The Market for Surprises: Selling Substitute Goods through Lotteries.*

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This version: October 2015

Abstract

In this paper we solve the revenue maximization problem of multi-product monopolist when the products are substitutes. We consider a Hotelling model with two horizontally differentiated goods located at the endpoints of the segment. Consumers are located uniformly on the segment, their valuations for each of the goods equal to base consumption value minus distance costs. We consider different specifications for the distance cost function: linear, concave, and convex. When base consumption value is high, the seller maximizes her expected profit by offering a menu of base and opaque goods. A continuum of type-specific opaque goods is optimal under convex costs, whereas a single half-half lottery over base goods is optimal under concave and linear costs. When base consumption value is low, only base goods are sold. Finally, when base-consumption value is intermediate, the optimal mechanism may entail the offering of lotteries with positive probability of no delivery. Our findings can explain the emergence of opaque goods sales (e.g. hotel bookings without complete description of the hotel by hotwire.com or priceline.com) as the outcome of the industries search for the optimal selling scheme.

Keywords: Price Discrimination; Optimal Selling Strategies; Lotteries
JEL classification: D42, D43, L11

*The first versions of some of the results of this paper appeared in chapter "Price discrimination through lotteries" of PhD theses of Filippo Balestrieri and Joao Leao at MIT. We are grateful to Ufuk Akcigit, Abhijit Banerjee, Suman Basu, Mathias Dewatripont, Glenn Ellison, Bengt Holmstrom, Joao Montez, Jean Tirole, Julie Ward, and participants of numerous seminars and conferences at which versions of this paper were presented for their comments and suggestions. All remaining errors are our own.

1 Introduction

In recent years, as internet access becomes more widespread, online retailers have been experimenting with novel business strategies to succeed in a wider and more heterogeneous market. One notable innovation is the design and sale of *opaque* goods. A good is opaque when at the moment of sale its characteristics are purposely not (fully) revealed by the seller to the buyer. We can explain the emergence of opaque goods as the outcome of industries' search for the optimal way to sell.

Opaque goods are especially popular in the traveling and hospitality industry for selling hotels, plane tickets and car rentals, with Hotwire and Priceline being the top two users of such strategies.[1] For instance, at hotwire.com, a customer can either book a room at the hotel of her choice or select an opaque option and pay less. When opting for the latter, the customer is asked to specify some preferences (i.e., a city neighborhood, rating, extra requirements: a four star hotel with a gym in the Union Square area of San Francisco) and learns the hotel's actual identity only after the payment. In the case of plane tickets, the opaque option conceals the name of the airline and the exact flight time, whereas, for car rentals, the opaque offer is such that the name of the rental company and pick-up location are hidden. Opaque-goods sellers have appeared also in the clothing retail business and the passenger transportation industry.[2]

The common element of these business applications is that an opaque good can be any of a few clearly identified base goods that are horizontally differentiated substitutes and that are independently available for purchase. An often mentioned reason for opaque sales is that they enable sale of perishable inventory in an anonymous fashion. However, opaque goods can be purchased long in advance and in both high and low seasons. Why to offer a cheaper way for customers to acquire the same goods?

In this paper, we show that opaque goods, specifically lotteries over the base goods, are a way to implement the revenue-maximizing selling mechanism for the multi-product monopolist. We solve the optimal problem for the monopolist in the market with two horizontally differentiated substitute goods, represented by a Hotelling model (Hotelling (1929)) with goods located at the opposite ends of the

[1]Green & Lomanno (2012, p. 95) report that in 2010 online travel agencies accounted for 9.4% of all hotel bookings in the US, a quarter of which (2.3%) involved opaque goods.

[2]Web-store swimoutlet.com offers their customers a "*grab bag*" option at a discounted price. By purchasing it, a customer buys a swimsuit which color and pattern is revealed only after payment is made. Uber offers customers two options for booking a ride in a luxury car in San Francisco: the customer can specify the model (UberBLACK) or not (UberSELECT). In the second case, the cost of the booking is lower.

segment. We show that the optimal mechanism entails the offer of base goods and *lotteries* (i.e. opaque goods). Lotteries are used to price-discriminate consumers based on how strongly they prefer one base good over the other. Consumers with strong idiosyncratic preferences buy their favorite base goods at higher prices, while more indifferent consumers choose lotteries in order to take advantage of the lower price. Hence, opaque goods or lotteries play the role of damaged goods (see Deneckere & McAfee (1996)).

Depending on the shape of the buyers' preferences, the number and format of the optimal lotteries change. Consumers' utility for each good is a function of a base-consumption value and a distance cost. We consider settings in which the distance cost function which is assumed to be concave, convex, or linear, respectively. These different characterizations of the distance cost function allow us to represent different type of markets. Given the symmetry of our environment, without loss of generality, we can focus on each half-segment separately.

When the costs are concave, the two conditions for standard price discrimination are verified (e.g., see Tirole (1994)). First, consumers located at the extremes of the Hotelling line have a higher willingness-to-pay for their favorite base good and for any lottery that yields it with higher probability than the other good. Second, consumers at the end-points are the least likely to change their purchasing decision in order to take advantage of a price difference. This implies that the single-crossing condition holds. Concave cost settings can be thought of as representing markets where high income consumers, with low marginal utility of income and low price-sensitivity, are willing to pay more for the benefit of having their favorite good instead of the lottery.

When the costs are convex, on the contrary, there is no universal ranking of high-valuation and low-valuation consumer types: each lottery implies a different ranking. Given a lottery, the consumers with the highest willingness-to-pay for it may be located in the middle of the Hotelling line, in the region of indifferent consumers. Still, as in the concave costs case, consumers at the extremes are the least likely to modify their purchasing decision because of price differences across lotteries. Therefore, under convex costs, high income consumers (i.e. with the highest willingness-to-pay for a lottery) may be also the most price-sensitive. If we accept that wealth is positively correlated with sophistication or information level,[3] convex cost settings may represent markets in which the difference between base products is overvalued by customers with less cognitive ability or information.[4] This can be the case because lower income and less sophisticated

[3] In this regard, see, for example, Ellison (2005) and Hausman & Sidak (2004).

[4] An instance may be the market of substitute generic drugs in which less sophisticated customers overvalue specific brands (see Bronnenberg, Dubé, Gentzkow & Shapiro (2015)).

consumers are less able to take advantage of complex price discrimination mechanisms that often require high ability to process information and familiarity with internet.

The optimal mechanism is such that the monopolist always sells the base goods. When the base-consumption value is low, the seller only sells the base goods. However, for higher base-consumption values, the seller finds it optimal to add opaque goods to her menu of offerings. When the distance costs are linear, one opaque good, i.e. a lottery with $\frac{1}{2}$ probability of winning each base good, is sufficient to maximize the seller's profits. The optimal lottery is *sure-prize*: the buyer receives a prize for sure.[5] The same three-items menu (i.e. the two base goods and the $\left(\frac{1}{2}, \frac{1}{2}\right)$ lottery) is generally optimal in settings in which the distance costs are concave. However, under certain conditions, a different, more complex mechanism guarantees higher expected profits. In this case, the optimal menu includes an additional continuum of non-sure-prize lotteries with only one base good as prize and lower than 1 probability of winning. Finally, when the distance cost is convex, it is optimal to offer a continuum of type contingent sure-prize lotteries.

To derive the optimal mechanism, we follow the method used in Balestrieri & Izmalkov (2014). They consider a seller who is privately informed about the characteristics of the only good on sale and find that the optimal mechanism may entail only partial and private information disclosure and probabilistic allocations. In our setting, instead, the seller does not hold any private information and has multiple goods on sale. This implies not only a different setting but leads to a different optimal mechanism.

In our environment we depart from more standard mechanism design settings of pure vertical differentiation, because the horizontal differentiation assumption implies that consumers are exposed to countervailing incentives: reporting a more distant location from a base good automatically implies a higher proximity to the other base good (see Maggi & Rodriguez-Clare (1995)).

Our analysis contributes to the literature of optimal contracts for multi-product monopolists. Whereas a take-it-or-leave-it offer is known to be the optimal selling strategy for a single-product monopolist (Myerson (1981), Riley & Zeckhauser (1983)), the question regarding the optimal mechanism for a multi-product monopolist remains open. The problem has been tackled mostly in environments in which consumers have additive utility over the goods and multi-dimensional types.[6] Under these assumptions, Stigler (1963) and Adams & Yellen (1976) have showed that the sale of bundles can be revenue improving.[7] McAfee

[5] We borrow the terminology from Pavlov (2011*b*).

[6] A survey of works on multidimensional screening is Rochet & Stole (2003).

[7] Adams & Yellen (1976) introduce the concepts of pure bundling: when a bundle is the only

& McMillan (1988) determined conditions under which the optimal mechanism problem can be solved by deriving the optimal price of base goods and bundles.[8] Since then, several works have focused on determining the optimal price of deterministic allocations in complex multi-dimensional settings.[9]

More recently, it was shown that stochastic allocations (i.e. lotteries) could be part of the optimal mechanism in multi-goods environments (see Rochet & Choné (1998)). This is in stark contrast with the findings of Riley & Zeckhauser (1983) in single-good settings. They were the first to examine the use of lotteries (with different probability of delivering the good) as devices to price-discriminate across consumers; they proved that lotteries are not part of the optimal selling strategy. Thanassoulis (2004) showed by numerical examples that the no-lottery result does not extend to a setting with two substitute goods and two-dimensional buyer's types: lotteries can be revenue improving for a monopolist. Other examples were later shown by Manelli & Vincent (2006), Manelli & Vincent (2007), Pycia (2006), and Hart & Reny (2015) in settings with multiple goods and additive utility.[10]

Pavlov (2011*a*) and Pavlov (2011*b*) considered environments that satisfy the conditions described in McAfee & McMillan (1988) and obtained some new results. Without deriving the optimal mechanism, Pavlov showed that sure-prize lotteries are part of it, contrary to McAfee & McMillan (1988). Furthermore, he presented examples in which multiple (but a finite number of) sure-prize lotteries are offered by the monopolist to maximize her profit. Briest, Chawla, Kleinberg & Weinberg (2015) followed Pavlov (2011*a*) by estimating the monopolist maximal potential benefit obtained with sure-prize lotteries using dynamic programming techniques; still, they did not provide an analytical solution for the optimal mechanism. They proved that when the number of goods is at least three, there is no finite bound on the ratio between the revenue obtained by pricing lotteries and the revenue obtained by pricing only the base goods. On the other hand, Chawla, Malec & Sivan (2015) showed that in specific environments (i.e. when the private valuations of a buyer for the multiple goods are either independent or subject to a specific kind of positive correlation) the gains from lotteries are limited.

