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"The Condorcet Principle Implies the Proxy Voting Paradox"

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Abstract

In this note, we formulate a condition describing the vulnerability of a social choice function to a specific kind of strategic behavior and show that two well known classes of choice functions suffer from it.

Classification JEL : D71, D72.

Key Words : Condorcet, Departing Voter Paradox, Backward Induction

*I decided to put in form these simple thoughts which were prompted by the discovery of Schelling's intringuing puzzle and later on by the reading of Sprumont's elegant analysis of a monotonicity property different but related to the issues tackled in this paper. I have discussed these ideas with many people over the years and I am grateful to all of them for their feedbacks and patience. In particular, I want to acknowledge my debt to Doug Creutz, a former graduate student of mine at Caltech, for useful insights on that puzzle and to Hannu Nurmi for an extremely careful reading of an early version of this note and for asking me to clarify the relationship between this question and the twin paradox formulated by Moulin (1988).

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1 Introduction

The analysis of voting paradoxes is one of the oldest pastimes of social choice theorists (Nurmi (1999)). Voting paradoxes are counteruintuitive, unexpected or unpleasant surprises in voting and elections. The outcomes sound bizarre, unfair or otherwise implausible, given the expressed opinions of voters. Typically, they suggest that something is wrong with the way individual opinions are being processed in voting. Voting paradoxes have an important role in the history of social choice theory.

The new paradox reported in this note is motivated by a puzzle formulated by Nalebuff (1988). In elections (either committees or mass elections) it is often the case that a voter who cannot participate in the election is permitted to transfer his vote to another voter. This choice is an alternative to abstention. Depending upon context, such a voter may need to explain/report the reasons for which he cannot participate to the election but even when it is so, the legal authorities never ask the voter to report verifiable evidence on these allegations. Therefore, such a choice is basically opened to any voter who wants to use that opportunity. Note however that such a transfer is always unconditional, that is the recipient of the voting power of the donator is free to make the use he wants of the extra ballot. If the vote is secret, there is no alternative to that possibility but even when the vote is not secret (this often happens in committees) freedom is the rule.

The purpose of this note is to examine the implications of that strategic option on the functioning of a voting rule. After all, sophisticated voters could possibly use that option to change the electoral outcome in a direction that is profitable to them but depart from the general interest as defined by the original social choice function. What social choice functions (voting rules) are (if any) vulnerable to this "manipulation" where manipulation refers to the fact that the electoral outcome resulting from the transfer operation between the two voters (the voter who gives his vote and the voter who accepts it) is prefered by these two voters to the original electoral outcome[1]? Hereafter, we refer to that possibility as the paradox of the proxy voting[2].

[1] This paradox should remind the paradox exhibited by Gale (1974) in the context of markets : by donating part of his endowment to another trader, a trader may change the walrasian outcome in such a way that these two traders are better off at the expense of the other traders. Similarly, the no show strategy studied by Moulin (1988) echoes the strategic destruction by a trader of a fraction of his endowments as abstention means that a voter refuses to use his political endowment.

[2] The Schelling's paradox reported in Nalebuff deals with another peculiarity. Indeed, it could be the case that, as a result of this transfer operation, the new electoral outcome goes up with respect to the donator' preference (he can even gets his top outcome) while it goes down with respect to the recipient preference (the outcome can move all the way from top to bottom). If we only require that there do not exist profiles and pair of individuals such that at least, the recipient is happy to receive the transfer, we are getting close to a monotonicity condition. Indeed that requirement means that for all possible profiles and all pair of individuals,

We show, in section 3, that all Condorcet consistent social choice functions are vulnerable to this manipulation. Then in section 4, we show that another important family of social choice functions implementable via backward induction suffers from the same weakness. We conclude by exhibiting a simple social choice function that is not vulnerable to that manipulation but which is, unfortunately, not implementable via backward induction.

2 Notations and Definitions

The collective choice problems considered in this paper are described by the following inputs: a finite set of alternatives A with $|A| \equiv m \geq 3$, a finite set of voters N with $|N| \equiv n \geq 2$ and a profile $P = (P_1, P_2, ..., P_n)$ of preferences where P_i denotes the preference of voter i. We assume that P_i is a linear order over A (i.e. a complete, transitive and anti-symmetric binary relation) and denote by $\mathcal{L}$ the set of linear orders over A. A social choice function is a mapping F from $\mathcal{L}^n$ into A: $F(P)$ denotes the social choice made by F when the profile of preferences is P.

