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"Numerical approximation of a cash-constrained firm value with investment opportunities."

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Numerical approximation of a cash-constrained firm value with investment opportunities.

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Abstract: We consider a singular control problem with regime switching that arises in problems of optimal investment decisions of cash-constrained firms. The value function is proved to be the unique viscosity solution of the associated Hamilton-Jacobi-Bellman equation. Moreover, we give regularity properties of the value function as well as a description of the shape of the control regions. Based on these theoretical results, a numerical deterministic approximation of the related HJB variational inequality is provided. We finally show that this numerical approximation converges to the value function. This allows us to describe the investment and dividend optimal policies.

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1 Introduction

In a frictionless capital market, Modigliani and Miller theorem demonstrates that firms can fund all valuable investment opportunities. However, if we introduce capital market imperfections, it is now a standard result that cash-constrained firms have to rely more on internal financial resources: cash holdings and credit line to fund investment opportunities. In recent years, there has been an increasing attention in the use of singular control techniques to model investment problems of a cash-constrained firm. As references for the theory of singular stochastic control, we may mention the pioneering works of Haussman and Suo [9] and [10] and for application to investment/dividend problems Jeanblanc and Shiryaev [15], Højgaard and Taksar [11], Asmussen, Højgaard and Taksar [1], Choulli, Taksar and Zhou [4], Paulsen [19] while more recent studies in corporate finance include Bolton, Chen and Wang [3], Décamps, Mariotti, Rochet and Villeneuve [6] and Hugonnier, Malamud and Morellec [14].
Singular control is an important class of problems in stochastic control theory. The associated HJB equation, which takes the form of variational inequalities with gradient constraints turns out to be very difficult to solve. In particular, the regularity of the solution are still not

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well understood. For concrete problems as those arising from dynamic corporate finance, it is thus important to propose a numerical approximation of the value function and to ensure that this numerical approximation converges to the targeted value function. It turns out from the paper by Barles and Souganidis [2] that when the value function of a singular control problem is the unique viscosity solution of the associated HJB variational inequality, a consistent, stable and monotone numerical scheme converges to the value function.
It is now well-established that there exist several approaches to approximate the value function of singular stochastic control problems. First, probabilistic methods based on Markov Chain approximations are essentially explicit finite difference schemes and thus suffer from the stability curse limiting the choice of time step (see [17]). On the other hand, analytical methods based on the tracking of control regions have been developed in [16]. They appear to be quite complex because they necessitate a good guess about the shape of control regions especially in the presence of multiple controls.
Our paper builds on the theoretical model of cash-constrained firms developed in [22] with the modification that the investment levels are here discrete. This assumption appears to be reasonable for big industries investing in capacity. The objective is to determine the firm value as well as the investment and dividend optimal policies leading to a singular control problem with regime switching where the regimes correspond to the different levels of production. Our main contributions are

- We prove that the value function is the unique viscosity solution of the HJB variational inequality. Moreover, we prove the regularity of the value function under a mild assumption about the existence of left and right derivative everywhere. Finally, we prove that it is optimal to pay dividends for high value of cash which allow us to set boundary conditions at right for our numerical scheme.

- We carry out a rigorous analysis of the direct control method proposed by [12], in the context of HJB variational inequality arising from cash management problem. Having proved a strong comparison theorem, we show that our direct control method is consistent, stable and monotone. In our context, the stability result appears to be a little bit tricky and its proof needs to prove a growth condition on the value function (see Lemma 2).

- Finally, the numerical approximation of the HJB variational inequality leads to the resolution of a linear system $AU = B$. We present a fixed-point iteration scheme similar to [12] for solving the linear system. To show the convergence of this iterative procedure, we need to prove that the tridiagonal block matrix A is a M-matrix (see Lemma 8) which necessitates an extension of the result proved in [12].

The paper is organized as follows: in section 2, we present the model and derive a standard analytical characterization of the value function in terms of viscosity solutions. Furthermore, we give regularity properties of the value function and a description of the shape of the control regions. Section 3 and 4 are devoted to the presentation of the numerical approximation and contains the convergence result which builds on an extension of the classical techniques developed in [12]. Section 5 concludes the paper with numerical illustrations.

2 The Model

We consider a firm characterized at each time t by the following balance sheet :
$K_t + M_t = L_t + X_t$ where

- K_t represents the firm's productive assets,
- M_t represents the amount of cash reserves or liquid assets,
- L_t represents the volume of outstanding debt,
- X_t represents the book value of equity.

We suppose that the firm is able to choose the level of its productive assets, by investment or disinvestment, in a range of N strictly positive levels : $K_t \in \{k_i\}_{i\in[1,N]}$. Without loss of generality, we suppose that $\{k_i\}_{i\in[1,N]}$ satisfy

$$\forall i \in [1, N], k_i = k_1 + (i-1)h.$$

The productive assets continuously generate cash-flows $(R_t)_{t\geq 0}$ over time. We assume

$$dR_t = \beta(K_t)(\mu dt + \sigma dB_t)$$

where μ and σ are positive constants and $(B_t)_{t\geq 0}$ is a standard Brownian motion on a complete probability space $(\Sigma, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$. In this model, we assume a decreasing-return-to-scale technology by introducing the increasing bounded concave function β. We denote $\bar{\beta}$ the maximum taken by the gain function on $\{k_i\}_{i\in[1,N]}$ i.e.

$$\bar{\beta} = \beta(k_N)$$

In order to finance its working capital requirement, we consider that the firm has access to a secured credit line. The collateral of the credit line is given by the market value of the firm assets. If we introduce $\gamma > 0$ the cost to disinvest the productive assets and M the level of cash, the credit line's depth is assumed to be

$$L_{\max} = (1-\gamma)K + M. \tag{1}$$

When this credit line limit is reached, the company is no longer able to meet its financial commitments and is therefore forced to go bankrupt. At this point, the manager liquidates the firm assets in order to refund the creditors with priority for debt holders over shareholders. However, we assume that there is a probability $1-\zeta$ that the firm does not manage to sell its assets at their market value . Introducing P the expected loss for the debt holders, we have

$$P = \zeta(1-\gamma)K.$$

This loss makes the credit line risky and thus it is subject to interest payments variable modeled by a function α depending only on the volume of debt the firm has issued. Finally, we suppose the access to equity market is excessively costly.

Assumption 1 *α is a strictly continuously differentiable convex function. Furthermore, it is assumed that the collateralized debt is risky, that is*

$$\forall x \geq 0, \alpha'(x) > r \text{ and } \alpha(0) = 0.$$

This assumption is consistent with what we observe in practice. The convexity of α allows us to model the fact that the interest payments asked by the creditors increase with the level of debt. Note that the model doesn't exclude a linear cost of the debt since α is not supposed to be strictly convex. In [22], it has been proved in a similar framework that it is optimal to use the credit line if and only if the cash reserves are depleted meaning that

$$L_t = (K_t - X_t)^+ \tag{2}$$

We therefore have the following dynamics for the book value of equity and the productive assets ([22]):

$$\begin{cases} dX_t = \beta(K_t)(\mu dt + \sigma dB_t) - \alpha((K_t - X_t)^+)dt - \gamma|dI_t| - dZ_t \\ dK_t = dI_t^+ - dI_t^- \end{cases}$$

where $Z = (Z_t)_{t\geq 0}$ is an increasing right-continuous $(\mathcal{F}_t)_t$ adapted process representing the cumulative dividend payments up to time t and $I^+ = (I_t^+)_{t\geq 0}$ (respectively I^-) is the cumulative investment process (respectively disinvestment). Here we suppose that the cost to investment is the same as the cost of disinvestment γ.
The manager acts in the interest of the shareholders and maximizes the expected discounted value of all future dividend payout. Shareholders are assumed to be risk-neutral and future cash-flows are discounted at the risk-free rate r. Thus, the objective is to maximize over the admissible control $\pi = (I^+, I^-, Z)$ the functional

$$V(x, k_i; \pi) = \mathbb{E}_{x,k_i}\left(\int_0^\tau e^{-rt} dZ_t^\pi\right)$$

where x and k_i are the initial values of equity capital and productive capital. τ is the time of bankruptcy and according to (1) and (2), we have

$$\tau = \inf_{t\geq 0}\{X_t^\pi \leq \gamma K_t^\pi\}$$

We denote by Π the set of admissible control variables and define the shareholders value functions by

$$\forall i \in [1, N], v_i(x) = v(x, k_i) = \sup_{\pi\in\Pi} V(x, k_i; \pi)$$

which are defined on the domains

$$\forall i \in [1, N], \Omega_i = [\gamma k_i, +\infty[$$

2.1 Viscosity solutions

The aim of this section is to determine the HJB variational inequality (HJB-VI) satisfied by the shareholders value functions $(v_i)_{i\in[1,N]}$. This analytical characterization will allow us to solve numerically the problem of optimal investment for a cash-constrained firm.

Proposition 1 *The shareholders value functions v_i are jointly continuous for every $i \in [1, N]$.*

Proof: Take $i \in [1, N]$, $x > \gamma k_i$ and $(x_n)_{n\in\mathbb{N}}$ a sequence of Ω_i that converges to x. We consider two admissible strategies :

- Strategy π_n^1 : from (x, k_i) wait until liquidation or the point (x_n, k_i). We denote $(X_t^{\pi_n^1}, K_t^{\pi_n^1})_{t\geq 0}$ the process controlled by π_n^1.
- Strategy π_n^2 : from (x_n, k_i) wait until liquidation or the point (x, k_i). We denote $(X_t^{\pi_n^2}, K_t^{\pi_n^2})_{t\geq 0}$ the process controlled by π_n^2.

We define

$$\theta_n^1 = \inf\{t \geq 0, (X_t^{\pi_n^1}, K_t^{\pi_n^1}) = (x_n, k_i)\}$$
$$\theta_n^2 = \inf\{t \geq 0, (X_t^{\pi_n^2}, K_t^{\pi_n^2}) = (x, k_i)\}$$

$$T_n^1 = \inf\{t \geq 0, X_t^{\pi_n^1} = \gamma K_t^{\pi_n^1})\}$$
$$T_n^2 = \inf\{t \geq 0, X_t^{\pi_n^2} = \gamma K_t^{\pi_n^2})\}$$

Dynamic programming principle and $v(X_{T_n^1}, K_{T_n^1}) = 0$ yield

$$\begin{aligned}
v_i(x) &\geq \mathbb{E}\left[\int_0^{\theta_n^1\wedge T_n^1} e^{-rt}dZ_t^{\pi_n^1} + e^{-r(\theta_n^1\wedge T_n^1)}1_{\{\theta_n^1<T_n^1\}}v(X_{\theta_n^1}, K_{\theta_n^1})\Big)\right] \\
&\geq \mathbb{E}\left[e^{-r\theta_n^1}1_{\{\theta_n^1<T_n^1\}}v_i(x_n)\right] \\
&\geq \Big(\mathbb{E}\big(e^{-r\theta_n^1}\big) - \mathbb{E}\big(e^{-r\theta_n^1}1_{\{\theta_n^1\geq T_n^1\}}\big)\Big)\, v_i(x_n) \\
&\geq \Big(\mathbb{E}\big(e^{-r\theta_n^1}\big) - \mathbb{P}\big(\theta_n^1 \geq T_n^1\big)\Big)\, v_i(x_n)
\end{aligned}$$

On the other hand, using $v(X_{T_n^2}, K_{T_n^2}) = 0$ we have

$$\begin{aligned}
v_i(x_n) &\geq \mathbb{E}\left[\int_0^{\theta_n^2\wedge T_n^2} e^{-rt}dZ_t^{\pi_n^2} + e^{-r(\theta_n^2\wedge T_n^2)}1_{\{\theta_n^2<T_n^2\}}v(X_{\theta_n^2}, K_{\theta_n^2})\Big)\right] \\
&\geq \mathbb{E}\left[e^{-r\theta_n^2}1_{\{\theta_n^2<T_n^2\}}v_i(x)\right] \\
&\geq \Big(\mathbb{E}\big(e^{-r\theta_n^2}\big) - \mathbb{E}\big(e^{-r\theta_n^2}1_{\{\theta_n^2\geq T_n^2\}}\big)\Big)\, v_i(x) \\
&\geq \Big(\mathbb{E}\big(e^{-r\theta_n^2}\big) - \mathbb{P}\big(\theta_n^2 \geq T_n^2\big)\Big)\, v_i(x)
\end{aligned}$$

In [22], the authors proved that

$$\lim_{n\to\infty} \mathbb{P}(\theta_n^1 \geq T_n^1) = 0$$
$$\lim_{n\to\infty} \mathbb{P}(\theta_n^2 \geq T_n^2) = 0$$

and

$$\lim_{n\to+\infty} \mathbb{E}(e^{-r\theta_n^1}) = \lim_{n\to+\infty} \mathbb{E}(e^{-r\theta_n^2}) = 1$$

Then

$$v_i(x) \geq \limsup_n v_i(x_n) \geq \liminf_n v_i(x_n) \geq v_i(x),$$

which proves the continuity of v_i. ⋄

Let $\mathcal{L}_i$ be the next differential operator:

$$\mathcal{L}_i\phi = (\beta(k_i)\mu - \alpha((k_i - x)^+))\phi'(x) + \frac{\beta(k_i)^2\sigma^2}{2}\phi''(x) - r\phi \tag{3}$$

The next lemma establishes a comparison principle which we shall use to prove a linear growth condition for the shareholders value function.

