Collective bounded rationality: theory and experiments

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Introduction

- This paper studies **collective** bounded rationality.
- Cognitive hierarchy models describe bounded rationality at individual level and explain well systematic deviations from equilibrium in certain games.
- However, individual bounded rationality may accumulate, and properties of collective decisions may differ qualitatively.
- Moreover, certain assumptions in the 'standard' bounded rationality models may not apply, e.g. overconfidence assumption.
- We suggest an **endogenous cognitive hierarchy** (ECH) model and study **asymptotic properties** of collective decision making.
- The tools of the Condorcet Jury Theorem are used to model group decision making.

Cognitive hierarchy models

- $g_k(h)$, level-k player's belief about proportion of level-h.

Assumption 1 (Cognitive limit)

$g_k(h) = 0$ for all $h > k$.

Assumption 2 (Overconfidence)

$g_k(k) = 0$ for all $k > 0$.

- **Level-k thinking** (L) - Nagel 1995, Stahl and Wilson 1995
  - $g_k(h) = 1$ iff $h = k - 1$.
- **Poisson cognitive hierarchy** (CH) - Camerar, Ho, and Chong 2004
  - Assume that f follows a Poisson distribution, and

$$g_k\left(h\right) = \frac{f_h}{\sum_{m=0}^{k-1} f_m} \text{ for } h = 0, \cdots, k-1.$$

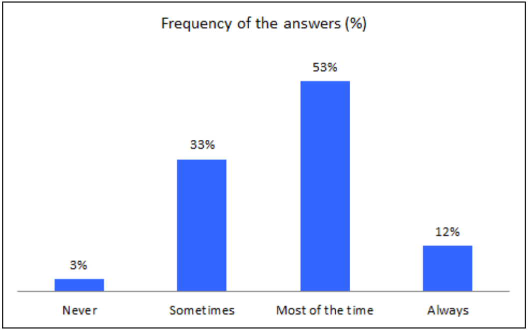
- In most games, detected levels are at most 2.

Overconfidence assumption

- Some psychological evidences *for* the overconfidence assumption (Camerer and Lovallo 1999)
- Advocates of the overconfidence assumption claim that the fixed-point problem is the main reason that players deviate from the equilibrium (e.g. Crawford, Costa-Gomes and Iriberri 2013).
- However, assuming complete lack of the ability of solving *any* fixed-point problem seems too extreme as a hypothesis.
- We rather think that the deviations arise from heterogeneity in the ability of solving fixed-point problems, which induces the players to form heterogeneous beliefs.
- Also, there are evidences *against* the overconfidence assumption.

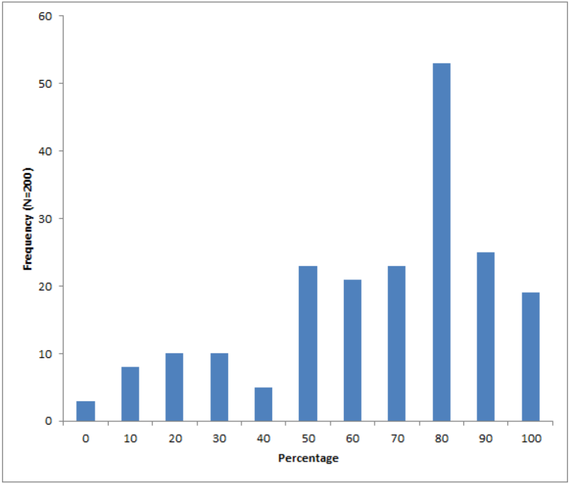
Endogenous cognitive hierarchy (ECH) model

- In our experiments, 194 out of 200 subjects gave a positive answer to the following question:
  - “When you made decisions, did you think that the other participants in your group used exactly the same reasoning as you did? - Never / Sometimes / Most of the time / Always”



Endogenous cognitive hierarchy (ECH) model

- "If answered yes in the previous question, what is the percentage of the other participants using the same reasoning, according to your estimation?"



The model

- Let $\langle N, S, u\rangle$ be a symmetric normal-form game where
  - $N = \{1, \cdots, n\}$ is the set of the players,
  - $S \subset \mathbb{R}$ is the set of pure strategies, and
  - $u : S^n \to \mathbb{R}^n$ is the payoff function.
- Let $f = (f_0, f_1 \cdots)$ be a distribution over $\mathbb{N}$. For each $k \in \mathbb{N}_+$, define $g_k = (g_k(0), \cdots, g_k(k))$ by:

$$g_k(i) = \frac{f_i}{\sum_{m=0}^{k} f_m} \text{ for } i = 0, \cdots, k.$$

  Then, a sequence of k-truncated distributions $g = (g_1, \cdots, g_k, \cdots)$ is uniquely defined from f.

