Multidimensional Uncertainty and Hypersensitive Asset Prices

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Abstract

We consider a dynamic asset pricing model with one asset, in which one informed trader trades against liquidity traders and competitive market makers. Informed trader has private information about the fundamental value of the asset as well as an exogenous demand shock on the market. We characterize the unique linear Markov equilibrium of the model. With just the private information about fundamentals the price converges to the fundamental value in a monotone way (Kyle [1985]). We show that the model with arbitrarily small demand shocks exhibits a price bubble. The bubble is created for strategic reasons by the informed trader, who follows a so called *pump-and-dump* strategy. He initially exacerbates the demand shock, trading at a loss (contrarian behavior), and gains later on by trading at an inflated price. Finally, both payoff relevant and payoff irrelevant information is revealed to the market.

1 Introduction

Why do asset price bubbles exist and how are they affected by rational traders? The predominant view among economists is that the strategic arbitraging behavior of buying low and selling high tends to stabilize asset prices around their underlying values. In the stringent neoclassical version, as exemplified by the *efficient market hypothesis* (Friedman [1953], Fama [1965]), the arbitraging precludes the departure of prices from the fundamental values. More recent research has attenuated this view by admitting the limits to the arbitraging behavior, due to the risk involved, short-selling and liquidity constraints, possibly aggravated by agency or coordination failures.[1] This may allow prices to be shaken away from the fundamentals, yet the informed arbitrageurs are expected to push them back, if not all the way, at least in the right direction.

While such arbitraging frictions render the price bubbles consistent with the presence of rational arbitrageurs, the origin of a bubble is still commonly attributed to animal spirits.[2] The sheer size of the most famous bubbles suggests the image of a market overtaken by bullish sentiment, with the arbitrageurs unable, in the face of the frictions, to bring the prices down. We would like to challenge this view in three progressive ways. We present a market microstructure model, in which the rational (risk-neutral and liquidity unconstrained) informed trader "rationally destabilizes" the price in equilibrium, by buying high and selling low. Second, this rational destabilization requires only an exogenous demand shock and not behavioral traders directly swayed by, say, trending prices. Third, the price may be destabilized almost exclusively by the rational arbitrageurs, that is endogenously, with hardly any exogenous shock.

In this paper we consider a dynamic asset pricing model with one asset, in which one informed trader trades against liquidity traders and competitive market makers. For the sake of simplicity we set it up as closely as possible to the seminal model by Kyle [1985]. The main difference is that in our case the informed trader has private information about the fundamental value of the asset as well as the exogenous persistent demand shock. The demand shock is the expected demand by the liquidity traders, at any point in time. While the first dimension of private information is standard (starting with Kyle [1985] and Glosten and Milgrom [1985]), the second dimension is meant to capture the "soft" information about the market, such as, for example, the extent of the common liquidity shock for the short lived rational market participants.[3]

The dynamic trade and the multidimensional private information offers a rich spec-

[1] See e.g. De long et al. [1990], De Long et al. [1990], Campbell and Kyle [1993] and Wang [1993] for the importance of risk involved, Miller [1977], Harrison and Kreps [1978], Duffie et al. [2002] for short-selling constraints, Grossman and Miller [1988], Shleifer and Vishny [1997], Brunnermeier and Pedersen [2009] for the liquidity constraints and Abreu and Brunnermeier [2003], Doblas-Madrid [2012] for the coordination failures; among others.

[2] By "price bubble" we refer to periods of large run-ups in asset prices followed by collapses, see e.g. Brunnermeier [2001] for a textbook treatment, and Kindleberger and Aliber [2011] for their classic account of the historical booms and busts. For a review of behavioral finance literature see Barberis and Thaler [2003], Shiller [2003] or Baker and Wurgler [2013]

[3] Kumar and Lee [2006] show that retail investor's tend to buy in concert, showing high correlation between the stocks they buy and the stocks that they sell, and only a small proportion of this investor sentiment can be predicated from market and macro variables. While Kumar and Lee [2006] look at within period correlation, Barber et al. [2009] show that this investor sentiment also persists over extended periods of time (more than a year).

trum of strategies to the informed trader. We show that in the unique linear equilibrium of the model the informed trader exacerbates rather than trades against the demand shocks. In this way he strategically destabilizes the price and, in the case of a positive demand shock, "pumps-up" the price bubble, often losing money in the short run. He rides the bubble for a while and reverts his position - starts "dumping" - some time before the price peak. Eventually all the private information is revealed to the market, so in particular, the price is driven back to the fundamentals. Finally, the model with just information about the fundamentals (Kyle [1985]) has expected price confined between the ex-ante expectation and the realized value. We show that even when the extent of the demand shock is arbitrarily small, it will trigger a significant price bubble, created for endogenous, strategic reasons.

In this paper a risk neutral informed trader learns at time zero both the value v of the asset as well as the extent of the demand shock m. At time one the value of the asset is revealed exogenously to the market. At times $0, \Delta, ..., 1-\Delta$ the informed trader submits his order flow and the liquidity traders submit theirs, which is drawn from a distribution with mean Δm. Market maker observes the total order flow and clears the market at a competitive price, equal to the expected value of the asset given the public history of the total order flows. Looking at the continuous-time limit, with all the variables believed to be normally distributed, we provide the analytical solution of the model. More precisely, we establish that the continuous time version of the model has a unique linear Markov equilibrium strategy of the informed trader, which is linear in the extents of market's biases about the value and the demand shock. The analytical characterization of this nonstationary dynamic equilibrium permits us a clean analysis in what follows.

As a starting point, we find that in the course of trading the informed trader reveals all the private information to the market: just before the end of trade the market essentially knows both the value of the asset and the demand shock. Thus, at least in the "long-run" the price aggregates the information about the asset, and so the market fulfills its informative role. That the trader reveals all private information about the value is fairly intuitive in our setting, and follows from the arbitrage argument: instead of ending with private information on his hands, the informed trader could have traded on it, making larger profits but also pushing the price closer to the value. That the trader can make money trading heavily on the knowledge about the demand shock - the payoff irrelevant information - just before the market shuts is less obvious. However, we show that while there is no arbitrage that yields higher profits at any point of time, there are dynamic strategies such that when, say, there is a positive demand shock, they result in net total sales and maintain the expected price above the fundamentals throughout. Crucially, they offer an arbitrage opportunity even in the last instants of trade.

With 1 dimensional uncertainty the pump-and-dump strategy is unprofitable. In our case, however, a demand shock allows the price to rise relatively quickly in the process of pumping, and fall slowly in the process of dumping. In the course of the whole "round-trip" trade the trader makes net sales, at inflated prices. Essentially, the informed trader who faces a positive demand shock chooses to pump the price up in order to service this demand at inflated prices.[4]

The informed trader's desire to exacerbate the demand shock in a dynamic setting directly contrasts the static version of the model where he acts to stabilizes the price. With a single period of trading an informed trader always acts to correct prices, buying if he expects the price to be below the asset value and selling if he expects the price to be above.[5] Thus he makes money by partially dampening the shock, selling at a price swung high by the positive demand shock and buying at a price swung low. That in the course of dynamic trading the expected price moves outside of the band between the ex-ante expected price and the realized value hinges on the informed trader using a destabilizing *pump-and-dump* strategy, as described above.

Importantly, the results carry over to the limit as we look at the sequence of models when the variance of the demand shock m shrinks to zero.[6] The price paths given a demand shock at the order of n standard deviations converge in law, as the standard deviation vanishes, to different distributions depending on n, and to a different distribution than the one in the model with $m = 0$ for sure (Kyle [1985]). Thus, an arbitrarily small demand shock can lead to a large persistent price bubble, when variance of m is small. The price is destabilized almost exclusively for strategic reasons. The empirical counterpart is that large price run-ups can be generated without the demand pressure from the liquidity traders.

There is ample evidence that some sophisticated traders pursue such destabilizing strategies. Brunnermeier and Nagel [2004] document how the most successful hedge funds, arguably the better informed and rational traders, would ride the dot com bubble by building up positions in appreciating and overpriced stocks, and eventually start

[4]Clearly, persistence of the demand shock is crucial for this argument. In a related two period model, an informed trader who knows upfront the realizations of independent demand shocks in both periods benefits by partially offsetting each and smoothing their effect across time. Prices are thus stabilized by the trader (Bernhardt and Taub [2008]).

[5]See Rochet and Vila [1994] and Lambert et al. [2014]

[6]The liquidity is not vanishing along the sequence, as we keep the per-period variance of the liquidity trade constant.

reverting position around the peak. Temin and Voth [2004] document the similar trading strategy of Hoarse Bank during the South Sea Bubble, which was purchasing the stock while at the same time refusing it as a loan collateral.[7] In contrast to the model with one dimensional uncertainty or a model with just arbitraging frictions, ours can yield the paths of asset prices and informed trader's holdings consistent with these empirical findings.

The destabilizing strategies akin to ours have been justified in dynamic asset pricing literature in models that include trend followers, or positive feedback traders: traders who expect prices to continue their trend (De long et al. [1990]). The results are related. One difference in our model is the alternative mechanism by which the rational traders benefit from pumping the price, in the absence of behavioral agents: they do not trigger the rule of thumb reaction that can be subsequently exploited, but front-run the common liquidity shock on the market. The second difference is the mechanism by which the competition between the rational agents does not wipe out the benefits of destabilization. We do not limit the competition in arbitrage by the risk involved but instead introduce asymmetric information about the demand shocks, between the informed trader and the competitive fringe of market makers.

Given a risk neutral market maker in our model, the price is the expectation of the value and so is believed to be a martingale, with no excess volatility beyond that of the value v. Thus, while the demand shock that sets off a large price bubble can be very small, it must be unexpected (see also Avery and Zemsky [1998]). One implication of the small magnitudes involved, however, is that if the distributions are estimated from the data, the discrepancies between the beliefs and the true distribution of the demand shocks seem hard to avoid. The result implies that, with small variance of m, small (in absolute terms) discrepancies between the beliefs and the true distribution of demand shocks will have large effects on the true distribution of prices, their volatility in particular.

Related Literature. A number of dynamic asset-pricing models consider the effect of multidimensional uncertainty. Most of this literature models the second dimension as uncertainty of the proportion of informed investors, uncertainty of their information quality, or uncertainty whether an information event occurred, as in Romer [1993], Avery and Zemsky [1998], Li [2012], Back et al. [2014], Banerjee and Green [2014].[8] While this

[7] See also Soros [2003].

[8] See also Rochet and Vila [1994] and recent papers by Ganguli and Yang [2009], Manzano and Vives [2011], Gao et al. [2013] and Lambert et al. [2014] for analysis of multidimensional uncertainty in static settings, and Bernhardt and Taub [2008] mentioned earlier in two period setting.

early literature emphasized the "bubble dynamics" of prices, it was due not to the strategic behavior of the informed trader, but the fact that a single price (or the order flow) cannot aggregate information from both dimensions: Second period order flow can reveal more information and bring about a large correction in price. Given no strategic effects, a small range of uncertainty can bring about only small bubbles. Also, our model shows that the multidimensional uncertainty of a related kind can be modeled in a tractable framework with normal distributions, as in the seminal paper by Kyle [1985].[9]

This paper also belongs to the literature on the asset price manipulation based solely on trade.[10] In some cases, an uninformed trader can make profits by pretending to be informed (Allen and Gale [1992] , Allen and Gorton [1992], Fishman and Hagerty [1995]). An informed trader may manipulate the price to slow down the release of his information. For example, in a paper that also builds on Kyle [1985], Foster and Viswanathan [1994] show that, in the presence of two informed traders with nested information, the better informed trader trades heavily on the common information early on to slow down the learning of the other trader about his private information (see also Chakraborty and Yılmaz [2004]). Guo and Ou-Yang [2014] show that when the informed trader has private information about the mean reverting (rather than persistent) liquidity demand, he trades against the liquidity trader's position, which slows down the release of the information to the market. In our model the manipulation has an additional feature that goes beyond the slowing down of the information release: Informed trader exacerbates the persistent demand shocks and thus pushes the price further away from the fundamentals, driving the bubble.

It is known that in the presence of risk-averse market making traders, prices lose their martingale property and exhibit much richer dynamic patterns. In particular, since risk-averse market makers only gradually absorb demand/supply shocks, prices can persistently depart from fundamentals (see e.g. De long et al. [1990], De Long et al. [1990], Campbell and Kyle [1993] and Wang [1993]). We have maintained the assumption of risk-neutrality both for analytical tractability and to isolate the analysis of the strategic destabilization, which does not require risk-averse market makers. Similarly, if the informed trader is risk-averse his trade will additionally depend on his privately known position, a different kind of "non-fundamental" private information (see Du and Zhu [2014], Guo and Ou-Yang [2014]; see also Vayanos [1999] for the model of trade based

[9] While the models differ, the crucial feature in all is that the market is trying to learn both whether, or to what extent, the order flow is driven by a demand from an informed trader, as well as well as how it reflects his private information.

[10] For other types of manipulation see Vives [2008], chapter 9.3, for an excellent textbook treatment.

solely on this insurance motive).