Lemma 1 *Suppose $(\varphi_i)_{i\in[1,N]}$ are N smooth functions on $(\gamma k_i, +\infty)$ such that $\varphi_i(\gamma k_i) \geq 0$ and*

$$\forall i \in [1, N], \forall x \geq \gamma k_i, \min\left[-\mathcal{L}_i\varphi_i(x), \varphi_i'(x) - 1, \varphi_i(x) - \max_{j\neq i}\varphi_j(x - \gamma|k_i - k_j|)\right] \geq 0 \tag{4}$$

then we have for all $i \in [1, N]$, $v_i \leq \varphi_i$.

Proof: Take $i \in [1, N]$, $X(0) = x \in \Omega_i$ and $\pi = (Z, I^+, I^-)$ an admissible control. We note $(I_n)_{n\geq 1}$ the moments of regime switching. Apply then Itô's formula to $e^{-rt}\varphi_i(X_t^\pi)$ between 0 and the stopping time $(I_1 \wedge \tau)$ noticing that for $0 \leq t < I_1 \wedge \tau$, the firm stays in the regime k_i.

$$\begin{aligned} e^{-r(I_1\wedge\tau)}\varphi_i(X^\pi_{(I_1\wedge\tau)^-}) =& \varphi_i(x) + \int_0^{(I_1\wedge\tau)^-} e^{-rs}\mathcal{L}_i\varphi_i(X_s^\pi)ds \\ &- \int_0^{(I_1\wedge\tau)^-} e^{-rs}\varphi'(X_s^\pi)dZ_s^{\pi,c} \\ &+ \int_0^{(I_1\wedge\tau)^-} e^{-rs}\varphi'(X_s^\pi)\sigma\beta(k_i)dW_s \\ &+ \sum_{0\leq t<I_1\wedge\tau} e^{-rs}[\varphi(X_s^\pi) - \varphi(X_{s^-}^\pi)] \end{aligned}$$

Noting that the integrand in the stochastic integral term is bounded we have

$$\mathbb{E}\left[e^{-r(I_1\wedge\tau)}\varphi_i(X^\pi_{(I_1\wedge\tau)^-})\right] = \varphi_i(x) + \mathbb{E}\left[\int_0^{(I_1\wedge\tau)^-} e^{-rs}\mathcal{L}_i\varphi_i(X^\pi_s)ds\right] - \mathbb{E}\left[\int_0^{(I_1\wedge\tau)^-} e^{-rs}\varphi'(X^\pi_s)dZ^{\pi,c}_s\right] + \mathbb{E}\left[\sum_{0\le t<I_1\wedge\tau} e^{-rs}[\varphi(X^\pi_s)-\varphi(X^\pi_{s^-})]\right]$$

Since $\varphi' \geq 1$, by the mean-value theorem we have

$$\varphi_i(X^\pi_s) - \varphi_i(X^\pi_{s^-}) \leq X^\pi_s - X^\pi_{s^-} = -\Delta Z_s$$

And so, by using the supersolution inequality of φ_i :

$$\mathbb{E}\left[e^{-r(I_1\wedge\tau)}\varphi_i(X^\pi_{(I_1\wedge\tau)^-})\right] \leq \varphi_i(x) - \mathbb{E}\left[\int_0^{(I_1\wedge\tau)^-} e^{-rs}dZ^\pi_s\right] \tag{5}$$

Now, using again the supersolution property, we have

$$\forall j \neq i, \varphi_i(X^\pi_{(I_1\wedge\tau)^-}) \geq \varphi_j(X^\pi_{(I_1\wedge\tau)^-} - \gamma|k_i - k_j|)$$

But by definition, if $I_1 \leq \tau$, there exists $j \in [1, N]\backslash\{i\}$ such that $X^\pi_{(I_1\wedge\tau)^-} - \gamma|k_i - k_j| = X^\pi_{(I_1\wedge\tau)}$ so we have in that case

$$\varphi_i(X^\pi_{(I_1\wedge\tau)^-}) \geq \varphi_j(X^\pi_{(I_1\wedge\tau)}) \tag{6}$$

If $T_1 > \tau$, note that we directly have $\varphi_i(X^\pi_\tau) = 0$ and so

$$\varphi_i(x) \geq \mathbb{E}\left[\int_0^\tau e^{-rs}dZ^\pi_s\right]$$

Plugging (6) into (5), we have

$$\varphi_i(x) \geq \mathbb{E}\left[\int_0^{(I_1\wedge\tau)^-} e^{-rs}dZ^\pi_s + \varphi_j(X^\pi_{(I_1\wedge\tau)})\right]$$

Again, we can prove the result for every $n \in \mathbb{N}^*$ and every φ_j where j is the regime for the process between $(I_n \wedge \tau)$ and $(I_{n+1} \wedge \tau)^-$.
So by iteration we have

$$\varphi_i(x) \geq \mathbb{E}\left[\int_0^{(I_n\wedge\tau)^-} e^{-rs}dZ^\pi_s + \varphi_j(X^\pi_{(I_n\wedge\tau)})\right] \geq \mathbb{E}\left[\int_0^{(I_n\wedge\tau)^-} e^{-rs}dZ^\pi_s\right]$$

By sending n to infinity, we obtain the required result from the arbitrariness of the control π. ◇

As a corollary, we prove a linear growth condition for the shareholders value functions v_i.

Lemma 2 *For all $i \in [1, N]$ and for all $x \in \Omega_i$, we have*

$$v_i(x) \leq x - \gamma k_i + \frac{\mu \bar{\beta}}{r}$$

Proof: For all $i \in [1, N]$, we define

$$\varphi_i(x) = x - \gamma k_i + \frac{\mu \bar{\beta}}{r}$$

We prove easily that $(\varphi_i)_{i \in [1,N]}$ are viscosity supersolutions of (4). Indeed,

$$\forall i \in [1, N], \varphi_i'(x) \geq 1$$

and

$$\varphi_i(x) - \varphi_j(x - \gamma|k_i - k_j|) = \gamma|k_i - k_j| - \gamma(k_i - k_j) \geq 0$$

and

$$-\mathcal{L}_i \varphi_i(x) = -(\beta(k_i)\mu - \alpha((k_i - x)^+)) + r(x - \gamma k_i) + \mu \bar{\beta} \geq 0$$

and we have $\varphi_i(\gamma k_i) = \frac{\mu \bar{\beta}}{r} > v_i(\gamma k_i)$ using that $v_i(\gamma k_i) = 0$. Lemma 1 proves the result. ◇

Proposition 2 *The shareholders value functions $(v_i)_{i \in [1,N]}$ are the unique continuous viscosity solutions to the HJB variational inequality :*

$$\begin{aligned} &\forall i \in [1, N], \forall x \geq \gamma k_i, \\ &\min \left\{ -\mathcal{L}_i v_i(x), v_i'(x) - 1, v_i(x) - \max_{j \neq i} v_j(x - \gamma|k_i - k_j|) \right\} = 0 \end{aligned} \tag{7}$$

with boundary conditions

$$\forall i \in [1, N], v_i(\gamma k_i) = 0 \tag{8}$$

Proof: The proof is postponed to the Appendix and is somehow similar to the one in [22]. ◇

Remark 1 *It is sufficient to impose the boundary condition (8) to have the uniqueness of the viscosity solution.*

Remark 2 *Given the property $v_i(x) \geq v_j(x - \gamma|k_i - k_j|)$, the HJB-VI is equivalent to*

$$\begin{aligned} &\forall i \in [1, N], \forall x \geq \gamma k_i, \\ &\min \Big\{ -\mathcal{L}_i v_i(x), v_i'(x) - 1, v_i(x) - \max \Big(v_{i-1}(x - \gamma h), v_{i+1}(x - \gamma h) \Big) \Big\} = 0 \end{aligned}$$

2.2 Regularity

We set for all $i \in [1, N]$,

$$\begin{aligned}
\mathcal{S}_{ij} &= \{x \in \Omega_i, v_i(x) = v_j(x - \gamma|k_i - k_j|)\} \\
\mathcal{S}_i^+ &= \cup_{j>i}\mathcal{S}_{ij} \\
\mathcal{S}_i^- &= \cup_{j<i}\mathcal{S}_{ij} \\
\mathcal{S}_i &= \mathcal{S}_i^+ \cup \mathcal{S}_i^-
\end{aligned}$$

which define the investment region ($\mathcal{S}_i^+$) and the disinvestment one ($\mathcal{S}_i^-$).

Before proving the main result of this section, we need to establish some preliminary results on the value function.

Lemma 3 *$\forall i \in [1, N]$, $v_i(x + h) \geq v_i(x) + h$*

Proof: It is obvious if we consider the sub-optimal strategy from initial state $(k_i, x + h)$ which consists on distributing h dollars as dividends at time $t = 0$ and following the optimal strategy hereafter. By the dynamic programming principle we have :

$$v_i(x + h) \geq v_i(x) + h$$

◇

Lemma 4 *$\forall i \in [1, N]$,*

1. *$\forall x \in \mathcal{S}_i^+, x - \gamma h \notin \mathcal{S}_{i+1}^-$*
2. *$\forall x \in \mathcal{S}_i^-, x - \gamma h \notin \mathcal{S}_{i-1}^+$*

Proof: We prove only the first assertion since the demonstration of the second assertion is similar. Suppose on the contrary that there exists $i \in [1, N]$ and $x \in S_i^+$ such that $x - \gamma h \in S_{i+1}^-$. Therefore,

$$v_i(x) = v_{i+1}(x - \gamma h) = v_i(x - 2\gamma h)$$

which is in contradiction with lemma 3. ◇

We proved in the previous section that the value function is continuous. It has been proved in [20] that the convexity of the value function is a sufficient condition for the regularity of its derivative. But, as proved in [22], the value function might be convex-concave when the credit line interest rate is high. Nonetheless, we will give below a regularity result under the following assumption about the existence of left and right derivatives.

Assumption 2 *The value function admits left (D^-) and right (D^+) derivatives on its definition domain.*

Proposition 3 *Under the assumption (2), the value functions v_i are C^1 for all $i \in [1, N]$, .*

Proof: Let be $i \in [1, N]$, $x_0 \in \Omega_i$ and suppose $D^+v_i(x_0) > D^-v_i(x_0)$. Then take some $q \in (D^-v_i(x_0), D^+v_i(x_0))$ and consider the function

$$\varphi_i(x) = v_i(x_0) + q(x - x_0) + \frac{1}{2\epsilon}(x - x_0)^2$$

with $\epsilon > 0$. Then x_0 is a local minimum of $v_i - \varphi_i$, with $\varphi'(x_0) = q$ and $\varphi''(x_0) = \frac{1}{\epsilon}$. Therefore, we get a contradiction by writing the supersolution inequality :

$$0 \leq -(\beta(k_i)\mu - \alpha((k_i - x)^+))q + r(v_i(x_0)) - \frac{\sigma^2\beta(k_i)^2}{2\epsilon}$$

and choosing ϵ small enough. So we have the inequality

$$D^+v_i(x_0) \leq D^-v_i(x_0)$$

Suppose now that there exists some $x_0 \notin \mathcal{S}_i$ such that $D^-v_i(x_0) > D^+v_i(x_0)$. We then fix some $q \in (D^+v_i(x_0), D^-v_i(x_0))$ and consider the function

$$\varphi(x) = v_i(x_0) + q(x - x_0) - \frac{1}{\epsilon}(x - x_0)^2$$

with $\epsilon > 0$. Then x_0 is a local maximum of $v_i - \varphi$ with $\varphi'(x_0) = q > D^+v_i \geq 1$. Since $x_0 \notin \mathcal{S}_i$, the subsolution inequality property implies

$$0 \geq rv_i(x_0) - (\mu\beta(k_i) - \alpha((k_i - x)^+))q + \frac{\sigma^2\beta(k_i)^2}{2\epsilon}$$

which leads to a contradiction by choosing ϵ sufficiently small. Therefore, we have that v_i is C^1 on the open set $\Omega_i \backslash \mathcal{S}_i$.
Let's prove now that v_i is still C^1 on $\mathcal{S}_i$. Fix $x_0 \in \mathcal{S}_i^+$ (the proof for $x_0 \in \mathcal{S}_i^-$ is similar) and take $j = \min\{l > i, x_0 - \gamma|k_i - k_l| \notin \mathcal{S}_l^+\}$. Then x_0 is a minimum of $v_i - v_j(. - \gamma(k_j - k_i))$, and so

$$D^-v_i(x_0) - D^-v_j(x_0 - \gamma(k_j - k_i)) \leq D^+v_i(x_0) - D^+v_j(x_0 - \gamma(k_j - k_i))$$

But, from the definition of j, $x_0 - \gamma(k_j - k_i) \notin S_j^+$ and from Lemma 4, $x_0 - \gamma(k_j - k_i) \notin S_j^-$ so $x_0 - \gamma(k_j - k_i)$ belongs to the open set $\Omega_j \backslash S_j$ and so $D^+v_j(x_0 - \gamma(k_j - k_i)) = D^-v_j(x_0 - \gamma(k_j - k_i))$ and thus

$$D^-v_i(x_0) \leq D^+v_i(x_0)$$

which proves the result since the reverse inequality has been proved previously. $\diamond$

Since we prove, under the Assumption 2, that the value function is C^1, we pose from now on :

$$\mathcal{D}_i = \{x \in \Omega_i, v_i'(x) = 1\}$$
$$\mathcal{C}_i = \Omega_i \backslash (\mathcal{D}_i \cup \mathcal{S}_i)$$

Proposition 4 *For all $i \in [1, N], v_i$ is C^2 on $\mathcal{C}_i$.*

Proof: In this open set, we have that v_i is a viscosity solution to

$$-\mathcal{L}_i v_i = 0, x \in \mathcal{C}_i \tag{9}$$

Now, for any arbitrary bounded interval $(x_1, x_2) \in \mathcal{C}_i$ consider the Dirichlet boundary linear problem :

$$\begin{aligned} & -\mathcal{L}_i w = 0 \\ & w(x_1) = v_i(x_1), \qquad w(x_2) = v_i(x_2) \end{aligned} \tag{10}$$

Classical results (see for instance [8]) provide the existence and uniqueness of a smooth C^2 function w solution on (x_1, x_2) to (10). In particular, this smooth function w is a viscosity solution to (9). From standard uniqueness results, we get $v_i = w$ on $(x_1, x_2) \in \mathcal{C}_i$ which proves that v_i is C^2 on $\mathcal{C}_i$ from the arbitrariness of (x_1, x_2). $\diamond$

2.3 Properties of the dividend region

At this point, we only have the boundary condition :

$$\forall i \in [1, N], v_i(\gamma k_i) = 0$$

However, to solve numerically the problem, we need another boundary condition on the right side. The next lemma gives us a property of the dividend region that will make the numerical scheme well-posed.