Definition (ECH)

Fix $K \in \mathbb{N}$. A sequence of symmetric strategies $\sigma = (\sigma_0, \cdots, \sigma_K)$ is called **endogenous cognitive hierarchy equilibrium** when there exists a distribution f over $\mathbb{N}$ under which

$$\operatorname{supp}(\sigma_k) \subset \arg\max_{s_i \in S} \mathbb{E}_{s_{-i}} \left[u(s_i, s_{-i}) \,|\, g_k, \sigma \right], \ \forall k \in \mathbb{N}_+ \tag{ECH}$$

where g_k is the k-truncated distribution induced by f, and the expectation over s_{-i} is drawn from a distribution

$$\gamma_k(\sigma) := \sum_{m=0}^{k} g_k(m)\, \sigma_m$$

for each player $j \neq i$.

- A standard assumption for the underlying distribution f is Poisson: $f^{\tau}(k) = \tau^k e^{-\tau} / k!$.

Condorcet Jury Theorem

- n players make a collective decision, $d \in \{-1, 1\}$.
- State: $\omega \in \{-1, 1\}$, with common prior $\Pr[\omega = 1] = 1/2$.
- Homogeneous utility: $\exists q \in [0, 1]$ s.t.

$$u(d, \omega) = \begin{cases} 0 & \text{if } d \neq \omega, \\ q & \text{if } d = \omega = 1, \\ 1 - q & \text{if } d = \omega = -1. \end{cases}$$

- Each player receives a signal $s_i \sim \mathcal{N}(\omega, \sigma)$, conditionally independent, private info.
- After observed the signal, each player submits a vote $v_i \in \mathbb{R}$.
- The collective choice is taken by the ex post efficient decision rule:

$$\mu(v) = \text{sgn}\left(\sum_{i \in N} v_i + \frac{\sigma^2}{2} \ln\left(\frac{q}{1-q} \right) \right).$$

Collective efficiency

- Due to the normality assumption in this example, best reply is a bias strategy: $v_i(s_i) = s_i + b_i$ where $b_i \in \mathbb{R}$.
- The best reply function is:

$$\beta(b_{-i}) = -\sum_{j \neq i} b_j.$$

- Now, let us compute level-1 and level-2 strategies for:
  1. the standard level-k model (L),
  2. the Poisson cognitive hierarchy model (CH), and
  3. the endogenous cognitive hierarchy model (ECH).

- For any level-0 strategy b^0, the level-1 strategy (L1) is:

$$b^{L1} = -(n-1)b^0.$$

- The level-2 strategy (L2) is:

$$b^{L2} = (n-1)^2 b^0.$$

- The level-1 startegy in CH model (CH1) is the same as L1:

$$b^{CH1} = b^{L1}.$$

- The level-2 startegy in CH model (CH2) is:

$$b^{CH2} = -(n-1)^2 \frac{1-\tau}{1+\tau} b^0 + O(n).$$

- The level-1 strategy in ECH model (ECH1) solves:

$$b^{ECH1} = -(n-1)\left(\frac{1}{1+\tau}b^0 + \frac{\tau}{1+\tau}b^{ECH1}\right) + const,$$

  hence

$$b^{ECH1} = -\frac{n-1}{n\tau+1}b^0 + O(n^{-1}).$$

- The level-2 strategy (ECH2) satisfies:

$$b^{ECH2} = O(n^{-1})b^0 + O(n^{-1}).$$

Collective efficiency

Theorem

As $n \to \infty$, probability of correct decision making converges to:

1. 1 in the symmetric Nash equilibrium.
2. 1/2 in the standard level-k (L) and the CH model.
3. 1 in the ECH model.

Asymptotic properties

- We compare asymptotic properties of the following three models of cognitive hierarchy, as $n \to \infty$:
  - The standard level-k model (L)
  - The Poisson cognitive hierarchy model (CH)
  - The endogenous cognitive hierarchy model (ECH)
- We consider a sequence of symmetric games $\Gamma = \{G(n)\}_{n=1}^{\infty}$, in which the number of players increases.
- For each n, let $G(n) = \langle n, \mathbb{R}, \pi^n \rangle$ be a symmetric game with n players where the set of pure strategies is $\mathbb{R}$, and $\pi^n : \mathbb{R}^n \to \mathbb{R}$ is the payoff function.
- We assume that there is a unique symmetric Nash equilibrium in each game and normalize $x_*^n = 0$ for all n.

