Growth and Mitigation Policies with Uncertain Climate Damage

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November 11, 2014

Abstract

Climate physics predicts that the intensity of natural disasters will increase in the future due to climate change. We present a stochastic model of a growing economy where natural disasters are multiple and random, with damages driven by the economy's polluting activity. We provide a closed-form solution and show that the optimal path is characterized by a constant growth rate of consumption and the capital stock until a shock arrives, triggering a downward jump in both variables. Optimum mitigation policy consists of spending a constant fraction of output on emissions abatement. This fraction is an increasing function of the arrival rate, polluting intensity of output, and the damage intensity of emissions. We subsequently extend the baseline model by adding climate-induced fluctuations around the growth trend and stock-pollution effects, demonstrating robustness of our results. In a quantitative assessment of our model we show that the optimal abatement expenditure at the global level may represent 0.9% of output, which is equivalent to a tax of $70 per ton carbon.

JEL Classification: O10, Q52, Q54

Key Words: Climate policy, uncertainty, natural disasters, endogenous growth.

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1 Introduction

1.1 Economics and the Climate

Understanding the economic effects of climate change is essential and urgent. The increase in global temperature is predicted to intensify the severity of natural disasters with significant adverse effects on development in different parts of the world. Ferocious tropical hurricanes, massive floods, droughts and landslides cause severe destruction of infrastructure, loss of physical and human capital, and undoubtedly result in a substantial setback in terms of economic growth. According to National Oceanic and Atmospheric Administration (2013), the cost of extreme weather events has risen from about \$20bn per event in 1980s to almost \$90bn in 2010. The recent Typhoon Haiyan in the Philippines was the strongest recorded storm to make landfall ever (see The Economist 2013). The surge swept away entire cities, at least 11m Filipinos have been affected, some killed, many displaced or left homeless. Although climate physicists are not unanimous on whether the frequency of natural disasters will increase in the future or not, the majority agrees that the intensity will get worse as the planet warms (see IPCC 2014). It is well understood that economic activities cause carbon and other greenhouse gas (GHG) emissions that alter the natural environment and lead to climate change. However, occurrence of climate shocks is not easily predictable and is typically viewed as a random event.

The complexity of both the economic and the ecological parts of the climate problem pose considerable modeling challenges, involving long time horizons and various sources of uncertainty. As a consequence, the vast majority of current economic climate frameworks consists of relatively complex numerical simulation models. These have provided many useful insights with respect to the costs and benefits of a climate policy but also produced diverging results. To gain further insights concerning the central mechanisms at work - and especially those related to the uncertain nature of climate change - a framework of investigation that relies on analytic solutions to provide clear-cut implications for the optimum climate policy can be very useful.

Within such a framework, a number of important questions need to be addressed. Given the uncertain nature of disasters caused by climate change, how should an economy appropriately balance its production, consumption, investment, and reduction of emissions? What is the optimal rate of output growth and the optimal emissions abatement in the uncertain environment? How do these key variables respond to changes in the underlying economic fundamentals? In the present paper we examine these questions within a model of a growing economy which features uncertainty about the arrival of climate shocks. We assume that the occurrence of a disaster (also referred to as an "event") follows a random process, and when it strikes, part of the economy's productive input (such as capital stock) is destroyed. Climate change induced natural catastrophes are large in scale and have a profound negative impact on the globalized economy. Unlike in the case of idiosyncratic shocks, the risk of such events cannot be insured.[1] The magnitude of the damage is assumed to be a positive function of polluting emissions. It follows that the capital accumulation process is both endogenous and stochastic. In our model, however, the world does not end after an environmental disaster, as it is often assumed in the literature on catastrophic events (see Section 1.3). We consider development with recurring shocks over time, which reflects a likely pattern of climate-induced events in the future. Optimal reduction in emissions, and the implied reduction in damages, can be achieved by appropriately balancing two types of activities: capital accumulation and abatement.

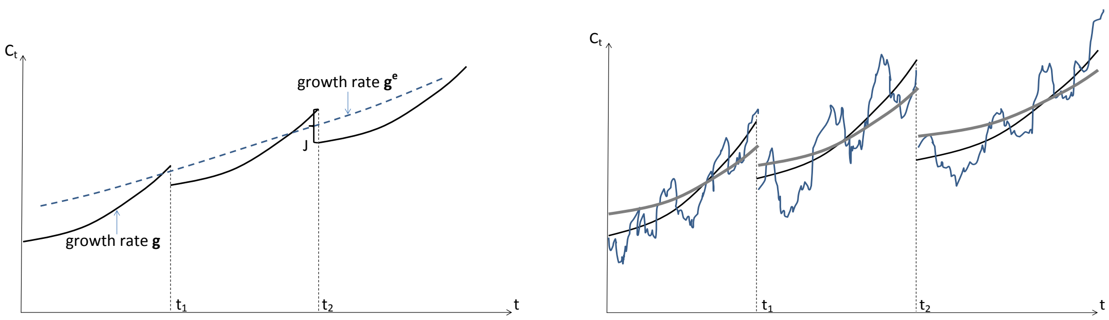
1.2 Main Findings

To the best of our knowledge the paper is the first to provide a clear-cut closed-form solution for the optimal abatement expenditure and the growth rate of the economy subject to random climate shocks with endogenous damages. We show that the optimal

[1] In a globalized world, large-scale natural disasters affect not only the economic activity of the country where they strike but also other economies by virtue of either close geographic location or trade relations, FDI, etc. When it comes to relatively small idiosyncratic shocks, they can be insured against by trading insurance claims within a group of regions subject to such shocks. Our focus, however, is on a global economy where an insurance contract against a large natural catastrophe simply cannot exist.

policy consists of devoting a constant fraction of output to emissions abatement. A more frequent occurrence of natural disasters (i.e., higher arrival rate) and a higher damage intensity have a negative impact on the optimal growth and call for more vigorous abatement policies. The dependence of the climate-policy instrument on the arrival rate points to the importance of relying on stochastic models when deriving meaningful and effective climate policies.

The optimal path is characterized by the consumption rate and the capital stock which grow at the same constant rate until an event arrives causing a downward jump in both variables. The size of the jump is endogenously determined and depends on the arrival rate, abatement efficiency, damage intensity and the intertemporal substitution elasticity. As an illustration of an optimal path, we show in Figure 1a the consumption rate as a function of time. The solid line represents the stochastic path, which exhibits a growth rate g in the absence of climate events. At times t_1 and t_2, negative environmental shocks are assumed to occur causing an immediate downward jump, followed by a subsequent period of growth at the previous rate. The dashed line shows the time profile of consumption under the *expected* growth scenario. There is a fundamental difference between the dashed and the solid curves in that the former smoothes out the jumps and discontinuities of the latter, creating an illusion of a perfect consumption smoothing and thereby ignoring the crucial effects of uncertainty. Moreover, the growth rate in the stochastic scenario (g) is unambiguously higher than under the expected equivalent



(a) Expected (dashed) vs. Poisson-driven (solid) growth.

(b) Poisson and Wiener uncertainty (thin gray) vs. stock effects (bold gray).

Figure 1: Time profile of consumption.

(g^e), reflecting a specific kind of the broadly known precautionary effect. Given that random shocks constitute a central part of the climate-change problem, they need to be taken into account within an appropriate modeling framework, which we propose in the present article.

In the second half of the paper, we provide two extensions of the baseline setup by introducing (i) climate change induced fluctuations around the growth trend, modeled by a Brownian motion, in addition to the Poisson-driven jumps and (ii) the link between climate damage size and the entire history of pollution. Figure 1b provides an illustration of the optimal consumption path under these two alternative scenarios, where the former is shown by the thin gray line and the latter by the bold gray line. We confirm that the results and intuitions of the baseline model continue to hold in these richer frameworks. In the former case, the optimal growth rate of the economy may either exceed or fall short of the growth rate of the baseline model. The share of output devoted to abatement is, however, unambiguously larger as compared to the baseline. In the second extension, the growth rate is unambiguously smaller, the jumps in the consumption rate and the capital stock are less pronounced, while the abatement share is larger than in the baseline.

1.3 Contribution to Literature

The present paper relates to several contributions in the field of climate economics. The framework is close to Pindyck and Wang (2013) who consider a growing economy subject to random disasters which cause random damages to the capital stock. There are, however, several important differences. First, Pindyck and Wang do not model climate change nor pollution dynamics. Second, we endogenize the damages by linking their size to the polluting activity and to abatement policy. Third, Pindyck and Wang focus on the society's willingness to pay for eliminating the possibility of a disaster, while we focus on the optimal emissions mitigation and the optimal consumption growth in the uncertain environment.

Golosov et al. (2014) derive a simple formula for the optimal carbon tax showing that it is proportional to GDP and depends on just a small number of parameters. The convenient expression for the tax arises due to two simplifying assumptions: first, the future path of climate damages is constant over time; and second, the optimal propensity to save is constant. Although their model is more detailed in several aspects than ours (e.g. energy production and carbon cycle), our results are similar, in the sense that our climate policy instrument can also be conveniently expressed as a fraction of output which depends on the economy's fundamental characteristics. The novelty and strength of our setup is to fully characterize the solution under uncertainty about large-scale natural disasters, which lies at the heart of the climate problem. Moreover, the damages in our framework are not only time-varying but fully endogenous.[2] We do not need to assume constancy of the saving propensity but, instead, we derive the optimal saving rate of the economy and show how it is affected by the climate change and the associated uncertainty.

There is a growing literature on random catastrophic events causing irreversible damage.[3] We believe, however, that irreversibility is a rather extreme assumption and that repeated shocks constitute a more realistic scenario. Tsur and Zemel (1998) are the first to explicitly analyze reversible events, although they focus on the optimal steady state policy and transitional dynamics of an economy which is not engaged in any investment activity. By contrast, we consider a growing economy which engages in both capital accumulation and in emissions control.[4] Van der Ploeg (2014) analyzes the optimal carbon tax in an economy subject to a random shock which reduces the nature's capacity to absorb greenhouse gases. The analysis brings forward the possibility of the hazard

[2]Ikefuji and Hori (2012) examine the optimal growth rate and the carbon tax in an economy where private capital is subject to stochastic depreciation due to climate change. They assume, however, that these stochastic shocks are idiosyncratic and reflect a large number of independent small climate events. With this setup, it becomes possible to work with the expected values, while in our approach we make a sharp distinction between the expected and stochastic outcomes. Soretz (2007) analyzes efficient pollution taxation within an endogenous growth model where environmental quality has a stochastic impact on factor productivity, which is driven by a Wiener process. Bretschger and Suphaphiphat (2014) study the implications of climate change affecting capital depreciation rate within a deterministic two-country model.

[3]For early contributions see Clarke and Reed (1994) and Tsur and Zemel (1996).