To each $P \in \mathcal{L}^n$ and $a, b \in A$ $\mathcal{L}^n$ with $a \neq b$,we attach the number $n(a, b, P) \equiv |\{i \in N : aP_ib\}|$. This defines the majority relation $M(P)$ as follows $aM(P)b$ iff $n(a, b, P) > n(b, a, P)$. An alternative a is a Condorcet winner for P if: $aM(P)b$ for all $b \in A, b \neq a$. A social choice function F is Condorcet consistent if $F(P) = a$ whenever a is a Condorcet winner for P.

A social choice function F is a binary procedure if there exists a binary tree i.e. a triple $\Delta = (M, \varphi, \theta)$ where M is a finite set of nodes with a distinguished node m_0 (the origin of the tree), φ is a mapping $M \to M$ (φ associates each node with its predecessor) such that $\varphi(m) = m$ iff $m = m_0$ and $\varphi^{-1}(m)$ is a set which is either the empty set or consists of exactly two elements (nodes wich have no successors are called terminal nodes and denoted by Z) and is an onto mapping $Z \to A$. If n is odd[3], a binary tree Δ induces a social choice function $F_\Delta . F_\Delta(P)$ i s computed recursively as follows : pick two terminal nodes m and m' with a common predecessor $m'' = \varphi(m) = \varphi(m')$ and concatenate m and m' into a single terminal node m'' with $\theta(m'') = \theta(m)$ if $\theta(m) M(P) \theta(m')$ and $\theta(m'') = \theta(m')$ if $\theta(m') M(P) \theta(m)$. Similarly, a social choice function F is implementable via backward induction if there exists a finite extensive game form[4] $\Gamma = (M, \prec \phi, \theta)$ such that for all $P \in \mathcal{L}^n$, $F(P)$ is equal to the

either a transfer leaves the outcome unchanged or it deteriorates the well being of the recipient. Note however that this threat is deterred as soon as the voters are rational and free to accept or refuse a proxy. The implications of a stronger form of a property of monotonicity of that kind is explored in a stimulating paper by Sprumont (1993).

[3]If n is odd, the majority relation $M(P)$ is complete. If the majority relation $M(P)$ is not complete, then an additional device has to be introduced to break the ties. We will not need it in our note.

[4]The definition of a finite extensive form game is space consuming and will be avoided here as we dont need it in this note. We refer the reader to Dutta and Sen (1993) for a complete definition. Strategic behavior in such setting is often refered to as sophisticated voting (Farquharson (1969)). Social choice functions constructed via

backward induction solution of Γat P. For all profiles $P \in \mathcal{L}^n$ and any proper subset $B \subset A$ containing at least two elements, we say that B is an adjacent set of P if for all $b, b' \in B$, for all $a \in A \backslash B$, for all $i \in N$, aP_ib iff aP_ib. Let $\mathcal{P}_B \subset \mathcal{L}^n$ the set of profiles having B as an adjacent set. A social choice function F satisfies *adjacency* if for all $B \subset A$, for all $P, P\prime \in \mathcal{P}_B$, if for all $i \in N$, P_i and P_i agree on $A \backslash B$, then: (i) $F(P) \in A \backslash B \Rightarrow F(P') = F(P)$ and (ii) $F(P) \in B \Rightarrow F(P') \in B$. It has been demonstrated by Dutta and Sen (1993) and Moulin (1986) that if F is implementable via backward induction, then F satisfies adjacency.

3 Schelling's Paradox of the Departing Voter

The story reported as puzzle 5 in Nalebuff (1988) and credited by him to Tom Schelling is described as follows: " A five-man board is to elect one of its members chairman by a procedure involving successive majority votes. Anderson, first in alphabetical order, will be paired against Barnes; the winner of that vote will be paired against Carlson, the winner then paired against Davis, and the winner of that paired against Evans. The winner of this fourth and final ballot will be declared chairman. Everyone knows everyone else's preferences. Everyone wants to be chairman.". In my notations, the set A contains five alternatives : A, B, C, D and E. The profile of preferences P is assumed to be as follows

Anderson (1): $AP_1BP_1CP_1DP_1E$

Barnes (2): $BP_2AP_2EP_2DP_2C$

Carlson (3): $CP_3DP_3AP_3EP_3B$

Davis (4): $DP_4BP_4AP_4EP_4C$

Evans (5): $EP_5DP_5BP_5CP_5A$

"All five committee members have perfect foresight and vote strategically. Who will win ?...Now, for the Schelling fillip, Anderson is forced to miss the meeting. He is allowed to transfer his vote to someone else. This transfer must be *unconditional* in that Anderson is not allowed to specify how the receiver must vote. *To whom should Anderson give his vote and what do you expect will happen* ?"

an extensive game form or a binary tree are dominance solvable as defined by Moulin (1979,1994).