Definitions

Definition (asymptotic expansion)

A sequence $\{f_n\}_{n=1}^{\infty}$ of functions $f_n : \mathbb{R}^n \to \mathbb{R}$ is an **asymptotic expansion** if $\exists \{c_n\}_{n=1}^{\infty}$, $M > 0$ and $n' \in \mathbb{N}$ such that $\forall n \geq n'$, $c_n > Mn$ and $\forall x, y \in \mathbb{R}^n$, $|f_n(x) - f_n(y)| \geq c_n \left| \frac{1}{n} \sum_i (x_i - y_i) \right|$.

Definition (asymptotic contraction)

A sequence $\{f_n\}_{n=1}^{\infty}$ of functions $f_n : \mathbb{R}^n \to \mathbb{R}$ is an **asymptotic contraction** if $\exists \{c_n\}_{n=1}^{\infty}$ and $n' \in \mathbb{N}$ such that $\forall n \geq n'$, $c_n < 1$ and $\forall x, y \in \mathbb{R}^n$, $|f_n(x) - f_n(y)| \leq c_n \left| \frac{1}{n} \sum_i (x_i - y_i) \right|$.

- We say that a sequence of games $\Gamma = \{G(n)\}_{n=1}^{\infty}$ is an asymptotic expansion (resp. contraction), if the sequence of the best reply functions is an asymptotic expansion (resp. contraction).

Asymptotic properties

- Let $b^k(n)$ denote the k-th level strategy measured by the distance from the symmetric Nash equilibrium in $G(n)$, under one of the three cognitive hierarchy models L, CH and ECH.

Theorem

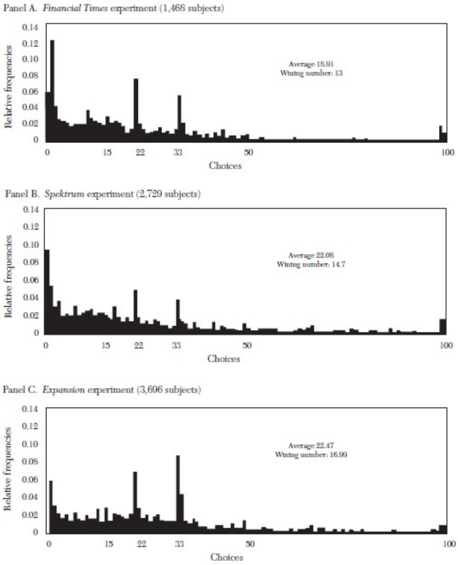
Consider an asymptotically expanding sequence of games $\Gamma = \{G(n)\}_{n=1}^{\infty}$. For any $b \neq 0$, let $b^0(n) = b$, $\forall n$. Then, $\left|b^k(n)\right|$ grows in the order of n^k in the L and the CH models, while it is bounded in the ECH model.

Theorem

Consider an asymptotically contracting sequence of games $\Gamma = \{G(n)\}_{n=1}^{\infty}$. For any $b \neq 0$, let $b^0(n) = b$, $\forall n$. Then, $\left|b^k(n)\right|$ is bounded in all the L, the CH and the ECH models.

Example: asymptotic contraction

- Bosch-Domènech et al. 2002

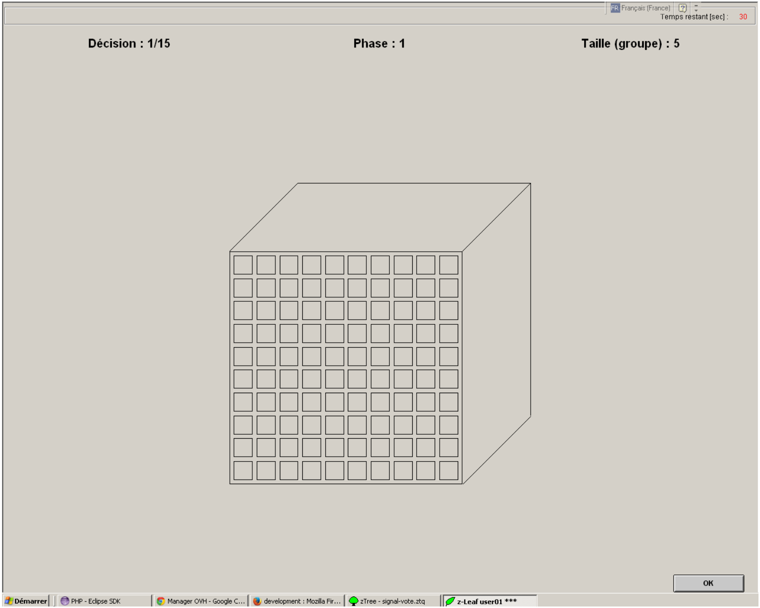


Our laboratory experiments

- Conducted at Ecole Polytechnique Experimental Lab.
- From November to December 2013
- 9 sessions, with 20 subjects in each session
- Subjects consist of students, graduate students, researchers, employees