[4]De Zeeuw and Zemel (2012) provide a dynamic characterization of optimal emission policy when the time of the regime switch from low to high damage is uncertain. One of their key findings is that, due to precautionary reasons, emissions in the low-damage regime may be lower than in the case where the system is already in the high-damage regime.

rate being a function of accumulated pollution.[5] Although this ingredient of the model imposes a partial equilibrium approach (with exogenous output and no capital accumulation), it allows to disentangle the components of the optimal tax which are driven by the presence of uncertainty and endogeneity of the hazard rate. We add to this literature by providing clear-cut analytical solutions for the growth rate of consumption and the optimal abatement policy in a general equilibrium model featuring random shocks with damages being driven by investment and abatement decisions.

The remainder of the paper is organized as follows. Section 2 develops our baseline framework. In Section 3, we present the main results with respect to the optimal growth rate, abatement, saving propensity and offer some quantitative implications. Section 4 extends the baseline model by introducing climate-induced fluctuations around the growth trend and stock-pollution effects. Finally, Section 5 concludes.

2 The Framework

2.1 General Model

We consider a global economy which produces a composite consumption good under constant returns to scale using as input broadly defined capital, denoted by K_t. The production process is polluting: every period t a flow of greenhouse gas (GHG) emissions, denoted by E_t, is released into the atmosphere. Emissions cause deterioration of the natural environment and global temperature increase, leading to a random occurrence of natural disasters. We assume that the arrival of a natural disaster (we shall also refer

[5] As we have mentioned earlier, there are no clear predictions about climate-induced changes in the disaster frequency, in some cases constant or even decreasing frequencies are predicted. The recent IPCC (2014) report states that "In the future, it is likely that the frequency of tropical cyclones globally will either decrease or remain unchanged, but there will be a likely increase in global mean tropical cyclone precipitation rates and maximum wind speed." (IPCC 2014, p.8) We therefore adopt a constant hazard rate assumption in our analysis below. It is clear that when the hazard rate increases in pollution stock, the abatement policy would need to be more stringent or, equivalently, the carbon tax would need to be higher than in a setting with a constant hazard. The extra positive term in the carbon-tax expression which arises due to endogeneity of the hazard rate is derived in van der Ploeg (2014).

to it as an "event") follows the Poisson process[6] with the constant mean arrival rate λ. When an event occurs, an endogenously-determined amount $\gamma_t \in [0, K_t]$ of the existing capital stock is destroyed. In fact, recent floods like the one in Pakistan in 2010 or in the Philippines in 2013 had a profound effect on the economies' infrastructure and the capital stock (both physical and human). According to the predictions of climate sciences, the magnitude of the damage is very likely to increase in the future due to climate change and hence we model it as a positive function of the economy's emissions, i.e., $\frac{\partial \gamma_t}{\partial E_t} > 0$.

The output, denoted by $Y_t(K_t)$, can be either spent on consumption, C_t, or invested. There are two types of non-consumption spending: (i) investment to augment the capital stock and (ii) financing of emissions abatement. Specifically, we assume that a share θ_t of output is spent on the latter, so that abatement expenditure is given by $I_t = \theta_t Y_t$. The remaining share $(1 - \theta_t)Y_t$ is split between consumption and capital accumulation. Total abatement, $Z(I_t)$, is a positive function of the abatement expenditure, $Z'(I_t) > 0$. The total per period emissions are then given by emissions stemming from the economic activity minus abatement. We assume that one unit of output causes ϕ units of pollution, so that total emissions are given by $E_t = \phi Y_t - Z(I_t)$.

The economy's objective is to maximize the expected discounted value of utility over an infinite planning horizon with respect to consumption, C_t, and the share of output devoted to abatement, θ_t, subject to the stochastic capital accumulation process. Specifically, the planner's programme is

$$\max_{C_t, \theta_t} \quad \mathbb{E}_0 \left\{ \int_0^\infty U(C_t) e^{-\rho t} dt \right\} \tag{1}$$

s.t.

$$dK_t = [(1 - \theta_t)Y_t(K_t) - C_t]dt - \gamma(E_t, K_t)dq_t, \tag{2}$$

$$E_t = \phi Y_t(K_t) - Z(I_t) \tag{3}$$

$$I_t = \theta_t Y_t(K_t), \tag{4}$$

where $\mathbb{E}_0$ is the expectations operator, dq_t is an increment of the Poisson process with

[6] Modeling occurrence of natural disasters by the Poisson process is quite standard, especially in the risk and insurance literature. See, e.g., Batabyal and Beladi (2001) and Baryshnikov et al. (2001).

a constant arrival rate λ and ρ is the constant rate of time preference. We also require that the capital stock, consumption and emissions are non-negative and $\theta_t \in [0, 1)$.

The baseline version of our model, presented in Sections 2 and 3, assumes that the damages to the productive input arise due to the *flow* of pollution. One may argue that pollution *stock* would be a more relevant source of damages. In our view, both possibilities exist, depending on the interpretation assigned to the productive input (for instance, human capital may be sensitive to both pollution flow and stock). In support of the pollution-stock argument, we present in Section 4.2 an extension of our baseline model which includes pollution dynamics and stock-driven damages. For the time being, however, we wish to focus on the simpler framework with flow-driven damages in order to make the reader acquainted with the mechanics of the model and key results. We will show in Section 4.2 that adding a second state variable (pollution stock) does complicate analytical derivations but does not fundamentally change any results of the baseline model, except for the effect of discounting.

2.2 Assumptions

We now introduce some useful functional forms and explain their motivation and consequences in the model.

Assumption 1: $Y_t = AK_t$

Following the mainstream of economic literature, we use the assumption of constant returns to scale in aggregate production. Since capital is the only input in the model, output is produced with an AK technology, where A is the constant factor productivity parameter and K_t is interpreted as a broad measure of capital in the economy, including physical and human capital, intangibles, etc.[7]

Assumption 2: $Z(I_t) = \sigma I_t$

[7] Despite its formal simplicity, the AK model unites all the desirable properties of an aggregate production function in a dynamic climate model. It generates sustained growth endogenously, results in the same implications for investment and growth as if we included different capital components like physical, human, and knowledge capital separately, and is fully consistent with the empirically observed strong positive relationship between investment rates and growth rates across countries and time periods (see McGrattan, 1998).

Total abatement is directly proportional to the resources allocated to emissions control, with the proportionality parameter $\sigma > 0$ representing the efficiency of abatement technology. In accordance with Assumption 1, it is sensible to equally assume constant returns in abatement activities.

Assumption 3: $\gamma(E_t, K_t) = \gamma_e E_t + \bar{\gamma} K_t$

We assume that the damage to the capital stock, $\gamma(E_t, K_t)$, consists of two terms: the first, which is directly proportional to emissions (the climate-change effect) and the second, proportional to the stock of capital exposed to destruction during a natural disaster (the exposure effect), which entails a loss of capital at the rate $\bar{\gamma} \in (0, 1)$ even in the absence of any polluting activity. We shall refer to the parameter γ_e as the damage intensity or the damage sensitivity to the economy's emissions. In Section 4.2 we introduce the link between the damages and the accumulated stock of pollution.

Assumption 4: $\phi < \sigma < (\lambda\gamma_e)^{-1}$

The assumption requires the abatement productivity, σ, to be sufficiently high (to exceed polluting intensity ϕ) but also to be bounded from above, given the severity of natural disasters (i.e., the Poisson arrival rate λ and the damage intensity of climate shocks γ_e). The second restriction prevents overly optimistic technology perspectives biasing the results in a too favorable direction.

Assumption 5: $U(C) = \frac{C^{1-\varepsilon}-1}{1-\varepsilon}$

The utility function takes a standard CRRA form, where $1/\varepsilon$ is the intertemporal substitution elasticity. We discuss in Section 3 the important role of the substitution elasticity in a climate context, which is absent when log-utility is assumed.

2.3 Solving the Model

Denoting by $V(K)$ the value function associated with the optimization problem described in (1) - (4), the Hamilton-Jacobi-Bellman (HJB) equation may be written as

$$\rho V(K) = \max \left\{ U(C) + V'(K)[(1-\theta)Y - C] + \lambda \left[\tilde{V}(\tilde{K}) - V(K) \right] \right\}, \tag{5}$$

where $\tilde{V}$ is the value function after the occurrence of the event which depends on the new capital stock $\tilde{K} = K - \gamma(E, K)$. Time subscripts are omitted when there is no ambiguity. The first-order conditions consist of

$$\begin{aligned}
C &: \quad U'(C) - V'(K) = 0, && (6)\\
\theta &: \quad -V'(K)Y + \lambda\tilde{V}'(\tilde{K})\gamma_e\sigma Y = 0, && (7)\\
K &: \quad \rho V'(K) = V''(K)[(1-\theta)Y - C] + V'(K)A(1-\theta) + \\
&\quad + \lambda\left(\tilde{V}'(\tilde{K})\left[1 - \gamma_e(\phi - \sigma\theta)A - \bar{\gamma}\right] - V'(K)\right). && (8)
\end{aligned}$$

The optimality conditions are complemented by the transversality condition for K, the non-negativity constraints on C, K, E, and the requirement $\theta \in [0, 1)$. Eqs. (6) - (8) allow us to obtain an explicit solution for the law of motion of the consumption rate (all the derivations in Section 2 are relegated to Appendix A)

$$\frac{dC}{C} = \frac{1}{\varepsilon}\left\{A\left(1 - \frac{\phi}{\sigma}\right) + \frac{1-\bar{\gamma}}{\sigma\gamma_e} - \rho - \lambda\right\} dt + \left(\frac{\tilde{C}}{C} - 1\right) dq, \tag{9}$$

where the new consumption rate at the time of the jump, $\tilde{C}$, is a constant fraction ω of the pre-jump rate:

$$\tilde{C} = \omega C, \quad \omega \equiv (\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}} \in [0, 1) \tag{10}$$

It follows that the last term on the RHS is negative and it represents the downward jump in consumption every time a natural disaster strikes.

The first term on the RHS of (9) represents what we label the "trend" consumption growth rate. Specifically, while the event has not arrived, consumption grows at the constant rate, defined as

$$g \equiv \frac{1}{\varepsilon}\left\{A\left(1 - \frac{\phi}{\sigma}\right) + \frac{1-\bar{\gamma}}{\sigma\gamma_e} - \rho - \lambda\right\}. \tag{11}$$

The expression reveals that the consumption rate is increasing over time if the effective discount rate, which includes not only the pure rate of time preference ρ but also the disaster arrival rate λ, is not too high, formally $g > 0 \Leftrightarrow A\left(1-\frac{\phi}{\sigma}\right)+\frac{1-\bar{\gamma}}{\sigma\gamma_e} > \rho+\lambda$. When an event occurs, consumption jumps down to the new level, $\widetilde{C}$, and then continues to grow at the rate g until the next event.