Lemma 5 *For all $i \in [1, N]$, we have*

$$b_i = \sup\{x \in \Omega_i, v_i'(x) > 1\} < +\infty$$

Proof: We note $\mathbb{I} = \{i \in [1, N], b_i < +\infty\}$ and we suppose that $\mathbb{I}^c = [1, N] \backslash \mathbb{I} \neq \emptyset$. For all $i \in [1, N]$, the function $x \to v_i(x) - x$ is an increasing bounded continuous function (see lemma 2) and therefore admits a limit $a_i = \lim_{x \to +\infty}(v_i(x) - x)$. We have for all $(i, j) \in [1, N] \times [1, N]$:

$$a_j - (a_i - \gamma|k_i - k_j|) = \lim_{x \to +\infty} (v_j(x) - v_i(x - \gamma|k_i - k_j|)) \geq 0$$

Take $j_0 \in \mathbb{I}^c$ such that $a_{j_0} = \max\{a_j, j \in \mathbb{I}^c\}$. In particular, we have for all $j \in \mathbb{I}^c \backslash \{j_0\}, a_{j_0} > (a_j - \gamma|k_{j_0} - k_j|)$. We prove easily that there exists $\bar{x} \in \mathbb{R}^+$ such that

$$\begin{cases} \bar{x} > k_{j_0} \\ r v_{j_0}(\bar{x}) > \mu \beta(k_{j_0}) \\ v_{j_0}(\bar{x}) > \bar{x} + \max\limits_{j \in \mathbb{I}^c \backslash \{j_0\}} (a_j - \gamma|k_{j_0} - k_j|) \\ \bar{x} > b_i + \gamma|k_i - k_{j_0}|, \forall i \in \mathbb{I} \end{cases}$$

We then define the function w such that

$$\forall x \leq \bar{x}, w(x) = v_{j_0}(x)$$
$$\forall x > \bar{x}, w(x) = v_{j_0}(\bar{x}) + x - \bar{x}$$

Then by definition, for $x \in [\gamma k_{j_0}, \bar{x}]$, w is a viscosity solution of

$$\min \left\{ -\mathcal{L}_{j_0} w, w'(x) - 1, w(x) - \max_{j \neq j_0} v_j(x - \gamma |k_{j_0} - k_j|) \right\} = 0$$

We still have to prove that w is a viscosity solution on $]\bar{x}, +\infty[$. First, for all $x \in]\bar{x}, +\infty[, w'(x) = 1$. Moreover,

$$\forall x > \bar{x}, -\mathcal{L}_{j_0} w = r(v_{j_0}(\bar{x}) + x - \bar{x}) - \mu\beta(k_{j_0})$$

So using :

$$r v_{j_0}(\bar{x}) > \mu\beta(k_{j_0})$$

we have that $-\mathcal{L}_{j_0} w > 0$. Finally, for all $j \in \mathbb{I}^c \backslash \{j_0\}$, we have

$$\forall x > \bar{x}, w(x) > x + a_j - \gamma|k_{j_0} - k_j| \geq v_j(x - \gamma|k_{j_0} - k_j|)$$

For $i \in \mathbb{I}$, as $\bar{x} - \gamma|k_i - k_{j_0}| > b_i$, for all $x > \bar{x}$, $v_i'(x - \gamma|k_i - k_{j_0}|) = 1$. Thereafter,

$$\begin{aligned}
\forall x > \bar{x}, v_i(x - \gamma|k_i - k_{j_0}|) - w(x) &= v_i(\bar{x} - \gamma|k_i - k_{j_0}) + x - \bar{x} - (v_{j_0}(\bar{x}) + x - \bar{x}) \\
&= v_i(\bar{x} - \gamma|k_i - k_{j_0}|) - v_{j_0}(\bar{x}) \\
&\leq 0
\end{aligned}$$

We proved that w is a viscosity solution to the variational inequality so by uniqueness, $w = v_{j_0}$, which is in contradiction with $j_0 \in \mathbb{I}^c$ and the result is proved. ◇

Lemma 5 ensures that if x is large enough, we have $v_i(x+h) = v_i(x) + h$. This property, with the left boundary condition at γk_i is enough to build a numerical scheme. However, we can prove more about the dividend region. Proposition 5 below specifies the form of the dividend region under certain assumption and builds on the two next lemmas.

Definition 1 *We say that $x \in \Omega_i$ is a left border (resp. right border) of a subset $\mathcal{E}$ if there exists $\epsilon > 0$ and $(x_n)_{n\in\mathbb{N}} \notin \mathcal{E}$ such that*

$$\lim_{n\to+\infty} x_n = x$$

and

$$\forall y \in]x, x + \epsilon[, y \in \mathcal{E}$$

Lemma 6 *For all $i \in [1, N]$, if a_i is a left border of $\mathcal{D}_i$ such that $a_i \in \mathcal{S}_{ij}$ then $a_i - \gamma|k_i - k_j|$ is a left border of $\mathcal{D}_j$.*

Proof: Take a_i a left border of $\mathcal{D}_i$ such that $a_i \in \mathcal{S}_{ij}$. There exists $\epsilon > 0$ such that

$$\forall x \in [a_i, a_i + \epsilon[, v_i(x) = v_i(a_i) + x - a_i$$

And $a_i \in \mathcal{S}_{ij}$ so :

$$\forall x \in [a_i, a_i + \epsilon[, v_i(x) = v_j(a_i - \gamma|k_i - k_j|) + x - a_i$$

Since

$$v_i(x) \geq v_j(x - \gamma|k_i - k_j|)$$

we have :

$$\forall x \in [a_i, a_i + \epsilon[, v_j(x - \gamma|k_i - k_j|) \leq v_j(a_i - \gamma|k_i - k_j|) + x - a_i$$

and using Lemma 3, we obtain

$$\forall x \in [a_i, a_i + \epsilon[, v_j(x - \gamma|k_i - k_j|) = v_j(a_i - \gamma|k_i - k_j|) + x - a_i$$

so $[a_i - \gamma|k_i - k_j|, a_i - \gamma|k_i - k_j| + \epsilon[\subset \mathcal{D}_j$. Moreover, if there exists $\delta > 0$ such that $[a_i - \gamma|k_i - k_j| - \delta, a_i - \gamma|k_i - k_j|] \subset \mathcal{D}_j$ then

$$\begin{aligned}\forall x \in]a_i - \gamma|k_i - k_j| - \delta, a_i - \gamma|k_i - k_j|], v_j(x) &= v_j(a_i - \gamma|k_i - k_j|) + x - a_i + \gamma|k_i - k_j| \\ &= v_i(a_i) + x - a_i + \gamma|k_i - k_j| \\ &> v_i(x + \gamma|k_i - k_j|)\end{aligned}$$

which is a contradiction and the result is proved. ◇

Lemma 7 *For all $i \in [1, N]$, if a_i is a left border of $\mathcal{D}_i$ then $a_i \geq k_i$. Moreover,*

$$a_i \in \bar{\mathcal{C}}_i \Rightarrow v_i(a_i) = \frac{\mu\beta(k_i)}{r}$$

Proof: Take a_i a left border of $\mathcal{D}_i$ such that $a_i < k_i$.

First case : $a_i \in \bar{\mathcal{C}}_i$.
Let's prove first that

$$v_i(a_i) = \frac{\mu\beta(k_i) - \alpha(k_i - a_i)}{r}$$

As $-\mathcal{L}_i v_i \geq 0$ and $v_i'(x) = 1$ on $[a_i, a_i + \epsilon[$, we have

$$\forall x \in [a_i, a_i + \epsilon[, -(\mu\beta(k_i) - \alpha((k_i - x)^+)) + rv_i(x) \geq 0$$

So,

$$v_i(a_i) \geq \frac{\mu\beta(k_i) - \alpha(k_i - a_i)}{r}$$

Moreover, as $a_i \in \bar{\mathcal{C}}_i$, it exists δ such that $]a_i - \delta, a_i[\subset \mathcal{C}_i$ and v_i is C^2 over this interval (see Proposition 4). Using Lemma 3, we have $v_i''(a_i^-) \leq 0$. So using the differential equation satisfied by v_i over $]a_i - \delta, a_i[$, we have

$$0 \geq rv_i(a_i) - \mu\beta(k_i) + \alpha(k_i - a_i)$$

So

$$v_i(a_i) = \frac{\mu\beta(k_i) - \alpha(k_i - a_i)}{r}$$

Because a_i is a left border of $\mathcal{D}_i$ and $a_i < k_i$, it exists $\epsilon > 0$ such that $]a_i, a_i+\epsilon[\in \mathcal{D}_i \cap]\gamma k_i, k_i[$. Then

$$\begin{cases} v_i(x) = v_i(a_i) + x - a_i \\ \quad\quad -\mathcal{L}_i v_i(x) \geq 0 \end{cases}$$

so,

$$-(\mu\beta(k_i) - \alpha(k_i - x)) + r\left(\frac{\mu\beta(k_i) - \alpha(k_i - x)}{r} + x - a_i\right) \geq 0$$

It follows that

$$\alpha(k_i - x) - \alpha(k_i - a_i) + r(x - a_i) \geq 0$$

which is a contradiction since $\alpha' > r$.

Second case : $a_i \in \mathcal{S}_{ij}$. Suppose for example that $j > i$. In this case, using Lemma 6, we know that $a_i - \gamma|k_i - k_j|$ is a left border of $\mathcal{D}_j$. Therefore, taking $l = \min\{j > i, a_i - \gamma|k_i - k_j| \notin \mathcal{S}_j^+\}$ we can use the first case and we have

$$a_i - \gamma|k_i - k_l| \geq k_l$$

which implies that

$$a_i \geq k_i$$

and the result is proved. ◇

Proposition 5 *For all $i \in [1, N]$, if $b_i \notin \mathcal{S}_i$ and $\mu\beta(k_i) > \alpha((1-\gamma)k_i)$ then $\mathcal{D}_i = [b_i, +\infty[\cup\mathcal{E}$ where $\mathcal{E}$ is a set with empty interior.*

Proof: Suppose there is another non-empty interior subset in $\mathcal{D}_i$, then it exists a right and a left border that we note d_i and g_i. We prove the result in two steps.

First step: Suppose $g_i = \gamma k_i$.
There exists $\epsilon > 0$ such that $[\gamma k_i, \gamma k_i + \epsilon] \subset \mathcal{D}_i$. So for all $x \in [\gamma k_i, \gamma k_i + \epsilon], v(x) = x - \gamma k_i$ and

$$-(\mu\beta(k_i) - \alpha(k_i - x)) + r(x - \gamma k_i) \geq 0$$

But

$$\lim_{x \to \gamma k_i} -(\mu\beta(k_i) - \alpha(k_i - x)) + r(x - \gamma k_i) = -\mu\beta(k_i) + \alpha((1 - \gamma)k_i) < 0$$

which is a contradiction.

Second step: $g_i > \gamma k_i$
Using Lemma 7 we know that $g_i \geq k_i$, so $d_i > k_i$. This means that there exists $\epsilon > 0$ such that $d_i - \epsilon \geq k_i$ and

$$\forall x \in]d_i - \epsilon, d_i], v_i(x) = v_i(d_i) + x - d_i$$

Using then that $-\mathcal{L}_i v_i(x) \geq 0$ over $]d_i - \epsilon, d_i]$, we have that

$$v_i(d_i) \geq \frac{\mu\beta(k_i)}{r}$$

Using again Lemma 7, as $b_i \notin \mathcal{S}_i$, then

$$v_i(b_i) = \frac{\mu\beta(k_i)}{r}$$

which is a contradiction since $b_i > d_i$ and v_i is a strictly increasing function. $\diamond$

3 Numerical Approximation

The aim of this section is to produce a grid and a discretization by means of central, forward and backward differencing of the free boundary problem (7). We prove that the scheme is monotone and therefore converges to the solution (see [7]).