The binary tree which is considered in Schelling's setting is the usual amendment procedure which is well documented in political science. The social choice $F(P)$ is the (unique) equilibrium outcome of the extensive form game resulting from the binary tree depicted on figure 1 and the profile P; equilibrium means iterative elimination of dominated strategies.

Insert figure 1 about here

Shepsle and Weingast (1984) have discovered a nice algorithm to compute this equilibrium. It only depends upon the majority relation $T(P)$ and works as follows in our case. The social choice $F(P)$ is the last element of the following finite sequence:

$$x_1 = E$$

$$x_2 = E \text{ if } ET(P)D \text{ and } x_2 = D \text{ if } DT(P)E$$

$$x_3 = C \text{ if } CT(P)x_2 \text{ and } CT(P)x_1.\text{Otherwise } x_3 = x_2.$$

$$x_4 = B \text{ if } BT(P)x_3, BT(P)x_2 \text{ and } BT(P)x_1.\text{Otherwise } x_4 = x_3.$$

$$x_5 = A \text{ if } AT(P)x_4, AT(P)x_3, AT(P)x_2 \text{ and } CT(P)x_1.\text{Otherwise } x_5 = x_4.$$

We observe that $T(P)$ despicted below admits D as a Condorcet winner.

Insert Figure 2 about here

It follows immediately that $F(P) = D$. Let us now compute $F(P^j)$ for each of the four profiles P^j $j = 2, 3, 4, 5$ obtained by replacing P_1 by either P_2, P_3, P_4 or P_5. The tournaments $T(P^2), T(P^3), T(P^4)$ and $T(P^5)$ are depicted on figures 3, 4 , 5 and 6 below.

Insert Figure 3 about here

Insert Figure 4 about here

Insert Figure 5 about here

Insert Figure 6 about here

Applying repeatedly the Shepsle-Weinsgast's algorithm lead to:

$$\begin{aligned}
F(P^2) &= F(P_2, P_2, P_3, P_4, P_5) = B \\
F(P^3) &= F(P_3, P_2, P_3, P_4, P_5) = D \\
F(P^4) &= F(P_4, P_2, P_3, P_4, P_5) = D \\
F(P^5) &= F(P_5, P_2, P_3, P_4, P_5) = A
\end{aligned}$$

We note that if the vote of Anderson is transfered to Evans, then Anderson gets elected: the best outcome for him but the worst outcome for Evans. I think that this is the paradoxical feature of the new equilibrium outcome that Schelling wanted to emphasize in his example. Paradoxical, as we intuitively expect that with one additional vote Evans's situation would see his situation improved. The *extreme* violation of an *expected monotonicity property* displayed by this outcome comes as a surprise. While intringuing, note however since Evans is assumed to be rational, he will anticipate the consequences of the transfer and refuse it.

When Anderson transfers his vote to Barnes, the new outcome is B which is the best for Barnes and second to best for Anderson. In that case, both A and B are happy with the electoral outcome resulting from the transfer. If we assume that a voter cannot be forced to vote on behalf of somebody else, then the transfer will be implemented iff the two parties find profitable to do so. This leads to the following definition. A social choice function F is vulnerable to the proxy voting paradox if there exists P in $\mathcal{L}^n$ and i, j in N such that :

$$F(P_j, P_{-i})P_iF(P) \text{ and } F(P_j, P_{-i})P_jF(P) \tag{1}$$

This reduced form definition[5] is easier to interpret when the social choice outcome is the equilibrium outcome of a game form as in the Schelling's setting. It simply means that it is never the case in this game that there exists a pair of players i and j such that j finds profitable to leave the game and let j play for him each time he was supposed to play. Profitable meaning that the new equilibrium outcome is preferred by i and j to the old one. In some sense, the transfer operation can also be seen as a manipulation of a very specific type by a coalition of size two.

We insist here on the fact that this vulnerability is bad news for those who expected to exit from the Gibbard-Satterthwaite impossibility result by considering dominance solvable game forms. Indeed, the analysis of Schelling's example as well as the results reported in the next section show that they are vulnerable to another form of manipulation involving two players

[5]This transfer operation is described by moving from a game form with n players to a game with $n-1$ players where one of the original player has inherated all the decision nodes of the departing player.

but quite straightforward to implement. The following simple example illustrates the proxy paradox. Suppose that there are three voters who have to decide who will act as chairman among the three. Their preferences display the usual Condorcet cycle pattern:

Anderson (1): AP_1BP_1C

Barnes (2): BP_2CP_2A

Carlson (3): CP_3AP_3B

Anderson is paired against Barnes; the winner of that vote is paired against Carlson. The winner of this second and final ballot will be declared chairman. The simple binary tree describing this game is depicted on figure 7 below.