It can be shown that the value function of the problem, satisfying the HJB equation and certain limiting conditions (see, e.g., Sennewald and Wälde 2006), is of the form

$$V(K) = \frac{\psi^{-\varepsilon}K^{1-\varepsilon}-1}{1-\varepsilon}, \tag{12}$$

where ψ is a function of the parameters of the model:

$$\psi \equiv \frac{1}{\varepsilon}\left\{\rho-(1-\varepsilon)\left[\frac{1-\bar{\gamma}-(\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}}}{\sigma\gamma_e}+A\frac{\sigma-\phi}{\sigma}\right]+\lambda\left[1-(\lambda\sigma\gamma_e)^{\frac{1-\varepsilon}{\varepsilon}}\right]\right\}.$$

Proposition 1: *The solution of the maximization problem given by (1) - (4) is characterized by the following:*
(i) optimal consumption is a constant fraction of the capital stock;
(ii) optimal abatement expenditure is a constant fraction of output;
(iii) consumption, capital stock, output, and abatement grow at the same constant rate, given by (11), between two subsequent shocks.

Proof: The result in (i) follows immediately from (6) and (12), so that:

$$C^* = \psi K. \tag{13}$$

Statement (ii) follows from (7) and (12); by combining the two expressions we find that optimal abatement share is given by:

$$\theta^* = \frac{\phi}{\sigma}-\frac{1-\bar{\gamma}-(\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}}}{A\sigma\gamma_e}. \tag{14}$$

The non-negativity constraint on E requires that $\theta^* \leqslant \frac{\phi}{\sigma}$ (see (4) and (3)). At the same time, θ^* must be non-negative, so that both conditions lead to the inequality

$0 \leqslant \frac{\phi}{\sigma} - \frac{1-\bar{\gamma}-(\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}}}{A\sigma\gamma_e} \leqslant \frac{\phi}{\sigma}$. After some rearrangements, we obtain

$$1 - \bar{\gamma} - A\phi\gamma_e \leqslant (\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}} \leqslant 1 - \bar{\gamma}, \tag{15}$$

which is the necessary restriction on the parameters of the model to ensure the existence of an interior solution.

To prove (iii), note that the stochastic time path of the capital stock can be solved for analytically by substituting the optimal controls (13) and (14) in (2) and solving the resulting stochastic differential equation

$$dK_t = [(1-\theta^*)A - \psi]K_t dt - [\gamma_e(\phi - \sigma\theta^*)A + \bar{\gamma}]\, K_t dq_t.$$

The solution is given by

$$K_t = K_0 e^{[(1-\theta^*)A-\psi]t + \ln[1-\bar{\gamma}-\gamma_e(\phi-\sigma\theta^*)A]q_t}.$$

We can verify that the term in the exponent involving the logarithm is well-defined since the argument of the logarithm is unambiguously positive and is equal to (using (14))

$$1 - \bar{\gamma} - \gamma_e(\phi - \sigma\theta^*)A = (\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}} > 0.$$

Substituting the solution for θ^* in $[(1-\theta^*)A - \psi]$, we obtain the following stochastic path of the capital stock

$$K_t = K_0 e^{gt + \frac{1}{\varepsilon}\ln(\lambda\sigma\gamma_e)q_t}, \tag{16}$$

where the term q_t in the exponent is responsible for the discontinuous downward jump at the time of a climate shock. The jump is downward since $\ln(\lambda\sigma\gamma_e)$, which multiplies q_t, is negative. When $q_t = 0$, $K_t = K_0 e^{gt}$, i.e. the capital stock grows at the constant rate g, so that consumption and capital grow at the same rate as long as an event has not arrived, in line with (13). Abatement expenditure, equal to a fraction θ of output, evolves over time according to

$$I_t = \theta A K_t = \left[\frac{A\phi}{\sigma} - \frac{1 - \bar{\gamma} - (\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}}}{\sigma\gamma_e} \right] K_0 e^{gt + \frac{1}{\varepsilon}\ln(\lambda\sigma\gamma_e) q_t}$$

showing that it grows at the trend rate g while $q_t = 0$. ■

In order to better understand the role of uncertainty, we also compute (see Appendix A.4) the *expected* consumption growth rate, defined as

$$g^e \equiv \frac{d\mathbb{E}_t C_t}{C_t} = \frac{1}{\varepsilon}\left\{ A\left(1 - \frac{\phi}{\sigma}\right) - \frac{\bar{\gamma}}{\sigma\gamma_e} - \rho + \lambda \left[\frac{1}{\lambda\sigma\gamma_e} + \varepsilon(\lambda\sigma\gamma_e)^{1/\varepsilon} - 1 - \varepsilon \right] \right\}. \quad (17)$$

It represents the expected percentage change in the consumption rate and thus takes into account the possibility of a jump due to a climate shock. It may also be interpreted as the average consumption growth rate. It can be easily verified that the expected consumption growth rate is smaller than the trend growth rate: $g^e < g$. The consumption paths corresponding to the growth rates g and g^e are illustrated in Figure 1a by the solid and the dashed lines, respectively. It is assumed in the figure that climate shocks arrive at times t_1 and t_2 causing instantaneous downward jumps followed by a next period of growth. The dashed line - the hypothetical time profile of consumption corresponding to the growth rate g^e - is flatter than the stochastic path as it smoothes out the jumps and discontinuities of the latter, mitigating the precautionary motive for saving. Pizer (1999) has already emphasized that drawing policy recommendations from the analysis of expected paths instead of the true stochastic paths will lead to an underestimation of both savings and emissions-control efforts. Our argument is parallel to that of Pizer, although he relied on numerical methods to support his statement, while we provide analytical support. Given that stochastic shocks constitute a central part of the economic analysis of climate change, they need to be taken into account within an appropriate modeling framework. The next sections propose a detailed characterization of the solution and an analysis of how the optimal growth rate and the abatement share respond to changes in the key parameters of the model.

3 Characterizing the Solution

3.1 Consumption Growth

We have established in Eq. (11) that the trend growth rate of consumption is given by g, which we rewrite as

$$g = \frac{1}{\varepsilon}\left\{A\left(1 - \frac{\phi}{\sigma}\right) - \rho + \lambda\left(\frac{1}{\lambda\sigma\gamma_e} - 1 - \frac{\bar{\gamma}}{\lambda\sigma\gamma_e}\right)\right\}. \tag{18}$$

The expression has a familiar Keynes-Ramsey form albeit with some modifications. The standard Keynes-Ramsey growth rate equals the difference between the real interest rate (usually the marginal product of capital) and the rate of pure time preference, adjusted by the elasticity of intertemporal consumption substitution. First, note that in Eq. (18) the economy's implicit real interest rate, given by the first term inside the parentheses, is not equal to just the marginal productivity of capital but is reduced by the emission intensity of output, adjusted by the abatement efficiency, i.e, the term ϕ/σ. It follows that in our framework, pollution has an unambiguously negative growth effect. It may be dampened by either increasing the abatement efficiency, σ, or decreasing the polluting intensity, ϕ.

Second, there is, of course, the effect of uncertainty, represented by the last term, which includes the exposure and the jump components. The exposure component, $\frac{\bar{\gamma}}{\lambda\sigma\gamma_e}$, is present due to our assumption that natural catastrophes may occur even in the absence of any polluting activity. In that case, arrival of a climate shock causes $\bar{\gamma}$ percent damage to the existing capital stock. This effect contributes to a growth slow-down. On the other hand, the jump component $\frac{1}{\lambda\sigma\gamma_e}$ translates into a *faster* trend consumption growth as compared to the standard Keynes-Ramsey growth. Since $\frac{1}{\lambda\sigma\gamma_e}$ is the ratio of marginal utilities of post- to pre-jump consumption, it is larger than unity (see also Eqs. (10) and (15)) and thus the term $\frac{1}{\lambda\sigma\gamma_e} - 1$ is positive. The optimal stochastic consumption path is therefore tilted counterclockwise, as compared to the consumption path in a deterministic Keynes-Ramsey model. Therefore, the economy starts with a relatively low consumption rate at the beginning of the planning horizon, which implies the presence of the precautionary-saving motive, including saving for financing of emissions control. The result is analogous to what has been found in the literature on precautionary savings under uncertainty.[8] The peculiarity of the current setting is that the gross savings are endogenously split between two purposes: capital accumulation and abatement, both of which serve to protect the economy from climate disasters. It is clear that abatement reduces emissions and therefore unambiguously contributes to a reduction in the damages. Capital accumulation, however, has a double-sided effect. On the one hand, more capital implies more output and more emissions. On the other hand, having more capital creates an "emergency buffer" for the rainy days - when a disaster strikes. In Section 3.3 we discuss in more detail the economy's optimal saving rate and how it is affected by the climate change.

The responses of the economy's optimal growth rate to changes in the fundamental parameters of the model are summarized in Proposition 2. It is important to distinguish between the effect of the expected frequency of natural disasters and the effect of the overall uncertainty. The former takes into account only the arrival rate λ. The latter includes both the arrival rate and the damage caused by the occurrence of an event, as reflected in the last term in Eq. (18).

Proposition 2: *The solution to the maximization problem (1) - (4) is characterized by the optimal trend consumption growth rate which is:*
(i) a decreasing function of the arrival rate, polluting intensity of production, and damage intensity,
(ii) an increasing function of the total factor productivity and
(iii) either an increasing or a decreasing function of abatement efficiency, depending on the parameter constellation.

[8]See, e.g., Wälde (1999), Toche (2001), Steger (2005).

Proof: Follows from comparative statics (Eq. (18)):

$$\frac{\partial g}{\partial \lambda} = -\frac{1}{\varepsilon} < 0, \quad \frac{\partial g}{\partial \phi} = -\frac{A}{\varepsilon\sigma} < 0, \quad \frac{\partial g}{\partial \gamma_e} = -\frac{1-\bar{\gamma}}{\varepsilon\sigma\gamma_e^2} < 0,$$
$$\frac{\partial g}{\partial A} = \frac{1}{\varepsilon}\left(1 - \frac{\phi}{\sigma}\right) > 0, \quad \frac{\partial g}{\partial \sigma} = \frac{1}{\varepsilon\sigma^2}\left(A\phi - \frac{1-\bar{\gamma}}{\gamma_e}\right) \gtrless 0. \quad \blacksquare$$

The effect of the arrival rate (λ) on the optimal growth rate is directly proportional to the negative of the elasticity of intertemporal consumption substitution. Although there is not a general consensus on the magnitude of this elasticity, the empirically plausible range of values lies between 1 and 3. This suggests that if the frequency of natural disasters were to rise in the future due to accentuated climate change, the economy may experience an important growth slowdown. The results on polluting intensity ϕ and damage intensity γ_e are intuitive and have already been discussed.

The fact that a higher abatement efficiency has an ambiguous bearing on economic growth is due to two effects - the emissions-reduction effect and the jump-smoothing effect - which work in opposite directions. On the one hand, an improvement in efficiency of abatement reduces total emissions and thus enhances the growth rate through the first term in Eq. (18). On the other hand, it increases the post-event consumption rate, shrinking the pre- to post-event consumption gap (see Eq. (10)) and thus contributes to a growth slowdown through the last term in Eq. (18). If the total factor productivity or polluting intensity or climate damage sensitivity are large, then the effect of σ on the trend growth rate is positive. This suggests that economies with a relatively high polluting intensity of production (higher ϕ) and with a high exposure to climate shocks (higher γ_e), such as developing economies, may enjoy substantial gains in terms of their growth rates by adopting (more) efficient abatement technologies.

