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"A Pseudo-Market Approach to Allocation with Priorities"

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Abstract

We propose a pseudo-market mechanism for no-transfer allocation of indivisible objects that honors priorities such as those in school choice. Agents are given token money, face priority-specific prices, and buy utility-maximizing assignments. The mechanism is asymptotically incentive compatible, and the resulting assignments are fair and constrained Pareto efficient. Hylland and Zeckhauser (1979)'s position-allocation problem is a special case of our framework, and our results on incentives and fairness are also new in their classical setting.

Keywords: *Priority-based allocation, Efficiency, Stability, Incentive Compatibility, Pseudo-Market Approach*

JEL Codes: C78, D82, I29

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1 Introduction

We study the allocation of indivisible objects in settings where monetary transfers are precluded and agents demand at most one object. Examples include student placement in public schools (where an object corresponds to a school seat and each object has multiple copies) and allocation of office and living space (where each object has exactly one copy). A common feature of these settings is that agents are prioritized. For example, students who live in a school's neighborhood or have siblings in the school may have admission priority in this school over those who do not, and the current resident may have priority over others in the allocation of the dormitory room he or she lives in.

The standard allocation mechanisms used and studied in these environments are ordinal: students are asked to rank schools or rooms, and the profile of rankings they submit determines the lottery over assignments. However, Miralles (2008) and Abdulkadiroglu, Che, and Yasuda (2011) pointed out that eliciting agents' cardinal utilities—that is their rates of substitution between probability shares in objects—may allow us to implement Pareto dominant assignments (cf. also Ergin and Sonmez (2006)). Furthermore, Liu and Pycia (2012) and Pycia (2014) showed that in large markets sensible ordinal mechanisms are asymptotically equivalent while mechanisms eliciting cardinal utilities maintain their efficiency advantage.[1]

This paper addresses the question of how to improve upon the ordinal mechanisms while honoring priorities. We construct an asymptotically incentive compatible and fair mechanism that honors priorities and is constrained efficient among stable and fair mechanisms. A mechanism honors priorities if no probability share of an object is given to an agent with lower priority at this object when a higher-priority agent prefers this object to some object in the support of his or her assignment. In the domain of deterministic assignments, this concept is known as stability or the elimination of justified envy (see e.g., Abdulkadiroglu and Sonmez (2003)); it is extended to random assignments and defined as *ex ante* stability by Kesten and Ünver (2015). We use the strong fairness concept, equal claim, proposed by

[1]The data on Boston and NYC school choice corroborates both the equivalence of ordinal mechanisms (see e.g., Pathak and Sonmez (2008) and Abdulkadiroglu, Pathak, and Roth (2009)) and the inefficiency of ordinal mechanisms (Abdulkadiroglu, Agarwal, and Pathak 2015). For analysis of ordinal mechanisms see the seminal work of Abdulkadiroglu and Sonmez (2003)) and Bogomolnaia and Moulin (2001). C.f. also the literature discussion below for other papers emphasizing the need to elicit cardinal information.

He, Li, and Yan (2015); a mechanism satisfies equal claim if agents with the same priority at an object are given the same opportunity to obtain it.

We refer to our construction as the pseudo-market mechanism (PM) since the mechanism extends the canonical Hylland and Zeckhauser (1979) mechanism by incorporating priorities.[2] The mechanism elicits cardinal preferences from agents and computes a random assignment that maximizes each agent's expected utility given prices and the agent's exogenous budget. The random assignment is a Walrasian equilibrium as in Hylland and Zeckhauser's work, except that we allow prices to depend on agents' priorities. For each object, there exists a cut-off priority group such that agents in priority groups strictly below the cut-off face an infinite price for the object (hence, they can never be matched with the object), while agents in priority groups strictly above the cut-off face zero price for the object.

We establish the existence of the above-mentioned Walrasian equilibrium, and the resulting PM mechanism is shown to be asymptotically incentive compatible in regular economies, where regularity guarantees that Walrasian prices are well defined as in the classical analysis of Walrasian equilibria (see e.g., Dierker (1974), Hildenbrand (1974), and Jackson (1992)). The latter result is also new in the original Hylland and Zeckhauser (1979) problem and proves the long-standing conjecture they formulated.[3] As in the setting without priorities (see e.g., Abdulkadiroglu, Che, and Yasuda (2011) and Pycia (2014)), the PM mechanism allows one to achieve higher social welfare than mechanisms eliciting only ordinal preferences such as the Probabilistic Serial and the Deferred-Acceptance mechanisms.

The PM mechanism honors priorities because of our design of the priority-specific prices. Given an object s and its cut-off priority group, whenever a lower-priority agent obtains a positive share of s, a higher-priority agent must face a zero price for s, and, therefore, is never assigned to an object they prefer less than s.

[2]The PM mechanism may also be interpreted as a generalization of the Gale-Shapley deferred-acceptance mechanism with ties broken endogenously and efficiently by the information on cardinal preferences. Agents with relatively higher cardinal preferences for an object obtain shares of that object before others who are in the same priority group.

[3]Stating the true preferences in the PM mechanisms is not always a dominant strategy for every agent. Hylland and Zeckhauser (1979) give an example where there are incentives for agents to misreport their preferences when objects do not rank agents. More generally, Roth (1982) and Zhou (1990) show strategy-proofness is in conflict with other desirable properties. In addition to proving the asymptotic incentive compatibility of the PM mechanism in regular economies, we also prove that it is limiting incentive compatible in the sense of Roberts and Postlewaite (1976).

We study fairness of the PM mechanism in the sense of *equal claim*, which requires that, for any given object, agents with the same priority are given the same opportunity to obtain this object.[4] Since prices for agents in the same priority group are by construction the same in the PM mechanism, we can conclude that equal claim is satisfied when agents are given equal budgets. Furthermore, we show that the PM mechanism in which agents have equal budgets is the only non-wasteful mechanism that honors priorities and satisfies equal claim.

Focusing on random assignment that honors priorities and equal-claim, we analyze efficiency: a random assignment is constrained Pareto efficient if no other assignment that honors priorities and satisfies equal claim dominates it in terms of agents' welfare.[5] An important corollary of our results is that the constrained Pareto efficient assignment is always an outcome of the PM mechanism with equal budgets.

Moreover, one may be interested in two-sided efficiency if the priority structure is closely related to object suppliers' preferences, e.g., when schools' priority ranking over students reflect a school district's preferences. A random assignment is *ex ante* two-sided Pareto efficient if it is not Pareto dominated by any other assignment with respect to both agents' expected utilities and objects' priorities treated as their ordinal preferences. When the welfare of objects is evaluated in terms of first-order stochastic dominance with respect to priorities (an object is better off if agents matched with this object in the new assignment first-order stochastically dominate those of the old one) then PM always delivers random assignments that satisfy *ex ante* two-sided efficiency.

Given these desirable properties, PM is a promising mechanism that can be used in school choice, dormitory room allocation, and other allocation problems based on priorities. Moreover, it is flexible enough to accommodate additional constraints such as group-specific quotas.

[4]See He, Li, and Yan (2015) for an analysis of this concept in the setting without priorities. Note that equal claim does not imply that same-priority agents at an object receive the same probability share at that object in the final assignment.

[5]The literature on ordinal mechanisms that follows Bogomolnaia and Moulin (2001) defines efficiency in terms of first-order stochastic dominance; since we study expected-utility-maximizing agents, we can use the standard Pareto efficiency concept. It should be noted that there are priority structures and priority-honoring assignments that are Pareto dominated by assignments that do not honor priorities; Abdulkadiroglu and Sonmez (2003) construct relevant examples in the ordinal setting, and the same examples remain valid in our setting.

Literature Review The early literature on school choice, the focal topic of priority-based allocation, e.g., Abdulkadiroglu and Sonmez (2003) and Ergin and Sonmez (2006), followed the two-sided matching literature where it is common to assume that both sides strictly rank the other side. Implicitly, weak priorities are augmented with random lotteries to create strict priorities. It has been noted that when priorities are coarse, some issues arise. For example, stability no longer implies Pareto efficiency (Erdil and Ergin 2006); and, more importantly, how ties are broken affects the welfare of agents since it introduces artificial constraints. Extending the Gale-Shapley Deferred-Acceptance mechanism (defined in Appendix A) Erdil and Ergin (2008) propose an algorithm for breaking priority ties and the computation of agent-efficient stable matchings when priority rankings are weak and only ordinal information is elicited. The two algorithms proposed by Kesten and Ünver (2015) offer further ways to break priority-ties in ordinal settings. However, Abdulkadiroglu, Pathak, and Roth (2009) and Kesten (2010) show that the inefficiency associated with a realized tie breaking in ordinal setting cannot be removed without harming student incentives.

Noting that agents may differ in their cardinal preferences, a strand of literature (e.g., Featherstone and Niederle (2008), Miralles (2008), Abdulkadiroglu, Che, and Yasuda (2011), Troyan (2012), Pycia (2014), Abdulkadiroglu, Che, and Yasuda (2015), and Ashlagi and Shi (Forthcoming)) emphasizes the importance of eliciting signals of cardinal preferences from agents in matching mechanisms.[6] Ties in priorities can be broken with such signals, although the space of preference profiles or signals considered in these papers is restricted. Our PM mechanism elicits the entire relevant utility information in a general setting. Moreover, compared to discrete signals of cardinal preferences such as those in the popular Boston mechanism (defined in Appendix A), the PM mechanism has the advantage of being (asymptotically) incentive compatible. It has been shown theoretically (e.g., Pathak and Sonmez (2008)), experimentally (e.g., Chen and Sonmez (2006)), and empirically (e.g., Abdulkadiroglu, Pathak, Roth, and Sonmez (2006), He (2012)), that strategic considerations may put less sophisticated agents at a disadvantage. More importantly, these effects do not disappear in large markets (Azevedo and Budish 2012). PM thus "levels the playing field" by

[6]In recent work, Lee and Yariv (2014) and Che and Tercieux (2014) show that when agent's utilities come from independent distributions, some ordinal mechanisms can be efficient.

eliminating this strategic concern while keeping the benefits of using cardinal preferences.

Our paper offers the first pseudo-market construction that honors priorities.[7] In addition, we also contribute to the growing literature on pseudo-market mechanisms in settings without priorities. The idea was first formulated by Hylland and Zeckhauser (1979). Miralles (2008) establishes a connection between the mechanism and the Boston mechanism in settings without priorities. Budish (2011) and Budish, Che, Kojima, and Milgrom (2013) extend the pseudo-market mechanism to multi-unit demand settings such as course scheduling. Miralles and Pycia (2014) show that every efficient assignment can be decentralized through prices, establishing the Second Welfare Theorem for the no-transfer setting without priorities. He, Li, and Yan (2015) make the point that any random assignment, not necessarily efficient, can be decentralized by personalized prices.

Our analysis of fairness is related to Ashlagi and Shi (Forthcoming) who study a model with a continuum of agents without priorities and show that the equal-budget pseudo market can implement any envy-free and Pareto efficient assignment. Envy-freeness is a weaker fairness property than equal claim, and the characterization of the equal-budget pseudo market in terms of envy-freeness and efficiency does not extend to large finite economies (see Miralles and Pycia (2015)).

Our analysis of the pseudo-market mechanism's asymptotic incentive compatibility addresses a long-standing open problem posed by Hylland and Zeckhauser (1979). We build on the classic literature on the price-taking behavior of agents in exchange economies, e.g., Roberts and Postlewaite (1976) and Jackson (1992). The only earlier analysis of incentive compatibility of PM without priorities is Azevedo and Budish (2012) who show that it satisfies the strategy-proofness-in-the-large criterion that they introduce provided that budgets are equal and the number of utility types is finite and stays bounded as the market grows. Our result does not hinge on either of these assumptions.[8]

[7]Notice that our paper subsumes He (2011), Miralles (2011), He and Yan (2012), and He, Miralles, and Yan (2012) who proposed this construction and proved it is well-defined. Subsequent work on personalized prices in pseudo-markets (e.g. Ashlagi and Shi (Forthcoming) and He, Li, and Yan (2015)) did not address the question of when personalized-price mechanisms honor priorities.

[8]Building on their work, one can show that PM satisfies their criterion in settings with priorities (details available upon request). Work on other related mechanisms includes Miralles (2012), Pycia (2014), and Hafalir and Miralles (2014) who study incentive compatible efficient mechanisms in specific parametric settings. Hashimoto (2013) constructs an *ex post* incentive-compatible mechanism that becomes efficient in

Organization of the Paper Section 2 sets up the model for the priority-based allocation problem. Section 3 defines the PM mechanism and establishes that it is well-defined. Section 4 investigate its incentive compatibility. We present fairness properties of PM and its characterization in Section 5. Section 6 discusses results on the efficiency advantage of PM relative to some well known mechanisms. The paper concludes in Section 7.

2 Model

We consider a priority-based allocation problem, or an economy, $\Gamma = \{\mathcal{S}, \mathcal{I}, Q, V, K\}$, where:

(i) $\mathcal{S} = \{s\}_{s=1}^{S}$ is a set of objects;

(ii) $\mathcal{I} = \{i\}_{i=1}^{I}$ is a set of agents, each of whom is to be matched with exactly one copy of an object;

(iii) $Q = [q_s]_{s=1}^{S}$ is a capacity vector, and $q_s \in \mathbb{N}$ is the supply of object s, $\forall s$. For simplicity, we assume that $\sum_{s=1}^{S} q_s = I$, i.e., there are just enough copies of objects to be allocated to agents; the extension to $\sum_{s=1}^{S} q_s \neq I$ is straightforward.

(iv) $V = [v_i]_{i\in\mathcal{I}}$, where $v_i = [v_{i,s}]_{s\in\mathcal{S}}$ and $v_{i,s} \in [0,1]$ is agent i's von Neumann-Morgenstern (vN-M) utility associated with object s.

(v) $K = [k_{s,i}]_{i\in\mathcal{I}, s\in\mathcal{S}}$, where $k_{s,i} \in \mathcal{K} \equiv \left\{1, 2, ..., \overline{k}\right\}$ is the priority group of agent i at object s, and $\overline{k}\,(\leq I)$ is the maximum number of priority groups.[9] Therefore, $k_{s,i} < k_{s,j}$ if and only if i has a higher priority at object s than j's. We allow both strict and coarse priority structures, in particular, the special case of interest when all agents have the same priority (the no-priority case).[10]

large markets. Nguyen, Peivandi, and Vohra (2015) introduce an optimization-based efficient mechanism that is strategy-proof-in-the-large. The asymptotic incentive properties of ordinal matching mechanisms have also been studied, e.g., Kojima and Pathak (2009), Kojima and Manea (2010), Kojima, Pathak, and Roth (2013), Liu and Pycia (2012), Lee (2014), and Ashlagi, Kanoria, and Leshno (2014).

