t + \eta_t \sum_{s \in S_t} \pi^s x(t, s) \ln \left(\frac{u(t, s)}{x(t, s)} \right) \right). \end{aligned}$$

Since this needs to hold for each $u \in U$, it follows that:

$$q_t^*(y) = x_t \exp \left(\frac{y - \chi_t - x_t\bar{\eta}_t}{x_t} \right) \text{ for each } y \in \mathbb{R},$$

$$\tilde{q}_t(u(t); x(t)) = x_t \exp \eta_t \left(\frac{1}{x_t} \sum_{s \in S_t} \pi^s x(t, s) \ln \left(\frac{u(t, s)}{x(t, s)} \right) \right).$$

Define $\psi_t \equiv \exp \eta_t \cdot \tilde{\psi}_t$ and $\bar{q}_t(u; x) = \exp(-\eta_t) \cdot \tilde{q}_t(u(t); x(t))$ to obtain the result. \square

The next step shows that q_t is differentiable.

Step 6. For each $t \in T$, q_t is differentiable.

Proof. By contradiction, assume q_t is not differentiable at $a_t \in \mathbb{R}_+$. This implies that $q'_t(a_t^-) \neq q'_t(a_t^+)$. Since v_t is continuous and monotonic, there exists a $a \in \mathbb{R}_+$ such that $a_t = \chi_t + x_t v_t(a)$; this is equivalent to $q_t(a_t) = q_t(\chi_t + \sum_{s \in S_t} \pi^s x(t, s) v_t(a))$. Let $t' \neq t$ and define $a_{t'} \equiv \chi_{t'} + x_{t'} v_{t'}(a)$.

By differentiability of v_t and $v_{t'}$ and by continuity and almost everywhere differentiability of q_t and $q_{t'}$, there exist a pair $u, \bar{u} \in U$ and a pair $b_t, b_{t'} \in \mathbb{R}_+$ such that: (i) $\frac{u(t, s)}{x(t, s)} = b_t > \frac{\bar{u}(t, s)}{x(t, s)} = a$ for each $s \in S_t$ and $a = \frac{\bar{u}(t', s)}{x(t', s)} > b_{t'} = \frac{u(t', s)}{x(t', s)}$ for each $s \in S_{t'}$; (ii) $u(\tilde{t}, s) = \bar{u}(\tilde{t}, s)$ for each $\tilde{t} \neq t, t'$ and each $s \in S_{\tilde{t}}$; (iii) $V(u; x) = V(\bar{u}; x)$; and (iv) q_t is differentiable at $(\chi_t + x_t v_t(b_t))$ and $q_{t'}$ is differentiable at $(\chi_{t'} + x_{t'} v_{t'}(b_{t'}))$.

Define $\Delta V \equiv V(u; x) - V(\bar{u}; x)$; by (iii):

$$\begin{aligned} \Delta V(b) &= [q_t(\chi_t + x_t v_t(b_t)) + q_{t'}(\chi_{t'} + x_{t'} v_{t'}(b_{t'}))] - \\ &\quad [q_t(\chi_t + x_t v_t(a)) + q_{t'}(\chi_{t'} + x_{t'} v_{t'}(a))] = 0. \end{aligned}$$

For each $b > 0$, define $\Delta V(b) \equiv V(u; bx) - V(\bar{u}; bx)$ and, substituting:

$$\begin{aligned} \Delta V(b) &= \left[q_t \left(b\chi_t + bx_t v_t \left(\frac{b_t}{b} \right) \right) + q_{t'} \left(b\chi_{t'} + bx_{t'} v_{t'} \left(\frac{b_{t'}}{b} \right) \right) \right] - \\ &\quad \left[q_t \left(b\chi_t + bx_t v_t \left(\frac{a}{b} \right) \right) + q_{t'} \left(b\chi_{t'} + bx_{t'} v_{t'} \left(\frac{a}{b} \right) \right) \right] = 0. \end{aligned}$$

By *proportionality*, $\Delta V(b) = 0$ for each $b > 0$ and differentiating $\Delta V(b)$ with respect to b , we deduce that $\left. \frac{\partial \Delta V(b)}{\partial b} \right|_{b=b^*} = 0$ for each $b^* > 0$. Using the fact that $q'_t(a_t^-) \neq q'_t(a_t^+)$

and that $a_t = \left(b\chi_t + bx_tv_t\left(\frac{a}{b}\right)\right)$ if $b = 1$, it follows that $\left.\frac{\partial\Delta V(b)}{\partial b}\right|_{b=1^-} \neq \left.\frac{\partial\Delta V(b)}{\partial b}\right|_{b=1^+}$.
A contradiction. \square

The next step proves that ψ_t (defining q_t) has the form of a mean of order ρ_t .