3.1 Finite difference method

First we make the following change of variable :

$$w_i(x) = v_i(x + \gamma k_i).$$

Clearly, w_i is the unique viscosity solution of the free boundary problem

$$\min\Big(-\mathcal{L}_i' w_i(x), w_i(x) - 1, w_i(x) - \max(w_{i-1}(x - 2\gamma h), w_{i+1}(x - 2\gamma h))\Big) = 0 \qquad (11)$$

with,

$$\mathcal{L}_i'\phi(x) = (\mu\beta(k_i) - \alpha(((1-\gamma)k_i - x)^+))\phi'(x) + \frac{\beta(k_i)^2\sigma^2}{2}\phi''(x) - r\phi$$

which we will continue thereafter to note $\mathcal{L}_i$. We also define the operator $\tilde{\mathcal{I}}$ for a function ϕ and a point x that gives the linear interpolation of ϕ at the point $x - 2\gamma h$.

Remark 3 *Remind that there isn't investment for $i = N$ neither disinvestent for $i = 1$. So in the precedent definition, we suppose that the condition $v_i(x) - v_{i-1}(x - \gamma h)$ (resp. $v_i(x) - v_{i+1}(x - \gamma h)$) fades away when $i = 1$ (resp. $i = N$).*

From the previous section, we know that for all $i \in [1, N]$, there exists b_i such that $w_i'(x) = 1$ over $[b_i, +\infty[$. So taking $x_{max} > \max_i\{b_i\}$ we have the boundaries conditions :

$$\forall i \in [1, N], w_i'(x_{max}) = 1$$

Problem (11) is therefore solved on the computational domain

$$(x, k) \in [0, x_{max}] \times [k_1, k_N] \qquad (12)$$

with the regular grid $(x_l)_{l\in[1,M]} = \{(l-1)\Delta x\}_{l\in[1,M]}$ with

$$\Delta x = \frac{x_{max}}{M-1}$$

in order to have $x_M = x_{max}$. Let $W_{l,i}$ be the approximate solution of equation (11) at (x_l, k_i) for every $i \in [1, N]$ and $l \in [1, M]$. We use a direct method similar to [12] to discretize equation (11) as well as central differencing as much as possible in order to improve the efficiency. Setting $\tilde{\mathcal{L}}_i$ the discretization of $\mathcal{L}_i$, we have:

$$\begin{aligned} &\rho_{l,i}\left[\theta_{l,i}\left(-\psi_{l,i}(\tilde{\mathcal{L}}W)_{l,i} + (1-\psi_{l,i})\left(\frac{W_{l,i}-W_{l-1,i}}{\Delta x} - 1\right)\right)\right] \\ &= -\,\rho_{l,i}(1-\theta_{l,i})(W_{l,i}-W_{l,i-1}) - (1-\rho_{l,i})(W_{l,i} - \tilde{\mathcal{I}}W_{l,i+1}) \end{aligned} \tag{13}$$

with,

$$\begin{aligned} \{\rho_{l,i}, \theta_{l,i}, \psi_{l,i}\} = argmin_{\substack{\rho\in\{0,1\}\\ \theta\in\{0,1\}\\ \psi\in\{0,1\}}} \Big\{ &\rho_{l,i}\left[\theta_{l,i}\left(-\psi_{l,i}(\tilde{\mathcal{L}}W)_{l,i} + (1-\psi_{l,i})\left(\frac{W_{l,i}-W_{l-1,i}}{\Delta x} - 1\right)\right)\right] \\ &+ \rho_{l,i}(1-\theta_{l,i})(W_{l,i}-W_{l,i-1}) + (1-\rho_{l,i})(W_{l,i} - \tilde{\mathcal{I}}W_{l,i+1}) \Big\} \end{aligned}$$

The terminal boundary condition $w_i'(x_M) = 1$ is classically discretized :

$$W_{M,i} = W_{M-1,i} + \Delta x, i \in [1, N]$$

and we also have

$$W_{0,i} = 0, i \in [1, N]$$

If we denote

$$\begin{cases} C_1(x_l, k_i) = \mu\beta(k_i) - \alpha(((1-\gamma)k_i - x_l)^+) \\ C_2(x_l, k_i) = \dfrac{\sigma^2\beta(k_i)^2}{2} > 0 \end{cases}$$

then to satisfy the positive coefficient condition and to maximize the efficiency, the discretized operator $\tilde{\mathcal{L}}_i$ is given by :

$$(\tilde{\mathcal{L}}W)_{l,i} = \begin{cases} C_2(x_l,k_i)\dfrac{W_{l+1,i}+W_{l-1,i}-2W_{l,i}}{\Delta x^2} + C_1(x_l,k_i)\dfrac{W_{l+1,i}-W_{l-1,i}}{2\Delta x} - rW_{l,i} \\ \text{if } 2C_2(x_l,k_i) \geq |C_1(x_l,k_i)|\Delta x \text{ (central differencing)} \\ C_2(x_l,k_i)\dfrac{W_{l+1,i}+W_{l-1,i}-2W_{l,i}}{\Delta x^2} + C_1(x_l,k_i)\dfrac{W_{l+1,i}-W_{l,i}}{\Delta x} - rW_{l,i} \\ \text{if } 2C_2(x_l,k_i) < |C_1(x_l,k_i)|\Delta x \text{ and } C_1(x_l,k_i) \geq 0 \text{ (forward differencing)} \\ C_2(x_l,k_i)\dfrac{W_{l+1,i}+W_{l-1,i}-2W_{l,i}}{\Delta x^2} + C_1(x_l,k_i)\dfrac{W_{l,i}-W_{l-1,i}}{\Delta x} - rW_{l,i} \\ \text{if } 2C_2(x_l,k_i) < |C_1(x_l,k_i)|\Delta x \text{ and } C_1(x_l,k_i) < 0 \text{ (backward differencing)} \end{cases}$$

Proposition 6 *The scheme is monotone, consistent and stable.*

Proof: The scheme is, as a finite difference scheme, consistent. Moreover, we check easily that in $-(\mathcal{L}W)_{i,l}$, the coefficients in front of $W_{l-1,i}, W_{l+1,i}, W_{l,i-1}$ are negatives. So are the coefficients in front of $W_{k,i+1}$ for k acting in the interpolation $\tilde{\mathcal{I}}W_{l,i+1}$. On the contrary, the coefficient in front of $W_{l,i}$ is positive which proves the monotony. We still have to prove the stability i.e. to prove that for all Δx, the schema has a solution $(W_{l,i})_{i,l}$ which is uniformly bounded independently of Δx. First, equation (13) implies that

$$\forall i \in [1, N], \forall l \in [2, M], W_{l,i} \geq W_{l-1,i} + \Delta x \geq W_{l-1,i} \tag{14}$$

so the sequence $l \to W_{l,i}$ is increasing. Let's prove that $W_{M,i}$ is bounded independently of Δx. We know by the terminal boundary condition that

$$\forall i \in [1, N], W_{M,i} = W_{M-1,i} + \Delta x.$$

Let's note $d = \max\{j \in [1, M], W_{j,i} > W_{j-1,i} + \Delta x\}$. By equation (13), we have one of the three next assertions which is true

1. $-(\tilde{\mathcal{L}}W)_{d,i} = 0.$
2. $W_{d,i} - W_{d,i-1} = 0.$
3. $W_{d,i} - \tilde{\mathcal{I}}W_{d,i+1} = 0.$

and by definition of d :

$$W_{d+1,i} = W_{d,i} + \Delta x. \tag{15}$$

Case 1 : Using the discretized operator, we have in the central differencing case :

$$-C_2(x_d, k_i)\frac{W_{d+1,i} + W_{d-1,i} - 2W_{d,i}}{\Delta x^2} - C_1(x_d, k_i)\frac{W_{d+1,i} - W_{d-1,i}}{2\Delta x} + rW_{d,i} = 0.$$

Then, using (15),

$$-C_2(x_d, k_i)\frac{W_{d-1,i} - W_{d,i} + \Delta x}{\Delta x^2} - C_1(x_d, k_i)\frac{W_{d,i} - W_{d-1,i} + \Delta x}{2\Delta x} + rW_{d,i} = 0.$$

Factoring,

$$rW_{d,i} = C_1(x_d, k_i) + \left(-\frac{C_2(x_d, k_i)}{\Delta x^2} + \frac{C_1(x_d, k_i)}{2\Delta x}\right)(W_{d,i} - W_{d-1,i} - \Delta x)$$

so, using that $W_{d,i} \geq W_{d-1,i} + \Delta x$ and the central differencing inequation, we have

$$W_{d,i} \leq \frac{C_1(x_d, k_i)}{r}.$$

Moreover, $C_1(x_d, k_i)$ is bounded independently of (x_d, k_i) by $\mu\bar{\beta}$. Then,

$$W_{d,i} \leq \frac{\mu\bar{\beta}}{r}.$$

But, by definition of d,

$$W_{M,i} = W_{d,i} + (M-d)\Delta x,$$

so

$$W_{M,i} \leq \frac{\mu\bar{\beta}}{r} + (M-d)\Delta x \leq \frac{\mu\bar{\beta}}{r} + x_M$$

The proof for the forward and backward differencing is similar and therefore omitted.

Case 2 : In this case, let's define $p = \max\{j \in [1, i-1], W_{d,j} - W_{d,j-1} > 0\}$. At this point, we necessarily have $-\tilde{\mathcal{L}}W_{d,p} = 0$. Let's prove that we also have

$$W_{d+1,p} = W_{d,p} + \Delta x$$

Suppose that $W_{d+1,p} > W_{d,p} + \Delta x$. Then, using that $W_{d,p} = W_{d,i}$

$$W_{d+1,p} > W_{d,i} + \Delta x.$$

But by definition of d, $W_{d,i} + \Delta x = W_{d+1,i}$, so

$$W_{d+1,p} > W_{d+1,i}$$

which is a contradiction since $p < i$. So we have

$$W_{d+1,p} = W_{d,p} + \Delta x$$

and

$$-\tilde{\mathcal{L}}W_{d,p} = 0$$

and we can use the first case to prove that

$$W_{d,i} \leq \frac{\mu\bar{\beta}}{r} + x_M$$

The proof in the third case is similar and is therefore omitted.
Finally, we have proved in all cases that

$$\forall i \in [1, N], \forall l \in [1, M], W_{l,i} \leq \frac{\mu\bar{\beta}}{r} + x_M$$

which is a bound independent of Δx and the result is proved.

$\diamond$

3.2 Matrix Form of the Discretized Equations

We denote U the vector of size $N \times M$ and $\hat{U}_i$ for $i \in [1, N]$ the vectors of size M such that

$$\hat{U}_i = (W_{l,i})_{l\in[1,M]}, \qquad U = (\hat{U}_1, \hat{U}_2, ..., \hat{U}_N)$$

With

$$j = l + (i-1)M$$

we have

$$\forall j \in [1, N \times M], U_j = W_{l,i}$$

Then we can write (13) in a linear matrix form as follows :

$$A(\rho, \psi, \theta)U + B(\rho, \psi, \theta) = 0 \tag{16}$$

where $A(\rho, \psi, \theta)$ is a square tridiagonal block matrix of size $N \times M$:

$$\overbrace{\begin{pmatrix} Z^1 & D^1 & & & & & \\ C^2 & \ddots & \ddots & & & \mathbf{0} & \\ & \ddots & \ddots & \ddots & & & \\ & & C^i & Z^i & D^i & & \\ & & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & & \ddots & \ddots & D^{N-1} \\ & & & & & C^N & Z^N \end{pmatrix}}^{N \times M}$$

with Z^i the tridiagonal matrix of size M given by,

$$Z^i = T^i + (\rho_{l,i}(1 - \theta_{l,i}) + (1 - \rho_{l,i}))Id$$

with the coefficients of T^i given, for all $l \in [2, M-1]$, by :

$$\begin{cases} t^i_{l,l} = \rho_{l,i}\theta_{l,i}\left(\psi_{l,i}\left(r + \dfrac{2C_2(x_l, k_i)}{\Delta x^2} + \dfrac{|C_1(x_l, k_i)|}{\Delta x}1_{\{2C_2(x_l,k_i)<\Delta x|C_1(x_l,k_i)|\}}\right) + \dfrac{(1-\psi_{l,i})}{\Delta x}\right) \\ t^i_{l,l+1} = \rho_{l,i}\theta_{l,i}\left(-\psi_{l,i}\left(\dfrac{C_2(x_l, k_i)}{\Delta x^2} + \dfrac{C_1(x_l, k_i)}{2\Delta x} + \dfrac{|C_1(x_l, k_i)|}{2\Delta x}1_{\{2C_2(x_l,k_i)<\Delta x|C_1(x_l,k_i)|\}}\right)\right) \\ t^i_{l,l-1} = \rho_{l,i}\theta_{l,i}\left(-\psi_{l,i}\left(\dfrac{C_2(x_l, k_i)}{\Delta x^2} - \dfrac{C_1(x_l, k_i)}{2\Delta x} + \dfrac{|C_1(x_l, k_i)|}{2\Delta x}1_{\{2C_2(x_l,k_i)<\Delta x|C_1(x_l,k_i)|\}}\right) - \dfrac{(1-\psi_{l,i})}{\Delta x}\right) \end{cases}$$