[9] It is innocuous to assume that every object has the same number of priority groups, as there might be no agent in a particular group of an object.

[10] Our results on incentive compatibility and fairness are also new in this classical case.

All objects and agents are acceptable to the other side, i.e., every agent considers every object better than being unassigned and is qualified to be assigned to any object, although the analysis can be extended to the setting with unacceptable objects/agents. Agents are assigned to objects under the unit-demand constraint such that each agent must be matched with exactly one object. In the following, unless otherwise stated, we require non-wastefulness such that all copies of every object are to be assigned to some agents. Given the acceptability of everyone on both sides, wastefulness clearly leads to Pareto inefficiency.

A *random assignment* is a matrix $\Pi = [\pi_i]_{i\in\mathcal{I}}$; $\pi_i = [\pi_{i,s}]_{s\in\mathcal{S}}$ and $\pi_{i,s} \in [0,1]$ is agent i's probability share of object s, or the probability that agent i is matched with object s; the unit-demand constraint implies that $\sum_{s\in\mathcal{S}} \pi_{i,s} = 1$ for all i, and non-wastefulness leads to $\sum_{i\in\mathcal{I}} \pi_{i,s} = q_s$ for all s. The set of all such random assignments is denoted as $\mathcal{A}$.

If there exists s_i for every i such that $\pi_{i,s_i} = 1$ and $\pi_{i,s} = 0$, $\forall s \neq s_i$, Π is a *deterministic assignment*. Every feasible random assignment can be decomposed into a convex combination of deterministic assignments and can therefore be resolved into deterministic assignments (Kojima and Manea 2010), which generalizes the Birkhoff-von Newmann theorem (Birkhoff 1946, von Neumann 1953). Notice that the convex combination may not be unique in general.

Given objects' priorities and supply, a matching mechanism is a mapping from agents' reported preferences, either cardinal or ordinal, to the space of random assignments, $\mathcal{A}$.

3 The Pseudo-Market Mechanism

Fix the structure of priorities K, the capacities Q, and the (reported) utilities $V = [v_i]_{i\in\mathcal{I}}$, the pseudo-market mechanism (PM) with budgets $[b_i]_{i\in\mathcal{I}}$, $b_i \in (0,1]$, calculates a feasible random assignment $[\pi_i]_{i\in\mathcal{I}}$, and a price vector $P = [p_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}} \in \mathcal{P} \equiv [0,+\infty]^{S\times\overline{k}}$, where $p_{s,k}$ is the price of object s for agents in s's priority group k, by solving the utility maximization problem for all i,[11]

$$\pi_i(v_i, P) \quad \in \quad \arg\max_{\pi_{i,s}} \sum_{s\in\mathcal{S}} \pi_{i,s} v_{i,s}$$

[11]If $p_{s,k} = +\infty$, we define $+\infty \cdot 0 = 0$ and $+\infty \cdot \pi_{i,s} = +\infty$ if $\pi_{i,s} > 0$. As in Hylland and Zeckhauser's pseudo-market, we can implement any feasible random assignment as a lottery over feasible deterministic assignments.

subject to:

(i) feasibility constraint: $\sum_{i \in \mathcal{I}} \pi_{i,s}(v_i, P^*) \leq q_s$ for all objects s;

(ii) budget constraint: $\sum_{s \in \mathcal{S}} p_{s,k_{s,i}} \pi_{i,s} \leq b_i$ and the stipulation that if there are multiple bundles maximizing her expected utility then a cheapest one is chosen.

(iii) priority constraint: $k^*(s)$ is the cut-off priority of object s if $\sum_{i \in \mathcal{I},\ k_{s,i} < k^*(s)} \pi_{i,s}(v_i, P^*) < q_s$ and $\sum_{i \in \mathcal{I},\ k_{s,i} \leq k^*(s)} \pi_{i,s}(v_i, P^*) = q_s$; moreover,

$$
\begin{aligned}
p^*_{s,k_{s,i}} &= 0, \text{ if } k_{s,i} < k^*(s), \\
p^*_{s,k_{s,i}} &\in [0, +\infty), \text{ if } k_{s,i} = k^*(s), \\
p^*_{s,k_{s,i}} &= +\infty, \text{ if } k_{s,i} > k^*(s).
\end{aligned}
$$

In the setting in which all agents have the same priority at all objects (i.e., no priorities), the priority constraint (iii) reduces to prices being non-negative and finite.

Remark 1 *PM accommodates personalized exogenous budgets, but to economize on notations, we focus on PM with equal budgets such that $b_i = 1$ for all i and refer to PM with equal budgets simply as PM. It should be noted that all results except fairness in Section 5 extend to PM with unequal budgets as long as budgets do not depend on reported utilities.*

Remark 2 *PM restricts the price structure such that prices are priority-specific and increase when we move down on the priority list. If s is consumed completely by agents in priority groups $k^*(s)$ and above, agents in s's priority groups strictly below $k^*(s)$ face an infinite price, while those in priority groups strictly above $k^*(s)$ face a zero price. Section 5 discusses the implications of such a price structure.*

Remark 3 *PM treats objects' priorities as agents' rights to obtain an object at a lower, sometimes zero, price. Whenever some agents with lower priorities can get a positive share of an object, an agent with a higher priority at that object can always get it for free. More importantly, agents can choose not to exercise the right if they wish, but they cannot trade priorities. This interpretation is similar to the consent in Kesten (2010) that allows agents* *to waive a certain priority at an object, but in contrast to the treatment in the top-trading-cycle mechanism which implicitly allows agents to trade their priorities (Abdulkadiroglu and Sonmez 2003).*

Our first main result is the existence of the PM outcome.

Theorem 1 *Given any reported preferences, there always exists an equilibrium price matrix in the PM mechanism.*

This key result shows that the PM mechanism is well-defined. The analogous result was proven by Hylland and Zeckhauser (1979) for the classical economy without priorities. The result is new for the case with priorities; the challenge in obtaining the result is the need to incorporate the priority condition (iii).[12] This condition is crucial in ensuring that the mechanism honors priorities (Section 5).

Note that an economy can have more than one PM equilibrium price matrix and multiple PM assignments, and therefore a complete specification of the mechanism must prescribe a price selection rule.[13] Our main results are robust to arbitrary selection rules, except those on incentive compatibility in the next section, which address the selection issue directly.

4 Asymptotic Incentive Compatibility

Our next analysis focuses on asymptotic incentive compatibility in sequences of replica economies,[14] and considers PM's incentive properties in large markets. For any base economy $\Gamma = \{\mathcal{S}, \mathcal{I}, Q, V, K\}$, we use $\Gamma^{(n)} = \left\{\mathcal{S}, \mathcal{I}^{(n)}, Q^{(n)}, V^{(n)}, K^{(n)}\right\}$ to denote an n-fold replica of Γ which is an economy such that: (i) for each $i \in \mathcal{I}$, there are n copies of i in $\mathcal{I}^{(n)}$ whose preferences and priorities are exactly the same as i; (ii) $\mathcal{S}$ is constant in all economies; and (iii) $Q^{(n)} = nQ$, or equivalently $q_s^{(n)} = nq_s$ for all s and n. In the sequence of replica

[12]Our paper subsumes He and Yan's and Miralles's work, who independently proposed the construction of priority-honoring pseudo markets.

[13]In the market design literature, Kovalenkov (2002) is another exception to explicitly consider selection rules in an approximate Walrasian mechanism.

[14]Given the impossibility result in Zhou (1990) and the example in Hylland and Zeckhauser (1979), it is known that agents may have incentives to misreport their preferences in any finite market. See the end of this section for an extension beyond replica economies, and Appendix C for an analysis of the limit incentive compatibility concept defined by Roberts and Postlewaite (1976).

economies $\left\{\Gamma^{(n)}\right\}_{n\in\mathbb{N}}$, each $\Gamma^{(n)}$ has n copies of the base economy Γ. Notice that the set of pseudo-market prices is constant along any sequence of replica economies, provided that all agents report truthfully.

We consider a natural analogue of regular economies from the general equilibrium literature (e.g., Dierker (1974), Hildenbrand (1974), Jackson (1992)).[15] To define this regularity concept, we use the Prohorov metric ρ to measure the distance between two distributions, μ and ν:

$$\rho(\mu,\nu) = \inf\left\{\varepsilon > 0 | \nu(E) \leq \mu(B_\varepsilon(E)) + \varepsilon \ \&\ \mu(E) \leq \nu(B_\varepsilon(E)) + \varepsilon,\ E \subseteq [0,1]^{S\times I} \text{ Borel}\right\}.$$

A distribution of utilities μ^* is **regular** if there exists a neighborhood B of μ^* and a finite number $m > 0$ of continuous functions $\psi_1, ..., \psi_m$ from B to $[0,+\infty]^{S\times\bar{k}}$ such that for every distribution $\mu \in B$ the set of PM equilibrium prices is $\{\psi_1(\mu), ..., \psi_m(\mu)\}$ and $\psi_i(\mu) \neq \psi_j(\mu)$ for every $i \neq j$. An economy Γ is regular if the corresponding distribution of utilities is regular. The proofs for this section show that if the base economy is regular then so is any replica economy.

Our second main result is the asymptotic incentive compatibility of PM. A mechanism is **asymptotically incentive compatible** on a sequence of replica economies $\Gamma^{(n)}$ if for every agent the utility gain from submitting a utility profile different from the truth vanishes along the sequence. That is, for every $\varepsilon > 0$, there exists n^* such that $n > n^*$ implies that the utility gain from unilateral misreporting for every agent in $\Gamma^{(n)}$ is bounded by ε when everyone else is truth-telling.

Theorem 2 *There always exists a selection of equilibrium prices in the PM construction such that the resulting PM mechanism is asymptotic incentive compatible on any sequence of replica economies whose base economy has a regular distribution of utilities.*

Note that Theorem 2 shows that the utility gain from unilateral misreporting is bounded for *all agents* in a large enough economy. An analogue of this result remains true beyond

[15] For simplicity we follow Jackson (1992) in defining regularity directly in terms of price behavior; alternatively we could express the definition of regularity in terms of properties of excess demand functions as in Dierker (1974) and Hildenbrand (1974).

replica economies: our proof of Theorem 2 also implies that the gain from manipulation for any agent who is present in all economies in a sequence vanishes as the economy grows, provided that the limit distribution of utilities is regular.[16]

Theorem 2 is new not only in the setting with priorities, but also in the canonical setting without priorities first studied by Hylland and Zeckhauser (1979).[17] While Hylland and Zeckhauser conjectured that their mechanism is asymptotically incentive compatible, their conjecture has so far remained open. The closest prior result was obtained Azevedo and Budish (2012) who introduced the concept of strategy-proofness-in-the-large and in a discrete setting proved that every envy-free mechanism is incentive compatible in their sense. In particular, their result implies that Hylland and Zeckhauser's mechanism with equal budgets is strategy-proof-in-the-large in large economies with a bounded number of utility types. Their approach hinges both on the equality of budgets and on there being a bounded number of possible utility types; in contrast our result is valid in the standard model that allows a continuum of utility types and it is valid for any profile of budgets.

5 Priority-Honoring and Fairness

We now discuss the priority-honoring and fairness properties of PM assignments. Given the priority structure, we show that the set of equal-budget PM assignments is equivalent to the set of assignments that honor priorities (or thos that are *ex ante* stable as in Kesten and Ünver (2015)) and satisfy "equal claim" as in He, Li, and Yan (2015).

5.1 Priority-Honoring

The key property of PM is that it honors priorities in the sense of *ex ante* stability introduced by Kesten and Ünver (2015). We say that a random assignment Π causes ***ex ante* justified envy** of $i \in \mathcal{I}$ toward $j \in \mathcal{I} \setminus \{i\}$ if $\exists s, s' \in \mathcal{S}$ such that $v_{i,s} > v_{i,s'}$, $k_{s,i} < k_{s,j}$, $\pi_{j,s} > 0$, and $\pi_{i,s'} > 0$. That is, a higher-priority agent i (at s) has justified envy towards lower-priority agent j (at s) if j has positive probability of an object s, while i has positive probability of an

[16]In addition, Appendix C shows that the PM mechanism is limiting incentive compatible in the sense of Roberts and Postlewaite (1976).

[17]See also e.g., Budish, Che, Kojima, and Milgrom (2013)

object that is worse than s in i's preferences. If a random assignment causes *ex ante* justified envy, it is not guaranteed that its implementation always leads to deterministic assignments that are justified-envy-free (also known as stability (Abdulkadiroglu and Sonmez 2003)).[18] A random assignment is ***ex ante* stable** if it does not cause any *ex ante* justified envy.

In defining PM we restrict the price structures to be such that the prices are 0 above the cut-off priority group and infinity below the cut-off. This restriction guarantees that PM is *ex ante* stable; in fact, such a restriction is also necessary for *ex ante* stability. To see the necessity and sufficiency of this restriction, we relax the PM construction by allowing the prices to be agent-specific (i.e., personalized) as in He, Li, and Yan (2015). With such personalized prices, we can normalize each agent's possibly-unequal budget to be 1 without loss of generality.

We first restrict ourselves to the set of non-wasteful assignments. Given an economy Γ, the set of all possible "equilibrium" personalized prices $\mathcal{P}_\Gamma$ and the associated set of non-wasteful assignments Π_Γ are defined as follows:

(i) For a given personalized price vector $P_i = [p_{i,s}]_{s\in\mathcal{S}} \in [0,+\infty]^S$ where $p_{i,s}$ is the price of object s for agent i, we construct the demand correspondence of agent i, $\pi_i^*(v_i, P_i)$, that maximizes $\sum_{s\in\mathcal{S}} \pi_{i,s} v_{i,s}$ subject to $\sum_{s\in\mathcal{S}} p_{i,s}\pi_{i,s} \le 1$ among feasible π_i such that $\sum_{s\in\mathcal{S}} \pi_{i,s} = 1$ and $\pi_{i,s} \ge 0$ for all objects s.

(ii) The set of "equilibrium" personalized prices:

$$\mathcal{P}_\Gamma \equiv \left\{ P^* = [P_i^*]_{i\in\mathcal{I}} \in [0,+\infty]^{I\times S} \,|\exists \pi_i \in \pi^*(v_i, P_i^*)\,,\ \sum_{i\in\mathcal{I}} \pi_{i,s}^* \le q_s,\, \forall i\in\mathcal{I},\, \forall s\in\mathcal{S} \right\}.$$

(iii) The set of associated assignments is $\Pi_\Gamma(P^*) \equiv \left\{[\pi_i]_{i\in\mathcal{I}} \in \mathcal{A} | \pi_i \in \pi^*(v_i, P_i^*)\,, \forall i \in \mathcal{I}\right\}$ for $P^* \in \mathcal{P}_\Gamma$, and $\Pi_\Gamma \equiv \cup_{P^*\in\mathcal{P}_\Gamma} \Pi_\Gamma(P^*)$.