Finally, Kyle [1985] and Back [1992] established, based on the arbitrage arguments, that in the course of trading all the private information is revealed to the market and the price converges to the fundamental value.[11] The results have been shown to be robust to the specification of the information structures (Ostrovsky [2012]). While those papers allowed for asymmetric information about the fundamental value only, we show that arbitrage argument can be extended to establish information revelation in the case of asymmetric information about both fundamental and non-fundamental information.

This paper is organised as follows. In section 2 we describe the model, then in section 3 we characterize the unique linear equilibrium. In section 4 we describe several properties of the equilibrium, such as the informed trader's destabilization strategy, and examine the limit when the demand stock becomes small. Section 5 then concludes. In the interest of flow all proofs have been left to the appendix.

2 The Model

There is a single risky asset with a value v, which will be publicly announced at time 1. At time 0 the single risk-neutral informed trader learns v as well as the demand/supply shock m. At any time t prior to 1 the asset is traded continuously among the noise (liquidity) traders, the informed trader and competitive risk-neutral market makers. The cumulative order from the liquidity traders Z_t follows[12]

$$dZ_t = mdt + dB_t,$$

where B_t is a one-dimensional Brownian Motion. The informed trader chooses his cumulative order S_t strategically. At any time t the cumulative total order $O_t = S_t + Z_t$ is publicly observed and the asset price P_t is determined as the expected value of the asset given O_t as well as the strategy of the informed trader.

The model is designed as a continuous time version of the seminal model of insider trading by Kyle [1985] (see Back [1992]). The difference is the private information in the hands of the informed trader: in our case he has private information not only about the payoff-relevant value of the asset ("fundamentals") but also the payoff-irrelevant demand

[11] Caldentey and Stacchetti [2010] establish that the information is revealed also in the case of random length of the trading window. Back et al. [2000] establish information revelation in the case of multiple symmetrically informed traders.

[12] We normalize the instantaneous volatility σ^2 to be equal to one. This is without loss of generality: the two relevant parameters for the model are the ratios of the variances of v and m over σ^2.

shock. Intuitively, private information about the fundamental value gives informed trader direct arbitrage opportunities: purchasing when the asset is undervalued or selling when it is overvalued. Private information about the demand shock allows him to service, or "trade against" it at favorable prices. This can alternatively be seen as a factor slowing down the release of information about the asset to the market: for example, selling given an unexpected demand shock is less likely to drive the price down.

Specifically, we assume that v and m are independently normally distributed with zero means and variances σ_v^2 and σ_m^2 respectively.[13] A strategy of the insider $S = \{S_t\}_{t \in [0,1)}$ is a continuous semimartingale[14] with respect to the filtration generated by Z_t, v and m. In order to exclude the "doubling strategies" (see Back [1992]) we also require a strategy to satisfy

$$\mathbb{E}\left[\int_{[0,1]} S_t^2 dt\right] < \infty.$$

In the paper we will focus on the equilibrium strategies in the following subclass. Let $\overline{V}_t$ and $\overline{M}_t$ be the processes of market expectations of v and m, which are measurable with respect to the filtration $\mathcal{F}_t^O$ generated by the total order process upto time t, and denote

$$X_t = v - \overline{V}_t, \quad Y_t = m - \overline{M}_t.$$

A *linear Markov* strategy S satisfies

$$dS_t = (\beta_t X_t + \delta_t Y_t)\, dt,$$

for some deterministic functions β_t and δ_t in $L_2([0,t])$ for any $t < 1$. Thus, a linear Markov strategy depends on the private information and the history of trade only through two state variables X_t and Y_t, which are the (negative) extents of market's biases about v and m. Moreover, at any point of time the order flow is linear in those biases. Let Σ_t

[13] We can easily extend the results to the case of correlated m and v. Independence is made for the sake of tractability and because it makes the interpretation of some of our results easier.

[14] S_t can be written as $S_t = S_{1,t} - S_{2,t} + M_t$, where $S_{i,t}$ are positive, increasing and continuous processes, and M_t is a continuous local martingale independent of B_t (see Karatzas [1991]). The fact that M_t and B_t are independent reflects the fact that at any given moment the order flow from the informed trader is independent of ("is submitted before") the demand from the liquidity traders. We exclude discontinuous strategies for the sake of tractability only: we comment in the proof of Lemma 3 that the informed trader would not benefit from discrete orders (see also Back [1992]).

be the market' posterior covariance matrix

$$\Sigma_t = \begin{bmatrix} \sigma_{11,t}^2 & \sigma_{12,t} \\ \sigma_{12,t} & \sigma_{22,t}^2 \end{bmatrix} = \mathbb{E}\left[\begin{array}{cc|c} X_t^2 & X_t Y_t & \\ X_t Y_t & Y_t^2 & \end{array} \mathcal{F}_t^O \right]. \tag{1}$$

We will assume that $P_t = \overline{V}_t$, which is motivated by the competitive fringe of the market makers. Finally, for a strategy S and a price process P the wealth of the informed trader at time $T \leq 1$ is defined as (see Back [1992])

$$W_T = \int_{[0,T]} (v - P_t)\, dS_t - [P, S]_T.$$

The term $[P, S]_T$ is the cross-variation process, whose differential is usually written as $dP_t dS_t$. It captures the fact that a large order - in the sense of having a positive quadratic variation - by the informed trader at time t affects the execution price at time t, equal to $P_{t-} + dP_t$, just like in discrete time models.[15] On the other hand, if the process S has differentiable paths, as in the case of a linear Markov strategy, we have $[P, S]_T = 0$.

2.1 Discrete Time Model

In the paper we will occasionally refer to the following discrete time version of the model. The model is parametrized by the period length $\Delta = 1/N$, $N \in \mathbb{N}$, together with the parameters of the continuous time model above, σ^2, σ_v^2 and σ_m^2. In each period $t = 0, \Delta, 1 - \Delta$ the timing of the game is as follows: First, the informed trader submits the order $s_t(\text{``} = dS_t'')$; then the liquidity traders submit their order $z_t(\text{``} = dZ_t'')$, which is drawn from $N(m\Delta, \Delta)$; finally at the end of period t the price P_t is set competitively to be $\overline{V}_t$, as in continuous time. A linear Markov strategy in period t thus takes the form

$$s_t = \beta_t \left(v - \overline{V}_{t-\Delta} \right) + \delta_t \left(m - \overline{M}_{t-\Delta} \right). \tag{2}$$

The wealth of informed trader at the end of period T is $W_T = \sum_{t=0}^{T} (v - P_t)\, s_t$.

We cannot hope to obtain tractable results in the discrete time framework for any period length. However, let us solve the simple static version of the model (with period

[15] If instead we assumed that the current order dX_t is executed at the "past" price P_{t-}, then the informed trader could freely destabilize the market estimates by adding a Brownian component to his strategy, and so freely create private information. Indeed, the value function in Theorem 1 is a semi-positive quadratic function in the state variables $v - \overline{V}_t$ and $m - \overline{M}_t$ (see also Kyle '85). Therefore, without the cross-variation component the informed trader would strictly prefer to destabilize the market estimates in this way.

length $\Delta = 1$) in order to develop some basic intuitions about the strategic behavior of the informed trader. The linear strategy in this setting is simply a strategy that is linear in the realized value and the demand shock, since the ex-ante expectations are fixed and equal to zero. An equilibrium strategy s_0 is a strategy that maximizes informed trader's payoff given that the price is equal to the expected value conditional on the realized order flow and the strategy s_0 being used.

Lemma 1 *In the static model there is a unique linear equilibrium strategy. The strategy, price and the wealth of the informed trader are given by*

$$\begin{aligned} s_0 &= \beta v + \delta m, \\ p &= \lambda \left(s_0 + m + \varepsilon \right), \\ \mathbb{E}\left[W_1 | S, v, m \right] &= \frac{\beta}{2} \left(v - \lambda m \right)^2, \end{aligned}$$

where

$$\lambda = \left(\frac{\sigma_v^2}{\sigma_m^2 + 4} \right)^{0.5}, \quad \beta = \frac{1}{2\lambda}, \quad \delta = -0.5.$$

A few properties are worth pointing out. First, the informed trader benefits from the known demand shocks. This is because his payoffs are proportional to the squared difference between the value of the asset v and the price λm that would result if he abstained from trading. The demand shocks destabilize this price, increasing his profits. It is also easy to verify the comparative statics result that informed trader's expected equilibrium profits are increasing in the variance of the demand shock σ_m^2.

Second, the informed trader benefits if the shocks to the value v and the demand m have opposite signs: when the asset is undervalued, in which case he wants to buy it, and there is a supply shock (or vice versa). One way to interpret it is that in this case his demand for the asset will be undetected by the market maker, and so will affect the price less. Alternatively, with v positive a supply shock will increase the difference between the value and the price λm that results if he abstains from trading.

Third, δ equal to negative half means that the informed trader trades against, or dampens the demand shocks. This again should be intuitive. If, say, $v = 0$, a demand shock will push the price above the value, and so in order to make profits the informed trader will submit a partially offsetting sell order. This offsetting by the informed trader means that fixing the magnitude of the demand shock $m + \varepsilon$, the larger the share of the unexpected part of the shock ε the bigger effect on the price it will have. On the other hand, the demand shocks are not fully offset by the informed trader, and so a demand shock will push the price up, and the supply shock - down.

One last aspect to consider is the information possessed by a liquidity trader. Each liquidity trader has an order $z_t \sim N(m\Delta, \Delta)$ which they service in the market. This order will allow the liquidity trader to update about the demand shock, m, or the value of the asset, v and potentially possess better information than the market. However, as Δ becomes small and we approach the continuous time model, each liquidity trader need have no additional information about the demand shock or the value of the asset. As the discrete model approaches continuous time, the idiosyncratic component of a liquidity trader's order overwhelms the systematic component generated by the demand shock, m. This means that in the limit the order of a single liquidity trader provides no additional information about the demand shock or the asset value that could be used in the market to make profit.[16]

3 Equilibrium

Let us focus back on the continuous-time model. We are interested in equilibrium strategies of the informed trader defined as follows. Such a strategy must be optimal, starting from any time t onwards (and conditional on any history), given that market maker assumes that the strategy is indeed being followed. More precisely, fix a strategy S and let $P_t = \overline{V}_t$ be the process of expected values if the market believes the informed trader uses S. We will say that S is an *equilibrium strategy* if for any other strategy $\widetilde{S}$ and time $t < 1$[17]

$$\mathbb{E}\left[\int_{[t,1]} (v - P_r)\, dS_r - [P,S]_{[t,1]} | S, v, m, \mathcal{F}_t^Z\right] \geq \mathbb{E}\left[\int_{[t,1]} (v - P_r)\, d\widetilde{S}_r - [P,\widetilde{S}]_{[t,1]} | \widetilde{S}, v, m, \mathcal{F}_t^Z\right]. \quad (3)$$

In the rest of this section we will prove the following result, which is one of the main results of the paper. We postpone the detailed analysis of the result until Section 4.

[16] To illustrate this, fix the market filtration, $\mathcal{F}_t$, at time t and consider a sequence of random variables z_n where $z_n \sim N\left(\frac{m}{n}, \frac{1}{n}\right)$. Let $F(v, m|.)$ be the conditional cdf of v and m, then as n goes to infinity $F(v, m|\mathcal{F}_t, z_n)$ converges almost surely to $F(v, m|\mathcal{F}_t)$.

[17] All the equalities and inequalities between the random variables in the paper are understood as holding almost everywhere. For example, the following inequality must hold for almost every history of the noise traders' order flow.