The game

- A CJT game, but all ‘political’ terms are avoided to exclude any psychological effect.
  - Subjects are randomly partitioned to groups of size n. (three phases: $n = 5, 9, 19$)
  - Each group faces a box either blue or yellow with probability $1/2$, but the color is unknown to the subjects.
  - Each box contains 100 cards either blue or yellow. 60 cards have the same color as the box, 40 the other.
  - At each period, 10 cards are drawn randomly (independently across subjects) and the colors are revealed.
  - After seeing the cards, each subject votes either for blue or yellow.
  - Majority decision is taken for each group.
  - If the group decision is correct, all members win pre-determined points. If incorrect, no point.
  - Biased/unbiased prior: if win by blue, the award is 900 (800, 500) points, by yellow, 200 (300, 500) points.
  - This is repeated for 15 periods (group is reformed each period).
  - Monetary reward is given at the end, according to the obtained points.



Français (France)
Temps restant [sec] : 30
Décision : 1/15
Phase : 1
Taille (groupe) : 5
OK
Démarrer
PHP - Eclipse SDK
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development : Mozilla Fir...
zTree - signal-vote.ztq
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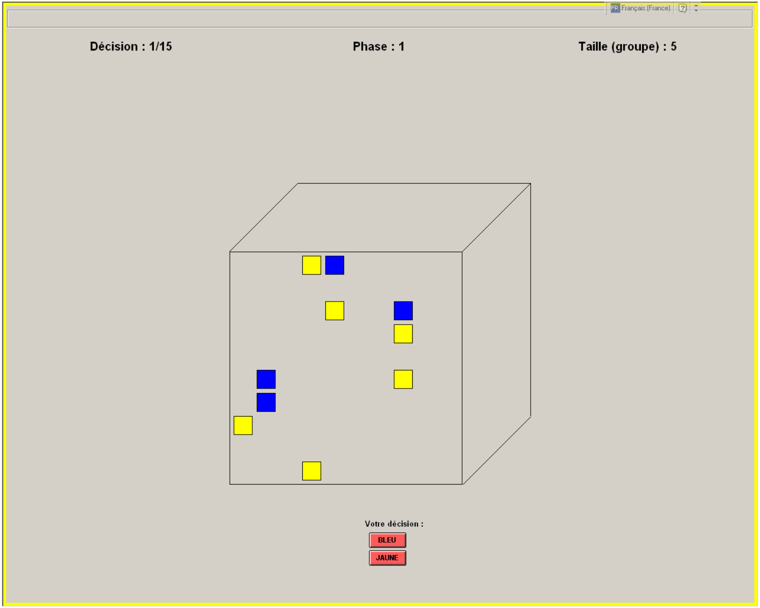
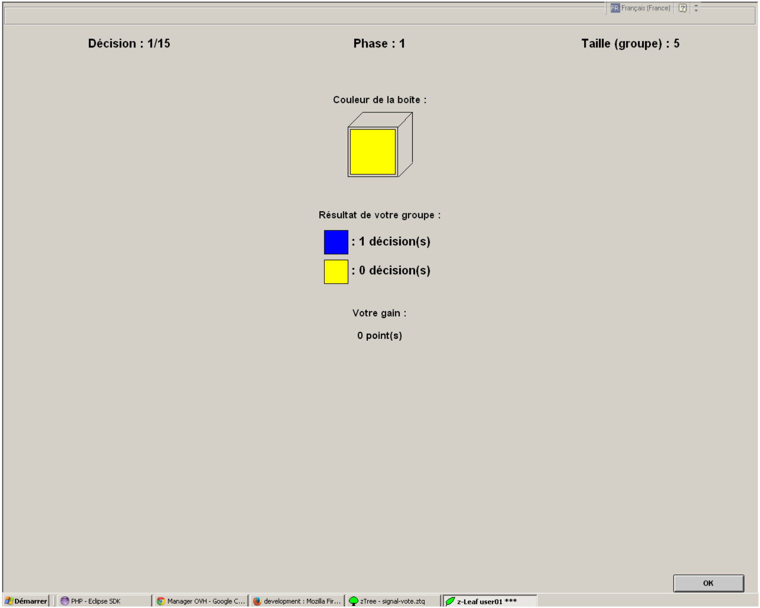

