Macroeconomics M1 2015–2016

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Very Preliminary (and Incomplete) Notes

Many Typos.

Slides on my webpage at TSE

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Introduction

Objectives of this course:

- Not a course in Dynamic Programming; Not a course in Time Series Econometrics; Not a course in Theoretical Macroeconomics; Not a course in Computational Economics; Just a Simple Mix of both!!!
- Organization of the Macro course in M1: one class "Advanced Macro for the doctoral track" (Robert Ulbrich), one class of "Advanced Macro" (Patrick Fève).
- 10 tutorials.
- This course is not (too much!!) technical (more advanced courses in M1 Macro-Doctoral and M2 Macro 1 and 2 courses in the doctoral track of TSE).

- The dynamic optimization problems and the dynamic general equilibrium models are considered in their simplest forms.

- This course tries to cover the major fields of modern macroeconomics (Dynamic Stochastic General Equilibrium models, labeled now DSGE models).

Organization of this course: Three main chapters (if we have enough time!!)

Chap 1: Two Period Equilibrium Models

1.1: An exchange Economy

- The model
- Equilibrium
- Central planner problem
- Discussion

1.2: A Production Economy

- The model
- Equilibrium
- Central planner problem
- Discussion

Chap 2: Models and Methods in Macroeconomic Dynamics

2.1: The Permanent Income Model

- The deterministic case
- The stochastic case

2.2: The Dynamic Labor Demand Model

- The deterministic case
- The stochastic case

Chap 3: Business Cycle Models

3.1: Two Simple DSGE Models

I - Habit Formation in a Simple Equilibrium Model

- The model
- Solution
- Quantitative implications

II - A Stochastic Growth Model

- The model
- Solution
- Quantitative implications

3.2: RBC Models

I - A RBC Model with Complete Depreciation

- The model
- Solution
- Quantitative implications

II- A prototypical RBC Model (Optional)

- The model
- Solution
- Quantitative implications

3.3: Beyond the RBC Model (Optional) in Progress

Rules of the game of this course:

- This course does not required additional materials, i.e. this course is almost self contained.
- BIBLIOGRAPHY : David Romer Advanced Macroeconomics 4th Edition, Mc-Graw Hill, 2012 (see chapters 5, 7, 8, 9, 12)
- Applications and exercises are contained in the homeworks: slight modifications (simpler and/or extensions) of the models expounded in the course.
- A mid–term examination (Two exercises on macroeconomic dynamics): 40 %
- A final examination: Two exercises on macroeconomic dynamics: 60%.

The quantitative approach to Economic Dynamics

1. Formulate a dynamic (deterministic or stochastic) model (partial equilibrium – households, firms – or general equilibrium): objectives, constraints, exogenous driving forces.

2. Determine the optimal decisions to the associated dynamic optimization problem.

3. Collect all the relevant equations that fully characterize the intertemporal equilibrium.

4. Solve the model analytically (in this course) or numerically if too complicated (not in this course, see my remark below).

5. Estimate and test the model from actual data. If the model fails (poor fit of the data), go to step 1 and redo the exercise. If not, go to the next step.

6. Simulate the model to evaluate its dynamics properties
7. Conduct counterfactual and policy experiments (fiscal and/or monetary policy shocks; change in institutions and regulation, ...)

This course only delivers analytical results from (very or maybe too) simple dynamic (deterministic and stochastic) models.

More realistic cases can be solved in the same way but they need computer intensive techniques in order to get solutions, estimations and simulations.

There exists now a bunch of softwares than can be used to solve and simulate dynamic stochastic general (or partial) equilibrium models. See DYNARE (free download under MATLAB programming) among many others (to obtain it, just write the name "DYNARE" in `Google`. It will appear first.), but we prefer to work with simpler models before any implementations with this useful free software). If you are very interesting in this software and the ways to implement our model economies in this setup, do not hesitate to ask me the codes!

Warning!!! This notes may contain many (Mathematical and English) typos!!!

Chap 2: A Two Period Equilibrium Model

2.1: An exchange Economy

- The model
- Equilibrium
- Central planner problem
- Discussion

2.2: A Production Economy

- The model

- Equilibrium
- Central planner problem
- Discussion

Chap 2: A Two Period Equilibrium Model

2.1: An exchange Economy

The Model

A very simple setup

two periods

a single (representative) consumer

An exchange (endowment) economy

Old idea (Irving Fisher): the allocation of goods over time $\Leftrightarrow$ allocation of resources to different goods at a given time

So goods at different dates are considered as different commodities.

Elements of the Exchange Economy:

1) Two commodities, denoted C_1 and C_2.

2) Endowments of these commodities: Y_1 and Y_2

An important point: Y is **non storable**

3) Preferences of the representative consumer represented by the following utility function

$$U(C_1, C_2)$$

We will specify this utility function latter

The elements 1) and 2) characterize the *feasible* allocations

$$C_1 \leq Y_1 \quad \text{and} \quad C_2 \leq Y_2$$

In addition, we are in a deterministic case.

The agent perfectly knows the quantities in second period.

Two important questions of interest:

i) How the consumer allocates consumption over time?

ii) What are the determinants of prices and quantities?

The competitive Equilibrium

We will characterize the equilibrium allocations in a competitive setup

(prices are given)

We denote P_1 and P_2 the price today and tomorrow, respectively.

The intertemporal budget constraint is given by

$$P_1 C_1 + P_2 C_2 \leq P_1 Y_1 + P_2 Y_2$$

In this economy, the prices P_1 and P_2 adjust such that

$$C_t = Y_t \quad \forall t = 1, 2$$

Definition of the competitive equilibrium

A competitive equilibrium is a system of prices $\{P_1, P_2\}$ and allocations $\{C_1, C_2\}$ such that:

1. The (representative) consumer maximizes the utility

$$U(C_1, C_2)$$

subject to the budget constraint

$$P_1 C_1 + P_2 C_2 \leq P_1 Y_1 + P_2 Y_2$$

for P_1 and P_2 given.

2. Supply of goods equals demand of goods each period:

$$C_1 = Y_1 \quad , \quad C_2 = Y_2$$

Application Let us assume the following utility function

$$U(C_1, C_2) = \log(C_1) + \beta \log(C_2)$$

This log–utility is separable (more general, not separable, Habit formation in consumption, see below)

The parameter $\beta > 0$ is a subjective discount factor.

$$\beta = \frac{1}{1+\theta}$$

where $\theta > 0$ is the subjective of preference for the present.

The household problem

$$\max_{C_1, C_2} \log(C_1) + \beta \log(C_2)$$

subject to the intertemporal budget constraint

$$P_1 C_1 + P_2 C_2 = P_1 Y_1 + P_2 Y_2$$

Solution: first we replace the intertemporal budget constraint

$$C_2 = \frac{P_1}{P_2}Y_1 + Y_2 - \frac{P_1}{P_2}C_1$$

into the objective

$$\log(C_1) + \beta \log\left(\frac{P_1}{P_2}Y_1 + Y_2 - \frac{P_1}{P_2}C_1\right)$$

and then maximize with respect to C_1.

FOC

$$\frac{1}{C_1} - \beta\frac{P_1}{P_2}\frac{1}{C_2} = 0$$

or equivalently

$$\frac{\beta/C_2}{1/C_1} = \frac{P_2}{P_1}$$

i.e. the marginal rate of substitution of consumption between two periods is equal to the relative price.

Now, use this optimality condition to express C_2:

$$C_2 = \beta \frac{P_1}{P_2} C_1$$

(rmk: if the equilibrium is such that $\beta \frac{P_1}{P_2} = 1$, perfect consumption smoothing)

Now, replace C_2 into the budget constraint

$$P_1 C_1 + P_2 C_2 = P_1 Y_1 + P_2 Y_2$$

$$P_1 C_1 (1 + \beta) = P_1 Y_1 + P_2 Y_2$$

Thus, we obtain

$$C_1 = \frac{1}{1 + \beta} \left(Y_1 + \frac{P_2}{P_1} Y_2 \right)$$

Now, using the optimality condition, we deduce C_2

$$C_2 = \frac{\beta}{1 + \beta} \left(\frac{P_1}{P_2} Y_1 + Y_2 \right)$$

The expressions for C_1 and C_2 determine the two demand functions.

Now using these two demand functions and market clearing condition of good market in each period $Y_1 = C_1$ and $Y_2 = C_2$ (Supply=Demand), we can deduce the (relative) equilibrium prices

$$Y_1 = \frac{1}{1+\beta}\left(Y_1 + \frac{P_2}{P_1}Y_2\right)$$

$\Leftrightarrow$

$$\frac{P_2}{P_1} = \beta\frac{Y_1}{Y_2}$$

Note that we obtain exactly the same equilibrium price if we use the condition

$$Y_2 = \frac{\beta}{1+\beta}\left(\frac{P_1}{P_2}Y_1 + Y_2\right)$$

Rmk1: The level of consumption each period is directly deduced from the resource constraint ($C_t = Y_t$, $t = 1, 2$).

This is because the good is perishable (it cannot be stored or invested, we will investigate this case below).

So, why considering an intertemporal model rather than a repeated static model? (i.e. max utility in period 1, max utility in period 2).

This is because the model allows to determine first the intertemporal allocations for P_1 and P_2 given at the household level and second, given the equilibrium conditions on good market, it allows to determine the prices.

In fact, for a given specification of utility, this is a model of intertemporal prices determination (real interest rate or asset prices).

A more general model with a storage technology (physical capital) will produce similar results (the computation of quantities is more complicated, see below).

Rmk2: The model cannot determine P_1 and P_2 separately. (the two demand functions give the same relative prices). The model only determines the relative prices P_1/P_2 (or P_2/P_1).

To see this, multiply both P_1 and P_2 by a scale factor $\lambda > 0$:

$$\tilde{P}_1 = \lambda P_1 \qquad \tilde{P}_2 = \lambda P_2$$

You will obtain the same allocation for C_1 and C_2 and the same equilibrium.

So, we can freely choose one price arbitrarily, say $P_1 = 1$ (the numeraire)

This means that the second price is measured in terms of unit of the first good.

So $P_2/P_1 = P_2$ is the intertemporal purchasing power of the consumer. One unit of consumption tomorrow costs P_2 unit of consumption today.

Finally, the allocation $\{C_1 = Y_1; C_2 = Y_2\}$ and the prices $\{P_1 = 1; P_2 = \beta(Y_1/Y_2)\}$ constitute a *competitive equilibrium* of the economy.

The Central Planner Problem

In this economy (without distortions), the two welfare theorems apply.

1) the competitive allocations (C_1, C_2) are those obtained by a central planner

2) The optimal allocations can be supported as a competitive equilibrium (with a proper system of prices)

The central planner must solve

$$\max_{C_1,C_2} \log(C_1) + \beta \log(C_2)$$

subject to the two budget constraints

$$C_1 \leq Y_1 \qquad C_2 \leq Y_2$$

Note that there is here no price, because the central planner fully internalizes all the market conditions.

The solution

$$C_1 = Y_1 \quad \text{and} \quad C_2 = Y_2$$

So we obtain the same allocations.

Now define the marginal utilities

$$\frac{\partial U}{\partial C_1} = \frac{1}{C_1} \quad , \quad \frac{\partial U}{\partial C_2} = \frac{\beta}{C_2}$$

Next compute the marginal rate of substitution

$$\frac{\frac{\partial U}{\partial C_2}}{\frac{\partial U}{\partial C_1}} = \beta \frac{Y_1}{Y_2}$$

In a competitive equilibrium, we have

$$\frac{\frac{\partial U}{\partial C_2}}{\frac{\partial U}{\partial C_1}} = \beta \frac{Y_1}{Y_2} \equiv \frac{P_2}{P_1}$$

So the two welfare theorems apply.

Discussion

1- The real interest rate

For the moment, we have considered two prices P_1 and P_2, but the model can be used to determine other form of intertemporal prices, i.e. the real interest rate.

With a slight modification, we can determine the equilibrium real interest rate.

<u>First Period</u> Let us define how the household can save

$$S = Y_1 - C_1$$

where S denotes saving ($S > 0$) or borrowing ($S < 0$).

<u>Second Period</u> Consume the disposable income $(1 + r)S + Y_2$,

$$C_2 = (1 + r)S + Y_2$$

where r denotes the real interest rate.

Now, maximize the intertemporal utility

$$\max_{C_1, C_2} \log(C_1) + \beta \log(C_2)$$

with respect to the constraints $C_1 = Y_1 - S$ and $C_2 = (1 + r)S + Y_2$.

Note that we can get the intertemporal budget constraint from two above constraints

$$C_2 = (1 + r)(Y_1 - C_1) + Y_2$$

Now, substitute this constraint into the objective

$$\max_{C_1} \log(C_1) + \beta \log((1+r)(Y_1 - C_1) + Y_2)$$

and maximize with respect to C_1.

FOC

$$\frac{1}{C_1} - \beta(1+r)\frac{1}{C_2} = 0$$

or equivalently

$$\frac{1}{C_1} = \beta(1+r)\frac{1}{C_2}$$

This is the **Euler equation** on consumption.

Now, use the equilibrium conditions (demand of goods=supply of goods)

$$C_1 = Y_1 \quad , \quad C_2 = Y_2$$

Note that at equilibrium, saving is equal to zero ($S = 0$).

This allows to determine the equilibrium real interest rate

$$\frac{1}{1+r} = \beta \frac{Y_1}{Y_2}$$

In the previous model version, we obtained

$$\frac{P_2}{P_1} = \beta \frac{Y_1}{Y_2}$$

So, we deduce

$$1 + r = \frac{P_1}{P_2}$$

With the normalization $P_1 = 1$,

$$1 + r = \frac{1}{P_2}$$

Rmk: If $Y_1 = Y_2$, we have

$$\frac{1}{1+r} = \beta \equiv \frac{1}{1+\theta}$$

so

$$r = \theta$$

Two main determinants for r: 1) preferences (β or θ) 2) endowments (Y_1 and Y_2)

2-Stochastic endowments

We have assumed perfect foresight in a deterministic setup.

What happens if Y is stochastic?

In this case, the consumer must expect future Y and thus future consumption.

A modification of our setup:

Period 1

Saving

$$A = Y_1 - C_1$$

Period 2

$$C_2 = Y_2 + (1 + r)A$$

To simplify, suppose that there is no uncertainty on the return r (a financial contracts that deliver the same return whatever the state of the nature). So, r is given.

Y_1 and Y_2 are now stochastic labor incomes in period 1 and 2.

The Euler equation on consumption becomes

$$\frac{1}{C_1} = \beta E_1 \left((1+r) \frac{1}{C_2} \right)$$

where E_1 is the expectation conditional on the information set in period 1, i.e. when agents must take their decisions.

Assume that $\beta(1+r) = 1$. The Euler equation rewrites

$$\frac{1}{C_1} = E_1\left(\frac{1}{C_2}\right)$$

By the Jensen inequality (Let $f(x)$ a convex function and x a random variable, then $E(f(x)) \geq f(E(x))$ or simply show it from a figure), we have

$$E_1\left(\frac{1}{C_2}\right) \geq \left(\frac{1}{E_1(C_2)}\right)$$

This implies that the consumption will lower with stochastic income, i.e. consumers try to save more (called as "precautionary" saving).

3 - Labor supply

We omit an important variable for the household: labor supply

We have assumed that labor is inelastic.

Here, we relax this assumption (allowing for elastic labor supply) in a two–period model (partial equilibrium).

Two opposite forces that dive labor supply: substitution effect and wealth effect.

The substitution effect: Labor supply increases after an increase in real wage.

Wealth effect: Labor supply decreases after an increase in real wage.

Utility function

$$U(C_1, C_2, N_1, N_2) = \log(C_1) + \beta \log(C_2) + \log(1 - N_1) + \beta \log(1 - N_2)$$

where N_1 and N_2 represent labor supply in period 1 and 2, respectively.

Time endowment is normalized to one.

The intertemporal budget constraint

$$C_1 + \frac{C_2}{R} \leq W_1 N_1 + \frac{W_2 N_2}{R}$$

where $R = 1 + r > 1$ and r is the real interest rate.

Now, form the Lagrangian

$$\mathcal{L} = U(C_1, C_2, N_1, N_2) - \lambda\left(C_1 + \frac{C_2}{R} - W_1 N_1 - \frac{W_2 N_2}{R}\right)$$

where $\lambda \geq 0$ is the Lagrange multiplier.