Insert Figure 7 here

The backward solution works as follows. If A passes the first round, then it is defeated at the second by C. If instead, B passes the second round it defeats C. So the three sophisticated voters understand that the real choice at the first round is between C and B. Since there is a majority in favor of B, B wins. Carlson is unhappy with that outcome. He would like to commit himself to vote A if A passes the first round but this promise is *not credible* if he participates to the vote as, once the second round round has been reached, his dominant strategy is to vote for C. But if by pretending being unable to participate to the meeting, he gives a proxy to Anderson, he eliminates that issue as A becomes a Condorcet winner and is elected. In some sense, proxy is a credible way to commit to another plan of action.

The Schelling's violation of monotonicity is something different but related to the proxy issue which motivated that paper. His monotonicity property is a weak version of a property introduced by Sprumont (1993) under the heading "Closer Preferences Better" (CPB). Informally, Sprumont defines a a social function to satisfy CPB if for any profile P and any pair i and j of voters, moving to a profile P' where all the preferences are unchanged, except the preference of voter j which is now closer (in a well defined sense) to the preference of voter i, does not hurt voter i. Schelling deals with the special case of identity which is the extreme form of closeness because the distance between the two voters is equal to 0. Let us say[6] that a social choice function is vulnerable to the Schelling/Sprumont monotonicity paradox if if there exists P in $\mathcal{L}^n$ and i, j in N such that :

[6]This terminology is mine.

$$F(P)P_jF(P_j,P_{-i}) \tag{2}$$

or even , a bit more demanding[7]:

$$F(P)P_jF(P_j,P_{-i})P_iF(P) \tag{3}$$

Once again, this monotonicity paradox has little to do with strategic behavior as we dont expect j to accept a proxy from i under the conditions (2) or (3). But it calls the attention on the fact that, in spite of appearances, it may not be a good idea for a voter to accept to vote on behalf of somebody else.

It is important also to point out[8] that the proxy paradox and the Schelling/Sprumont paradox are logically unrelated to the *twin paradox* introduced by Moulin(1988). The difference lies in the fact that to be properly defined, the twin paradox involves social choice functions over a variable electorate. Without being too formal, say that a social choice function suffers from the twin paradox if there exists n, $P \in \mathcal{L}^n$ and i such that $F(P)P_iF(P,P_i)$: it means that given P, voter i does not welcome the arrival/partipation of a voter who has identical preferences to him. There exist social choice functions which are suffering from the proxy paradox or Schelling/Sprumont paradox but not from the twin paradox: a nice family is the class of scoring functions. Indeed, Sprumont shows that scoring functions suffer from the Schelling/Sprumont paradox. However, they dont suffer from the twin paradox. There also exist social choice functions who do not suffer from the Schelling/Sprumont or Proxy paradox but suffer from the twin paradox. While artificial, here is an example. Consider two voters, say 1 and 2, and define $F(P)$ to be the top choice of P_1 is n is even and the top choice of P_2 if n is odd. The function is immuned to the Schelling/Sprumont paradox and proxy paradox because for a fixed value of n, F is dictatorial. But it is not immuned to the twin paradox since a voter may change the identity of the dictator by participating or not in the election. If he is a twin of voter 1 but n is odd, the best thing he has to do is not to show up.

4 Condorcet

The main purpose of this section is to prove that the situation exhibited in the preceding section is not exceptional as soon as the social choice function is Condorcet consistent

[7]Consistent with Schelling's story, we could even consider this stronger form : there exists P in $\mathcal{L}^n$ and i,j in N such that $F(P_j,P_{-i})$ is the top element of P_i and $F(P_j,P_{-i})$ is the bottom element of P_j. In Schelling's story, $F(P)$ is also almost the best element of P_j !

[8]I am grateful to Hannu Nurmi for raising this question.

Proposition 1 If $n \geq 15$, then all Condorcet social choice functions are vulnerable to the proxy voting paradox

Proof[9]

Let us prove that under the stated conditions, the social choice function is vulnerable to the proxy voting paradox.

Let $a, b, c \in A$ and for any profile $P \in \mathcal{L}^n$ where for every voter the alternatives a, b and c occupy the first three positions, denote:

$$N^1(P) \equiv \{i \in N : aP_ibP_ic\}$$

$$N^2(P) \equiv \{i \in N : bP_iaP_ic\}$$

$$N^3(P) \equiv \{i \in N : bP_icP_ia\}$$

$$N^4(P) \equiv \{i \in N : cP_ibP_ia\}$$

$$N^5(P) \equiv \{i \in N : cP_iaP_ib\}$$

$$N^6(P) \equiv \{i \in N : aP_icP_ib\}$$

Let $n^j(P)$ be the size of $N^j(P)$ for $j = 1, ..., 6$.