3.2 Abatement

How much of the current resources to devote to emissions control is a key policy question. We have shown in the previous section that it is optimal to allocate a specific constant fraction of output to abatement activities. The solution for the abatement share θ^* is reproduced from Eq. (14) for convenience

$$\theta^* = \overbrace{\frac{\phi}{\sigma}}^{100\%\text{ clean}} - \overbrace{\left(\frac{1-\bar{\gamma}}{A\sigma\gamma_e} - \frac{(\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}}}{A\sigma\gamma_e}\right)}^{\text{adj. cons. jump}}. \tag{19}$$

The first term, labeled "100% clean," indicates that if $\theta = \phi/\sigma$, all emissions are eliminated. The presence of the last two terms, labeled "adjusted consumption jump," indicates that in general it is not optimal for the economy to abate all emissions. If we ignore the exposure component of the damages for the moment by setting $\bar{\gamma} = 0$, we see that a 100% abatement policy is optimal only if the intertemporal substitution elasticity is zero (or coefficient of relative risk aversion is infinite). For finite ε, the optimal abatement share falls short of 100% due to the "jump" effect. In fact, by bringing all the terms in (19) to the common denominator, we see that the optimal θ depends on the difference between the marginal damage caused by an extra unit of accumulated capital ($A\phi\gamma_e$) and the magnitude of the jump ($1 - (\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}}$) in the capital stock (and also consumption) when a shock occurs. The jump effect works to reduce θ^*. The presence of the exposure component ($\bar{\gamma}$) works in the opposite direction to increase θ^*. The following proposition summarizes the effects of the fundamental parameters of the model on the optimal abatement share.

Proposition 3: *The solution to the maximization problem (1) - (4) is characterized by the optimal fraction of output devoted to emissions abatement which is:*
(i) an increasing function of the event arrival rate, total factor productivity, polluting intensity of output, and damage intensity,
(ii) either a decreasing or an increasing function of abatement efficiency, depending on the parameter constellation.

Proof: The results can be obtained from the following comparative statics, using Eq. (19)

$$\frac{\partial\theta^*}{\partial\lambda} = \frac{(\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}-1}}{A\varepsilon} > 0, \quad \frac{\partial\theta^*}{\partial A} = \frac{1-\bar{\gamma}-(\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}}}{A^2\sigma\gamma_e} > 0, \quad \frac{\partial\theta^*}{\partial\phi} = \frac{1}{\sigma} > 0,$$
$$\frac{\partial\theta^*}{\partial\gamma_e} = \frac{1+\frac{1-\varepsilon}{\varepsilon}(\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}}-\bar{\gamma}}{A\sigma\gamma_e^2} > 0, \quad \frac{\partial\theta^*}{\partial\sigma} = \frac{1-A\phi\gamma_e+\frac{1-\varepsilon}{\varepsilon}(\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}}-\bar{\gamma}}{A\sigma^2\gamma_e} \gtrless 0. \quad \blacksquare$$

The statements in (i) are intuitive. An increasing event arrival rate (λ) requires more abatement in order to better protect the economy against climate damages. If policy makers happened to misperceive the true arrival rate λ, the abatement policy would be sub-optimal. Specifically, if the predicted λ is lower than the true λ, there is too little abatement. This might happen if climate change induces a regime switch from low to high event frequency but general expectations, if based on past experience, may lag.

The total factor productivity (A) fosters pollution by raising output and thus acts in the same direction as the pollution parameters, such as polluting intensity of output (ϕ) and damage intensity (γ_e).

The statement in (ii) warrants some further comments. The reason for the ambiguous sign in $\partial\theta^*/\partial\sigma$ is that there are three effects which operate in different directions. They can be analyzed by examining the expression in (19). First, there is a direct effect of σ on the optimal abatement share, operating through the first term on the RHS of (19): Better abatement technology requires a smaller expenditure on emissions reduction, all else equal. Second, a better abatement efficiency has a positive effect on the economy's growth rate (provided $A\phi\gamma_e > 1-\bar{\gamma}$), which in turn calls for a larger abatement expenditure to compensate for an increase in polluting activities. If $A\phi\gamma_e < 1-\bar{\gamma}$, the reverse is true. Finally, abatement efficiency also affects the size of the downward jump in the consumption rate and in the capital stock when an adverse event occurs (the last term in (19)). The direction of this latter effect, however, depends on the intertemporal substitution elasticity, $1/\varepsilon$. When it is relatively high (resp., low), i.e., above (resp., below) unity, the effect of σ on the downward jump is positive (resp., negative). Overall, the first (direct) effect contributes to a decrease in abatement share; the second (growth) effect contributes to an increase or a decrease in abatement share; while the third (jump) effect can also be either positive or negative, depending on the intertemporal substitution elasticity.

Lemma 1: *If the intertemporal substitution elasticity is above (below) unity,*
(i) the optimal abatement share is convex (concave) in the arrival rate;
(ii) the response of the abatement share to a change in the arrival rate is more (less) pronounced when abatement technology is more (less) efficient and when climate-damage intensity is larger (smaller).

Proof: Follows directly from

$$\frac{\partial^2\theta^*}{\partial\lambda^2} = \left(\frac{1}{\varepsilon}-1\right)\frac{\lambda^{\frac{1}{\varepsilon}-2}(\sigma\gamma_e)^{\frac{1}{\varepsilon}-1}}{A\varepsilon} \gtrless 0 \Leftrightarrow \frac{1}{\varepsilon} \gtrless 1,$$
$$\frac{\partial^2\theta^*}{\partial\lambda\partial\sigma} = \left(\frac{1}{\varepsilon}-1\right)\frac{\sigma^{\frac{1}{\varepsilon}-2}(\gamma_e\lambda)^{\frac{1}{\varepsilon}-1}}{A\varepsilon} \gtrless 0 \Leftrightarrow \frac{1}{\varepsilon} \gtrless 1,$$
$$\frac{\partial^2\theta^*}{\partial\lambda\partial\gamma_e} = \left(\frac{1}{\varepsilon}-1\right)\frac{\gamma_e^{\frac{1}{\varepsilon}-2}(\sigma\lambda)^{\frac{1}{\varepsilon}-1}}{A\varepsilon} \gtrless 0 \Leftrightarrow \frac{1}{\varepsilon} \gtrless 1. \quad \blacksquare$$

The Lemma implies that, when the frequency of natural disasters is already relatively high, a further increase in the frequency should be associated with a more (less) than proportional increase in abatement if the intertemporal substitution elasticity is greater (smaller) than unity.

Lemma 2: *If the intertemporal substitution elasticity is*
(i) below 3, then the optimal abatement share is concave in the damage intensity;
(ii) below 2, then the response of the abatement share to a change in the damage intensity is less pronounced when abatement technology is more efficient. (These conditions are sufficient but not necessary.)

Proof: Follows directly from

$$\frac{\partial^2\theta^*}{\partial\gamma_e^2} = \frac{(\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}}(1-3\varepsilon)+2\varepsilon^2\left[(\sigma\lambda\gamma_e)^{\frac{1}{\varepsilon}}-(1-\bar{\gamma})\right]}{A\sigma\gamma_e^3} \gtrless 0,$$
$$\frac{\partial^2\theta^*}{\partial\gamma_e\partial\sigma} = \left(\frac{1-\varepsilon}{\varepsilon}\right)^2\frac{(\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}}}{A\sigma^2\gamma_e^2} - \frac{1-\bar{\gamma}}{A\sigma^2\gamma_e^2} \gtrless 0. \quad \blacksquare$$

These results formally support the argument that it is optimal to increase abatement activities when the magnitude of climate change related damages or the expected frequency of natural disasters increase. Our model predicts that the optimal increase in abatement share should be more (less) than proportional to an increase in the frequency of events if the intertemporal substitution elasticity is relatively high (low). The intuition here is straightforward. A higher elasticity of intertemporal consumption substitution implies that the economy is easily willing to forgo current consumption in exchange for more consumption in the future and thus an increase in the current abatement expenditure is less burdensome. In the limiting case $\varepsilon = 1$ (logarithmic utility), θ^* is linear in λ and monotone-increasing and concave in the damage intensity, $\frac{\partial^2\theta^*}{\partial\gamma_e^2} = -\frac{2(1-\bar{\gamma})}{A\sigma\gamma_e^3} < 0$. It can be either monotone-decreasing and convex or monotone-increasing and concave in abatement efficiency, depending on whether $A\phi\gamma_e \gtrless 1-\bar{\gamma}$.

3.3 Implications for Propensity to Save

The ratio of gross savings to output represents the economy's propensity to save (PTS), which we denote by s. The fraction of output devoted to abatement is a part of PTS. The effect of climate change on s is of interest from the macroeconomic perspective. In our model, the gross savings support two types of expenditures: investment in capital accumulation and abatement activities, with the optimal split between the two being endogenously determined. Knowing how climate change affects θ and s allows us to deduce its impact on capital accumulation. Using Eq. (13), we may express s as

$$s = 1 - \frac{\psi}{A} = \frac{1}{A\varepsilon}\left\{A - \rho + (1-\varepsilon)\frac{1-\bar{\gamma}-(\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}} - A\phi\gamma_e}{\sigma\gamma_e} - \lambda\left[1-(\lambda\sigma\gamma_e)^{\frac{1-\varepsilon}{\varepsilon}}\right]\right\}. \tag{20}$$

First, note that s depends only on the parameters of the model and does not depend on time. The optimality of a time-invariant PTS is a useful result which justifies the constant propensity to save assumption in Golosov et al. (2014), allowing the authors to derive their simple formula for the carbon tax. Second, when log-utility is assumed ($\varepsilon \to 1$), the expression simplifies to $1 - \frac{\rho}{A}$ and thus excludes the climate-related parameters altogether. This simplified preference structure implies that climate change would only cause a reallocation between capital investment and abatement, but not between consumption and gross savings. When ε is different from unity, the effects of the key climate parameters on s are as follows:

$$\begin{aligned}
\frac{\partial s}{\partial \lambda} &= \frac{1}{A\varepsilon}\left\{(\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}-1}-1\right\} \gtrless 0 \Leftrightarrow \frac{1}{\varepsilon} \lessgtr 1, \\
\frac{\partial s}{\partial \phi} &= -\frac{1-\varepsilon}{\varepsilon\sigma} \gtrless 0 \Leftrightarrow \frac{1}{\varepsilon} \lessgtr 1, \\
\frac{\partial s}{\partial \gamma_e} &= \frac{(1-\varepsilon)}{A\varepsilon}\frac{(\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}}}{\sigma\gamma_e^2} \gtrless 0 \Leftrightarrow \frac{1}{\varepsilon} \gtrless 1, \\
\frac{\partial s}{\partial \sigma} &= \frac{(1-\varepsilon)}{A\varepsilon}\frac{(\lambda\sigma\gamma_e)^{\frac{1}{\varepsilon}}+A\phi\gamma_e}{\sigma^2\gamma_e} \gtrless 0 \Leftrightarrow \frac{1}{\varepsilon} \gtrless 1.
\end{aligned}$$

The value of the intertemporal substitution elasticity appears to be crucial for resolving the ambiguity in the climate effects. For instance, an increase in λ causes an unambiguous increase in θ but may lead to a decline in s if $1/\varepsilon > 1$ and to an increase in s if $1/\varepsilon < 1$. It follows that, when the elasticity is relatively high, the optimal response of the economy to an increase in disaster frequency is to increase both its abatement expenditure and current consumption - at the expense of capital accumulation. By contrast, when the elasticity is relatively low, an increase in the abatement share is accompanied by a *reduction* in both consumption and capital accumulation ($\frac{\partial s}{\partial \lambda} < \frac{\partial \theta}{\partial \lambda}$, see the exact expression for $\frac{\partial \theta}{\partial \lambda}$ in the proof of Proposition 2). Moreover, when ε approaches unity, the derived impact of all the parameters - and most importantly those related to climate change, λ and γ_e - are at the lower bound of the empirically plausible impact range.