Maximize this Lagrangian with respect to C_1, C_2, N_1 and N_2

FOCs

$$1/C_1 - \lambda = 0$$

$$\beta/C_2 - \lambda/R = 0$$

$$-1/(1 - N_1) + \lambda W_1 = 0$$

$$-\beta/(1 - N_2) + \lambda W_2/R = 0$$

and the exclusion restriction

$$\lambda\left(C_1 + \frac{C_2}{R} - W_1 N_1 - \frac{W_2 N_2}{R}\right) = 0$$

Note that $\lambda > 0$ (because $\lambda = 1/C_1$) and then the intertemporal budget constraint binds

$$C_1 + \frac{C_2}{R} = W_1 N_1 + \frac{W_2 N_2}{R}$$

We impose $\beta R = 1$.

We then deduce

$$C_2 = C_1$$

$$N_1 = 1 - C_1/W_1$$

$$N_2 = 1 - C_1/W_2$$

Next substitute C_2, N_1 and N_2 into the budget constraint. We obtain

$$C_1 = \frac{R}{2(1+R)}(W_1 + W_2/R)(= C_2)$$

$$N_1 = 1 - \frac{R}{2(1+R)}(1 + (1/R)(W_2/W_1))$$

$$N_2 = 1 - \frac{R}{2(1+R)}((W_1/W_2) + (1/R))$$

If W_1/W_2, then N_1 increases and N_2 decreases (the substitution effect dominates the wealth effect).

Permanent versus transitory changes in real wage

Permanent change

Let $\tilde{W}_1 = \mu W_1$ and $\tilde{W}_2 = \mu W_2$ with $\mu > 1$ (a permanent increase).

We see that N_1 and N_2 are unaffected.

At the same time the consumption increases a lot (see the permanent income model)

This rise in consumption can be sustained by the increases in the real wage (the wealth effect perfectly cancels out the substitution effect)

Transitory change

Let $\tilde{W}_1 = \mu W_1$ and $\tilde{W}_2 = W_2$ with $\mu > 1$ (a transitory increase).

So, N_1 increases and N_2 decreases. At the same time, consumption increases very little because the increase in real wage is perceived as transitory.

4 - Ricardian equivalence

For the moment, we have ignored the government.

We use a simple extension of our exchange economy.

Our objective are

i) we want to describe how the definition of the economy is affected by the introduction of government spending and taxes.

ii) we want to explore the effect of government policy on prices and quantities.

The government

The government does two things:

1) the government consumes quantities $\{G_1, G_2\}$ of goods in each period;

2) the government collects lump-sum taxes $\{T_1, T_2\}$ from consumer to finance its own consumptions.

Government consumption and taxes in period t $(t = 1, 2)$ are measured in unit of the date t good.

Taxes and spending decisions are related through the government's (intertemporal) budget constraint.

$$P_1G_1 + P_2G_2 = P_1T_1 + P_2T_2$$

The present value of taxes equals the present value of government spending (remind that $P_2/P_1 = 1/(1+r)$)

An important assumption: the government spending has no effect on utility (and in a production economy, no effect on production, i.e. unproductive government spending, purely wasteful)

The consumer

Same as before, but the intertemporal budget constraint becomes

$$P_1 C_1 + P_2 C_2 = P_1(Y_1 - T_1) + P_2(Y_2 - T_2)$$

because the consumer must now pay taxes in each period.

Definition of the competitive equilibrium

A competitive equilibrium consists of prices $\{P_1, P_2\}$, private decisions $\{C_1, C_2\}$ and government policies $\{G_1, G_2, T_1, T_2\}$ such that:

- The consumer maximizes utility, given prices and taxes, subject to the budget constraint;
- The government's decisions satisfy its budget constraint;
- Supply=Demand for each good:

$$C_1 + G_1 = Y_1 \qquad C_2 + G_2 = Y_2$$

An illustration

Endowments: $\{Y_1, Y_2\}$

Utility: $\max_{C_1,C_2} \log(C_1) + \beta \log(C_2)$

Government spending: $\{G_1, G_2\}$ are given

<u>Consumer</u>

Replace the new budget constraint

$$C_2 = \frac{P_1}{P_2}(Y_1 - T1) + (Y_2 - T_2) - \frac{P_1}{P_2}C_1$$

into the objective. Next, maximize the utility with respect to C_1 (for

P_1, P_2, T_1 and $T-2$ given). This yields

$$\frac{1}{C_1} - \beta\frac{P_1}{P_2}\frac{1}{C_2} = 0$$

Same equation as before.

Equilibrium

$$C_1 = Y_1 - G_1 \qquad C_2 = Y_2 - G_2$$

Notice that

$$\frac{dC_1}{dG_1} = \frac{dC_2}{dG_2} = -1$$

Perfect crowding out of private consumption by public spending ,

because the endowment are given and invariant to G.

Rmk productive government spending

Example: public expenditures on infrastructure (road, port, communication systems), research, basic provision of education and health.

A simple representation

$$Y_1 = \bar{Y}_1 + \eta G_1 \qquad Y_2 = \bar{Y}_2 + \eta G_2$$

where $\eta \geq 0$.

With this slight modification, we obtain

$$C_1 = \bar{Y}_1 + (\eta - 1)G_1 \geq \bar{Y}_1 - G_1 \qquad C_2 = \bar{Y}_2 + (\eta - 1)G_2 \geq \bar{Y}_2 - G_2$$

and

$$\frac{dC_1}{dG_1} = \frac{dC_2}{dG_2} = \eta - 1$$

When $\eta > 0$ no perfect crowding–out effect, if η is sufficiently large, crowding–in. End of the remark.

Now, from C_1 and C_2, we can determine the intertemporal prices

$$\frac{P_2}{P_1} = \beta\frac{C_1}{C_2} \equiv \beta\frac{Y_1 - G_1}{Y_2 - G_2}$$

Note that neither prices, nor quantities depends on taxes and their timing. Taxes are adjusted in order to satisfy the government' s budget constraint. We have the following main result.

Result (Ricardian Equivalence) If prices $\{P_1, P_2\}$, private decisions $\{C_1, C_2\}$ and government policies $\{G_1, G_2, T_1, T_2\}$ constitute a competitive equilibrium, then a new tax policy $\{\tilde{T}_1, \tilde{T}_2\}$ satisfying

$$P_1G_1 + P_2G_2 = P_1\tilde{T}_1 + P_2\tilde{T}_2$$

will constitute a competitive equilibrium with the **same prices, private decisions and government spending**.

Proof: First note that $P_1G_1 + P_2G_2 = \bar{G}$ is constant and does not vary across the two equilibria since the prices and government spending are the same in each equilibrium. The consumer's intertemporal budget constraint only depend on the present value of taxes $P_1T_1 + P_2T_2$, not T_1 and T_2 separately. From the government budget constraint, the present values of taxes are equal to $\bar{G}$ and is thus constant. So, the consumer's budget constraint remains unaffected, and so the optimal decision on C_1 and C_2. Then the intertemporal prices are the same. This completes the proof.■

Because the timing of taxes have no effect on equilibrium prices and quantities, debt financing or tax financing are equivalent (Ricardian equivalence) in this economy.

The debt of the government in period 1

$$B = G_1 - T_1$$

The previous result says that for any level of debt B will produce the same equilibrium.

If B increase, then we need to increase taxes tomorrow to satisfy the intertemporal government budget constraint

$$T_2 = G_2 + \frac{P_1}{P_2} B \equiv (1 + r) B$$

A important point This result does not mean the the government' decisions have no effect on the economy (see the discussion above). It just states that the timing of taxation does not matter.

Departures from Ricardian Equivalence

- Short–lived consumer.
- Distortionary taxation.
- Imperfect financial market.

2.2: A Production Economy

- The model
- Equilibrium
- Central planner problem
- Discussion

In the previous setup of an endowment economy, the real interest rate only depends on preferences and endowments.

This is because the agents have no opportunity to produce goods.

We consider now that agents can have access to a technology that allows to transform current consumption into future consumption.

The Model

The agent can now invest in physical capital.

Period 1 The agent invests K units of physical capital.

Period 2 At the beginning of the period, $F(K)$ units of goods are produced from the K units.

Next, the K is fully destroyed (full depreciation).

So, agents receive $F(K)$ in period 2.

Assumptions on the production $F(K)$:

i) $F(0) = 0$

ii) $F' > 0$ and $F'' < 0$ (increasing and concave)

iii) Inada conditions: $F'(0) = \infty$ and $F'(\infty) = 0$.

With the investment in physical capital and the production technology, we can now consider a new actor in the economy: A Firm.

The firm can buy (or rent) the capital from the consumer at a price, denoted Q, per unit of capital and then can use it to produce extra output $F(K)$ in period 2.

The Firm's decision

The firm chooses optimally the level of K in period 2 such that she maximizes the profit:

$$\Pi = P_2 F(K) - QK$$

The Consumer's decision

The representative consumer owns the capital and the firm.

The new budget constraint

Let X is the amount of capital rented to the firm and Π the profit that she receives from owning the firm.

The consumer 's decision is to choose C_1, C_2 and X to maximize the intertemporal utility

$$U(C_1, C_2)$$

under the (new) intertemporal budget constraint

$$P_1 C_1 + P_2 C_2 \leq P_1(Y_1 - X) + P_2 Y_2 + QX + \Pi$$

The competitive equilibrium

Definition of the equilibrium

A competitive equilibrium is a price system $\{P_1, P_2, Q\}$ and allocations $\{C_1, C_2, X, K\}$ such that

i) The consummer maximizes the intertemporal utility given prices and subject to the intertemporal budget constraint.

ii) The firm maximizes profit given prices and the technology.

iii) Supply=Demand for each goods

$$C_1 + K = Y_1 \quad \text{(good market in period 1)}$$

$$C_2 = Y_2 + F(K) \quad \text{(good market in period 2)}$$

$$X = K \quad \text{(capital market)}$$

Application

Utility function

$$U(C_1, C_2) = \log(C_1) + \beta \log(C_2)$$

Endowments

$$Y_1 = \bar{Y} \qquad Y_2 = 0$$

(the production allows to consume tomorrow without endowment)

$$F(K) = K^{\alpha}$$

(+ full depreciation of physical capital)

The consumer

$$\max_{C_1, C_2, X} \log(C_1) + \beta \log(C_2)$$

under the intertemporal budget constraint. First, substitute the constraint

$$C_2 = \frac{P_1}{P_2}\bar{Y} - \left(\frac{P_1}{P_2} - \frac{Q}{P_2}\right) X + \frac{\Pi}{P_2} - \frac{P_1}{P_2} C_1$$

FOCs

with respect to C_1

$$\frac{1}{C_1} - \beta\frac{P_1}{P_2}\frac{1}{C_2} = 0 \Leftrightarrow \frac{1}{C_1} = \beta\frac{P_1}{P_2}\frac{1}{C_2} \quad \text{(Euler equation)}$$

with respect to X

$$-\beta\frac{1}{C_2}\frac{P_1}{P_2} + \beta\frac{1}{C_2}\frac{Q}{P_2} = 0 \Leftrightarrow P_1 = Q$$

The firm

$$\max_{K} \Pi P_2 K^{\alpha} - QK$$

FOC (Marginal productivity of capital is equal to its real unit cost)

$$\alpha K^{\alpha-1} = \frac{Q}{P_2}$$

Capital demand

$$K = \left(\frac{Q}{\alpha P_2}\right)^{1/(\alpha-1)}$$

Equilibrium

Good market in period 1

$$C_1 = \bar{Y} - K$$

Good market in period 2

$$C_2 = K^\alpha$$

Capital market (+ the condition $Q = P_1$)

$$X = K = \left(\frac{P_1}{\alpha P_2}\right)^{1/(\alpha-1)}$$

From the Euler equation on consumption, we determine the relative price

$$\frac{P_2}{P_1} = \beta\frac{C_1}{C_2} \equiv \beta\frac{\bar{Y} - K}{K^\alpha}$$

From the firm's optimality condition, we have

$$\frac{P_2}{P_1} = \frac{K}{\alpha K^\alpha}$$

We deduce

$$\beta\frac{\bar{Y} - K}{K^\alpha} = \frac{K}{\alpha K^\alpha} \Leftrightarrow \alpha\beta(\bar{Y} - K) = K$$

and we obtain the level of the physical capital

$$K = \omega\bar{Y}$$

where

$$\omega = \frac{\alpha\beta}{1+\alpha\beta} \in (0,1)$$

We then deduce the consumptions in period 1 and 2

$$C_1 = (1-\omega)\bar{Y} \qquad C_2 = (\omega\bar{Y})^\alpha$$

Note that the consumer does not consume all the endowment in first period, but choose to save (in the form of physical capital).

The saving is then use to consume tomorrow (with any endowment in period 2), according to the technology (K^α)

The parameter ω can be interpreted as the saving rate ($\omega = Y_1/K$).

The model provides micro-foundations for this parameter.

It depends on two "deep" parameters reflecting the preferences (β) and the technology (α).

The saving is an increasing function of β and α:

$$\frac{\partial \omega}{\partial \alpha} = \frac{\beta}{(1+\alpha\beta)^2} \qquad \frac{\partial \omega}{\partial \beta} = \frac{\alpha}{(1+\alpha\beta)^2}$$

More patient agents (β larger) save more. More efficient technology

(α larger) creates incentives to invest.

Finally, from C_1 and C_2, we can determine the relative prices

$$\frac{P_2}{P_1} = \beta\frac{C_1}{C_2} \equiv \beta\frac{(1-\omega)\bar{Y}}{(\omega\bar{Y})^{\alpha}}$$

Note that we can define as previously the real interest rate using the relation

$$\frac{P_2}{P_1} = \beta\frac{1}{1+r}$$

The central planner

$$\max_{C_1, C_2} U(C_1, C_2)$$

under the two constraints

$$C_1 + K \leq Y_1 \qquad C_2 \leq Y_2 + F(K)$$

or equivalently

$$C_2 \leq Y_2 + F(Y_1 - C_1)$$

Application: Using the above assumption, this problem becomes

$$\max_{C_1,C_2,K} \log(C_1) + \beta \log(C_2)$$

After replacement of the constraint into the objective, we obtain

$$\max_K \log(\bar{Y} - K) + \beta \log(K^\alpha) \Leftrightarrow \max_K \log(\bar{Y} - K) + \alpha\beta \log(K)$$

FOC

$$-\frac{1}{\bar{Y} - K} + \alpha\beta\frac{1}{K} = 0$$

This yields

$$K = \omega\bar{Y} \qquad C_1 = (1-\omega)\bar{Y} \qquad C_2 = (\omega\bar{Y})^\alpha$$

The same allocation than the one of the competitive equilibrium.

Chap 2: Models and Methods in Macroeconomic Dynamics

We use two useful models to introduce methods and concepts of modern macrodynamics.

2.1: The Permanent Income Model

- The deterministic case
- The stochastic case

2.2: The Dynamic Labor Demand Model

- The deterministic case
- The stochastic case

We do not consider here two other useful models:

A model of (physical) capital/investment decisions with adjustment costs (see the exercise)

A model of dynamic labor supply (see the next chapter in a simple two-period model)

Chap 2: Models and Methods in Macroeconomic Dynamics

2.1: The Permanent Income Model

- The deterministic case
- The stochastic case

The Keynesian consumption function (key function in the Keynesian model, i.e. the value of the government spending multiplier)

$$C_t = C_o + cY_t$$

C_t is the real consumption and Y_t the real (after tax) income.

C_o denotes a positive constant and $c \in (0, 1)$ is the marginal propensity to consume.

[Insert a figure]

Empirical studies (after the WWII) have questioned the empirical relevance of this function (for ex., with a long data sample, estimation results suggest $c \simeq 1$) + Theoretical extensions: relative income approaches, including reference income and ratchet effect (see Duesenberry in 50'), and basic formulations of habit persistence (see Brown in 1954).

More recently, the permanent income model allows to disconnect the current consumption from the current labor income.

Here, the permanent income model with an infinite horizon, following Hall (1978) and all the recent developments on the consumption functions.

Simpler forms: a two period model of consumption (already done in Micro courses and in partial equilibrium; we will investigate below a simple two–period setup, but in a general equilibrium setup)

Understanding intertemporal consumption decisions is essential in modern Macroeconomic Dynamics.

Here, we use a (too!! and probably not realistic) simple and tractable setup from which we can obtain analytical solutions (and then we can analyze them extensively)

The Deterministic Case

The setup

A (representative) household (or consumer) will seek to maximize (every period) the following objective function

$$\sum_{i=0}^{\infty} \beta^i U(C_{t+i})$$

under a sequence of (instantaneous) budget constraint

$$A_{t+1} = (1 + r)A_t + Y_t - C_t \quad \forall t$$

where $U(C_t)$ is the instantaneous utility function (specified below).

Rmk: the utility function in period t only depends on the current consumption (extension: non-separable utility function, ex. habit formation in consumption)

$\beta \in (0, 1)$ is the subjective discount rate (preferences)

r the real (constant) interest rate (we assume $r > 0$), given

A_t is the level of (real) financial wealth

Y_t is the real (net of taxation) labor income, here supposed to be known in each period (we will consider latter the stochastic case)

C_t is the real consumption

Status of the variables

A_t is a state (or pre-determined) variable, which summarizes the past decisions on consumption, given the (present and past) realizations of the labor income.