Case 1 : n is a multiple of 3

Let P be such that $n^1(P) = n^3(P) = n^5(P) = \frac{n-3}{3}$ and $n^2(P) = n^4(P) = n^6(P) = 1$. Suppose that $F(P) = a$. Let P' be the profile obtained from P by shifting one voter, say i, from $N^1(P)$ to $N^6(P)$. We must have $F(P') = a$. Otherwise, at profile P', if i transfers his vote to any voter j still in $N^1(P)$, we obtain : $aP_iF(P')$ and $aP_jF(P')$. By shifting stepwise any voter in $N^1(P)$ until only one is left, we obtain a profile P'' such that:

$F(P'') = a$, $n(c, a, P'') = \frac{n-3}{3} + 1 + \frac{n-3}{3} = \frac{2n-3}{3}, n(c, b, P'') = \frac{n-3}{3} + 1 + \frac{n-3}{3} + 1 - 1 = \frac{2n-3}{3}$ and $n(c, x, P'') = n$ for all $x \in A \backslash \{a, b, c\}$. Since $n \geq 15$, we deduce that c is a Condorcet winner and therefore $F(P'') = c$. Contradiction.

We can conduct a similar reasoning in the case where $F(P) = b$ or $F(P) = c$. When $F(P) = x$

[9] This simple but elegant argument is due to Sprumont (1993, proposition B) and reproduced here for the sake of completeness. It shows that under the stated conditions the Condorcet social choice functions are vulnerable the Schelling/Srumonts monotonicity paradox.

Case 2: $n-1$ is a multiple of 3

Let P be such that $n^1(P) = \frac{n-7}{3}$ and $n^3(P) = n^5(P) = \frac{n-4}{3}$, $n^2(P) = n^6(P) = 2$ and $n^4(P) = 1$. If $F(P) = a$, then we construct P'' as in case 1 by shifting all but one voters from $N^1(P)$ to $N^6(P)$. Then we obtain : $F(P'') = a$. However, $n(c, a, P'') = \frac{n-4}{3} + 1 + \frac{n-4}{3} = \frac{2n-5}{3}$, $n(c, b, P'') = 1 + \frac{n-4}{3} + 2 + \frac{n-7}{3} - 1 = \frac{2n-5}{3}$ and $n(c, x, P'') = n$ for all $x \in A \backslash \{a, b, c\}$. Since $n \geq 15$, we deduce that c is a Condorcet winner and therefore $F(P'') = c$. A contradiction.

If $F(P) = b$, we construct a profile P'' as in case 1 by shifting all but one voters from $N^3(P)$ to $N^2(P)$. Then, we obtain : $F(P'') = b$. However, $n(a, b, P'') = \frac{n-7}{3} + \frac{n-4}{3} + 2 = \frac{2n-5}{3}$, $n(a, c, P'') = \frac{n-8}{3} + 2 + \frac{n-8}{3} - 1 + 2 = \frac{2n-7}{3}$ and $n(c, x, P'') = n$ for all $x \in A \backslash \{a, b, c\}$. Since $n \geq 15$, we deduce that a is a Condorcet winner and therefore $F(P'') = a$. Contradiction.

If $F(P) = c$, we construct a profile P'' as in case 1 by shifting all but one voters from $N^5(P)$ to $N^4(P)$. Then, we obtain : $F(P'') = c$. However, $n(b, a, P'') = 2 + \frac{n-4}{3} + 2 + 1 + \frac{n-4}{3} - 1 = \frac{2n+3}{3}$, $n(b, c, P'') = \frac{n-7}{3} + 2 + \frac{n-4}{3} = \frac{2n-5}{3}$ and $n(c, x, P'') = n$ for all $x \in A \backslash \{a, b, c\}$. Since $n \geq 15$, we deduce that b is a Condorcet winner and therefore $F(P'') = b$. Contradiction.

Case 3: $n-2$ is a multiple of 3

Let P be such that $n^1(P) = n^3(P) = \frac{n-8}{3}$, $n^5(P) = \frac{n-5}{3}$, $n^2(P) = 3$ and $n^4(P) = n^6(P) = 2$. If $F(P) = a$, then we construct P'' as in case 1 by shifting all but one voters from $N^1(P)$ to $N^6(P)$. Then we obtain : $F(P'') = a$. However, $n(c, a, P'') = \frac{n-8}{3} + 2 + \frac{n-5}{3} = \frac{2n-7}{3}$, $n(c, b, P'') = 2 + \frac{n-5}{3} + 2 + \frac{n-8}{3} - 1 = \frac{2n-4}{3}$ and $n(c, x, P'') = n$ for all $x \in A \backslash \{a, b, c\}$. Since $n \geq 15$, we deduce that c is a Condorcet winner and therefore $F(P'') = c$. A contradiction.