In other words, $\mathcal{P}_\Gamma$ is the set of all possible personalized prices that can rationalize some assignment as a result of agents' utility maximization (given budgets); $\Pi_\Gamma(P^*)$ is the set of assignments corresponding to P^*; and Π_Γ is the set of all possible assignments that can be

[18] Many school districts insist on avoiding justified envy, for example, NYC (Abdulkadiroglu, Pathak, and Roth 2005) and Boston (Abdulkadiroglu, Pathak, Roth, and Sonmez 2005).

supported as a result of agents' utility maximization. Of course, every feasible assignment can be represented in this way, i.e., $\Pi_\Gamma = \mathcal{A}$ (see He, Li, and Yan (2015) for details).

By definition and given that $\sum_{s\in\mathcal{S}} q_s = I$, PM restricts prices to be in the following set:

$$\mathcal{P}_\Gamma^{Stable} \equiv \left\{ P^* \in \mathcal{P}_\Gamma | \forall s, \forall \pi \in \Pi_\Gamma(P^*), \exists k', p^*_{i,s} = \left\{ \begin{array}{l} 0 \quad \text{if} \sum\limits_{j\in\mathcal{I} \text{ s.t. } k_{s,j}\leq k'} \pi_{j,s} < q_s \ \& \ k_{s,i} < k' \\ +\infty \text{ if} \sum\limits_{j\in\mathcal{I}, \text{ s.t. } k_{s,j}\leq k'} \pi_{j,s} = q_s \ \& \ k_{s,i} > k' \end{array} \right. \right\}.$$

By our Theorem 1, $\mathcal{P}_\Gamma^{Stable} \neq \varnothing$, and thus the set of assignments $\Pi_\Gamma^{Stable} \equiv \cup_{P^*\in\mathcal{P}_\Gamma^{Stable}} \Pi_\Gamma(P^*)$ is also non-empty. Furthermore, Π_Γ^{Stable} corresponds exactly to the set of *ex ante* stable assignments.

Proposition 1 Π_Γ^{Stable} *is the set of all non-wasteful ex ante stable random assignments.*

In particular, given the construction of PM:

Corollary 1 *Every PM assignment is ex ante stable.*

Remark 4 *It should be emphasized that the above result is also true for PM with unequal budgets. Since prices are either zero or infinite except for those in the cut-off priority groups, a lower or higher budget would only change the probability shares that one can obtain in the objects where he or she is in the cut-off group. This certainly does not lead to violations of the requirement of ex ante stability.*

Remark 5 *The above construction can be naturally extended to possibly wasteful assignments, i.e.,* $\sum\limits_{i\in\mathcal{I}} \pi_{i,s} = q'_s \leq q_s$ *for all schools. The real capacities* (q_s) *are replaced by "wasteful" ones* (q'_s) *to form the "market-clearing" conditions, exactly satisfying the wastefulness.*

5.2 Equal claim

The PM mechanism satisfies the strong fairness criterion of equal claim, introduced by He, Li, and Yan (2015) in a setting without priorities.[19]

Definition 1 *An ex ante stable random assignment* Π *satisfies equal claim if and only if* $\exists P^* \in \mathcal{P}_\Gamma^{Stable}$ *such that* $\Pi \in \Pi_\Gamma(P^*)$ *and that for any* s, $p^*_{i,s} = p^*_{j,s}$ *whenever* $k_{s,i} = k_{s,j}$.

[19] See their work for a motivating discussion of this property.

That is, an *ex ante* stable assignment Π satisfies equal claim if Π is an expected-utility-maximization outcome as if everyone has an equal budget and those in any given priority group of s face an equal price of s. Note that this definition allows Π to be wasteful. Nonetheless, the definition of PM, Theorem 1, and Corollary 1 together imply that in any economy Γ there exists a non-wasteful *ex ante* stable random assignment satisfying equal claim.

5.3 Characterization

In the characterization, we again require non-wastefulness, i.e., there is no positive probability share of any object being unassigned should there exist an agent willing to obtain it within the unit-demand constraint.

Theorem 3 *The set of PM assignments is equivalent to the set of non-wasteful assignments satisfying both ex ante stability and equal claim.*

In the special case where agents' preferences and objects' priorities are both strict, we have the following result.

Theorem 4 *If both agents and objects rank the other side strictly, the set of PM assignments is equivalent to the set of stable assignments.*

6 Efficiency

The results of the previous section imply that no mechanism that honors priorities and is fair in the sense of equal claim can dominate PM in efficiency terms. We now illustrate via examples how PM can dominate other mechanisms.[20] Given our incentive compatibility result, we assume everyone is truth-telling under PM.

A random assignment $\Pi' \in \mathcal{A}$ is *ex ante* **Pareto dominated for agents** by another random assignment $\Pi \in \mathcal{A}$ if

$$\sum_{s \in \mathcal{S}} \pi_{i,s} v_{i,s} \geq \sum_{s \in \mathcal{S}} \pi'_{i,s} v_{i,s}, \forall i \in \mathcal{I},$$

and at least one inequality is strict. A random assignment is *ex ante* **agent-efficient** if it is not Pareto dominated for agents by any other feasible random assignment. The definition can be readily

[20] The fact that some of the mechanisms we study can be dominated is known, see Ergin and Sonmez (2006), Abdulkadiroglu, Che, and Yasuda (2011), Troyan (2012), and Pycia (2014).

extended to deterministic assignments, and every deterministic assignment in any decomposition of an *ex ante* agent-efficient random assignment is Pareto optimal for agents.

In general, PM cannot achieve *ex ante* agent-efficiency due to the priority structure, and Theorem 3 implies that any agent-efficient assignment satisfying priority-honoring and equal claim is a PM assignment.

The unique feature of PM is that it elicits and uses cardinal preferences to make the assignment. This implies that the outcome has the potential to be more efficient than ordinal mechanisms. In a one-sided setting (i.e., no priorities), Abdulkadiroglu, Che, and Yasuda (2011) show cardinal mechanisms can dominate ordinal ones, and, building on subsequent analysis by Pycia (2014), we extend this result to the setting with priorities.

The following definition is useful for the comparison.

Definition 2 *A random assignment* Π^* *is ordinally efficient if there does not exist* $\Pi \neq \Pi^*$ *such that:*

$$\sum\nolimits_{s' \ s.t. \ v_{i,s'} \geq v_{i,s}} \pi^*_{i,s'} \leq \sum\nolimits_{s' \ s.t. \ v_{i,s'} \geq v_{i,s}} \pi_{i,s'}, \forall s \in \mathcal{S}, i \in \mathcal{I},$$

where at least one inequality is strict. Π^* *is symmetric ordinal efficient if furthermore* $\pi^*_{i,s} = \pi^*_{j,s}$, $\forall s$*, whenever* i *and* j *have the same ordinal preferences.*

6.1 The Cost of Ordinality

The following example, based on Pycia (2014), illustrates the extent to which restricting ourselves to ordinal mechanisms may result in an efficiency loss.

Example 1 *Let us consider the following economy with four agents* $(i_1, ..., i_4)$ *and four objects* $(s_1, ..., s_4)$ *with one copy of each available:*

Cardinal Preferences

	Objects			
Agent	s_1	s_2	s_3	s_4
i_1	1	ϵ	ϵ^2	0
i_2	1	$1-\epsilon$	ϵ^2	0
i_3	1	ϵ^2	$1-\epsilon$	0
i_4	1	ϵ^2	ϵ	0

$0 < \epsilon < 0.5$

Priority Structure

	Objects			
Agent	s_1	s_2	s_3	s_4
i_1	1	2	2	1
i_2	1	2	1	1
i_3	1	1	2	1
i_4	1	2	2	2

Smaller number means higher priority.

Note that no pair of agents has the same priorities at all objects. The following prices and assignment is an equilibrium outcome of PM:

PM Priority-Specific Prices

Agent	s_1	s_2	s_3	s_4
	Objects			
i_1	2	1	1	0
i_2	2	1	0	0
i_3	2	0	1	0
i_4	2	1	1	0

PM Assignment

	Objects				*Expected*
Agent	s_1	s_2	s_3	s_4	*Utility*
i_1	1/2	0	0	1/2	1/2
i_2	0	1	0	0	$1-\epsilon$
i_3	0	0	1	0	$1-\epsilon$
i_4	1/2	0	0	1/2	1/2

Like in Pycia (2014), we can replicate this example and compare PM with "best" ordinal mechanisms ignoring the priority constraint because Liu and Pycia (2012) showed that in large economies, all regular, asymptotically strategy-proof, asymptotically symmetric, and asymptotically efficient ordinal mechanisms deliver outcomes asymptotically equivalent to the symmetric ordinal efficient assignments.

PS: Symmetric Ordinally Efficient Assignment

	Objects				*Expected*
Agent	s_1	s_2	s_3	s_4	*Utility*
i_1	1/4	1/2	0	1/4	$(1+2\epsilon)/4$
i_2	1/4	1/2	0	1/4	$(3-2\epsilon)/4$
i_3	1/4	0	1/2	1/4	$(3-2\epsilon)/4$
i_4	1/4	0	1/2	1/4	$(1+2\epsilon)/4$

Can be achieved by the Probabilistic Serial.

The above assignment can be implemented by the Probabilistic Serial (PS) whose definition is in Appendix A. Given that $0 < \epsilon < 0.5$, the above PS assignment is Pareto dominated by the PM assignment in terms of agent welfare, despite the fact that the PS assignment ignores priorities.[21]

[21]For PS that takes priorities into account, see Afacan (2015). Such extensions of PS yield assignments that are dominated by those from standard PS.

6.2 Comparison with the Gale-Shapley Deferred-Acceptance Mechanism

The Gale-Shapley Deferred-Acceptance mechanism (DA), whose definition is also available in Appendix A, is a mechanism that has attracted the most attention both in the literature as well as in practice. When it is implemented in settings where priorities are coarse/weak, some tie-breaking rule is needed. For example, following reforms in NYC and Boston, the school choice program uniformly randomly chooses a single tie-breaking order for equal-priority students at each school and then employs the student-proposing DA using the modified priority structure.

From the perspective of tie-breaking, one may view PM as a version of DA with coarse priorities on one side. The unique feature of PM is that the ties are broken endogenously according to cardinal preferences. The following example shows a case where PM dominates DA.

Example 2 *In the same setting as in Example 1, the random assignment from DA with single tie-breaking (DA-STB) is as follows:*

The DA-STB Assignment

	Objects				*Expected*
Agent	s_1	s_2	s_3	s_4	*Utility*
i_1	1/4	1/6	1/12	1/2	$(3+2\epsilon+\epsilon^2)/12$
i_2	1/4	7/24	11/24	0	$(13-7\epsilon+11\epsilon^2)/24$
i_3	1/4	11/24	7/24	0	$(13-7\epsilon+11\epsilon^2)/24$
i_4	1/4	1/12	1/6	1/2	$(3+2\epsilon+\epsilon^2)/12$

DA-STB: The DA mechanism with single tie-breaking.

It is Pareto dominated by the PM assignment in terms of agents' expected utility for $\epsilon \in (0, 0.5)$.

It should be noted that Kesten and Ünver (2015) extend DA and propose two variants to deal with the tie-breaking on the object side. Since their mechanisms still rely on ordinal preferences of agents, the cost of ordinality (Section 6.1) still applies. Empirically, Abdulkadiroglu, Agarwal, and Pathak (2015) use data from the NYC high school match to show that possible improvements upon DA from various ordinal mechanisms are rather limited. In fact, the best outcomes that the mechanisms in Kesten and Ünver (2015) can achieve are constrained ordinal efficiency, which are necessarily dominated by ordinal efficient outcomes.[22]

We also note a special case in which agents' preferences and objects' priorities are both strict. In this case, it must be that $\overline{k} = I$ and that there is exactly one agent in each priority group of any object. Noting that any DA assignment when agents report true ordinal preferences is stable, we have the following result as a corollary of Theorem 4:

Corollary 2 *If both agents and objects rank those on the other side strictly, any DA assignment when agents report true ordinal preferences is a PM assignment.*

6.3 Comparison with the Boston Mechanism

The PM mechanism is closely related to another commonly used mechanism, the Boston mechanism (BM), whose definition is available in Appendix A. It has been noted in the literature that BM elicits signals of agents' cardinal preferences, and indeed sometimes BM can yield PM assignments in Baysian Nash equilibrium.

Proposition 2 *A PM assignment is also a Bayesian Nash equilibrium assignment of BM, if every agent has strict preferences and consumes a bundle that either includes only free objects (according to her own prices), or includes one object with a positive price in* $(1, +\infty)$ *(according to her own price) and all others free to all agents.*

Note that the above result is a sufficient condition, and the following example shows there are other cases where PM and BM coincide.

Example 3 *In the same setting as in Example 1, one can verify that the following strategies constitute a Nash equilibrium under BM (with single tie-breaking), and the equilibrium outcome is exactly the PM assignment.*

BM Equilibrium Strategies

Agent	*Rank-Order List*			
i_1	s_1	s_4	$\ldots$	$\ldots$
i_2	s_2	$\ldots$	$\ldots$	$\ldots$
i_3	s_3	$\ldots$	$\ldots$	$\ldots$
i_4	s_1	s_4	$\ldots$	$\ldots$

"..." indicates an arbitrary school.

BM Equilibrium Assignment

	Objects				*Expected*
Agent	s_1	s_2	s_3	s_4	*Utility*
i_1	1/2	0	0	1/2	1/2
i_2	0	1	0	0	$1-\epsilon$
i_3	0	0	1	0	$1-\epsilon$
i_4	1/2	0	0	1/2	1/2

[22] In recent work, Che and Tercieux (2014) provide modifications of DA to improve asymptotic efficiency.

When agents do not have strict preferences, or at least one of them spends her budget on more than one object with positive and finite prices, in general, a PM assignment is not an equilibrium outcome of BM. More importantly, in addition to BM's disadvantages discussed in the literature review, not every equilibrium outcome of BM is a PM outcome. The following example show this clearly.