Y_t is an exogenous variable

Rmk1: no decision on labor supply, or labor is constant. If not, solution is more complicated.

Rmk2: the elasticity of labor supply is subject to empirical controversies (macro studies, large; micro studies, very small)

C_t is the control variable

An Important Property Controlling C_t is equivalent to controlling A_{t+1}.

Proof directly deduced from the budget constraint

$$A_{t+1} + C_t = (1 + r)A_t + Y_t$$

and the above mentioned status of the variables.

Consequence We can replace the constraint into the objective in order to simply get the optimal decision on consumption.

Rmk Note that we can formulate the problem in a simple recursive way. Let $\mathcal{V}_t$ the value of this dynamic problem in period t

$$\mathcal{V}_t = \sum_{i=0}^{\infty} \beta^i U(C_{t+i}) \equiv U(C_t) + \beta U(C_{t+1}) + \beta^2 U(C_{t+2}) + ...$$

now, rewrite this value in period $t+1$ (just manipulate the time index)

$$\mathcal{V}_{t+1} = \sum_{i=0}^{\infty} \beta^i U(C_{t+1+i}) \equiv U(C_{t+1}) + \beta U(C_{t+2}) + \beta^2 U(C_{t+3}) + ...$$

Now multiply $\mathcal{V}_{t+1}$ by β

$$\beta\mathcal{V}_{t+1} = \beta \sum_{i=0}^{\infty} \beta^i U(C_{t+1+i}) \equiv \beta U(C_{t+1}) + \beta^2 U(C_{t+2}) + \beta^3 U(C_{t+3}) + ...$$

and then subtract $\beta\mathcal{V}_{t+1}$ from $\mathcal{V}_t$

$$\mathcal{V}_t - \beta\mathcal{V}_{t+1} = U(C_t) \Leftrightarrow \mathcal{V}_t = U(C_t) + \beta\mathcal{V}_{t+1}$$

So, the value of this intertemporal decision problem today is equal to the current utility plus the discounted value tomorrow.

Given the optimal decision on consumption C_t, we can obtain easily compute the welfare function of the consumer (useful to evaluate the welfare implication of various economic policy, for example fiscal policy)

Two equivalent Economies

i) A Small Open Economy

The equilibrium on the good market

$$Y_t = C_t + I_t + G_t + X_t - IM_t$$

where Y_t, C_t, I_t, G_t, X_t and IM_t denote output, consumption, investment, government spending, exports and imports, respectively (all variables are expressed in real terms).

The current account equation

$$B_{t+1} = (1 + r)B_t + TB_t$$

where B_t is the net wealth of the domestic economy with respect to the rest of the world. The (constant) real interest rate ($r > 0$) is given (by the rest of the world).

Let us define TB_t the trade balance (or net exports)

$$TB_t = X_t - IM_t$$

No relative price (a single homogenous good is produced both home and abroad)

Combining the previous equations yields

$$B_{t+1} = (1 + r)B_t + (Y_t - I_t - G_t) - C_t$$

where $Y_t - I_t - G_t$ is assumed to be exogenous

Equivalence: $A = B$ and Y is now defined as net of private investment and public spending $(Y_t - I_t - G_t)$

ii) A Growth Model

Equilibrium on the good market

$$Y_t = C_t + I_t$$

where Y_t, C_t and I_t denote output, consumption and investment, respectively.

Law of motion of physical capital K_t

$$K_{t+1} = (1 - \delta)K_t + I_t$$

where $\delta \in [0, 1]$ is a constant depreciation rate.

The technology

$$Y_t = Z_t + aK_t$$

where Z_t represent changes in the production, independently from the level of K_t. $a > 0$ is a scale parameter (see the exercise and chap. 2).

Note that the capital stock today is pre-determined and investment decision today will become productive tomorrow.

If we combine the three equations above, we get:

$$K_{t+1} = (1 - \delta + a)K_t + Z_t - C_t$$

Equivalence: $A = K$, $r = a - \delta$ (the real interest rate is equal the marginal productivity of capital minus the depreciation rate, i.e. the real return of physical capital) and $Y_t = Z_t$.

Optimality Condition

We assume that the utility takes the following form

$$U(C_t) = \alpha_0 C_t - \frac{\alpha_1}{2} C_t^2$$

where $\alpha_0, \alpha_1 > 0$.

The marginal utility is linear and decreasing in C

$$U'(C_t) = \alpha_0 - \alpha_1 C_t \qquad (U'' = -\alpha_1 < 0)$$

To simply get the optimal decision on consumption, replace the constraint into the objective

$$\sum_{i=0}^{\infty} \beta^i U(C_{t+i}) = U(C_t) + \beta U(C_{t+1}) + \beta^2 U(C_{t+2}) + ...$$

where

$$C_t = (1+r)A_t + Y_t - A_{t+1}$$

$$C_{t+1} = (1+r)A_{t+1} + Y_{t+1} - A_{t+2}$$

$$C_{t+2} = (1+r)A_{t+2} + Y_{t+2} - A_{t+3}$$

and so on.

Don't forget: controlling $C_t \Leftrightarrow$ controlling A_{t+1} every period (increasing current consumption is equivalent to reduce saving!)

$$U(C_t) + \beta U(C_{t+1}) + \beta^2 U(C_{t+2}) + ... = U((1+r)A_t + Y_t - A_{t+1}) +$$

$$\beta U((1+r)A_{t+1} + Y_{t+1} - A_{t+2})$$

$$+\beta^2 U((1+r)A_{t+2} + Y_{t+2} - A_{t+3}) + ...$$

Then maximize the objective with respect to A_{t+1}. This will give you the optimal consumption/saving decision in period t.

You can always redo the exercise for other periods (see the discussion below).

First Order Condition (denoted FOC, hereafter) in period t.

$$-U'(C_t) + \beta(1 + r)U'(C_{t+1}) = 0$$

This equation is called the **Euler equation** on consumption.

This decision is intertemporal because increasing consumption today will reduce saving and then consumption tomorrow.

The agent will thus face a trade–off (intertemporal substitution) between consumption today and consumption tomorrow.

Equivalently

$$U'(C_t) = \beta(1+r)U'(C_{t+1}) \quad \Leftrightarrow \quad \frac{U'(C_t)}{U'(C_{t+1})} = \beta(1+r)$$

The marginal rate of substitution of consumption between period t and period $t+1$ is equal to the discounted real interest rate (relative price).

Rmk: this optimality condition is satisfied every period (if we redo this optimization problem at other periods $t+h$).

$$U'(C_{t+h}) = \beta(1+r)U'(C_{t+h+1})$$

for $h = 0, 1, ...$ So, up the time index, the optimal plan for consumption remains the same.

The consumption decision is thus *time consistent*.

In other words, given the structure of the theoretical model, there is no incentive for the consumer to modify their optimal plans over time.

Now, we assume

$$\beta(1 + r) = 1$$

Subjective discount factor = Objective discount rate

To see this, let $\beta = 1/(1 + \theta)$, where $\theta > 0$ is the subjective rate of time preference;

θ (or β) is a deep parameter representing a key intertemporal behavior: θ large (β small) means an impatient agent, whereas θ small (β close to one) means a patient agent.

It follows that

$$\beta(1+r) = 1 \Leftrightarrow r = \theta$$

Now, we use the marginal utility

$$U'(C_t) = \alpha_0 - \alpha_1 C_t$$

and we substitute this marginal utility into the Euler equation

$$\alpha_0 - \alpha_1 C_t = \beta(1+r)(\alpha_0 - \alpha_1 C_{t+1})$$

Because $r = \theta$, this Euler equation reduces to

$$C_t = C_{t+1}$$

The optimal decision for the (representative) household is to maintain (perfect smoothing) constant its consumption over time.

Again, this optimal decision is satisfied every period

$$C_{t+h} = C_{t+h+1}$$

for $h = 0, 1, ...$ This consumption decision is thus *time consistent*.

The Euler equation defines the *structural form* of the model, but this is not the solution.

Definition: The solution of any dynamic model expresses the control variable(s) as a (linear) function of the state variable(s) and the exogenous variable(s).

In our case, the solution for the consumption will be of the following linear form

$$C_t = \phi_1 A_t + \phi_2 Y_t$$

where the parameters ϕ_1 and ϕ_2 are determined by the model's properties and a function of the structural parameters (see our computations below).

Computation of the solution (forward substitutions)

$\Rightarrow$ From the instantaneous budget constraint to the intertemporal budget constraint

Let us start from the instantaneous budget constraint in period t:

$$A_{t+1} = (1+r)A_t + Y_t - C_t$$

Because $r > 0$ and thus $1 + r > 1$, the first order difference equation is unstable backward (current wealth as a function of past wealth).

But it is stable forward (current wealth as a function of future wealth)

[Insert figures that illustrate this property]

$$A_t = \frac{1}{1+r}A_{t+1} + \frac{1}{1+r}C_t - \frac{1}{1+r}Y_t$$

Now, write this equation in period $t+1$

$$A_{t+1} = \frac{1}{1+r}A_{t+2} + \frac{1}{1+r}C_{t+1} - \frac{1}{1+r}Y_{t+1}$$

and then substitute this budget constraint in period $t+1$ into the budget constraint in period t

$$A_t = \frac{1}{(1+r)^2} A_{t+2} + \frac{1}{(1+r)^2} C_{t+1} - \frac{1}{(1+r)^2} C_{t+1} + \frac{1}{1+r} C_t - \frac{1}{1+r} Y_t$$

Again, write the budget constraint in period $t+2$ and then substitute this expression into A_t, ... and continue.

This yields

$$A_t = \lim_{T\to\infty} \frac{1}{(1+r)^T} A_{t+T} + \frac{1}{1+r} \lim_{T\to\infty} \sum_{i=0}^{T-1} \frac{1}{(1+r)^i} (C_{t+i} - Y_{t+i})$$

Now, impose the “transversality” condition

$$\lim_{T\to\infty} \frac{1}{(1+r)^T} A_{t+T} = 0$$

meaning technically that the wealth can not explode at some rate that exceeds r.

With this restriction on A_t, the intertemporal budget constraint becomes

$$A_t = \frac{1}{1+r} \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} (C_{t+i} - Y_{t+i})$$

or equivalently

$$A_t + H_t = \frac{1}{1+r} \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} C_{t+i}$$

where H_t defines the non–financial wealth (or "human" wealth), i.e.

all that you can deterministically earn during your (infinite) participation on the labor market.

$$H_t = \frac{1}{1+r}\sum_{i=0}^{\infty}\frac{1}{(1+r)^i}Y_{t+i}$$

To obtain the consumption function, we first need to compute

$$\frac{1}{1+r}\left(C_t + \frac{1}{1+r}C_{t+1} + \frac{1}{(1+r)^2}C_{t+2} + ...\right)$$

From the Euler equation on consumption, we know that the optimal decision satisfies

$$C_{t+h} = C_{t+h+1}$$

for $h = 0, 1, 2, ...$

It follows that

$$C_t = C_{t+1} = C_{t+2} = ...$$

If we replace this into the above equation, we obtain

$$\frac{1}{1+r}\left(C_t + \frac{1}{1+r}C_{t+1} + \frac{1}{(1+r)^2}C_{t+2} + ...\right) =$$

$$\frac{1}{1+r}C_t\left(1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + ...\right)$$

Because $r > 0$, then $1/(1+r) < 0$, so the sequence

$$\left(1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + ...\right) = \sum_{i=0}^{\infty} \frac{1}{(1+r)^i}$$

converges to a finite number.

We have

$$\sum_{i=0}^{\infty} \frac{1}{(1+r)^i} = \frac{1}{1-\frac{1}{1+r}} \equiv \frac{1+r}{r}$$

We obtain

$$\frac{1}{1+r} \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} C_{t+i} = \frac{C_t}{r}$$

Now, if we replace this expression into

$$A_t + H_t = \frac{1}{1+r} \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} C_{t+i}$$

we get

$$C_t = r\,(A_t + H_t)$$

The consumption function defines a linear relation between the current consumption and the financial and human wealth.

Very different from the Keynesian consumption function that expresses the current consumption as a linear function of the current income).

The variable H_t defines the non–financial (or human) wealth.

It represents all the income that an agent can obtain from its labor income over its (infinite) lifetime.

Here, we assume perfect foresight (the deterministic case), so the agent perfectly knows the sequence

$$Y_t, \quad Y_{t+1}, \quad Y_{t+2}, \ldots$$

For simplicity, assume that the labor income is constant

$$Y_t = \bar{Y} \quad \forall t$$

Putting this into the non–financial wealth

$$H_t = \frac{1}{1+r} \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} Y_{t+i},$$

we obtain

$$H_t = \frac{1}{1+r} Y_t \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} = \frac{1}{1+r} Y_t \frac{1+r}{r} \equiv \frac{Y_t}{r}$$

Now, using this expression, we can deduce the consumption function

$$C_t = r\,(A_t + H_t) \equiv r A_t + Y_t$$

The current consumption C_t is a linear function of current income Y_t

+ financial revenues $r A_t$.

Now, replace this expression into the instantaneous budget constraint

$$A_{t+1} = (1 + r)A_t + Y_t - C_t \Leftrightarrow A_{t+1} = A_t$$

The consumption let the level of the financial wealth constant.

In this economy, the optimal decision on consumption allows to smooth the consumption and to let constant the financial wealth.

Rmk: Another way to compute the solution is the method of *undetermined coefficients*.

The basic idea is to identify the formula for the parameters ϕ_1 and ϕ_2 in the postulated equation

$$C_t = \phi_1 A_t + \phi_2 Y_t$$

that satisfied both the Euler equation on consumption (optimal decisions) and the budget constraint (restrictions on the intertemporal allocation of consumption).

First replace the postulated equation into the Euler equation on consumption

$$\phi_1 A_{t+1} + \phi_2 Y_{t+1} = \phi_1 A_t + \phi_2 Y_t$$

we know that

$$Y_t = \bar{Y} \qquad \forall t$$

Then,

$$\phi_1 A_{t+1} + \phi_2 \bar{Y} = \phi_1 A_t + \phi_2 \bar{Y} = \Leftrightarrow A_{t+1} = A_t$$

Now use the budget constraint

$$A_{t+1} = (1 + r)A_t + Y_t - C_t$$

and substitute the postulated equation for C_t

$$(1 + r)A_t + Y_t - C_t = A_t \Leftrightarrow (1 + r)A_t + Y_t - \phi_1 A_t - \phi_2 Y_t = A_t$$

Identify the terms

$$(1 + r) - \phi_1 = 1 \quad , \quad 1 - \phi_2 = 0$$

This gives us the two parameters $\phi_1 = r$ and $\phi_2 = 1$.

The Stochastic Case

The setup

A (representative) household will seek to maximize (every period)

$$E_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i})$$

under a sequence of (instantaneous) budget constraint (as before)

$$A_{t+1} = (1+r)A_t + Y_t - C_t$$

The status and the definition of the variables are the same as before, but now the variable Y_t is stochastic (for example, stochastic after tax real wage).

The agent knows the stochastic process of this variable (she knows the probability distribution of Y_t).

The operator E_t denotes the conditional expectation operator.

Optimality condition

The same as before, up to the conditional expectation operator E_t.

Rmk Rational Expectations:

Here, we assume rational expectations

What are rational expectations?

Expectations lie at the core of economic dynamics (individual behavior, but most importantly equilibrium properties).

Long tradition in economics, since Keynes, but put forward by the new classical economy (Lucas, Prescott, Sargent in the 70' and 80').

The term “rational expectations” is most closely associated to Robert

Lucas (Nobel Laureate, University of Chicago), but the rationality of expectations deeply examined before (see Muth, 1960)

We consider the following definition.

Def *Agents formulate expectations in such a way that their subjective probability distribution of economic variables (conditional on the available information) coincides with the objective probability distribution of the same variable (according to a measure of the state of nature) in an equilibrium.*

Expectations should be consistent with the model $\Rightarrow$ Solving the model is finding an expectation function (see below when we will solve the model.)

With this definition (and some restrictions on the model, i.e. recursive representation), we assume that agents know the model and the probability distribution of exogenous variables (or shocks) that hit the economy.

$E_t \equiv E(./I_t)$ where I_t denotes the information set in period t, i.e. when the consumer (or firm) agent makes its decision on consumptions (or labor/capital demand).