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purchasing option offered by the seller to the buyer; and mixed bundling: when both the bundle and individual goods are offered.

[8]Such conditions were revised in a more restrictive sense by Manelli & Vincent (2006).

[9]Prominent examples are McAfee, McMillan & Whinston (1989), Armstrong (1996), Armstrong (1999), McAdams (1999), and Cai & Daskalakis (2015).

[10]Thanassoulis & Rochet (2015) offered a characterization of the optimal mechanism for a settings with two goods and additive utility. They showed that conditional stochastic bundles are part of the optimal mechanism: if the buyer buys a first good, he receives also a lottery according to which the second good is delivered to him with some probability.

We propose an alternative approach. Instead of considering different distributions over the buyer's valuations for the base-goods, we take advantage of the well-known structure of the Hotelling model and examine different types of markets by considering different distance cost functions. Our optimal mechanism changes sharply as the cost function switches from concave to convex and presents features (e.g. non-sure-prize lotteries, continuum of lotteries) that are novel in the literature.[11] We prove that lotteries are optimal even in a setting with uni-dimensional buyer's types. In other words, the multi-dimensionality of the buyers' type space is not a necessary condition for lotteries to be optimal price-discrimination devices. Furthermore, we show that restricting the search for the optimal mechanism to sure-prize lotteries as in Pavlov (2011*b*) and Briest et al. (2015) may lead to forfeited profits. Finally, we identify precise conditions under which Thanassoulis (2004) conjecture is true: a simple menu with the two base goods and the $\left(\frac{1}{2}, \frac{1}{2}\right)$ lottery is optimal. In our setting this is the case for sufficiently high base consumption value and under linear or concave cost functions.

To the best of our knowledge, we are the first to fully characterize the optimal mechanism for a monopolist selling two substitute goods. We do so in a setting with a natural structure of consumers' preferences, i.e. the Hotelling model.[12] Our modeling choice is consistent with a relatively extensive literature that studies the properties of opaque goods as price-discrimination devices. Examples are Jiang (2007), Fay & Xie (2008), and Fay & Xie (2010). In all these works buyers preferences are modeled according to Hotelling models with linear costs and only one opaque good, the $\left(\frac{1}{2}, \frac{1}{2}\right)$ lottery, is considered. Instead, we do not impose any restriction on the mechanism space available to the monopolist and we consider a more general set of preferences. In our analysis, the types of opaque goods emerge as an endogenous decision of the profit maximizing seller, rather than being an exogenous attribute of the environment.

The rest of the paper is organized as follows. We present the model in Section 2. In Section 3 we set-up the optimization problem and in Section 4 we present the solution under different assumptions over the shape of the cost function. In Section 5 we consider several extensions of the model and offer an interpretation of our results for the product line design problem. Conclusions are in Section 6; proofs are in the Appendix.

[11] Daskalakis, Deckelbaum & Tzamos (2015) present a two-good example in which a continuum of lotteries is optimal. In their environment buyers have additive utilities and their valuations for the goods are independent and distributed according to very specific Beta distributions.

[12] Variations of our environment are considered in Thanassoulis (2004), Pavlov (2011*a*), Briest et al. (2015).

2 Model

We consider a variation of the Hotelling (1929) model of horizontal differentiation. There are two goods, indexed by $i = \{0, 1\}$, located at the two endpoints of a segment $[0, 1]$. There is a continuum of unit demand consumers uniformly distributed along the segment. The utility that a consumer (he) receives from consuming a good is represented as a function of his distance from that good. The utility of consuming good 0 for a consumer located at x, where $x \in [0, 1]$, is given by

$$U_0(x) = V - c(x) - p,$$

where V is a positive constant value, p is the price, and $c(x)$ is a generic transportation cost function. We assume that $c(\cdot)$ and $c'(\cdot)$ are continuous functions, $c(0) = 0$, and $c'(x) > 0$. Similarly, the utility of consuming good 1 is

$$U_1(x) = V - c(1 - x) - p.$$

Not consuming any good represents the outside option and its utility is zero. The consumer type, given by its location x, is private information. We consider both concave and convex transportation cost functions.

A monopolist (she) sells both goods with the objective of maximizing revenues. We assume that the marginal cost of producing each good is the same, and, without loss of generality, we set it equal to zero.

3 Optimization Problem

We want to identify the monopolist's optimal selling scheme. By the direct revelation principle, for any equilibrium of any selling mechanism there exists an outcome-equivalent equilibrium of a direct mechanism in which each player reports truthfully his private information. Then, without loss of generality, we can limit our attention to direct mechanisms, where each consumer reveals his location x. In order for truthtelling to be an equilibrium, individual rationality and incentive compatibility constraints have to hold.

Any direct mechanism μ consists of an allocation and a payment function, and can be represented as $\mu = (q_0, q_1, p)$, where $q_i(x)$ is the probability of obtaining good i, for $i \in \{0, 1\}$, and $p(x)$ is the required payment for the consumer reporting type x.

As consumers have unit demand, without loss of generality we can consider only feasible allocations such that

$$\forall x \in [0, 1], \qquad q_0(x) \geq 0, \quad q_1(x) \geq 0, \quad q_0(x) + q_1(x) \leq 1. \tag{F}$$

These probabilities can be thought of as parameters of a lottery with the two goods as prizes. Thus, we refer to a probabilistic allocation $(q_0(x), q_1(x))$ as a *lottery* $l(x)$. Lottery $l(x)$ is *sure prize* if $q_0(x) + q_1(x) = 1$.

Given mechanism $\mu = (l, p)$, let $U(y \mid x)$ be the utility of a consumer of type x reporting y, and $U(x) = U(x \mid x)$. By definition,

$$U(y \mid x) = q_0(y)(V - c(x)) + q_1(y)(V - c(1-x)) - p(y).$$

Individual rationality requires that the utility from reporting his true type is higher than the outside option.

$$\forall x \in [0,1], \qquad U(x) \geq 0. \tag{IR}$$

Incentive compatibility requires that each consumer prefers to report the truth when all other consumers are truthful.

$$\forall x, y \in [0,1], \qquad U(x) \geq U(y \mid x). \tag{IC}$$

The monopolist's problem is then

$$\begin{aligned} &\max_{q_0, q_1, p} \int_0^1 p(x)\, dx, \\ &\text{s.t.} \quad \text{IC, IR, F.} \end{aligned} \tag{MP}$$

A solution $\mu = (l, p)$ to (MP) is symmetric if $q_0(x) = q_1(1-x)$ for all $x \in [0,1]$.

Proposition 1. *There exists a symmetric solution to (MP).*

Proof. Existence of any solution follows from the fact that the set of mechanisms satisfying (IC), (IR), and (F) is convex, non-empty, and bounded. Clearly, if μ_1 and μ_2 are solutions, then $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ is also a solution for any $0 \leq \lambda \leq 1$. Finally, if $\mu = (l, p)$ is a solution, define $\mu' = (l', p'_1)$ as $q'_0(x) = q_1(1-x)$, $q'_1(x) = q_0(1-x)$, and $p'(x) = p(1-x)$. Then, $\frac{1}{2}\mu + \frac{1}{2}\mu'$ is a symmetric solution. $\square$

Lemma 1. *Any symmetric solution satisfies*

$$\forall x \in \left[0, \frac{1}{2}\right], \quad q_0(x) \geq q_1(x). \tag{1}$$

Proof. Follows from IC constraints for types x and $1-x$. $\square$

Therefore, to find a solution to (MP) it suffices to consider only symmetric mechanisms and solve (MP) over segment $\left[0, \frac{1}{2}\right]$ under additional constraint (1).

To solve the general problem of revenue maximization we proceed as in Balestrieri & Izmalkov (2014), where the optimal mechanism is derived for a seller who is privately informed about the attributes of a single good. In Balestrieri & Izmalkov (2014), the good can be either of type 0 or of type 1 and allocation probabilities q_i are defined as contingent on the realization of good i. In our setting, instead, both goods are simultaneously present and allocation probabilities are subject to feasibility constraint $q_0(x) + q_1(x) \leq 1$.

Consider the direct mechanism $\mu = (l, p)$. The utility function of type x

$$U(x) = q_0(x)(V - c(x)) + q_1(x)(V - c(1-x)) - p(x). \tag{2}$$

Following Myerson (1981) approach, using local IC constraints we can show that the derivative of $U(x)$ exists almost everywhere, and when it exists, it is defined by

$$U'(x) = -q_0(x)c'(x) + q_1(x)c'(1-x). \tag{3}$$

Accordingly, we can express $U(x)$ as

$$U(x) = U(0) - \int_0^x q_0(t)c'(t)\,dt + \int_0^x q_1(t)c'(1-t)\,dt. \tag{4}$$

Using (2) and (4), $p(x)$ can be expressed as

$$\begin{aligned} p(x) &= q_0(x)(V - c(x)) + q_1(x)(V - c(1-x)) \\ &\quad - U(0) + \int_0^x q_0(t)c'(t)\,dt - \int_0^x q_1(t)c'(1-t)\,dt. \end{aligned} \tag{5}$$

If we let x^* be a type with the lowest utility from mechanism μ, then using (4), we can express

$$U(0) = U(x^*) + \int_0^{x^*} q_0(t)c'(t)dt - \int_0^{x^*} q_1(t)c'(1-t)dt. \tag{6}$$

Then, using (5), (6), and collecting the double integrals, we can express the maximization of the expected revenue in the incentive compatible mechanism $(l(x), p(x))$ as

$$
\begin{aligned}
\max_{p,q_0,q_1} ER &= \int_0^1 p\,(x)\,dx \\
&= -\,U(x^*) + \int_0^{x^*} \left[q_0(x)A(x) + q_1(x)C(x)\right] dx \\
&\quad + \int_{x^*}^1 \left[q_0(x)B(x) + q_1(x)D(x)\right] dx, \\
&\quad \text{s.t. IC, IR, F.}
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
A\,(x) &= V - c\,(x) - c'\,(x)\,x, & B(x) &= A(x) + c'(x), \\
C(x) &= V - c(1-x) + c'(1-x)x, & D(x) &= C(x) - c'(1-x).
\end{aligned} \tag{8}
$$

Notice that $A\,(x) = D\,(1-x)$, $B\,(x) = C\,(1-x)$, and $A\left(\frac{1}{2}\right) = D\left(\frac{1}{2}\right)$, $C\left(\frac{1}{2}\right) = B\left(\frac{1}{2}\right)$.