Step 7. For each $t \in T$, there exist a $\rho_t \in \mathbb{R}$ and $\xi_t \in \mathbb{R}$ such that:

$$\begin{aligned} q_t(\cdot) &= \frac{\xi_t}{\rho_t} \bar{q}_t(u; x, r_t)^{\rho_t} \text{ if } \rho_t \neq 0, \\ q_t(\cdot) &= \xi_t \ln \bar{q}_t(u; x, r_t) \text{ if } \rho_t = 0. \end{aligned}$$

Proof. As shown in Step 5, $q_t(u(t); x(t)) \geq q_t(\bar{u}(t); x(t))$ if and only if $q_t(u(t); ax(t)) \geq q_t(\bar{u}(t); ax(t))$ for each $a > 0$. Thus, there exists a function $\lambda: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\lambda(1) = 1$ and $\lambda(a) \neq 0$ such that for each $a > 0$:

$$q_t(u(t); x(t)) - q_t(\bar{u}(t); x(t)) = \lambda(a) [q_t(u(t); ax(t)) - q_t(\bar{u}(t); ax(t))]. \quad (20)$$

Substituting for $q_t(u(t); x(t)) = \psi_t[\bar{q}_t(u; x, r_t)]$ and since $\bar{q}_t(u; x, r_t)$ is positively linearly homogeneous in x , gives:

$$\psi_t[\bar{q}_t(u; x, r_t)] - \psi_t[\bar{q}_t(\bar{u}; x, r_t)] = \lambda(a) [\psi_t[a\bar{q}_t(u; x, r_t)] - \psi_t[a\bar{q}_t(\bar{u}; x, r_t)]].$$

Define $\delta \equiv \bar{q}_t(u; x, r_t) - \bar{q}_t(\bar{u}; x, r_t)$ and substitute. Dividing by δ and taking the limit for $\delta \rightarrow 0$, yields:

$$\psi'_t(\bar{q}_t(u; x, r_t)) = \lambda(a) a \psi'_t(a\bar{q}_t(u; x, r_t)). \quad (21)$$

Take the log transformation and rearrange as:

$$\ln \psi'_t(a\bar{q}_t(u; x, r_t)) - \ln \psi'_t(\bar{q}_t(u; x, r_t)) = -\ln \lambda(a) - \ln a$$

Divide by $\ln a$ and take the limit for $a \rightarrow 1$. Differentiability of q_t implies differentiability of ψ_t . The latter implies that the limit of the RHS exists and is finite. Let $\rho_t \equiv -\lim_{a \rightarrow 1} \frac{\ln \lambda(a)}{\ln a}$ and define the transformation $\bar{g}(\ln y) \equiv \psi'_t(y)$ for each $y > 0$. Substituting gives:

$$\lim_{a \rightarrow 1} \frac{\ln \bar{g}(\ln a + \ln y) - \ln \bar{g}(\ln y)}{\ln a} = -\lim_{a \rightarrow 1} \frac{\ln \lambda(a)}{\ln a} - 1 = \rho_t - 1$$

As in Step 4, the LHS is a derivative:

$$\frac{d \ln \bar{g}(\ln y)}{d \ln y} = \rho_t - 1.$$

Integrating with respect to y (let $\ln \xi_t$ be the integrating constant) gives:

$$\bar{g}(\ln y) = \psi'_t(y) = \xi_t y^{\rho_t - 1} \quad (22)$$

Further integrating implies that for each $y > 0$ $\psi_t(y) = \frac{\xi_t}{\rho_t} y^{\rho_t}$ if $\rho_t \neq 0$ and $w_t(y) = \xi_t \ln y$ otherwise. Remark that the integrating constant is left out as ordinarily irrelevant for the representation. Substituting for $y = \bar{q}_t(u; x, r_t)$ gives the result. \square

Step 8. For each $t \in T$, $\rho_t = \rho$.