Here, I_t includes all the histories of C and Y, i.e. $\{C_t, C_{t-1}, ...; Y_t, Y_{t-1}, ...\}$

$E_t C_{t+1}$ is thus the linear projection of

C_{t+1} on $\{C_t, C_{t-1}, ...; Y_t, Y_{t-1}, ...\}$

Properties of rational expectations

Property 1: No systematic bias. Let the expectation error $\varepsilon^c_{t+1} = C_{t+1} - E_t C_{t+1}$. This error term satisfies:

$$E_t \varepsilon^c_{t+1} = 0$$

Proof: Straightforward (using $E_t(A+B) = E_t A + E_t B$ and $E_t E_t A = E_t A$):

$$E_t \varepsilon^c_{t+1} = E_t(C_{t+1} - E_t C_{t+1}) = E_t C_{t+1} - E_t E_t C_{t+1}$$

$$= E_t C_{t+1} - E_t C_{t+1} = 0$$

Property 2: Expectation errors do not exhibit any serial correlation.

Proof: Straightforward using the conditional auto–covariance function (see the previous chapter for a formal definition of this function):

$$\begin{aligned} Cov_t(\varepsilon^c_{t+1}, \varepsilon^c_t) &= E_t(\varepsilon^c_{t+1}\varepsilon^c_t) - E_t(\varepsilon^c_{t+1})E_t(\varepsilon^c_t) \\ &= E_t(\varepsilon^c_{t+1})\varepsilon^c_t - E_t(\varepsilon^c_{t+1})\varepsilon^c_t \\ &= 0 \end{aligned}$$

An example: an AR(1) process for $\tilde{Y}_t$ (see below the specification of labor income wages)

$$\tilde{Y}_t = \rho \tilde{Y}_{t-1} + \varepsilon_t^y$$

The information set in period t is given by all the realizations of the random variable $\tilde{Y}$ from period t (all the history at period t of the variable $\tilde{Y}$), i.e.

$$I_t = \{\tilde{Y}_t, \tilde{Y}_{t-1}, ...\}$$

From this definition, we get

$$E_t\tilde{Y}_{t+1} = E(\rho\tilde{Y}_t + \varepsilon_t^y) = \rho E_t\tilde{Y}_t + E_t\varepsilon_{t+1}^y = \rho\tilde{Y}_t + E_t\varepsilon_{t+1}^y$$

Since ε_{t+1}^y is an innovation, it is orthogonal to the information set and thus $E_t\varepsilon_{t+1}^y = 0$.

It follows that

$$E_t\tilde{Y}_{t+1} = \rho\tilde{Y}_t$$

The expectation errors

$$\varepsilon^y_{t+1} = \tilde{Y}_{t+1} - E_t\tilde{Y}_{t+1} = \tilde{Y}_{t+1} - \rho\tilde{Y}_t$$

thus satisfies

$$E_t\varepsilon^y_{t+1} = 0$$

This property of rational expectations coincides with the optimal forecast of an econometrician who use the observations and the AR(1) process to formulate an optimal forecast of $\tilde{Y}$ (one step–ahead)

We now go back to our stochastic setup

$$E_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}) = E_t \left(U(C_t) + \beta U(C_{t+1}) + \beta^2 U(C_{t+2}) + ... \right)$$

where

$$C_t = (1+r)A_t + Y_t - A_{t+1}$$

$$C_{t+1} = (1+r)A_{t+1} + Y_{t+1} - A_{t+2}$$

$$C_{t+2} = (1 + r)A_{t+2} + Y_{t+2} - A_{t+3}$$

Again, don't forget: controlling $C_t \Leftrightarrow$ controlling A_{t+1} every period

(increasing current consumption is equivalent to reduce saving!)

$$E_t \left(U(C_t) + \beta U(C_{t+1}) + \beta^2 U(C_{t+2}) + ... \right) = E_t(U((1+r)A_t + Y_t - A_{t+1})$$

$$+\beta U((1 + r)A_{t+1} + Y_{t+1} - A_{t+2})$$

$$+\beta^2 U((1+r)A_{t+2} + Y_{t+2} - A_{t+3}) + ...)$$

Then maximize the objective with respect to A_{t+1}.

Rmk: because A_t is pre-determined and Y_t is known in period t (and $r > 0$ is constant), the level of financial wealth in period $t+1$ is known!

$$E_t\left(-U'(C_t) + \beta(1+r)U'(C_{t+1})\right) = 0$$

This is the **Euler equation** on consumption, but now in expectations. We assume the same utility function as before and we impose the same restriction $\beta(1+r) = 1$. This yields

$$E_t(-C_t + C_{t+1}) = 0$$

By the linearity (in consumption) of the Euler equation

$$E_t(-C_t + C_{t+1}) = -E_t C_t + E_t C_{t+1}$$

and using the fact that the current consumption is known in period t ($E_t C_t = C_t$), we deduce $C_t = E_t C_{t+1}$

or equivalently

$$E_t \Delta C_{t+1} = 0$$

where $\Delta C_{t+1} = C_{t+1} - C_t$. We can also write

$$C_{t+1} = C_t + \varepsilon_{t+1}^c$$

where the random variable ε_{t+1}^c satisfies

$$E_t \varepsilon_{t+1}^c = 0$$

Important Quantitative Implications

1) The current consumption is an optimal predictor of future consumption.

2) The change in consumption is unpredictable and does not display serial correlation.

3) The consumption follows a random walk.

These quantitative implications can be tested easily from actual data (see the discussion below)

Again, we must solve the model, i.e. we must derive the consumption function.

We start from the instantaneous budget constraint in period t

$$A_{t+1} = (1 + r)A_t + Y_t - C_t$$

All the variables are known in period t.

This means that if we apply the conditional expectation operator, we get the same formulation.

Again, we formulate a forward representation of this constraint

$$A_t = \frac{1}{1+r}A_{t+1} + \frac{1}{1+r}C_t - \frac{1}{1+r}Y_t$$

Now, write this equation in period $t+1$

$$A_{t+1} = \frac{1}{1+r}A_{t+2} + \frac{1}{1+r}C_{t+1} - \frac{1}{1+r}Y_{t+1}$$

The level of wealth in period $t+1$ is known in period t, but the variables C_{t+1}, Y_{t+1} and A_{t+2} are unknown.

This is because the income is a stochastic variable and thus agent must formulate (rational) expectations about the future realizations of this variable.

Then, we apply the conditional expectation E_t on this budget constraint

$$A_{t+1} = \frac{1}{1+r} E_t A_{t+2} + \frac{1}{1+r} E_t C_{t+1} - \frac{1}{1+r} E_t Y_{t+1}$$

and then substitute this expected budget constraint in period $t+1$ into the budget constraint in period t

$$A_t = \frac{1}{(1+r)^2} E_t A_{t+2} + \frac{1}{(1+r)^2} E_t C_{t+1} - \frac{1}{(1+r)^2} E_t C_{t+1} + \frac{1}{1+r} C_t - \frac{1}{1+r} Y$$

Again, write the budget constraint in period $t+2$, take expectations conditional on the information set in period t and then substitute this expression into A_t, ... and continue.

This yields

$$A_t = \lim_{T\to\infty} E_t \frac{1}{(1+r)^T} A_{t+T} + \frac{1}{1+r} \lim_{T\to\infty} E_t \sum_{i=0}^{T-1} \frac{1}{(1+r)^i} (C_{t+i} - Y_{t+i})$$

Now, impose the "transversality" condition (in expectations)

$$\lim_{T\to\infty} E_t \frac{1}{(1+r)^T} A_{t+T} = 0$$

meaning technically that the wealth can not explode in expectations

at some rate that exceeds r.

The intertemporal budget constraint is given by

$$A_t = \frac{1}{1+r} E_t \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} (C_{t+i} - Y_{t+i})$$

or equivalently

$$A_t + H_t = \frac{1}{1+r} E_t \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} C_{t+i}$$

where H_t defines the non–financial wealth (or "human" wealth), i.e. all that you can expect to earn during your (infinite) participation on the labor market.

$$H_t = \frac{1}{1+r} E_t \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} Y_{t+i}$$

To obtain the consumption function, we first need to compute

$$\frac{1}{1+r} E_t \left(C_t + \frac{1}{1+r} C_{t+1} + \frac{1}{(1+r)^2} C_{t+2} + \ldots \right)$$

From the Euler equation on consumption, we know that the optimal decision satisfies

$$C_{t+h} = E_{t+h} C_{t+h+1}$$

for $h = 0, 1, 2, ...$

More precisely, we have

$$E_t C_{t+1} = C_t$$

$$E_{t+1}C_{t+2} = C_{t+1}$$

$$E_{t+2}C_{t+3} = C_{t+2}$$

and so on. Now use, the second Euler equation (Euler equation in period $t + 1$)

$$E_{t+1}C_{t+2} = C_{t+1}$$

and apply the conditional expectations operator E_t

$$E_t E_{t+1} C_{t+2} = E_t C_{t+1}$$

By definition of rational expectations, the difference between the information set in period t and $t + 1$ is unpredictable, so

$$E_t E_{t+1} C_{t+2} = E_t C_{t+2}$$

We then deduce

$$E_t C_{t+2} = E_t C_{t+1} = C_t$$

If we perform the same exercise on the third Euler equation, we get

$$E_t C_{t+3} = E_t C_{t+2} = E_t C_{t+1} = C_t$$

More generally, we have

$$E_tC_{t+h} = E_tC_{t+h-1} = ... = E_tC_{t+1} = C_t$$

for some positive integer h.

From the above computations, we can easily determine

$$\frac{1}{1+r}E_t\left(C_t + \frac{1}{1+r}C_{t+1} + \frac{1}{(1+r)^2}C_{t+2} + ...\right) =$$

$$\frac{1}{1+r}C_t\left(1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + ...\right)$$

Because $r > 0$, then $1/(1+r) < 0$, so the sequence

$$\left(1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + ...\right) = \sum_{i=0}^{\infty}\frac{1}{(1+r)^i}$$

converges to a finite number.

We have

$$\sum_{i=0}^{\infty} \frac{1}{(1+r)^i} = \frac{1}{1-\frac{1}{1+r}} \equiv \frac{1+r}{r}$$

We obtain

$$\frac{1}{1+r} E_t \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} C_{t+i} = \frac{C_t}{r}$$

Now, if we replace this expression into

$$A_t + H_t = \frac{1}{1+r} E_t \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} C_{t+i}$$

we get

$$C_t = r\left(A_t + H_t\right)$$

We must now determine the non–financial wealth H_t.

To do that, we need to specify a stochastic process for Y_t.

This stochastic process is perfectly known by the consumer.

We assume that Y_t is determined by the following process

$$Y_t = \bar{Y} + \tilde{Y}_t$$

where $\bar{Y}$ is a known positive constant (the deterministic permanent component) and the stochastic (possibly highly persistent) component is given by

$$\tilde{Y}_t = \rho \tilde{Y}_{t-1} + \varepsilon_t^y$$

where $\rho \in [0, 1]$. The random term satisfies

$$E_t \varepsilon^y_{t+1} = 0$$

The non–financial wealth is given by

$$H_t = \frac{1}{1+r} E_t \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} Y_{t+i},$$

So, we need to compute

$$E_t Y_t, E_t Y_{t+1}, E_t Y_{t+2}, ...$$

From the above process for Y_t, we deduce

$$E_t Y_t = Y_t \equiv \bar{Y} + \tilde{Y}_t$$

$$E_t Y_{t+1} = E_t \left(\bar{Y} + \tilde{Y}_{t+1} \right) = \bar{Y} + E_t \tilde{Y}_{t+1}$$

$$= \bar{Y} + E_t(\rho \tilde{Y}_t + \varepsilon^y_{t+1}) = \bar{Y} + \rho \tilde{Y}_t + E_t \varepsilon^y_{t+1}$$

$$= \bar{Y} + \rho \tilde{Y}_t$$

$$E_t Y_{t+2} = E_t \left(\bar{Y} + \tilde{Y}_{t+2} \right) = \bar{Y} + E_t \tilde{Y}_{t+2}$$

$$= \bar{Y} + E_t(\rho \tilde{Y}_{t+1} + \varepsilon^y_{t+2}) = \bar{Y} + \rho E_t \tilde{Y}_{t+1} + E_t \varepsilon^y_{t+2}$$

$$= \bar{Y} + \rho^2 \tilde{Y}_t$$

and we continue...

The non–financial wealth is then deduced

$$H_t = \frac{1}{1+r}\bar{Y}\sum_{i=0}^{\infty}\frac{1}{(1+r)^i} + \frac{1}{1+r}\tilde{Y}_t\sum_{i=0}^{\infty}\frac{\rho^i}{(1+r)^i}$$

Since $\rho \in [0,1]$ and $r > 0$, it follows that

$$\frac{\rho}{(1+r)} < 1$$

and the sequence $\frac{\rho^i}{(1+r)^i}$ is convergent.

We obtain

$$H_t = \frac{\bar{Y}}{r} + \frac{1}{1+r-\rho}\tilde{Y}_t$$

From this expression, we see that the process for Y_t matters a lot for the non–financial wealth.

This is because agents use the stochastic process to make their forecast about their future income.

This means that non–financial wealth is not invariant to the parameters of the process.

Using H_t, we can deduce the consumption function

$$C_t = rA_t + \bar{Y} + \frac{r}{1+r-\rho}\tilde{Y}_t$$

Rmk: The *undetermined coefficients* in a stochastic setup.

The basic idea is to identify the formula for the parameters ϕ_0, ϕ_1 and ϕ_2 in the postulated (linear) equation

$$C_t = \phi_0 + \phi_1 A_t + \phi_2 \tilde{Y}_t$$

that satisfied both the Euler equation on consumption (optimal decisions) and the budget constraint (restrictions on the intertemporal allocation of consumption).

First replace the postulated equation into the Euler equation on consumption

$$\phi_0 + \phi_1 E_t A_{t+1} + \phi_2 E_t \tilde{Y}_{t+1} = \phi_0 + \phi_1 A_t + \phi_2 \tilde{Y}_t$$

We know that

$$E_t A_{t+1} = A_{t+1} \quad \text{because } A \text{ is pre-determined}$$

and

$$\tilde{Y}_{t+1} = \rho \tilde{Y}_t$$

Then,

$$\phi_1 A_{t+1} + \phi_2 \rho \tilde{Y}_t = \phi_1 A_t + \phi_2 \rho \tilde{Y}_t$$

Now use the budget constraint

$$A_{t+1} = (1+r)A_t + \bar{Y} + \tilde{Y}_t - C_t$$

and substitute the postulated equation for C_t

$$\phi_1(1+r)A_t + \phi_1\bar{Y} + \phi_1\tilde{Y}_t - \phi_1 C_t = \phi_1 A_t + \phi_2\tilde{Y}_t$$

$$\phi_1(1+r)A_t + \phi_1\bar{Y} + \phi_1\tilde{Y}_t - \phi_1\phi_0 - \phi_1^2 A_t - \phi_1\phi_2\tilde{Y}_t = \phi_1 A_t + \phi_2\tilde{Y}_t$$

Identify the terms

$$\phi_1(1+r) - \phi_1^2 = \phi_1 \quad \phi_1\bar{Y} - \phi_1\phi_0 = 0 \quad \phi_1 - \phi_1\phi_2 + \phi_2\rho = \phi_2$$

This gives us the three parameters $\phi_0 = \bar{Y}$, $\phi_1 = r$ and $\phi_2 = r/(1 + r - \rho)$.

The Lucas Critique

Lucas (1976) claims that standard macroeconometric models can not be used for policy evaluation, because that the consumption function (among other private behaviors) are not invariant to policy parameters.

In other words, these macroeconometric models wrongly assume that the parameters of the consumption function does not change when agents modify their expectations after a policy change.

To see this, suppose that the econometrician wrongly assumes that the consumption function of the form:

$$C_t = c_0 + c_1 \tilde{Y}_t + c_2 A_t$$

This specification is richer than the standard keynesian consumption function, because it includes some wealth effects.

But, this specification assumes that the parameter c_1 is policy invariant.

From the model's solution, this parameter is given by

$$\frac{\partial C_t}{\partial \tilde{Y}_t} = \frac{r}{1 + r - \rho}$$

This parameter is not invariant to ρ, the parameters that governs the expectations about future income.

To illustrate this lack of invariance, assume pure transitory income shock ($\rho = 0$)

$$\frac{\partial C_t}{\partial \tilde{Y}_t} = \frac{r}{1+r}$$

The sensitivity of consumption to income is very small (r=0.01 per quarter, so sensitivity around 1%).

This is because agents expects only transitory changes in the labor income.

Because they want to smooth their consumption over time, they almost disconnect their consumption from transitory incomes.

Assume now permanent income shocks ($\rho = 1$). The sensitivity is now

$$\frac{\partial C_t}{\partial \tilde{Y}_t} = 1$$

So, the response is now one-to-one, because agents expect now permanent changes.