Theorem 1 *There exists a unique linear Markov equilibrium strategy S. The equilibrium is characterized by*

$$
\begin{aligned}
dS_t &= (\beta_t X_t + \delta_t Y_t)dt, \\
dX_t &= -\lambda_t \left(dS_t + Y_t dt + dB_t\right), \\
dY_t &= -\phi_t \left(dS_t + Y_t dt + dB_t\right), \\
\mathbb{E}\left[W_1 | S, v, m, \mathcal{F}_t^Z\right] &= aX_t^2 + 2b_t X_t Y_t + cY_t^2 + d_t,
\end{aligned}
$$

where the deterministic parametres $a, b_t, c, d_t, \beta_t, \delta_t, \lambda_t$ and ϕ_t are such that:

(Value) *$b_t = (t/2 - 1/4)$, a and c are the unique solution with $a, c > 0, ac > 1/16$ to*

$$
\frac{1}{\sqrt{ac} - \frac{1}{4}} - \frac{1}{\sqrt{ac} + \frac{1}{4}} + \frac{1}{\sqrt{ac}}\left[ln(\sqrt{ac} - \frac{1}{4}) - ln(\sqrt{ac} + \frac{1}{4})\right] = 4\sigma_m^2, \tag{5}
$$

$$
\frac{1}{\sqrt{ac} - \frac{1}{4}} - \frac{1}{\sqrt{ac} + \frac{1}{4}} - \frac{1}{\sqrt{ac}}\left[ln(\sqrt{ac} - \frac{1}{4}) - ln(\sqrt{ac} + \frac{1}{4})\right] = \frac{4a}{c}\sigma_v^2,
$$

and d_t is the solution to $d_1 = 0$ and $dd_t = -\left(a\lambda_t^2 + 2b_t\lambda_t\phi_t + c\phi_t^2\right) dt$;

(Learning) *λ_t and ϕ_t are defined as*

$$
\lambda_t = \frac{c}{2(ac - b_t^2)}, \ \phi_t = \frac{-b_t}{2(ac - b_t^2)}; \tag{6}
$$

while the market's posterior covariance matrix Σ_t is the unique solutions of the ordinary differential equation

$$
d\Sigma_t = -\begin{bmatrix} \lambda_t^2 & \lambda_t\phi_t \\ \lambda_t\phi_t & \phi_t^2 \end{bmatrix} dt, \tag{7}
$$

with the initial conditions $\sigma_{11,0}^2 = \sigma_v^2$, $\sigma_{22,0}^2 = \sigma_m^2$ and $\sigma_{12,0} = 0$;

(Strategy) *β_t and δ_t are the solutions to*

$$
\begin{bmatrix} \lambda_t \\ \phi_t \end{bmatrix} = \Sigma_t \begin{bmatrix} \beta_t \\ \delta_t + 1 \end{bmatrix}. \tag{8}
$$

The proof of the Theorem proceeds in several steps. First, Lemma 2 is fairly standard and characterizes the Bayesian learning by the market makers when the informed trader follows a linear Markov strategy.

Lemma 2 *Fix a linear Markov strategy S with parameters β_t and δ_t that market believes* *the informed trader is using. If the informed trader uses a strategy $\widetilde{S}_t$ then the market biases follow equations*

$$dX_t = -\lambda_t(d\widetilde{S}_t + Y_t dt + dB_t), \tag{9}$$
$$dY_t = -\phi_t(d\widetilde{S}_t + Y_t dt + dB_t).$$

The gain functions λ_t and ϕ_t are defined by (8), where the market makers posterior covariance matrix Σ_t is the unique soluton to (7) with the initial conditions $\sigma^2_{11,0} = \sigma^2_v$, $\sigma^2_{22,0} = \sigma^2_m$ and $\sigma_{12,0} = 0$.

In the case when $\widetilde{S} \equiv S$ the result is an application of the Kalman - Bucy filter (see Liptser and Shiryaev [2000]). For any other strategy $\widetilde{S} \neq S$ the result follows from the Girsanov's Theorem (see Karatzas [1991]). Therefore, instead of thinking of a deviation strategy $\widetilde{S}$ as changing the measure over the paths of total order O_t and the biases X_t and Y_t (as in (3)), one can simply think of it as steering or changing the drift of those biases, thought of as state variables. Put differently, the informed trader chooses his trading strategy to maximize the integral of of his flow payoffs $X_t d\widetilde{S}_t - dP_t d\widetilde{S}_t$, given the law of motions in (9), for fixed λ_t and ϕ_t.

For any strategy $\widetilde{S}$ let $[\widetilde{S}]_t$ be its quadratic variation process. Note that for any linear Markov strategy $\widetilde{S}$ we have $[\widetilde{S}] \equiv 0$.

Lemma 3 *Fix a linear Markov equilibrium strategy S and let $T^* \leq 1$ be the first time when X_t and Y_t are perfectly correlated, i.e., $\det \Sigma_t = 0$. Then there are $a, b, c > 0$ such that with $b_t = b + t/2$*

i) for any strategy $\widetilde{S}$ and any $t < T < T^*$ we have

$$\mathbb{E}\left[\int_{[t,T]} (v - P_r)\, d\widetilde{S}_r - [P, \widetilde{S}]_{[t,T]} | \widetilde{S}, v, m, \mathcal{F}^Z_t\right] = aX_t^2 + 2b_t X_t Y_t + cY_t^2 + d_t - \tag{10}$$
$$- \mathbb{E}\left[aX_T^2 + 2b_T X_T Y_T + cY_T^2 + d_T | \widetilde{S}, v, m, \mathcal{F}^Z_t\right] - \mathbb{E}\left[\int_{[t,T]} \frac{\lambda_r}{2} d[\widetilde{S}]_r | \widetilde{S}, v, m, \mathcal{F}^Z_t\right];$$

ii) at any $t < T^*$ the gain functions satisfy (6), where d_t is defined by $d_T = 0$ and $dd_t = -\sigma^2\,(a\lambda_t^2 + 2b_t\lambda_t\phi_t + c\phi_t^2)\, dt$.

The main point of the Lemma is that in equilibrium, as long as the biases are not perfectly correlated, the informed trader must be indifferent between any strategies with $[\widetilde{S}] = 0$ that reveal the same amount of information to the market, in the sense that they result in the same covariance matrix of X_T and Y_T (compare Kyle [1985], Back [1992]). It also shows that using a strategy with $[\widetilde{S}] \neq 0$ is suboptimal.

The indifference constrains the gain parameters λ_t and ϕ_t as well as the parameters a, c and b_t in the payoff function. For example, when $Y_t = 0$ and the informed trader abstains from trading the market biases have no drift. Thus, indifference between trading now and abstaining for an instant requires a to be constant. The case of c and b_t is slightly more complicated and relies on the fact that the state variable Y_t and $d\widetilde{S}_t$ affect the biases in the same way. For example, in the case when only $X_t = 0$ and the informed trader abstains from trading the market biases will drift. However, the effect of this drift on the state variables and flow payoff is the same as the effect of $d\widetilde{S}_t$, and so must be equal to zero. Thus, to have the informed trader indifferent between trading now and abstaining from trade c must be constant. In Section 4 we discuss the properties of the equilibrium strategy in more detail.

For the rest of the proof we will use the following technical Lemma.[18] The implication is that in any linear Markov equilibrium, as long as the market does not believe m and v to be perfectly correlated (and so the gain parameters λ_t and ϕ_t are not colinear - see (6)), the informed trader can drive the biases X_t and Y_t to any arbitrary vector.

Lemma 4 *Fix two bounded functions f_t and g_t on $[0,1]$ and consider two processes X_t and Y_t that solve*

$$dX_t = f_t\left(u_t dt + dB_t\right),$$
$$dY_t = g_t\left(u_t dt + dB_t\right),$$

where u_t is a $\mathcal{F}_t^{X,Y}$-measurable control and B_t is a Brownian Motion. Suppose that for every $T < 1$ f_t and g_t are not colinear on $[T, 1]$. Then for every $x, y \in \mathbb{R}$ and $\varepsilon > 0$ there exists a control u_t such that X_t and Y_t satisfy

$$\mathbb{E}[X_1] = x,\ \mathbb{E}[(X_1 - x)^2] < \varepsilon,$$
$$\mathbb{E}[Y_1] = y,\ \mathbb{E}[(Y_1 - y)^2] < \varepsilon.$$

The following Lemma shows that in linear Markov equilibrium the market cannot believe before the end of trading that m and v are perfectly correlated. Otherwise, the

[18] It would not be difficult to strengthen the result to avoid ε in the statement and have convergence a.e. The weaker result below is sufficient and, in our view, more instructive. Also, the generalization to more dimensions is straightforward.

informed trader could make arbitrarily high profits by first fooling the market that, say, the demand shock is greater than the truly realized one (see Lemma 4). Then, once the market stops learning independently about the demand shock, the trader would drive the price arbitrarily low for a long period of time and purchase the asset undetected by the market, proxying in for the demand shock.

Lemma 5 *Fix a linear Markov equilibrium strategy S and let Σ_t be the market makers' posterior covariance matrix (see (1)). Then, for every $t < 1$ $det\Sigma_t \neq 0$, i.e., the equilibrium biases X_t and Y_t are not colinear.*

The following Theorem is one of the first main result of this section.

Theorem 2 *In any linear Markov equilibrium all the information is revealed by the end of trading*

$$\lim_{t\to 1} \Sigma_t = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

That the informed trader reveals all the information about the value ($\sigma^2_{11,1} = 0$) is fairly intuitive. Note that private information about v is directly payoff relevant and offers easy arbitrage: If, say, the informed trader knows that the asset is undervalued he can use this information and make money by buying the asset. By deviating to a strategy that trades more aggressively on this information just before the end of trade he can increase his flow profits at any point of time.[19]

The argument why the information about the payoff irrelevant demand shock is revealed as well ($\sigma^2_{22,1} = 0$) is more complicated and, to the best of our knowledge, new. Note that the demand shock is payoff irrelevant, and so private information about it does not offer direct arbitrage opportunities as above. In particular, when just the information about the value is revealed, there is no deviating arbitrage strategy that offers strictly higher expected flow payoff at any time.[20] There exists, however, an opportunity for a more complicated *dynamic* arbitrage that constitutes an overall profitable deviation, while decreasing flow profits at some points of time. One of the main results of the

[19] More formally, under the simple arbitraging strategy with the flow trade DX_t for $D >> 0$, the expected flow payoff and so the drift of $-(a_t X_t^2 + 2b_t X_t Y_t + c_t Y_t^2)$ equals DX_t^2. Thus, in the case when not all the information is revealed the simple arbitraging just before the end of trade constitutes a profitable (while typically not the optimal) deviation.

[20] A deviating strategy will increase expected flow profits exactly when it reveals information about v quicker. Achieving this with probability one is impossible, since the original strategy reveals all the information about v.

paper (Proposition 1) shows that such a strategy is used by the insider in equilibrum throughout the whole trade, and we postopone the discussion of it until Section 4.

We are now ready to establish Theorem 1. We show below that there is a unique linear Markov strategy that satisfies the conditions of Lemmas 2, 3 and the Theorem 2 - indifference, no colinearity and information revelation. On the other hand, we verify that this is indeed an equilibrium strategy.

Verification. It is easy to verify that the equations (5) are equivalent to

$$\sigma_v^2 = \int_{[0,1]} \lambda_t^2 dt = \int_{[0,1]} \frac{c^2}{4(ac - b_t^2)^2} dt, \qquad (11)$$
$$\sigma_m^2 = \int_{[0,1]} \phi_t^2 dt = \int_{[0,1]} \frac{b_t^2}{4(ac - b_t^2)^2} dt.$$

In other words, given (7) in Lemma 2, the strategy S reveals all the information to the market (if it is believed to be folowed). Also, $[S] \equiv 0$. Therefore it follows from the formula for the expected profits in part i) of Lemma 3 and the positive definiteness of the matrix $\begin{bmatrix} a & b_1 \\ b_1 & c \end{bmatrix}$ that the strategy S is optimal.

Uniqueness. Lemma 2 establishes that β_t and δ_t is uniquely pinned down by λ_t, ϕ_t and Σ_t by (8), and that Σ_t in turn is pinned down by λ_t and ϕ_t. Lemmas 3 and 5 establish the form of the value function as well as λ_t and ϕ_t for some a, c, b_t with $db_t = 0.5dt$ and d_t as in the Theorem. Consequently, the three "free" parameters, through which all the other parameters in the equilibrium are defined, are a, b_0 and c. (7) in Lemma 2 and Theorem 2 imply that those parameters must solve (11) together with

$$0 = \int_{[0,1]} \lambda_t \phi_t dt = \int_{[0,1]} \frac{-cb_t}{4(ac - b_t^2)^2} dt.$$

It is easy to see that for the above equation to be satisfied it must be that $b_0 = -1/4$, and so $b_t = (t/2 - 1/4)$. Given that, equations (11) are equivalent to equations (5). Lemma 7 in the Appendix shows that for the given σ_v^2 and σ_m^2 equations (5) have a unique solution (a, c) with $a, c > 0, ac > 1/16$. This concludes the proof of the uniqueness and the theorem.

4 Rational Destabilization

In this section we investigate in detail the strategic response of the informed trader to the demand shocks, and its implications on prices and asset holdings.

Just as in the static model, the informed trader benefits if the demand shocks are large and if v and m have opposite signs. To see the effects of large shocks lets focus on the strategy that abstains from trading until (just before) time 1 (see Lemma 3). Demand shock Y_0 adds a drift to the price, which can be eventually exploited by the informed trader. Indeed, conditional on no trading by the informed trader, we have $E[X_1^2]$,$E[X_1 Y_1]$ and $E[Y_1^2]$ linear in Y_0^2, with all of the coefficients strictly positive.

Let us introduce the following change of variables. Fix the equilibrium strategy S from Theorem 1 together with the covariance matrix function Σ_t. Let $H_t = \frac{\sigma_{12,t}}{\sigma_{11,t}^2}$ and define

$$\widehat{Y}_t = Y_t - \frac{\sigma_{12,t}}{\sigma_{11,t}^2} X_t$$

to be the orthogonalized part of Y. In other words, we have $\mathbb{E}[\widehat{Y}_t X_t | \mathcal{F}_t^O] = 0$, for any $t < 1$.