Proof. Combining Steps 5 and 7, it follows that ρ_t is the degree of homogeneity of $q_t(u(t); x(t))$ with respect to x . Then, homotheticity of $V(u; x)$ with respect to x (consequence of *proportionality*), requires that $\rho_t = \rho$ for each $t \in T$. \square

Summarizing the previous results, for each $u \in U$, $q_t(u(t); x(t))$ assumes one of the following forms depending on ρ and $r_t \leq 1$:

$$\begin{aligned} q_t(u(t); x(t)) &= \xi_t \frac{1}{\rho} \left[x_t \left[\frac{1}{x_t} \sum_{s \in S_t} \pi^s x(t, s) \left(\frac{u(t, s)}{x(t, s)} \right)^{r_t} \right]^{\frac{1}{r_t}} \right]^\rho \text{ if } \rho, r_t \neq 0 \\ q_t(u(t); x(t)) &= \xi_t \frac{1}{\rho} \left[x_t \exp \left(\frac{1}{x_t} \sum_{s \in S_t} \pi^s x(t, s) \ln \left(\frac{u(t, s)}{x(t, s)} \right) \right) \right]^\rho \text{ if } \rho \neq 0, r_t = 0 \\ q_t(u(t); x(t)) &= \xi_t \ln \left[x_t \left[\frac{1}{x_t} \sum_{s \in S_t} \pi^s x(t, s) \left(\frac{u(t, s)}{x(t, s)} \right)^{r_t} \right]^{\frac{1}{r_t}} \right] \text{ if } \rho = 0, r_t \neq 0 \\ q_t(u(t); x(t)) &= \xi_t \ln \left[x_t \exp \left(\frac{1}{x_t} \sum_{s \in S_t} \pi^s x(t, s) \ln \left(\frac{u(t, s)}{x(t, s)} \right) \right) \right] \text{ if } \rho, r_t = 0 \end{aligned} \quad (23)$$

The next step uses *intergenerational equity* to determine restrictions on ρ and the parameters ξ_t .

Step 9. The following parameter restrictions holds: $\rho \leq 1$ and $\xi_t = x_t^{1-\rho}$ for each $t \in T$.

Proof. Let a pair $u, \bar{u} \in U$ be such that for some $t, t' \in T$ and $a, b, \delta \in \mathbb{R}_+$:

- (i) $\frac{u(t, s)}{x(t, s)} = \frac{\bar{u}(t, s)}{x(t, s)} - \frac{\delta}{\pi_t x(t, s)} = a \geq 1$ for each $s \in S_t$;
- (ii) $\frac{u(t', s)}{x(t', s)} = \frac{\bar{u}(t', s)}{x(t', s)} + \frac{\delta}{\pi_{t'} x(t', s)} = b \leq 1$ for each $s \in S_{t'}$;
- (iii) $u(\tilde{t}, s) = \bar{u}(\tilde{t}, s)$ for each $(\tilde{t}, s) \in \overline{TS}$ with $\tilde{t} \neq t, t'$.

By *intergenerational equity*, $u R_x \bar{u}$, or equivalently, $V(u; x) - V(\bar{u}; x) \geq 0$ and, using (iii):

$$q_t(u(t); x(t)) - q_t\left(u(t) + \frac{\delta}{\pi_t}; x(t)\right) + q_{t'}(u(t'); x(t')) - q_{t'}\left(u(t') - \frac{\delta}{\pi_{t'}}; x(t')\right) \geq 0.$$

Divide by δ and take the limit for $\delta \rightarrow 0$. Then:

$$\frac{1}{\pi_t} \sum_{s \in S_t} \frac{\partial q_t(u(t); x(t))}{\partial u(t, s)} \leq \frac{1}{\pi_{t'}} \sum_{s \in S_{t'}} \frac{\partial q_{t'}(u(t'); x(t'))}{\partial u(t', s)}. \quad (24)$$

Then, simple computation from the definition of $q_t(u(t); x(t))$ shows that, for each r_t and each ρ (including when these are 0):

$$\frac{\partial q_t(u(t); x(t))}{\partial u(t, s)} = \xi_t x_t^{\rho-1} a^{\rho-1} \pi^s.$$

Substituting in (24) both for t and t' , gives:

$$\xi_t x_t^{\rho-1} a^{\rho-1} \leq \xi_{t'} x_{t'}^{\rho-1} b^{\rho-1}. \quad (25)$$

Equation (25) holds true for each $a \geq 1 \geq b$. Thus $\xi_t = x_t^{1-\rho}$, $\xi_{t'} = x_{t'}^{1-\rho}$, and $\rho \leq 1$. \square

The last step combines the previous results.