Function $A(x)$ is the virtual valuation or marginal revenue from selling good 0 to consumer x assuming that all consumers between 0 and x also purchase good 0. Function $C\,(x)$ can be interpreted as the lost marginal revenue from not selling good 1 to consumers between 0 and x. A similar interpretation can be applied to functions $D(x)$ and $B\,(x)$. Integrands $q_0(x)A(x) + q_1(x)C(x)$ and $q_0(x)B(x) + q_1(x)D(x)$ are the marginal expected revenues from the consumer x for $x < x^*$ and $x > x^*$, respectively, for an incentive compatible direct mechanism $\mu = (q_0, q_1)$ with worst type x^*.

We define function $W\,(x)$ as

$$
W\,(x) = A\,(x) + \frac{c'\,(x)}{c'\,(1-x)} C\,(x)\,. \tag{9}
$$

Function $W\,(x)$ is important for deriving and presenting the optimal mechanism. It captures the effect of local trade-off (at and near of a specific type x) in marginal (expected) revenue while keeping utility constant. Indeed, under constraint $U'(x) = 0$, from (3) we obtain that goods' quantities have to satisfy

$$
q_1\,(x) = q_0\,(x)\,\frac{c'\,(x)}{c'\,(1-x)}. \tag{10}
$$

Under this constraint and from (8), the marginal expected revenues for type x in (7) are such that

$$
q_0(x)A(x) + q_1(x)C(x) = q_0(x)B(x) + q_1(x)D(x) = W(x)q_0(x).
$$

We assume that functions $A(x)$, $C(x)$, and $A(x)+C(x)$ are regular, that is, considering the interval $\left[0, \frac{1}{2}\right]$, $A(x)$ is strictly decreasing, $C(x)$ is strictly increasing, and $A(x)+C(x)$ crosses the x-axis at most twice.[13] Depending on the shape of the cost function, some of these assumptions are automatically satisfied. Strict concavity guarantees that $C(x)$ is increasing, strict convexity entails that $A(x)$ is decreasing. Under linear costs, all assumptions on $A(x)$, $C(x)$, and $A(x)+C(x)$ are satisfied.

Definition 1. *Let x_A, x_C, and x_W be solutions to $A(x_A)=0$, $C(x_C)=0$, and $W(x_W)=0$, respectively. Let $x_=$ be the value x such that $A(x_=)=C(x_=)$, and V^{AC} be the value V at which $A(x_=)=C(x_=)=0$.*

For arbitrary V and $c(x)$ these equations may not have a solution or have multiple solutions. We will be using x_A, x_C, and x_W only when solutions exist and are unique (on the interval of interest). Type $x_=$ exists for all V, as $A(0)>C(0)$ and $A\left(\frac{1}{2}\right) < C\left(\frac{1}{2}\right)$, and is independent of V. When $V=V^{AC}$, it is that $x_= = x_A = x_C = x_W$.

Given the regularity conditions, the following Lemma holds.

Lemma 2. *For any transportation cost function $c(x)$, we have*

$$x_= < \frac{1}{2}, \quad \frac{\partial x_A}{\partial V} > 0, \quad \frac{\partial x_C}{\partial V} < 0,$$

for any interior $x_A(V)$ and $x_C(V)$. Thus, if $V > V^{AC}$, then $x_C < x_= < x_A$; if $V < V^{AC}$, then $x_A < x_= < x_C$.

4 The solution

To solve the general problem of revenue maximization, as in Balestrieri & Izmalkov (2014), we first guess one value (or a set of values) for x^*. Then we compute $q_0(x)$ and $q_1(x)$ that maximize the expected revenue (7) type-by-type on $x \leq \frac{1}{2}$, subject to the feasibility constraints (F) and symmetry (1). Even though functions A, B, C, D capture implications of local IC constraints, there is no guarantee that the computed solution satisfies global IC constraints or the IR constraints of types x other than x^*. We verify if any of such constraints are violated and, if so, we recompute $q_0(x)$ and $q_1(x)$ taking into account violated constraints. Finally, we optimize over x^* (if needed).

[13] Irregular environments can be dealt with in a similar fashion as in the optimal auction problem for the single good (Myerson (1981)): one has to select among several local maxima. This is a straightforward albeit a cumbersome exercise.

The optimal mechanism takes different shapes depending on whether the transportation costs function is concave or convex. We examine each case separately. But first we consider the environment with linear costs, which is a common assumption in the literature and the limit case for both concave and convex costs.

4.1 Linear Costs

When costs are linear, i.e. $c(x) = x$, the utility of any lottery $l = (q, q)$ is constant across all consumers: $V - qx - q(1-x) = V - q$. Given that we look for a symmetric solution, from (1) it is clear that $x^* = \frac{1}{2}$.

The optimization problem (7) by symmetry and setting $U(x^*) = 0$, can be rewritten as

$$\max_{q_0, q_1, p} \int_0^{\frac{1}{2}} \left[q_0(x)A(x) + q_1(x)C(x)\right] dx, \quad (11)$$
$$\text{s.t. IR, IC, F}$$

where $U(x^*) = U\left(\frac{1}{2}\right) = 0$.

Note that for $x < x_=$ we have $A(x) > C(x)$, which implies that the marginal revenue of good $i = 0$ is higher than the marginal revenue of any lottery, given by $q_0A(x) + q_1C(x)$. Hence, for $x < x_=$, it is optimal to sell only good $i = 0$ whenever $A(x) > 0$. For $x_= < x \leq \frac{1}{2}$, we have $C(x) > A(x)$. The pointwise maximization of the integrand in (11) implies setting q_1 as high as possible, that is, offering lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$, whenever the (double) of its marginal revenue is positive, $A(x) + C(x) = 2V - 1 > 0$.

Finally, if $V > V^{AC} = \frac{1}{2}$, then both $x_A > x_=$ and $A(x) + C(x) > 0$. This implies that it is optimal to sell good 0 for $x < x_=$ and lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ for $x \in (x_=, \frac{1}{2})\#$; and if $V < V^{AC}$, the opposite inequalities are true and hence it is optimal to sell good 0 for $x < x_A$ and nothing for $x \in (x_A, \frac{1}{2})$

Given that, we introduce two mechanisms that are optimal for different base consumption values V. Such mechanisms are relevant not only when the costs are linear, but also in other settings (i.e. concave and convex). Therefore, for the sake of generality, we offer definitions with respect to a generic function $c(x)$ and threshold y. To ease the notation and for clarity, in the description and derivation of the optimal mechanisms we will omit the specification of allocation and prices for threshold values: one can take either the left or the right limit.

Mechanism $\mu^{bg}(y)$ (base good) for a threshold value $y < \frac{1}{2}$ is such that the monopolist sells the closest good (i.e. the degenerate lotteries $l = (1, 0)$ and $l = (0, 1)$) to the consumers close to the extremes of the Hotelling line and nothing to the consumers close to the middle.

$$\mu^{bg}(y) = \begin{cases} l(x) = (1,0), \quad p(x) = V - c(y), & \text{for } x < y, \\ l(x) = (0,0), \quad p(x) = 0, & \text{for } y < x < 1-y, \\ l(x) = (0,1), \quad p(x) = V - c(y), & \text{for } x > 1-y. \end{cases}$$

Mechanism $\mu^l(y)$ (lottery) for the threshold value $y < \frac{1}{2}$ is such that the monopolist sells the base goods to consumers close to the extremes of the segment and lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ to consumers close to the middle. Under the assumption of linear costs, all consumers have the same willingness to pay $U(x) = V - \frac{1}{2}$ for the lottery. Therefore, the sale of the lottery is such that the market is *fully-covered*: all consumers buy something (i.e. a good or the lottery) from the monopolist.

$$\mu^l(y) = \begin{cases} l(x) = (1,0), \quad p(x) = p^g, & \text{for } x < y, \\ l(x) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad p(x) = V - c\left(\frac{1}{2}\right), & \text{for } y < x < 1-y, \\ l(x) = (0,1), \quad p(x) = p^g, & \text{for } x > 1-y, \end{cases}$$

where $p^g = V - \frac{1}{2}c(y) + \frac{1}{2}c(1-y) - \frac{1}{2}$. The prices in both mechanisms are determined by the IC constraint for threshold type y and the IR constraint for the worst type $x^* = \frac{1}{2}$.

We have established

Proposition 2. *For the case of linear costs $V^{AC} = \frac{1}{2}$ and hence the optimal mechanism is*

$$\mu^* = \begin{cases} \mu^l\left(\frac{1}{4}\right), & \textit{for } V > \frac{1}{2}, \\ \mu^{bg}\left(\frac{V}{2}\right), & \textit{for } V < \frac{1}{2}. \end{cases}$$

Thus, the monopolist uses lotteries once the base-consumption value is sufficiently high, and only one lottery is sufficient to maximize the expected revenues. If the lottery is offered, the market is fully covered, and no surplus is left to the consumers who buy the lottery; whereas the ones who buy the base goods earn informational rents.

While offering lotteries brings more profits to the monopolist relative to the optimal posted prices for $V > \frac{1}{2}$, the welfare implications are ambiguous. There are two opposite effects from using lotteries: (i) an increase in market coverage, which is positive; and (ii) allocating not the most preferred good, which is clearly negative. The combined effect is zero at the threshold value $V = \frac{1}{2}$, when lotteries appear. If V increases, then the total welfare impact of the lotteries grows at first when the market coverage effect dominates, but then decreases and becomes negative after $V = \frac{5}{6}$, when the market coverage effect disappears.