If $F(P) = b$, we construct a profile P'' as in case 1 by shifting all but one voters from $N^3(P)$ to $N^2(P)$. Then, we obtain : $F(P'') = b$. However, $n(a, b, P'') = \frac{n-8}{3} + \frac{n-5}{3} + 2 = \frac{2n-7}{3}$, $n(a, c, P'') = \frac{n-7}{3} + 2 + \frac{n-4}{3} - 1 + 2 = \frac{2n-2}{3}$ and $n(c, x, P'') = n$ for all $x \in A \backslash \{a, b, c\}$. Since $n \geq 15$, we deduce that a is a Condorcet winner and therefore $F(P'') = a$. Contradiction.

If $F(P) = c$, we construct a profile P'' as in case 1 by shifting all but one voters from $N^5(P)$ to $N^4(P)$. Then, we obtain : $F(P'') = c$. However, $n(b, a, P'') = 3 + \frac{n-8}{3} + 2 + \frac{n-5}{3} - 1 = \frac{2n-1}{3}$, $n(b, c, P'') = \frac{n-8}{3} + 3 + \frac{n-8}{3} = \frac{2n-7}{3}$ and $n(c, x, P'') = n$ for all $x \in A \backslash \{a, b, c\}$. Since $n \geq 15$, we deduce that b is a Condorcet winner and therefore $F(P'') = b$. Contradiction. □

The proposition does not say that $n \geq 15$ is necessary for the result to hold true for the proxy voting paradox . As seen at the end of the preceding section, the difficulty shows up already with $n = 3$. On the other hand, some minimal value of n is needed for the Schelling monotonicity paradox to show up. For $n = 3$, no Condorcet consistent social choice function is vulnerable to that paradox. Indeed, if one voter transfer his vote to another voter, then the top outcome of the recipient is a Condorcet winner.

5 Extensions and Open Problem

The Condorcet consistent social choice functions are not the only ones vulnerable to the proxy voting paradox. In this section, we show that another important class of social choice functions known as voting by successive veto (Moulin (1983)) is also vulnerable to this paradox[10]. Then, we show that there are social choice function which are immuned to that paradox for any number of voters and alternatives. But we also show that the particular social choice function exhibited to make that point is not implementable via backward induction. We conclude by an open problem.

Voting by successive veto is defined as follows. Let σ be a mapping from $\{1, 2, ..., m-1\}$ into N. To σ is attached the perfect information extensive game form defined informally as follows : player $\sigma(1)$ starts by vetoing one alternative out of the m alternatives, player $\sigma(2)$ moves then by vetoing one alternative among the $m-1$ remaining alternatives, player $\sigma(3)$ moves next by vetoing one alternative among the $m-2$ remaining alternatives and so on. After $m-2$ rounds, one player is left with a veto among the remaining two alternatives. For each P , the backward induction solution $V_\sigma(P)$ is well defined. In what follows , we will use the following theorem demonstrated by Moulin (1981) and known as the mirror image theorem. Let $\overline{\sigma}$ be the sequence from $\{1, 2, ..., m-1\}$ into N defined as follows : $\overline{\sigma}(1) = \sigma(m-1)$, $\overline{\sigma}(2) = \sigma(m-2), ..., \overline{\sigma}(m-1) = \sigma(1)$. Let $S_{\overline{\sigma}}(P)$ be the sincere solution attached to P and defined as follows: voter $\overline{\sigma}(1)$ eliminates his worst alternative, $\overline{\sigma}(2)$ eliminates his worst alternative among the $m-1$ remaining alternatives and so on. The mirror image theorem states that $V_\sigma(P) = S_{\overline{\sigma}}(P)$.

Proposition 2 If $|\sigma\{1, 2, ..., m-1\}| \geq 3$ (which implies $n \geq 3$ and $m \geq 4$) then all voting by successive vetos social choice functions are vulnerable to the proxy voting paradox.

Proof. Without loss of generality, let $\sigma(1) = 1$ and k be the smallest integer such that $\sigma(k) \neq 1$ and assume without loss of generality that $\sigma(k) = 2$. Let k' be the smallest integer larger than k and such that $\sigma(k') \neq 1$ and $\sigma(k') \neq 2$. Assume without loss of generality that $\sigma(k') = 3$. Finally, let us denote by q_i the cardinality of the sets $\{j \in \{1, 2, ..., m-1\} : \sigma(j) = i \text{ and } j > k'\}$ for $i = 1, 2, 3$ and let $\{B_1, B_2, B_3, B_4\}$ be a collection of pairwise disjoint sets in $A \setminus \{a_1, a_2, a_3, a_4\}$[11] where $\{a_1, a_2, a_3, a_4\}$ is an arbitrary quadruple in A such that:

$$|B_i| = q_i \text{ for } i = 1, 2, 3$$

$$|B_4| = |\{j \in \{1, 2, ..., m-1\} : \sigma(j) \notin \{1, 2, 3\}\}|$$

[10] We could also import proposition C in Sprumont stating that all strict scoring methods are exposed to the Schelling/Sprumont paradoxes to show that they also suffer from the proxy paradox.