3.4 Quantitative Implications

In this subsection we illustrate the quantitative implications of our model by calibrating the optimal fraction of output which is devoted to emissions control and by comparing it with recent findings in the literature. For instance, Golosov et al. (2014) calculate the optimal carbon tax of \$56.9 and \$496 per ton carbon, assuming alternative discount rates and the world output of \$70 trillion in 2010. Given the total world emissions of about 9.7 bn tons of carbon in that year, the total tax proceeds would amount to either 0.8% of world output (with the \$56.9 tax) or to 7% of world output (with the \$496 tax).

The abatement share in our model depends on a set of parameters, some of which may be more easily calibrated than others. We will remain cautious about choosing values pertaining to the world economy as a whole. The reference unit of time is set to one year and averages over the past few decades are used to calibrate specific parameters. We set the exposure component of capital depreciation $\bar{\gamma}$ to 0.001 and gross capital return A to 0.04. For calculating the growth rate we assume the rate of time preference of 1.5% per year, as in Golosov et al. (2014) and Nordhaus (2008). The former article also assumes a unitary elasticity of marginal utility, $\varepsilon = 1$ (and thus a unitary intertemporal substitution elasticity), which we adopt here as a starting point for the purpose of having a meaningful comparison. The statistics for large-scale natural disasters over the last two decades suggest that the arrival rate $\lambda \approx 0.099$ - an event arrives slightly more often than once in ten years. Damages from such catastrophes constitute on average approximately 0.9% of output.[9] This proxies for the consumption drop equal to $1 - (\lambda\sigma\gamma_e)^{1/\varepsilon}$ in our model. We can then calibrate $\sigma\gamma_e$ as $1 - (\lambda\sigma\gamma_e)^{1/\varepsilon} = 0.009 => \sigma\gamma_e = (0.991)^{\varepsilon}/\lambda$. With $\lambda = 0.099$ and $\varepsilon = 1$, we obtain $\sigma\gamma_e = 9.9798$. The last parameter to calibrate is the polluting intensity of output ϕ. Polluting intensities vary considerably not only across countries but also across industries and even firms in the same industry within a given country. As a starting point, we use a rather optimistic value of $\phi = 0.05$ and test robustness of our results with respect to variations in this and other parameters.

Under the benchmark calibration described above, we obtain the optimal abatement share θ of 0.8968 percent and the trend growth rate of consumption of 3.4552 percent. Our value of θ of $\approx 0.9\%$ is slightly higher, although comparable, to the 0.8% implied by the carbon tax in Golosov et al. (2014).

[9] For example, the Indian Ocean Tsunami in 2004 caused at least \$10 bn worth of damage and affected mainly six countries: Indonesia, India, Maldives, Sri Lanka, Somalia, and Thailand. The damage amounts to 0.86% of the sum of GDPs in 2004 of the affected countries (Somalia not included due to lacking GDP data in WDI). Hurricane Katrina in 2005 caused \$108 bn damage which amounts to 0.825% of GDP in the USA. Typhoon Haiyan in the Philippines in 2013 caused \$2.8 bn damage, equivalent to 1.05% of GDP.

A drawback of this preliminary calculation is that the damage intensity and the abatement efficiency are not disentangled and, in fact, their calibration depends on the value of λ and ε. In an attempt to provide a more rigorous estimation of θ, we turn to the statistics on global carbon emissions and damages caused by major natural disasters. According to Reuters and the data by the World Bank, global CO_2 emissions in 2013 amounted to 36 bn metric tons or equivalently to (36/3.67 =) 9.8092 bn tons of carbon. With overall damages from severe natural disasters climbing over \$90 bn, the damage intensity is calibrated as $\gamma_e = 9.175$ (= 90/9.8092) dollars per ton carbon. With the global world output in 2013 at \$74.17172 trillion, the polluting intensity is calibrated as $\phi = 0.1322$ tons carbon per thousand dollars worth of output. The total factor productivity is set at 7% and $\sigma = 1$. This calibration implies an average world growth rate of 5.43% and an abatement share of 0.9316%, which is equivalent to a tax of \$70.44 per ton carbon. With a more precise calibration of the climate damage intensity and polluting intensity, we obtain slightly higher estimates of θ and g. Further, we are interested in how these estimates react to small variations in the climate-related parameters, such as event frequency, polluting intensity and damage intensity. We find that an increase in the climate-event frequency by as little as 1% leads to more than doubling of the optimal abatement share, although the effect on the optimal trend growth rate is relatively small, only a 0.1 percentage point drop. When the damage intensity is increased by 1%, θ rises from 0.93% to 2.45%, while the global trend growth rate declines from 5.43 to 5.32%. A 5% increase in polluting intensity leads to a 0.66 percentage point increase in θ and a 0.05 pp drop in g. These results point to a relatively high sensitivity of the optimal abatement efforts to changes in the main climate parameters of the model.

We have argued in the previous section that the logarithmic utility assumption and the implied unitary elasticity of intertemporal consumption substitution may not be appropriate. We therefore examine the implications of assuming a higher value of ε in our calibration. It turns out that with $\varepsilon = 1.5$, the optimal abatement share increases to 5% of output (with the implied carbon tax of \$378), while with $\varepsilon = 3$ it jumps to 10% (the growth rate of consumption falls to 1.8% in the latter case), holding other parameters unchanged.

As noted by Pindyck (2013), ε may be viewed as either "behavioral parameter (reflecting behavior of consumers, investors or firms) or a policy parameter (reflecting opinions and objectives of policymakers)." If ε is assigned the former interpretation, its estimates may range from 1.5 to 4. If the latter interpretation is adopted, concern for intergenerational consumption inequality will call for a smaller value, say, between 1 and 3. The intertemporal substitution elasticity exerts two opposing effects on future welfare (and thus current climate policy). On the one hand, a larger ε implies that the marginal utility drops more quickly with an increase in consumption. If consumption is expected to grow, one extra unit in the future will yield a smaller marginal utility. On the other hand, ε reflects aversion to risk. So if future welfare is uncertain, its value will be smaller the larger is ε. Pindyck (2013) writes: "Most models show that unless risk aversion is extreme (e.g., η is above 4), the first effect dominates, which means an increase in η (say, 1 to 4) will reduce the benefits from an abatement policy." (The parameter η in his analysis corresponds to our ε). This sounds like rather bad news for climate policy if one uses an empirically plausible calibration for ε. However, the above reasoning does not take into account two important considerations. First, consumption growth depends on ε and, second, it also depends on the optimal climate policy. A higher ε leads to a decline in the optimal consumption growth and therefore to smaller future consumption rates as compared to the case where the dependency of the growth rate on ε is not taken into account. Since g falls, the former effect described by Pindyck (the fall in marginal utility) - which is supposed to dominate unless ε is extremely high - is mitigated. In our framework the effect of an increase in ε on the optimal abatement share is clearly positive.[10]

The numerical examples above illustrate the properties of the model and the importance of the climate change related parameters. However, the numbers should be taken as suggestive and interpreted with caution. Varying crucial environmental parameters,

[10]The exact expression is $\frac{d\theta}{d\varepsilon} = -\frac{(\lambda\sigma\gamma_e)^{1/\varepsilon}}{\varepsilon^2 A\sigma\gamma_e}\ln(\lambda\sigma\gamma_e) > 0$ since the argument of the logarithm is less than unity.

such as pollution intensity and the arrival rate of climate events, has a drastic impact on optimal policies. In a similar way, we confirm that the curvature of the utility function matters significantly for climate policy under uncertainty. Specifically, the assumption of log-utility might limit the interpretation and applicability of the implied results in climate economics.

4 Model Extensions

4.1 Adding Trend Fluctuations

In this Section we extend the baseline model to a more general specification of the stochastic process for the capital stock. We shall assume that harmful emissions and the associated climate change cause not only jumps but also fluctuations around the trend. The latter are modeled by the Wiener process. Formally, the stochastic law of motion for the capital stock reads

$$dK_t = [(1-\theta_t)Y_t - C_t]dt + b(E_t)dz_t - \gamma(E_t)dq_t,$$

where dz is an increment of the standard Brownian motion, i.e., z_t has mean zero and variance t, and $b_t = b_z E_t$, $b_z \in (0,1)$. The magnitude of the random fluctuations is linearly proportional to emissions with the proportionality parameter or "amplifier" b_z, assumed to be a small number. Given that z_t has a normal distribution with mean zero, the deviations from the trend may be either positive or negative. The mechanisms by which GHG emissions may cause downward deviations from the trend are rather intuitive. They may be indirectly related to negative externalities caused by pollution to the economy's technology, total factor productivity, health status of the workforce, etc. Upward deviations from the trend, however, may also occur when (polluting) economic activity generates positive spillovers which cause the capital stock to increase. We do not model explicitly either of these deviations but take a shortcut through adopting the

Brownian motion component in the stochastic representation of the capital stock path. The jump component remains the same as in the baseline setup, except that we set the exposure parameter $\bar{\gamma} = 0$ for simplicity, so that $\gamma(E_t) = \gamma_e E_t$, $\gamma_e > 0$.