Example 4 *Let us consider the following economy with three agents* $(i_1, ..., i_3)$ *and three objects* $(s_1, ..., s_3)$ *with one copy of each available. Moreover, there are no priorities. The unique PM equilibrium is as follows:*

Cardinal Preferences

Agent	s_1 (*Objects*)	s_2	s_3
i_1	1	0.9	0
i_2	1	0.6	0
i_3	1	0.1	0

PM Prices

Agent	s_1 (*Objects*)	s_2	s_3
i_1	15/8	9/8	0
i_2	15/8	9/8	0
i_3	15/8	9/8	0

PM Assignment

Agent	s_1 (*Objects*)	s_2	s_3
i_1	0	8/9	1/9
i_2	7/15	1/9	19/45
i_3	8/15	0	7/15

Note that in the PM equilibrium, i_2 *spends a positive amount on both* s_1 *and* s_2*. Moreover, the BM Bayesian Nash equilibrium, which is unique in terms of outcomes, is that* i_1 *top ranks* s_2*, while* i_2 *and* i_3 *top ranking* s_1*, leading to an assignment different from the PM assignment:*

BM Equilibrium Strategies

Agent	*Rank-Order List*		
i_1	s_2	$\dots$	$\dots$
i_2	s_1	s_3	$\dots$
i_3	s_1	s_3	$\dots$

"..." indicates an arbitrary school.

BM Assignment

Agent	s_1 (*Objects*)	s_2	s_3
i_1	0	1	0
i_2	1/2	0	1/2
i_3	1/2	0	1/2

6.4 "Welfare" of Both Sides

Usually, it is natural to care only about the welfare of agents, as objects' priority ranking over agents do not necessarily reflect any underlying preferences of their suppliers. However, there are exceptions, and needless to say, priorities are usually not randomly chosen. For example, in school choice, priority rules may reflect preferences of the local constituency such as minimizing transportation costs (distance-based priorities) and/or encouraging investment in studying (test-score-based priorities).

When one is interested in taking the welfare of both sides into account, PM is two-sided Pareto efficient in terms of both the preferences of agents and those defined by priorities. In this context, we say that a random assignment $\Pi' \in \mathcal{A}$ is ***ex ante* two-side dominated** by another random assignment $\Pi \in \mathcal{A}$ if

$$\begin{aligned}\sum_{s\in\mathcal{S}} \pi_{i,s} v_{i,s} &\geq \sum_{s\in\mathcal{S}} \pi'_{i,s} v_{i,s}, \forall i \in \mathcal{I},\\ \sum_{i\in\{k_{s,i}\leq k\}} \pi_{i,s} &\geq \sum_{i\in\{k_{s,i}\leq k\}} \pi'_{i,s},\ \forall s \in \mathcal{S}, \forall k \in \mathcal{K},\end{aligned}$$

and at least one inequality is strict. That is, every agent has a weakly higher expected utility in Π, and, for each object s, the assignment Π first-order stochastically dominates Π' with respect to the priority structure. A random assignment is ***ex ante* two-sided efficient** if it is not *ex ante* two-side dominated by any other random assignment. We then obtain:

Theorem 5 *Every PM assignment is ex ante two-sided efficient.*

Remark 6 *If the problem is indeed two-sided, i.e., objects' priorities represent some underlying possibly-weak preferences, our results then make PM a promising candidate for two-sided matching with weak preferences.*

7 Concluding Remarks

This paper studies the allocation of indivisible goods that honors priorities when monetary transfers are not possible and agents have unit demand. We propose a pseudo-market mechanism, PM, in which agents are endowed with budgets of token money and purchase bundles of probability shares in objects to maximize their expected utility. The prices in the mechanism are calculated from agents' reported cardinal preferences and are priority-specific. More specifically, everyone in any given priority group of an object faces the same price, while those who are in higher priority groups of an object face a lower, sometimes zero, price of that object.

We show that the mechanism has desirable properties. After establishing the existence of PM equilibrium prices, we prove that it is asymptotically incentive compatible for agents in a sequence of replica economies. Moreover, the mechanism delivers a random assignment that honors priorities, i.e., satisfies *ex ante* stability or eliminates *ex ante* justified envy. The structure of the price system also guarantees that everyone in the same priority group of an object has an equal claim to that object, when budgets are equal. The mechanism can achieve all assignments that are not dominated by other assignments satisfying the above criteria. Because of the explicit use of cardinal preferences, PM has an efficiency advantage over other popular mechanisms.[23]

These properties of the mechanism make it a promising candidate for real-life applications to settings such as school choice. Schools often prioritize student applications, and the priority structure is usually determined by the school district or local laws. In most cases, a school's priority ranking over students is not strict, which makes the PM mechanism a natural candidate to run seat assignment. The mechanism guarantees that the random assignment honors priorities and thus that it can be implemented as a lottery over deterministic assignments that honor priorities. Furthermore, as Abdulkadiroglu, Che, and Yasuda (2011) point out, in settings such as school choice, students may have similar ordinal preferences. Therefore, without information on cardinal preferences, the efficiency that a mechanism can achieve may be limited.[24] Indeed, using data from the high school match in NYC, Abdulkadiroglu, Agarwal, and Pathak (2015) show the significant welfare-improving potential of eliciting cardinal utilities. By explicitly using students' cardinal preferences, the PM mechanism allows school districts to achieve such efficiency gains.[25]

[23]A concern in using pseudo-market-like mechanisms is the potential difficulty of eliciting cardinal preferences from agents (see e.g. Bogomolnaia and Moulin (2001)). The experimental evidence in Budish and Kessler (2014) suggests that pseudo-market mechanisms perform well despite this difficulty. Furthermore, Chen and He (2015) show that the need to report cardinal preferences is an incentive for agents to investigate whether an object is a good fit for them, and that such an investigation can be welfare-improving.

[24]Whereas ordinal inefficiency may vanish in large markets (Che and Kojima 2010), the cardinal inefficiency of ordinal mechanisms persists (Pycia 2014).

[25]Note that one can accommodate group-specific quotas within the PM similarly to how they might be accommodated within the Deferred-Acceptance mechanism (see e.g., Abdulkadiroglu and Sonmez (2003)): to accommodate such quotas, one can divide each school into multiple sub-schools each of which has a quota equal to the one for the corresponding group and gives that group the highest priority.

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Appendices

A Alternative Mechanisms

This appendix gives the definitions of three mechanisms: the Probabilistic Serial, the Boston mechanism (also known as the immediate-acceptance mechanism), and the Gale-Shapley deferred-acceptance mechanism.

The **Probabilistic Serial** is defined by the following symmetric simultaneous eating algorithm. It is proposed for one-sided matching where objects do not rank agents. Each object s is considered as an infinitely divisible good with supply q_s that agents eat in the time interval $[0, 1]$.

Round 1. Each agent eats away from her favorite object at the same unit speed, and the algorithm proceeds to the next step when an object is completely exhausted.

Generally, in:

Round k (k>1). Each agent eats away from her most-preferred object among the remaining ones at the same unit speed, and the algorithm proceeds to the next step when an object is completely exhausted.

The process terminates after any round k when every agent has eaten exactly one total unit of objects (i.e., at time 1). The random assignment of an agent i is then given by the amount of each object she has eaten during the run of the algorithm.

The **Boston mechanism** solicits rank-ordered lists of objects from agents, uses pre-defined rules, including tie-breaking rules, to determine objects' strict ranking over agents, and has multiple rounds:

Round 1. Each object considers all the agents who rank it first and assigns its copies in order of their priority at that object until either there are no copies of the object left or no such agents left.

Generally, in:

Round (k>1). The k-th choice of the agents who have not yet been assigned is considered. Each object that still has available copies assigns the remaining copies to agents who rank it as kth choice in order of their priority at that object until either there are no copies of that object left or no such agent left.

The process terminates after any round k when every agent is assigned a copy of some object, or if the only agents who remain unassigned listed no more than k choices.

The **Gale-Shapley Deferred-Acceptance mechanism** (DA) can be agent-proposing or object-proposing. In the former, the mechanism collects objects' supplies and their priority structure over agents, as well as agents' submitted rank-ordered lists of objects. When necessary, tie-breaking rules are applied to form strict rankings of objects over agents. The process then has several rounds:

Round 1. Every agent applies to her first choice. Each object rejects the least preferred agents in excess of its supply and *temporarily holds* the others.

Generally, in:

Round (k>1). Every agent who is rejected in Round $(k-1)$ applies to the next choice on her list. Each object pools new applicants and those who are held from Round $(k-1)$ together and rejects the least preferred agents in excess of its supply. Those who are not rejected are *temporarily held* by the objects.

The process terminates after any Round k when no rejections are issued. Each object is then matched with agents it is currently holding. The object-proposing DA is similarly defined.

B Proofs

B.1 Proof of Theorem 1

First, we transform the price space from $[0,+\infty]^{S\times\bar{k}}$ to $\mathcal{Z}\equiv[0,\pi/2]^{S\times\bar{k}}$ such that $\forall P\in[0,+\infty]^{S\times\bar{k}}$, there is a $Z\in\mathcal{Z}$ and $Z=[z_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}}=[\arctan(p_{s,k})]_{s\in\mathcal{S},k\in\mathcal{K}}$, with $\arctan(+\infty)\equiv\pi/2$ and $\tan(\pi/2)\equiv+\infty$.[26] Since the arctangent function, arctan, is a positive monotonic transformation, the reverse statement is also true such that $\forall Z\in\mathcal{Z}$, there is a $P\in[0,+\infty]^{S\times\bar{k}}$ and $P=\mathcal{TAN}(Z)\equiv[\tan(z_{s,k})]_{s\in\mathcal{S},k\in\mathcal{K}}$.

A price-adjustment process for Γ is defined as,

$$\begin{aligned}&H[Z,G(\mathcal{TAN}(Z),u)]\\ \equiv\ &\left\{Y=[y_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}}\;\middle|\;\begin{array}{c}y_{s,k}\left([d_{s,k}]_{k\in\mathcal{K}}\right)=\min\left\{\frac{\pi}{2},\max\left[0,z_{s,k}+\left(\sum_{\kappa=1}^{k}d_{s,\kappa}-\frac{q_s}{I}\right)\right]\right\}\\ \forall\,[d_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}}\in G(\mathcal{TAN}(Z),u)\end{array}\right\}.\end{aligned}$$

where $u=(u_1,\cdots,u_I)$ are agents' reports, and $G(\mathcal{TAN}(Z),u)$ is the per capita demand correspondence for each priority group of each object.

Since G is the average of individual demand correspondences, it is then upper hemicontinuous and convex-valued, and thus $H[Z,G]$ has the same properties. $H[Z,G]$ satisfies all the conditions of Kakutani's fixed-point theorem, and there must exist a fixed point Z^* such that $Z^*\in H[Z^*,G(\mathcal{TAN}(Z^*),u)]$.

Given Z^*, there also exists $[d_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}}\in G$ such that $\forall s$ and $\forall k$,

$$z^*_{s,k}=\min\left\{\frac{\pi}{2},\max\left[0,z^*_{s,k}+\left(\sum_{\kappa=1}^{k}d_{s,\kappa}-\frac{q_s}{I}\right)\right]\right\}.$$

This implies that $\forall s$, $\sum_{\kappa=1}^{\bar{k}}d_{s,\kappa}=q_s/I$ and there exists $k^*(s)$ for each s such that $\sum_{\kappa=1}^{k^*(s)}d_{s,\kappa}=\frac{q_s}{I}$, $z^*_{s,k^*(s)}\in\left[0,\frac{\pi}{2}\right)$, and $d_{s,k^*(s)}>0$; if $k<k^*(s)$, $\sum_{\kappa=1}^{k}d_{s,\kappa}<q_s/I$ and $z^*_{s,k}=0$; and if $k>k^*(s)$, $d_{s,k}=0$, and $z^*_{s,k}\in[0,\pi/2]$.

Moreover, if $d_{s,k}=0$, and $z^*_{s,k}\in[0,\pi/2)$ for some $k>k^*(s)$, there must exist another Z^{**} such that $Z^{**}\in H[Z^{**},G(\mathcal{TAN}(Z^{**}),u)]$ and that if $k\le k^*(s)$, $z^{**}_{s,k}=z^*_{s,k}$; and if $k>k^*(s)$, $z^{**}_{s,k}=\pi/2$.

In summary, $P^{**}=\mathcal{TAN}(Z^{**})$ satisfies the structure of PM prices and indeed clears the market. Therefore, an equilibrium price matrix in PM exists.

B.2 Proof of Theorem 2

Let us represent each economy by a probability measure. Let $T=[0,1]^S\times\mathcal{K}^S$ be the compact space of utility-priority profiles endowed with the standard Euclidean distance. For any profile

[26] We could alternatively work in the original space $[0,+\infty]^{S\times\bar{k}}$. Here and in the following, with some abuse of notation, π, without subscript, is the mathematical constant, i.e., the ratio of a circle's circumference to its diameter.

$(v,k) \in T$ and scalar $\varepsilon > 0$, let $B_\varepsilon(v,k)$ be the ball of profiles within distance ε of (v,k). Let $\mathcal{M}$ be the space of compact-support Borel probability measures on T. An economy can be conveniently represented by a probability measure μ on T, where $\mu(v,k)$ is the proportion of agents with utility-priority profile (v,k) in the economy. Therefore, each of the sequence of replica economies can be represented by the same measure. We extend our use of the Prohorov metric ρ to measure the distance between measures on T,

$$\rho(\mu,\nu) = \inf\{\varepsilon > 0 | \nu(E) \leq \mu(B_\varepsilon(E)) + \varepsilon \text{ and } \mu(E) \leq \nu(B_\varepsilon(E)) + \varepsilon, \ \forall E \subset T\}.$$

Notice that the entire set of regular economies can be partitioned into open and disjoint subsets such that for every subset B there is a finite number $m > 0$ of continuous functions $\psi_1, ..., \psi_m$ from B to $[0,+\infty]^{S\times\bar{k}}$ such that the set of transformed PM price matrices $\Psi(\mu) = \{\psi_1(\mu), ..., \psi_m(\mu)\}$ for every $\mu \in B$. Indeed, consider an open ball of regular economies around each regular economy. Non-disjoint balls must have the same set of price functions. Taking a union of open sets with the same set of price functions gives us an open set with these price functions that is disjoint from regular economies with other price sets.

Let us set $\psi^{(n)}(\mu) = \psi(\mu) = \psi_1(\mu)$ for regular economies, and set both $\psi^{(n)}(\mu)$ and $\psi(\mu)$ to be an arbitrary price vector otherwise. By construction, this price function is continuous at every regular economy.