Lemma 6 *For the equilibrium strategy S as in Theorem 1 we have*

$$\mathbb{E}\left[W_1 | S, v, m, \mathcal{F}_t^Z\right] = \widehat{a}_t X_t^2 + 2\widehat{b}_t X_t \widehat{Y}_t + c\widehat{Y}_t^2 + d_t, \tag{12}$$

where $\widehat{a}_t = a + 2b_t H_t + cH_t^2$ and $\widehat{b}_t = b_t + cH_t$, with a, b_t, c as in Theorem 1. In particular,

$$\widehat{b}_t < 0, \ \forall t < 1.$$

The logic behind $\widehat{b}_t < 0$ is the same as in the static setting. When the asset is, say, undervalued and the demand from the liquidity traders overestimated, the informed trader can satisfy his demand with relatively little response of the price. An easy corollary to the Lemma is that the informed trader does not fully absorb the demand shock, just as in the static model.

Corollary 1 *In the equilibrium strategy as in Theorem 1 we have $\delta_t > -1$, for all t.*

Recall that in the static setting the informed trader was making money by stabilizing the demand shocks, $\delta < 0$. The dynamic setting, however, offers him many more ways to respond to the shock. In particular, while in the static case the magnitudes and the signs of the market biases are a matter of luck, in the dynamic setting the informed trader can choose to "drive" the biases of the market in the desired regions. For example, instead of trading against the demand shock, as in the static setting, the trader can choose to first exacerbate the shock and thus destabilize the price even further. The following Proposition is the main result of this section.

Proposition 1 *In the equilibrium strategy as in Theorem 1 we have $\delta_t > 0$, for all t.*

To provide the intuition for the proof let us consider a discrete time model with two periods, $t = 0, 1/2$. It will be useful to write down the linear Markov strategy s_t, $t = 0, 1/2$, as

$$s_t = \widetilde{\beta}_t(v - \widetilde{P}_t) + \widetilde{\delta}_t(m - \widetilde{M}_t), \tag{13}$$

where $\widetilde{P}_t$ and $\widetilde{M}_t$ are the expected price and the demand shock estimate at the end of period t if the informed trader abstains from trading,

$$\widetilde{P}_t = P_{t-\Delta} + \lambda_t(m - \overline{M}_{t-\Delta}),$$
$$\widetilde{M}_t = \overline{M}_{t-\Delta} + \phi_t(m - \overline{M}_{t-\Delta}).$$

The reparametrization is motivated by the continuous time model, in which "$\widetilde{P}_t = P_{t-\Delta}$" and "$\widetilde{M}_t = \overline{M}_{t-\Delta}$" (as long as the strategy has no quadratic variation). Thus, say, the continuous time state variable X_t can be alternatively interpreted as $v - P_t$ or $v - \widetilde{P}_t$.

First, it is easy to see that $\widetilde{\delta}_{1/2} = 0$. This is because with this reparametrization the static benefits from trading against the demand shock $m - \widetilde{M}_t$ disappear. The same is true in the continuous time model.

Second, just as in the static model analyzed in section 2.1, the expected payoff at the beginning of the second period is proportional to the squared difference between the value of the asset v and the price $\widetilde{P}_{1/2}$

$$\mathbb{E}\left[W_1 | v, m, \varepsilon_0, s_0, s_{1/2}\right] = a_{1/2}(v - \widetilde{P}_{1/2})^2,$$

where $a_{1/2} > 0$. In equilibrium both the demand shock m and the order flow s_0 in the first period typically move the price $\widetilde{P}_{1/2}$ in the same direction.[21] Thus, in the dynamic

[21]The price is given by

$$\widetilde{P}_{1/2} = \lambda_0\left(s_0 + m + \varepsilon_0\right) + \lambda_{1/2}\left(m - M_{1/2}\right) =$$
$$= \left(\lambda_0 - \lambda_{1/2}\phi_0\right)\left(s_0 + m + \varepsilon_0\right) + \lambda_{1/2} m,$$

model the informed trader has an additional benefit from accentuating the demand shock in the first period, since it helps further destabilize the price $\widetilde{P}_{1/2}$ in the second period. This effect is absent in the static model. Here it results in delta decreasing over time, and so $\widetilde{\delta}_0 > \widetilde{\delta}_{1/2} = 0$.

In continuous time, the proof is more complicated for the following reason. The intensity of trade on private information, β_t and δ_t, must be proportional to the inverse of $\det \Sigma_t$ and so, roughly speaking, the amount of the outstanding private information. In particular, since in continuous time the private information must be revealed at the end (and so $\det \Sigma_t = 0$, see Theorem 2), δ_t and β_t converge to infinity at the end of trading. This is the "Brownian Bridge" property of the prices familiar from Kyle [1985]. This statistical effect counterbalances the strategic force described above, which pushed delta to be decreasing. The proof of the Proposition establishes that δ_t scaled by the positive $\det \Sigma_t$ is decreasing and equal to zero at the end of trade.

An immediate implication of $\delta_t > 0$ is that for a fixed level of a demand shock, its effect on the price is increasing in the extent to which it was expected by the informed trader. More precisely, for the linear Markov equilibrium strategy S, any time $t < 1$ and any magnitude of the positive order flow from the liquidity traders $dZ_t > 0$

$$dP_t|S, dZ_t, m > dP_t|S, dZ_t, m',$$

for any two levels of expectations by the informed trader $m > m' > 0$. In other words, in our model with private information monopolized by a single informed trader, the arbitraging behavior exacerbates instead of stabilizing the demand shocks' effect on price, in the short term.

The informed trader's equilibrium strategy of accentuating the demand shocks is reminiscent of the *pump-and-dump* strategy. Consider the event when $v = 0$ but there is a positive demand shock, $m > 0$. Figure 1 illustrates the price paths and the informed trader's asset holdings. The price starts at 0 and climbs up, in response to the exogenous demand shock. Initially the trader purchases the asset, pumping the price even more. Once the price climbs sufficiently high, the negative component in the strategy $\beta_t(v - P_t)$ corresponding to the trade on the fundamental information dwarfs the positive component $\delta_t(m - \overline{M}_t)$ that corresponds to the trade on the non-fundamental information. In other words, the informed trader starts dumping the asset. Note that the informed trader starts

and so this will be true as long as $(\lambda_0 - \lambda_{1/2}\phi_0) > 0$.. It can be verified that $(\lambda_0 - \lambda_{1/2}\phi_0) > 0$ for the linear Markov equilibrium strategy.

selling *before* the price reaches its peak (pushed along by the shock), as in the trading pattern of hedge fund portfolios during the tech bubble documented by Brunnermeier and Nagel [2004]. The selling pressure drives the price down, eventually back to the true realized value $v = 0$ (see Theorem 2). Overall, the price is inflated for two reasons. One is the exogenous demand shock; the other is the the strategic response by the informed trader.

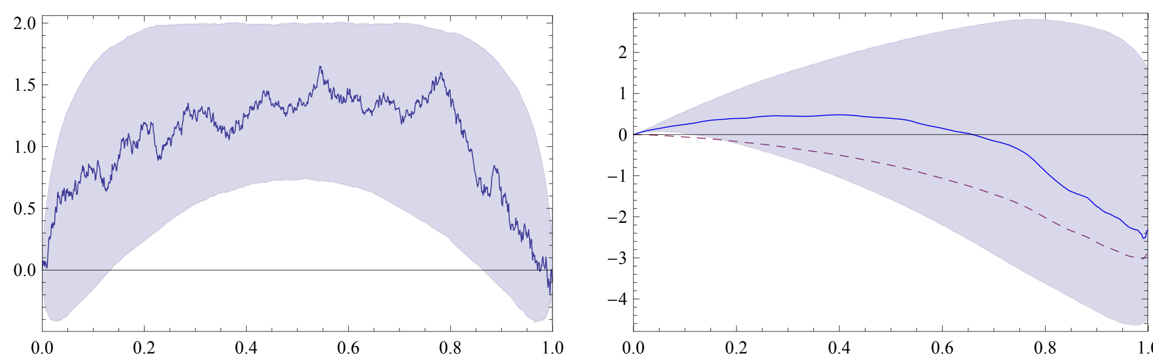

Figure 1: The left panel shows a realized price path and the 99.8% confidence interval for $v = 0$ and $m = 3$. The right panel shows the corresponding asset holdings by the informed investor (solid) and the 99.8% confidence interval and the asset holdings by the informed investor in the model with one dimensional uncertainty (dashed). $\sigma_v^2 = \sigma_m^2 = 1$

The rationale behind the pump-and-dump strategy can be summarized as follows. The informed trader, knowing about the price insensitive demand shock[22] buys along and pushes the price even higher, in a sense frontrunning it. Doing so allows him to service the exogenous demand shock, or sell against it later, at inflated prices. More precisely, given the exogenous positive demand shock the price rises relatively quickly during the pumping, helped by the shock, and falls slowly during the dumping. In the course of the whole trade the informed trader makes net sales, and all along at prices above the value of the asset. Let us also point out that without the second dimension of uncertainty (see Kyle [1985]) such a pump-and-dump strategy is not profitable. In this case the net sales are zero and the losses incurred in the initial period of driving the price up are exactly offset by the additional profits of selling the asset later on.

The pump-and-dump strategy used by the informed trader has clear empirical implications on the joint process of prices and his asset holdings or profits. Recall that in the

[22]See XXXX last section.

model with one dimension of uncertainty, for the realized value v the informed trader always trades to correct the mispricing of the asset (Kyle [1985], Theorem 3):

$$\frac{dS_t^K}{dt} = \beta_t^K (v - P_t),\ \beta_t^K > 0, \text{ for } t \in [0, 1).$$

This implies that with probability one the change in the asset holdings has the opposite sign to the market bias $(P_t - v)$ or, equivalently, the difference between P_t and P_1 (see Theorem 2)

$$\frac{dS_t^K}{dt} \times (P_t - v) = -\beta_t^K (v - P_t)^2 < 0, \text{ for } t \in [0, 1). \tag{14}$$

In particular, the bubble ("inverse-U") paths of prices that exceed the value of the asset have the informed trader deepening his short position on the asset throughout (see Figure 1).

This relationship of prices and asset holdings can be restated in terms of the flow profits. Formula (14) says that in the model with one dimension of uncertainty the flow profits of the informed trader are always positive.

In our model the informed trader no longer trades just to correct the mispricing and his flow trade can have the same sign as the market bias. In particular, for the bubble paths of prices exceeding the asset value the informed trader might initially be "riding the bubble", buying the asset in the process of pumping (see Figure 1). In terms of profits, this exhibits a persistent form of contrarian behavior, as he is incurring short term losses in the initial periods. The losses are recovered later on in the process of dumping, when he starts reverting his position.

Formally, while Theorem 1 fully characterizes the joint distribution of prices and asset holdings, we have the following implication:

Proposition 2 *Let S be the linear Markov equilibrium strategy.*

i) At any time t there is a positive probability that the informed trader rides the bubble:

$$P\left(\frac{dS_t}{dt} \times (P_t - v) > 0 | S\right) > 0;$$

ii) Fix v and m. If $|m|$ is sufficiently high relative to $|v|$ then there is $T > 0$ such that the informed trader is expected to ride the bubble until time T :

$$\mathbb{E}\left[\frac{dS_t}{dt} \times (P_t - v) | S, v, m\right] > 0, \text{ for } t \in (0, T].$$

4.1 Vanishing Second Dimension of Uncertainty

In this section we consider models with vanishing asymmetric information about the mean demand from the liquidity traders. While we maintain the assumption that only the informed trader knows the extent of the demand shock (asymmetric information), we send the variance of demand shocks σ_m^2 to zero. Thus, in some weak sense, the models approximate the model with only one dimension of uncertainty (Kyle [1985]). The players are *almost sure* that the order from liquidity trader follows *almost* a Brownian Motion (while there is never *common knowledge* that it is *exactly* so). We will see that even negligible amount of asymmetric information will have discontinuous effect on the price paths and asset holdings.

Consider models parametrized by the variance of the demand shock σ_m^2, with the asset value v and the demand shock $m = \alpha\sigma_m$, for some α. The α effectively measures how surprising the shock is, irrespectively of its magnitude. Note that for any α, as $\sigma_m \to 0$ the "real" effect of the shock vanishes and the process of cumulative order from the liquidity traders Z_t converges in distribution to a Brownian Motion.

In the absense of asymmetric information, the effects of the demand shocks are negligible when σ_m is small. In fact, recall that in the model with one dimension of uncertainty the process of the equilibrium (negative) market bias $X_t^K = v - \overline{V}_t = v - P_t$ follows (Kyle [1985], Theorem 3)

$$dX_t^K = -\left(\frac{1}{\sigma_v(1-t)}X_t^K dt + \sigma_v dZ_t\right), \quad X_0^1 = v. \tag{15}$$

If the demand from liquidity traders follows $dZ_t = \alpha\sigma_m dt + dB_t$, the process of market biases X_t^K (and so prices) has an additional drift of $-\alpha\sigma_v\sigma_m dt$, and so converges to the process in the equilibrium of the Kyle model, as $\sigma_m \to 0$.

However, asymmetric information about the demand shocks and the strategic response by the informed trader upsets this continuity result.