The HJB equation becomes

$$\rho V(K) = \max_{C,\theta} \left\{ u(C) + V'(K)[(1-\theta)Y - C] + \frac{1}{2}V''^2(K) + \lambda[V(\widetilde{K}) - V(K)] \right\}$$

which now has an extra component - the second derivative of the value function - reflecting the presence of the Brownian motion. The optimality conditions with respect to the control and state variables are

$$C : u'(C) - V'(K) = 0, \tag{21}$$

$$\theta : -V'(K)Y - \frac{1}{2}V''(K)2b_z^2(\phi - \sigma\theta)Y^2\sigma + \lambda V'(\widetilde{K})\gamma_e\sigma Y = 0, \tag{22}$$

$$\begin{aligned} K : \rho V'(K) &= V''(K)[(1-\theta_t)Y - C] + V'(K)(1-\theta)A + \\ &+ \frac{1}{2}\left[V'''^2(K) + V''(K)2b(\phi - \sigma\theta)b_z A\right] + \lambda\left[V'(\widetilde{K})[1 - \gamma_e(\phi - \sigma\theta)A] - V'(K)\right]. \end{aligned} \tag{23}$$

We show in Appendix B that the optimality conditions yield

$$\frac{dC}{C} = \frac{1}{\varepsilon}\left\{A\left(1 - \frac{\phi}{\sigma}\right) - \rho - \lambda + \frac{1}{\sigma\gamma_e} - \frac{\varepsilon}{\widetilde{\psi}}b_z^2\frac{(\phi - \sigma\theta)}{\gamma_e} + \frac{\varepsilon(1+\varepsilon)}{2\widetilde{\psi}^2}b_z^2(\phi - \sigma\theta)^2\right\}dt +$$

$$+ \frac{b_z(\phi - \sigma\theta)}{\widetilde{\psi}}dz + \left(\frac{\widetilde{C}}{C} - 1\right)dq,$$

where

$$\widetilde{\psi} = \frac{\psi^b}{A}, \quad \psi^b = \frac{C}{K} = \frac{1}{\varepsilon}\left[\rho - (1-\varepsilon)(1-\theta)A + \frac{1}{2}\varepsilon(1-\varepsilon)b_z^2(\phi - \sigma\theta)^2A^2 - \lambda\gamma_e A(\phi - \sigma\theta)\right]. \tag{24}$$

We define the trend growth rate as

$$g^b \equiv \frac{1}{\varepsilon}\left\{A\left(1 - \frac{\phi}{\sigma}\right) - \rho - \lambda + \frac{1}{\sigma\gamma_e} - \frac{\varepsilon}{\widetilde{\psi}}b_z^2\frac{(\phi - \sigma\theta)}{\gamma_e} + \frac{\varepsilon(1+\varepsilon)}{2\widetilde{\psi}^2}b_z^2(\phi - \sigma\theta)^2\right\}, \tag{25}$$

where the last two terms inside the parentheses depend on the abatement share θ. The optimal abatement share solves

$$\varepsilon b_z^2(\phi - \sigma\theta)\sigma A + \lambda\sigma\gamma_e[1 - \gamma_e(\phi - \sigma\theta)A]^{-\varepsilon} = 1, \tag{26}$$

which is a non-linear equation in θ with two positive roots. Setting b_z to zero, we are back to our baseline model with the unique solution for θ given by θ^* in Eq. (14) (with $\bar{\gamma}$ set to zero). Defining $x = \phi - \sigma\theta$, we can rewrite (26) as

$$1 - \varepsilon b_z^2 x\sigma A = \lambda\sigma\gamma_e(1 - \gamma_e x A)^{-\varepsilon}. \tag{27}$$

To ensure the existence of a meaningful solution for a wide range of ε, we require that the term in the parentheses on the RHS is positive[11] or $x < \frac{1}{\gamma_e A}$.

Proposition 4: *When the evolution of the economy's capital stock is affected by climate change induced random jumps (via Poisson process) and fluctuations around the trend (via Wiener process),*
(i) the optimal abatement expenditure represents a constant share of output;
(ii) the economy's trend growth rate is constant;
(iii) the abatement share is larger, while the growth rate is either higher or lower as compared to the baseline scenario (with only random jumps).

Proof: Follows from the discussion below. ■

Given that eq. (27) contains only constant terms, the implied abatement share, call it θ^b, depends only on the parameters of the model and is therefore not time dependent. The solution to (27) is illustrated in Figure 2, where the left-hand side is shown by the downward-sloping straight line with the intercept at 1 and the right-hand side is shown by the hyperbola with the asymptote at $\frac{1}{A\gamma_e}$ and an intercept at $\lambda\sigma\gamma_e$. The two roots are given by the intersection of the straight line with the two parts of the hyperbola

[11] If the term happens to be negative and $\varepsilon \in (0, 1)$, the RHS may become a complex number, an outcome which we do not wish to study further.

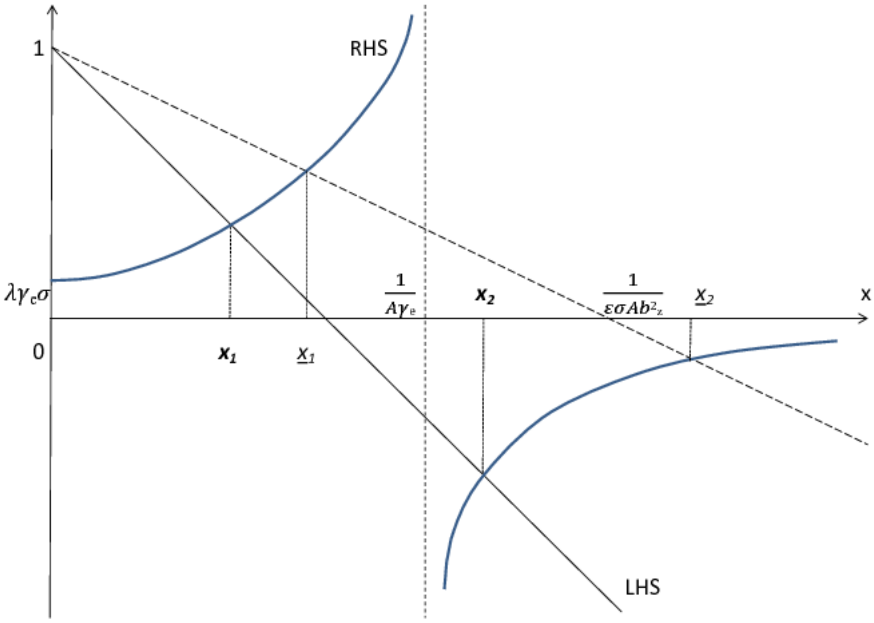


Figure 2: Solution to non-linear equation with two positive roots.

and are shown by x_1 and x_2. Note that, depending on the parameter values, the slope of the LHS may be either greater or smaller than unity, the latter case shown by the dashed line intersecting the x-axes at $\frac{1}{\varepsilon\sigma A b_z^2}$. The roots are as indicated by $\underline{x}_1$ and $\underline{x}_2$. Regardless of the slope value of LHS, one of the roots always lies in $(0, \frac{1}{A\gamma_e})$, while the other root is always greater than $\frac{1}{A\gamma_e}$ and is therefore not considered further.

A constant θ^b implies a constant ψ^b and thus the optimal consumption is a constant share of the capital stock (and of output as well). The optimal trend consumption growth rate (g^b) is then also constant. Comparing the trend growth rates of our baseline model with the current solution we have:

$$g^b \gtrless g \quad \Leftrightarrow \quad \theta^b \lessgtr \frac{\phi}{\sigma} - \frac{2\psi^b}{A\gamma_e\sigma(1+\varepsilon)},$$

where θ^b is the solution of (27).

In the case of logarithmic utility (setting $\varepsilon = 1$) the roots can be found explicitly by solving the quadratic equation

$$\gamma_e (Ab_z)^2 \sigma x^2 - (\gamma_e + b_z^2 \sigma) A x + 1 - \lambda \sigma \gamma_e = 0 \tag{28}$$

with

$$x_1 = \frac{\gamma_e + b_z^2\sigma - \sqrt{(\gamma_e - b_z^2\sigma)^2 + 4(\gamma_e b_z \sigma)^2 \lambda}}{2\gamma_e A b_z^2 \sigma}, \ x_2 = \frac{\gamma_e + b_z^2\sigma + \sqrt{(\gamma_e - b_z^2\sigma)^2 + 4(\gamma_e b_z \sigma)^2 \lambda}}{2\gamma_e A b_z^2 \sigma}. \tag{29}$$

Given our requirement $x < \frac{1}{\gamma_e A}$, the second root is eliminated, so that the corresponding solution for the abatement share is $\theta^b = \frac{\phi - x_1}{\sigma}$. It is thus possible to characterize the range of values of the fluctuations amplifier b_z such that g^b is greater or less than g. Using (29) and the definition of ψ^b, we obtain

$$g^b \gtrless g \quad \Leftrightarrow \quad \gamma_e + b_z^2\sigma - \frac{2Ab_z^2\sigma\rho}{1+\lambda} \gtrless \sqrt{(\gamma_e - b_z^2\sigma)^2 + 4(\gamma_e b_z \sigma)^2 \lambda}. \tag{30}$$

In general, there are two cases to distinguish, depending on whether the LHS of (30) is positive or negative, i.e., $\gamma_e(1+\lambda) + b_z^2\sigma(1+\lambda - 2A\rho) \gtrless 0$. In Case (1), the parameter constellation is such that the inequality is strictly negative. This occurs when

$$b_z^2 > \frac{\gamma_e(1+\lambda)}{\sigma[2A\rho - (1+\lambda)]} \equiv \tau_1,$$

which defines the threshold τ_1. If (the square of) the amplifier is above this threshold, g^b is unambiguously lower than g. Intuitively, when polluting emissions have a relatively strong effect on the magnitude of fluctuations in the path of the capital stock, the economy's growth rate is lower than in a scenario without such fluctuations. Given that the amplifier is bounded from above by unity, the parameter space is restricted to $(1+\lambda)(1+\frac{\gamma_e}{\sigma}) < 2A\rho$. This condition also ensures that $\tau_1 > 0$. We note that this parameter constellation is realistically rather unlikely (although theoretically possible) since the LHS is a product of two numbers greater than unity, while the RHS is 2 times a product of relatively small numbers - the marginal productivity of capital and the rate of time preference.

We turn next to Case (2) such that the LHS of (30) is positive, i.e., $\gamma_e(1+\lambda) + b_z^2\sigma(1+\lambda-2A\rho) > 0$. Raising both sides of (30) to the power of two and solving for b_z yields another threshold value, τ_2:

$$g^b \gtrless g \quad \Leftrightarrow \quad b_z^2 \lessgtr \frac{\gamma_e(1+\lambda)[(1-\gamma_e\sigma\lambda)(1+\lambda)-A\rho]}{A\rho\sigma(1+\lambda-A\rho)} \equiv \tau_2$$

It can be verified that $\tau_2 < \tau_1$ if they are both positive. Two subcases are possible, depending on the parameter constellation: either $0 < \tau_2 < \tau_1 < 1$ or $\tau_1 < 0 < \tau_2 < 1$.