Take the n-replica regular economy $\Gamma^{(n)}$ that is represented by $\mu^{(n)}(=\mu)$. Suppose agent i submits a report u instead of v_i, and the resulting measure on utility profiles is $\mu_u^{(n)}$. By definition of the Prohorov metric, $\mu_u^{(n)}$ is close to $\mu^{(n)}$ in which everyone is truth-telling. For large enough n we have that $\mu_u^{(n)}$ is in the same price-function-ball as $\mu^{(n)} = \mu$. Since $\psi^{(n)}$ is continuous on each price-function ball, agent i can affect prices by only a small amount: given every $\varepsilon > 0$, for every n sufficiently large and for all u_i,

$$\left|\arctan\left(\psi^{(n)}\left(u_i, V_{-i}^{(n)}\right)\right) - \arctan\left(\psi^{(n)}\left(v_i, V_{-i}^{(n)}\right)\right)\right| < \varepsilon.$$

We therefore specify a price selection rule for PM; since agents' utilities are continuous in prices, Theorem 2 follows.

For completeness, we provide below a detailed analysis of the latter statement, including a useful technical lemma.

Let $\mathcal{P}_{u_i}^{(n)}$ denote the set of PM equilibrium prices when one copy of i reports u_i while all others reporting truthfully ($V_{-i}^{(n)}$) in $\Gamma^{(n)}$. Therefore, $\mathcal{P}_{v_i}^{(n)}$ is the set of equilibrium prices when everyone in $\Gamma^{(n)}$ is truth-telling, and $\cup_{u_i\in[0,1]^S}\mathcal{P}_{u_i}^{(n)}$ is the set of prices that i can obtain through unilateral manipulation of her reports. Furthermore, $\cup_{u_i\in[0,1]^S}\mathcal{P}_{u_i}^{(1)}$ is the set of obtainable prices associated with the base economy Γ. Similar to the lemma in Roberts and Postlewaite (1976), we have the following:

Lemma B1 *Given the sequence of replica economies, $\left\{\Gamma^{(n)}\right\}_{n\in\mathbb{N}}$, and a agent i,*

(i) $\cup_{u_i\in[0,1]^S}\mathcal{P}_{u_i}^{(n)}$ is closed for all n.

(ii) The sets of prices that i can obtain by unilateral manipulation in $\left\{\Gamma^{(n)}\right\}_{n\in\mathbb{N}}$ have a nesting structure: $\cup_{u_i\in[0,1]^S}\mathcal{P}_{u_i}^{(n)} \subseteq \cup_{u_i\in[0,1]^S}\mathcal{P}_{u_i}^{(n')}$ for all $n > n'$.

(iii) If $P \notin \mathcal{P}_{v_i}^{(1)}$, there exists n^* such that $n > n^*$ implies $P \notin \mathcal{P}_{u_i}^{(n)}$, and thus $\mathcal{P}_{v_i}^{(1)} = \cap_{n\in\mathbb{N}}\left(\cup_{u_i\in[0,1]^S}\mathcal{P}_{u_i}^{(n)}\right)$.

Proof of Lemma B1. We prove the lemma step by step.

(i) $\cup_{u_i\in[0,1]^S}\mathcal{P}^{(n)}_{u_i}$ is closed.

Consider a sequence of price matrices $P^{(m)} \to \bar{P}$, where $P^{(m)} \in \cup_{u_i\in[0,1]^S}\mathcal{P}^{(n)}_{u_i}$. That is, for each m, there is a sequence of $u_i^{(m)}$ such that $P^{(m)} \in \mathcal{P}^{(n)}_{u_i^{(m)}}$. Since $u_i^{(m)}$ is bounded, there must exist a convergent subsequence, which is also denoted as $u_i^{(m)} \to \bar{u}_i$. Besides, the corresponding subsequence of price matrices, still denoted as $P^{(m)}$, converges to $\bar{P}$. This implies:

$$\pi^{(m)}\left(u_i^{(m)}, P^{(m)}\right) + (n-1)\,\pi^{(m)}\left(v_i, P^{(m)}\right) + n\sum_{j\neq i}\pi^{(m)}\left(v_j, P^{(m)}\right) = nQ,$$

where $\pi^{(m)}\left(u_i, P^{(m)}\right)$ denotes an element in the set $\pi\left(u_i, P^{(m)}\right)$. Due to their boundedness, there is a subsequence of $\left\{\pi^{(m)}\left(u_i^{(m)}, P^{(m)}\right), \pi^{(m)}\left(v_i, P^{(m)}\right)\right\}$ that converges to $\{\bar{\pi}_{u_i}, \bar{\pi}_{v_i}\}$.

The maximum theorem implies that $\pi(u_i, P)$ is upper hemicontinuous in (u_i, P), and therefore $\bar{\pi}_{u_i} \in \pi\left(\bar{u}_i, \bar{P}\right)$, $\bar{\pi}_{v_i} \in \pi\left(v_i, \bar{P}\right)$, and $\bar{\pi}_{v_j} \in \pi\left(v_j, \bar{P}\right)$. The equality above leads to:

$$\bar{\pi}_{u_i} + (n-1)\,\bar{\pi}_{v_i} + n\sum_{j\neq i}\pi^{(m)}\bar{\pi}_{v_j} = nQ,$$

which proves that $\bar{P} \in \cup_{u_i\in[0,1]^S}\mathcal{P}^{(n)}_{u_i}$ and hence that $\cup_{u_i\in[0,1]^S}\mathcal{P}^{(n)}_{u_i}$ is closed.

(ii) The nesting structure of $\cup_{u_i\in[0,1]^S}\mathcal{P}^{(n)}_{u_i}$.

To simplify notations, in the following, let us assume that the demand correspondence $\pi(u_i, P)$ is single-valued for all i, all u_i, and all P. The proof can easily be extended to allow $\pi(u_i, P)$ to be set-valued.

$P \in \cup_{u_i\in[0,1]^S}\mathcal{P}^{(n)}_{u_i}$ means that there exists $u_i^{(n)}$ such that P clears the market given reports $\left(u_i^{(n)}, v_i\right)$:

$$\pi\left(u_i^{(n)}, P\right) + (n-1)\,\pi(v_i, P) + n\sum_{j\neq i}\pi(v_j, P) = nQ.$$

To have P as an equilibrium price in $\Gamma^{(n')}$, there has to exist some $u_i^{(n')} \in [0,1]^S$ such that:

$$\pi\left(u_i^{(n')}, P\right) + \left(n'-1\right)\pi(v_i, P) + n'\sum_{j\neq i}\pi(v_j, P) = n'Q.$$

Differencing the two equations and rearranging the terms lead to:

$$\pi\left(u_i^{(n')}, P\right) = \frac{n'}{n}\pi\left(u_i^{(n)}, P\right) + \frac{n-n'}{n}\pi(v_i, P).$$

Since $\pi\left(u_i^{(n)}, P\right)$ and $\pi(v_i, P)$ are affordable to i, the convex combination of the two must be affordable to i. Therefore, there must exist some $u_i^{(n')}$ such that the above equation is satisfied.

(iii) $\mathcal{P}^{(1)}_{v_i} = \cap_{n\in\mathbb{N}}\left(\cup_{u_i\in[0,1]^S}\mathcal{P}^{(n)}_{u_i}\right)$

It is straightforward to verify that $\mathcal{P}^{(1)}_{v_i} \subseteq \mathcal{P}^{(n)}_{v_i} \subseteq \cup_{u_i\in[0,1]^S}\mathcal{P}^{(n)}_{u_i}$ for all n. We then show that for any $P \notin \mathcal{P}^{(1)}_{v_i}$, there exists n^* such that $n > n^*$ implies $P \notin \mathcal{P}^{(n)}_{u_i}$.

Suppose that $P \notin \mathcal{P}^{(1)}_{v_i}$ but the statement in the lemma is false. The nesting structure implies that $P \in \cup_{u_i \in [0,1]^S} \mathcal{P}_{u_i}^{(n)}$, for all n. Therefore, there exists a sequence of reports by the given copy of agent i, $\left\{u_i^{(n)}\right\}_{n \in \mathbb{N}}$, such that the market clears at P:

$$\pi\left(u_i^{(n)}, P\right) + (n-1)\,\pi\left(v_i, P\right) + n\sum_{j \neq i} \pi\left(v_j, P\right) = nQ.$$

Rearranging the above equation yields:

$$\pi\left(u_i^{(n)}, P\right) - \pi\left(v_i, P\right) = n\left(Q - \sum_{j \in \mathcal{I}} \pi\left(v_j, P\right)\right),$$

where the left-hand-side term is bounded due to the unit demand constraint. Moreover, $P \notin \mathcal{P}_{v_i}^{(1)}$ implies $Q - \sum_{j \in \mathcal{I}} \pi\left(v_j, P\right) \neq 0$, which means the right-hand-side of the equation diverges when n increases. Therefore, there must exist $\bar{n}$ such that the above equation cannot be satisfied for $n > \bar{n}$. This contradiction proves the lemma. ■

We are now ready to finish the proof of our main incentive compatibility theorem.

Proof of Theorem 2. Suppose that for a copy of agent type i, also denoted as i, there exists a subsequence of replica economies $\left\{\Gamma^{(n_m)}\right\}_{n_m \in \mathbb{N}}$ where she gains at least ε by unilateral misreporting. Let $P^{(m)} = \psi^{(n_m)}\left(u_i^{(n_m)}, V_{-i}^{(n_m)}; K^{(n)}\right)$ where $P^{(m)}$ is the price matrix with which PM implements the assignment in economy $\Gamma^{(n_m)}$ after i's unilateral manipulation. Since $\left\{\arctan\left(P^{(m)}\right)\right\}_{n_m \in \mathbb{N}}$ is bounded, there is a subsequence (also denoted as $\left\{\arctan\left(P^{(m)}\right)\right\}_{n_m \in \mathbb{N}}$) converging to some $\arctan\left(\bar{P}\right)$. Because $\cup_{u_i \in [0,1]^S} \mathcal{P}_{u_i^{(n_m)}}^{(n_m)}$ and thus $\arctan\left(\cup_{u_i \in [0,1]^S} \mathcal{P}_{u_i^{(n_m)}}^{(n_m)}\right)$ are closed (Lemma B1), we have:

$$\arctan\left(\bar{P}\right) \in \arctan\left(\cap_{n \in \mathbb{N}}\left(\cup_{u_i \in [0,1]^S} \mathcal{P}_{u_i^{(n_m)}}^{(n_m)}\right)\right) = \arctan\left(\mathcal{P}_{v_i}^{(1)}\right),$$

which, together with the continuity of $\psi^{(n_m)}\,(=\psi)$ as shown at the beginning of this subsection, further implies $\bar{P} = \psi\left(V, K\right)$. In other words, $\bar{P}$ is the PM equilibrium price matrix in Γ selected by ψ when everyone is truth-telling.

We define the indirect utility function $W_{u_i}\left(P\right)$ as the expected utility (with respect to true preferences v_i) that i can obtain when reporting u_i given price P. By the maximum theorem, i's utility maximization problem implies that $W_{u_i}\left(P\right)$ is continuous in P. Moreover, the utility from manipulation, $W_{u_i}\left(P^{(m)}\right)$, is always bounded above by $W_{v_i}\left(P^{(m)}\right)$, and $W_{v_i}\left(P^{(m)}\right)$ goes to $W_{v_i}\left(\bar{P}\right)$ when m goes to infinite. Therefore, the (sub)sequence of $W_{u_i}\left(P^{(m)}\right)$ is bounded above by the utility from truth-telling:

$$\limsup_{m \to \infty} W_{u_i}\left(P^{(m)}\right) \leq \limsup_{m \to \infty} W_{v_i}\left(P^{(m)}\right) = W_{v_i}\left(\bar{P}\right).$$

This contradiction proves that the statement in the theorem is true for a given copy of i.

To prove the statement holds true for each copy of each agent type, we note that there is a finite number of agent types in $\Gamma^{(n)}$. There thus must exist n^* such that $n > n^*$ implies that the utility gain from unilateral misreporting for any agent is uniformly bounded by ε given that everyone else is truth-telling. ■

B.3 Other Proofs

Proof of Proposition 1. Given an *ex ante* stable assignment Π, $\forall s \in \mathcal{S}$, all the priority groups belong to one of the three categories:

(a) cut-off group, i.e., $k^*(s)$ such that $\sum_{j\in\mathcal{I},\ k_{s,j}<k^*(s)} \pi_{j,s} < q_s$, $\sum_{j\in\mathcal{I},\ k_{s,j}\leq k^*(s)} \pi_{j,s} = q_s$, and $\sum_{j\in\mathcal{I},\ k_{s,j}>k^*(s)} \pi_{j,s} = 0$;

(b) groups that have higher priority than $k^*(s)$ at s in Π, i.e., a set $\overline{\mathcal{K}}_s \subset \mathcal{K}$ such that $k \in \overline{\mathcal{K}}_s$ iff $k < k^*(s)$;

(c) groups that have lower priority than $k^*(s)$ at s in Π, i.e., a set $\underline{\mathcal{K}}_s \subset \mathcal{K}$ such that $k \in \underline{\mathcal{K}}_s$ iff $k > k^*(s)$.

Note that $k^*(s)$ always exists and is unique for all s and for any given Π, while $\overline{\mathcal{K}}_s$ or $\underline{\mathcal{K}}_s$ may be empty. As long as there are at least two priority groups, $\overline{\mathcal{K}}_s = \varnothing$ implies $\underline{\mathcal{K}}_s \neq \varnothing$, and vice versa.

(i) We first show that every Π in Π_Γ^{Stable} is *ex ante* stable.

If $P \in \mathcal{P}_\Gamma^{Stable}$, then

$$p_{i,s} = \begin{cases} 0 & \text{if } k_{s,i} \in \overline{\mathcal{K}}_s \\ \in [0,+\infty] & \text{if } k_{s,i} = k^*(s) \\ +\infty & \text{if } k_{s,i} \in \underline{\mathcal{K}}_s \end{cases}$$

Fix $\Pi \in \Pi_\Gamma(P)$ for some $P \in \mathcal{P}_\Gamma^{Stable}$. $\forall i,j \in \mathcal{I}$, $\forall s,s' \in \mathcal{S}$ such that $v_{i,s} > v_{i,s'}$ and $k_{s,i} < k_{s,j}$, if $\pi_{j,s} > 0$, we must have $\pi_{i,s'} = 0$ since $p_{i,s} = 0$ according to the definition of $\mathcal{P}_\Gamma^{Stable}$. Equivalently, $\pi_{i,s'} > 0$ is not optimal for i facing $(p_{i,1}, \cdots, p_{i,S})$, which proves every Π in Π_Γ^{Stable} is *ex ante* stable.

Proof of Theorem 3. By Proposition 1 as well as Corollary 1, PM assignments are *ex ante* stable. Moreover, by the definition of equal claim among *ex ante* stable assignment, PM assignments also satisfy equal claim.