Model'Solution

If we replace the consumption function into the instantaneous budget constraint in period t, we obtain

$$A_{t+1} = A_t + \frac{1-\rho}{1+r-\rho}\tilde{Y}_t$$

In period $t-1$, we also have

$$A_t = A_{t-1} + \frac{1-\rho}{1+r-\rho}\tilde{Y}_{t-1}$$

Now, define $\Delta A_t = A_t - A_{-1}$ and take the first difference of the consumption function

$$\begin{aligned}
\Delta C_t &= \Delta A_t + \frac{r}{1+r-\rho}\Delta\tilde{Y}_t \\
&= \frac{1-\rho}{1+r-\rho}\tilde{Y}_{t-1} + \frac{r}{1+r-\rho}\Delta\tilde{Y}_t \\
&= \frac{r}{1+r-\rho}\tilde{Y}_t - \frac{r\rho}{1+r-\rho}\tilde{Y}_{t-1}
\end{aligned}$$

$$\Delta C_t = \frac{r}{1+r-\rho}\varepsilon_t^y$$

So the consumption function rewrites (a random walk)

$$C_t = C_{t-1} + \frac{r}{1+r-\rho}\varepsilon_t^y$$

Note that when computing the consumption function, we determine the relationship between an innovation on consumption ε_t^c and an innovation on stochastic income ε_t^y

$$\varepsilon_t^c = \frac{r}{1+r-\rho}\varepsilon_t^y$$

From the above equation, we can determine the dynamic responses of consumption to an unexpected shock to stochastic labor income. This is given by the following expression (IRF for Impulse Response Function):

$$IRF_c(h) = \frac{\partial C_{t+h}}{\partial \varepsilon_t^y} \qquad \text{for} \ \ h = 0, 1, 2, ...$$

[Insert a figure]

We deduce

$$IRF_c(0) = \frac{r}{1+r-\rho}\ , \quad IRF_c(1) = \frac{r}{1+r-\rho}\ , \quad IRF_c(2) = \frac{r}{1+r-\rho}\ ,$$

The IRF is an increasing function of ρ, consumption is more sensitive to an innovation to labor income.

A Special Case

Unit root in the income process ($\rho = 1$), supported by the actual data in industrialized countries.

$$\tilde{Y}_t = \tilde{Y}_{t-1} + \varepsilon_t^y$$

So, the change in labor income ΔY_t is given by

$$\Delta Y_t = \varepsilon_t^y$$

We deduce from the above formula

$$C_t = rA_t + Y_t$$

$$A_{t+1} = A_t$$

$$\Delta C_t = \Delta Y_t$$

Quantitative implications

1) Volatility

$$\sigma(\Delta C_t) = \sigma(\Delta Y_t)$$

In the actual data (in industrialized countries, where C is measured as the sum of non–durable goods and services)

$$\sigma(\Delta C_t) = 0.7\sigma(\Delta Y_t)$$

"Excess Smoothness Puzzle" (Deaton, 1987)

2) Persistence

$$\Delta C_t = \varepsilon_t^y$$

Consumption changes do not display serial correlation.

In the data,

$$Corr(\Delta C_t, \Delta C_{t-1}) \simeq 0.3$$

So consumption changes are predictable. Other empirical studies have shown that a set of relevant variables (past consumption, pas income) help to predict future consumption.

Extensions

- Habit persistence in consumption

- Borrowing constraint

2.2: The Dynamic Labor Demand Model

Standard theory only consider short–run and long–run labor demand without any explicit adjustment process between these two situations.

The theory of labor demand with adjustment costs offers an opportunity to fill the gap between theses two situations.

Question: Why adjustment costs on labor??

Possible Explanations:

Cost of adjusting employment

$$\Delta L_t = H_t - F_t$$

where H_t denotes hirings and F_t denotes firings.

- Hiring costs: training costs, re–organization costs
- Firing costs: legislation (employment protection)

Empirical studies suggest

- In US, hiring costs are high and outstrip the cost of separation
- In countries where strong legal measures are in place to ensure

job security (many countries of continental Europe), the costs of separation far outstrip recruitment costs.

The theoretical setup: a dynamic labor demand model

Possible forms of adjustment costs.

Two versions:

- Deterministic setup.
- Stochastic setup.

The Costs of Labor Adjustments

Quadratic Costs

Let us consider the first specification

$$\mathcal{C}(L_t, L_{t-1}) = \frac{b}{2}(L_t - L_{t-1} - a)^2$$

where $a, b > 0$.

Advantage: this function introduces an asymmetry between the cost of positive and negative variations in employments, since $a > 0$.

Limit: when L is constant, there exists strictly positive costs of adjusting employment (!!!), since $a > 0$.

To eliminate this problem, set $a = 0$ in the previous specification

$$\mathcal{C}(L_t, L_{t-1}) = \frac{b}{2}(L_t - L_{t-1} - a)^2$$

where $a > 0$.

But now, costs are convex and **symmetric**.

Asymmetric Convex Costs:

$$\mathcal{C}(L_t, L_{t-1}) = -1 + \exp(L_t - L_{t-1}) - a(L_t - L_{t-1}) + \frac{b}{2}(L_t - L_{t-1})^2$$

Depending on the value of a, the marginal cost of an increase in employment is greater (or smaller) than that of a reduction.

When $a = 0$, we retrieve the quadratic case.

Note that when L is constant, the cost of adjustment is zero.

Another form:

$$\mathcal{C}(L_t, L_{t-1}) = \frac{b_h}{2}(L_t - L_{t-1})^2 \qquad \text{if } L_t - L_{t-1} \geq 0$$

and

$$\mathcal{C}(L_t, L_{t-1}) = \frac{b_f}{2}(L_t - L_{t-1})^2 \qquad \text{if} \;\; L_t - L_{t-1} \leq 0$$

where $b_h, b_f > 0$

Linear Costs:

$$\mathcal{C}(L_t, L_{t-1}) = b_h\,(L_t - L_{t-1}) \qquad \text{if } \; L_t - L_{t-1} \geq 0$$

and

$$\mathcal{C}(L_t, L_{t-1}) = -b_f\,(L_t - L_{t-1}) \qquad \text{if } \; L_t - L_{t-1} \leq 0$$

where $b_h, b_f > 0$

The parameter b_h represents the unit cost of a hiring and b_f is the unit cost of firing.

The adjustment of employment is asymmetric since $b_h \neq b_f$

Fixed Costs:

Firms face a strictly positive cost when

$$L_t - L_{t-1} \neq 0,$$

but they are not subject to any cost if

$$L_t - L_{t-1} = 0$$

The Deterministic Setup

The value of the firm $\mathcal{V}_t$ is the discounted sum of instantaneous profits

$$\mathcal{V}_t = \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i \Pi_{t+i}$$

where $r > 0$ is a constant real interest rate and the instantaneous profit is given by

$$\Pi_t = Y_t - W_t L_t - \mathcal{C}(L_t, L_{t-1})$$

where W_t is the real wage (the price of Y_t is used as a *numeraire*.

Rmk: As before, we can define the value of the firm in a recursive way.

$$\mathcal{V}_t = \Pi_t + \left(\frac{1}{1+r}\right)\mathcal{V}_{t+1}$$

The adjustment cost function is given by

$$\mathcal{C}(L_t, L_{t-1}) = \frac{b}{2}(L_t - L_{t-1})^2$$

where $b \geq 0$ is the adjustment cost parameter.

W_t is assumed to be perfectly known (no shocks)

The production function is given by

$$Y_t = \alpha_0 L_t - \frac{\alpha_1}{2} L_t^2$$

where $\alpha_0, \alpha_1 > 0$. The production function satisfies

$$\frac{\partial Y}{\partial Y} = \alpha_0 - \alpha_1 L_t > 0$$

$\Leftrightarrow L_t < (\alpha_0/\alpha_1)$ (we assume that this restriction holds)

$$\frac{\partial^2 Y}{\partial Y^2} = -\alpha_1 < 0$$

Decreasing return to scale.

If $\alpha_1 = 0$, constant return to scale and the marginal labor productivity is independent from the level of output (or employment).

Intertemporal decisions

The decision on L_t today affect profits today: production $(F(L_t))$, labor cost (W_t), cost of adjustment $((b/2)(L_t - L_{t-1})^2)$

The decision on L_t today affect profits tomorrow: cost of adjustment $((b/2)(L_{t+1} - L_t)^2)$

So, choices on L_t are interdependent.

If $b = 0$, the value of the firm is the discounted sum of independent profit functions (a sequence of static conditions)

$$L_t = \frac{\alpha_0 - W_t}{\alpha_1}$$

Status of the variables

- W_t is an **exogenous** deterministic variable.
- L_{t-1} is a pre-determined variable
- $\{W_t, L_{t-1}\}$ are the state variables (the solution will be a linear function of theses two variables)
- L_t is a the choice variable.

FOC (wrt L_t)

$$\frac{\partial \mathcal{V}_t}{\partial L_t} = 0$$

$\Leftrightarrow$

$$\frac{\partial \Pi}{\partial L_t} + \left(\frac{1}{1+r}\right)\frac{\partial \Pi_{t+1}}{\partial L_t} = 0$$

$\Leftrightarrow$

$$\alpha_0 - \alpha_1 L_t - W_t - b(L_t - L_{t-1}) + \beta b(L_{t+1} - L_t) = 0$$

where $\beta = 1/(1+r) \in (0,1)$.

Rmk If $\alpha_1 = 0$ (constant return to scale, see the exercise in tutorial class), the FOC reduces to

$$\alpha_0 - W_t - bNH_t + \beta bNH_{t+1} = 0$$

where $NH_t = \Delta L_t$ are the net hirings. $\Delta L_t > 0$ (¡0) $\rightarrow NH_t > 0$ (¡0).

We study the case $\alpha_1 > 0$

Let $\bar{L}$ the steady state value of L_t and $\hat{L}_t = L_t - \bar{L}$.

The steady–state of L solves:

$$\alpha_0 - \alpha_1 \bar{L} - \bar{W} - b(\bar{L} - \bar{L}) + \beta b(\bar{L} - \bar{L}) = 0$$

which reduces to

$$\alpha_0 - \alpha_1 \bar{L} - \bar{W} = 0$$

So,

$$\bar{L} = (\alpha_0 - \bar{W})/\alpha_1$$

the “long–run” labor demand.

We have

$$\frac{\partial L}{\partial \bar{W}} = -\frac{1}{\alpha_1} < 0$$

A permanent increase in the real wage (the real cost of labor) reduce long-run employment.

With our definition of $\hat{L}_t$, the FOC rewrites

$$-\alpha_1 \hat{L}_t - \hat{W}_t - b(\hat{L}_t - \hat{L}_{t-1}) + \beta b(\hat{L}_{t+1} - \hat{L}_t) = 0$$

or

$$\beta b \hat{L}_{t+1} - (\alpha_1 + b + \beta b)\hat{L}_t + b\hat{L}_{t-1} = \hat{W}_t$$

$$\hat{L}_{t+1} - \frac{(\alpha_1 + b + \beta b)}{\beta b}\hat{L}_t + \frac{1}{\beta}\hat{L}_{t-1} = \frac{1}{\beta b}\hat{W}_t$$

The second order difference equation admits the following multivariate autoregressive representation

$$\begin{bmatrix} \hat{L}_{t+1} \\ \hat{L}_t \end{bmatrix} = \begin{bmatrix} \frac{(\alpha_1+b+\beta b)}{\beta b} & -\frac{1}{\beta} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{L}_t \\ \hat{L}_{t-1} \end{bmatrix} + \begin{bmatrix} \frac{1}{\beta b} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{W}_t \\ 0 \end{bmatrix}$$

Now, concentrate only on the dynamic properties of $\hat{L}_t$ and ignore $\hat{W}_t$ (or set $\hat{W}_t = 0$, i.e. W_t is always at its steady–state value)

$$\begin{bmatrix} \hat{L}_{t+1} \\ \hat{L}_t \end{bmatrix} = \mathcal{A} \begin{bmatrix} \hat{L}_t \\ \hat{L}_{t-1} \end{bmatrix}$$

where

$$\mathcal{A} = \begin{bmatrix} \frac{(\alpha_1+b+\beta b)}{\beta b} & -\frac{1}{\beta} \\ 1 & 0 \end{bmatrix}$$

To study the dynamic properties of the model, we must characterize the eigenvalues of $\mathcal{A}$.

These are obtained from

$$det(\mathcal{A} - \lambda I_2) = 0$$

In our notations

$$\begin{vmatrix} \frac{(\alpha_1+b+\beta b)}{\beta b} - \lambda & -\frac{1}{\beta} \\ 1 & -\lambda \end{vmatrix} = 0$$

This gives

$$\lambda^2 - \lambda T + D = 0$$

where $T = trace(\mathcal{A})$ and $D = det(\mathcal{A})$.

Given $\mathcal{A}$, we obtain

$$T = \frac{(\alpha_1 + b + \beta b)}{\beta b} \quad \text{and} \quad D = \frac{1}{\beta}$$

We can now define the characteristic equation

$$P(\lambda) = \lambda^2 - \lambda T + D \equiv (\lambda - \lambda_1)(\lambda - \lambda_2)$$

Let $\lambda = 0$. We have

$$P(0) = D \equiv \frac{1}{\beta} > 1$$

We also know that

$$P(0) = \lambda_1 \lambda_2$$

At least one root (say λ_2) is greater than one.

Let us define $P(1)$.

We obtain

$$P(1) = 1 - T + D \equiv 1 - \frac{(\alpha_1 + b + \beta b)}{\beta b} + \frac{1}{\beta}$$

It follows that

$$P(1) = \frac{-\alpha_1}{\beta b} < 0$$

We know that

$$P(1) = (1 - \lambda_1)(1 - \lambda_2)$$

As $P(1) < 0$, we deduce $\lambda_1 < 1$ and $\lambda_2 > 1$.

Property The labor demand model with adjustment costs displays a saddle path property, i.e. it exists a unique path toward the steady–state.

Computation of the solution We know that $\lambda_1 < 1$ and $\lambda_2 > 1$ and here we assume $\hat{W}_t = 0$.

Now, consider the backshift operator B, such that

$$BL_t = L_{t-1}$$

The FOC writes

$$\hat{L}_{t+1} - \frac{(\alpha_1 + b + \beta b)}{\beta b}\hat{L}_t + \frac{1}{\beta}\hat{L}_{t-1} = 0$$

or equivalently

$$\hat{L}_{t+1}\left(1 - \frac{(\alpha_1 + b + \beta b)}{\beta b}B + \frac{1}{\beta}B^2\right) = 0$$

Using the characteristic polynomial, we obtain

$$\hat{L}_{t+1}(1 - \lambda_1 B)(1 - \lambda_2 B) = 0$$

We deduce

$$\left(\tilde{\hat{\hat{L}}}_{t+1}(1 - \lambda_2 B)\right) = 0$$

where

$$\tilde{\hat{\hat{L}}}_{t+1} = (1 - \lambda_1 B)\hat{L}_{t+1} = \hat{L}_{t+1} - \lambda_1 \hat{L}_t$$

The above equation rewrites

$$\tilde{\hat{\hat{L}}}_{t+1} - \lambda_2 \tilde{\hat{\hat{L}}}_t = 0$$

$$\tilde{\hat{\hat{L}}}_{t+1} = \lambda_2 \tilde{\hat{\hat{L}}}_t$$

$$\tilde{\hat{\hat{L}}}_t = \frac{1}{\lambda_2} E_t \tilde{\hat{\hat{L}}}_{t+1}$$

Since $\lambda_2 > 1$, we have $1/\lambda_2 < 1$. So the model must be solved forward, i.e. by successive iterations on the future.

After imposing the “transversality” condition

$$\lim_{T \to \infty} \left(\frac{1}{\lambda_2} \right)^T \tilde{\hat{\hat{L}}}_{t+T} = 0$$

we deduce

$$\tilde{\hat{\hat{L}}}_t = 0$$

Now, using the definition of $\tilde{L}_t$, we get

$$\hat{L}_t = \lambda_1 \hat{L}_{t-1}$$

Now using the definition of $\hat{L}_t$, we deduce

$$L_t = (1 - \lambda_1)\bar{L} + \lambda_1 L_{t-1}$$

or

$$L_t = (1 - \lambda_1)\frac{(\alpha_0 - \bar{W})}{\alpha_1} + \lambda_1 L_{t-1}$$

So, the labor demand can depart from its long–run (”frictionless”) value, due to the costs of adjusting employment between two periods.

The speed of adjustment toward $\bar{L}$ will depend on the adjustment costs parameter.

For b small, the cost is small too, and the discrepancy between L and $\bar{L}$ is not persistent.

Conversely, when b is large, the adjustment process toward the long run value of employment can be very persistent.

The Effect of on increase in W

Let us suppose two real wages W_1 and W_2,

$$W_1 < W_2$$

It follows that

$$\bar{L}_1 > \bar{L}_2$$

Now, we want to evaluate the transition path for L_t from $\bar{L}_1$ to $\bar{L}_2$.