To fulfill the Dirichlet conditions, we force :

$$\forall i \in [1, N], \forall l \in [1, M], \begin{cases} Z^i_{1,l} = \delta_{1l} \\ D^i_{1,l} = C^i_{1,l} = 0 \end{cases}$$

Also to satisfy the right boundary conditions, the controls are fixed for $l = M$:

$$\forall i \in [1, N], \begin{cases} \rho_{M,i} = 1 \\ \theta_{M,i} = 1 \\ \psi_{M,i} = 0 \end{cases}$$

By doing this we make sure that at the right boundary of the grid, we are in the dividend region.
C^i is the diagonal matrix of size M with

$$\forall i \in [1, N], \forall l \in [1, M], c^i_{l,l} = -(1 - \theta_{l,i})\rho_{l,i}$$

The form of the matrix D^i is given by the operator $\tilde{\mathcal{I}}$. To have a monotone scheme, we need to use a linear operator. With a constant step, the form of the matrix is as follows :

$$D^i = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -(1-\rho_{3,i})\lambda & -(1-\rho_{3,i})(1-\lambda) & 0 & & & \\ & \ddots & \ddots & \ddots & & \mathbf{0} \\ & \mathbf{0} & \ddots & \ddots & \ddots & \\ & & & -(1-\rho_{M,i})\lambda & -(1-\rho_{M,i})(1-\lambda) & 0 \end{pmatrix}$$

The offset d with respect to the diagonal in the matrix D^i, i.e the number of zero lines at the beginning of D^i, is equal to

$$d = 1 + E\left(\frac{2\gamma h}{\Delta x}\right)$$

where E is the floor function. We also define the fractional part :

$$\lambda = \frac{2\gamma h}{\Delta x} - E\left(\frac{2\gamma h}{\Delta x}\right)$$

Last, the vector B, is of size $N \times M$ and satisfies

$$\forall j \in [1, N \times M], b_j = -\rho_{l,i}\theta_{l,i}(1 - \psi_{l,i}), \qquad j = l + (i-1)M$$

Definition 2 *A matrix A is a M-matrix if A is non singular, the off-diagonal coefficients of A are negatives and $A^{-1} \geq 0$.*

Remark 4 *A matrix such that $A^{-1} \geq 0$ satisfies*

$$X \geq 0 \Leftrightarrow A^{-1}X \geq 0$$

Lemma 8 *The matrices $Z^i(\rho, \psi, \theta) = (z^i_{l,j})_{l,j}$ are M-matrices.*

Proof: Take $i \in [1, N]$. We observe that the matrix Z^i satisfies for all l, $z^i_{ll} > 0$ and for all $l \neq j$, $z^i_{lj} \leq 0$. Moreover Z^i is a diagonally dominant matrix. Indeed,

$$\forall l \in [2, M], z^i_{ll} - \sum_{j \neq l} |z^i_{l,j}| = \begin{cases} r, & \psi_{li}\rho_{l,i}\theta_{l,i} = 1 \\ 0, & \psi_{li} = 0 \text{ et } \rho_{l,i}\theta_{l,i} = 1 \\ 1, & \rho_{li}\theta_{li} = 0 \end{cases}$$

For $l = 1$, Dirichlet condition dictates that

$$z^i_{11} - \sum_{j \neq 1} |z^i_{1,j}| = 1$$

However the matrix is not a strict diagonally dominant matrix since when there is distribution of dividends (i.e. $\psi_{li} = 0$ and $\rho_{l,i}\theta_{l,i} = 1$), the sum of the line coefficients is equal to zero and thus the classical technique used in [12] does not apply. Another way to prove that $Z^i(\rho, \psi, \theta)$ is a M-matrix is to find a M-size vector W such that $W > 0$ and $Z^iW > 0$ (see [21]). Let's prove that W the M-size vector given by

$$\forall l \in [1, M], W_l = 1 + l\epsilon$$

with

$$\epsilon = \frac{r\Delta x}{\mu\bar{\beta}} > 0$$

satisfies this condition for all $i \in [1, N]$. First, as Z^i is a tridiagonal matrix, we have

$$\forall l \in [2, M-1], (Z^iW)_l = z^i_{l,l-1}(1 + (l-1)\epsilon) + z^i_{l,l}(1 + l\epsilon) + z^i_{l,l+1}(1 + (l+1)\epsilon)$$

For $l = 1$

$$(Z^iW)_1 = (1 + \epsilon)$$

and for $l = M$

$$(Z^iW)_M = \frac{1 + M\epsilon}{\Delta x} - \frac{1 + (M-1)\epsilon}{\Delta x}$$

Then, for all $l \in [2, M]$, we have :

$$(Z^iW)_l = \begin{cases} (1 + (l-1)\epsilon)r + \epsilon(z^i_{ll} + 2z^i_{l,l+1}), & \rho_{li}\theta_{li}\psi_{li} = 1 \\ \dfrac{\epsilon}{\Delta x}, & \rho_{li}\theta_{li} = 1 \text{ et } \psi_{li} = 0 \\ 1 + l\epsilon, & \rho_{li}\theta_{li} = 0 \end{cases}$$

Moreover, when $\rho_{li}\theta_{li}\psi_{li} = 1$, we have

$$z^i_{ll} + 2z^i_{l,l+1} = r - \frac{C_1(x_l, k_i)}{\Delta x}$$

So,

$$\begin{aligned}(1+(l-1)\epsilon)r+\epsilon(z^i_{ll}+2z^i_{l+1,l}) &= (l-1)r\epsilon+r+\epsilon\left(r-\frac{C_1(x_l,k_i)}{\Delta x}\right)\\ &\geq lr\epsilon+r-\epsilon\frac{C_1(x_l,k_i)}{\Delta x}\end{aligned}$$

Using the definition of ϵ and the fact that

$$\forall i\in[1,N],\forall l\in[1,M],C_1(x_l,k_i)\leq\mu\bar{\beta}$$

we have that

$$(1+(l-1)\epsilon)r+\epsilon(z^i_{ll}+2z^i_{l+1,l})\geq lr\epsilon$$

So

$$\forall l\in[1,M],(Z^iW)_l>0$$

And we conclude that Z^i is a M-matrix. ◇

Corollary 1 *The matrix $A(\rho,\theta,\psi)$ is a M-matrix.*

Proof: To prove the result we use the same technique as in Lemma 8. Choose

$$\epsilon=\frac{r\Delta x}{\mu\bar{\beta}}>0$$

and

$$0<\eta<(d+\lambda)\epsilon$$

Thus, we define W the $N\times M$-size vector as follows

$$\forall i\in[1,N],\forall l\in[1,M],W_{l+(i-1)M}=1+l\epsilon+i\eta$$

Then, using the results of Lemma 8, we obtain that for all $i\in[1,N]$ and for all $l\in[2,M-1]$,

$$\rho_{li}\theta_{li}=1\Rightarrow(AW)_{l+(i-1)M}\geq\min\left(\frac{\epsilon}{\Delta x},lr\epsilon\right)+(z^i_{l,l}+z^i_{l,l-1}+z^i_{l,l+1})i\eta$$

Using that the matrices Z^i are diagonally dominant, we have that

$$\rho_{li}\theta_{li}=1\Rightarrow(AW)_{l+(i-1)M}>0$$

In the case $(\theta_{li},\rho_{l,i})=(0,1)$, we have

$$(AW)_{l+(i-1)M}=1+l\epsilon+i\eta-(1+l\epsilon+(i-1)\eta)=\eta>0$$

and in the case $\rho_{l,i}=0$, we have

$$\begin{aligned}(AW)_{l+(i-1)M}=&1+l\epsilon+i\eta-(1-\lambda)[1+(l-d)\epsilon+(i+1)\eta]\\ &-\lambda[1+(l-1-d)\epsilon+(i+1)\eta]\\ =&(d+\lambda)\epsilon-\eta\\ >&0\end{aligned}$$

Thus, we display a vector $W>0$ such that, for all $i\in[1,N]$, for all $l\in[1,M]$, $(AW)_{l+(i-1)M}>0$ which proves he result.

◇

4 Convergence of the scheme

The main result of this section is that U is the solution of the following Newton algorithm :

$$A^q U^{q+1} + B^q = 0 \tag{17}$$

with

$$A^q = A(\rho^q, \theta^q, \psi^q)$$
$$B^q = B(\rho^q, \theta^q, \psi^q)$$

and is resumed in Theorem 1.

Algorithm 1 Policy Iteration

$(\rho^0, \theta^0, \psi^0) = (1, 1, 1)$

$q = 0$

while Error $> \epsilon$ **do**

Solve W^{q+1} solution of

$$\rho^q_{l,i}\left[\theta^q_{l,i}\left(-\psi^q_{l,i}(\tilde{\mathcal{L}}W^{q+1})_{l,i} + (1-\psi^q_{l,i})\left(\frac{W^{q+1}_{l,i} - W^{q+1}_{l-1,i}}{\Delta x} - 1\right)\right)\right]$$
$$= -\rho^q_{l,i}(1-\theta^q_{l,i})(W^{q+1}_{l,i} - W^{q+1}_{l,i-1}) - (1-\rho^q_{l,i})(W^{q+1}_{l,i} - \tilde{\mathcal{I}}(W^{q+1}_{i+1}))$$

$(\rho^{q+1}, \theta^{q+1}, \psi^{q+1})$ solution of

$$\begin{aligned}(\rho^{q+1}, \theta^{q+1}, \psi^{q+1}) \;=\; & argmin_{(\rho,\theta,\psi)\in\{0,1\}}\Big\{\rho_{l,i}\theta_{l,i}\Big[-\psi_{l,i}(\tilde{\mathcal{L}}W^{q+1})_{l,i} \\ & +(1-\psi_{l,i})\left(\frac{W^{q+1}_{l,i} - W^{q+1}_{l-1,i}}{\Delta x} - 1\right)\Big] \\ & +\rho_{l,i}(1-\theta_{l,i})(W^{q+1}_{l,i} - W^{q+1}_{l,i-1}) + (1-\rho_{l,i})(W^{q+1}_{l,i} - \tilde{\mathcal{I}}(W^{q+1}_{i+1}))\Big\}\end{aligned}$$

Error$= \frac{||\rho^{q+1}-\rho^q||}{||\rho^q||} + \frac{||\theta^{q+1}-\theta^q||}{||\theta^q||} + \frac{||\psi^{q+1}-\psi^q||}{||\psi^q||}$

$q = q + 1$

end while

Theorem 1 *Under the conditions*

i) The matrices A^q are M-matrix

ii) $\|(A^q)^{-1}\|$ and $\|B^q\|$ are bounded regardless of q.

the scheme in Algorithm 1 converges to the unique solution of equation (16).

Proof: The proof is classic and is posponed in appendix. ⋄

Lemma 9 *The sequence $(U^q)_{q\geq 0}$ is bounded.*

Proof: To prove the result, we have to prove that the matrices $(A^q)^{-1}$ and B^q are bounded, regardless of q. First, by definition, we have that for all $q \geq 0$ and for all $j \in [1, N \times M]$, $b_j^q \leq 1$, so the matrix B^q is bounded. Moreover, since ρ^q, θ^q and ψ^q take discrete values :

$$\forall q \geq 0, \forall l \in [1, M], \forall i \in [1, N], \rho_{l,i}^q, \theta_{l,i}^q, \psi_{l,i}^q = 0 \text{ or } 1$$

we have a finite number of invertible matrices A^q so also a finite number of $(A^q)^{-1}$ and taking the maximum over all the possible combinations lead to a supremum of $(A^q)^{-1}$ regardless of q and the result is poved. $\diamond$

5 Numerical results

5.1 Description of the optimal regions

In chapter 2.3, we proved that for all $i \in [1, N]$ there exists $b_i > k_i$ such that the dividend region contains at least $[b_i, +\infty[$, which allows us to define in the numerical scheme a border condition. The numerical results give us much more information about the optimal control regions. Next Proposition resumes those results :

Proposition 7 *The optimal control regions satisfy*

1. $\forall i \in [1, N], \mathcal{D}_i = [b_i, +\infty[$.
2. $\forall i \in [2, N], \exists d_i \in \Omega_i, \mathcal{S}_i^- = [\gamma k_i, d_i]$
3. $\exists k^* \in [1, N], \forall i \geq k^*, \mathcal{S}_i^+ = \emptyset$
4. $\forall i < k^*, \exists a_i \in]d_i, b_i], \mathcal{S}_i^+ = [a_i, +\infty[$

Figure 1 illustrates Proposition 7. The results are obtained using a linear debt and an exponential gain function :

- Linear debt : $\alpha(x) = \lambda x$
- Exponential gain function : $\beta(x) = \bar{\beta}\left(1 - \exp(-\frac{\eta}{\bar{\beta}}x)\right)$

and the next values for the different parameters :

$$[\mu = 0.25, \sigma = 0.40, r = 0.02, \lambda = 0.10, \bar{\beta} = 2, \eta = 1, \gamma = 1e^{-3}, N = 20, M = 1e^5]$$