Step 10. *The social ordering is reference-dependent utilitarian.*

Proof. Substitute $\xi_t = x_t^{1-\rho}$ in (23). Restrict the parameters as to satisfy $\rho \leq 1$ and, for each $t \in T$, $r_t \leq 1$. Then the reference dependent utilitarian social welfare function $W(u; x, \mathbf{r}, \rho)$ defined in (1)-(4) immediately follows. \square

A.3. Proposition 1

Proof. Let $I \in \mathcal{I}$. Define $U^{RE} \subseteq U^I$ as the subset of feasible prospects satisfying *recursive equity*. Let the expected utility at 0 be $u_0 \equiv \mu^{-1} \left[\sum_{s \in S_0} \frac{\pi^s}{\pi_0} \mu(u(0, s)) \right]$ and define $U_0 \equiv \{\bar{u}_0 \in \mathbb{R}_+ \mid u_0 = \bar{u}_0 \text{ for some } u \in U^{RE}\}$. The set U_0 is non-empty: by assumption $U^I \neq \emptyset$ and by continuity and no free lunch of technology, there exists $u \in U^I$ and $k > 0$ such that $u(t, s) = k$ for each $(t, s) \in \overline{TS}$; thus $u_0 = k \in U_0$. The set U_0 is bounded: this immediately follows from U^I being bounded. The set U_0 is compact: this follows from the continuity of technology F and the transformation μ . Let $u^* \in U^{RE}$ be such that u_0^* is the maximal element of U_0 . By construction, u^* satisfies *recursive equity*. By contradiction, assume that u^* does not satisfy *maximality*: then there exists $u' \in U^I$ such that $u' > u^*$. By assumption on technology, there exists a $u'' \in U^{RE}$ such that $u'' \gg u^*$, contradicting u_0^* being a maximal element of U_0 . This implies that $\phi^{R\mu}$ is well-defined. \square

A.4. Corollary 1

Proof. An immediate consequence of the previous result is that the selected reference is strictly positive: $x(t, s) > 0$ for each $(s, t) \in \overline{TS}$ and each $x \in \phi^{R\mu}(I)$. I focus on

uniqueness. By contradiction, assume for some $I \in \mathcal{I}$, $\phi^{R\mu}(I)$ is not a singleton, i.e. there exist a pair $u, \bar{u} \in \phi^{R\mu}(I)$ with $u \neq \bar{u}$. Let $t \in T$ be the first period for which $u(t, s) \neq \bar{u}(t, s)$ for some $s \in S_t$. If $t = 0$, $u_0 \geq \bar{u}_0$ and the same argument for the proof of Proposition 1 leads to a contradiction of *maximality*.

Assume $t > 0$. For each $s \in S_t$, let:

$$U^{RE}(t, s) \equiv \{u' \in U^{RE} \mid u'(t', s') = u(t', s') = \bar{u}(t', s') \text{ for each } s' \in S_{t'} \text{ with } t' < t\},$$

$$U(t, s) \equiv \{\bar{u}(t, s) \in \mathbb{R}_+ \mid \bar{u}(t, s) = \tilde{u}(t, s) \text{ for some } \tilde{u} \in U^{RE}(t, s)\}.$$

Then, the same reasoning as for U_0 in the proof of Proposition 1 leads to $u(t, s) = \bar{u}(t, s)$. This shows that the recursive rule $\phi^{R\mu}$ selects a unique prospect for each intergenerational problem. \square

B. Alternative rules for the selection of the reference

I here discuss alternative ways to identify the reference. By means of examples, I argue that the recursive rule defines a compelling reference even when the alternative rules do not. For the sake of brevity, I omit the proofs of the characterization of these rules which are straightforward.