Proposition 3 *i) In the equilibrium strategy as in Theorem 1 we have*

$$\delta_t = \frac{1}{\sigma_m} \times \frac{\sqrt{3}}{(1-t)^2} + O(1), \qquad \beta_t = \frac{1}{\sigma_v} \times \frac{1+2t}{(1-t)^2} + O(\sigma_m),$$
$$\lambda_t = \sigma_v + O(\sigma_m), \qquad \phi_t = \sqrt{3}\sigma_m(1-2t) + o(\sigma_m),$$
$$a = \frac{1}{2\sigma_v} + O(\sigma_m), \qquad c = \frac{\sigma_v}{4\sqrt{3}\sigma_m} + O(1).$$

ii) Consider models with the asset value v and the demand shock $m = \alpha\sigma_m$, for some fixed α. As $\sigma_m \to 0$ the processes of the equilibrium market biases X_t and Y_t/σ_m converge in distribution to the solutions X_t^* and Y_t^* of

$$dX_t^* = -\sigma_v \left(\left(\frac{(1+2t)}{\sigma_v (1-t)^2} X_t^* + \frac{\sqrt{3}}{(1-t)^2} Y_t^* \right) dt + dB_t \right), \qquad X_0^* = v$$

$$dY_t^* = -\sqrt{3}\,(1-2t) \left(\left(\frac{(1+2t)}{\sigma_v (1-t)^2} X_t^* + \frac{\sqrt{3}}{(1-t)^2} Y_t^* \right) dt + dB_t \right). \quad Y_0^* = \alpha$$

Corollary 2 *i) Fix v and α as in Proposition (3). The expectations $\overline{X}_t$ and $\overline{Y}_t$ of the limits of the equilibrium market biases X_t^* and Y_t^* are given by*

$$\overline{X}_t = v - (v + \sqrt{3}\alpha)t + \sqrt{3}\alpha t^2,$$
$$\overline{Y}_t = \alpha - \sqrt{3}(v + \sqrt{3}\alpha)t + (\sqrt{3}v + 6\alpha)t^2 - 4\alpha t^2.$$

In particular, for $v = 0$ and $\alpha > 0$ the expected limit price path $p_t = -\overline{X}_t$ is a strictly concave function with $p_0 = p_1 = 0$.

ii) Fix value v of the asset. The process X_t^K of the market biases in the model with one dimensional uncertainty has a different distribution than the limit process X_t^* of the equilibrium market biases, for any α.

Part ii) of the Proposition says that even as the exogenous shocks become negligible (σ_m small), they have significant influence on the distribution of prices. This is because as the shocks vanish the informed trader accentuates them more ($\delta_t \sim \frac{1}{\sigma_m}$, part i) of the Proposition), and so their overall effect on the order flow, which is proportional to $\delta_t \times \sigma_m$, does not disappear. Put otherwise, for small σ_m the trade, and so the prices can be destabilized for purely strategic reasons.

The intuition for $\delta_t \sim \frac{1}{\sigma_m}$ is in turn as follows. If instead δ_t was of order one the market would not be able to learn the demand shock. This is because the order flow would be dominated by the trade based on the mispricing of the asset (formally, the learning parameters λ_t and ϕ_t would be determined by β_t - see (8)). But then this would open the door to a profitable deviation to a pump-and-dump strategy: first inflate the price and then trade against the (undetected) demand shock. Put otherwise, in

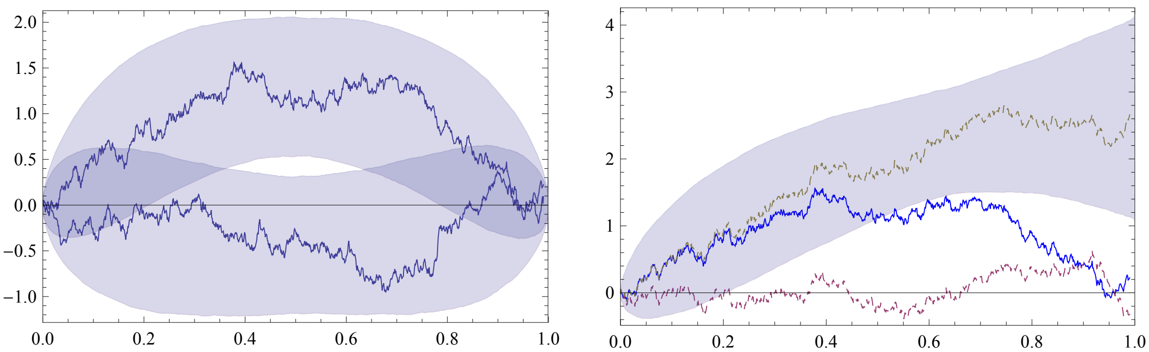

Figure 2: The left panel shows realized price paths and the 99.8% confidence intervals for $v = 0$ and either $m = 3\sigma_m$ (above) or $m = -\sigma_m$ (below). The right panel shows realized price path (solid) and the corresponding asset holdings by the liquidity traders in our model (lower dashed) for $v = 0$ and $m = 3\sigma_m$. Compare with the liquidity traders' holdings in the model with one dimensional uncertainty: for the same price path (upper dashed) and the paths within the 99.8% confidence interval. $\sigma_m \approx 0, \sigma_v = 1$.

equilibrium the market must always entertain the possibility that high prices are due to an (exacerbated) demand shock.

Figure 2, left panel, illustrates the paths of prices for different realizations of tiny persistent demand shocks and $v = 0$. An immediate implication of the price destabilized strategically is that the large deviations of the price from the fundamental value do not require large demand from liquidity traders. It follows from the formula (15) that in the model with one dimensional uncertainty (Kyle [1985]) this would be the only way to destabilize the price.[23] (see Figure 2, right panel).

As we pointed out in the introduction, given price setting by competitive risk neutral market maker our model cannot generate asset prices more volatile than the fundamentals, and so $\mathbb{E}[X_t^2] \leq \sigma_v^2$, for any t. However, this need not be true if the markets' beliefs are misspecified.[24] The model with tiny demand shocks seems particularly susceptible to misspecification errors, given difficulties in potential estimation of such shocks from the data. Most direct reason is the large impact of small measurement errors. Even with no errors, estimating the extent of a demand shock, or a drift of a Brownian diffusion of liquidity traders' demand in the case when the values of the drift are dwarved by the volatility ($\sigma_m << 1$) is practically impossible. **XX add citationXX**. Figure 3 illustrates

[23] More precisely, to generate $P_t = n\sigma_v$ it must be that $Z_t \geq n$.

[24] The informed trader knows the realization of v and m so his beliefs about their distribution are irrelevant; we assume that the beliefs of the market are common knowledge.

the expected variance of the price in cases when players have correct beliefs about the variance of the fundamentals, but the market slightly underestimates, in absolute terms, the variance of tiny demand shocks.

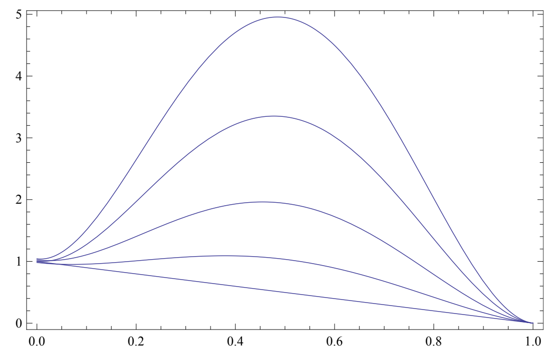

Figure 3: Expected variance of prices when the market believes that $\sigma_m = \varepsilon$ while the true σ_m equals 1ε (lowest), 2ε, 3ε, 4ε and 5ε (highest). $\varepsilon \approx 0, \sigma_v = 1$.

The asymmetric information affects discontinuously also the paths of the asset holdings of the informed trader and how they correlate with the prices. It is a simple corollary to Proposition 3 that as $\sigma_m \to 0$ the linear equilibrium strategy of the informed trader converges in distribution to S^*, where

$$dS_t^* = \left(\frac{1}{\sigma_v} \times \frac{1+2t}{(1-t)^2} X_t^* + \frac{\sqrt{3}}{(1-t)^2} Y_t^* \right) dt, \tag{16}$$

for X_t^* and Y_t^* as in Proposition 3. Thus, the strategy inherits the qualitative feature of pump-and-dump, as if it was exagerrating the demand shocks with variance 1 with the explicitly computed β_t and $\delta_t = \frac{\sqrt{3}}{(1-t)^2} > 0$. Formulas in Proposition 3 and (16) explicitly characterize the joint limit distribution of price paths and asset holdings. In particular, we get the following implication, strenghtening the results of Proposition 2. We note that in the model with one dimensional uncertainty we have $\frac{dS_t^K}{dt} \times (P_t - v)$ distributed as $-\sigma_v N^2$, for a standard Normal variable N, and so independent of time t.

Proposition 4 *Let S^* be the limit of the linear Markov equilibrium strategies as $\sigma_m \to 0$, as in (16).*

i) $\frac{dS_t^*}{dt} \times X_t^*$ has distribution independent of t such that:

$$\frac{dS_t^*}{dt} \times X_t^* \sim -\sigma_v(N_1^2 + \sqrt{3} N_1 N_2),$$

for independent standard Normally distributed variables N_1 and N_2. In particular $P(\frac{dS_t^*}{dt} \times X_t^* > 0) > 0$.

ii) Fix v and α. If $|\alpha|$ is sufficiently high relative to $|v|$ then there is $T > 0$ such that the informed trader is expected to ride the bubble until time T :

$$\mathbb{E}\left[\frac{dS_t^*}{dt} \times X_t^* | S^*, v, \alpha\right] > 0, \text{ for } t \in (0, T].$$

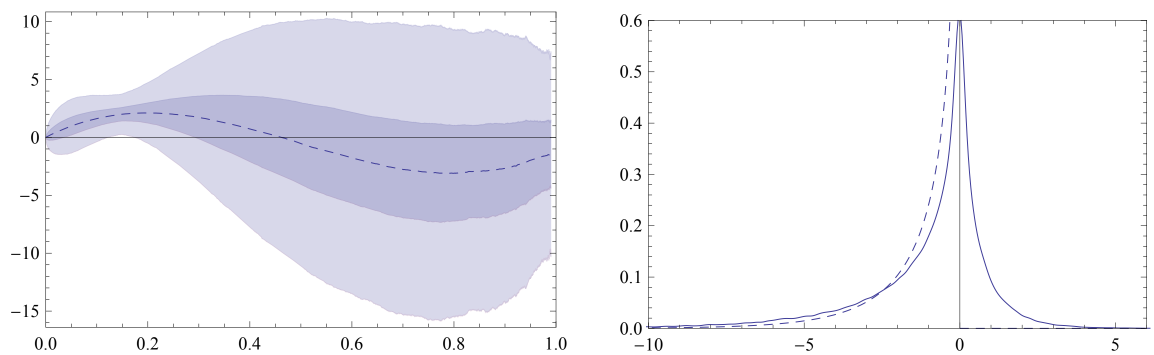

Figure 4: The left panel shows the mean and the 99.8% and 70% confidence intervals for the price times the change in the informed trader's asset holdings for $v = 0$ and $m = 3\sigma_m$. The right panel shows the unconditional time-independent density functions in our (solid) and the model with one dimensional uncertainty (dashed). $\sigma_m \approx 0, \sigma_v = 1$.

5 Conclusions

In the paper we presented a model of dynamic asset trading, in which the informed trader has multidimensional private information about both the fundamentals and the exogenous demand shock. In order to isolate the strategic "rational destabilization" effect in this setting and its implications on price and asset holdings we strived to make the model parsimonious, basing it on the seminal paper by Kyle (Kyle [1985]). The simplicity allowed us to establish uniqueness and characterize the parameters of the (non-stationary) equilibrium analytically in closed form (Theorem 1). This facilitated a clean analysis of the equilibrium and the following results. Both payoff relevant and payoff irrelevant information is revelaed to the market in the process of trading, and so at least eventually the markets perform their informative role (Theorem 2). In the process of trading the informed trader "rationally destabilizes" the price by exacerbating the exogenous demand shocks via pump-and-dump strategy (Proposition 1), which leads to persistent price departures from the fundamental value and the pattern of asset holdings of "riding a bubble", much as documented empirically. Finally, due to the extreme "rational destabilization" persistent and significant price departures from the fundamental value can be triggered by negligible exogenous demand shocks (3).

6 Appendix

6.1 Proofs for Section 3

Proof. (Lemma 3) Fix a linear equilibrium strategy S and thus the gain functions λ_t and ϕ_t as in Lemma 2. For any differentiable a_t, b_t, c_t, d_t, when the informed trader uses a linear strategy $\widetilde{S}$ with drift θ_t then we have from Lemma 2 and Ito's formula

$$\begin{aligned} & d\left(a_t X_t^2 + 2b_t X_t Y_t + c_t Y_t^2 + d_t\right) + X d\widetilde{S}_t = \\ & = X_t\theta_t - 2X_t(\theta_t + Y)(a_t\lambda_t + b_t\phi_t) - 2Y_t(\theta_t + Y)(b_t\lambda_t + c_t\phi_t) + \\ & + da_t X_t^2 + db_t X_t Y_t + dc_t Y_t^2 + h_t dB_t + dd_t + \left(a_t\lambda_t^2 + 2b_t\lambda_t\phi_t + c_t\phi_t^2\right)(dt + d[\widetilde{S}]_t), \end{aligned} \tag{17}$$

for $h_t = -2(a_t\lambda_t X_t + b_t(\phi_t X_t + \lambda_t Y_t) + c_t\phi_t Y_t)$.