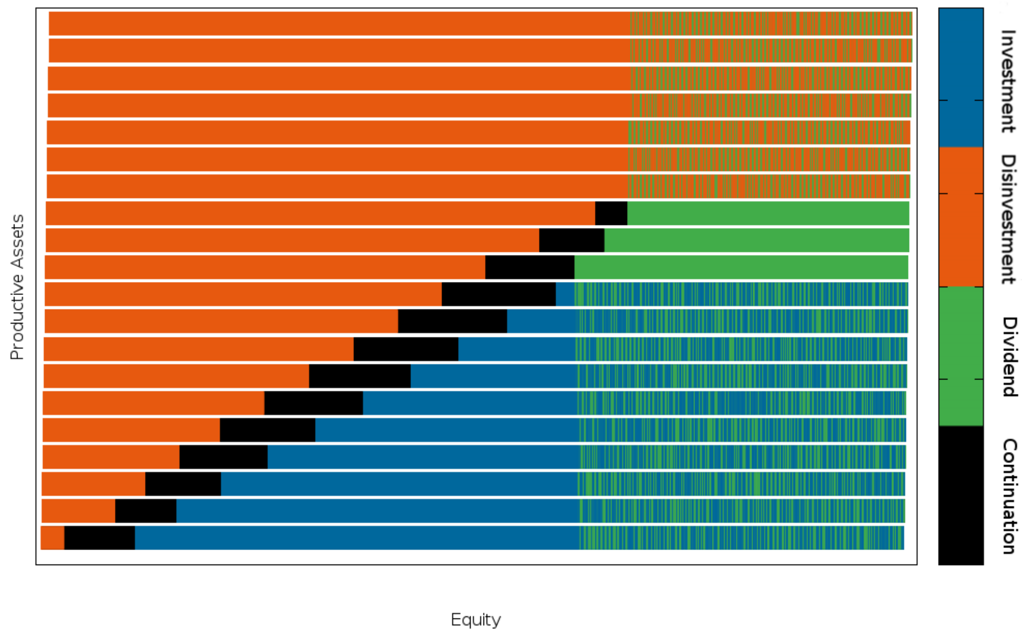

Figure 1: Optimal control regions.

In figure 1, we differentiate six areas :

1. $\mathcal{S}_i^-$ (Orange zone) : Disinvestment area where the book value of equity is low in comparison to the level of the firm's productive assets. In this zone, it is optimal to disinvest to lower the risk of bankruptcy.

2. $\mathcal{S}_i^+$ (Blue area) : Investment area where the ratio book value of equity over firm's productive assets is high. It is optimal to invest to increase the rentability via the gain function. The risk increases proportionnaly but the cash reserves protect the company against bankrupcy.

3. $\mathcal{C}_i$ (Black area) : In between, there is the continuation area where it is optimal to not activate the controls.

As proved in chapter 2.3, we observe that on the right side of figure 1, corresponding to a high level of equity, it is always optimal to pay dividends leading to three different areas :

4. $\mathcal{S}_i^+ \cap \mathcal{D}_i$ (Green and blue area) : for a low level of productive assets. In this zone, it is optimal to pay dividends and invest until reach the optimal level k^*.

5. $\mathcal{S}_i^- \cap \mathcal{D}_i$ (Green and orange area) : for a high level of productive assets. In this zone, it is optimal to disinvest until a maximum level of productive assets $k_{\max}$ in order to distribute dividends.

6. $\mathcal{D}_i$ (Green area) : in between, it isn't optimal to invest neither to disinvest but just to pay dividends.

Those results are consistent with the economic theory. Furthermore, they bring to light two meaningful conclusions :

- It is optimal to pay dividends only once the company has reached a optimal size depending of its sector.
- There exists a maximum size that the company shouldn't exceed.

Those conclusions are directly attributable to the characteristics of the gain function chosen. Indeed, the rentability increases with the gain function which is concave with a finite limit at the infinity. So at some point, the marginal gain is small compared to the value for the shareholders to receive dividends.

5.2 Impact of the cost of investment

The cost of investment γ (and of disinvestment) plays an important role in the form of the switching regions. Figure 2 to 4, present the optimal control regions for different values of γ and the same parameters than figure 1.

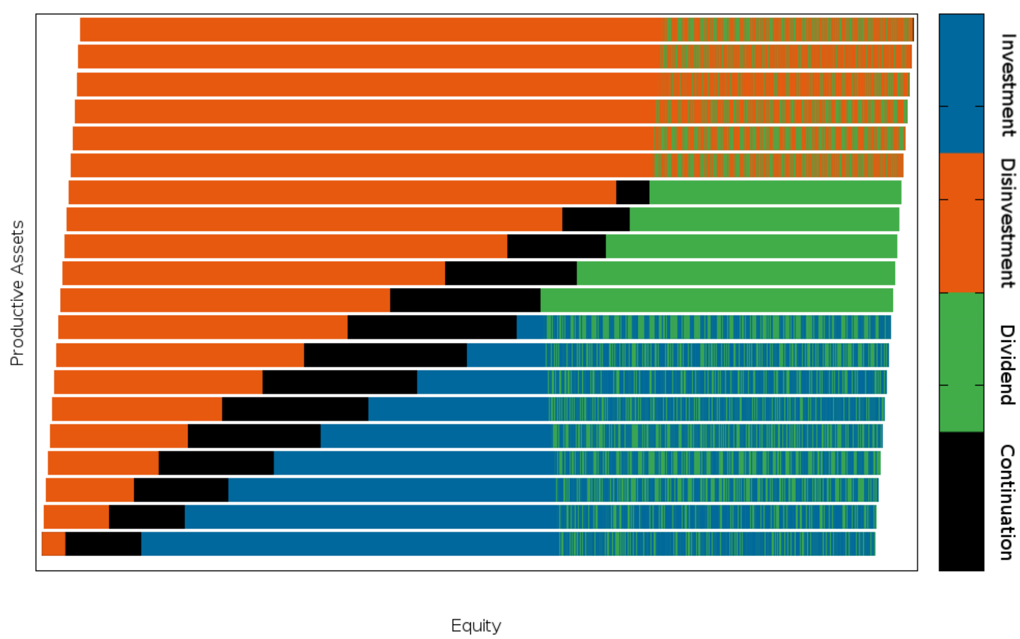

Figure 2: Optimal control regions for $\gamma = 0.05$.

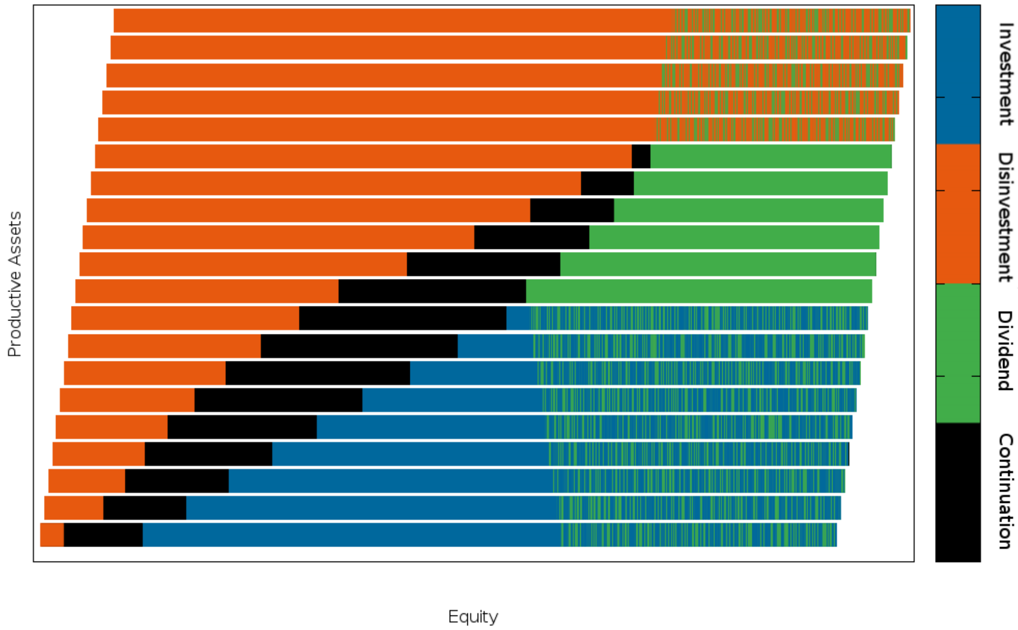

Figure 3: Optimal control regions for $\gamma = 0.1$.

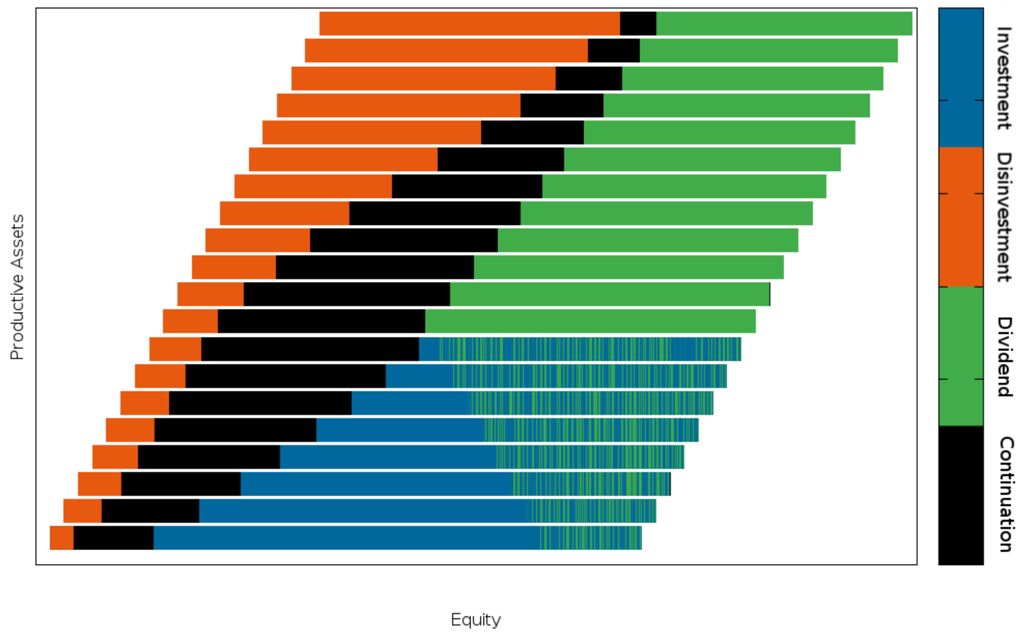

Figure 4: Optimal control regions for $\gamma = 0.5$.

We observe that the higher γ is, the wider the continuation region is. Which is consistent since if the cost is low the manager is prone to invest since he knows that he could desinvest at lower prices.

6 Appendix

6.1 Proof of Proposition 2

Supersolution

Let $i \in [1, N]$, $\bar{x} \in \Omega_i$ and φ a C^2 function such that $\bar{x}$ is a minimum of $v_i - \varphi$ and $v_i(\bar{x}) = \varphi(\bar{x})$.

First, let us consider the admissible control where the manager decide to invest at time $t = 0$. From the dynamic programming principle, we have

$$v_i(\bar{x}) \geq v_{i-1}(\bar{x} - \gamma h)$$

In the same way, we also have

$$v_i(\bar{x}) \geq v_{i+1}(\bar{x} - \gamma h)$$

Second, let us consider the admissible control π from $\bar{x}$ where the manager decide neither to invest (or disinvest) nor paying dividends until the liquidation, and $(X_t^\pi, k_i)_{t\geq 0}$ the process controlled by π. Then, from the dynamic programming principle, we have between $[0, \tau \wedge h]$ with $h > 0$:

$$\begin{aligned}\varphi_i(\bar{x}) = v_i(\bar{x}) &\geq \mathbb{E}\left(\int_0^{\tau\wedge h} e^{-rt} dZ_t^\pi + e^{-r(\tau\wedge h)} v(X_{\tau\wedge h}^\pi, K_{\tau\wedge h}^\pi)\right) \\ &\geq \mathbb{E}(e^{-r(\tau\wedge h)} v_i(X_{\tau\wedge h}^\pi)) \\ &\geq \mathbb{E}(e^{-r(\tau\wedge h)} \varphi_i(X_{\tau\wedge h}^\pi)).\end{aligned} \tag{18}$$

Applying Itô's formula to the process $e^{-rt}\varphi_i(X_t)$ between 0 and $\tau \wedge h$ and taking the expectation, we have

$$\mathbb{E}(e^{-r(\tau\wedge h)}\varphi_i(X_{\tau\wedge h}^\pi)) = \varphi_i(\bar{x}) + \mathbb{E}\left(\int_0^{\tau\wedge h} \mathcal{L}\varphi_i(X_t^\pi) dt\right) + \mathbb{E}\left[\sum_{t\leq\tau\wedge h} e^{-rt}(\varphi_i(X_t^\pi) - \varphi_i(X_{t^-}^\pi))\right].$$

We then observe that X^π is continuous on $[0, \tau \wedge h]$, so

$$\mathbb{E}(e^{-r(\tau\wedge h)}\varphi_i(X_{\tau\wedge h}^\pi)) = \varphi_i(\bar{x}) + \mathbb{E}\left(\int_0^{\tau\wedge h} \mathcal{L}\varphi_i(X_t^\pi) dt\right) \tag{19}$$

Therefore using (18) and (19)

$$0 \geq \mathbb{E}\left(\int_0^{\tau\wedge h} \mathcal{L}\varphi_i(X_t^\pi) dt\right)$$

By dividing the above inequality by h with $h \to +\infty$, we conclude that

$$-\mathcal{L}\varphi_i(\bar{x}) \geq 0$$

Proceeding analogously, we prove that $\varphi_i'(x) - 1 \geq 0$ and the supersolution property is proved.