The simplest rule is the one that assigns the largest and identical utility level to each potential generation. This strongly egalitarian rule, however, does not distribute all the available resources and violates *maximality*. Moreover, by *proportionality*, such rule would not convey any information to the reference-dependent utilitarian criterion and would lead to rankings of alternatives that are independent of the intergenerational problem faced by society. I thus focus on rules that satisfy *maximality*.

One possibility is to introduce an equity principle similar to the one by Hammond (1979). A prospect cannot be considered egalitarian if there is a feasible alternative that assigns more utility to the worst-off potential agent.

Weak equity. For each $I \in \mathcal{I}$, $x \in \phi(I)$ implies that no $u \in U^I$ exists such that $x(t, s) < u(t, s) \leq u(t', s') < x(t', s')$ for each pair $(t, s), (t', s') \in \overline{TS}$.

When combined with *maximality*, *weak equity* characterizes the **leximin rule** ϕ^{lex} . For each intergenerational problem, the leximin rule selects the prospect that lexicographically maximizes the utility of the worst-off potential agent across generations and states of nature.¹⁷

Proposition 6. *On the domain \mathcal{I} , the leximin rule ϕ^{lex} is well-defined and uniquely characterized by maximality and weak equity.*

¹⁷See Fn.13 for a definition of lexicographic ordering.

The leximin rule gives full priority to the worst-off potential agent. Nevertheless, it might select a prospect with unacceptably large intergenerational inequities. The following example highlights this issue.

Example 1. Consider an intergenerational problem $I_1 \in \mathcal{I}$ with: two periods $T \equiv \{0, 1\}$, two states of nature $S \equiv \{s, s'\}$ revealed at period 1, and no extinction. Feasibility requires that $u(0, s) + \frac{1}{10}u(1, s) \leq 2$ at state s ; $u(0, s') + u(1, s') \leq 2$ at state s' ; and $u(0, s) = u(0, s') \equiv u_0$. Then, $x \in \phi^{lex}(I_1)$ is such that $x_0 = x(1, s') = 1$ and $x(1, s) = 10$. Further aggravating the inequity of this prospect, the probability of state s' does not influence the selection. The utility of generation 0 cannot exceed the lowest utility achieved by the later generation at the worst state of nature, regardless of how small, but nevertheless positive, the corresponding probability is.

An intuitive way to avoid this problem is to treat generations equally in expected terms. For Example 1, this requires the degenerate lottery of generation 0, that is (x_0, x_0) , to be considered as desirable as the lottery of generation 1, that is $(x(1, s), x(1, s'))$. This idea is closely related to an ex-ante concern for equity introduced in the Harsanyi setting by Diamond (1967) and Epstein and Segal (1992). According to this view, society should avoid the inequalities that emerge when some agents are given better lotteries than others.

The following axiom formalizes this ex-ante equity principle. As in Section 3, the concave function μ accommodates any degree of aversion to risk.

Intergenerational (μ -expected) equity. For each $I \in \mathcal{I}$, $x \in \phi(I)$ implies that for each $t, t' \in T$:

$$\sum_{s \in S_t} \frac{\pi^s}{\pi_t} \mu(x(t, s)) = \sum_{s \in S_{t'}} \frac{\pi^s}{\pi_{t'}} \mu(x(t', s)).$$

This requirement is identical to Condition (i) of *recursive equity*. Let the ex-ante equality rule ϕ^{ea} satisfy *maximality* and *intergenerational equity*.

Proposition 7. *On the domain \mathcal{I} , the ex-ante equality rule ϕ^{ea} is well-defined.*

Unfortunately, the reference selected by this rule might not be compelling. This is illustrated in the following example.

Example 2. Consider an intergenerational problem $I_2 \in \mathcal{I}$ with: two periods $T \equiv \{0, 1\}$, two equally likely states of nature $S \equiv \{s, s'\}$ revealed at period 0, and no extinction. Feasibility requires that $u(0, s) + \frac{1}{10}u(1, s) \leq 1$ at state s and $\frac{1}{10}u(0, s') + u(1, s') \leq 1$ at state s' . Then, $x \in \phi^{ea}(U_2^f)$ is such that $0 \leq x(0, s') = x(1, s) < x(0, s) = x(1, s') \leq 10$. Even though the true state of nature is already known when assigning utilities, inequalities occur. Is it morally acceptable to assign a higher utility at $(0, s)$ based on the fact that

the later generation would have been compensated in a state of nature that is known to be impossible?¹⁸

This rhetorical question points to another alternative proposed in the literature, related to ex-post concern for fairness (see Adler and Sanchirico (2006) and Fleurbaey (2010)). The idea is to aggregate potential generations first over time, for each state of nature, and then across states of nature. Society should avoid inequalities that occur at each possible state of nature.

