TOULOUSE SCHOOL OF ECONOMICS, M1, 2015–2016
Macroeconomics – Patrick Feve

PROBLEM SET 1 - TWO PERIOD MODELS

Exercice I: The Elasticity of Labor Supply

Consider the following dynamic optimization two-period problem:

$$\max \left\{ \log\left(c_1 - \frac{\eta}{1+\nu} n_1^{1+\nu}\right) + \beta \log\left(c_2 - \frac{\eta}{1+\nu} n_2^{1+\nu}\right) \right\}$$

s.t.

$$c_1 + \frac{c_2}{1+r} = w_1 n_1 + \frac{w_2 n_2}{1+r}$$

where $\beta \in (0,1)$ is the discount factor, c_t is consumption ($t = 1, 2$), n_t represents labor supply, η and ν are positive parameters and w_t and r denote the real wage and the real interest rate, respectively.

1. Determine the optimal labor supply (for given prices).
2. Compute the elasticity of labor supply with respect to the wage.
3. What's the effect on labor supply of a change in the real wage profile (permanent or transitory changes)? Explain.

Exercice II: Liquidity constraint and Consumption Behavior

Consider a two-period deterministic economy. We assume that the representative agent faces a liquidity constraint in period 1 of the form

$$s = y - c_1 \geq 0$$

where y and c_1 are the labor income and consumption in period 1, respectively.

Consider the following dynamic optimization two-period problem:

$$\max \log(c_1) + \beta \log(c_2)$$

under the two constraints

$$c_1 + \frac{c_2}{1+r} \leq y + \frac{y}{1+r}$$
$$c_1 \leq y$$

where $r > 0$ is the (constant) real interest rate. Note that the labor income is constant over periods 1 and 2. The subjective discount factor β can be expressed as

$$\beta = \frac{1}{1+\theta}$$

where $\theta > 0$ is the rate of preference for the present.

1. Determine the optimal consumption behavior.
2. Assume $s = y - c_1 > 0$. Compute the consumptions c_1 and c_2. Determine the threshold value for r (say $\hat{r}$) such that $s = y - c_1 > 0$ when $r > \hat{r}$. Discuss.
3. What are the consumptions c_1 and c_2 when $r < \hat{r}$? Discuss.

Exercice III: The Effect of Fiscal Rule

Consider a two-period stochastic economy. The inter-temporal utility is given by

$$E_1 \left\{ (\alpha_0 c_1 - (\alpha_1/2)c_1^2) + \beta(\alpha_0 c_2 - (\alpha_1/2)c_2^2) \right\}$$

where $\alpha_0, \alpha_1 > 0$ and E_1 is the expectation conditional on the information set in period 1. c_1 and c_2 denote the consumption in period 1 and 2. The inter-temporal budget constraint is given by

$$c_1 + E_1 \frac{c_2}{1+r} = y(1-t_1) + E_1 \frac{y(1-t_2)}{1+r}$$

where y denotes a constant labor income and t_1 and t_2 are the levels of taxation in period 1 and 2. We assume that $\beta(1+r) = 1$.

1. Find the Euler equation on consumption.
2. Determine the consumption function c_1.
3. Assume that the government announces in period 1 the following taxation rule for period 2

   $$t_2 = \lambda t_1 + \varepsilon_2$$

   where ε_2 has zero mean and finite variance. In addition, we have $E_1 \varepsilon_2 = 0$, so that it is unpredictable given the information available in period 1. λ is a given parameter and it is known by the households. Discuss this rule. Find the relationship between c_1 and t_1. Show that the solution illustrates the *Lucas critique*.

Exercice IV: Labor Supply and Capital Accumulation

Consider a two–period production economy with a variable labor supply. The economy is deterministic. The inter-temporal utility is given by:

$$\log(c_1) + \beta \log(c_2) + \beta \log(1 - n_2)$$

where $\beta \in (0, 1)$ is the discount factor. Here, we assume that the representative household can supply labor (n_2) in second period at a wage w. The household receives an endowment $y_1 = \bar{y} > 0$ in first period, but no endowment in the second period ($y_2 = 0$). Moreover, she can allocate goods inter-temporally. The representative household owns both the capital stock and the firm and we define the intertemporal budget constraint as follows. Let x be the amount of capital rented to the firms and Π be profits that she receives from owning the firm. Then the household chooses consumptions c_1 and c_2 (at given prices p_1 and p_2) and capital rentals x (at a given price q) to maximize the utility subject to the intertemporal budget constraint

$$p_1 c_1 + p_1 x + p_2 c_2 = p_1 \bar{y} + qx + wn_2 + \Pi$$

or equivalently

$$p_1 c_1 + p_2 c_2 = p_1(\bar{y} - x) + qx + wn_2 + \Pi$$

It is clear that we must have $q = p_1$ in equilibrium. If not the consumer would choose x to plus infinity or minus infinity.