We start with $L_t = \bar{L}_1$. Now using the solution for L_t with the new real wage, we deduce

$$L_{t+1} = (1 - \lambda_1)\bar{L}_2 + \lambda_1 L_t \equiv (1 - \lambda_1)\bar{L}_2 + \lambda_1 \bar{L}_1$$

One period after

$$L_{t+2} = (1 - \lambda_1)\bar{L}_2 + \lambda_1 L_{t+1} \equiv (1 - \lambda_1)\bar{L}_2 + \lambda_1((1 - \lambda_1)\bar{L}_2 + \lambda_1 \bar{L}_1)$$

At the limit (when $h \to \infty$), we obtain

$$\lim_{h\to\infty} L_{t+h} = \lim_{h\to\infty} \lambda_1^h \bar{L}_1 + (1 - \lambda_1)\bar{L}_2 \lim_{h\to\infty} \sum_{i=0}^{h} \lambda_1^h$$

Since $\lambda_1 < 1$, we have

$$\lim_{h\to\infty} \lambda_1^h \bar{L}_1 = 0$$

and

$$\lim_{h\to\infty} \sum_{i=0}^{h} \lambda_1^h = \frac{1}{1 - \lambda_1}$$

We deduce

$$\lim_{h \to \infty} L_{t+h} = \bar{L}_2$$

The labor monotonically converge toward its new steady–state.

[INSERT A FIGURE]

The Stochastic Setup

The value of the firm $\mathcal{V}_t$ is the expected discounted sum of profit.

$$\mathcal{V}_t = E_t \sum_{i=1}^{\infty} \left(\frac{1}{1+r} \right)^i \Pi_{t+i}$$

where E_t is the conditional expectations operator.

$r > 0$ (the constant real interest rate) and the instantaneous profit is given by

$$\Pi_t = Y_t - W_t L_t - \frac{b}{2} (L_t - L_{t-1})^2$$

where W_t is the real wage (the price of Y_t is used as a *numeraire* and $b \geq 0$ is the adjustment cost parameter.

The production function is given by

$$Y_t = \alpha_0 L_t - \frac{\alpha_1}{2} L_t^2$$

where $\alpha_0, \alpha_1 > 0$.

Rmk1 Properties of the production function

$$\frac{\partial Y}{\partial Y} = \alpha_0 - \alpha_1 L_t > 0$$

$\Leftrightarrow L_t < (\alpha_0/\alpha_1)$ (we assume that this restriction holds for all the states of the nature)

$$\frac{\partial^2 Y}{\partial Y^2} = -\alpha_1 < 0$$

Decreasing return to scale. If $\alpha_1 = 0$, constant return to scale and the marginal labor productivity is independent from the level of output (or employment).

Rmk2 Shocks

Here, we assume that only real wage is a random variable. Without difficulty we can also include a TFP shock into the production function

$$Y_t = \alpha_{0,t} L_t - \frac{\alpha_1}{2} L_t^2$$

where $\alpha_{0,t}$ is now a random variable

We specify a stochastic process for W_t. We assume that W_t can be decomposed into two elements: one is deterministic, one is stochastic

$$W_t = \bar{W} + \tilde{W}_t$$

$\bar{W}$ deterministic, $\tilde{W}_t$ stochastic.

The stochastic component $\tilde{W}_t$ follows an Autoregressive process of order one (AR(1)):

$$\tilde{W}_t = \rho_w \tilde{W}_{t-1} + \sigma_{\varepsilon,w} \varepsilon_{w,t}$$

where $\rho_w \in [0, 1]$, $\sigma_{\varepsilon,w} > 0$ and $\varepsilon_{w,t}$ is iid with zero mean and unit variance.

In addition, it verifies:

$$E_t \varepsilon_{w,t+1} = 0$$

Rmk 3 Again, we assume here rational expectations

Rmk4 Intertemporal decisions

The decision on L_t today affect profits today: production (F(L)), labor cost (W), cost of adjustment $((b/2)(L_t - L_{t-1})^2)$

The decision on L_t today affect profits tomorrow: cost of adjustment $((b/2)(L_{t+1} - L_t)^2)$

So, choices on L_t are interdependent.

But, because W_t is stochastic, this creates a decision about an expected variable $E_t L_{t+1}$

If $b = 0$, the value of the firm is the discounted sum of independent profit functions (a sequence of static conditions)

Rmk5 Status of the variables

- W_t is an **exogenous** stochastic variable.
- L_{t-1} is a pre-determined variable
- $\{W_t, L_{t-1}\}$ are the state variables (the solution will be a linear function of theses two variables)
- L_t is a the choice variable.

FOC (wrt L_t)

$$\frac{\partial \mathcal{V}_t}{\partial L_t} = 0$$

$\Leftrightarrow$

$$\frac{\partial \Pi}{\partial L_t} + \left(\frac{1}{1+r}\right) E_t \frac{\partial \Pi_{t+1}}{\partial L_t} = 0$$

$\Leftrightarrow$

$$\alpha_0 - \alpha_1 L_t - W_t - b(L_t - L_{t-1}) + \beta b E_t(L_{t+1} - L_t) = 0$$

where $\beta = 1/(1+r)_i n(0,1)$

Rmk If $\alpha_1 = 0$ (crs), the FOC reduces to

$$\alpha_0 - W_t - bNH_t + \beta b E_t NH_{t+1} = 0$$

where $NH_t = \Delta L_t$ are the net hirings. $\Delta L_t > 0$ (¡0) $\rightarrow NH_t > 0$ (¡0). See the Homework.

Here, we solve the model in the more general case where $\alpha_1 > 0$

Second order difference equation with stochastic real wage

$$\beta b E_t L_{t+1} - (\beta b + b + \alpha_1) L_t + b L_{t-1} = (W_t - \alpha_0)$$

Divide by βb

$$E_t L_{t+1} - \frac{(\beta b + b + \alpha_1)}{\beta b} L_t + \frac{1}{\beta} L_{t-1} = \frac{(W_t - \alpha_0)}{\beta b}$$

As in the previous section, define labor and real wage at deterministic steady state ($W_t = \bar{W}$).

From the above equation, we immediately obtain

$$\bar{L} = \frac{\alpha_0 - \bar{W}}{\alpha_1}$$

Then each variable in the FOC can be expressed in deviation from their steady–state values:

$$E_t\hat{L}_{t+1} - \frac{(\beta b + b + \alpha_1)}{\beta b}\hat{L}_t + \frac{1}{\beta}\hat{L}_{t-1} = \frac{1}{\beta b}\hat{W}_t$$

Notice that

$$\hat{W}_t = W_t - \bar{W} \equiv \tilde{W}_t$$

Now, consider again the backshift operator B, such that

$$BL_t = L_{t-1}$$

This allows to rewrite

$$E_t\left(\hat{L}_{t+1}\left(1-\frac{(\beta b+b+\alpha_1)}{\beta b}B+\frac{1}{\beta}B^2\right)\right)=\frac{1}{\beta b}\tilde{W}_t$$

The dynamics of labor adjustment is exactly the same as in the deterministic case.

So, we have one root larger than unity (say $\lambda_2 > 1$) and the second between 0 and 1 (say $\lambda_1 \in [0, 1)$).

Up to the shock $\tilde{W}_t$ and conditional expectations, the model is very similar to the one obtained in the deterministic case.

Now rewrite the expectations

$$E_t\left(\tilde{\hat{L}}_{t+1}(1 - \lambda_2 B)\right) = \frac{1}{\beta b}\tilde{W}_t$$

where

$$\tilde{\hat{\hat{L}}}_{t+1} = (1 - \lambda_1 B)\hat{L}_{t+1} = \hat{L}_{t+1} - \lambda_1 \hat{L}_t$$

The above equation rewrites

$$E_t\left(\tilde{\hat{\hat{L}}}_{t+1} - \lambda_2 \tilde{\hat{\hat{L}}}_t\right) = \frac{1}{\beta b}\tilde{W}_t$$

$$E_t \tilde{\hat{\hat{L}}}_{t+1} = \lambda_2 \tilde{\hat{\hat{L}}}_t + \frac{1}{\beta b}\tilde{W}_t$$

$$\tilde{\hat{\hat{L}}}_t = \frac{1}{\lambda_2} E_t \tilde{\hat{\hat{L}}}_{t+1} - \frac{1}{\beta b} \frac{1}{\lambda_2} \tilde{W}_t$$

Since $\lambda_2 > 1$, we have $1/\lambda_2 < 1$. So the model must be solved forward, i.e. by successive iterations on the future.

Write this equation in $t+1$ (take care, don't forget to adjust the time index on the available information set!)

$$\tilde{\hat{\hat{L}}}_{t+1} = \frac{1}{\lambda_2} E_{t+1} \tilde{\hat{\hat{L}}}_{t+2} - \frac{1}{\beta b} \frac{1}{\lambda_2} \tilde{W}_{t+1}$$

and then substitute this equation into the previous one

$$\tilde{\hat{\hat{L}}}_t = \left(\frac{1}{\lambda_2}\right)^2 E_t E_{t+1} \tilde{\hat{\hat{L}}}_{t+2} - \frac{1}{\beta b} \frac{1}{\lambda_2} \tilde{W}_t - \frac{1}{\beta b} \left(\frac{1}{\lambda_2}\right)^2 E_t \tilde{W}_{t+1}$$

and continue..... (remark that $E_t E_{t+1} = E_t$ or $E_t E_{t+1} E_{t+q} = E_t$)

At the limit we obtain

$$\tilde{\hat{\hat{L}}}_t = \lim_{T\to\infty} \left(\frac{1}{\lambda_2}\right)^T E_t \tilde{\hat{\hat{L}}}_{t+T} - \frac{1}{\beta b} \lim_{T\to\infty} E_t \frac{1}{\lambda_2} \sum_{i=0}^{T} \left(\frac{1}{\lambda_2}\right)^i \tilde{W}_{t+i}$$

or

$$\tilde{\hat{\hat{L}}}_t = -\frac{1}{\beta b} E_t \frac{1}{\lambda_2} \sum_{i=0}^{\infty} \left(\frac{1}{\lambda_2}\right)^i \tilde{W}_{t+i}$$

as we impose the "transversality" condition

$$\lim_{T \to \infty} \left(\frac{1}{\lambda_2}\right)^T E_t \tilde{\hat{\hat{L}}}_{t+T} = 0$$

To solve this equation, we must compute the successive expectations

$$E_t \tilde{W}_t \quad , \quad E_t \tilde{W}_{t+1} \quad , \quad E_t \tilde{W}_{t+2} \quad , \quad E_t \tilde{W}_{t+3} \quad , \quad \ldots.$$

Now, we use the stochastic process for $\tilde{W}_t$

$$E_t\tilde{W}_t = \tilde{W}_t$$

$$E_t\tilde{W}_{t+1} = \rho_w\tilde{W}_t$$

$$E_t\tilde{W}_{t+2} = \rho_w^2\tilde{W}_t$$

$$E_t\tilde{W}_{t+3} = \rho_w^3\tilde{W}_t$$

and continue...

After, discount each expectation

$$i = 0 : 1 \times E_t \tilde{W}_t = \tilde{W}_t$$

$$i = 1 : \left(\frac{1}{\lambda_2}\right) \times E_t \tilde{W}_{t+1} = \left(\frac{\rho_w}{\lambda_2}\right) \tilde{W}_t$$

$$i = 2 : \left(\frac{1}{\lambda_2}\right)^2 \times E_t \tilde{W}_{t+2} = \left(\frac{\rho_w}{\lambda_2}\right)^2 \tilde{W}_t$$

....

$$i = q : \left(\frac{1}{\lambda_2}\right)^q \times E_t \tilde{W}_{t+q} = \left(\frac{\rho_w}{\lambda_2}\right)^q \tilde{W}_t$$

Then collect these computations

$$\tilde{\hat{\hat{L}}}_t = -\frac{1}{\beta b}\frac{1}{\lambda_2}\tilde{W}_t\left(\sum_{i=0}^{\infty}\left(\frac{\rho_w}{\lambda_2}\right)^i\right)$$

Since

$$\left(\frac{\rho w}{\lambda_2}\right) < 1$$

The series is convergent and its sum is given by

$$\sum_{i=0}^{\infty}\left(\frac{\rho w}{\lambda_2}\right)^{i} = \frac{1}{1-\frac{\rho w}{\lambda_2}} \equiv \frac{\lambda_2}{\lambda_2 - \rho w} > 0$$

Now, replace them in $\tilde{\hat{\hat{L}}}_t$

$$\tilde{\hat{\hat{L}}}_t = -\frac{1}{\beta b}\frac{1}{\lambda_2}\frac{\lambda_2}{\lambda_2 - \rho w}\tilde{W}_t$$

This simplifies

$$\tilde{\hat{\hat{L}}}_t = -\frac{1}{\beta b}\frac{\tilde{W}_t}{\lambda_2 - \rho_w}$$

Now using the definition of $\tilde{\hat{\hat{L}}}_t$, we deduce

$$\hat{L}_t = \lambda_1 \hat{L}_{t-1} - \frac{1}{\beta b} \frac{\tilde{W}_t}{\lambda_2 - \rho_w}$$

and the definition of $\hat{L}_t$ and $\hat{W}_t$

$$L_t = (1 - \lambda_1)\bar{L} + \lambda_1 L_{t-1} - \frac{1}{\beta b} \frac{W_t - \bar{W}}{\lambda_2 - \rho_w}$$

where

$$W_t - \bar{W} = \tilde{W}_t$$

$$\tilde{W}_t = \rho_w \tilde{W}_{t-1} + \sigma_{\varepsilon,w} \varepsilon_{w,t}$$

where $\rho_w \in [0, 1]$, $\sigma_{\varepsilon,w} > 0$ and $\varepsilon_{w,t}$ is iid with zero mean and unit variance.

In addition, $\bar{L}$ is the long–run ("frictionless") labor demand:

$$\bar{L} = \frac{\alpha_0 - \bar{W}}{\alpha_1}$$

Similar to the deterministic case, but now it includes a random variable, $\tilde{W}_t$ that represents the excess real wage (with respect to the long–run real wage).

When it takes large and persistent values, the labor demand will decrease persistently.

Another Important Aspect The LUCAS CRITIQUE (AGAIN!!)

The labor demand equation

$$L_t = (1 - \lambda_1)\bar{L} + \lambda_1 L_{t-1} - \frac{1}{\beta b}\frac{W_t - \bar{W}}{\lambda_2 - \rho_w}$$

is not invariant to the form of the stochastic process $\tilde{W}_t$, i.e. it is very sensitive to ρ_w.

If changes in the real wages are transitory $\rho_w = 0$, the reduced form for labor demand is

$$L_t = (1 - \lambda_1)\bar{L} + \lambda_1 L_{t-1} - \frac{1}{\beta b}\frac{W_t - \bar{W}}{\lambda_2}$$

When these changes are very persistent ($\rho_w \simeq 1$), the labor demand rewrites

$$L_t = (1 - \lambda_1)\bar{L} + \lambda_1 L_{t-1} - \frac{1}{\beta b}\frac{W_t - \bar{W}}{\lambda_2 - 1}$$

It is easy to verify

$$\frac{1}{\lambda_2} < \frac{1}{\lambda_2 - 1}$$

So, labor demand is less sensitive to real wage, when their changes are perceived as transitory (compared to very persistent).

Computation of Dynamic Responses:

The IRFs (Impulse Response Function) are given by

$$\frac{\partial L_{t+h}}{\partial \varepsilon_{w,t}}$$

for $h \geq 0$ is the horizon of the response.

Let us consider

$$L_t = (1 - \lambda_1)\bar{L} + \lambda_1 L_{t-1} + \mu \tilde{W}_t$$

where μ is given by

$$\mu = -\frac{1}{\beta b}\frac{1}{\lambda_2 - \rho_w}$$

We normalize $\sigma_{\varepsilon,w}$ to one.

$h = 0$

$$\frac{\partial L_t}{\partial \varepsilon_{w,t}} = \mu \frac{\partial \tilde{W}_t}{\partial \varepsilon_{w,t}} = \mu$$

$h = 1$

$$\frac{\partial L_{t+1}}{\partial \varepsilon_{w,t}} = \frac{\partial L_{t+1}}{\partial L_t} \frac{\partial L_t}{\partial \varepsilon_{w,t}} + \mu \frac{\partial \tilde{W}_{t+1}}{\partial W_t} \frac{\partial \tilde{W}_t}{\partial \varepsilon_{w,t}}$$

$$\frac{\partial L_{t+1}}{\partial \varepsilon_{w,t}} = \lambda_1 \mu + \rho_w \mu = \mu(\lambda_1 + \rho_w)$$

Rmk: the response when $h = 1$ can be greater than on impact ($h = 0$) when

$$\lambda_1 + \rho_w > 1$$

For any h, the chain rule differenciation

$$\frac{\partial L_{t+h}}{\partial \varepsilon_{w,t}} = \frac{\partial L_{t+h}}{\partial L_{t+h-1}} \cdots \frac{\partial L_t}{\partial \varepsilon_{w,t}} + \mu \frac{\partial \tilde{W}_{t+h}}{\partial W_{t+h-1}} \cdots \frac{\partial \tilde{W}_t}{\partial \varepsilon_{w,t}}$$

[INSERT A FIGURE WITH A HUMP–SHAPED RESPONSE]

Summary of the results

For convenience, most of empirical works uses a quadratic specification for adjustment costs.