First, if $\widetilde{S} = S$ and so $\theta_t = \beta_t X_t + \delta_t Y_t$ and $[\widetilde{S}] \equiv 0$, as long as the functions a_t, b_t, c_t, d_t satisfy the equations

$$\begin{aligned} da_t + \beta_t(1 - 2a_t\lambda_t - 2b_t\phi_t)dt &= 0, \\ dc_t - 2(\delta_t + 1)(b_t\lambda_t + c_t\phi_t)dt &= 0, \\ 2db_t + \delta_t(1 - 2a_t\lambda_t - 2b_t\phi_t) - 2\beta_t(b_t\lambda_t + c_t\phi_t)]dt - 2(a_t\lambda_t + b_t\phi_t) &= 0, \\ dd_t - \left(a_t\lambda_t^2 + 2b_t\lambda_t\phi_t + c_t\phi_t^2\right)dt &= 0, \end{aligned} \tag{18}$$

]then the drift of the term in (17) is zero. Also, it follows from the "no doubling" restriction that for any strategy $\widetilde{S}$[25]

$$\mathbb{E}\left[\int_{[t,T]} h_t dB_r | \widetilde{S}, v, m, \mathcal{F}_t^Z\right] = 0. \tag{19}$$

and so (10) holds.

Second, (17) implies that

$$(1 - 2a_t\lambda_t - 2b_t\phi_t) = (b_t\lambda_t + c_t\phi_t) = 0. \tag{20}$$

Otherwise the informed trader could profitably deviate from the strategy S: for example, if $(1 - 2a_t\lambda_t - 2b_t\phi_t) > 0$ then choosing the drift $\theta = (\beta_t + \gamma) X_t dt$, with $\gamma > 0$, in any event when $|X_t|$ is large and Y_t is equal or sufficiently close to zero would increase the expected wealth at time T relative to strategy S (note that we are using here that $t < T$, and so Y_t and X_t are not perfectly correlated). The necessary condition (20) implies (6) as well as, together with (18), that $da_t = dc_t = 0$ and $db_t = 0.5dt$.

Finally, for an arbitrary strategy $\widetilde{S}$ we have

$$\begin{aligned}
& d\left(a_t X_t^2 + 2b_t X_t Y_t + c_t Y_t^2 + d_t\right) + X d\widetilde{S}_t - dP_t d\widetilde{S}_t = \\
& = h_t dB_t + \left(a_t\lambda_t^2 + 2b_t\lambda_t\phi_t + c_t\phi_t^2\right) d[\widetilde{S}]_t - dP_t d\widetilde{S}_t = \\
& = h_t dB_t + \left(a_t\lambda_t^2 + 2b_t\lambda_t\phi_t + c_t\phi_t^2 - \lambda_t\right) d[\widetilde{S}]_t = \\
& = h_t dB_t - \frac{\lambda_t}{2} d[\widetilde{S}]_t.
\end{aligned}$$

The first equality follows from (17), (18) and (20), the second from $dP_t d\widetilde{S}_t = -dX_t d\widetilde{S}_t = \lambda_t d[\widetilde{S}]_t$ and the last equality from $a_t\lambda_t^2 + 2b_t\lambda_t\phi_t + c_t\phi_t^2 = \frac{\lambda_t}{2}$. The formula, together with (19) establishes (10) and concludes the proof of the Lemma.[26] ■

Proof. (Lemma 4) We construct the control u_t as follows. Fix $T < 1$ such that

[25] Given that dX_t and dY_t are linear in $d\widetilde{S}_t, Y_t$ and dB_t and the linear coefficients λ_t and ϕ_t are uniformly bounded, it follows from the Ito formula that the "no doubling" restriction $\mathbb{E}\left[\int_{[0,1]}^2 \widetilde{S}_t^2 dt\right] < \infty$ implies that $\mathbb{E}\left[\int_{[0,1]} X_t^2 dt | \widetilde{S}\right] < \infty$. The last inequality is sufficient for the expectation in the text to be equal to zero (see KS). We leave the details to the reader.

[26] Observe also that the analogous equality would hold if we allowed discontinuous strategies: the only change would be an additional term $-\frac{\lambda_r}{2}\Delta[\widetilde{S}]_t$, where $\Delta[\widetilde{S}]_t$ are the squared differences of the cumulative order at the points of discontinuity (compare Back 92).

$\int_{[T,1]} f_t^2 dt, \int_{[T,1]} g_t^2 dt < \varepsilon$. Consider functions f_t^T and g_t^T defined on $[T,1]$ such that

$$\int_{[T,1]} f_t f_t^T = 1, \quad \int_{[T,1]} g_t f_t^T = 0,$$
$$\int_{[T,1]} g_t g_t^T = 1, \quad \int_{[T,1]} f_t g_t^T = 0.$$

The existence of such functions follows from the fact that f_t and g_t are not colinear: For example, f_t^T can be defined as the appropriately rescaled difference between f_t and the projection of g_t on f_t

$$f_t^T = C \times \left(f_t - \frac{\int_{[T,1]} f_t g_t dt}{\int_{[T,1]} g_t^2 dt} g_t \right).$$

Now, define the control u_t as follows

$$u_t = 0,\, t \leq T,$$
$$u_t = (x - X_T)\, f_t^T + (y - Y_T)\, g_t^T,\, t > T.$$

Given the control u_t we have

$$X_1 = X_T + \int_{[T,1]} f_t((x - X_T)\, f_t^T + (y - Y_T)\, g_t^T)dt + f_t dB_t = x + \int_{[T,1]} f_t dB_t,$$
$$Y_1 = Y_T + \int_{[T,1]} g_t((x - X_T)\, f_t^T + (y - Y_T)\, g_t^T)dt + g_t dB_t = y + \int_{[T,1]} g_t dB_t,$$
$$\mathbb{E}[X_1] = x,\, \mathbb{E}[(X_1 - x)^2] = \int_{[T,1]} f_t^2 dt < \varepsilon,$$
$$\mathbb{E}[Y_1] = y,\, \mathbb{E}[(Y_1 - y)^2] = \int_{[T,1]} g_t^2 dt < \varepsilon,$$

where the inequalities in the last two lines follow from the choice of T. This concludes the proof of the Lemma. ■

Proof. (Lemma 5) We will establish that if the market makers were convinced that X_t and Y_t are perfectly correlated from some time $T < 1$ onwards, this would allow the informed trader to obtain arbitrarily high profits. Suppose by contradiction that on an nonempty interval $[T,1]$ we have $Y_t = \alpha X_t$ for some constant α, and so the equilibrium strategy takes the form $dS_t = \widetilde{\beta}_t X_t dt$. As in the proof of Lemma 3 on this interval we have, for any differentiable $\widetilde{a}_t$ and $\widetilde{d}_t$

$$\begin{aligned} & d(\widetilde{a}_t X_t^2 + \widetilde{d}_t) + X_t dS_t = \\ & = -2\widetilde{a}_t \lambda_t X_t (\beta_t X_t + \alpha X_t) + X_t^2 d\widetilde{a}_t + d\widetilde{d}_t - \sigma^2 \widetilde{a}_t \lambda_t^2 + \beta_t X_t^2 dt - 2\widetilde{a}_t \lambda_t X_t dB_t = \\ & = X_t^2(\beta_t(1 - 2\widetilde{a}_t\lambda_t) + d\widetilde{a}_t - 2\alpha\widetilde{a}_t\lambda_t) + dd_t - \sigma^2\widetilde{a}_t\lambda_t^2 - 2\widetilde{a}_t\lambda_t X_t dB_t, \end{aligned}$$

and so

$$\mathbb{E}\left[\int_{[t,1]} (v - P_r)\, dS_r | S, v, m, \mathcal{F}_t^Z\right] = \widetilde{a}_t X_t^2 + \widetilde{d}_t - \mathbb{E}\left[\widetilde{a}_{t1} X_1^2 | S, v, m, \mathcal{F}_t^Z\right],$$

for any $\widetilde{a}_t$ and $\widetilde{d}_t$ that solve $\beta_t(1 - 2\widetilde{a}_t\lambda_t) + d\widetilde{a}_t + 2\alpha\widetilde{a}_t\lambda_t = 0$, and $d\widetilde{d}_t - \sigma^2\widetilde{a}_t\lambda_t^2$, $\widetilde{d}_1 = 0$. On the other hand, if at time t the informed trader deviates to a linear Markow strategy $\widetilde{S}$ with drift θdt then

$$d(\widetilde{a}_t X_t^2 + \widetilde{d}_t) + X_t d\widetilde{S}_t = X_t\theta_t(1 - 2\widetilde{a}_t\lambda_t) + d\widetilde{d}_t - \sigma^2\widetilde{a}_t\lambda_t^2 - 2\widetilde{a}_t\lambda_t X_t dB_t,$$

and so to make sure that the informed trader has no profitable deviation it must be that $1 - 2\widetilde{a}_t\lambda_t = 0$. This implies, just as in Lemma 3, that for any strategy $\widetilde{S}$ with $[\widetilde{S}] \equiv 0$

$$\mathbb{E}\left[\int_{[t,1]} (v - P_r)\, d\widetilde{S}_r | \widetilde{S}, v, m, \mathcal{F}_t^Z\right] = \widetilde{a}_t X_t^2 + \widetilde{d}_t - \mathbb{E}\left[\widetilde{a}_1 X_1^2 | \widetilde{S}, v, m, \mathcal{F}_t^Z\right]. \tag{21}$$

This establishes that conditional on the equilibrium play on the interval $[0, T]$, for any linear Markov strategy $\widetilde{S}$ the expected profits from trading on the interval $[T, 1]$ depend only on $\mathbb{E}\left[X_1^2 | \widetilde{S}, v, m, \mathcal{F}_t^Z\right]$.

Lets construct a deviation strategy $\widetilde{S}$ that yields arbitrarily high expected payoffs as follows. Intuitively, the strategy first "fools" the market to believe that there is a larger demand shock than the truly relized one. Then, on the interval $[T, 1]$ the strategy drives the price arbitrarily low (X_t arbitrarily high) for a long period of time, at which price the informed trader purchases the asset undetected by the market. More precisely, equations (6) imply that the conditions of Lemma 4 are satisfied, and so there is a strategy such that (with high probability) X_T is very close to 0, and Y_T is very close to $-N$, for some $N > 0$. Notice that for any t in $[0, T]$, any public history η on $[T, t]$ and any two strategies $\widetilde{S}$ and $\widehat{S}$ we have

$$\mathbb{E}\left[dX_t | \widetilde{S}, X_T = 0, Y_T = -N, \eta\right] = \mathbb{E}\left[dX_t | \widehat{S}, X_T = 0, Y_T = 0, \eta\right]$$

exactly when $d\widetilde{S}_t = d\widehat{S}_t + N$. From this and (21) follows that conditional on $X_T = 0, Y_T = -N$ the expected payoff on the interval $[T, 1]$ of a strategy $\widetilde{S}$ that results in $X_1 = 0$ is equal to

$$\widetilde{d}_t + N \times \mathbb{E}\left[\int_{[T,1]} X_t dt | \widetilde{S}, X_T = 0, Y_T = -N\right].$$

Therefore, arbitrarily high expected profits can be made by a strategy $\widetilde{S}$ that results in $X_1 = 0$ and at the same time in the arbitrarily high expectation of $\int_{[T,1]} X_t dt$. This establishes the desired contradiction and so the proof of the Lemma. ■

Proof. (Theorem 2) Fix an equilibrium strategy S and let a, b_t and c be the parameters in the expected payoff function (10). We claim that it is suffient to show that the matrix $\mathbb{P}_1$,

$$\mathbb{P}_1 := \begin{bmatrix} a & b_1 \\ b_1 & c \end{bmatrix},$$

is strictly positive definite. This is because, from Lemma 3, for any strategy $\widetilde{S}$ the expected flow payoff equals the drift of $-(a_t X_t^2 + 2b_t X_t Y_t + c_t Y_t^2)$ under $\widetilde{S}$, and so maximizing expected payoffs requires minimizing $a_1 X_1^2 + 2b_1 X_1 Y_1 + c_1 Y_1^2$. If the matrix $\mathbb{P}_1$ is strictly positive definite but Σ_1 was not the zero matrix under tentative equilibrium strategy S, a strategy $\widetilde{S}$ such that $\mathbb{E}[X_t^2|\widetilde{S}], \mathbb{E}[Y_t^2|\widetilde{S}] < \varepsilon$ (see (6) and Lemma 4) would result in strictly smaller $a_1 X_1^2 + 2b_1 X_1 Y_1 + c_1 Y_1^2$ and constitute a profitable deviation, for ε sufficiently small.