Français (France)
Décision : 1/15
Phase : 1
Taille (groupe) : 5
Votre décision :
BLEU
JAUNE


Français (France)
Décision : 1/15
Phase : 1
Taille (groupe) : 5
Couleur de la boîte :
Résultat de votre groupe :
: 1 décision(s)
: 0 décision(s)
Votre gain :
0 point(s)
OK
Démarrer
PHP - Eclipse SDK
Manager OVH - Google C...
development : Mozilla Fir...
zTree - signal-vote.ztq
z-Leaf user01 ***

Analysis: cutoff strategies

- Given any belief on other players' strategy, the best reply is a cutoff strategy (with logit errors).

Payoff	$n = 5$	$n = 9$	$n = 19$
900 : 200	4.43	4.67	4.84
800 : 300	4.66	4.84	4.93
500 : 500	5	N/A	N/A

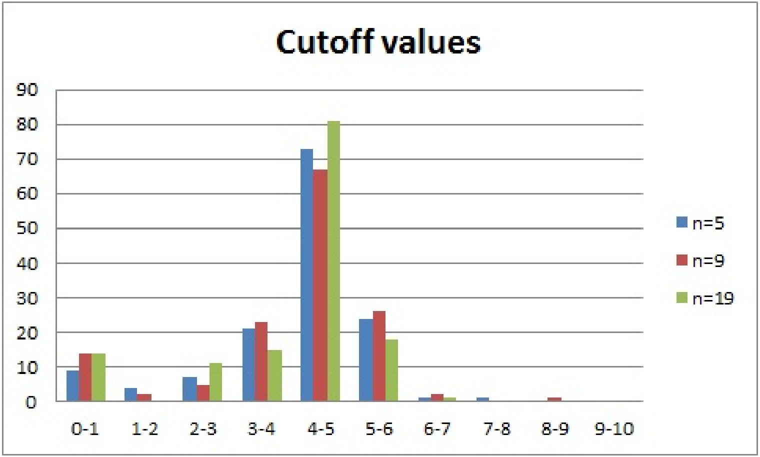
Table: Symmetric Nash Equilibria (NE) cutoff strategies

Payoff	$n = 5$	$n = 9$	$n = 19$
900 : 200	.833	.913	.980
800 : 300	.868	.935	.987
500 : 500	.879	N/A	N/A

Table: Predicted accuracy of group decisions (NE)

Analysis: cutoff strategies

- For each session, each phase, cutoff strategies are estimated.



Graph: histogram of the cutoff values, (payoffs 900:200, N=140)

Analysis: observed strategies

Payoff	$n = 5$	$n = 9$	$n = 19$	# obs
900 : 200	4.06	3.99	3.92	140
800 : 300	4.26	4.46	4.34	40
500 : 500	4.85	N/A	N/A	180

Table: Average of estimated cutoff strategies

Payoff	$n = 5$	$n = 9$	$n = 19$
900 : 200	0.776	0.833	0.774
800 : 300	0.813	0.833	0.958
500 : 500	0.806	N/A	N/A

Table: Average of observed frequencies of correct group decisions

Analysis

- Nash behavior is **not** detected, either symmetric or asymmetric.
- Observed values are more biased toward the prior, and the bias is intensified as the bias increases.
- Group accuracy is worse than theoretical predictions in all cases. However, the differences are not significant (difference in proportions test).
- Condorcet properties are **not** confirmed by our data.
- Decreasing accuracy with larger juries (Guarnaschelli et al. 2000).

Level-k estimation

- We set the level-0 strategy as 0 (always vote for the choice with prior bias).
- L1 strategy is 10 (the upper bound), L2 strategy is 0 (the lower bound).
- The standard level-k argument is not appealing for the games in which the best reply function is an expansion mapping.