Subsolution
We prove the subsolution property by contradiction. Suppose that the claim is not true. Then, there exists $i \in [1, N]$, $\bar{x} \in \Omega_i$ and $\varphi \in C^2$ such that $\varphi(\bar{x}) = v_i(\bar{x})$, $\bar{x}$ is a maximum of $v_i - \varphi$ and

$$\min\{-\mathcal{L}_i\varphi(\bar{x}), \varphi'(\bar{x}) - 1, v_i(\bar{x}) - v_{i-1}(\bar{x} - \gamma h), v_i(\bar{x}) - v_{i+1}(\bar{x} - \gamma h)\} > 0.$$

Otherwise

$$\begin{aligned}&-\mathcal{L}_i\varphi(\bar{x}) > 0 \\ &\varphi'(\bar{x}) - 1 > 0 \\ &v_i(\bar{x}) - v_{i-1}(\bar{x} - \gamma h) > 0 \\ &v_i(\bar{x}) - v_{i+1}(\bar{x} - \gamma h) > 0\end{aligned}$$

But $\varphi \in C^2$ and we proved (proposition 1) that v_i is continuous so it exists $\eta > 0$ and $\epsilon > 0$ such that

$$\begin{aligned}\forall x \in]\bar{x} - \epsilon, \bar{x} + \epsilon[, &-\mathcal{L}_i\varphi(x) > \eta \\ &\varphi'(x) - 1 > \eta \\ &v_i(x) - v_{i-1}(x - \gamma h) > \eta \\ &v_i(x) - v_{i+1}(x - \gamma h) > \eta.\end{aligned} \tag{20}$$

Let π be an admissible strategy and (X_t^π, K_t^π) the process controlled by π from $(\bar{x}, k_i)$. Consider the exist times

$$\begin{aligned}\tau_\epsilon &= \inf\{t \geq 0, X_t^\pi \notin]x-\epsilon, x+\epsilon[\}\\ \tau_1 &= \inf\{t \geq 0, K_t^\pi \neq k_i\}\end{aligned}$$

Applying Itô's formula to the process $e^{-rt}\varphi(X_t^\pi)$ between 0 and $\tau^- = (\tau_\epsilon \wedge \tau_1)^-$, we get

$$\begin{aligned}\mathbb{E}[e^{-r\tau^-}\varphi(X_{\tau^-}^\pi)] =& \varphi(\bar{x}) + \mathbb{E}\left[\int_0^{\tau^-} e^{-rt}\mathcal{L}_i\varphi(X_t^\pi)dt\right]\\ &- \mathbb{E}\left[\int_0^{\tau^-} e^{-rt}\varphi'(X_t^\pi)dZ_t^{\pi,c}\right]\\ &+ \mathbb{E}\left[\sum_{0\leq t<\tau} e^{-rt}[\varphi(X_t^\pi) - \varphi(X_{t^-}^\pi)]\right]\end{aligned} \tag{21}$$

By the mean value theorem and using that for $t < \tau$, we have $\Delta X_t^\pi = -\Delta Z_t^\pi$, where ΔX_t^π is the jump in the process X_t^π at time t, there is some $\rho \in]0,1[$ such that

$$\begin{aligned}\varphi(X_t^\pi) - \varphi(X_{t^-}^\pi) &= \Delta X_t^\pi \varphi'(X_t^\pi + \rho\Delta X_t^\pi)\\ &= -\Delta Z_t^\pi \varphi'(X_t^\pi + \rho\Delta X_t^\pi).\end{aligned}$$

So using (20),

$$\varphi(X_t^\pi) - \varphi(X_{t^-}^\pi) \leq -(1+\eta)\Delta Z_t^\pi.$$

Putting into (21) and using again(20), we have

$$\begin{aligned}v_i(\bar{x}) = \varphi(\bar{x}) \geq& \mathbb{E}\left[\int_0^{\tau^-} e^{-rt}dZ_t^\pi + e^{-r\tau^-}\varphi(X_{\tau^-}^\pi)\right]\\ &+ \eta\mathbb{E}\left[\int_0^{\tau^-} e^{-rt}dt + \int_0^{\tau^-} e^{-rt}dZ_t^\pi\right]\\ \geq& \mathbb{E}\left[\int_0^{\tau^-} e^{-rt}dZ_t^\pi + e^{-r\tau_\epsilon^-}\varphi(X_{\tau_\epsilon^-}^\pi)1_{\tau_\epsilon<\tau_1} + e^{-r\tau_1^-}\varphi(X_{\tau_1^-}^\pi)1_{\tau_\epsilon\geq\tau_1}\right]\\ &+ \eta\mathbb{E}\left[\int_0^{\tau^-} e^{-rt}dt + \int_0^{\tau^-} e^{-rt}dZ_t^\pi\right]\end{aligned}$$

Because $X_{\tau_\epsilon}^\pi$ is not in $]\bar{x}-\epsilon, \bar{x}+\epsilon[$, there exists $\theta \in [0,1]$ such that

$$\begin{aligned}X^{(\theta)} &= X_{\tau_\epsilon^-}^\pi + \theta\Delta X_{\tau_\epsilon}^\pi\\ &= X_{\tau_\epsilon^-}^\pi - \theta\Delta Z_{\tau_\epsilon}^\pi\end{aligned}$$

with $X^{(\theta)}$ in the frontier of $]\bar{x}-\epsilon, \bar{x}+\epsilon[$. So as before,

$$\varphi(X^{(\theta)}) - \varphi(X_{\tau_\epsilon^-}^\pi) \leq -\theta(1+\eta)\Delta Z_{\tau_\epsilon}^\pi$$

Observe that

$$X^{(\theta)} = X^\pi_{\tau_\epsilon} + (1-\theta)\Delta Z^\pi_{\tau_\epsilon}$$

so by the dynamic programming principle

$$v_i(X^{(\theta)}) \geq v_i(X^\pi_{\tau_\epsilon}) + (1-\theta)\Delta Z^\pi_{\tau_\epsilon}$$

We then have using that $v_i - \varphi \leq 0$,

$$\varphi(X^\pi_{\tau_\epsilon^-}) \geq v_i(X^\pi_{\tau_\epsilon}) + (1+\theta\eta)\Delta Z^\pi_{\tau_\epsilon}.$$

Therefore,

$$\begin{aligned} v_i(\bar{x}) \geq & \mathbb{E}\left[\int_0^{\tau^-} e^{-rt} dZ^\pi_t + e^{-r\tau_\epsilon} v_i(X^\pi_{\tau_\epsilon}) 1_{\tau_\epsilon < \tau_1} + e^{-r\tau_1^-} \varphi(X^\pi_{\tau_1^-}) 1_{\tau_\epsilon \geq \tau_1}\right] \\ & + \eta \mathbb{E}\left[\int_0^{\tau} e^{-rt} dt + \int_0^{\tau^-} e^{-rt} dZ^\pi_t + \theta e^{-r\tau_\epsilon} \Delta Z^\pi_\epsilon 1_{\tau_1 > \tau_\epsilon}\right] \\ & + \mathbb{E}[e^{-r\tau_\epsilon} \Delta Z^\pi_{\tau_\epsilon} 1_{\tau_1 > \tau_\epsilon}] \end{aligned}$$

For $\tau_\epsilon \geq \tau_1$, we have for all $t \leq \tau_1$, $X^\pi_t \in]x-\epsilon, x+\epsilon[$, so using again (20),

$$\varphi(X_{\tau_1^-}) \geq v_i(X_{\tau_1^-}) > \eta + v_j(X_{\tau_1^-} - \gamma h)$$

where $j = i+1$ ou $j = i-1$. Then

$$\begin{aligned} v_i(\bar{x}) \geq & \mathbb{E}\left[\int_0^{\tau^-} e^{-rt} dZ^\pi_t + e^{-r\tau_\epsilon} v_i(X^\pi_{\tau_\epsilon}) 1_{\tau_\epsilon < \tau_1} + e^{-r\tau_1} v_j(X^\pi_{\tau_1}) 1_{\tau_\epsilon \geq \tau_1}\right] \\ & + \eta \mathbb{E}\left[\int_0^{\tau} e^{-rt} dt + \int_0^{\tau^-} e^{-rt} dZ^\pi_t + e^{-r\tau_1} 1_{\tau_1 \leq \tau_\epsilon} + \theta e^{-r\tau_\epsilon} \Delta Z^\pi_\epsilon 1_{\tau_1 > \tau_\epsilon}\right] \\ & + \mathbb{E}[e^{-r\tau_\epsilon} \Delta Z^\pi_{\tau_\epsilon} 1_{\tau_1 > \tau_\epsilon}] \end{aligned} \tag{22}$$

We now claim that there is a constant $c_0 > 0$ such that for every admissible strategy π

$$\mathbb{E}\left[\int_0^{\tau} e^{-rt} dt + \int_0^{\tau^-} e^{-rt} dZ^\pi_t + e^{-r\tau_1} 1_{\tau_1 \leq \tau_\epsilon} + \theta e^{-r\tau_\epsilon} \Delta Z^\pi_\epsilon 1_{\tau_1 > \tau_\epsilon}\right] \geq c_0 \tag{23}$$

Let us consider the function C^2, $\psi(x) = c_0 \left[1 - \frac{(x-\bar{x})^2}{\epsilon^2}\right]$, with

$$0 < c_0 \leq \min\left\{\frac{\epsilon}{2}, 1, \left(r + \frac{2(\mu\beta(k_i) - \alpha(k_i))}{\epsilon} + \frac{\sigma^2\beta^2(k_i)}{\epsilon^2}\right)^{-1}\right\}$$

Then ψ satisfies

$$\begin{cases} \min\{\mathcal{L}_i\psi + 1, 1-\psi', -\psi+1\} \geq 0, x \in]\bar{x}-\epsilon, \bar{x}+\epsilon[\\ \psi = 0, x \in \partial(]\bar{x}-\epsilon, \bar{x}+\epsilon[) \end{cases}$$

Applying Itô's formula to the process $e^{-rt}\psi(X_t^\pi)$ between 0 and τ^-, we obtain

$$\begin{aligned} 0 < c_0 = \psi(\bar{x}) \leq \mathbb{E}\left[e^{-r\tau}\psi(X^\pi_{\tau^-})\right] \\ + \mathbb{E}\left[\int_0^{\tau^-} e^{-rt}dt + \int_0^{\tau^-} e^{-rt}dZ_t^\pi\right] \end{aligned} \tag{24}$$

Moreover $\psi'(x) \leq 1$, so

$$\psi(X^\pi_{\tau_\epsilon^-}) - \psi(X^{(\theta)}) \leq (X^\pi_{\tau_\epsilon^-} - X^{(\theta)})$$

which is equivalent, using that $\psi(X^{(\theta)}) = 0$, to

$$\psi(X^\pi_{\tau_\epsilon^-}) \leq \theta \Delta Z_{\tau_\epsilon}.$$

Then, plugging into (24), we have

$$\begin{aligned} 0 < c_0 \leq \mathbb{E}\left[e^{-r\tau_1}\psi(X^\pi_{\tau_1^-})1_{\tau_1 \leq \tau_\epsilon}\right] \\ + \mathbb{E}\left[\int_0^{\tau^-} e^{-rt}dt + \int_0^{\tau^-} e^{-rt}dZ_t^\pi\right] \\ + \mathbb{E}\left[\theta e^{-r\tau_\epsilon}\Delta Z_{\tau_\epsilon}1_{\tau_\epsilon < \tau_1}\right]. \end{aligned}$$

Using that $\psi(x) \leq 1$ for all $x \in]\bar{x} - \epsilon, \bar{x} + \epsilon[$, we have the result (23). Finally, by taking the supremum over all the admissible strategies and using the dynamic programming principle, (22) implies that $v_i(\bar{x}) \geq v_i(\bar{x}) + \eta c_0$ which is a contradiction. So the subsolution property of v_i is proved.