The firm buys capital from households at price q per unit and uses it to produce extra output in the second period. The firm also uses the labor input n_2 and pays a wage w. The firm's production function is given by:

$$k^{\alpha} n_2^{1-\alpha}$$

where $\alpha \in (0, 1)$. The firm chooses optimally the levels of capital and labor (denoted k and n_2) in period 2 such that she maximizes the profit:

$$\Pi = p_2 k^{\alpha} n_2^{1-\alpha} - qk - wn_2$$

where p_2 is the price of output in period 2, q the rental rate of capital and w the nominal wage. The capital is fully destroyed at the end of period 2.

1. Define a competitive equilibrium of this economy.

2. Solve for a competitive equilibrium. First solve the household problem (i.e. find the optimal consumptions c_1, c_2, labor supply n_2 and saving x, for given prices). Next, determine the optimal capital and labor demands, for given prices. After, use the equilibrium conditions on goods market (in period 1 and 2), on capital (financial) market and labor market and combine them with the optimality conditions in order to obtain the equilibrium allocations and prices).

3. Solve the central planner problem and compare the optimal allocations with those of the competitive equilibrium.

Exercice I: The Permanent Income Model with Capital Accumulation

We consider an economy in which an homogenous good Y_t can be either consumed C_t or invested I_t:

$$Y_t = C_t + I_t$$

The good Y_t is produced with the following technology

$$Y_t = Z_t + aK_t$$

where $a > 0$ and K_t is the capital stock in period t. Z_t is a stochastic variable. This stochastic variable evolves according to

$$Z_t = \rho Z_{t-1} + \varepsilon_t$$

where $|\rho| \leq 1$. The random variable ε_t satisfies $E_{t-1}\varepsilon_t = 0$, where E_{t-1} is the expectation operator conditional on the information set in period $t-1$. The capital stock evolves according to the following law of motion:

$$K_{t+1} = (1-\delta)K_t + I_t$$

where $\delta \in (0,1)$ is a constant depreciation rate. The representative household seeks to maximize the following intertemporal expected utility function:

$$E_t \sum_{i=0}^{\infty} \beta^i u(C_{t+i})$$

where

$$u(C_t) = \alpha_0 C_t - \frac{\alpha_1}{2}(C_t)^2$$

and α_0, α_1 are positive real numbers. The parameter $\beta \in (0,1)$ is a subjective discount factor. E_t is the expectation operator conditional on the information set in period t (i.e. when consumption decisions are made).

1. Determine the FOCs of the social planner optimization problem.

2. We impose $\beta(1+a-\delta) = 1$. Interpret this restriction. Show that the Euler equation on consumption has the following form

$$E_t \Delta C_{t+1} = 0$$

3. Compute the solution, i.e. express the the choice variable C_t in terms of the pre-determined variable K_t and the exogenous variable Z_t.

4. Show that the solution illustrates the *Lucas critique*.

5. Determine the value of the ratio $\sigma_{\Delta_c}/\sigma_{\Delta_y}$ when $\rho = 1$.

6. Compute the dynamic responses of consumption after a positive shock to the technology.

Exercice II: News Shocks and Stochastic Permanent Income

This exercise is an adaptation of Hall (1978) to news shocks on disposable labor income. A given household seeks to maximize

$$E_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}) \quad \text{where the discount factor satisfies} \quad \beta \in (0,1)$$

under the budget constraint

$$A_{t+1} = (1+r)A_t + Y_t^d - C_t$$

where A is the financial wealth, Y^d the disposable labor income and C the real consumption. The real interest rate r is strictly positive and constant. E_t is the conditional expectations operator. The instantaneous utility $U(C_t)$ is given by:

$$U(C_{t+i}) = \alpha_0 C_t - \frac{\alpha_1}{2} C_t^2$$

where the parameters α_0 and α_1 are strictly positive. In addition, we assume $\beta(1+r) = 1$

1. Determine the optimal decision on consumption.

2. Define the intertemporal budget constraint and thus determine the consumption decisions in terms of financial A_t and non–financial wealth (the expected discounted sum of net disposable labor income).