I formalize this ex-post equity principle by a condition that mirrors *intergenerational equity*. However, as uninsurable risks might prevent equality across states, I express the principle in a weaker form. Similarly to the above *weak equity*, this ensures compatibility with *maximality*.

For each $I \in \mathcal{I}$ and each $s \in S$, let $T_s \equiv \{t \in T \mid (t, s) \in \overline{TS}\}$. For each $u \in U$ let $w^s(u; \mu) \equiv \mu^{-1} [\sum_{t \in T_s} \mu(u(t, s))]$ be the equally distributed equivalent utility at state s with respect to the non-linear average based on transformation μ . A prospect is ex-post egalitarian if it is not possible to improve the equally distributed equivalent utility at the worst state of nature.

Weak interstate (μ -expected) equity. For each $I \in \mathcal{I}$, $x \in \phi(U^f)$ implies that there exists no $u \in U^I$ such that $w^s(x; \mu) < w^s(u; \mu) \leq w^{s'}(u; \mu) < w^{s'}(x; \mu)$ for each $s, s' \in S$.

Let the ex-post equity rule ϕ^{ep} satisfy *maximality* and *weak interstate equity*.

Proposition 8. *On the domain \mathcal{I} , the ex-post equality rule ϕ^{ep} is well-defined.*

Once again, the reference selected by this rule might not be compelling, as the next example shows.

Example 3. Consider an intergenerational problem $I_3 \in \mathcal{I}$ with: two periods $T \equiv \{0, 1\}$, two equally likely states of nature $S \equiv \{s, s'\}$ revealed at period 1, and no extinction. Resource scarcity is such that $\frac{1}{10}u(0, s) + u(1, s) \leq 1$ at states s , $\frac{1}{10}u(0, s') + u(1, s') \leq 1$ at state s' , and $u(0, s) = u(0, s')$. Then, $x \in \phi^{ep}(I_3)$ is such that $10 \geq x(0, s) = x(0, s') > x(1, s) = x(1, s') \geq 0$. Since the feasibility constraint is equal across states of nature, society should consider which state eventually occurs ethically irrelevant. Nevertheless, the selected prospect is characterized by possibly large inequalities and generation 0 might exhaust the utility possibilities of the later generation.

These examples should not be surprising. They demonstrate the root of the difficulty when introducing both ex-ante and ex-post concerns to inequality in the assessment of risky social situations (see Fleurbaey (2010)). In a dynamic setting, however, additional

¹⁸This conclusion remains valid when the domain of economies is restricted to have a single initial node, but requires a 3 periods example.

information is available: the timing of disclosure of risk. This information can be used to select prospects that satisfy both concerns jointly. As in Examples 1 and 3, expectations across states of nature capture the ex-ante concern for equity and help to judge the appeal of lotteries assigned to generations in states of nature that can still occur. However, when states of nature are known to be impossible, as in Example 2, these expectations should be avoided. In contrast, equality in each state of nature captures the ex-post concern and ensures that all generations are treated alike when the state of nature is already known, as in Example 2. However, it might lead to unacceptable inequalities when the state of nature is yet unknown, as in Example 3.

By using the information about the time disclosure of risk, the recursive rule $\phi^{R\mu}$ selects appealing prospects for each of the introduced examples. For the intergenerational problem I_1 of Example 1, $x \in \phi^{R\mu}(I_1)$ is such that $\mu(x_0) = \pi^s \mu(10(2 - x_0)) + \pi^{s'} \mu(2 - x_0)$. The assignment of generation 0 is the certainty equivalent of the lottery assigned to generation 1. More precisely, for any degree of concavity of μ , it follows that $x(1, s) > x_0 > x(1, s') > 0$, with x_0 becoming smaller as the concavity of μ increases. For the intergenerational problem I_2 of Example 2, $x \in \phi^{R\mu}(I_2)$ requires equality across time for states s and s' respectively as these states are known when policy choices are made, i.e. $x(0, s) = x(1, s)$ and $x(0, s') = x(1, s')$. The same conclusion also holds for the intergenerational problem I_3 of Example 3. The selected prospect $x \in \phi^{R\mu}(I_3)$ requires the assigned utilities to be equal across all potential agents.