4.2 Concave Costs

We consider $c(x)$ such that $c' > 0$ and $c'' < 0$ for all $x \in [0, 1]$. When the cost function is concave, in any symmetric incentive compatible mechanism, $U(x) \geq U\left(\frac{1}{2} \mid x\right) \geq U\left(\frac{1}{2}\right)$; therefore $x^* = \frac{1}{2}$ is surely a type for whom IR binds. Thus, as in the linear costs case, the optimization problem (MP) becomes (11).

Here we present a sketch of the solution, formulate the main proposition, leaving the complete proof and all technical details to the Appendix.

Unlike the case of linear costs, straightforward type-by-type optimization of (11) does not yield an incentive compatible mechanism for all V, and a careful treatment of IC constraints is needed.

Definition 2. *Let x_{+1} and x_{+2} be solutions to $A(x)+C(x) = 0$, so that $A'(x_{+1})+C'(x_{+1}) < 0$ and $A'(x_{+2}) + C'(x_{+2}) > 0$. Let*

$$R^* = \int_{x_=}^{\frac{1}{2}} A(x) + C(x)dx \quad \textit{computed at } V = V^{AC}.$$

As for other threshold types, we will be using x_{+1} and x_{+2} only when such solutions exist and are unique (on $x \leq \frac{1}{2}$). For the concave costs case, $A(x)+C(x)$ is strictly decreasing at $x = 0$ and is strictly increasing at $x = \frac{1}{2}$. Thus, given our regularity assumption, $A(x)+C(x)$ crosses 0 at most twice, and if it does so, we have $x_{+1} < x_{+2}$.

In Figure 1 we present the typical positioning of functions $A(x)$, $C(x)$, $A(x)+C(x)$ and value R^* for the base consumption value $V = V^{AC}$. As shown, $x_{+1} = x_=$, but this is not necessary. It can also be the case that $x_{+2} = x_=$, in which case $R^* > 0$ surely.

Consider the case shown in the Figure. As $A(x)+C(x)$ dips below 0, pointwise optimization yields selling lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ for $x > x_{+2}$ and selling nothing for intermediate consumer types $x \in (x_{+1}, x_{+2})$. This is not incentive compatible, as these intermediate types get positive utility from the lottery. Value R^* constitutes the (double of) expected revenue from selling lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ to all consumers between $x_=$ and $\frac{1}{2}$. If the lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ were the only lottery that can be offered, then, at $V = V^{AC}$ the optimal decision would be simple: if $R^* > 0$, that is if the gains exceed the losses, sell the lottery to types $x > x_=$; if $R^* < 0$, sell nothing to types $x > x_=$. However, the monopolist has a richer set of choices at

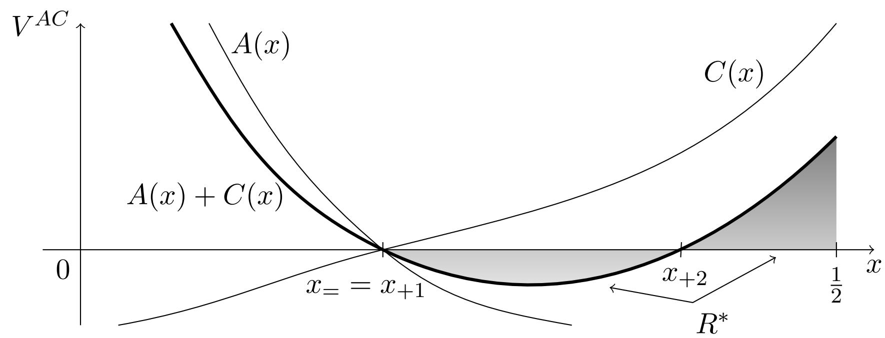

Figure 1: Functions $A(x)$, $C(x)$, $A(x)+C(x)$, and the value of R^* at $V = V^{AC}$.

his disposal, and so, depending on the sign of R^*, there are two main cases to consider.

Case 1: $R^* > 0$. As we shall see, for $V > V^{AC}$ it is optimal to sell lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ for all consumers $x > x_{=}$.

For $V < V^{AC}$, we show that if lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ is offered, then the best way to minimize losses from the intermediate types depends on the sign of $W(x)$. If $W(x) > 0$, it is optimal to maximize q_1 in the trade-off between q_0 and q_1 under tight global IC constraint implied by $l = \left(\frac{1}{2}, \frac{1}{2}\right)$; and if $W(x) < 0$, it is optimal to maximize q_0 in this trade-off. Thus, selling type-specific not-sure-prize one-good lotteries $l(x) = (\beta(x), 0)$ with $\beta(x) = \frac{1}{2}\left(1 - \frac{c'(1-x)}{c'(x)}\right)$ to consumers located in the interval (x_A, x_W) generates a smaller marginal loss (negative marginal revenue) than selling them lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$. We want to emphasize that a positive "fraction" of good $i = 0$ has to be offered to these consumers to satisfy global IC constraint given that lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ is also offered.

The decision on whether to offer any lotteries depends on the combined expected revenue from lotteries,

$$ER^*_{x>x_A} = \int_{x_A}^{x_W} \frac{1}{2}\beta(x)A(x)\,dx + \int_{x_W}^{\frac{1}{2}} \frac{1}{2}\left(A(x)+C(x)\right)\,dx. \tag{12}$$

Such value is well defined for $V < V^{AC}$.

Definition 3. *Let $V^{\#}$ be the value of V at which $ER^*_{x>x_A} = 0$.*

Altogether, we will show that for $V < V^{\#}$ mechanism $\mu^{bg}(x_A)$ is optimal, and for $V^{\#} < V < V^{AC}$ the following multiple lotteries mechanism $\mu^{ml}\left(x_A, x_W\right)$ is optimal.

Mechanism $\mu^{ml}(y,z)$ (multiple lotteries). Consider thresholds $y < z < \frac{1}{2}$. For consumers $x \in (y,z)$ a type-specific one good lottery is offered, such that the probability of receiving the good is less than 1. The lottery $l = \left(\frac{1}{2},\frac{1}{2}\right)$ is targeted for consumers $x \in (z, 1-z)$.

$$\mu^{ml}(y,z) = \begin{cases} l(x) = (1,0)\,,\ p(x) = p^{g}, & \text{for } x < y, \\ l(x) = (\beta(x),0)\,,\ p(x) = p^{l}(x), & \text{for } y < x < z, \\ l(x) = \left(\frac{1}{2},\frac{1}{2}\right)\,,\ p(x) = V - c\left(\frac{1}{2}\right), & \text{for } z < x < 1-z, \\ l(x) = (0,\beta(1-x))\,,\ p(x) = p^{l}(1-x), & \text{for } 1-z < x < 1-y, \\ l(x) = (0,1)\,,\ p(x) = p^{g}, & \text{for } x > 1-y, \end{cases}$$

where $\beta(x) = \frac{1}{2}\left(1 - \frac{c'(1-x)}{c'(x)}\right)$, $p^{g} = \frac{1}{2}c(1-y) - \frac{1}{2}c(y) + \left(V - c\left(\frac{1}{2}\right)\right)$, and $p^{l}(x) = \beta(x)V + \left(\frac{1}{2} - \beta(x)\right)c(x) + \frac{1}{2}c(1-x) - c\left(\frac{1}{2}\right)$.

The fact that mechanism $\mu^{ml}(y,z)$ entails the sale of non-sure-prize lotteries is noteworthy. To the best of our knowledge, we are the first to identify settings in which such lotteries are optimal.

Riley & Zeckhauser (1983) considered non-sure-prize lotteries in environments with one good and they proved that they are not optimal. Pavlov (2011*a*) shows that a two-good monopolist uses only sure-prize lotteries to maximize her revenues and such result is presented as an extension of Riley & Zeckhauser (1983): it is optimal to always guarantee the delivery of some good to any buyer. In our setting this is not the case. This is due to the combined effect of concave costs, as long as $R^{*} > 0$, and countervailing incentives. Indeed, given any lottery $l = (q_0, q_1)$, customers' willingness-to-pay decreases, whereas its marginal revenue (i.e. $q_0 A(x) + q_1 C(x)$) is not monotonic. When one good is on sale and its marginal revenue is non-monotonic, the optimal solution is derived by applying ironing techniques. With one control variable, the probability of sale, the outcome is one of the extremes, either sell with probability 1 or not. When two goods are on sale, there are two control variables with different marginal benefits ($A(x)$ and $C(x)$). Richer possibilities allow for non-extreme solutions to be optimal.

Case 2: $R^{*} < 0$. In this case it is not optimal to sell any lotteries for $V \leq V^{AC}$. For $V > V^{AC}$, if $x_{=} < x_{+1} < \frac{1}{2}$, there is a choice whether to offer lottery $l = \left(\frac{1}{2},\frac{1}{2}\right)$ to all consumers $x > x_{=}$ or only to $x \in (x_{=}, x_{+1})$. This decision depends on the difference in expected revenues from these two choices,

$$ER^{*}_{x>x_{+1}} = \int_{x_{+1}}^{\frac{1}{2}} \frac{1}{2}\left(A(x) + C(x)\right)\, dx. \tag{13}$$

Such value is well defined for $V > V^{AC}$.

Definition 4. *Let $V^{\#\#}$ be the value of V at which $ER^*_{x>x_{+1}} = 0$.*

Altogether, we will show that for $V > V^{\#\#}$ mechanism $\mu^l(x_=)$ is optimal, while for $V^{AC} < V < V^{\#\#}$ the following mechanism with no sales region in the middle $\mu^{lns}(x_=, x_{+1})$ is optimal.

Mechanism $\mu^{lns}(y, z)$ (lottery with no sale region) for $y < z < \frac{1}{2}$ is a modification of $\mu^l(y)$. The lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ is priced in such a way that a "no sale" region appears in the middle of the Hotelling line.

$$\mu^{lns}(y,z) = \begin{cases} l(x) = (1,0), \quad p(x) = p^g, & \text{for } x < y, \\ l(x) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad p(x) = p^l, & \text{for } y < x < z, \\ l(x) = (0,0), \quad p(x) = 0, & \text{for } z < x < 1 - z, \\ l(x) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad p(x) = p^l, & \text{for } 1 - z < x < 1 - y, \\ l(x) = (0,1), \quad p(x) = p^g, & \text{for } x > 1 - y, \end{cases}$$

where $p^g = V - \frac{1}{2}(c(y) + c(z)) + \frac{1}{2}(c(1-y) - c(1-z))$, and $p^l = V - \frac{1}{2}c(z) - \frac{1}{2}c(1-z)$.