Case (2a): $g^b < g \quad \Leftrightarrow \quad \tau_2 < b_z^2 < \tau_1, \quad \text{and} \quad g^b > g \quad \Leftrightarrow \quad 0 < b_z^2 < \tau_2, \quad \text{when}$

$$\left(\frac{1+\lambda}{2}\right)\left(1+\frac{\gamma_e}{\sigma}\right) < A\rho < (1+\lambda)(1-\gamma_e\sigma\lambda).$$

Case (2b): $g^b > g \quad \Leftrightarrow \quad 0 < b_z^2 < \tau_2, \quad \text{and} \quad g^b < g \quad \Leftrightarrow \quad \tau_2 < b_z^2 < 1, \quad \text{when}$

$$\begin{cases} A\rho < \min\left\{\frac{1+\lambda}{2}, (1+\lambda)(1-\gamma_e\sigma\lambda)\right\} \\ A\rho \in (z_1, z_2), \; z_{1,2} = \frac{(1+\lambda)\left[\gamma_e+\sigma\pm\sqrt{(\gamma_e-\sigma)^2+4\sigma^2\gamma_e^2\lambda}\right]}{2\sigma}. \end{cases}$$

Intuitively, when climate change induced jumps and trend fluctuations are both present, the growth rate of the economy is smaller than under the baseline scenario if the fluctuations amplifier is relatively large - above τ_1 in Case (1), between τ_2 and τ_1 in Case (2a), and between τ_2 and unity in Case (2b). If the amplifier is relatively small, so that emissions do not cause large swings in the capital stock, the trend growth rate of the economy is enhanced.

Even though the optimal growth rate may be either greater or smaller than in the baseline, the optimal abatement share, θ^b, is unambiguously greater than θ^*. To see this, first note that (27) is decreasing in θ. Then insert θ^* in the equation and find that the LHS becomes smaller than the RHS. In order for the two to be equal again, θ must increase. Thus, $\theta^b > \theta^*$. The intuition is that now θ^b must absorb an additional source of uncertainty. Since the fluctuations are amplified by greenhouse gas emissions, there is a need to increase the resources devoted to abatement.

Total differentiation of (27) allows to draw insights on the optimal response of the abatement share to changes in the climate parameters of the model

$$\begin{aligned}
\frac{d\theta^b}{db_z} &= \frac{2b_z x}{\sigma[b_z^2 + \lambda\gamma_e^2(1-\gamma_e x A)^{-\varepsilon-1}]} > 0, \\
\frac{d\theta^b}{d\gamma_e} &= \frac{\lambda(1-\gamma_e x A)^{-\varepsilon-1}[1-\gamma_e x A(1-\varepsilon)]}{\varepsilon\sigma A[b_z^2 + \lambda\gamma_e^2(1-\gamma_e x A)^{-\varepsilon-1}]} > 0, \\
\frac{d\theta^b}{d\lambda} &= \frac{\gamma_e(1-\gamma_e x A)^{-\varepsilon}}{\varepsilon\sigma A[b_z^2 + \lambda\gamma_e^2(1-\gamma_e x A)^{-\varepsilon-1}]} > 0.
\end{aligned}$$

As expected, an increase in both damage-intensity parameters, b_z and γ_e, as well as the arrival rate, requires an unambiguous increase in the optimal abatement share.

4.2 Introducing Pollution Stock

In our baseline model we assumed that climate change induces natural disasters and the associated damages are larger, the larger is the flow of polluting emissions. In this extension we explore the implications of the assumption that damages depend on the accumulated stock of pollution, labeled P_t. We now model damages to the capital stock by the function $\gamma(P_t, K_t) = \gamma_p P_t + \bar{\gamma} K_t$, with γ_p being the new damage intensity parameter which may, in general, be either greater or smaller than γ_e of the baseline model, although it is safe to assume that the marginal damage impact of the stock should be at least as large as that of the flow, implying $\gamma_p \geqslant \gamma_e$. The dynamics of the pollution stock are described by the differential equation

$$dP_t = (E_t - \alpha P_t)dt. \tag{31}$$

The parameter $\alpha \in [0, 1)$ represents the natural absorption rate of greenhouse gases and E_t stands for the current emissions as before. The new stochastic law of motion for the capital stock is then given by

$$dK_t = [(1-\theta)Y_t - C_t]dt - \gamma(P_t, K_t)dq_t \tag{32}$$

and the after-shock capital stock is $\widetilde{K} = K - \gamma(P, K)$. We generalize the representative agent's utility function to include the negative impact of accumulated pollution. For simplicity, we assume a standard CRRA additively-separable function where consumption and pollution enter symmetrically,[12] $U(C, P) = \frac{C^{1-\varepsilon} - \chi P^{1-\varepsilon}}{1-\varepsilon}$. The parameter χ represents the relative weight of pollution and is non-negative (we allow for a possibility $\partial U/\partial P \to 0$ if $\chi \to 0$).

The problem is to maximize the expected present discounted utility subject to (31), (32) and the two constraints of the baseline model, reproduced for convenience

$$E_t = \phi Y_t(K_t) - \sigma I_t, \tag{33}$$

$$I_t = \theta Y_t(K_t), \tag{34}$$

where θ is the share of output devoted to abatement. Since the problem involves two state variables, the value function in the Hamilton-Jacobi-Bellman equation has now two arguments - the capital and the pollution stocks:

$$\rho V(K, P) = \max_{C,\theta} \left\{ U(C, P) + V_k[(1-\theta)Y - C] + V_p[(\phi - \theta\sigma)Y - \alpha P] + \lambda(\widetilde{V} - V) \right\},$$

where we omitted the time subscripts, substituted (33) in (31), and used the notation

[12]The symmetric structure is convenient for obtaining an explicit analytical solution. This structure also implies that the disutility of pollution is increasing at a lower incremental rate in pollution stock, which may be interpreted as a "getting used to" effect.

$V_k \equiv \partial V(K,P)/\partial K$, $V_p \equiv \partial V(K,P)/\partial P$. The first-order conditions include:

$$
\begin{aligned}
C: &\quad U_c - V_k = 0, \\
\theta: &\quad -V_k Y - V_p \sigma Y = 0, \\
K: &\quad \rho V_k = V_{kk}[(1-\theta)Y - C] + V_k(1-\theta)A + V_p(\phi - \sigma\theta)A + V_{pk}[(\phi-\sigma\theta)Y - \alpha P] + \\
&\quad +\lambda\left(\widetilde{V}_k(1-\bar{\gamma}) - V_k\right), \\
P: &\quad \rho V_p = U_p + V_{pp}[(\phi-\sigma\theta)Y - \alpha P] + V_{kp}[(1-\theta)Y - C] - \lambda\gamma_p\widetilde{V}_k - \alpha V_p + \\
&\quad +\lambda\left(\widetilde{V}_p - V_p\right).
\end{aligned}
$$

Computing the differentials of V_k and V_p allows us to obtain a system of two equations in two unknowns - the differential of consumption and the ratio of marginal utilities of consumption before and after the jump. We define the post-to-pre-jump consumption ratio as $\widetilde{\omega}$ so that:

$$\frac{\tilde{U}_c}{U_c} = \widetilde{\omega}^{-\varepsilon} = \frac{A(1-\frac{\phi}{\sigma}) + \alpha + \sigma\frac{U_p}{U_c}}{\lambda(\sigma\gamma_p + \bar{\gamma})}, \tag{35}$$

$$g^p \equiv \frac{dC}{C} = \frac{1}{\varepsilon}\left\{A\left(1-\frac{\phi}{\sigma}\right) - \rho + \lambda\left[\widetilde{\omega}^{-\varepsilon}(1-\bar{\gamma}) - 1\right]\right\}, \tag{36}$$

where g^p defines the trend consumption growth rate.

It can be shown that the value function is $V(K,P) = \frac{X_1 K^{1-\varepsilon}}{1-\varepsilon} - \frac{X_2 P^{1-\varepsilon}}{1-\varepsilon}$. The two constants X_1 and X_2 are functions of the parameters of the model and are determined by solving the following system of two non-linear equations, where $x \equiv \left(\frac{X_2}{X_1}\right)^{1/\varepsilon}$ is the relative weight of pollution in the value function:

$$
\begin{aligned}
x\left(\chi\sigma^{\frac{1-\varepsilon}{\varepsilon}} X_2^{-1} - \rho - (1-\varepsilon)\alpha\sigma^{\frac{1-\varepsilon}{\varepsilon}}\right) = \varepsilon x X_2^{-\frac{1}{\varepsilon}} + (1-\varepsilon)A\left(1-\frac{\phi}{\sigma}\right) - (\rho+\lambda) + \\
+\lambda\left(1-\bar{\gamma} - \gamma_p\sigma^{1/\varepsilon}x\right)^{1-\varepsilon},
\end{aligned}
\tag{37}
$$

$$\frac{A(1-\frac{\phi}{\sigma}) + \alpha - \frac{\chi}{X_2}}{\lambda(\sigma\gamma_p + \bar{\gamma})} = \left(1-\bar{\gamma} - \gamma_p x\sigma^{1/\varepsilon}\right)^{-\varepsilon}. \tag{38}$$

Proposition 5: *The solution of the maximization problem described by (1), (31) - (34) is characterized by the following:*

(i) optimal consumption is a constant fraction of the capital stock;

(ii) optimal abatement expenditure is a constant fraction of output;

(iii) consumption, capital and pollution stocks, output, and abatement grow at the same constant rate.

Proof: follows directly from the first-order conditions, Eqs. (35) - (36), and the optimal value function. The consumption-capital ratio is given by $X_1^{-1/\varepsilon}$. ■

Lemma 3: *Compared to the baseline setting where the climate damage is a function of emissions flow, in a setting where the climate damage depends on the whole history of pollution,*

(i) the downward jump in consumption induced by a climate-shock is smaller and

(ii) the trend consumption growth rate is smaller.

Proof: (i) The optimal ratio of post- to pre- jump consumption rate is obtained by substituting the optimal controls for U_p/U_c in Eq. (35)

$$\frac{\widetilde{C}}{C} = \left[\frac{\lambda(\sigma\gamma_p + \bar{\gamma})}{A\left(1 - \frac{\phi}{\sigma}\right) + \alpha - \frac{\chi}{X_2}} \right]^{1/\varepsilon}. \tag{39}$$

It is clear that the term in the denominator is between zero and unity, while in the numerator $\gamma_p \geqslant \gamma_e$, and therefore $\widetilde{\omega} > \omega$, implying that the consumption jump is smaller than under the baseline scenario of Section 2.

(ii) Comparing the expression for the trend growth rate in Eq. (36) with Eq. (18), we find that $g^p < g$ since $\widetilde{\omega}^{-\varepsilon} < \omega^{-\varepsilon}$. ■

It follows from Lemma 3 that the optimal consumption under "pollution-stock scenario" exhibits a time profile with less pronounced jumps and less vigorous growth (illustrated by the bold gray line in figure 1b). This is because in this case, as opposed to the baseline, arrival of a climate shock causes a larger damage to the capital stock, since the damage depends on the accumulated pollution and not just on the current level.

Therefore, a relatively slow accumulation of pollution, associated with a slower growth, becomes optimal.