For any given assignment that satisfies *ex ante* stability and equal claim, Proposition 1 and the definition of equal claim imply that the assignment can be rationalized by prices that satisfy the PM construction. Therefore, the assignment is also an equilibrium outcome of PM. ■

Proof of Theorem 4. Given a stable matching, for each object s, we may find $k^*(s) = \max_{i\in\{j\in\mathcal{S}|j \text{ is matched with } s\}}\{k_{s,i}\}$, which is the lowest priority group of s among those who are matched with s. We may then define the following price system:

$$p_{s,k_{s,i}} = \begin{cases} 0, \text{ if } k_{s,i} \leq k^*(s) \\ +\infty, \text{ if } k_{s,i} > k^*(s) \end{cases}, \forall s.$$

This price system satisfies the requirement of the PM mechanism. We need to show that agents maximize their expected utility given the prices.

The only possible deviation for an agent i is to choose some object s which is free to her. That is, she is in a higher priority group of s than someone who is already accepted by s. If this deviation is profitable to i, (i,s) forms a blocking pair (or i has justified envy at s). By the definition of stability, there is no such pair. This proves that every stable matching is an equilibrium assignment of the PM mechanism.

Similarly, for any PM assignment, there exists a corresponding price matrix that guarantees that prices are either zero or infinite, which implies that the assignment is deterministic. For deterministic assignments, *ex ante* stability is equivalent to stability, and Theorem 3 implies that PM assignments in this case are stable. ■

Proof of Proposition 2. Let P^* be an equilibrium price matrix of PM.

Suppose that $s_{i,1}$ is the non-free object (according to her own price) on which agent i spends her budget, and that $s_{i,2}$ is her most preferred object among all free ones. By assumption, $1 < p^*_{s_{i,1},k_{s_{i,1},i}} < +\infty$. Since each agent has strict preferences over objects, $s_{i,2}$ is unique and $v_{i,s_{i,1}} > v_{i,s_{i,2}}$. By assumption, if i's consumption includes a positive probability share of $s_{i,2}$, $s_{i,2}$ must be free to everyone. i's random assignment $\left\{\pi^*_{i,s}\right\}_{s\in\mathcal{S}}$ must be such that:

$$\pi^*_{i,s_{i,1}} = 1/p^*_{s_{i,1},k_{s_{i,1},i}},\ \pi^*_{i,s_{i,2}} = 1 - \pi^*_{i,s_{i,1}},\ \text{and}\ \pi^*_{i,s} = 0,\ \forall s \neq s_{i,1}, \neq s_{i,2};$$

Alternatively, if i does not spend any budget on any non-free objects,

$$\pi^*_{i,s'_{i,2}} = 1,\ \text{and}\ \pi^*_{i,s} = 0\ \forall s \neq s'_{i,2}.$$

Note that such $s'_{i,2}$ may or may not be free to every agent.

Consider that agent i's submitted rank-order list in BM is $L^*_i = (s_{i,1}, s_{i,2})$ if i spends some of her budget or $L^*_i = \left(s'_{i,2}\right)$ if she does not spend any budget at all. It can be verified that given these rank-order lists, BM clears the market in two rounds and delivers the same random assignment as the PM mechanism. The only thing left to check is that this is a Nash equilibrium.

(i) If $L^*_i = \left(s'_{i,2}\right)$, suppose there exists s' s.t. $v_{i,s'} > v_{i,s'_{i,2}}$. If not, there is no profitable deviation for i, as she is matched with her most preferred object already. If i ranks s' above $s'_{i,2}$, she cannot be matched with s', because all those top ranking s' must be in a higher priority group of s'. Otherwise, s' would cost i a finite amount, which would allow her to purchase some shares in s' under the PM mechanism. Certainly, ranking s' below $s'_{i,2}$ does not change the assignment. Similarly, i cannot benefit by ranking objects less preferable than $s'_{i,2}$ in her list.

(ii) Now suppose $L^*_i = (s_{i,1}, s_{i,2})$ and L'_i is a profitable deviation for i. Given the assumptions, we have the following results:

(a) Object $s_{i,1}$ is not available after the first round of BM;

(b) i may obtain a positive share of s' by ranking it first if in PM $p_{s',k_{s',i}} < +\infty$ (i.e., i's priority at object s' is at least as high as the cut-off group).

(c) Only objects available in the second round and rounds later are those ranked as second choice by some agents. In other words, they are those who have zero prices for everyone in the PM mechanism.

Therefore, if L'_i still has $s_{i,1}$ as her first choice, it cannot be profitable, because she can get $1/p^*_{s_{i,1},k_{s_{i,1},i}}$ of $s_{i,1}$ and at best $1 - 1/p^*_{s_{i,1},k_{s_{i,1},i}}$ shares of $s_{i,2}$.

If L'_i has s' ($s' \neq s_{i,1}$) as her first choice, to be profitable, $v_{i,s'} > v_{i,s_{i,2}}$ and s' cannot be of zero price or infinite price to i in PM. If s' is of zero price to i, i could have obtained s' instead of $s_{i,2}$ in PM; if s' is of infinite price to i, i could never obtain any shares of s'. i thus must be in the cut-off priority group of s'. Given L^*_{-i} and the rules of BM, by ranking s' as first choice, i can obtain:

$$\pi'_{i,s'} = \frac{q_{s'} - \sum_{j\in\left\{j\in\mathcal{I}:k_{s',j}<k_{s',i}\right\}} \pi^*_{j,s'}}{p^*_{s',k_{s',i}}\left(q_{s'} - \sum_{j\in\left\{j\in\mathcal{I}:k_{s',j}<k_{s',i}\right\}} \pi^*_{j,s'}\right) + 1},$$

where $q_{s'} - \sum_{j \in \left\{j \in \mathcal{I}: k_{s',j} < k_{s',i}\right\}} \pi^*_{j,s'}$ is the remaining quota at s' after those who are in higher priority groups claim their shares; and $p^*_{s',k_{s',i}} \left(q_{s'} - \sum_{j \in \left\{j \in \mathcal{I}: k_{s',j} < k_{s',i}\right\}} \pi^*_{j,s'}\right)$ is the total expenditure on s' by agents who are in the same priority group of s' as i; and more importantly it is the total number of such agents other than i who have ranked s' as first choice given L^*_{-i}. This is because everyone spends her budget on at most one object and $p^*_{s',k_{s',i}} > 1$ by assumption. This lead to $p^*_{s',k_{s',i}} \pi'_{i,s'} < 1$, which implies that $\pi'_{i,s'}$ is affordable to i in PM.

Moreover, given L^*_{-i} and any L'_i, besides the first-choice object (s'), i can only obtain some shares in objects that are free to everyone in the PM. Therefore, the assignment resulting from a potentially deviation is still affordable to i in PM, which implies that it cannot be profitable.

This complete the proof that $\left(L^*_i, L^*_{-i}\right)$ is a Bayesian Nash equilibrium in BM. ■

(ii) We show that if $\Pi \in \mathcal{A}$ is *ex ante* stable, then $\exists P^* \in \mathcal{P}^{Stable}_{\Gamma}$ such that $\Pi \in \Pi_{\Gamma}(P^*)$. It suffices to show that $\forall i \in \mathcal{I}$, $[\pi_{i,s}]_{s \in \mathcal{S}}$ is the optimal choice facing $\left[p^*_{i,s}\right]_{s \in \mathcal{S}}$ and $\left[p^*_{i,s}\right]_{s \in \mathcal{S}}$ is in $\mathcal{P}^{Stable}_{\Gamma}$.

Given Π, we can still define three sets of priorities, $\overline{\mathcal{K}}_s$, $\{k^*(s)\}$, and $\underline{\mathcal{K}}_s$. Across agents, the only restriction on prices in $\mathcal{P}^{Stable}_{\Gamma}$ is that prices for agents with priorities in $\overline{\mathcal{K}}_s \cup \underline{\mathcal{K}}_s$ and not in cut-off groups must be the same (either zero or infinite). An immediate finding is that $\forall k_{s,i} \in \underline{\mathcal{K}}_s$, we can set $p^*_{i,s} = +\infty$ since $\pi_{i,s} = 0$ for all such i and s, which satisfies the property of $\mathcal{P}^{Stable}_{\Gamma}$.

Given Π, we can further group the objects into three distinct sets for agent i, $\mathcal{S} = \underline{\mathcal{S}}_i \cup \mathcal{S}^c_i \cup \overline{\mathcal{S}}_i$:

$$\underline{\mathcal{S}}_i = \{s \in \mathcal{S} | k_{s,i} \in \underline{\mathcal{K}}_s\}; \mathcal{S}^c_i = \{s \in \mathcal{S} | k_{s,i} = k^*(s)\}; \overline{\mathcal{S}}_i = \left\{s \in \mathcal{S} | k_{s,i} \in \overline{\mathcal{K}}_s\right\}.$$

Also note that $\forall i \in \mathcal{I}$, $\mathcal{S} \backslash \underline{\mathcal{S}}_i \neq \varnothing$, and we consider the following possibilities:

(a) $\mathcal{S}^c_i = \varnothing$: The *ex ante* stability implies that i is matched with her most-preferred object within $\mathcal{S} \backslash \underline{\mathcal{S}}_i = \overline{\mathcal{S}}_i$ with probability 1, thus $p^*_{i,s} = 0$ $\forall s \in \mathcal{S} \backslash \underline{\mathcal{S}}_i = \overline{\mathcal{S}}_i$ supports this assignment as a utility-maximization outcome and satisfies the properties of $\mathcal{P}^{Stable}_{\Gamma}$.

(b) $\overline{\mathcal{S}}_i = \varnothing$: This implies that $\mathcal{S} \backslash \underline{\mathcal{S}}_i = \mathcal{S}^c_i$. By adjusting the prices of objects in $\mathcal{S}^c_i$, one can make $[\pi_{i,s}]_{s \in \mathcal{S}}$ an optimal choice of i. This is feasible because there are no restrictions on prices of objects in $\mathcal{S}^c_i$.

(c) $\mathcal{S}^c_i \neq \varnothing$ and $\overline{\mathcal{S}}_i \neq \varnothing$: We denote the most-preferred object within $\overline{\mathcal{S}}_i$ for i as $\overline{s}_i$, then the *ex ante* stability implies that $\forall s \in \mathcal{S} \backslash \underline{\mathcal{S}}_i$, $\pi_{i,s} = 0$ if $v_{i,\overline{s}_i} > v_{i,s}$. Let us set $p^*_{i,s} = 0$ for $\forall s \in \overline{\mathcal{S}}_i$, which satisfies the properties of $\mathcal{P}^{Stable}_{\Gamma}$.

Denote $\mathcal{S}^c_i(\overline{s}_i) \equiv \{s \in \mathcal{S}^c_i | v_{i,s} \geq v_{i,\overline{s}_i}\}$. If $\pi_{i,\overline{s}_i} = 0$, i must only consume objects in $\mathcal{S}^c_i(\overline{s}_i)$. Given zero prices for all objects in $\overline{\mathcal{S}}_i$ and infinite prices for objects in $\underline{\mathcal{S}}_i$, one can find a vector of personalized prices for all objects in $\mathcal{S}^c_i(\overline{s}_i)$ to make $[\pi_{i,s}]_{s \in \mathcal{S}}$ i's optimal choice. Note that this can be done independently for all agents. If instead $\pi_{i,\overline{s}_i} > 0$, it implies that i only consumes objects in $\{\overline{s}_i\} \cup \mathcal{S}^c_i$. Similarly, one can find a price vector for objects in $\mathcal{S}^c_i(\overline{s}_i)$ to make $[\pi_{i,s}]_{s \in \mathcal{S}}$ i's optimal choice.

This proves that there always exists a price matrix $P^* \in \mathcal{P}^{Stable}_{\Gamma}$ such that each $\left[p^*_{i,s}\right]_{s \in \mathcal{S}}$ supports $[\pi_{i,s}]_{s \in \mathcal{S}}$ if Π is *ex ante* stable. ■

Proof of Theorem 5. We define the following rules regarding infinity:

$$0 * +\infty = 0; +\infty \geq +\infty.$$

Suppose a PM random assignment, $\left[\pi^*_{i,s}\right]_{i \in \mathcal{I}, s \in \mathcal{S}}$, is *ex ante Pareto dominated* by another random assignment $[\pi_{i,s}]_{i\in\mathcal{I},s\in\mathcal{S}}$, i.e.,

$$\begin{aligned}\sum_{s\in\mathcal{S}}\pi_{i,s}v_{i,s} &\geq \sum_{s\in\mathcal{S}}\pi^*_{i,s}v_{i,s},\forall i\in\mathcal{I}, &(1)\\ \sum_{i\in\{k_{s,i}\leq k\}}\pi_{i,s} &\geq \sum_{i\in\{k_{s,i}\leq k\}}\pi^*_{i,s},\ \forall s\in\mathcal{S},\forall k\in\mathcal{K}, &(2)\end{aligned}$$

and at least one inequality is strict.

For any agent whose most preferred object is free or has the associated price no more than one, she obtains that object for sure, and there is no other assignment that makes her better off. If for agent i, $\sum_{s\in\mathcal{S}}\pi_{i,s}v_{i,s} > \sum_{s\in\mathcal{S}}\pi^*_{i,s}v_{i,s}$, it must be such that $\sum_{s\in\mathcal{S}}p_{s,k_{s,i}}\pi_{i,s} > 1$ and $\sum_{s\in\mathcal{S}}\pi^*_{i,s}p_{s,k_{s,i}} = 1$. Otherwise $\left[\pi^*_{i,s}\right]_{s\in\mathcal{S}}$ would not be optimal for i.

Moreover, for agents other than i who do not obtain their most preferred objects, it must be that $\sum_{s\in\mathcal{S}}p_{s,k_{s,j}}\pi_{j,s} \geq \sum_{s\in\mathcal{S}}p_{s,k_{j,s}}\pi^*_{j,s}$, since $\left[\pi^*_{j,s}\right]_{s\in\mathcal{S}}$ is the cheapest among bundles delivering the same expected utility. Therefore,

$$\sum_{s\in\mathcal{S}}p_{s,k_{s,i}}\pi_{i,s}+\sum_{j\neq i}\sum_{s\in\mathcal{S}}p_{s,k_{s,j}}\pi_{j,s} > \sum_{s\in\mathcal{S}}p_{s,k_{s,i}}\pi^*_{i,s}+\sum_{j\neq i}\sum_{s\in\mathcal{S}}p_{s,k_{s,j}}\pi^*_{j,s}.$$

However, because prices are higher for agents in lower priority groups, equation (2) implies that:

$$\sum_{j\in\mathcal{I}}\sum_{s\in\mathcal{S}}p_{s,k_{s,j}}\pi_{j,s} \leq \sum_{j\in\mathcal{I}}\sum_{s\in\mathcal{S}}p_{s,k_{s,j}}\pi^*_{j,s},$$

which leads to a contradiction.