3. Now assume that the disposable labor income follows the process

$$\Delta Y_t^d = \varepsilon_{t-1}$$

   where $\Delta Y_t^d = Y_t^d - Y_{t-1}^d$. The random term satisfies $E(\varepsilon_{t-1}) = 0$ and $V(\varepsilon_{t-1}) = \sigma^2_{\varepsilon_y}$. This shocks is also iid, *i.e.* $E_t \varepsilon_{t+i} = 0$ for all $i \geq 1$. Determine the consumption function. Compare to the case where

$$\Delta Y_t^d = \varepsilon_t$$

4. Show that the model implies $Var(\Delta C_t) < Var(\Delta Y_t^d)$.

5. Discuss the quantitative properties in terms of *Granger causality*

   (see Wikipedia http://en.wikipedia.org/wiki/Granger_causality).

Exercice III: Habits in Consumption

This exercise is an adaptation of Hall (1978) to external habits in consumption. A given household seeks to maximize

$$E_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}) \quad \text{where the discount factor satisfies} \quad \beta \in (0,1)$$

under the budget constraint

$$A_{t+1} = (1+r)A_t + Y_t^d - C_t$$

where A is the financial wealth, Y^d the disposable labor income and C the real consumption. The real interest rate r is strictly positive and constant. E_t is the conditional expectations operator. The instantaneous utility $U(C_t)$ is given by:

$$U(C_{t+i}) = \alpha_0 C_t^\star - \frac{\alpha_1}{2} {C_t^\star}^2$$

The variable $C_t^\star$ is given by

$$C_t^\star = C_t - b\bar{C}_{t-1}$$

where $b \in [0,1)$ is the habit parameter ($b = 0$ corresponds to Hall, 1978). The variable $\bar{C}_{t-1}$ denotes the aggregate (or the reference for the group) consumption, which is considered as given at the individual level. At the aggregate level (or the group level), we have $\bar{C}_{t-1} = C_{t-1}$. So, $\bar{C}_{t-1}$ acts as an externality (optimal decisions at individual level does not internalize this past consumption, but the aggregate consumption is a function of past consumption). The parameters α_0 and α_1 are strictly positive. In addition, we assume $\beta(1+r) = 1$

1. Determine the optimal decision on consumption at the individual level and then at the aggregate level.
2. Now assume that the disposable labor income follows the process

   $$\Delta Y_t^d = \varepsilon_t$$

   where $\Delta Y_t^d = Y_t^d - Y_{t-1}^d$. The random term satisfies $E(\varepsilon_t) = 0$ and $V(\varepsilon_t) = \sigma^2_{\varepsilon_y}$. This shocks is also iid, *i.e.* $E_t\varepsilon_{t+i} = 0$ for all $i \geq 1$. Define the intertemporal budget constraint and thus determine the consumption function.
3. From the reduced form, compute the dynamic responses of the consumption to an innovation ε_t in period t.
4. Show that the model can solve the excess smoothing puzzle (remind that with industrialized countries, we have $\sigma_{\Delta c}/\sigma_{\Delta y^d} \simeq 0.7$). What is the critial value on b that allows to solve the puzzle?

Exercice IV: Quadratic Adjustment Costs on Labor with a Linear Technology

The value of the firm $\mathcal{V}_t$ is the expected discounted sum of profit.

$$\mathcal{V}_t = E_t \sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i \Pi_{t+i}$$

where E_t is the conditional expectations operator. $r > 0$ (the constant real interest rate) and the instantaneous profit is given by

$$\Pi_t = Y_t - wL_t - \frac{b}{2}\left(L_t - L_{t-1}\right)^2$$

where w is a constant real wage and $b \geq 0$ is the adjustment cost parameter.

The production function is given by

$$Y_t = (a + a_t)L_t$$

where $a > 0$ is a scale parameter and a_t a technology shock. This stochastic variable evolves according to

$$a_t = \rho a_{t-1} + \varepsilon_t$$

where $|\rho| \leq 1$. The random variable ε_t satisfies $E_{t-1}\varepsilon_t = 0$, where E_{t-1} is the expectation operator conditional on the information set in period $t-1$.

1. Determine the FOC of the firm's optimization problem.

2. Compute the solution, i.e. express the the choice variable L_t in terms of the pre-determined variable L_{t-1} and the exogenous variable Z_t. First use successive forward substitutions. Second, use the method of undetermined coefficients.

3. Show that the solution illustrates the *Lucas critique*.

4. Compute the dynamic responses of labor after a positive shock to the technology.

5. Assume now a new process for a_t

$$a_t = \rho a_{t-1} + \varepsilon_{t-1}$$

Comment this specification for the technology shock. Redo the exercise (FOC, solution).