Uniqueness property

Suppose that $(v_i)_{i\in[1,N]}$ are continuous viscosity subsolutions of (7) and $(w_i)_{i\in[1,N]}$ continuous viscosity supersolutions, satisfying the boundary conditions

$$u_i(\gamma k_i) \leq w_i(\gamma k_i)$$

and the linear growth

$$\forall x \in [\gamma k_i, +\infty[, |u_i(x)| + |w_i(x)| < C_1 + C_2 x$$

We will show that $u_i(x) \leq w_i(x)$ for $x \in [\gamma k_i, +\infty[$.
Step 1
We first construct strict supersolutions to (7) with pertubations of w_i. Set :

$$\forall x \geq \gamma k_i, p(x) = A + Bx + x^2$$

with

$$A = \frac{1 + \mu\bar{\beta}B + \sigma^2\bar{\beta}^2}{r} + \sum_{i\in[1,N]} w_i(\gamma k_i)$$

$$B = 2 + 2\gamma h + \frac{2\mu\bar{\beta}}{r}$$

Remark 5 *The constant A is expressed as the sum of two terms. The first one assures the strict supersolution property while the second one, the sum of $w_i(\gamma k_i)$, assures in the second step of the demonstration that for all $i \in [1, N]$, $\lim_{x\to\gamma k_i}(u_i(x) - w_i^\theta(x)) < 0$.*

We then define for all $\theta \in]0, 1[$, the next continuous functions on $[\gamma k_i, +\infty[$:

$$\forall i \in [1, N], w_i^\theta = (1 - \theta)w_i + \theta p$$

First, we observe that

$$\begin{aligned} p(x) - p(x - \gamma h) &= A + Bx + x^2 - \big(A + B(x - \gamma h) + (x - \gamma h)^2\big) \\ &\geq [2\gamma hx + \gamma hB - \gamma^2 h^2] \\ &\geq \gamma^2 h^2 \end{aligned}$$

Moreover, we have

$$p_i'(x) - 1 = B + 2x - 1 \geq 1.$$

It remains to prove that

$$-\mathcal{L}_i p(x) > 0.$$

By definition,

$$\begin{aligned} -\mathcal{L}_i p(x) &= rp(x) - (\beta(k_i)\mu - \alpha((k_i - x)^+))p'(x) - \frac{\beta(k_i)^2\sigma^2}{2}p''(x) \\ &= r(A + Bx + x^2) - (\beta(k_i)\mu - \alpha((k_i - x)^+))(B + 2x) - \beta(k_i)^2\sigma^2 \\ &= \big(rA - \beta(k_i)^2\sigma^2 - (\beta(k_i)\mu - \alpha((k_i - x)^+))B\big) + \big(rB - 2(\beta(k_i)\mu - \alpha((k_i - x)^+))\big)\, x + rx^2 \end{aligned}$$

so using the values of A and B :

$$-\mathcal{L}_i p(x) \geq 1$$

We then have

$$\min\{-\mathcal{L}_i p(x), p'(x) - 1, p(x) - p(x - \gamma h)\} \geq \min\{1, \gamma^2 h^2\} \tag{25}$$

Finally, take $\bar{x} > \gamma k_i$ and $\varphi_i \in C^2$ such that $\bar{x}$ is a minimum of $w_i^\theta - \varphi_i$. Then with $\varphi_2 = \frac{\varphi_i - \theta p}{1-\theta}$, $\bar{x}$ is also a minimum of $w_i - \varphi_2$. So using that w_i is a viscosity supersolution of (7) and the result (25) we have that w_i^θ is a supersolution of the system

$$\min\big\{-\mathcal{L}_i\varphi(x), \varphi'(x) - 1, w_i^\theta(x) - w_j^\theta(x - \gamma h)\big\} \geq \theta \min\{1, \gamma^2 h^2\} = \delta$$

Step 2

Assume by a way of contradiction that there exists $\theta \in]0, \tilde{\theta}[$ and $i \in [1, N]$ such that

$$\lambda = \sup_{\substack{i\in[1,N] \\ x\geq\gamma k_i}} (u_i - w_i^\theta)(x) > 0$$

Because u_i and w_i have linear growth, we have $\lim_{x\to+\infty}(u_i(x)-w_i^\theta(x)) = -\infty$ and $\lim_{x\to\gamma k_i}(u_i(x)-w_i^\theta(x)) < 0$. So by continuity of the functions u_i et w_i^θ, there exists $x_0 \in]\gamma k_i, +\infty[$ such that

$$\lambda = u_i(x_0) - w_i^\theta(x_0)$$

For $\epsilon > 0$, let us consider the functions

$$\Phi_\epsilon(x,y) = u_i(x) - w_i^\theta(y) - \phi_\epsilon(x,y)$$
$$\phi_\epsilon(x,y) = \frac{1}{4}|x-x_0|^4 + \frac{1}{2\epsilon}|x-y|^2$$

By standard arguments in comparison principle of the viscosity solution theory, we know that the function Φ_ϵ attains a maximum in a point (x_ϵ, y_ϵ), which converges to (x_0, y_0) when ϵ goes to 0. Moreover,

$$\lim_{\epsilon\to 0} \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon} = 0$$

Applying theorem 3.2 in Crandall Ishii Lions ([5]), we get the existence of $M_\epsilon, N_\epsilon \in \mathbb{R}$ such that

$$(p_\epsilon, M_\epsilon) \in J^{2,+}u_i(x_\epsilon)$$
$$(q_\epsilon, N_\epsilon) \in J^{2,-}w_i^\lambda(y_\epsilon)$$

and,

$$\begin{pmatrix} M_\epsilon & 0 \\ 0 & -N_\epsilon \end{pmatrix} \leq D^2\phi_\epsilon(x_\epsilon, y_\epsilon) + \epsilon(D^2\phi(x_\epsilon, y_\epsilon))^2 \tag{26}$$

with

$$p_\epsilon = D_x\phi_\epsilon(x_\epsilon, y_\epsilon) = \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) + (x_\epsilon - x_0)^3,$$
$$q_\epsilon = -D_y\phi_\epsilon(x_\epsilon, y_\epsilon) = \frac{1}{\epsilon}(x_\epsilon - y_\epsilon)$$

and

$$D^2\phi_\epsilon(x_\epsilon, y_\epsilon) = \begin{pmatrix} 3(x_\epsilon - x_0)^2 + \frac{1}{\epsilon} & -\frac{1}{\epsilon} \\ -\frac{1}{\epsilon} & \frac{1}{\epsilon} \end{pmatrix}$$

so

$$D^2\phi_\epsilon(x_\epsilon, y_\epsilon) + \epsilon(D^2\phi_\epsilon(x_\epsilon, y_\epsilon))^2 = \begin{pmatrix} \frac{3}{\epsilon} + 9\epsilon(x_\epsilon - x_0)^4 + 9(x_\epsilon - x_0)^2 & -\frac{3}{\epsilon} - 3(x_\epsilon - x_0)^2 \\ -\frac{3}{\epsilon} - 3(x_\epsilon - x_0)^2 & \frac{3}{\epsilon} \end{pmatrix}$$

Therefore multiplying by the vector $(1,1)$ and its transpose, (26) implies that

$$M_\epsilon - N_\epsilon \leq 3(x_\epsilon - x_0)^2(1 + 3\epsilon(x_\epsilon - x_0)^2) \tag{27}$$

Because u_i et w_i^θ are respectively subsolution and strict supersolution, we have

$$\begin{aligned}\min\Big\{ &-\left(\frac{1}{\epsilon}(x_\epsilon - y_\epsilon) + (x_\epsilon - x_0)^3\right)(\mu\beta(k_i) - \alpha((k_i - x)^+)) - \frac{\sigma^2\beta^2(k_i)}{2}M_\epsilon + ru_i(x_\epsilon), \\ &\left(\frac{1}{\epsilon}(x_\epsilon - y_\epsilon) + (x_\epsilon - x_0)^3\right) - 1, \\ &u_i(x_\epsilon) - u_j(x_\epsilon - \gamma h)\Big\} \leq 0\end{aligned} \tag{28}$$

$$\begin{aligned}\min\Big\{ &-\left(\frac{1}{\epsilon}(x_\epsilon - y_\epsilon)\right)(\mu\beta(k_i) - \alpha((k_i - x)^+)) - \frac{\sigma^2\beta^2(k_i)}{2}N_\epsilon + rw_i^\theta(y_\epsilon), \\ &\frac{1}{\epsilon}(x_\epsilon - y_\epsilon) - 1, \\ &w_i^\theta(y_\epsilon) - w_j^\theta(y_\epsilon - \gamma h)\Big\} \geq \delta\end{aligned} \tag{29}$$

We then distinguish the following cases :

- *Case 1* : If $\frac{1}{\epsilon}(x_\epsilon - y_\epsilon) + (x_\epsilon - x_0)^3 - 1 \leq 0$ then using (29), we have $\delta + (x_\epsilon - x_0)^3 \leq 0$ which is a contradiction when ϵ goes to 0.

- *Case 2* : If $u_i(x_\epsilon) - u_j(x_\epsilon - \gamma h) \leq 0$ then when ϵ goes to 0, we obtain

  $$u_i(x_0) \leq u_j(x_0 - \gamma h)$$

  And using (29), we have

  $$w_i^\theta(y_\epsilon) - w_j^\theta(y_\epsilon - \gamma h) \geq \delta$$

  So when ϵ goes to 0 and using the continuity of w_i^θ we obtain

  $$w_i^\theta(x_0) \geq w_j^\theta(x_0 - \gamma h) + \delta.$$

  Therefore

  $$\begin{aligned}\lambda = u_i(x_0) - w_i^\theta(x_0) &\leq u_j(x_0 - \gamma h) - w_j^\theta(x_0 - \gamma h) - \delta \\ &\leq \lambda - \delta\end{aligned}$$

  which is a contradiction.

- *Case 3* : If $-\left(\frac{1}{\epsilon}(x_\epsilon - y_\epsilon) + (x_\epsilon - x_0)^3\right)(\mu\beta(k_i) - \alpha((k_i - x)^+)) - \frac{\sigma^2\beta^2(k_i)}{2}M_\epsilon + ru_i(x_\epsilon) \leq 0$. Then using (29), we have

  $$-\left(\frac{1}{\epsilon}(x_\epsilon - y_\epsilon)\right)(\mu\beta(k_i) - \alpha((k_i - x)^+)) - \frac{\sigma^2\beta^2(k_i)}{2}N_\epsilon + rw_i^\theta(y_\epsilon) \geq \delta$$

  So,

  $$-(x_\epsilon - x_0)^3(\mu\beta(k_i) - \alpha((k_i - x)^+)) - \frac{\sigma^2\beta^2(k_i)}{2}(M_\epsilon - N_\epsilon) + r(u_i(x_\epsilon) - w_i^\theta(y_\epsilon)) \leq -\delta$$

But according to (27), we have

$$\frac{\sigma^2\beta^2(k_i)}{2}(M_\epsilon - N_\epsilon) \le \frac{3\sigma^2\beta^2(k_i)}{2}(x_\epsilon - x_0)^2(1+3\epsilon(x_\epsilon - x_0)^2)$$

Therefore

$$r(u_i(x_\epsilon)-w_i^\theta(y_\epsilon)) \le (x_\epsilon-x_0)^3(\mu\beta(k_i)-\alpha((k_i-x)^+))-\delta+\frac{3\sigma^2\beta^2(k_i)}{2}(x_\epsilon-x_0)^2(1+3\epsilon(x_\epsilon-x_0)^2)$$

so when ϵ goes to 0 and using the continuity of the functions, we have the contradiction

$$r\lambda \le -\delta < 0.$$

Thus we have proved that for all

$$\forall\theta \in]0,\tilde{\theta}[, \forall i \in [1,N], \sup_{x\in[\gamma k_i,+\infty[}(u_i - w_i^\theta) \le 0$$

and the final result is proved by sending θ to 0.

6.2 Proof of Theorem 1

Using the approximations q and $q+1$, the system (17) can be written as :

$$A^{q+1}(U^{q+2}-U^{q+1}) + A^{q+1}U^{q+1} - A^qU^{q+1} = -B^{q+1} + B^q$$
$$A^{q+1}(U^{q+2}-U^{q+1}) = A^qU^{q+1} + B^q - (A^{q+1}U^{q+1} + B^{q+1})$$

We know that $(\rho^{q+1}, \psi^{q+1}, \theta^{q+1})$ minimize $A(\rho,\psi,\theta)U^{q+1} + B(\rho,\psi,\theta)$ so

$$A^{q+1}(U^{q+2}-U^{q+1}) \ge 0$$

Then using that A^{q+1} is a M-matrix we have

$$U^{q+2}-U^{q+1} \ge 0$$

Therefore the scheme is non-decreasing. Morevover $(A^q)^{-1}$ and B^q are bounded regardless of q so using that

$$U^{q+1} = -(A^q)^{-1}B^q$$

we know that U^q is also bounded so the scheme is convergent. We note U^* the limit of $(U^q)_{q\in\mathbb{N}}$. We still have to prove that the limit is unique and independent of U^0. Suppose there exists U^* and $\bar{U}^*$ two limits. U^* and $\bar{U}^*$ are both solutions of (13) so

$$A^*U^* + B^* = 0$$
$$\bar{A}^*\bar{U}^* + \bar{B}^* = 0$$

then subtracting the two equations we have,

$$\bar{A}^*(\bar{U}^* - U^*) = B^* + A^*U^* - (\bar{B}^* + \bar{A}^*U^*)$$

But $(\bar{\rho}^*, \bar{\psi}^*, \bar{\theta}^*)$ minimize $\bar{A}\bar{U}^* + B$ so

$$\bar{A}^*(\bar{U}^* - U^*) \leq 0$$

Then using that $\bar{A}^*$ is a M-matrix we have

$$\bar{U}^* - U^* \leq 0$$

We prove in the same way that

$$\bar{U}^* - U^* \geq 0$$

which achieves the demonstration.

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