Suppose instead that for object s, equation (2) is satisfied for all k, and $\exists\underline{k}\in\left\{1,...,\overline{k}-1\right\}$, such that

$$\sum_{i\in\{k_{s,i}\leq\underline{k}\}}\pi_{i,s} > \sum_{i\in\{k_{s,i}\leq\underline{k}\}}\pi^*_{i,s}.$$

This implies,

$$\sum_{j\in\mathcal{I}}p_{s,k_{s,j}}\pi_{j,s} < \sum_{j\in\mathcal{I}}p_{s,k_{s,j}}\pi^*_{j,s},$$

again because prices are higher for agents in lower priority group. Aggregating over all objects,

$$\sum_{j\in\mathcal{I}}\sum_{s\in\mathcal{S}}p_{s,k_{s,j}}\pi_{j,s} < \sum_{j\in\mathcal{I}}\sum_{s\in\mathcal{S}}p_{s,k_{s,j}}\pi^*_{j,s}.$$

However, based on the same arguments as above, equation (1) implies that $\sum_{s\in\mathcal{S}}p_{s,k_{s,j}}\pi_{j,s} \geq \sum_{s\in\mathcal{S}}p_{s,k_{s,j}}\pi^*_{j,s}$, $\forall j\in\mathcal{I}$, and thus,

$$\sum_{j\in\mathcal{I}}\sum_{s\in\mathcal{S}}p_{s,k_{s,j}}\pi_{j,s} \geq \sum_{j\in\mathcal{I}}\sum_{s\in\mathcal{S}}p_{s,k_{s,j}}\pi^*_{j,s}.$$

This leads to another contradiction.

Therefore, $\left[\pi^*_{i,s}\right]_{i\in\mathcal{I},s\in\mathcal{S}}$, must be two-sided *ex ante efficient*. ■

C Limiting Individual Incentive Compatibility of PM

This appendix proves that PM satisfies the concept of limiting individual incentive compatibility as in Roberts and Postlewaite (1976).

Definition C1 *Let $\left\{\Gamma^{(n)}\right\}_{n\in\mathbb{N}}$ be a sequence of economies and let i be an agent in each $\Gamma^{(n)}$. A mechanism is limiting individually incentive compatible for i in $\left\{\Gamma^{(n)}\right\}_{n\in\mathbb{N}}$ if for any ε there exists n^* such that $n > n^*$ implies that for each π_i attainable by i in $\Gamma^{(n)}$ there exists a competitive assignment π_i^* to i in $\Gamma^{(n)}$ (everyone is truth-telling) such that $\sum_{s\in\mathcal{S}}\pi_{i,s}^*v_{i,s} > \sum_{s\in\mathcal{S}}\pi_{i,s}v_{i,s} - \varepsilon$.*

Therefore, this concept focuses on the incentive for an individual agent to misreport while everyone else is truth-telling. In particular, it does not require a price selection rule, because only the existence of such a truth-telling equilibrium is required. The following shows that PM satisfies this property in a sequence of economies.

C.1 Sequence of Economies

We first define per capita demand functions and take into account that agents in different priority groups face different prices, and thus the per capita demand is priority-specific. Let $F_i(P)$ be the augmented set of feasible consumption bundles for agent i,

$$F_i(P) \equiv \begin{cases} \left\{\pi_i = [\pi_{i,s}]_{s\in\mathcal{S}} \;\middle|\; \begin{array}{c} \pi_{i,s} \geq 0,\, \forall s,\ \sum_{s\in\mathcal{S}}\pi_{i,s} = 1, \\ \text{and } \sum_{s\in\mathcal{S}}\pi_{i,s}p_{s,k_{s,i}} \leq 1 \end{array}\right\}, \text{ if } p_{s,k_{s,i}} \leq 1 \text{ for some } s; \\ \left\{\pi_i = [\pi_{i,s}]_{s\in\mathcal{S}} \;\middle|\; \begin{array}{c} \pi_{i,s} \geq 0,\, \forall s,\ \sum_{s\in\mathcal{S}}\pi_{i,s} = \frac{1}{\min_{t=1,\ldots,S}\left\{p_{t,k_{t,i}}\right\}}, \\ \text{and } \sum_{s\in\mathcal{S}}\pi_{i,s}p_{s,k_{s,i}} \leq 1 \end{array}\right\}, \text{ if } p_{s,k_{s,i}} > 1,\, \forall s. \end{cases}$$

When there are no affordable bundles such that $\sum_{s\in\mathcal{S}}\pi_{i,s} = 1$, the second part of the definition assumes that every agent is allowed to spend all their money on the cheapest objects. $F_i(P)$ is then non-empty, closed, and bounded.[27]

Let $U_i = \sum_{s\in\mathcal{S}}\pi_{i,s}v_{i,s}$ be i's expected utility function. Define $G_i(P, v_i)$ as the set of bundles that i would choose from $F_i(P)$ to maximize U_i. Formally,

$$G_i(P, v_i) = \left\{\pi_i \in F_i(P) \;\middle|\; \begin{array}{c} \forall \pi_i' \in F_i(P),\, U_i(\pi_i) > U_i(\pi_i'), \\ \text{or } U_i(\pi_i) \geq U_i(\pi_i') \text{ and } \sum_{s\in\mathcal{S}}\pi_{i,s}p_s \leq \sum_{s\in\mathcal{S}}\pi_{i,s}'p_s \end{array}\right\}.$$

Since $G_i(P, v_i)$ is obtained from the closed, bounded, and non-empty set $F_i(P)$ by maximizing (and minimizing) continuous functions, $G_i(P, v_i)$ must be non-empty. $G_i(P, v_i)$ is a convex set, because $U_i(\pi_i)$ and $\sum_{s\in\mathcal{S}}\pi_{i,s}p_{s,k_{s,i}}$ are linear functions of π_i. Define $G(P, v)$ as the set of per capita demand for each priority group for each object that can emerge when prices equal P and each agent i chooses a vector in $G_i(P, v_i)$, that is, $\forall P \in \mathcal{P}$:

$$G(P, V) = \left\{D = [d_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}} \;\middle|\; \begin{array}{c} d_{s,k} = \frac{1}{|\mathcal{I}|}\sum_{\{i\in\mathcal{I}|k_{s,i}=k\}}\pi_{i,s},\, \forall s, \forall k \\ [\pi_{i,s}]_{s\in\mathcal{S}} \in G_i(P, v_i) \end{array}\right\}.$$

It can be verified that $G(P, V)$ is also closed, bounded, and upper hemicontinuous.

The following definition is needed to define the sequence of economies.

[27] It is important to note that P cannot be an equilibrium whenever the second part of $F_i(P)$'s definition is invoked.

Definition C2 *A sequence of correspondences $f^{(n)}(P)$ uniformly converge to $f(P)$ if and only if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that when $n \geq N$,*

$$\sup_P d_H\left(f^{(n)}(P), f(P)\right) \leq \varepsilon,$$

where d_H is Hausdorff distance, i.e.,

$$d_H\left(f^{(n)}(P), f(P)\right) = \max\left\{ \begin{array}{c} \sup_{Y\in f(P)} \inf_{Y^{(n)}\in f^{(n)}(P)} \left\|Y^{(n)} - Y\right\|, \\ \sup_{Y^{(n)}\in f^{(n)}(P)} \inf_{Y\in f(P)} \left\|Y^{(n)} - Y\right\| \end{array} \right\},$$

where $\|\cdot\|$ is the Euclidean distance.

Let $\left\{\Gamma^{(n)}\right\}_{n\in\mathbb{N}}$ be a sequence of economies where $\Gamma^{(n)} = \left\{\mathcal{S}, \mathcal{I}^{(n)}, Q^{(n)}, V^{(n)}, K^{(n)}\right\}$ and $\forall n \in \mathbb{N}$:

(i) $\mathcal{I}^{(n)} \subset \mathcal{I}^{(n')}$ and $q_s^{(n)} < q_s^{(n')}$ for all s if $n < n'$; $\left|\mathcal{I}^{(n)}\right| = \sum_{s\in\mathcal{S}} q_s^{(n)}$; and $q_s^{(n)} / \left|\mathcal{I}^{(n)}\right| = q_s/I$;

(ii) $K^{(n)}$ is such that the associated priority groups satisfy $\left|\left\{i \in \mathcal{I}^{(n)} | k_{s,i} = k\right\}\right| / \left|\mathcal{I}^{(n)}\right| = C_{s,k}$, for all k and s, where $C_{s,k}$ is a constant.

(iii) the number of objects, $S = |\mathcal{S}|$, is constant;

(iv) the corresponding per capita demand $G^{(n)}\left(P, V_{-i}^{(n)}\right) \to g(P)$ uniformly as $n \to \infty$.

Remark C1 *Analogous to the regularity imposed in the main text, the above restrictions on the sequence of economies can also be interpreted as regular conditions.*

Remark C2 *$g(P)$ is a convex-valued, closed, bounded, and upper hemicontinuous correspondence, since $G^{(n)}\left(P, V^{(n)}\right)$ has these properties. This definition includes two special cases: (i) a sequence of replica economies where $G^{(n)}\left(P, V^{(n)}\right) = g(P)$, for all $n \in \mathbb{N}$; and (ii) a sequence of economies in which agents' preference-priority profiles are i.i.d. drawn from a joint distribution of preferences and priorities, while holding constant the relative size of each priority group at each object.*

C.2 Results and Proofs

We first present a result on the set of equilibrium prices and then another on the limiting incentive compatibility.

Lemma C2 *In the sequence of economies $\left\{\Gamma^{(n)}\right\}_{n\in\mathbb{N}}$, let $\mathcal{P}_{u_i}^{(n)} \subset [0, +\infty]^{S\times\overline{k}}$ be the set of PM equilibrium prices given $\left(u_i, V_{-i}^{(n)}\right)$. Then $\lim_{n\to\infty} d_H\left(\mathcal{P}_{v_i}^{(n)}, \mathcal{P}_{u_i}^{(n)}\right) = 0$, $\forall u_i \in [0,1]^S$, for any i in all $\mathcal{I}^{(n)}$.*

Proof. This is proven by the following three steps.

(1) Misreporting cannot affect per capita demand by priority groups in the limit.

First, recall that per capita demand of each priority group at each object is $G(P, v)$ for $P \in [0, +\infty]^{S\times\overline{k}} \equiv \mathcal{P}$ and v is the tuple of all agents' preferences.

Since each agent can increase or decrease the total demand of a priority group of an object at most by one copy, $\forall \left[d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \in G^{(n)}\left(P,\left(u_i,V_{-i}^{(n)}\right)\right)$, there must exist $\left[d'_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \in G^{(n)}\left(P,\left(v_i,V_{-i}^{(n)}\right)\right)$, such that, $\forall s$, $\forall k$,

$$d'_{s,k} - \frac{1}{\left|\mathcal{I}^{(n)}\right|} \leq d_{s,k} \leq d'_{s,k} + \frac{1}{\left|\mathcal{I}^{(n)}\right|}.$$

Similarly, $\forall \left[d'_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \in G^{(n)}\left(P,\left(v_i,V_{-i}^{(n)}\right)\right)$, there exists $\left[d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \in G^{(n)}\left(P,\left(u_i,V_{-i}^{(n)}\right)\right)$, such that $\forall s$, $\forall k$,

$$d_{s,k} - \frac{1}{\left|\mathcal{I}^{(n)}\right|} \leq d'_{s,k} \leq d_{s,k} + \frac{1}{\left|\mathcal{I}^{(n)}\right|}.$$

Therefore, given any P,

$$\sup_{u_i\in[0,1]^S} d_H\left(G^{(n)}\left(P,\left(u_i,V_{-i}^{(n)}\right)\right), G^{(n)}\left(P,\left(v_i,V_{-i}^{(n)}\right)\right)\right) \leq \frac{\sqrt{S\bar{k}}}{\left|\mathcal{I}^{(n)}\right|},$$

which implies that, given any P,

$$\lim_{n\to\infty}\sup_{u_i\in[0,1]^S} d_H\left(G^{(n)}\left(P,\left(u_i,V_{-i}^{(n)}\right)\right), G^{(n)}\left(P,\left(v_i,V_{-i}^{(n)}\right)\right)\right) = 0. \tag{3}$$

By definition, $G^{(n)}\left(P,\left(v_i,V_{-i}^{(n)}\right)\right) \to g(P)$ uniformly. Therefore, Equation (3) implies that $G^{(n)}\left(P,\left(u_i,V_{-i}^{(n)}\right)\right)$ converges to $g(P)$ uniformly as $n\to\infty$.

(2) Price Adjustment Process

Similar to the proof for Theorem 1, define $Z \equiv \left[z_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \in [0,\pi/2]^{S\times\bar{k}} \equiv \mathcal{Z}$, where $z_{s,k} = \arctan\left(p_{s,k}\right)$, $\forall s$, $\forall k$.

A price adjustment process for $\Gamma^{(n)}$ is defined as,

$$\begin{aligned} & H\left[Z, G^{(n)}\left(\mathcal{TAN}(Z),\left(v_i,V_{-i}^{(n)}\right)\right)\right] \\ \equiv & \left\{ Y = \left[y_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \;\middle|\; \begin{array}{c} y_{s,k}\left(\left[d_{s,k}\right]_{k\in\mathcal{K}}\right) = \min\left\{\pi/2, \max\left[0, z_{s,k} + \left(\sum_{\kappa=1}^{k} d_{s,\kappa} - q_s/I\right)\right]\right\} \\ \forall \left[d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \in G^{(n)}\left(\mathcal{TAN}(Z),\left(v_i,V_{-i}^{(n)}\right)\right) \end{array} \right\}, \end{aligned}$$

where, $\mathcal{TAN}(Z) \equiv \left[\tan\left(z_{s,k}\right)\right]_{s\in\mathcal{S},k\in\mathcal{K}}$. It is straightforward to verify that the correspondence H is a mapping from $\mathcal{Z}$ to $\mathcal{Z}$, given $\left(v_i,V_{-i}^{(n)}\right)$. Similarly,

$$\begin{aligned} & H\left[Z, g\left(\mathcal{TAN}(Z)\right)\right] \\ \equiv & \left\{ Y = \left[y_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \;\middle|\; \begin{array}{c} y_{s,k}\left(\left[d_{s,k}\right]_{k\in\mathcal{K}}\right) = \min\left\{\pi/2, \max\left[0, z_{s,k} + \left(\sum_{\kappa=1}^{k} d_{s,\kappa} - q_s/I\right)\right]\right\} \\ \forall \left[d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \in g\left(\mathcal{TAN}(Z)\right); \end{array} \right\}. \end{aligned}$$

Claim: $H\left[Z, G^{(n)}\left(\mathcal{TAN}(Z),\left(v_i,V_{-i}^{(n)}\right)\right)\right] \to H\left[Z, g\left(\mathcal{TAN}(Z)\right)\right]$ uniformly as $n\to\infty$.