Assuming that L and W are observable, the parameters of the model are obtained from the equation

$$L_t = (1 - \lambda_1)\bar{L} + \lambda_1 L_{t-1} - \frac{W_t - \frac{1}{\beta b}\bar{W}}{\lambda_2 - \rho_w}$$

where λ_1 and λ_2 are functions of the "deep" parameters β, b and α_1 and $\bar{L}$ is a function of $\bar{L}$, α_0 and α_1.

At the same time, the process of W_t is given by

$$W_t - \bar{W} = \tilde{W}_t$$

$$\tilde{W}_t = \rho_w \tilde{W}_{t-1} + \sigma_{\varepsilon,w} \varepsilon_{w,t}$$

The econometrician must account for the cross equation restrictions between the equation of L and the equation of W (see the parameter ρ_w).

This equation of L is obtained after solving the model (computation of the reduced form).

To obtain this reduced form, it is necessary to specify the process of W.

Another approach, using the FOC of the firm

$$E_t\left(\beta b L_{t+1} - (\beta b + b + \alpha_1) L_t + b L_{t-1} - (W_t - \alpha_0)\right) = 0$$

Let Z_t a set of instrumental variables

$$Z_t = \{L_t, L_{t-1}, ..., L_{t-q}, W_t, W_{t-1}, ..., W_{t-q}, Cste\}$$

a subset of the information set used by the firm. The FOC can be rewritten

$$E\left[\left(\beta b L_{t+1} - (\beta b + b + \alpha_1) L_t + b L_{t-1} - (W_t - \alpha_0)\right) | Z_t\right] = 0$$

This suggests that the model's parameters can be estimated using an IV estimator (GMM, Hansen (1982)).

This approach can be generalized to more complex adjustment costs functions.

- The quadratic adjusted costs function is less and less supported by the data (symmetry and convexity).
- Data favor asymmetric, piecewise linear and fixed costs approaches.
- The speed of adjustment of labor demand is relatively high in US.
- The result is not affected if we consider multiple inputs.
- Firms adjust hours of work more rapidly than the number of workers (adjustment costs are higher for workers than for hours)
- Employment more rapidly in US than anywhere else.

- More rapid adjustment in Europe than in Japan.

A difficulty Identification of adjustment costs parameter.

Remind that the solution takes the form

$$\hat{L}_t = \lambda_1 \hat{L}_{t-1} + \mu \tilde{W}_t$$

where $\mu = -\frac{1}{\beta b}\frac{1}{\lambda_2 - \rho_w}$.

For simplicity, normalize the parameter $\mu = 1$.

This yields

$$\hat{L}_t = (\lambda_1 + \rho_w)\hat{L}_{t-1} - (\lambda_1 \rho_w)\hat{L}_{t-2} + \sigma_{\varepsilon,w}\varepsilon_{w,t}$$

Suppose that the econometrician does not observe the real wage W_t

but only the labor input.

The (unconstrained) reduced form is given by

$$\hat{L}_t = \beta_1 \hat{L}_{t-1} + \beta_2 \hat{L}_{t-2} + u_t$$

Let $\hat{\beta}_1$ and $\hat{\beta}_2$ the estimated values of β_1 and β_2.

From

$$\beta_1 = \lambda_1 + \rho_w \qquad \beta_2 = -\lambda_1 \rho_w$$

we deduce

$$\hat{\lambda}_1 = \frac{\hat{\beta}_1 \pm \sqrt{\hat{\beta}_1^2 + 4\hat{\beta}_2}}{2} \qquad \hat{\rho}_w = \hat{\beta}_1 - \hat{\lambda}_1$$

Two opposite conclusions:

- $\hat{\lambda}_1$ large and $\hat{\rho}_w$ small: large adjustment costs and small persistence of shocks.
- $\hat{\lambda}_1$ small and $\hat{\rho}_w$ large: small adjustment costs and large persistence of shocks.

Summary and Conclusion

- There exists substantial costs of adjusting employment (hiring versus firing costs, US versus continental Europe)
- When adjustment costs are quadratic, the firm gradually adjusts the labor input.
- In the stochastic case, the reduced form on employment is not invariant to the specification of the exogenous real wages.

Chap 3: Business Cycle Models

3.1: Two Simple DSGE Models

I - Habit Formation in a Simple Equilibrium Model

- The model
- Solution
- Quantitative implications
- Taking the model to the data

II - A Stochastic Growth Model

- The model
- Solution
- Quantitative implications
- Taking the model to the data

3.2: RBC Models

I - A RBC Model with Complete Depreciation

- The model
- Solution
- Quantitative implications
- Taking the model to the data

II- A prototypical RBC Model (Optional)

- The model
- Solution
- Quantitative implications
- Taking the model to the data

3.3: Beyond the RBC Model (Optional)

3.1: Two Simple DSGE Models

I - Habit Formation in a Simple Equilibrium Model

Habit Persistence in consumption

Another form of habit persistence: leisure (Eichenbaum, Hanse, Singleton (1988), Wenn (1998), ...)

Two models:

A) Habit persistence in the permanent income model: Homework

B) Habit persistence in a simple DSGE model

B) Habit persistence in a simple DSGE model

A simple DSGE model without capital accumulation

Equilibrium on the good market

$$Y_t = C_t$$

Production economy.

Technology used by firms (CRS with labor as the sole variable input)

$$Y_t = Z_t h_t$$

where Z_t is a technology (TFP) perturbation (the only one we consider in this simple economy)

Optimality condition for the representative competitive firm:

$$Z_t = w_t$$

where w_t is the real wage.

Household problem

$$E_t \sum_{i=0}^{\infty} \log(C_t - bC_{t-1}) - h_t$$

Comments:

- log utility in consumption

- habit persistence in consumption ($b \geq 0$)

- linear utility in leisure (or labor supply)

Meaning of habit persistence Marginal utility in period t

$$\frac{\partial U}{\partial C_t} = \frac{1}{C_t - bC_{t-1}}$$

The marginal utility is decreasing in C_t but increasing in C_{t-1} provided $b > 0$. This creates intertemporal complementarity in consumption decision.

In other words, if the consumer wants to maintain its utility, for a given value of its past consumption, he must choose a high value of consumption today.

So higher consumption yesterday translates into even more higher consumption today.

Budget constraint

$$C_t = w_t h_t + \Pi_t$$

where Π_t are the profits received from the firms (rmk: at equilibrium these profits are zero)

Two controls: C and h

Now, replace the budget constraint into the objective $\Rrightarrow$ h is the only one control

$$E_t \log(w_t h_t - b w_{t-1} h_{t-1}) - h_t + \beta(\log(w_{t+1} h_{t+1} - b w_t h_t) - h_{t+1}) + \dots$$

FOC

$$\frac{w_t}{C_t - bC_{t-1}} - \beta b E_t \frac{w_t}{C_{t+1} - bC_t} = 1$$

$$w_t \left(\frac{1}{C_t - bC_{t-1}} - \beta b E_t \frac{1}{C_{t+1} - bC_t} \right) = 1$$

$$\frac{1}{C_t - bC_{t-1}} - \beta b E_t \frac{1}{C_{t+1} - bC_t} = \frac{1}{Z_t}$$

Non–linear forward looking equation with rational expectations.

Difficult (not impossible, in this simple setup) to compute the solution.

An simple approximation:

Log–linear approximation (extensively used for more complex models in the DSGE literature)

Let the following function

$$y = f(x_1, x_2, ..., x_n)$$

Linear approximation:

$$y = f(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) + \sum_{i=1}^{n} \frac{\partial y}{\partial x_i}\bigg|_{x=\bar{x}} (x_i - \bar{x}_i)$$

where

$$\bar{y} = f(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$$

This rewrites

$$y - \bar{y} = \sum_{i=1}^{n} \left.\frac{\partial y}{\partial x_i}\right|_{x=\bar{x}} (x_i - \bar{x}_i)$$

Now divide both side by $\bar{y}$:

$$\frac{y - \bar{y}}{\bar{y}} = \sum_{i=1}^{n} \left.\frac{\partial y}{\partial x_i}\right|_{x=\bar{x}} \frac{1}{\bar{y}}(x_i - \bar{x}_i)$$

or equivalently

$$\frac{y - \bar{y}}{\bar{y}} = \sum_{i=1}^{n} \left.\frac{\partial y}{\partial x_i}\right|_{x=\bar{x}} \frac{\bar{x}_i}{\bar{y}} \frac{x_i - \bar{x}_i}{\bar{x}_i}$$

The log–linear approximation of y is then given by

$$\hat{y} = \sum_{i=1}^{n} \frac{\partial y}{\partial x_i}\bigg|_{x=\bar{x}} \frac{\bar{x}_i}{\bar{y}} \hat{x}_i$$

where

$$\hat{y} = (y - \bar{y})/\bar{y} \simeq \log(y) - \log(\bar{y})$$

and

$$\hat{x}_i = (x_i - \bar{x}_i)/\bar{x}_i \simeq \log(x_i) - \log(\bar{x}_i)$$

Examples:

1) Production function

$$y_t = z_t h_t^{\alpha}$$

Y is the output, Z_t the TFP and h_t the labor input.

The log–linear approximation

$$\hat{y}_t = h^{\alpha}(z/y)\hat{z}_t + z\alpha h^{\alpha-1}(h/y)\hat{h}_t$$

From $y = zh^{\alpha}$, this reduces to:

$$\hat{y}_t = \hat{z}_t + \alpha\hat{h}_t$$

2) Equilibrium condition

Closed economy

$$y_t = c_t + i_t + g_t$$

The log–linear approximation

$$\hat{y}_t = \frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{i}_t + \frac{g}{y}\hat{g}_t$$

c/y, i/y and g/y represent the average shares ($c/y + i/y + g/y = 1$ by construction) or steady–state values in the model's language.

3) Euler equation on consumption

log utility and stochastic returns (see asset pricing models)

$$\frac{1}{C_t} = \beta E_t R_{t+1} \frac{1}{C_{t+1}}$$

where R is the gross return on assets between periods t and $t+1$.

Steady state

$$\beta R = 1$$

Log–linearization:

$$-\hat{C}_t = E_t \hat{R}_{t+1} - E_t \hat{C}_{t+1}$$

or

$$\hat{C}_t = -E_t\hat{R}_{t+1} + E_t\hat{C}_{t+1}$$

Application to the model with habit:

$$\frac{1}{C_t - bC_{t-1}} - \beta b E_t \frac{1}{C_{t+1} - bC_t} = \frac{1}{Z_t}$$

Deterministic steady–state

$$\frac{1 - \beta b}{C(1 - b)} = \frac{1}{Z}$$

Variable definition

$$X_t = \frac{1}{C_t - bC_{t-1}}$$

with steady state

$$X = \frac{1}{C(1-b)}$$

The model becomes

$$X_t = \beta b E_t X_{t+1} + \frac{1}{Z_t}$$

Log–linearization

$$\hat{X}_t = \beta b E_t \hat{X}_{t+1} - (1 - \beta b)\hat{Z}_t$$

Forward substitutions

$$\hat{X}_t = -(1-\beta b)E_t \sum_{i=0}^{\infty} (\beta b)^i \hat{Z}_{t+i}$$

Log–normal technology shocks

$$\hat{Z}_t = \rho_z \hat{Z}_{-1} + \varepsilon_t^z$$

where

$$E(\varepsilon_t^z) = 0 \qquad V(\varepsilon_t^z) = \sigma_z^2$$

It follows that

$$\hat{X}_t = -\frac{1 - \beta b}{1 - \beta b \rho_z} \hat{Z}_t$$

Now, we compute the log–linearization of X

$$\hat{X}_t = -\frac{1}{1-b}\hat{C}_t + \frac{b}{1-b}\hat{C}_{t-1}$$

After substitution, we obtain

$$\hat{C}_t = b\hat{C}_{t-1} + \frac{(1-b)(1-\beta b)}{1-\beta b\rho_z}\hat{Z}_t$$

So, the consumption follows an AR(2) process because Z follows an AR(1) process

$$(1 - bB)(1 - \rho_z B)\hat{C}_t = \frac{(1-b)(1-\beta b)}{1 - \beta b \rho_z}\varepsilon_t^z$$

where B is the backshift operator, i.e. $B\hat{C}_t = \hat{C}_{t-1}$.

From this reduced form, we can compute moments (volatility, ACFs) and IRFs

The Lucas Critique Again and Again: a change in ρ_z will affect the reduced form!

The behavior of hours: We have

$$\hat{Y}_t = \hat{C}_t = \hat{Z}_t + \hat{h}_t$$

After a substitution into the consumption function, one gets

$$\hat{Z}_t + \hat{h}_t = b(\hat{Z}_{t-1} + \hat{h}_{t-1}) + \frac{(1-b)(1-\beta b)}{1-\beta b \rho_z}\hat{Z}_t$$

For a very persistent technology shock ($\rho_z \rightsquigarrow 1$), we have

$$\hat{h}_t = b\hat{h}_{t-1} - b\varepsilon_t^z$$

The technology shocks has a negative and persistent effect on hours ($b > 0$).

In this case, the IRFs are:

$$IRF(0) = -b \quad , \quad IRF(1) = -b^2, \ldots \quad , \quad IRF(h) = -b^{h+1}$$

Why?: Consumption is "sticky" due to habit persistence.

A permanent increase in the real wage leads to put the extra resources on leisure (the income effects dominate the substitution effect).

So, a perfectly competitive model of the business cycle can produce a decrease in hours (due to strong labor supply effects)

In contradiction with Gali (199) who claimed that the standard neaclassical model cannot explained this feature (found in the data).

Conversely, a Keynesian model can replicate this.

Reason? Simple. In the short–run, the level of output is determined by the level of demand.

Suppose a positive technology shock Z_t

$$Y_t = Z_t h_t$$

The output Y_t does not change in the short run.

So, an increase in Z reduces the level of h, because the same quantity Y can be produced with less labor input. (See the figure)

The case of random walk technology shocks with drift:

$$\Delta \hat{Z}_t = \gamma_z + \varepsilon_t^z$$

where $\gamma_z > 1$ is the gross growth rate of technical progress (and thus growth rate of the economy along the balanced growth path

$$\gamma_z = \gamma_y = \gamma_c$$

Notice that hours do not display this trend pattern (hours are first–

order and second–order stationary).

Thus, a stationary version of this economy would be obtained by deflating the equilibrium conditions by Z.

All variables are then defined in deviation from the stochastic trend Z.

See the computation of the solution in a more general case with capital accumulation.

II - Stochastic AK Model

A representative household will seek to maximize

$$E_t \sum_{i=0}^{\infty} \log(C_{+i})$$

(no leisure choices) under the constraints

Capital accumulation

$$K_{t+1} = (1 - \delta)K_t + I_t$$

Aggregate resources constraint

$$Y_t = C_t + I_t$$

Technology

$$Y_t = Z_t K_t$$

K is the sole input; constant return to scale in the reproducible factor.

Z_t is a TFP shock.

Putting these three equations together yields

$$K_{t+1} = (1 - \delta)K_t + Z_t K_t - C_t$$

or

$$K_{t+1} = (1 - \delta + Z_t)K_t - C_t$$

Here we do not impose full depreciation, i.e. $\delta \in [0, 1)$.

After substitution of this constraint into the objective, the FOC is given by

$$\frac{1}{C_t} = \beta E_t(1 - \delta + Z_{t+1})\frac{1}{C_{t+1}}$$

Here $(1-\delta+Z_t)$ represent the real interest rate (marginal productivity of capital net of depreciation)

Solving the model:

K_t is predetermined and thus $E_t K_{t+1} = E_t((1-\delta)K_t + Z_t K_t - C_t) = K_{t+1}$.

Now divide and multiply by K_{t+1} the right–hand side of the Euler equation.