References

- ADLER, M. D. AND C. W. SANCHIRICO (2006): “Inequality and uncertainty: Theory and legal applications,” *University of Pennsylvania Law Review*, 155, 279–377.
- ARROW, K. J. AND A. C. FISHER (1974): “Environmental preservation, uncertainty, and irreversibility,” *The Quarterly Journal of Economics*, 88, 312–319.
- ASHEIM, G. B. AND K. A. BREKKE (2002): “Sustainability when capital management has stochastic consequences,” *Social Choice and Welfare*, 19, 921–940.
- ASHEIM, G. B. AND S. ZUBER (2015): “Evaluating intergenerational risks: Probability adjusted rank-discounted utilitarianism,” *mimeo*, available at <http://folk.uio.no/gasheim/rduRi14.pdf>.
- ATKINSON, A. B. (1970): “On the measurement of inequality,” *Journal of Economic Theory*, 2, 244–263.
- BLACKORBY, C., W. BOSSERT, AND D. J. DONALDSON (2005): *Population issues in social choice theory, welfare economics, and ethics*, 39, Cambridge University Press.
- BLACKORBY, C. AND D. DONALDSON (1982): “Ratio-scale and translation-scale full interpersonal comparability without domain restrictions: Admissible social-evaluation functions,” *International Economic Review*, 249–268.
- BROOME, J. (1984): “Uncertainty and fairness,” *Economic Journal*, 94, 624–632.
- (1991): *Weighing goods: Equality, uncertainty and time*, Basil Blackwell Oxford.

- DALTON, H. (1920): "The Measurement of the Inequality of Incomes," *The Economic Journal*, 30, 348–361.
- DASGUPTA, P. (2008): "Discounting climate change," *Journal of Risk and Uncertainty*, 37, 141–169.
- DASGUPTA, P. S. AND G. HEAL (1979): *Economic theory and exhaustible resources*, Cambridge University Press.
- DEKEL, E., B. L. LIPMAN, AND A. RUSTICHINI (1998): "Standard state-space models preclude unawareness," *Econometrica*, 159–173.
- DHILLON, A. AND J.-F. MERTENS (1999): "Relative utilitarianism," *Econometrica*, 67, 471–498.
- DIAMOND, P. A. (1967): "Cardinal welfare, individualistic ethics, and interpersonal comparison of utility: Comment," *The Journal of Political Economy*, 75, 765–766.
- EBERT, U. AND P. MOYES (2003): "Equivalence scales reconsidered," *Econometrica*, 71, 319–343.
- EPSTEIN, L. G. AND U. SEGAL (1992): "Quadratic social welfare functions," *Journal of Political Economy*, 100, 691–712.
- EPSTEIN, L. G. AND S. E. ZIN (1989): "Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework," *Econometrica*, 57, 937–969.
- FLEURBAEY, M. (2010): "Assessing Risky Social Situations," *Journal of Political Economy*, 118, 649–680.
- FLEURBAEY, M. AND S. ZUBER (2015a): "Discounting, beyond utilitarianism," *Economics: The Open-Access, Open-Assessment E-Journal*, 9, 1–52.
- (2015b): "Discounting, Risk and Inequality: A General Approach," *Journal of Public Economics*, forthcoming.
- GERLAGH, R. AND M. LISKI (2012): "Carbon prices for the next thousand years," *CESifo Working Paper No. 3855*.
- GOLLIER, C. (2002): "Discounting an uncertain future," *Journal of Public Economics*, 85, 149–166.
- GORMAN, W. (1968): "Conditions for additive separability," *Econometrica*, 36, 605–609.
- GRANT, S., A. KAJII, B. POLAK, AND Z. SAFRA (2010): "Generalized utilitarianism and Harsanyi's impartial observer theorem," *Econometrica*, 78, 1939–1971.
- HAMMITT, J. K., R. J. LEMPERT, AND M. E. SCHLESINGER (1992): "A sequential-decision strategy for abating climate change," *Nature*, 357, 315–318.
- HAMMOND, P. J. (1979): "Equity in two person situations: some consequences," *Econometrica: Journal of the Econometric Society*, 1127–1135.
- HANEMANN, W. M. (1989): "Welfare evaluations in contingent valuation experiments with discrete response data: reply," *American Journal of Agricultural Economics*, 71, 1057–1061.
- HARSANYI, J. C. (1955): "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility," *Journal of Political Economy*, 63, 309–321.
- HEAL, G. M. AND A. MILLNER (2014): "Agreeing to disagree on climate policy," *Proceedings of the National Academy of Sciences*, 111, 3695–3698.
- HENRY, C. (1974): "Investment decisions under uncertainty: the irreversibility effect," *The*