The uniform convergence of $G^{(n)}\left(P,\left(v_i,V_{-i}^{(n)}\right)\right)$ to $g(P)$ means that $\forall\varepsilon>0$, $\exists N\in\mathbb{N}$, such that when $n>N$, $\forall P\in\mathcal{P}$, i.e., $\forall Z\in\mathcal{Z}$,

$$\begin{aligned}
\sup_{\left[d_{s,k}^{(n)}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in G^{(n)}\left(P,\left(v_i,V_{-i}^{(n)}\right)\right)}\ \inf_{\left[d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in g(P)}\left\|\left[d_{s,k}^{(n)}-d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\right\| &\le \varepsilon,\text{ and}\\
\sup_{\left[d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in g(P)}\ \inf_{\left[d_{s,k}^{(n)}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in G^{(n)}\left(P,\left(v_i,V_{-i}^{(n)}\right)\right)}\left\|\left[d_{s,k}^{(n)}-d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\right\| &\le \varepsilon.
\end{aligned}$$

By the definition of the Euclidean distance, the first inequality implies that,

$$\sup_{\left[d_{s,k}^{(n)}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in G^{(n)}}\ \inf_{\left[d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in g(P)}\left\|\left[\begin{array}{c}\min\left\{\frac{\pi}{2},\max\left[0,\arctan\left(p_{s,k}\right)+\left(\sum_{\kappa=1}^{k}d_{s,\kappa}^{(n)}-\frac{q_s}{I}\right)\right]\right\}\\ -\min\left\{\frac{\pi}{2},\max\left[0,\arctan\left(p_{s,k}\right)+\left(\sum_{\kappa=1}^{k}d_{s,\kappa}-\frac{q_s}{I}\right)\right]\right\}\end{array}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\right\|$$
$$\le\varepsilon.$$

Or, equivalently,

$$\sup_{Y^{(n)}\in H\left[Z,G^{(n)}\left(\mathcal{TAN}(Z),\left(v_i,V_{-i}^{(n)}\right)\right)\right]}\ \inf_{Y\in H[Z,g(\mathcal{TAN}(Z))]}\left\|Y^{(n)}-Y\right\|\le\varepsilon. \tag{4}$$

Similarly, we have,

$$\sup_{Y\in H[Z,g(\mathcal{TAN}(Z))]}\ \inf_{Y^{(n)}\in H\left[Z,G^{(n)}\left(\mathcal{TAN}(Z),\left(v_i,V_{-i}^{(n)}\right)\right)\right]}\left\|Y^{(n)}-Y\right\|\le\varepsilon. \tag{5}$$

Since (4) and (5) are satisfied for all $n>N$ and $\forall Z\in\mathcal{Z}$, $H\left[Z,G^{(n)}\left(\mathcal{TAN}(Z),\left(v_i,V_{-i}^{(n)}\right)\right)\right]$ converges to $H\left[Z,g\left(\mathcal{TAN}(Z)\right)\right]$ uniformly.

From the proof for Theorem 1, $H\left[Z,G^{(n)}\right]$ is upper hemicontinuous and convex-valued and thus satisfies all the conditions of Kakutani's fixed-point theorem.

Claim: Given $\left(v_i,V_{-i}^{(n)}\right)$ and any equilibrium price $P\in\mathcal{P}$, its positive monotonic transformation $Z\in\mathcal{Z}$ is a fixed point of $H\left[Z,G^{(n)}\left(\mathcal{TAN}(Z),\left(v_i,V_{-i}^{(n)}\right)\right)\right]$.

If P^* is an equilibrium price, there must exist a unique $k^*(s)\in\mathcal{K}$ for each s such that, for some $\left[d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in G^{(n)}\left(P^*,\left(v_i,V_{-i}^{(n)}\right)\right)$,
(i) $p_{s,k^*(s)}^*\in[0,+\infty)$ and $\sum_{\kappa=1}^{k^*(s)}d_{s,\kappa}=\frac{q_s}{I}$,
(ii) $\sum_{\kappa=1}^{k}d_{s,\kappa}<\frac{q_s}{I}$ and $p_{s,k}^*=0$ if $k<k^*(s)$, and
(iii) $d_{s,k}=0$ and $p_{s,k}^*=+\infty$ if $k>k^*(s)$.

Let $P^* = \mathcal{TAN}(Z^*)$, given the same $[d_{s,k}]_{s\in\mathcal{S},k\in\mathcal{K}}$, we must have

$$\begin{aligned}
\min\left\{\frac{\pi}{2}, \max\left[0, z^*_{s,k} + \left(\sum_{\kappa=1}^{k} d_{s,\kappa} - \frac{q_s}{I}\right)\right]\right\} &= 0 = z^*_{s,k}, \text{ if } k < k^*(s);\\
\min\left\{\frac{\pi}{2}, \max\left[0, z^*_{s,k} + \left(\sum_{\kappa=1}^{k} d_{s,\kappa} - \frac{q_s}{I}\right)\right]\right\} &= z^*_{s,k}, \text{ if } k = k^*(s);\\
\min\left\{\frac{\pi}{2}, \max\left[0, z^*_{s,k} + \left(\sum_{\kappa=1}^{k} d_{s,\kappa} - \frac{q_s}{I}\right)\right]\right\} &= \frac{\pi}{2} = z^*_{s,k}, \text{ if } k > k^*(s).
\end{aligned}$$

Therefore, $Z^* \in H\left[Z^*, G^{(n)}\left(\mathcal{TAN}(Z^*), \left(v_i, V^{(n)}_{-i}\right)\right)\right]$.

Note that not every fixed point of H is a PM equilibrium price matrix as the proof for Theorem 1 has discussed, while the transformation of any equilibrium price matrix is a fixed point.

Similarly, when agent i reports u_i, $H\left[Z, G^{(n)}\left(\mathcal{TAN}(Z), \left(u_i, V^{(n)}_{-i}\right)\right)\right]$ has the same properties and converges to $H[Z, g(\mathcal{TAN}(Z))]$ uniformly, since $G^{(n)}\left(P, \left(u_i, V^{(n)}_{-i}\right)\right)$ converges to $g(P)$ uniformly. In the same manner, the transformations of all the equilibrium prices can be found as a fixed point of $H\left[Z, G^{(n)}\left(\mathcal{TAN}(Z), \left(u_i, V^{(n)}_{-i}\right)\right)\right]$.

Denote $\mathcal{P}^{(\infty)}_{v_i}$ as the set of equilibrium prices corresponding to the subset of fixed points of $H[Z, g(\mathcal{TAN}(Z))]$ which all have PM price properties (i.e., the structure of priority-specific prices).

(3) Asymptotic Equivalence of $\mathcal{P}^{(\infty)}_{v_i}$ and $\mathcal{P}^{(n)}_{u_i}$.

In equilibrium, some prices may be $+\infty$ for some s and k. We supplement the definition of Euclidean distance by defining the following for $+\infty$:[28]

$$\begin{aligned}
|(+\infty) - (+\infty)| &= 0;\ \sqrt{+\infty} = +\infty;\ (+\infty)^2 = +\infty;\\
|(+\infty) - x| &= |x - (+\infty)| = +\infty, \forall x \in [0, +\infty);\\
\text{and } (+\infty) + x &= +\infty, \forall x \in [0, +\infty].
\end{aligned}$$

For any $\widehat{P}^{(n)} \in \mathcal{P}^{(n)}_{u_i}$, by definition, $\exists \left[d^{(n)}_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \in G^{(n)}\left(\widehat{P}^{(n)}, \left(u_i, V^{(n)}_{-i}\right)\right)$, such that $q_s/I = \sum_{\kappa=1}^{\overline{k}} d^{(n)}_{s,\kappa}$, $\forall s$. Since $G^{(n)}\left(P, \left(u_i, V^{(n)}_{-i}\right)\right) \to g(P)$ uniformly as $n \to \infty$,

$$\lim_{n\to\infty} \inf_{\left[d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \in g\left(\widehat{P}^{(n)}\right)} \left\| [q_s/I]_{s\in\mathcal{S}} - \left[\sum_{\kappa=1}^{\overline{k}} d_{s,\kappa}\right]_{s\in\mathcal{S}} \right\| = 0,$$

[28] $p_{s,k} = +\infty$ means that there is no supply for the preference group k at school s. It therefore makes sense to define the distance between $+\infty$ and $+\infty$ as 0.

which implies that $Z = \mathcal{TAN}^{-1}\left(\widehat{P}^{(n)}\right)$ has to be a fixed point of $H\left[Z, g\left(\mathcal{TAN}\left(Z\right)\right)\right]$ in the limit. Therefore for some $P^* \in \mathcal{P}_{v_i}^{(\infty)}$,

$$\lim_{n\to\infty}\left\|P^* - \widehat{P}^{(n)}\right\| = 0,$$

which means that, more precisely,
(i) when n is large enough, there is $\left[k^*\left(s\right)\right]_{s\in\mathcal{S}} \in \mathcal{K}^S$ such that $\forall s$, $0 \leq p^*_{s,k^*(s)}, \widehat{p}^{(n)}_{s,k^*(s)} < +\infty$; $p^*_{s,k} = \widehat{p}^{(n)}_{s,k} = 0$ if $k < k^*_s$; $p^*_{s,k} = \widehat{p}^{(n)}_{s,k} = +\infty$ if $k > k^*_s$;
(ii) $\lim_{n\to\infty}\left\|\left[p^*_{s,k^*(s)}\right]_{s\in\mathcal{S}} - \left[\widehat{p}^{(n)}_{s,k^*(s)}\right]_{s\in\mathcal{S}}\right\| = 0$.

Since this is true $\forall \widehat{P}^{(n)} \in \mathcal{P}_{u_i}^{(n)}$,

$$\lim_{n\to\infty}\sup_{\widehat{P}^{(n)}\in\mathcal{P}_{u_i}^{(n)}}\inf_{P^*\in\mathcal{P}_{v_i}^{(\infty)}}\left\|P^* - \widehat{P}^{(n)}\right\| = 0. \tag{6}$$

On the other hand, for any $P^* \in \mathcal{P}_{v_i}^{(\infty)}$, by definition, $\exists\left[d_{s,k}\right]_{s\in\mathcal{S},k\in\mathcal{K}} \in g\left(P^*\right)$, such that $q_s/I = \sum_{\kappa=1}^{\overline{k}} d_{s,\kappa}$, $\forall s$. Since $G^{(n)}\left(P, \left(u_i, V_{-i}^{(n)}\right)\right)$ converges to $g\left(P\right)$ uniformly,

$$\lim_{n\to\infty}\inf_{\left[d_{s,k}^{(n)}\right]_{s\in\mathcal{S},k\in\mathcal{K}}\in G^{(n)}\left(P^*,\left(u_i,V_{-i}^{(n)}\right)\right)}\left\|\left[q_s/I\right]_{s\in\mathcal{S}} - \left[\sum_{\kappa=1}^{\overline{k}} d_{s,\kappa}^{(n)}\right]_{s\in\mathcal{S}}\right\| = 0,$$

which implies that P^* is an asymptotic equilibrium price for $\left(u_i, V_{-i}^{(n)}\right)$, i.e.,

$$\lim_{n\to\infty}\inf_{\widehat{P}^{(n)}\in\mathcal{P}_{u_i}^{(n)}}\left\|P^* - \widehat{P}^{(n)}\right\| = 0.$$

Thus the above two properties (i) and (ii) are satisfied. Since this is true for all $P^* \in \mathcal{P}_{v_i}^{(\infty)}$, therefore

$$\lim_{n\to\infty}\sup_{P^*\in\mathcal{P}_{v_i}^{(\infty)}}\inf_{\widehat{P}^{(n)}\in\mathcal{P}_{u_i}^{(n)}}\left\|P^* - \widehat{P}^{(n)}\right\| = 0. \tag{7}$$

Combining (6) and (7), we have $\lim_{n\to\infty} d_H\left(\mathcal{P}_{v_i}^{(\infty)}, \mathcal{P}_{u_i}^{(n)}\right) = 0$, $\forall u_i \in [0,1]^S$ and for any i in all $\mathcal{I}^{(n)}$.

Furthermore, $\lim_{n\to\infty} d_H\left(\mathcal{P}_{v_i}^{(\infty)}, \mathcal{P}_{v_i}^{(n)}\right) = 0$ and, therefore, $\lim_{n\to\infty} d_H\left(\mathcal{P}_{v_i}^{(n)}, \mathcal{P}_{u_i}^{(n)}\right) = 0$, $\forall u_i \in [0,1]^S$ and for any i in all $\mathcal{I}^{(n)}$. ■

Proposition C1 *Suppose i is in every economy of the sequence $\left\{\Gamma^{(n)}\right\}_{n\in\mathbb{N}}$. PM is limiting individually incentive compatible for i.*

Proof. By Lemma C2, for any $\xi > 0$, there exists n^* such that for $n > n^*$ and for every price in $P_{u_i} \in \mathcal{P}_{u_i}^{(n)}$ there exists a price $P_{v_i} \in \mathcal{P}_{v_i}^{(n)}$ such that $\left|P_{v_i} - P_{u_i}\right| < \xi$.

We define the indirect utility function $W_{u_i}\left(P\right)$ as the expected utility (with respect to true preferences v_i) that i can obtain when reporting u_i given price P. By the maximum theorem, i's utility maximization problem implies that $W_{u_i}\left(P\right)$ is continuous in P. Moreover, the utility from manipulation, $W_{u_i}\left(P\right)$, is always bounded above by $W_{v_i}\left(P\right)$. Therefore, $W_{u_i}\left(P_{u_i}\right) \leq W_{v_i}\left(P_{u_i}\right)$.

When ξ is set small enough, the continuity of $W_{v_i}(.)$ implies that we can find $P_{v_i} \in \mathcal{P}_{v_i}^{(n)}$ in all large enough economies ($n > n^*$) such that:

$$|W_{v_i}(P_{u_i}) - W_{v_i}(P_{v_i})| < \varepsilon.$$

Therefore,

$$W_{u_i}(P_{u_i}) \leq W_{v_i}(P_{u_i}) < W_{v_i}(P_{v_i}) + \varepsilon.$$

Or equivalently,

$$W_{v_i}(P_{v_i}) > W_{u_i}(P_{u_i}) - \varepsilon,$$

which proves that PM is limiting individually incentive compatible for i. ■