This implies

$$\frac{1}{C_t} = \beta E_t(1 - \delta + Z_{t+1})\frac{K_{t+1}}{K_{t+1}}\frac{1}{C_{t+1}}$$

The numerator is

$$(1-\delta+Z_{t+1})K_{t+1} = (1-\delta)K_{t+1}+Z_{t+1}K_{t+1} = (1-\delta)K_{t+1}+Y_{t+1}$$

Now, use the equilibrium condition on the good market in period $t+1$

$$Y_{t+1} = C_{t+1} + I_{t+1}$$

and the law of motion on the capital stock

$$K_{t+2} = (1-\delta)K_{t+1} + I_{t+1}$$

$$K_{t+2} = (1-\delta)K_{t+1}+Y_{t+1}-C_{t+1} \Leftrightarrow K_{t+2}+C_{t+1} = (1-\delta)K_{t+1}+Y_{t+1}$$

This yields

$$\frac{K_{t+1}}{C_t} = \beta E_t \frac{C_{t+1} + K_{t+2}}{C_{t+1}}$$

or

$$\frac{K_{t+1}}{C_t} = \beta E_t \left(1 + \frac{K_{t+2}}{C_{t+1}}\right)$$

Let $X_t = \frac{K_{t+1}}{C_t}$, the Euler equation rewrites

$$X_t = \beta E_t(1 + X_{t+1})$$

Since $\beta \in (0, 1)$, the solution is obtained by forward substitutions:

$$X_t = \frac{\beta}{1 - \beta}$$

This yields

$$C_t = \frac{1-\beta}{\beta} K_{t+1}$$

Now, substitute this equation into the law of motion of capital

$$K_{t+1} = (1 - \delta + Z_t) K_t - C_t$$

This yields

$$K_{t+1} = \beta(1 - \delta + Z_t) K_t$$

All the other aggregate variables can be then deduced.

Output Dynamics We have

$$Y_t = Z_t K_t$$

So

$$\frac{Y_t}{Y_{t-1}} = \frac{Z_t}{Z_{t-1}} \frac{K_t}{K_{t-1}}$$

Now, using the process of K, it comes

$$\frac{Y_t}{Y_{t-1}} = \frac{Z_t}{Z_{t-1}} \beta(1 - \delta + Z_{t-1})$$

Taking logs (lower letters), we have

$$\Delta y_t = \Delta z_t + \log(\beta) + \log(1 - \delta + Z_{t-1})$$

Non–linear in Z (except when $\delta = 1$), so no explicit solution for the moments.Complicated likelihood function.

Conversely, the model is easy to simulate and thus moments can be obtained by simulation.

Another approach: compute an approximate solution by linearization of the model (see the previous model).

3.2: RBC Models

I - RBC Model with complete depreciation

We consider an economy with a representative household and a representative firm. The firm produces an homogenous good with the following production function

$$Y_t = Z_t K_t^{1-\alpha} h_t^{\alpha}$$

with $\alpha \in (0, 1)$. K_t is the capital stock and h_t denotes the labor input. Z_t is a random variable that represents changes in Total Factor

Productivity (TFP). The capital stock evolves according to

$$K_{t+1} = (1 - \delta)K_t + I_t$$

where δ is the depreciation rate. I_t denotes investment. Here, we assume complete depreciation ($\delta = 1$)

$$K_{t+1} = I_t$$

The representative household consumes C_t and works h_t every period. The household seeks to maximize

$$E_t \sum_{i=0}^{\infty} \beta^i \left(\log(C_{t+i}) - V(h_{t+i})\right)$$

$V(.)$ is a convex function. $\beta \in (0,1)$ is the discount factor and E_t denotes the conditional expectations operator. The equilibrium condition on the good market is given by

$$Y_t = C_t + I_t$$

The dynamic optimization problem

Pre-determined variable: K_t

Choice variable: C_t and h_t

Rmk: Controlling C_t is equivalent to control K_{t+1}

$$\max E_t \sum_{i=0}^{\infty} \beta^i \left(\log(C_{t+i}) - V(h_{t+i})\right)$$

subject to

$$K_{t+1} = I_t$$

$$Y_t = C_t + I_t$$

$$Y_t = Z_t K_t^{1-\alpha} h_t^{\alpha}$$

Combining the three equations yields

$$K_{t+1} = Z_t K_t^{1-\alpha} h_t^{\alpha} - C_t$$

Now, replace the constraint

$$C_t = Z_t K_t^{1-\alpha} h_t^{\alpha} - K_{t+1}$$

into the objective

$$E_t \sum_{i=0}^{\infty} \beta^i \left(\log(Z_{t+i} K_{t+i}^{1-\alpha} h_{t+i}^{\alpha} - K_{t+i+1}) - V(h_{t+i}) \right)$$

and maximize this function with respect to K_{t+1} and h_t. One obtains

$$-\frac{1}{C_t} + \alpha\beta E_t \left[\left(\frac{Y_{t+1}}{K_{t+1}} \right) \left(\frac{1}{C_{t+1}} \right) \right] = 0$$

or equivalently

$$\frac{1}{C_t} = \alpha\beta E_t \left[\left(\frac{Y_{t+1}}{K_{t+1}} \right) \left(\frac{1}{C_{t+1}} \right) \right]$$

Thus is the Euler equation on consumption (in a DSGE model)

$$V'(h_t) = \frac{1}{C_t}(1-\alpha)\frac{Y_t}{h_t}$$

This represent the marginal rate of substitution between labor supply and consumption.

Resolution

Forward substitutions

With complete depreciation, the physical capital evolves according to

$$K_{t+1} = I_t$$

Substitute into the the Euler equation on consumption

$$\frac{1}{C_t} = \alpha\beta E_t \left[\left(\frac{Y_{t+1}}{I_t} \right) \left(\frac{1}{C_{t+1}} \right) \right]$$

I_t is known in period t. So, we can move out the investment from the right hand side of the Euler equation

$$\frac{I_t}{C_t} = \alpha\beta E_t \left[\frac{Y_{t+1}}{C_{t+1}}\right]$$

From the equilibrium condition on the good market in period $t+1$

$$Y_{t+1} = C_{t+1} + I_{t+1}$$

we obtain

$$\frac{I_t}{C_t} = \alpha\beta E_t \left[\frac{C_{t+1} + I_{t+1}}{C_{t+1}}\right]$$

or

$$\frac{I_t}{C_t} = \alpha\beta E_t \left[1 + \frac{I_{t+1}}{C_{t+1}}\right]$$

Forward substitutions ($\alpha\beta < 1$)

Solution

$$\frac{I_t}{C_t} = \frac{\alpha\beta}{1 - \alpha\beta}$$

$$I_t = \frac{\alpha\beta}{1 - \alpha\beta} C_t$$

Using the equilibrium condition on the good market

$$Y_t = C_t + I_t \Rightarrow Y_t = \frac{C_t}{1 - \alpha\beta} \Leftrightarrow \frac{C_t}{Y_t} = 1 - \alpha\beta (\text{constant})$$

$$\frac{I_t}{Y_t} = \alpha\beta(\text{constant saving rate})$$

Labor

$$V'(h_t) = \frac{1}{C_t}(1-\alpha)\frac{Y_t}{h_t} \Leftrightarrow h_t V'(h_t) = (1-\alpha)\frac{Y_t}{C_t}$$

So, labor is constant (we denote $\bar{h}$).

From the law of motion on physical capital

$$K_{t+1} = Y_t - C_t \Leftrightarrow \alpha\beta Y_t$$

and the production function

$$Y_t = Z_t K_t^{1-\alpha} \bar{h}^{\alpha}$$

we obtain

$$K_{t+1} = \alpha\beta Z_t K_t^{1-\alpha} \bar{h}^{\alpha}$$

Analyzing the dynamic properties

Now, take this equation in logs (all variables are positive)

$$k_{t+1} = c_k + z_t + \alpha k_t$$

where c_k is a constant term that depends on the deep parameters of the model.

The equation for output (in logs) is

$$y_t = c_y + z_t + \alpha k_t$$

where c_y is a constant.

For simplicity (without loss of generality), set this two constant terms to zero.

Now, introduce the backshift operator B, i.e.

$$Bk_{t+1} = k_t \quad \text{or} Bk_t = k_{t-1} \quad \text{or} By_t = y_{t-1}$$

$$k_{t+1} = z_t + \alpha k_t \Leftrightarrow k_{t+1} = z_t + \alpha B k_{t+1}$$

$$\Leftrightarrow (1 - \alpha B) k_{t+1} = z_t \Leftrightarrow k_{t+1} = \frac{z_t}{1 - \alpha B}$$

$$\Leftrightarrow k_t = \frac{z_{t-1}}{1 - \alpha B}$$

Now, insert this equation into the output (in log) equation

$$y_t = z_t + \alpha \frac{z_{t-1}}{1 - \alpha B} \Leftrightarrow (1 - \alpha B) y_t = (1 - \alpha B) z_t + \alpha z_{t-1}$$

$$y_t = \alpha y_{t-1} + z_t$$

Now assume that z_t is an iid shock (purely transitory)

$$z_t = \varepsilon_t^z$$

Impulse response of output (in log, so this is an elasticity) to a technology innovation ε_t^z in period t.

$$IRF_y(h) = \frac{\partial y_{t+h}}{\partial \varepsilon_t^z}$$

with $h = 0, 1, 2, ...$

From the above equation, we obtain

$$IRF_y(h) = \alpha^h$$

[INSERT FIGURE]

Rmk 1: the response of consumption and investment (in logs) are exactly the same.

So, a transitory technology improvement has a long lasting positive effect on real variables.

Rmk 2: no effect on labor .

Rmk 3: The Lucas critique does not apply here. The solution (I mean the reduced form parameters) is not affected by the parameters that summarize the stochastic process of z_t.

II - RBC Model with Linear Utility in Leisure

For simplicity, the model includes only a random walk in productivity (Z_t).

Similar results are obtained with a stationary labor wedge shock.

The intertemporal expected utility function of the representative household is given by

$$E_t \sum_{i=0}^{\infty} \log(C_{+i}) - \chi V(H_t)$$

where $\chi > 0$, $\beta \in (0, 1)$ denotes the discount factor and E_t is the expectation operator conditional on the information set available as of time t.

C_t is the consumption at t and H_t represents the household's labor supply.

The representative firm uses capital K_t and labor H_t to produce the homogeneous final good Y_t.

The technology is represented by the following constant returns–to–scale Cobb–Douglas production function

$$Y_t = K_t^{\alpha} \left(Z_t H_t\right)^{1-\alpha},$$

where $\alpha \in (0, 1)$. Z_t is assumed to follow an exogenous process of the form

$$\Delta \log(Z_t) = \sigma_z \varepsilon_{z,t},$$

where $\varepsilon_{z,t}$ is *iid* with zero mean and unit variance.

The capital stock evolves according to the law of motion

$$K_{t+1} = (1 - \delta) K_t + I_t,$$

where $\delta \in (0, 1)$ is the constant depreciation rate.

Finally, the final good can be either consumed or invested

$$Y_t = C_t + I_t.$$

FOCs + Equilibrium conditions + Exogenous variable

$$\frac{1}{C_t} = \beta E_t \left(1 - \delta + \alpha \frac{Y_{t+1}}{K_{t+1}}\right) \frac{1}{C_{t+1}}$$

$$V'(H_t)1 = \frac{1}{C_t}(1-\alpha)\frac{Y_t}{H_t}$$

$$K_{t+1} = (1-\delta)K_t + K_t^{\alpha}\left(Z_t H_t\right)^{1-\alpha} - C_t$$

$$\Delta \log(Z_t) = \sigma_z \varepsilon_{z,t},$$

In this model, the technology shock Z_t induces a stochastic trend into output, consumption, investment and capital.

Accordingly, to obtain a stationary equilibrium, these variables must be detrended as follows

$$\breve{y}_t = \frac{Y_t}{Z_t}, \quad \breve{c}_t = \frac{C_t}{Z_t}, \quad \breve{i}_t = \frac{I_t}{Z_t}, \quad \breve{k}_{t+1} = \frac{K_{t+1}}{Z_t}.$$

With these transformations, the approximate solution of the model is computed from a log–linearization of the stationary equilibrium conditions around this deterministic steady state.

Here we assume

$$V'(H_t) = 1$$

i.e. the utility is linear.

$\Rightarrow$ **More details (FOC, stationary version of the model, Steady–state, log–linearization)**

The log–linearization of equilibrium conditions around the deterministic steady state yields

$$\widehat{\breve{k}}_{t+1} = (1-\delta)(\widehat{\breve{k}}_t - \sigma_z \varepsilon_{z,t}) + \frac{y}{k}\widehat{\breve{y}}_t - \frac{c}{k}\widehat{\breve{c}}_t \tag{1}$$

$$\widehat{h}_t = \widehat{\breve{y}}_t - \widehat{\breve{c}}_t \tag{2}$$

$$\widehat{\breve{y}}_t = \alpha(\widehat{\breve{k}}_t - \sigma_z \varepsilon_{z,t}) + (1-\alpha)\widehat{\breve{h}}_t \tag{3}$$

$$E_t \widehat{\breve{c}}_{t+1} = \widehat{\breve{c}}_t + \alpha\beta\frac{y}{k}E_t(\widehat{\breve{y}}_{t+1} - \widehat{\breve{k}}_{t+1} - \sigma_z \varepsilon_{z,t+1}) \tag{4}$$

where $y/k = (1-\beta(1-\delta))/(\alpha\beta)$ and $c/k = y/k - \delta$. After substitution of (2) into (3), one gets

$$\widehat{\breve{y}}_t - \widehat{\breve{k}}_t = -\sigma_z \varepsilon_{z,t} - \frac{1-\alpha}{\alpha}\widehat{\breve{c}}_t$$

Now, using the above expression, (1) and (4) rewrite

$$E_t \widehat{\breve{c}}_{t+1} = \varphi \widehat{\breve{c}}_t \quad \text{with} \quad \varphi = \frac{\alpha}{1-\beta(1-\alpha)(1-\delta)} \in (0,1) \tag{5}$$

$$\widehat{\breve{k}}_{t+1} = \nu_1 \widehat{\breve{k}}_t - \nu_1 \sigma_z \varepsilon_{z,t} - \nu_2 \widehat{\breve{c}}_t$$
$$\text{with } \nu_1 = \frac{1}{\beta\varphi} > 1 \quad \text{and} \quad \nu_2 = \frac{1-\beta(1-\delta(1-\alpha^2))}{\alpha^2\beta} \tag{6}$$

As $\nu_1 > 1$, (6) must be solved forward

$$\widehat{\breve{k}}_t = \sigma_z \varepsilon_{z,t} + \left(\frac{\nu_2}{\nu_1}\right) \lim_{T \to \infty} E_t \sum_{i=0}^{T} \left(\frac{1}{\nu_1}\right)^i \widehat{\breve{c}}_{t+i} + \lim_{T \to \infty} E_t \left(\frac{1}{\nu_1}\right)^T \widehat{\breve{k}}_{t+T}$$

Excluding explosive pathes, *i.e.* $\lim_{T \to \infty} E_t \left(1/\nu_1\right)^T \widehat{\breve{k}}_{t+T} = 0$, and using (5), one gets the decision rule on consumption:

$$\widehat{\breve{c}}_t = \left(\frac{\nu_1 - \varphi}{\nu_2}\right) \left(\widehat{\breve{k}}_t - \sigma_z \varepsilon_{z,t}\right) \tag{7}$$

After substituting (7) into (6), the dynamics of capital is given by:

$$\widehat{\breve{k}}_{t+1} = \varphi \left(\widehat{\breve{k}}_t - \sigma_z \varepsilon_{z,t} \right) \tag{8}$$

The persistence properties of the model is thus governed by the parameter $\varphi \in (0, 1)$.

The decision rules of the other (deflated) variables are similar to equation (7).

The consumption to output ratio is given by

$$
\begin{aligned}
\widehat{\breve{c}}_t - \widehat{\breve{y}}_t &= \nu_{cy} \left(\widehat{\breve{k}}_t - \sigma_z \varepsilon_{z,t} \right) \\
&= \nu_{cy} \left(-\frac{\varphi}{1 - \varphi L} \sigma_z \varepsilon_{z,t-1} - \sigma_z \varepsilon_{z,t} \right) \\
&= -\nu_{cy} \left(\frac{\sigma_z \varepsilon_{z,t}}{1 - \varphi L} \right)
\end{aligned}
$$

where $\nu_{cy} = \alpha(\nu_1 - \varphi - \nu_2)/\nu_2$ is positive.

The latter expression shows that the consumption to output ratio follows exactly the same stochastic process (an autoregressive process of order one) as the deflated capital $\log(K_t/Z_{t-1})$ in equation (8).

From the above expression, we can directly deduce the behavior of hours:

$$(1 - \varphi L)\widehat{h}_t = \nu_{cy}\sigma_z\varepsilon_{z,t} \tag{9}$$

Hours worked increase after a technology improvement and displays a monotonic dynamic response.

3.3: Beyond the RBC Model (Optional) In progress