Exercice V: Dynamic Factor Demand

Let a representative firm, endowed with a linear-quadratic production function and subject to quadratic adjustment costs, choose the capital stock so as to maximize the present discounted value of its profits

$$\max_{k_t} E_t \sum_{i=0}^{\infty} \beta^i \left[(a + a_t) k_{t+i} - \frac{\alpha}{2} k_{t+i}^2 - \frac{b}{2} (k_{t+i} - k_{t+i-1})^2 \right]$$

where E_t is the expectation operator conditional on the information set in period t. $\beta \in (0,1)$ is a constant discount factor. The technology parameters a and α are positive. The adjustment cost parameter is $b \geq 0$. We assume that the productivity shock a_t follows an autoregressive process of order one,

$$a_t = \rho_a a_{t-1} + \varepsilon_t$$

where $|\rho_a| \leq 1$. The random variable ε_t satisfies $E_{t-1}\varepsilon_t = 0$.

1. Determine the FOC of the firm's optimization problem.

2. Compute the solution, i.e. express the choice variable k_t in terms of the pre-determined variable k_{t-1} and the exogenous variable a_t.

3. Show that the solution illustrates the *Lucas critique*.

4. Compute the dynamic responses of capital after a positive shock to the technology.

5. Assume now a new process for a_t

$$a_t = \rho_a a_{t-1} + \varepsilon_{t-1}$$

Comment this specification for the technology shock. Redo the exercise (FOC, solution).

Exercice I: Government Spending and Economic Activity

Consider a discrete time economy populated with a large number of infinitely-lived, identical agents. The model considered here is repeated static because there is no saving for the household and no physical capital for the firm. The representative household seeks to maximize

$$\log(c_t + \alpha_g g_t) - \frac{\eta}{1+\nu} n_t^{1+\nu}$$

subject to the sequence of budget constraints

$$c_t + T_t = w_t n_t$$

The variable c_t is the private consumption, g_t denotes public expenditures, n_t is the labor supply, w_t is the real wage rate, and T_t denotes lump-sum taxes. The Frisch elasticity of labor supply is $1/\nu$ and $\eta > 0$ is a scale parameter. The parameter α_g, in turn, accounts for the complementarity/substitutability between private consumption c_t and public spending g_t. If $\alpha_g \geq 0$, government spending substitutes for private consumption, with perfect substitution if $\alpha_g = 1$. When the parameter $\alpha_g < 0$, government spending complements private consumption.

The representative firm produces a homogeneous final good y_t using labor as the sole input, according to the constant returns-to-scale technology

$$y_t = n_t.$$

Government purchases are entirely financed by taxes,

$$T_t = g_t.$$

Finally, the market clearing condition on the goods market writes

$$y_t = c_t + g_t.$$

1. Determine the optimal consumption–labor supply decision.
2. Compute the equilibrium output, i.e. determine the level of output as a non–linear function of public spending.
3. Log–linearize the equilibrium output around the steady-state and discuss the aggregate effects of government spending, i.e. the value of the government spending multiplier with respect to the parameters α_g and ν. Discuss.

Exercice II: A RBC Model

We consider an economy with a representative household and a representative firm. The firm produces an homogenous good with the following production function

$$Y_t = K_t^{1-\alpha} N_t^{\alpha}$$

with $\alpha \in (0, 1)$. K_t is the capital stock and N_t denotes the labor input. The capital stock evolves according to

$$K_{t+1} = (1 - \delta)K_t + I_t$$

where δ is the depreciation rate. I_t denotes investment. Here, we assume complete depreciation ($\delta = 1$)

$$K_{t+1} = I_t$$

The representative household consumes C_t and works N_t every period. The household seeks to maximize

$$E_t \sum_{i=0}^{\infty} \beta^i \left(\log(C_{t+i} + G_{t+i}) - V(N_{t+i})\right)$$

where G_t are the government expenditures, that are supposed to be stochastic. Notice that G_t enters linearly in the utility. $V(.)$ is a convex function. $\beta \in (0, 1)$ is the discount factor and E_t denotes the conditional expectations operator. The equilibrium condition on the good market is given by

$$Y_t = C_t + I_t + G_t$$

1. Set up the optimization problem for the social planer. Derive the first-order conditions. Explain the meaning of each condition.
2. Show that the ratio $(C_t + G)/Y_t$ and I_t/Y_t are constant at equilibrium.
3. Show that labor supply is constant at equilibrium.
4. Express the solution in a log–linear form (omit the constant terms). Comment.
5. Study the response of the economy to a government spending shock.