First, no eigenvalue e of the matrix $\mathbb{P}_1$ can be strictly negative. Otherwise, if (x_e, y_e) is the associated eigenvector, a strategy $\widetilde{S}$ resulting in $(\mathbb{E}[X_t|\widetilde{S}], \mathbb{E}[Y_t|\widetilde{S}])$ sufficiently close to (Mx_e, My_e) would yield arbitrarily high profits as $M \to \infty$, in view of (10). Second, no eigenvalue of the matrix can be equal to zero. Otherwise, when $ac - b_1^2 = 0$, we would have, say,

$$d\sigma_{11,1}^2 = -\lambda_1^2 \sigma^2 = -\frac{c^2 \sigma^2}{4(ac - b_1^2)^2} = -\infty,$$

and it is easy to verify that $\sigma_{11,t}^2 = -\int_t^1 d\sigma_{11,1}^2 + \sigma_{11,1}^2$ would have to be infinite for any $t < 1$. ■

Lemma 7 *For given parameters σ_v^2 and σ_m^2 equations (5) have a unique solution (a, c) with $a, c > 0, ac > 1/16$.*

Proof. If $x = \sqrt{ac} - \frac{1}{4}$ then the LHS of the first equation in (5) becomes

$$G(x) = \frac{1}{x} - \frac{1}{x+0.5} + \frac{1}{x+0.25}\left[ln\left(x\right) - ln\left(x+0.5\right)\right]$$

Clearly $G(x) \to \infty$ as $x \downarrow 0$ and $G(x) \to 0$ as $x \to \infty$: The first part follows from $\frac{1}{x}$ heading to infinity faster than $-ln(x)$ when $x \to 0$. The second part follows as all three terms converge to zero as $x \to \infty$.

The next step is to show that $G(x)$ is strictly decreasing. Taking the first derivative, $G'(x)$ for $x > 0$:

$$\begin{aligned} G'(x) &= -\frac{1}{x^2} + \frac{1}{\left(x+0.5\right)^2} + \frac{ln\left(1+0.5x^{-1}\right)}{\left(x+0.25\right)^2} + \frac{1}{\left(x+0.25\right)}\left[\frac{1}{x} - \frac{1}{x+0.5}\right] \\ &= \frac{-0.25}{x+0.25}\left(\frac{1}{x^2} + \frac{1}{\left(x+0.5\right)^2} - \frac{4ln\left(1+0.5x^{-1}\right)}{\left(x+0.25\right)}\right) \\ &\leq \frac{-0.25}{x+0.25}\left(\frac{1}{x^2} + \frac{1}{\left(x+0.5\right)^2} - \frac{4\left(0.5x^{-1}\right)}{\left(x+0.25\right)}\right) \\ &\leq \frac{-0.25}{x+0.25}\left(\frac{1}{x^2} + \frac{1}{\left(x+0.5\right)^2} - \frac{2}{x\left(x+0.5\right)}\right) < 0 \end{aligned}$$

Thus, $G(x)$ monotonically declines from ∞ to 0 and so there is a unique solution to $4s_m^2 = G(x)$, which thus pins down unique $ac > 0$.

The LHS of the second equation in (5) is strictly greater than that of the first equation and thus strictly positive. Thus the second equation pins down unique $a, c > 0$ for a fixed value of $ac > 0$. ■

6.2 Proof for Section 4

Proof. (Lemma 6) The form of the coefficients in the expected wealth function follows easily from comparing with the expected wealth function under the original variables X_t and Y_t in Theorem 1

Let us establish here that $\widehat{b}_t < 0$ for every $t < 1$. Since $H_0 = 0$ we have $\widehat{b}_0 = -1/4$. On the other hand, since $\sigma_{12,t}, \sigma_{11,t}^2 \to 0$ as $t \to 1$ we have from L'Hopital's rule that as $t \to 1$

$$\widehat{b}_t \to b_1 + c \times \frac{\sigma'_{12,1}}{\sigma^{2\prime}_{11,1}} = b_1 + c \times \frac{-\lambda_1\phi_1}{-\lambda_1^2} = b_1 + c \times \frac{-b_1}{c} = 0,$$

Also, we have

$$\widehat{b}_t = b_t + cH_t = \frac{2\,(ac - b_t^2)}{\sigma_{11,t}^2}[\lambda_t \sigma_{12,t} - \phi_t \sigma_{11,t}^2],$$

$$\widehat{b}_t' = \frac{1}{2} + cH_t' = \frac{1}{2} + c\frac{\lambda_t}{\sigma_{11,t}^4}[\lambda_t \sigma_{12,t} - \phi_t \sigma_{11,t}^2] = \frac{1}{2} + c\frac{\lambda_t}{\sigma_{11,t}^4}[\lambda_t \sigma_{12,t} - \phi_t \sigma_{11,t}^2] =$$

$$= \frac{1}{2} + c\frac{\lambda_t}{\sigma_{11,t}^4}\frac{\sigma_{11,t}^2}{2\,(ac - b_t^2)}\widehat{b}_t = \frac{1}{2} + \frac{\lambda_t^2}{\sigma_{11,t}^2}\widehat{b}_t.$$

This establishes that $\widehat{b}_t > 0$ implies $\widehat{b}_t' > 0$. Together with $\widehat{b}_0 < 0$ and $\widehat{b}_1 = 0$ this establishes the proof of the lemma. ■

Proof. (Corollary 1) It is easy to verify that $d\widehat{Y}_t = \frac{(\delta_t+1)}{\det \Sigma_t}(d\widetilde{S}_t + Y_t + dB_t)$. Consequently, if $\delta_t \leq -1$ a strategy such that $d\widetilde{S}_t + Y_t < 0$ both weakly increases the drift of $\widehat{Y}_t$ and decreases that of X_t. Given the form of the expected value function in (12) and in particular $\widehat{b}_t < 0$ this contradicts the indifference of the informed trader in the case when $X_t = 0$. ■

Proof. (Proposition 1) From the projection Theorem (see (6)) we have:

$$\delta_t + 1 = \frac{(\sigma_{11,t}^2 \phi_t - \sigma_{12,t}^2 \lambda_t)}{\sigma_{11,t}^2 \sigma_{22,t}^2 - \sigma_{12,t}^2}.$$

Let h_t be the difference between the numerator and the denominator,

$$h_t = (\sigma_{11,t}^2 \phi_t - \sigma_{12,t}^2 \lambda_t) - (\sigma_{11,t}^2 \sigma_{22,t}^2 - \sigma_{12,t}^2).$$

Also, from the definition of λ_t and ϕ_t in (6) we have,

$$\lambda_t' = -2\lambda_t \phi_t,$$
$$\phi_t' = -(\frac{a}{c}\lambda_t^2 + \phi_t^2).$$

Therefore

$$\begin{aligned} h_t' &= (\sigma_{11,t}^2\phi_t' - \phi_t\lambda_t^2 + \phi_t\lambda_t^2 - \sigma_{12,t}\lambda_t') - (-\sigma_{11,t}^2\phi_t^2 - \sigma_{22,t}^2\lambda_t^2 + 2\sigma_{12,t}\lambda_t\phi_t) = \\ &= -\sigma_{11,t}^2(\frac{a}{c}\lambda_t^2 + \phi_t^2) + 2\sigma_{12,t}\lambda_t\phi_t + \sigma_{11,t}^2\phi_t^2 + \sigma_{22,t}^2\lambda_t^2 - 2\sigma_{12,t}\lambda_t\phi_t = \\ &= -\frac{\lambda_t^2}{c}(a\sigma_{11,t}^2 - c\sigma_{22,t}^2) = -\frac{\lambda_t^2}{c}\int_t^1 (a\lambda_s^2 - c\phi_s^2)ds = \\ &= -\frac{\lambda_t^2}{4c}\int_t^1 \frac{ac^2 - cb_s^2}{(ac-b_s^2)^2}ds = -\frac{\lambda_t^2}{4c}\int_t^1 \lambda_s ds < 0. \end{aligned}$$

On the other hand, since $\sigma_{11,1}^2 = \sigma_{22,1}^2 = \sigma_{12,1} = 0$ and λ_1, ϕ_1 are finite we have that $h_1 = 0$. Together this implies that $h_t > 0$ for $t < 1$ and so establishes the proof. ∎

Proof. (Proposition 2)

i) Since $\frac{dS_t}{dt} \times (P_t - v) = \beta_t X_t^2 + \delta_t X_t Y_t$, the result follows from the fact that X_t and Y_t have a bivariate Normal distribution and are not perfectly correlated.

ii) From Ito's formula we have

$$\begin{aligned} &\frac{d}{dt}\mathbb{E}\left[\frac{dS_t}{dt} \times (P_t - v)|S, v, m\right] = \\ &= -\frac{d}{dt}\mathbb{E}[\beta_t X_t^2 + \delta_t X_t Y_t|S, v, m] = \\ &= -\mathbb{E}[\beta_t' X_t^2 + \delta_t' X_t Y_t + 2\beta_t X_t dX_t + \delta_t(X_t dY_t + Y_t dX_t) + \beta_t(dX_t)^2 + \delta_t dX_t dY_t|S, v, m] = \\ &= -\mathbb{E}[A_t X_t^2 + B_t X_t Y_t + C_t Y_t^2 + D_t|S, v, m]dt, \end{aligned}$$

with appropriate continuous functions A_t, B_t, C_t and D_t, and in particular

$$C_t = -\delta_t(\delta_t + 1)\phi_t.$$

Since $\mathbb{E}[X_t^2|S, v, m], \mathbb{E}[X_t Y_t|S, v, m]$ and $\mathbb{E}[Y_t^2|S, v, m]$ are continuous, the function $\mathbb{E}\left[\frac{dS_t}{dt} \times (P_t - v)|S, v, m\right]$ is continuously differentiable. On the other hand, it follows from Proposition 1 that $C_0 < 0$, and so the derivative of $\mathbb{E}\left[\frac{dS_t}{dt} \times (P_t - v)|S, v, m\right]$ at $t = 0$ is strictly positive if $|m|$ is sufficiently large relative to $|v|$. This establishes the proof. ∎

Proof. (Proposition 3**)** i) To simplify the notation, for the parameters a, b_t and c of the value function defined in Theorem 1 (line (5)), denote

$$\xi_t = \frac{ac}{ac - b_t^2},$$

ξ_t depends on the parameters of the model σ_m^2 and σ_v^2 - the dependence that we do not make explicit, for the sake of tractability. The approximations of the parameters in the Proposition are based on the approximation $\xi_t \approx 1$, which is justified since, as we will see, $\sigma_m \downarrow 0$ implies $ac \uparrow \infty$.

Since Σ_t satisfies the differential equation (7) and Σ_1 is the null matrix, we have

$$\begin{aligned}
\lambda_t &= \frac{1}{2a}\xi_t = \frac{1}{2a}\left(1+\varepsilon_1\right), \\
\phi_t &= \frac{-b_t}{2ac}\xi_t = \frac{1}{4ac}(\frac{1}{2}-t)\left(1+\varepsilon_1\right) \\
\sigma_{11,t}^2 &= \int_{[t,1]} \lambda_s^2 ds = \int_{[t,1]} \frac{c^2}{4(ac-b_t^2)^2} ds = \frac{1}{4a^2}\int_{[t,1]} \xi_s^2 ds = \frac{1-t}{4a^2}\left(1+\varepsilon_2\right), \\
\sigma_{12,t} &= \int_{[t,1]} \lambda_s \phi_s ds = \frac{1}{4a^2 c}\frac{t(t-1)}{4}\left(1+\varepsilon_3\right), \\
\sigma_{22,t}^2 &= \int_{[t,1]} \phi_s^2 ds = \frac{1}{4a^2c^2}\int_{[t,1]} b_s^2 \xi_s^2 ds = \frac{1}{16a^2c^2}\frac{\left(1-t\right)\left(1-2t+4t^2\right)}{12}\left(1+\varepsilon_4\right),
\end{aligned}$$

where $\varepsilon_1 \in [0, \max_t \xi_t - 1]$ and $\varepsilon_i \in [0, (\max_t \xi_t)^2 - 1]$, $i = 2,3,4$ (we are not making explicit the dependence of the error terms on time). In particular, the last equation evaluated at $t = 0$ yields

$$ac = \frac{1}{\sigma_m 8\sqrt{3}} + \frac{\sqrt{1+\varepsilon_4}-1}{\sigma_m 8\sqrt{3}}.$$

Consequently

$$\begin{aligned}
ac &= \frac{1}{\sigma_m 8\sqrt{3}} + O\left(1\right), \quad a = \frac{1}{2\sigma_v} + O\left(\sigma_m\right), \\
\varepsilon_i &= O\left(\sigma_m\right). \quad i = 1, ..., 4
\end{aligned}$$