Proposition 3. *For the case of concave costs the optimal mechanism is*

$$\mu^* = \begin{cases} \left.\begin{array}{ll} \mu^l(x_=), & \textit{for } V > V^{AC}, \\ \mu^{ml}(x_A, x_W), & \textit{for } V^{\#} < V < V^{AC}, \\ \mu^{bg}(x_A), & \textit{for } V < V^{\#}, \end{array}\right| & \textit{if } R^* > 0; \\ \left.\begin{array}{ll} \mu^l(x_=), & \textit{for } V > V^{\#\#}, \\ \mu^{lns}(x_=, x_{+1}), & \textit{for } V^{AC} < V < V^{\#\#}, \\ \mu^{bg}(x_A), & \textit{for } V < V^{AC}, \end{array}\right| & \textit{if } R^* < 0. \end{cases}$$

The proof of Proposition 3 is in Appendix A.

On Figure 2 we show the optimal allocation function for good $i = 0$, $q_0(x)$, for different values of V and depending on R^*. Allocation function $q_1(x)$ can be obtained by symmetric reflection of $q_0(x)$ over $x = \frac{1}{2}$ axis (dashed lines).

When the transportation cost function is concave the monopolist uses lotteries to price discriminate among consumer's types based on their degree of indifference

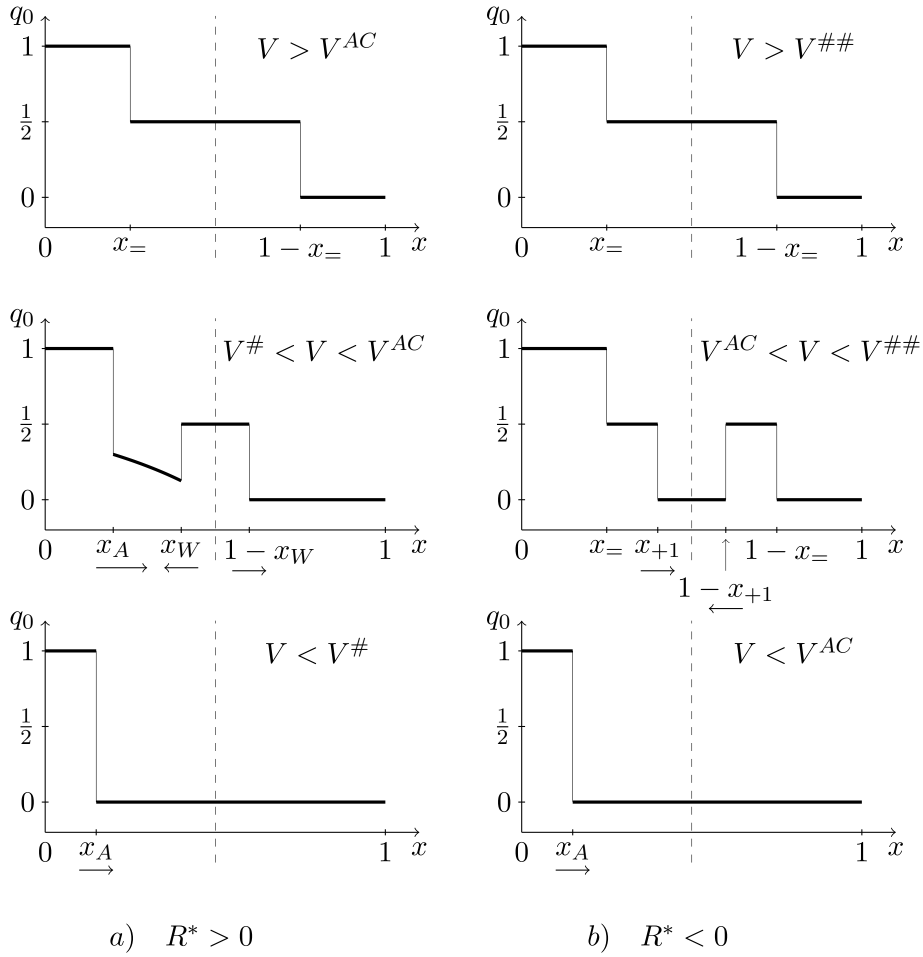

Figure 2: Optimal allocation function $q_0(x)$ for concave costs depending on R^*. Allocation function $q_1(x)$ can be obtained by symmetric reflection of $q_0(x)$ over $x = \frac{1}{2}$ axis (dashed lines). Horizontal arrows show how the relevant thresholds move when V increases.

between goods. For V high enough, the optimal mechanism involves charging a relatively high price for the base goods and a lower price for a lottery over the substitute goods. The base goods are sold to the consumer's types with strong preferences for each base good, who are located closer to the extremes of the Hotelling segment. The lottery is sold to consumers that have a relatively higher indifference between the substitute goods, located closer to the center of the segment.

Compared to the linear case, there are three notable differences: (i) lotteries do not necessarily cover the entire market; (ii) lottery buyers are left with some surplus; and (iii) even though whenever lotteries are offered lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ is always on sale, different (non-sure-prize) lotteries may be also components of the optimal menu.

4.3 Convex Costs

We consider $c(x)$ such that $c' > 0$ and $c'' > 0$ for all $x \in [0, 1]$. As for the case of concave costs, here we offer a sketch of the solution, formulate the main proposition, leaving details to Appendix A.2..

Unlike the linear and concave costs cases, when the with transportation costs function is convex, we cannot say in advance which consumer type is the worst type x^* in any incentive compatible mechanism. The lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ is the most preferred by consumers in the middle, the base goods are most preferred by the consumers at the extremes. Thus, we can decompose the whole optimization problem into two components: optimize given x^* and then optimize over x^*.

For a symmetric mechanism, for $x^* < \frac{1}{2}$, the objective of (7) can be rewritten as

$$\begin{aligned} &2 \int_0^{x^*} \left[q_0(x)A(x) + q_1(x)C(x)\right] dx \\ &+ \int_{x^*}^{\frac{1}{2}} \left[q_0(x)\left(A\left(x\right) + B\left(x\right)\right) + q_1(x)\left(C\left(x\right) + D\left(x\right)\right)\right] dx. \end{aligned} \tag{14}$$

Let us take a look at $x < x^*$ first. If $x^* > x_=$ and $A(x) + C(x) > 0$ for $x \in (x_=, x^*)$, then the type-by-type maximization of the first integrand of (14) gives $l(x) = \left(\frac{1}{2}, \frac{1}{2}\right)$. This allocation violates IR constraint for these customers. Optimization of the first part of (14) under the binding IR constraint, $U(x) = U(x^*)$ on $x \in (x_=, x^*)$ produces selling multiple type specific sure prize lotteries $l(x) = (\gamma(x), \gamma(1-x))$, where $\gamma\left(x\right) = \frac{c'(1-x)}{c'(1-x)+c'(x)}$.

Considering $x > x^*$, it can be shown that the second integrand of (14) under the binding IR constraint can be rewritten as

$$q_0(x)\left(A\left(x\right)+B\left(x\right)\right)+q_1(x)\left(C\left(x\right)+D\left(x\right)\right)=2W\left(x\right)q_0\left(x\right).$$

As we show in the Appendix, $W(x) > 0$ on $x < x_W$. Thus, if $x^* < x_W$, then the type-by-type maximization of the second integrand gives $l(x) = (1, 0)$. However, this allocation violates IR constraint for customers $x \in (x^*, x_W)$. Once the binding IR constraint is taken into account, lotteries $l(x) = (\gamma(x), \gamma(1-x))$ are optimal for customers for which $W(x) > 0$.

Given these observations, we can show that type specific lotteries are offered if the base consumption value is sufficiently high, so that $x_A < x_=$, for all customer types $x > x_=$ for which $W(x) > 0$. In turn, all of these customer types have their surpluses fully extracted.

Accordingly, we define mechanisms $\mu^{ml2}(y)$ and $\mu^{mlns}(y)$. Mechanism $\mu^{ml2}(y)$ for a threshold $y < \frac{1}{2}$ is such that the monopolist price discriminates consumers by selling a set of type-specific sure prize lotteries.

$$\mu^{ml2}\left(y\right)=\begin{cases} l\left(x\right)=\left(1,0\right), \quad p\left(x\right)=V-c\left(y\right), & \text{for } x<y, \\ l(x)=\left(\gamma\left(x\right),1-\gamma\left(x\right)\right), \quad p\left(x\right)=p^{l}\left(x\right), & \text{for } y<x<1-y, \\ l\left(x\right)=\left(0,1\right), \quad p\left(x\right)=V-c\left(y\right), & \text{for } x>1-y, \end{cases}$$

where $\gamma\left(x\right)=\frac{c'(1-x)}{c'(1-x)+c'(x)}$, $p^l(x) = V - \gamma(x)c\left(x\right) - \gamma(1-x)c\left(1-x\right)$.

Mechanism $\mu^{mlns}(y)$ for two thresholds $y < z < \frac{1}{2}$ is a variation of mechanism $\mu^{ml2}(y)$ in which a "no-sale" region is introduced. Indeed, in equilibrium some consumers do not buy neither a lottery nor the base good.

$$\mu^{mlns}\left(y,z\right)=\begin{cases} \mu^{ml2}\left(y\right), & \text{for } x<z \text{ and } x>1-z, \\ l(x)=(0,0),\ p(x)=0, & \text{for } z<x<1-z. \end{cases}$$

Before we formulate the proposition, we would like to note that differently from the setting with concave costs, when $c(x)$ is strictly convex, threshold V^{AC} is such that $V^{AC} < c\left(\frac{1}{2}\right)$.