[11] If $k' = m-1$, and therefore in particular when $m = 4$, these sets are empty.

Let A_1, A_2 be two disjoint sets in $A \backslash \{a_1, a_2, a_3, a_4\} \cup B_1 \cup B_2 \cup B_3 \cup B_4$ such that:

$$|A_i| = |\{j \in \{1, 2, ..., m-1\} : \sigma(j) = i \text{ and } k > k'\}| - 1 \text{ for } i = 1, 2$$

Consider the profile:

$$a_1 P_1 a_2 P_1 a_3 P_1 A_2 P_1 A_1 P_1 a_4 P_1 B_4 P_1 B_1 P_1 B_2 P_1 B_3$$

$$a_4 P_2 a_3 P_2 A_1 P_2 a_1 P_2 A_2 P_2 a_2 P_2 B_4 P_2 B_2 P_2 B_3 P_2 B_1$$

$$a_1 P_3 a_3 P_3 a_4 P_3 A_1 P_3 A_2 P_3 a_2 P_3 B_4 P_3 B_2 P_3 B_1 P_3 B_2$$

$$\text{Arbitrary } P_i B_4 \text{ for } i \geq 4$$

From the mirror image theorem, we deduce that:

$$V_\sigma(P) = a_3$$

Now let player 3 transfers his vote to voter 1 i.e. let $P' = (P_1, P_2, P_1, P_4, ..., P_n)$. Using the mirror image theorem again, we obtain:

$$V_\sigma(P') = a_1$$

Since $a_1 P_1 a_3$ and $a_1 P_3 a_3$, we have shown that V_σ is vulnerable to the proxy voting paradox

□

In the case where $m = 4$ and $n = 3$, the successive veto game form is depicted on figure 8.

Insert Figure 8 here

The above profile writes here:

$$a_1 P_1 a_2 P_1 a_3 P_1 a_4$$

$$a_4 P_2 a_3 P_2 a_1 P_2 a_2$$

$$a_1 P_3 a_3 P_3 a_4 P_3 a_2$$

What is happening here is that by ceding his veto to 1, 3 has now a credible threat *not* to veto a_2 with the last veto. Thus voter 2 *must* veto a_2. If instead voter 3 does not cede his veto and participate to the procedure, the other voters anticipate that a_2 will be vetoed.

It is interesting to remark that voting by successive vetos is also vulnerable to the Schelling monotonicity paradox. If instead of the profile considered above, we consider the profile P below:

$$a_1 P_1 a_2 P_1 a_3 P_1 a_4$$

$$a_3 P_2 a_2 P_2 a_1 P_2 a_4$$

$$a_2 P_3 a_4 P_3 a_1 P_3 a_3$$

we check that $V_\sigma(P) = a_1$ while if voter 3 transfers his vote to voter 1 i.e. if we move to $P' = (P_1, P_2, P_1)$, then $V_\sigma(P') = a_2$: 3 gets his best outcome while voter 1 is worse off.

We may wonder to which extend there exists for all n and m, social choice functions F such that there does not exist profile P and individuals i and j for which there does not exist P in $\mathcal{L}^n$ and i, j in N such that $F(P_j, P_{-i})P_i F(P)$ and $F(P_j, P_{-i})P_j F(P)$. That such social exist is illustrated by the following simple construction. Let $\gg$ be a linear order over the set $\mathcal{L}$ of $m!$ linear orders and let $F_\gg(P)$ be defined as the top alternative of the largest (according to $\gg$) order among the orders P_i $i = 1, ..., n$. Clearly $F_\gg$ is proxy-proof. Note also that it is efficient and anonymous. But it is not implementable via backward induction as it does not satisfy[12] the adjacency property which is a necessary property for implementation via backward induction (Dutta and Sen (1993), Moulin (1986)).

We leave as open problems the following two questions. Do there exist social functions which are anonymous, with a range of cardinality greater than 2[13], implementable via backward induction and not vulnerable to the proxy voting paradox ? Do there exist anonymous, with a range of cardinality greater than 2 social choice functions F such that for all for all $P \in \mathcal{L}^n$ and all $i, j \in N$: either $F(P_j, P_{-i}) = F(P)$ or $F(P_j, P_{-i})P_j F(P)$?

[12] The argument is relegated in the appendix.