By substituting those values we obtain the formulas for λ_t and ϕ_t. From the Bayes formula (8) in Lemma 2 we have

$$
\begin{aligned}
\delta_t + 1 &= \frac{\left(\sigma_{11,t}^2 \phi_t - \sigma_{12,t}^2 \lambda_t\right)}{\det \Sigma_t} = \frac{\frac{1-t}{4a^2}\left(1+\varepsilon_2\right)\frac{1}{4ac}\left(\frac{1}{2}-t\right)\left(1+\varepsilon_1\right) - \frac{1}{4a^2c}\frac{t(t-1)}{4}\left(1+\varepsilon_3\right)\frac{1}{2a}\left(1+\varepsilon_1\right)}{\frac{1-t}{4a^2}\left(1+\varepsilon_2\right)\frac{1}{16a^2c^2}\frac{(1-t)(1-2t+4t^2)}{12}\left(1+\varepsilon_4\right) - \left[\frac{1}{4a^2c}\frac{t(1-t)}{4}\left(1+\varepsilon_3\right)\right]^2} = \\
&= ac \times \frac{24}{(1-t)^2} \times \left(1 + O\left(\sigma_m\right)\right) = \frac{\sqrt{3}}{\sigma_m (1-t)^2} + O(1); \\
\beta_t &= \frac{\left(\sigma_{22,t}\lambda_t - \sigma_{12,t}\phi_t\right)}{\det \Sigma_t} = \\
&= \frac{\frac{1}{16a^2c^2}\frac{(1-t)\left(1-2t+4t^2\right)}{12}\left(1+\varepsilon_4\right)\frac{1}{2a}\left(1+\varepsilon_1\right) - \frac{1}{4a^2c}\frac{t(t-1)}{4}\left(1+\varepsilon_3\right)\frac{1}{4ac}\left(\frac{1}{2}-t\right)\left(1+\varepsilon_1\right)}{\frac{1-t}{4a^2}\left(1+\varepsilon_2\right)\frac{1}{16a^2c^2}\frac{(1-t)(1-2t+4t^2)}{12}\left(1+\varepsilon_4\right) - \left[\frac{1}{4a^2c}\frac{t(1-t)}{4}\left(1+\varepsilon_3\right)\right]^2} = \\
&= a \times \frac{2(1+2t)}{(1-t)^2} \times \left(1 + O\left(\sigma_m\right)\right) = \frac{1+2t}{\sigma_v (1-t)^2} + O\left(\sigma_m\right).
\end{aligned}
$$

ii) It follows from part i) and the continuous dependence of the solutions of stochastic ODE on the parameters (see Karatzas [1991]). ∎

Proof. (Corollary 2) i) Since the stochastic differential equations in Proposition 3 are linear in states X_t^* Y_t^*, the expectations $\overline{X}_t$ and $\overline{Y}_t$ must solve the corresponding system of linear ordinary differential equations, with truncated stochastic part "dB_t''. It is easy to verify that the functions in Corollary 2 are the solution.

ii) Lack of convergence for $\alpha \neq 0$ follows for example from comparing the drifts at $t = 0$. In the case $\alpha = 0$, the drift of X_t^K depends only on the value of X_t^K, while the drift of X_t^* depends both on X_t^* and Y_t^*. The result thus follows form the fact that X_t^* and Y_t^* are not perfectly correlated. ∎

Proof. (Proposition 4) i) From Proposition 3 it follows that

$$
\begin{aligned}
\mathbb{E}[X_t^{*2}] &= \int_t^1 \sigma_v^2 ds = \sigma_v^2(1-t), \\
\mathbb{E}[Y_t^{*2}] &= \int_t^1 3(1-2s)^2 ds = 1 - 3t + 6t^2 - 4t^3, \\
\mathbb{E}[X_t^* Y_t^*] &= \int_t^1 \sigma_v \sqrt{3}(1-2s) ds = \sigma_v \sqrt{3} t(t-1).
\end{aligned}
$$

Consequently,

$$\begin{aligned}\frac{dS_t}{dt} \times (P_t - v) &= -\frac{1+2t}{\sigma_v (1-t)^2} X_t^{*2} - \frac{\sqrt{3}}{(1-t)^2} X_t^* Y_t^* \sim \\ &\sim -\frac{1+2t}{\sigma_v (1-t)^2} X_t^{*2} - \frac{\sqrt{3}}{(1-t)^2} X_t^* \left(\frac{\mathbb{E}[X_t^* Y_t^*]}{\mathbb{E}[X_t^{*2}]} X_t^* + \left(\mathbb{E}[Y_t^{*2}] - \frac{(\mathbb{E}[X_t^* Y_t^*])^2}{\mathbb{E}[X_t^{*2}]} \right)^{1/2} N_2 \right) \sim \\ &\sim AN_1 + BN_1 N_2,\end{aligned}$$

where N_1 and N_2 are independent standard Normally distributed random variables, and

$$\begin{aligned}A &= -\mathbb{E}[X_t^{*2}] \left(\frac{1+2t}{\sigma_v (1-t)^2} + \frac{\sqrt{3}}{(1-t)^2} \frac{\mathbb{E}[X_t^* Y_t^*]}{\mathbb{E}[X_t^{*2}]} \right) = -\sigma_v, \\ B &= -\frac{\sqrt{3}}{(1-t)^2} \sqrt{\mathbb{E}[X_t^{*2}]\mathbb{E}[Y_t^{*2}] - (\mathbb{E}[X_t^* Y_t^*])^2} = -\sqrt{3}\sigma_v \sigma_m.\end{aligned}$$

■

References

Dilip Abreu and Markus K Brunnermeier. Bubbles and crashes. *Econometrica*, 71(1): 173–204, 2003.

Franklin Allen and Douglas Gale. Stock-price manipulation. *Review of financial studies*, 5(3):503–529, 1992.

Franklin Allen and Gary Gorton. Stock price manipulation, market microstructure and asymmetric information. *European Economic Review*, 36(2-3):624–630, 1992.

Christopher Avery and Peter Zemsky. Multidimensional uncertainty and herd behavior in financial markets. *American economic review*, pages 724–748, 1998.

Kerry Back. Insider trading in continuous time. *Review of financial Studies*, 5(3):387–409, 1992.

Kerry Back, C. Henry Cao, and Gregory A. Willard. Imperfect Competition among Informed Traders. *Journal of Finance*, 55(5):2117–2155, October 2000.

Kerry Back, Kevin Crotty, and Tao Li. Estimating the order- ow component of security returns. *SSRN*, 2367203, 2014.

Malcolm Baker and Jeffrey Wurgler. Behavioral corporate finance: An updated survey. *Handbook of the Economics of Finance SET: Volumes 2A & 2B: Corporate Finance and Asset Pricing*, page 357, 2013.

Snehal Banerjee and Brett Green. Signal or noise? uncertainty and learning about whether other traders are informed. *SSRN*, 2139771, 2014.

Brad M. Barber, Terrance Odean, and Ning Zhu. Systematic noise. *Journal of Financial Markets*, 12(4):547 – 569, 2009.

Nicholas Barberis and Richard Thaler. A survey of behavioral finance. *Handbook of the Economics of Finance*, 1:1053–1128, 2003.

Dan Bernhardt and Bart Taub. Front-running dynamics. *Journal of Economic Theory*, 138(1):288–296, 2008.

Markus K Brunnermeier. *Asset pricing under asymmetric information: Bubbles, crashes, technical analysis, and herding.* Oxford University Press, 2001.

Markus K. Brunnermeier and Stefan Nagel. Hedge funds and the technology bubble. *Journal of Finance*, 59(5):2013–2040, 2004.

Markus K Brunnermeier and Lasse Heje Pedersen. Market liquidity and funding liquidity. *Review of Financial studies*, 22(6):2201–2238, 2009.

René Caldentey and Ennio Stacchetti. Insider trading with a random deadline. *Econometrica*, 78(1):245–283, 2010.

John Y Campbell and Albert S Kyle. Smart Money, Noise Trading and Stock Price Behaviour. *Review of Economic Studies*, 60(1):1–34, January 1993.

Archishman Chakraborty and Bilge Yılmaz. Informed manipulation. *Journal of Economic Theory*, 114(1):132–152, 2004.

J Bradford De long, Andrei Shleifer, Lawrence H Summers, and Robert J Waldmann. Positive feedback investment strategies and destabilizing rational speculation. *the Journal of Finance*, 45(2):379–395, 1990.

J Bradford De Long, Andrei Shleifer, Lawrence H. Summers, and Robert J. Waldmann. Noise Trader Risk in Financial Markets.

Journal of Political Economy, 98(4):703–38, August 1990. URL `http://ideas.repec.org/a/ucp/jpolec/v98y1990i4p703-38.html`.

Antonio Doblas-Madrid. A robust model of bubbles with multidimensional uncertainty. *Econometrica*, 80(5):1845–1893, 2012.

Songzi Du and Haoxiang Zhu. Welfare and optimal trading frequency in dynamic double auctions. *SSRN*, 2040609, 2014.

Darrell Duffie, Nicolae Garleanu, and Lasse Heje Pedersen. Securities lending, shorting, and pricing. *Journal of Financial Economics*, 66(2-3):307–339, 2002.

Eugene F Fama. The behaviour of stock market prices. *Journal of Business*, pages 34–105, 1965.

Michael J Fishman and Kathleen M Hagerty. The incentive to sell financial market information. *Journal of Financial Intermediation*, 4(2):95–115, 1995.

F Douglas Foster and S Viswanathan. Strategic trading with asymmetrically informed traders and long-lived information. *Journal of financial and Quantitative Analysis*, 29 (04):499–518, 1994.

Milton Friedman. The case for flexible exchange rates. In *Essays in Positive Economics*. University of Chicago Press, Chicago, IL, 1953.

Jayant Vivek Ganguli and Liyan Yang. Complementarities, multiplicity, and supply information. *Journal of the European Economic Association*, 7(1):90–115, 2009.

Feng Gao, Fengming Song, and Jun Wang. Rational expectations equilibrium with uncertain proportion of informed traders. *Journal of Financial Markets*, 16(3):387–413, 2013.

Lawrence R. Glosten and Paul R. Milgrom. Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *Journal of Financial Economics*, 14(1): 71–100, 1985.

Sanford J Grossman and Merton H Miller. Liquidity and market structure. *the Journal of Finance*, 43(3):617–633, 1988.

Ming Guo and Hui Ou-Yang. Feedback trading between fundamental and nonfundamental information. *Review of Financial Studies*, page hhu057, 2014.

J Michael Harrison and David M Kreps. Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations. *The Quarterly Journal of Economics*, 92(2):323–36, May 1978.

Ioannis Karatzas. *Brownian motion and stochastic calculus*, volume 113. springer, 1991.

Charles P Kindleberger and Robert Z Aliber. *Manias, panics and crashes: a history of financial crises*. Palgrave Macmillan, 2011.

Alok Kumar and Charles M.C. Lee. Retail investor sentiment and return comovements. *The Journal of Finance*, 61(5):2451–2486, 2006.

Albert S Kyle. Continuous Auctions and Insider Trading. *Econometrica*, 53(6):1315–35, 1985.

Nicolas S Lambert, Michael Ostrovsky, and Mikhail Panov. Strategic trading in informationally complex environments. In *Proceedings of the fifteenth ACM conference on Economics and computation*, pages 3–4. ACM, 2014.

Tao Li. Insider trading with uncertain informed trading. *SSRN*, 946324, 2012.

Robert S Liptser and Albert N Shiryaev. *Statistics of Random Processes I*, volume 1. Springer, 2000.

Carolina Manzano and Xavier Vives. Public and private learning from prices, strategic substitutability and complementarity, and equilibrium multiplicity. *Journal of Mathematical Economics*, 47(3):346–369, 2011.

Edward M. Miller. Risk, uncertainty, and divergence of opinion. *The Journal of Finance*, 32(4):pp. 1151–1168, 1977. ISSN 00221082.

Michael Ostrovsky. Information aggregation in dynamic markets with strategic traders. *Econometrica*, 80(6):2595–2647, 2012.

Jean-Charles Rochet and Jean-Luc Vila. Insider trading without normality. *The Review of Economic Studies*, 61(1):131–152, 1994.

David Romer. Rational Asset-Price Movements without News. *American Economic Review*, 83(5):1112–30, 1993.

Robert J Shiller. From efficient markets theory to behavioral finance. *Journal of economic perspectives*, pages 83–104, 2003.

Andrei Shleifer and Robert W Vishny. The limits of arbitrage. *The Journal of Finance*, 52(1):35–55, 1997.

George Soros. *The alchemy of finance*. John Wiley & Sons, 2003.

Peter Temin and Hans-Joachim Voth. Riding the south sea bubble. *American Economic Review*, 94(5), 2004.

Dimitri Vayanos. Strategic trading and welfare in a dynamic market. *The Review of Economic Studies*, 66(2):219–254, 1999.

Xavier Vives. *Information and Learning in Markets*. Princeton University Press, New York, first edition, 2008.

Jiang Wang. A model of intertemporal asset prices under asymmetric information. *The Review of Economic Studies*, 60(2):249–282, 1993.