Proposition 4. *For the case of convex costs the optimal mechanism is*

$$\mu^{***}=\begin{cases} \mu^{ml2}\left(x_=\right), & \textit{for } V>c\left(\frac{1}{2}\right), \\ \mu^{mlns}(x_=,x_W), & \textit{for } V^{AC}<V<c\left(\frac{1}{2}\right), \\ \mu^{bg}(x_A), & \textit{for } V<V^{AC}. \end{cases}$$

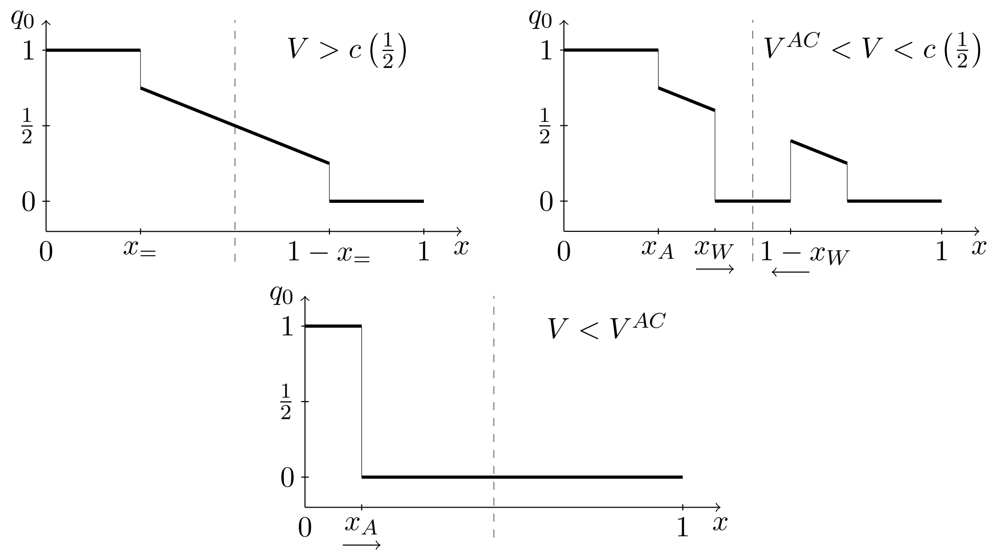

Figure 3: Optimal allocation function $q_0(x)$ for convex costs (drawn for $c(x) = \frac{1}{2}x^2$). Allocation function $q_1(x)$ can be obtained by symmetric reflection of $q_0(x)$ over $x = \frac{1}{2}$ axis (dashed lines). Horizontal arrows show how the relevant thresholds move when V increases.

The proof of Proposition 4 is in Appendix A.2.

On Figure 3 we show the optimal allocation function for good $i = 0$, $q_0(x)$, for different values of V and depending on R^*. Allocation function $q_1(x)$ is symmetric to $q_0(x)$ around $\frac{1}{2}$.

When $V > c\left(\frac{1}{2}\right)$, the marginal revenue of lottery $l(x) = (\gamma(x), 1 - \gamma(x))$ is positive for all x in $\left(0, \frac{1}{2}\right)$. Therefore, it is optimal for the monopolist to fully cover the market by selling the base goods to consumers located in the interval $(0, x_=)$ and type-specific lotteries $l(x) = (\gamma(x), 1 - \gamma(x))$ to each consumer x such that $x \in \left(x_=, \frac{1}{2}\right)$.

When $V^{AC} < V < c\left(\frac{1}{2}\right)$, it is shown that the marginal revenue of lottery $l(x) = (\gamma(x), 1 - \gamma(x))$ is positive for $x < x_W$. Therefore, in this case it is optimal to sell lotteries for $x \in (x_=, x_W)$ and sell nothing for $x \in (x_W, \frac{1}{2})$.

When $V < V^{AC}$, the marginal revenue from lottery $l(x) = (\gamma(x), 1 - \gamma(x))$ is negative for $x > x_A$, and so no lotteries are offered.

As in the concave case, the optimal mechanism is such that the monopolist price discriminates based on consumers' degree of indifference between goods. In this setting, however, the price discrimination is not obtained through one simple lottery (i.e. lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$). On the contrary, the optimal mechanism entails a set of type-specific *sure-prize* lotteries. This result depends crucially on the convexity of the transportation costs function.

As in the linear case, lotteries are priced in order to extract the full surplus from consumers. However, in a setting with convex costs, it may or may not be optimal for the monopolist to fully cover the market.

5 Extensions and an Application

The optimality of lotteries for multi-product monopolists is a robust result that persists in symmetric environments with more than two products and asymmetric settings.

The common extension of the Hotelling model to study multiple products is the Salop model (Salop (1979)). Such model is represented as a circle and the symmetric products can be interpreted as equally distant locations on it. Such model implies that each customer can fully rank all products in the market and can be indifferent between two products at most. In order to derive the optimal mechanism in such context, it is sufficient to notice that each arc connecting any pair of consecutive products is isomorphic to the Hotelling line that we studied in the previous sections. Given that, if the number of products is n, then the optimal mechanism entails n two-product lotteries. Allocation probabilities and prices associated with the lotteries depend on the base-consumption value and the shape of the transportation cost function as derived for the Hotelling model.

Another type of extension implies different assumptions on the information regarding the base consumption value V. If such value is assumed to be seller's private information, then, as in Yilankaya (1999) and Mylovanov & Tröger (2012), it is optimal for the seller to fully disclose ex-ante the value of V. This is because the seller's expected revenue is increasing in V and full separation is the only equilibrium. Such result is indeed consistent with the observed behavior of opaque good sellers assuring buyers about the equivalent quality of all base goods in the opaque good's support (e.g. grouping together only hotels with the same rating or flights with the same number of stops). Hence, our optimal mechanism is also the optimal solution in such "informed seller" setting.

Alternatively, we can assume that V is buyer's private information. In such environment buyer's types are bi-dimensional: a type is defined as the tuple (V, x). It is useful to observe that if types (V, x) and (V', x) for $x \neq \frac{1}{2}$ are both assigned sure prize lotteries, their allocations and prices have to be the same.

Indeed, the incentive compatibility constraint for type (V, x) relative to $(V'x)$ is

$$q_0(V,x)(V-c(x))+q_1(V,x)(V-c(1-x))-p(V,x)\geq \\ q_0\left(V',x\right)(V-c(x))+q_1\left(V',x\right)(V-c(1-x))-p\left(V',x\right).$$

When $q_0 + q_1 = 1$, it can be rewritten as

$$q_0(V,x)(c(1-x)-c(x))-p(V,x)\geq q_0\left(V',x\right)(c(1-x)-c(x))-p\left(V',x\right).$$

Similarly, for type (V', x):

$$q_0\left(V',x\right)(c(1-x)-c(x))-p\left(V',x\right)\geq q_0(V,x)(c(1-x)-c(x))-p(V,x).$$

The incentive compatibility constraint is independent of V and, therefore, types (V, x) and (V', x) can be assigned the same allocation q_0 (and ergo q_1) and payment p.

The value of V matters only for its role in the individual rationality constraint. When the cost function is linear (i.e. $c(x) = x$) and V is distributed according to a uniform distribution over a segment $\left[V^L, V^H\right]$ where V^L is a sufficiently high value (so that the restriction on considering only sure-prize lotteries is not binding), then the optimal mechanism is

$$\mu^*\left(V^*,\frac{1}{4}\right)=\begin{cases} l=(1,0), & p=p^g, \quad \text{for } (V,x): V>V^* \text{ and } x<\frac{1}{4}, \\ l=\left(\frac{1}{2},\frac{1}{2}\right), & p=V^*-\frac{1}{2}, \quad \text{for } (V,x): V>V^* \text{ and } \frac{1}{4}<x<\frac{3}{4}, \\ l=(0,1), & p=p^g, \quad \text{for } (V,x): V>V^* \text{ and } x>\frac{3}{4}, \\ l=(0,0), & p=0, \quad \text{for } (V,x): V<V^* \text{ and } \forall x, \end{cases}$$

where $p^g = V^* - \frac{1}{4}$ and

$$V^*=\arg\max_V\left(\int_0^{\frac{1}{4}}\left(V-\frac{1}{4}\right)dx+\int_{\frac{1}{4}}^{\frac{1}{2}}\left(V-\frac{1}{2}\right)dx\right)\left(\frac{V^H-V}{V^H-V^L}\right).$$

Similarly, when the cost function is concave, as long as the base consumption value is sufficiently high, the optimal mechanism is derived by maximizing the expected seller's revenue with respect to V. The case of convex cost is more complicated, because, in the pointwise (with respect to V) optimal mechanism, the individual rationality constraint defines the allocation and payment of multiple types. As a result, V and x are not separable.

A slightly more complicated extension is the one regarding asymmetric settings. Several sources of asymmetry can be considered (e.g. different base-consumption levels across goods, asymmetric cost functions, non-uniform distributions of customers, etc.). They can be treated with different degrees of difficulties with the same methodology as considered here. Without symmetry, one would have to "guess" x^* and optimize on both $x < x^*$ and $x > x^*$ separately.

As an example, consider a setting with asymmetric locations. We propose a slight variation of the two-good model: we replace good 0 with a good k that is located at $x = k$, where $k \in (0, 1)$. In this new framework, the subset $[0, k]$ represents a sort of captive market of consumers most interested in good k. The asymmetry of the environment implies that the solution is also asymmetric. As in our original model, the monopolist realizes price discrimination by offering lotteries targeted to the consumers who are relatively more indifferent between the goods. However, the base good k commands a higher price than the base good 1 as its demand is relatively less elastic. Furthermore, the monopolist still includes in its menu the symmetric $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ lottery.

5.1 Product Line Design

The single lottery mechanism is often directly observed in the real world. For example, in the market for hotels, hotwire typically only offers one lottery for each city area and rating level. Instead, consumers are not typically offered a selection among multiple lotteries options as it would be prescribed by the optimal mechanism with convex costs. However, an alternative interpretation of our setting can offer a way to reconcile what we derived with business practice.

As described in Lancaster (1971), goods can be studied as collections of different attributes. Consumers have preferences for attributes rather than goods. Therefore, the process of determining which goods to market becomes equivalent to the selection of which attributes to mix together (and in which proportions). Then, multi-product mechanism design models with additive utility can be used also to solve optimal product line design problems:[14] goods are reinterpreted as attributes and bundles (and lotteries) become products.[15]

To reinterpret our setting with substitute goods in terms of product design choices (as in Pavlov (2011*a*)), few adjustments are necessary. The endpoints of the Hotelling line can be seen as representing two ideal product configurations. Buyers have different willingness-to-pay for them. Moreover, their preferences are negatively correlated: the more a buyer is willing to pay for one configuration,

[14]The seminal paper Mussa & Rosen (1978) set up the optimal product line design problem. More recent works are Johnson & Myatt (2014) and Anderson & Celik (2014).