In the special case of logarithmic utility, the system (37) - (38) can be solved explicitly for the constants X_1 and X_2. For notational convenience, define the economy's implicit interest rate as $r \equiv A\left(1-\frac{\phi}{\sigma}\right)+\alpha$. Then from (37)

$$X_2 = \frac{x(\chi-1)}{\rho(x-1)}$$

and (38) becomes a quadratic equation in x only:

$$x^2\left[\rho\chi-(\chi-1)r\right]\sigma\gamma_p+x\left\{(\chi-1)[r(1-\bar{\gamma})-\lambda(\bar{\gamma}+\sigma\gamma_p)]-\rho\chi(1-\bar{\gamma}+\sigma\gamma_p)\right\}+\rho\chi(1-\bar{\gamma})=0.$$

Focusing on the unique root and setting $\bar{\gamma}=0$, $\chi=1/2$ to save on notation, we have

$$x=\frac{r+\rho+\sigma\gamma_p(\rho-\lambda)}{2\sigma\gamma_p(r+\rho)},$$

$$X_1=\frac{\sigma\gamma_p(r+\rho)}{\rho\left[(r+\rho)(2\sigma\gamma_p-1)-\sigma\gamma_p(\rho-\lambda)\right]},\quad X_2=\frac{r+\rho+\sigma\gamma_p(\rho-\lambda)}{2\rho\left[(r+\rho)(2\sigma\gamma_p-1)-\sigma\gamma_p(\rho-\lambda)\right]}.$$

Inserting the solution for X_2 into (39) and the result in (36), we obtain the optimal trend growth rate of consumption

$$g^p=r-\alpha-(\rho+\lambda)+\frac{r+\rho-\sigma\gamma_p(\rho+\lambda)}{\sigma\gamma_p[r+\rho+\sigma\gamma_p(\rho-\lambda)]}$$

The responses of the optimal growth rate to changes in the parameters are provided below:

$$
\begin{aligned}
\frac{dg^p}{d\alpha} &= \frac{2\rho}{\big[r+\rho+\sigma\gamma_p(\rho-\lambda)\big]^2} > 0,\\
\frac{dg^p}{d\phi} &= -\left\{1+\frac{2\rho}{\big[r+\rho+\sigma\gamma_p(\rho-\lambda)\big]^2}\right\}\frac{A}{\sigma} < 0,\\
\frac{\partial g^p}{\partial\lambda} &= -1-\frac{2\rho\sigma\gamma_p}{\big[r+\rho+\sigma\gamma_p(\rho-\lambda)\big]^2} < 0,\\
\frac{\partial g^p}{\partial\gamma_p} &= -\frac{(r+\rho)\big[r+\rho+\sigma\gamma_p(\rho-\lambda)\big]+\sigma\gamma_p(\rho-\lambda)\big[r+\rho-\sigma\gamma_p(\rho+\lambda)\big]}{\sigma\gamma_p^2\big[r+\rho+\sigma\gamma_p(\rho-\lambda)\big]^2} < 0,\\
\frac{dg^p}{d\sigma} &= \frac{(A\phi\gamma_p-\sigma^2)\big[r+\rho+2\sigma\gamma_p(\rho-\lambda)\big](r+\rho)+(\sigma\gamma_p)^2(\rho-\lambda)\big[\rho(A\phi\gamma_p+\sigma^2)-\lambda(A\phi\gamma_p-\sigma^2)\big]}{\sigma^2\gamma_p\big[r+\rho+\sigma\gamma_p(\rho-\lambda)\big]^2} \gtrless 0.
\end{aligned}
$$

Similarly to the baseline model, a higher pollution intensity of production, a larger disaster arrival rate and a higher damage intensity reduce the growth potential, while the abatement efficiency has an ambiguous effect. The ambiguity is due to the two counteracting forces. The first is the positive effect of σ on the implicit interest rate r, which translates into the effect on growth via $\frac{\partial g^p}{\partial r}\frac{\partial r}{\partial \sigma} = \left\{1+\frac{2\rho}{\big[r+\rho+\sigma\gamma_p(\rho-\lambda)\big]^2}\right\}\frac{A\phi}{\sigma^2} > 0$. The second is the negative effect of σ on the consumption jump - a better abatement efficiency reduces the magnitude of the consumption drop at the time of the shock. This effect is given by the term $\frac{\partial g^p}{\partial \sigma} = -\frac{\sigma^2\gamma_p(\rho-\lambda)\big[2(r+\rho)-\sigma\gamma_p(\rho+\lambda)\big]+(r+\rho)^2}{\gamma_p\big[r+\rho+\sigma\gamma_p(\rho-\lambda)\big]^2} < 0$. If the magnitude of the positive effect of σ on the real interest rate is larger than the inverse of the damage intensity, i.e., $\frac{A\phi}{\sigma^2} > \frac{1}{\gamma_p}$, and the arrival rate does not exceed the rate of time preference, i.e., $\rho \geqslant \lambda$, then the overall effect $dg^p/d\sigma$ is unambiguously positive.

We turn next to the optimal share of output devoted to abatement activities. The optimality condition for the choice of θ and the optimal value function imply that the trend growth rates of both capital and pollution stocks are the same. The growth rate of the pollution stock is easily obtained from Eq. (31), so that we have

$$
g^p = \left(\frac{\phi}{\sigma}-\theta^p\right)\frac{A}{x}-\alpha,
$$

where θ^p stands for the optimal share of output spent on emissions mitigation. After rearranging and using the solution for x,

$$\theta^p = \frac{\phi}{\sigma} - \frac{r+\rho+\sigma\gamma_p(\rho-\lambda)}{2\sigma\gamma_p(r+\rho)A}\left(g^p+\alpha\right). \tag{40}$$

If we evaluate the solution for θ in Eq. (14) at $\varepsilon = 1$ and set $\bar{\gamma} = 0$, we can directly compare θ and θ^p. One may show that the optimal abatement share in the baseline model (i.e. with only pollution flow affecting the magnitude of damages) is smaller than the abatement share in the current setting (with pollution stock effect), assuming that the damage intensity of the pollution stock is at least as large as that of the flow ($\gamma_p \geqslant \gamma_e$). Intuitively, since pollution stock at time t represents the accumulation of flows over the period $[0, t]$ and the damages are proportional to the stock, the share of GDP devoted to a reduction of the stock must be larger than that pertaining to the flow.

The effects of the key climate parameters on θ^p have the same sign as those of the baseline model and their magnitudes are as follows:

$$\frac{d\theta^p}{d\lambda} = \frac{2+\sigma\gamma_p(r-\rho-\lambda)}{2A\sigma\gamma_p(r+\rho)} > 0,$$
$$\frac{d\theta^p}{d\gamma_p} = \frac{1+r+\rho+\sigma\gamma_p(\rho-\lambda)}{2A\sigma^2\gamma_p^3} > 0.$$

The abatement efficiency σ affects θ^p through three channels: First, directly through the first term in Eq. (40) - the negative effect; second, through the relative weight of pollution in the value function x - the negative effect; and, finally, through the growth rate - either positive or negative effect. Therefore, the overall effect of σ on the abatement share is ambiguous.

The extension of the baseline model to a setting where the entire pollution history determines the magnitude of climate damages is justified from the perspective of natural scientists. The baseline model, however, is more appealing as it has the advantages of being more tractable and more easily amendable. Moreover, it admits closed-form solutions for any range of the intertemporal substitution elasticity. Importantly, the main qualitative conclusions stemming from the baseline model remain unaltered in the more complex model. A simple intuition behind the similarity of results is that pollution stock is nothing else but an accumulation of flows over time (adjusted by the natural absorption capacity, a negligible effect). It follows that the impact of a pollution stock on the share of capital subject to destruction during a shock is more amplified as compared to the impact of a flow. This, in turn, calls for a more stringent abatement policy and thus a slower economic expansion.

The truly important difference between the flow and the stock model lies in the effect of discounting. Obviously, discounting plays a more prominent role in the latter by affecting both the trend growth rate and optimum abatement efforts. The role of discounting in the climate-policy debate, however, is already well understood and we need not emphasize it here.

5 Conclusions

An increase in the global temperature is predicted to intensify the severity of natural disasters such as tropical storms, hurricanes, tsunamis, floods, droughts, etc. These calamities have a profound negative impact on an economy's infrastructure, physical and human capital, and they undoubtedly represent a set-back in terms of economic growth and development. An appropriate and timely climate policy is necessary in order to limit the damages from these devastating shocks.

In the present article we propose a model of a growing economy subject to random natural disasters (driven by the Poisson process) which destroy part of the economy's productive input. An important feature of our model is that the extent of the damage is endogenously determined through the interaction of capital accumulation process and an appropriate emissions abatement policy. We show that the optimal time path of consumption is characterized by a constant growth rate until a disaster strikes causing a downward jump in both consumption and capital stock. After the shock, the economy continues to grow at the same constant rate until the next one strikes. We believe that this scenario, with recurring shocks over time, is more realistic than an extreme scenario with a so-called tipping point - a doomsday without future. The optimal climate policy consists of devoting a constant fraction of output to emissions abatement. We provide clear closed-form solutions for the optimal consumption growth rate and the climate policy instrument in terms of the model's parameters. A higher arrival rate of natural disasters and a larger damage intensity unambiguously reduce consumption growth rate and call for a more stringent climate policy. Moreover, these two key climate-change parameters affect the economy's optimal propensity to save. The direction of the effect, however, is determined by the value of the intertemporal substitution elasticity relative to unity. In the empirically-plausible range of the intertemporal substitution elasticity, a higher arrival rate and a larger damage intensity of natural disasters are associated with a larger saving propensity. The increase in propensity to save is, however, smaller than the increase in the abatement share, so that emissions mitigation takes place at the expense of both current consumption and capital accumulation. In the case of a unitary elasticity of substitution (log-utility), often used in the literature on the grounds of better tractability, the propensity to save is independent of the climate-change parameters, so that only a reallocation between capital accumulation and abatement takes place, while gross savings remain unaffected. In light of this, the log-utility assumption seems to impose significant limitations on the credibility of the findings in the existing literature.

We also provide some quantitative results by calibrating our model to the recent data on global carbon emissions, output, frequency of large natural catastrophes and their damages. We find that the share of output which should be devoted to emissions control is approximately 0.9% when log-utility is assumed. This number is slightly higher but nonetheless comparable to what has been found in the recent literature. However, when we depart from the log-utility assumption and use empirically-supported values for the intertemporal substitution elasticity, we find that the abatement share increases drastically as we consider higher degrees of concavity of the utility function. We emphasize that the numerical results should rather be taken as suggestive and interpreted with caution. The key message from our quantitative analysis is that optimal abatement efforts react sharply to relatively small changes in the arrival rate, damage intensity, and the intertemporal substitution elasticity. If one subscribes to the notion that climate change may cause an increase in disaster frequency (perhaps in addition to the damage intensity), then an even stronger argument in favor of a more stringent climate policy is warranted.

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