- American Economic Review*, 64, 1006–1012.
- JACKSON, M. O. AND L. YARIV (2014): “Collective Dynamic Choice: The Necessity of Time Inconsistency,” *American Economic Journal: Microeconomics*, forthcoming.
- KAHNEMAN, D. AND A. TVERSKY (1979): “Prospect theory: An analysis of decision under risk,” *Econometrica*, 47, 263–291.
- KARNI, E. AND M.-L. VIERØ (2013): ““Reverse Bayesianism”: A choice-based theory of growing awareness,” *The American Economic Review*, 103, 2790–2810.
- KARP, L. (2005): “Global warming and hyperbolic discounting,” *Journal of Public Economics*, 89, 261–282.
- KOOPMANS, T. C. (1960): “Stationary ordinal utility and impatience,” *Econometrica*, 28, 287–309.
- KOSZEGI, B. AND M. RABIN (2006): “A model of reference-dependent preferences,” *The Quarterly Journal of Economics*, 121, 1133–1165.
- KREPS, D. M. AND E. L. PORTEUS (1978): “Temporal resolution of uncertainty and dynamic choice theory,” *Econometrica*, 46, 185–200.
- LAIBSON, D. (1997): “Golden eggs and hyperbolic discounting,” *The Quarterly Journal of Economics*, 112, 443–478.
- OK, E. A., P. ORTOLEVA, AND G. RIELLA (2015): “Revealed (P) Reference Theory,” *American Economic Review*, 105, 299–321.
- PIACQUADIO, P. G. (2014): “Intergenerational Egalitarianism,” *Journal of Economic Theory*, 153, 117–127.
- PIGOU, A. C. (1912): *Wealth and Welfare*, London: Macmillan.
- PINDYCK, R. S. (2000): “Irreversibilities and the timing of environmental policy,” *Resource and energy economics*, 22, 233–259.
- RAWLS, J. (1971): *A Theory of Justice*, Harvard University Press.
- ROTHSCHILD, M. AND J. E. STIGLITZ (1970): “Increasing risk: I. A definition,” *Journal of Economic Theory*, 2, 225–243.
- STERNER, T. AND U. M. PERSSON (2008): “An even sterner review: Introducing relative prices into the discounting debate,” *Review of Environmental Economics and Policy*, 2, 61–76.
- THOMSON, W. (2011): “Chapter Twenty-One - Fair Allocation Rules,” in *Handbook of Social Choice and Welfare*, ed. by A. S. Kenneth J. Arrow and K. Suzumura, Elsevier, vol. 2 of *Handbook of Social Choice and Welfare*, 393–506.
- TRAEGER, C. P. (2012): “Once Upon a Time Preference - How Rationality and Risk Aversion Change the Rationale for Discounting,” *CESifo Working Paper Series No. 3793*.
- (2014): “Why uncertainty matters: discounting under intertemporal risk aversion and ambiguity,” *Economic Theory*, 56, 627–664.
- WEITZMAN, M. L. (2001): “Gamma discounting,” *American Economic Review*, 91, 260–271.
- ZUBER, S. (2011): “The aggregation of preferences: can we ignore the past?” *Theory and Decision*, 70, 367–384.
- ZUBER, S. AND G. B. ASHEIM (2012): “Justifying social discounting: the rank-discounted utilitarian approach,” *Journal of Economic Theory*, 147, 1572–1601.