[15]See McAfee & McMillan (1988), Rochet & Choné (1998), and Pycia (2006), for example.

the less he is willing to pay for the other. Accordingly, each lottery represents a *mixed* good of inferior quality which specifications have elements from the two ideal configurations.[16] Following this interpretation of the model, our optimal mechanism offers insights on product assortment strategies based exclusively on a price-discrimination argument.

Our solution shows that it is always optimal for the monopolist to sell goods that perfectly match the ideal configurations (or *pure* goods). When the indifferent buyers have the highest willingness-to-pay for *mixed* goods (i.e. the costs are convex), then the monopolist maximizes her profits by adding a continuum of products to his stock; when, instead, the indifferent consumers have the lowest willingness-to-pay for mixed goods (i.e. the costs are concave), then the seller optimally offers only one, unique mixed good. Our solution seems to offer a way to rationalize the fact that we observe different firms successfully adopting opposite SKU (Stock Keeping Units) management practices at the same time. The convex case justifies firms' choice of marketing a wide array of different SKUs, whereas the concave case explains how some firms prefer to keep a short product list.[17]

6 Conclusion

In this paper we fully characterize the optimal selling mechanism for a two-product monopolist. The solution depends on the value of the base-consumption and on the shape of transportation costs. In general, as long as the base-consumption value is sufficiently high, we show that the monopolist uses lotteries to price discriminate consumers based on their degree of indifference between the two substitute goods. The shape of the transportation costs determine which lotteries are optimal. By considering different transportation cost functions, we

[16]An example is the market of high-end PCs that is divided between consumers that are either "gamers" or "business analysts". The PC device is size-constrained and different groups request different components (e.g. a powerful graphic processor versus a big data storage disk). There are two ideal configurations that cater to the two different segments; any mixed-specs product is perceived as inferior by both segments. Alternatively, we can think of digital TV subscriptions for soccer fans. A soccer fan is usually interested in watching all the season games of his/her favorite team (i.e. the ideal configuration). Subscription options that mix games of different teams are sold at cheaper prices (as lower quality goods).

[17]In 2013 Amazon USA was selling over 200 million products in the USA categorized into 35 departments. There were almost 5 million items in the Clothing department, almost 20 million in Sports & Outdoors, and over 4 million Office Products. There were 7 million items in the Amazon Jewelry department, 24 million in Electronics, 1.4 million products in the Beauty department, 570 thousand Baby products, and 600 thousand Grocery items. In 2012 Costco, a US major retailer, was reported to hold only 3950 SKUs in total.

model different kinds of markets.

We discuss the generality of the main features of our optimal mechanism considering environments with vertical differentiation, multiple goods and asymmetric settings. In particular, we stress the robustness of our results regarding lotteries: it is optimal for a multi-product monopolist to sell lotteries in a variety of settings.

After examining the optimality of lottery-based mechanisms in a setting with a multi-product monopolist, we are interested in analyzing the appearance of lotteries when there is competition. In our companion paper, we consider a setting in which each substitute good is sold by an independent firm. All firms compete with each other in a market with unit-demand consumers. In such context, we look at the possibility for intermediaries to enter the market and offer the service to organize (and sell) multi-good lotteries. We study the competitive equilibria that arise under different conditions in terms of base-consumption and transportation cost function.

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A Appendix

A.1 Concave Costs

In Section 4.2 we have already pointed out two essential elements of the proof. First, since $A(x) + C(x)$ can dip below 0 (as illustrated in Figure 1) for some values of V, the point-wise optimization of the integrand in (11) yields selling nothing to types x for which $A(x) + C(x) < 0$ and $A(x) < 0$, but that would violate global IC constraint for these types if there are some other types closer to $x = \frac{1}{2}$, for which $A(x) + C(x) > 0$, and which are offered a lottery $l \neq (0, 0)$. Second, the optimal way to account for global IC constraints depends on the sign of function $W(x)$.

Thus, before we proceed with the Proof of Proposition 3, we first establish some properties of functions $A(x)+C(x)$, $W(x)$, and the relevant threshold types and values.

Lemma 3. *When $c(x)$ is strictly concave, $V^{AC} > c\left(\frac{1}{2}\right)$.*

Proof. We follow Balestrieri & Izmalkov (2014). By strict concavity $c\left(\frac{1}{2}\right) - c(x_=) < c'(x_=)\left(\frac{1}{2} - x_=\right)$ and $c\left(\frac{1}{2}\right) - c(1 - x_=) < c'(1 - x_=)\left(x_= - \frac{1}{2}\right)$. By definition, $V^{AC} = c(x_=) + c'(x_=)\,x_=$ and $V^{AC} = c(1 - x_=) - c'(1 - x_=)\,x_=$. This implies that $c\left(\frac{1}{2}\right) - V^{AC} < c'(x_=)\left(\frac{1}{2} - 2x_=\right)$ and that $c\left(\frac{1}{2}\right) - V^{AC} < c'(x_=)\left(2x_= - \frac{1}{2}\right)$ for any $x_=$. This can only be if $V^{AC} > c\left(\frac{1}{2}\right)$. $\square$

Lemma 4. *When $c(x)$ is strictly concave:*

i) type x_W is well defined (exists and is unique) for values $c\left(\frac{1}{2}\right) < V < V^{AC}$ and is decreasing in V;

ii) $W(x) > 0$ *on* $x_W < x < \frac{1}{2}$*;*
iii) $x_W < x_A \Leftrightarrow V > V^{AC}$.

Proof. i) By differentiating (9), we obtain

$$W'(x) = w'(x)\left(V - c(1-x)\right), \qquad w(x) = \frac{c'(x)}{c'(1-x)}. \tag{15}$$

Since for strictly concave transportation costs $w'(x) < 0$, we have $W'(x) < 0$ iff $V > c(1-x)$. Thus, for $V > c\left(\frac{1}{2}\right)$, $W'\left(\frac{1}{2}\right) < 0$. As $W\left(\frac{1}{2}\right) = A\left(\frac{1}{2}\right) + C\left(\frac{1}{2}\right) > 0$ for $V > c\left(\frac{1}{2}\right)$, function $W(x)$ can cross 0 at most once and from below.

For $V = V^{AC}$, $W(x_=) = 0$, and $x_W = x_= = x_A = x_C$. For $V = c\left(\frac{1}{2}\right)$, $W\left(\frac{1}{2}\right) = 0$. Thus, for each value $c\left(\frac{1}{2}\right) < V < V^{AC}$, $W(x)$ crosses 0 at $x_W \in \left[x_=, \frac{1}{2}\right]$, and $W'(x_W) > 0$. By total differentiation of $W(x_W) = 0$, we obtain

$$\frac{\partial x_W}{\partial V} = -\frac{\frac{\partial W}{\partial V}}{W'(x_W)} < 0.$$

iii) By Lemma 2, x_A is increasing in V, while x_W is decreasing, at $V = V^{AC}$, $x_A = x_W$. ii) Straightforward. □

Proof of Proposition 3. The simple cases are those for which the pointwise optimization of the integrand in (11) satisfies IC constraints. This happens in two extreme cases: (i) very low V: if $V < V^{AC}$ and $A(x) + C(x) < 0$ for all $x > x_A$, in which case mechanism $\mu^{bg}(x_A)$ is optimal; and (ii) very high V: if $V > V^{AC}$ and $A(x) + C(x) > 0$ for all $x > x_=$, in which case mechanism $\mu^{l}(x_=)$ is optimal.

In all other cases (with strict inequalities), the pointwise optimization violates IC constraints, which has to be taken into account. Out treatment of the global IC constraints follows the one by Balestrieri & Izmalkov (2014), accounting for the differences in the setup.

A way to proceed is to account for the IC constraint explicitly. For any pair x, z, the combination $U(x) \geq U(z|x)$ and $U(z) \geq U(x|z)$ gives

$$[U(x) - U(x|z)] \geq [U(z|x) - U(z)]$$

$$(q_0(x) - q_0(z))(c(z) - c(x)) + (q_1(x) - q_1(z))(c(1-z) - c(1-x)) \geq 0. \tag{16}$$

Let $r(x) = q_0(x) - q_1(x)$. By (1), $r(x) \geq 0$. Rearranging terms, the local IC constraints can be written as

$$r(x) + q_1(x)\delta(x,z) \geq r(z) + q_1(z)\delta(x,z), \tag{17}$$

where $\delta(x,z) = 1 - \frac{c(1-x) - c(1-z)}{c(z) - c(x)}$, $x < z \leq \frac{1}{2}$. It is that $\delta(x,z) \in (0,1)$.

However, taking into account local IC is insufficient for the case of concave costs, global IC constraints need to be accounted for any $x \leq z \leq \frac{1}{2}$ with binding $U(x) = U(z|x)$. Since global IC implies that for all $x \leq z \leq \frac{1}{2}$, $U(x) \geq U(z|x)$, and as $U(x)$ is continuous and differentiable almost everywhere, we must have

$$U'(x) \leq U'_x(z|x). \tag{18}$$

for any $x \leq z \leq \frac{1}{2}$ with $U(x) = U(z|x)$. That is, decreasing x, the utility of truthtelling has to be increasing at least as fast as the utility of pretending to be z. Constraint (18) can be written as

$$r(x) + q_1(x)\delta(x,x) \geq r(z) + q_1(z)\delta(x,x), \tag{19}$$

where $\delta(x,x) = 1 - \frac{c'(1-x)}{c'(x)}$. In the optimal solution both constraints (17) and (19) must hold.

Now, suppose the cost function and the value are such that $A(x) + C(x)$ dips below 0 and we need to account for IC constraints, so that $x_{+2} > x_{=}$, and $A(z) + C(z) > 0$ on $z \in \left(x_{+2}, \frac{1}{2}\right)$, while $A(x) + C(x) < 0$ and $A(x) < 0$ for some $x < x_{+2}$. Suppose further that lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ is offered to each such z.