[13] Without anonymity, it is easy to construct non dictatorial social choice functions such that the two paradoxes can be avoided. Pick two fixed individuals, say 1 and 2 and let $F(P)$ be the top element of P_1 among all the $m-1$ alternatives that remain after the elimination of the bottom element of P_2. On the other hand, if the range has only two elements then all monotonic social choice functions are immuned to the two paradoxes.

6 Appendix : Proof that the Social Choice Function $F_\gg$ Violates the Adjacency Property

Let $\gg$ be an arbitrary ordering of $\mathcal{L}$ and let a, b and c be three alternatives. Consider the six preferences where a, b, c are on top and the alternatives in $A \backslash \{a, b, c\}$ are below in a prescribed order. Hereafter, we restrict to profiles with preferences limited to these six types. Suppose that the social choice function $F_\gg$ described above satisfies the adjacency property. Without loss of generality suppose that that the order abc comes first in $\gg$

If cab is second then $F_\gg(P) = c$ whenever the profile P contains exclusively preference abc and preferences cab. The preference bac comes after abc and cab. If P' is a profile where the voters with preference abc in P have preference bac in P' then $F_\gg(P') = b$. This contradicts adjacency as $\{a, b\}$ is an adjacent set for P and P'.

If cba is second then $F_\gg(P) = c$ whenever the profile P contains exclusively preference abc and preferences cba. The preference bac comes after abc and cab. If P' is a profile where the voters with preference abc in P have preference bac in P' then $F_\gg(P') = b$. This contradicts adjacency as $\{a, b\}$ is an adjacent set for P and P'.

If bca is second then $F_\gg(P) = b$ whenever the profile P contains exclusively preference abc and preferences bca. The preference acb comes after abc and cab. If P' is a profile where the voters with preference abc in P have preference acb in P' then $F_\gg(P') = a$. This contradicts adjacency as $\{b, c\}$ is an adjacent set for P and $P\prime$

Therefore, only the orders acb and bac can occupy the second position

Case 1 *acb is second*

We have four possibilities for the third position

- bac is third. Then cab is after abc, acb and bac. Take a profile P with preferences either acb or bac. then $F_\gg(P) = b$. Now construct a profile P' where voters with acb in P have now cab. Then $F_\gg(P') = c$. This contradicts adjacency as $\{a, c\}$ is an adjacent set.

- bca is third. Then cab is after abc, acb and bca. Take a profile P with preferences either acb or bca. then $F_\gg(P) = b$. Now construct a profile P' where voters with acb in P have now cab. Then $F_\gg(P') = c$. This contradicts adjacency as $\{a, c\}$ is an adjacent set.

- cab is third. Then bac is after abc, acb and cab. Take a profile P with preferences either abc orcab. then $F_\gg(P) = c$. Now construct a profile P' where voters with abc in P have now bac. Then $F_\gg(P') = b$. This contradicts adjacency as $\{a, b\}$ is an adjacent set.

- cba is third. Then bac is after abc, acb and cba. Take a profile P with preferences either abc or cab. then $F_\gg(P) = c$. Now construct a profile P' where voters with abc in P have now bac. Then $F_\gg(P') = b$. This contradicts adjacency as $\{a, b\}$ is an adjacent set.

Case 2 *bac is second*

We have four posibilities for the third position.

- *acb* is third. Then bca is after *abc*, *bac* and *acb*. Take a profile P with preferences either *bac* or *acb*. then $F_{\gg}(P) = a$. Now construct a profile P' where voters with *bac* in P have now *bca*. Then $F_{\gg}(P') = b$. This contradicts adjacency as $\{a, c\}$ is an adjacent set.

- *cab* is third. Then *bca* is after *abc*, *bac* and *cab*. Take a profile P with preferences either *bac* or *cab*. then $F_{\gg}(P) = c$. Now construct a profile P' where voters with *bac* in P have now *bca*. Then $F_{\gg}(P') = b$. This contradicts adjacency as $\{a, c\}$ is an adjacent set.

- *cba* is third. Then *acb* is after *abc*, *bac* and *cba*. Take a profile P with preferences either *abc* or *cba*. then $F_{\gg}(P) = c$. Now construct a profile P' where voters with *abc* in P have now *acb*. Then $F_{\gg}(P') = a$. This contradicts adjacency as $\{b, c\}$ is an adjacent set.

- *bca* is third. Then *acb* is after *abc*, *bac* and *bca*. Take a profile P with preferences either *abc* or *bca*. then $F_{\gg}(P) = b$. Now construct a profile P' where voters with *abc* in P have now *acb*. Then $F_{\gg}(P') = a$. This contradicts adjacency as $\{b, c\}$ is an adjacent set.

7 References

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