CH estimation

	$n=5$		$n=9$		$n=19$	
Session	*CH*2	*LL*	*CH*2	*LL*	*CH*2	*LL*
1	3.243	-80.877	3.191	-101.250	2.961	-116.377
2	2.286	-70.727	1.938	-81.505	0.945	-105.269
3	3.085	-71.839	2.981	-84.109	2.675	-101.445
4	2.754	-76.815	2.577	-79.094	2.172	-88.452
5	3.249	-138.549	3.198	-80.788	2.976	-99.057
6	2.824	-60.806	2.662	-75.287	2.302	-88.340
7	3.098	-65.756	2.998	-78.415	2.692	-93.122

Table: The *CH*2 strategies and the log-likelihood values.

- CH0 strategy is 0.
- Then CH1 strategy is the same as L1: 10 (the upper bound).

ECH, $n = 5$

Session	τ^*	Level 0	Level 1	Level 2	LL
1	4.5	0	4.650	4.519	-26.070
2	2.2	0	4.636	4.425	-39.647
3	4.25	0	4.641	4.497	-25.417
4	2.0	0	4.726	4.519	-32.101
5	10.0	0	4.567	4.492	-21.459
6	10.0	0	4.510	4.424	-25.349
7	6.75	0	4.585	4.477	-32.687

Table: ECH model with $n = 5$

ECH, $n = 9$

Session	τ^*	Level 0	Level 1	Level 2	LL
1	4.75	0	4.960	4.721	-33.044
2	3.5	0	4.950	4.660	-33.621
3	10.0	0	4.810	4.685	-25.058
4	2.1	0	5.175	4.792	-35.984
5	10.0	0	4.839	4.709	-26.683
6	4.5	0	4.949	4.687	-37.910
7	6.5	0	4.888	4.696	-39.595

Table: ECH model with $n = 9$

ECH, $n = 19$

Session	τ^*	Level 0	Level 1	Level 2	LL
1	10.0	0	5.031	4.870	-21.948
2	3.5	0	5.324	4.853	-37.363
3	10.0	0	5.030	4.850	-38.118
4	3.0	0	5.404	4.890	-38.449
5	10.0	0	5.031	4.870	-35.075
6	10.0	0	5.020	4.830	-34.164
7	6.5	0	5.120	4.858	-33.342

Table: ECH model with $n = 19$

Group Accuracy under ECH

Phase	1	2	3	4	5	6	7	ave.
$n = 5$	0.824	0.730	0.824	0.752	0.844	0.822	0.832	0.804
$n = 9$	0.904	0.852	0.903	0.839	0.918	0.887	0.908	0.887
$n = 19$	0.979	0.946	0.977	0.957	0.984	0.969	0.978	0.970

Table: Predicted group accuracy: according to the best-fit ECH models

Analysis: ECH

- As n increases, the best-fit ECH strategies moves toward higher cutoffs.
- If computational burden increases as n increases, we may expect the best-fit τ^* to be decreasing in n. We do not observe such tendencies. (caveat: legitimacy of the Poisson assumption)
- Predicted group accuracy is higher than the actual data, especially in large groups.

Comparison

	$n=5$			$n=9$			$n=19$		
Session	CH	ECH	NE	CH	ECH	NE	CH	ECH	NE
1	-80.88	-26.07	-53.31	-101.25	-33.04	-58.31	-116.38	-21.95	-34.43
2	-70.73	-39.65	-46.52	-81.51	-33.62	-45.40	-105.27	-37.36	-51.35
3	-71.84	-25.42	-36.74	-84.11	-22.06	-31.08	-101.45	-38.12	-35.85
4	-76.82	-32.10	-53.92	-79.09	-35.98	-72.29	-88.45	-38.45	-70.05
5	-138.55	-21.46	-21.32	-80.79	-26.68	-24.06	-99.06	-35.08	-47.56
6	-60.81	-25.35	-25.11	-75.29	-37.91	-51.13	-88.34	-34.16	-40.14
7	-65.76	-32.69	-41.86	-78.42	-39.60	-49.08	-93.12	-33.34	-54.27

- Regardless the group size, ECH performs better than Nash in most cases.
- L and CH models do not explain the data at all.

Conclusion

- The overconfidence assumption in the standard level-k and the Poisson-CH model is too restrictive, especially in the games with expanding best reply functions.
- We suggest an endogenous cognitive hierarchy model.
- Predicted behaviors in the ECH better capture the idea of cognitive hierarchy in large games.
- Group decision making of the Condorcet Jury Theorem:
    - A game in which the sequence of the best reply functions is an asymptotical expansion.
    - The ECH fits better than the standard level-k, CH, and Nash.
    - Decreasing rationality (with respect to the group size) is *not* detected.

Extensions

- Non-Poisson estimations
- Overfitting
- Subjects' profile
- Learning