Then, constraint (19) becomes $r(x) + q_1(x)\delta(x,x) = \frac{1}{2}\delta(x,x)$. Therefore, we can express $r(x)$ as

$$r(x) = \left(\frac{1}{2} - q_1(x)\right)\delta(x,x) \tag{20}$$

for $x \in (x_A, x_{+2})$. In that interval, the expected revenue at x is

$$\begin{aligned} & r(x)\,A(x) + q_1(x)\,(A(x) + C(x)) \\ = & \left(\frac{1}{2} - q_1(x)\right)\delta(x,x)\,A(x) + q_1(x)\,(A(x) + C(x)). \end{aligned}$$

Rearranging, we have

$$\frac{1}{2}\delta(x,x)\,A(x) + q_1(x)\,\frac{c'(1-x)}{c'(x)}W(x).$$

Then, optimization of the expected revenue subject to global IC constraints (assuming $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ is offered) yields: if $W(x) > 0$, set $q_1(x) = \frac{1}{2}$ and $r(x) = 0$, that is, offer $l(x) = \left(\frac{1}{2}, \frac{1}{2}\right)$; if $W(x) < 0$, set $q_1(x) = 0$ and $r(x) = \frac{1}{2}\delta(x,x) = \beta(x)$, that is, offer $l(x) = (\beta(x), 0)$.

Ultimately, whether lottery $l = \left(\frac{1}{2}, \frac{1}{2}\right)$ is to be offered depends on the comparison of gains from types z relative to losses from types x. Before we compare the gains with losses, note that it would be suboptimal to offer any other lottery

$l(z)$ besides $(0,0)$ and $\left(\frac{1}{2},\frac{1}{2}\right)$ to any $z > x_{+2}$. Indeed, as lottery $l = \left(\frac{1}{2},\frac{1}{2}\right)$ means $(r, q_1) = \left(0, \frac{1}{2}\right)$, offering $r(z) > 0$ for some types z is suboptimal as it both generates negative revenue $A(z)r(z)$ from z and puts an incentive "overhead" on types x in consideration. Offering different $q_1(z)$ for different z is suboptimal as incentive "overhead" would be the same as if $q_1^{\max} = \max_{z > x_{+2}} q_1(z)$ is offered for all $z > x_{+2}$, while less revenue is generated. And, gains and losses are proportional to $q_1^{\max}$.[18]

As described in Section 4.2, we have two cases to consider depending on R^* (computed at $V = V^{AC}$).

Case 1: $R^* > 0$. If $V > V^{AC}$, by Lemmata 2 and 4, we have $x_W < x_= < x_A$. Then, for each type $x_= < x < x_{+2}$ (if exists), it is optimal to offer $l(x) = \left(\frac{1}{2},\frac{1}{2}\right)$ if such lottery is offered for $z > x_{+2}$. Gains clearly exceed the losses as their sum is increasing in V, and so they are higher than R^*. The optimal mechanism is $\mu^l(x_=)$.

If $V < V^{AC}$, by Lemmata 2 and 4, $x_A < x_= < x_W$. If lottery $l = \left(\frac{1}{2},\frac{1}{2}\right)$ is offered, then the expected revenue on $x > x_A$ is $ER^*_{x > x_A}$ (see (12)), which is strictly increasing in V. The relevant threshold is $V^{\#}$. For $V < V^{\#}$ mechanism $\mu^{bg}(x_A)$ is optimal, and for $V^{\#} < V < V^{AC}$ mechanism $\mu^{ml}(x_A, x_W)$ is optimal.

Case 2: $R^* < 0$. If $V < V^{AC}$, mechanism $\mu^{bg}(x_A)$ is optimal as expected revenue $ER^*_{x > x_A}$ is lower than R^*. If $V > V^{AC}$ and lottery $l = \left(\frac{1}{2},\frac{1}{2}\right)$ is offered, then the expected revenue on $x > x_{+1}$ is $ER^*_{x > x_{+1}}$ (see (13)), which is also strictly increasing in V. Note that on $x \in (x_=, x_{+1})$ and as long as expected revenue in (13) remains negative we have $A(x) + C(x) > 0$. Then, on $V^{AC} < V < V^{\#\#}$ mechanism with no sales for $x > x_{+1}$, $\mu^{lns}(x_=, x_{+1})$, is optimal, while for $V > V^{\#\#}$ mechanism $\mu^l(x_=)$ is optimal.

For all the mechanisms derived prices are determined using IR constraint $U\left(\frac{1}{2}\right) = 0$, indifference conditions at threshold types, and (5). □

A.2 Convex costs

Before we proceed with the Proof of Proposition 4, we first establish some properties of functions $A(x) + C(x)$, $W(x)$, and the relevant threshold types and values.

Lemma 5. *When $c(x)$ is strictly convex, $V^{AC} < c\left(\frac{1}{2}\right)$.*

Proof. Similarly to the proof of Lemma 4, by strict convexity $c\left(\frac{1}{2}\right) - V^{AC} > c'(x_=)\left(\frac{1}{2} - 2x_=\right)$ and $c\left(\frac{1}{2}\right) - V^{AC} > c'(x_=)\left(2x_= - \frac{1}{2}\right)$ have to hold simultaneously. This can only happen if $V^{AC} < c\left(\frac{1}{2}\right)$. □

[18] For a more detailed treatment see Balestrieri & Izmalkov (2014).

Lemma 6. *When $c(x)$ is strictly convex:*

i) $W'(x) < 0$, $W(x) > 0$ for all $x \leq \frac{1}{2}$ on $V > c\left(\frac{1}{2}\right)$;

ii) type x_W is well defined and is increasing in V for $V^{AC} < V < c\left(\frac{1}{2}\right)$;

iii) $x_W < x_A \Leftrightarrow V < V^{AC}$.

Proof. i) $W'(x)$ is given by (15). Note that $w'(x) > 0$ because $c(x)$ is convex. $V - c(1-x) < 0$ for all $x \in \left(0, \frac{1}{2}\right)$ by Lemma 5. The second statement is true as for $V = c\left(\frac{1}{2}\right)$, $W\left(\frac{1}{2}\right) = 0$.

ii) For $V = V^{AC}$, by expression (9), $W(x_{=}) = 0$. We have

$$\frac{\partial x_W}{\partial V} = -\frac{\frac{\partial W}{\partial V}}{W'(x_W)} > 0.$$

iii) By Lemma 2, $x_A < x_C$ for $V < V^{AC}$. For $x < x_A$, since $C(x) < 0$ we have $A(x) > W(x)$. Therefore, $x_W < x_A$ given $W'(x) < 0$. □

Proof of Proposition 4. We proceed in four steps. First, we put a lower bound on the worst possible types in the optimal solution. Second, we give a partial characterization of the optimal solution given the lowest worst type. Third, we identify the lowest worst possible type depending on the primitives of the setup. Finally, we derive the optimal mechanism.

Step 1. Let x^{**} be the smallest x among the types with lowest utility in the optimal solution. Then, $x^{**} \leq \min\{x_A, x_{=}\}$.

Suppose not, then consider two cases. Case (i): if $x_A < x_{=}$, then for $x \in (x_A, x_{=})$ we have $A(x) + C(x) < 0$. Optimization of the first integrand in (14) gives $l(x) = (0, 0)$, which means $U(x) = 0 = U(x^{**})$, contradicting the supposition. Case (ii): if $x_A > x_{=}$, then for $x \in (x_{=}, x_A)$, we have $A(x) + C(x) > 0$. The optimal unconstrained solution is $l(x) = \left(\frac{1}{2}, \frac{1}{2}\right)$, and so $U(x) < U(x^{**}|x) \leq U(x^*)$. Thus, offering the lottery to consumer x violates IR constraint. This means that the optimization under the binding IR constraint for the worst type x^{**} on $x \in (x_{=}, x^{**})$ has to result in binding IR constraint for at least some $x < x^{**}$, again contradicting the supposition. If $V = V^{AC}$ and so $x_{=} = x_A$, then there exist $x \in (x_{=}, x^{**})$ with either $A(x) + C(x) > 0$ or $A(x) + C(x) < 0$, and the same argumentation leading to the contradictions applies.

Step 2. Considering the problem (14) on $x > x^* = x^{**}$, note that $A(x) + B(x) > C(x) + D(x)$ for $x < \frac{1}{2}$. Therefore, the optimal solution given x^* is to maximize the probability $q_0(x)$ without violating the IR constraint for types $x > x^*$. Under constraint (10), the second integrand of (14) can be expresses as

$$q_0(x)\left[A(x) + B(x)\right] + q_1(x)\left[C(x) + D(x)\right] = 2W(x)q_0(x).$$

Hence, when $W(x) > 0$ it is optimal to set $q_0(x) = \gamma(x) = \frac{c'(1-x)}{c'(1-x)+c'(x)}$ and $q_1(x) = 1 - \gamma(x)$, and when $W(x) < 0$ – to set $l(x) = (0, 0)$.

Step 3. For $V > V^{AC}$, $x^{**} = x_=$. Suppose not, that is $x^{**} < x_=$ (by Step 1 and Lemma 2 we know that $x^{**} \leq x_= < x_A$). Since $W(x) > 0$ for $x = x_=$ and $W'(x) < 0$ by Lemma 6, $W(x) > 0$ for $x \in (x^{**}, x_=)$. Thus, the optimal solutions computed given $x^* = x^{**}$ and $x^* = x_=$ differ only on $x \in (x^{**}, x_=)$. Note that $U(x_=) = 0$ for both of them. The optimal solution computed given $x^* = x_=$ generates strictly more revenue, as the unconstrained maximization of the first integrand in (14) on $x \in (x^{**}, x_=)$ gives $l(x) = (1, 0)$, which together with the rest of the solution is IC and IR. For $V < V^{AC}$, $x^{**} = x_A$ by a similar argument.

Step 4. For $V < V^{AC}$, we have $x^{**} = x_A$. By Lemma 6, $x_W < x_A$, and so the optimal mechanism is $\mu^{bg}(x_A)$.

For $V^{AC} < V < c\left(\frac{1}{2}\right)$, we have $x^{**} = x_=$ and $x_= < x_W < \frac{1}{2}$ (follows from Lemma 6). Thus, the optimal mechanism is $\mu^{mlns}(x_=, x_W)$.

For $V > c\left(\frac{1}{2}\right)$, we have $x^{**} = x_=$ and by Lemma 6, $W(x) > 0$ on $x < \frac{1}{2}$. Thus, the optimal mechanism is $\mu^{ml2}(x_=)$.

For all the mechanisms derived prices are determined using IR constraint $U(x^*) = 0$, indifference conditions at threshold types, and (5). $\square$