Université Toulouse 1 Capitole
GREMAQ

Habilitation à diriger des recherches

spécialité "Mathématiques appliquées et applications des mathématiques"

par

Adrien Blanchet

Méthodes variationnelles appliquées en biologie et en économie

HDR soutenue le lundi 3 décembre 2012 devant le jury composé de :

Henri Berestycki	CAMS – EHESS
Yann Brenier	CMLS – École Polytechnique
Pierre Degond	IMT – Université Paul Sabatier
Jean Dolbeault	CEREMADE – Université Paris Dauphine
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Après avis de :

Andrea Bertozzi	University of California, Los Angeles (USA)
Yann Brenier	CMLS – École Polytechnique
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If I have seen farther it is by standing on ye sholders of Giants
Newton, 1676

I would like to thank Andrea Bertozzi who kindly accepted to refer this manuscript at this extremely busy time of the year. Andrea is very occupied by various important responsabilities and leads an active group exploring the boundaries of mathematics. Her enthousiasm is very inspiring.

Peter Markowich warmly welcomed me in his group at the University of Cambridge in 2008. This 5 year Senior position, which I unfortunately had to leave after 5 months, boosted my self-confidence at a time when I needed it. It was very generous of him to accept to refer this HDR, writing such a precise report.

Yann Brenier est un des chercheurs les plus aventuriers scientifiquement qu'il m'ait été donné de rencontrer. Je me souviens de ce sujet dont il m'avait parlé en 2006 en me mettant en garde "c'est le type de problème un peu incertain, dont tu peux te rendre compte dans 5 ans que cela a déjà été fait ou que c'est trivial, mais si ce n'est pas le cas ce serait une très belle connexion complètement inattendue". Les incitations à court terme deviennent de plus en plus la norme en recherche et ne laissent que peu de place pour des chercheurs explorateurs. Yann a réussi à s'affranchir de ces contraintes et a eu plusieurs contributions majeures et élégantes en EDP. Sa démarche scientifique me fait rêver. C'est toujours un plaisir d'écouter Yann s'amuser à développer des idées originales et provocantes aussi bien en mathématique que sur la vie en général. Yann est un aristocrate de la recherche, jouisseur de la vie de qualité et je suis très honoré de l'avoir eu pour rapporteur et enchanté de sa lecture fine de mon travail.

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Pierre Degond est un leader fédérateur de la recherche à Toulouse et au delà. En plus d'être un grand mathématicien, il est constamment à l'affût de projets novateurs à la frontière des disciplines et sait parfaitement reconnaître les chercheurs des autres disciplines avec lesquels il serait bénéfique de travailler. Nous n'avons pas encore réussi à nous retrouver concrètement sur un projet tourné vers les questions économiques mais j'ai bon espoir que l'on y pervienne prochainement pour y faire de belles avancées.

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ciales qui stimulent la réflexion. C'est dans cette philosophie que nous avons entrepris ces dernières années de développer une approche multidisciplinaire de la modélisation des phénomènes humains. J'ai énormément appris de nos séances de travail entre biologistes, économistes, mathématiciens et physiciens. Pour toutes ces réunions, ces échanges nuit et week-end je remercie Ingela Alger, Etienne Danchin, Paul Seabright, Guy Theraulaz et Clément Sire.

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Contents

Introduction

This "habilitation à diriger des recherches" (HDR) thesis is interested in understanding emergence behaviours in biology and social sciences. Emergence properties have always intrigued philosophers and scientists, dating back to Aristotle in *Metaphysics*, where he defines these properties as "the totality is not, as it were, a mere heap, but the whole is something besides the parts": an emergent property of a system is one that is not a property of any component of that system, but is a feature of the system as a whole. The concept of emergence is a subject of considerable controversy within the field of philosophy, and interest in the question has developed across many other disciplines, ranging from chemistry to psychology through biology into economics and others, often with applications merging together across disciplines. A precise definition is thus still lacking and the distinction between weak and strong emergence remains unclear: *weak emergence* describes the property of a system that can be deduced from the dynamics of its parties, whereas *strong emergence* describes phenomena in which there is more causal links in the emergent structure than between its parties. In [147], Bedau and Humphreys observe:

> *Although strong emergence is logically possible, it is uncomfortably like magic. How does an irreducible but supervenient downward causal power arise, since by definition it cannot be due to the aggregation of the micro-level potentialities? Such causal powers would be quite unlike anything within our scientific ken. This not only indicates how they will discomfort reasonable forms of materialism. Their mysteriousness will only heighten the traditional worry that emergence entails illegitimately getting something from nothing.*

The very existence of strong emergences is still not a consensus. One could argue that the characterisation of strong emergence phenomena could be the result of the Human wisdom limits. After all, the properties of crystal, tension, pressure or even temperature first appeared to be emergent properties before being deduced from microscopic quantities. Indeed, most of the laws of physics themselves appear to have emerged during the course of time, making emergence the most fundamental principle in the universe, and raising the question of what might be the most fundamental law of physics from which all others emerged. To a leader in the field of materials science, who urged the participants at a meeting dedicated to "fundamental problems in condensed matter physics" to accept that "nothing was left but extensive science", in [6] the Nobel prize of physics Anderson answers

> *The behavior of large and complex aggregates of elementary particles, it turns out, is not to be understood in terms of a simple*

> *extrapolation of the properties of a few particles. Instead, at each level of complexity entirely new properties appear, and the understanding of the new behaviors requires research which I think is as fundamental in its nature as any other.*

As a matter of fact, properties like the emergence of life on Earth or the consciousness in Human beings cannot be deduced from molecule interactions.

Evolution of life is the process describing the emergence of complex living beings. In [80], Corning observes:

> *[In] evolutionary processes, causation is iterative; effects are also causes. And this is equally true of the synergistic effects produced by emergent systems. In other words, emergence itself... has been the underlying cause of the evolution of emergent phenomena in biological evolution; it is the synergies produced by organized systems that are the key.*

Emergent structures are a common strategy elected during the evolution of many animal groups: colonies of ants, mounds built by termites, swarms of bees, schools of fish, flocks of birds, and herds of mammals. As will be described here, the Keller-Segel system is a very simple model which serves as a first step towards understanding, in evolution, how a uni-cellular organism can form a multi-cellular organism with emergent properties.

Systems with emergent structures may appear to defy entropic principles and the second law of thermodynamics, because they form and increase order despite the lack of command and central control. Having a large number of interactions is not enough by itself to guarantee emergent behaviour; many of the interactions may be negligible or irrelevant, or may cancel each other out, as is the case when the law of large numbers applies. In some cases, a large number of interactions can in fact work against the emergence of interesting behaviour, by creating a lot of noise to drown out any emerging signal; the emergent behaviour may need to reach enough critical mass to be self-supporting, as is the case in the Keller-Segel system we will present here.

However, groups of human beings tend to produce spontaneous order, rather than the meaningless chaos often feared. Indeed, whenever you have a multitude of individuals interacting with one another, there often comes a moment when disorder gives way to order and something new emerges: a pattern, a decision, a structure, or a change in direction. According to the economist and philosopher F. von Hayek, see [112], spontaneous order arises when multiple actors spontaneously adopt a set of actions that provides them with a competitive advantage, and this behaviour creates a pattern that is self-sustaining, attracting more actors and consolidating the pattern. Hayek dismisses philosophies that do not adequately recognise the emergent nature of society, and which describe it as the conscious creation of a rational agent (be it God, the Sovereign, or any kind of personified body politic, such as Hobbes' Leviathan). The idea of laws and markets as emergent phenomena comes fairly naturally to an economist, and was indeed present in the works of early economists such as Bernard Mandeville, David Hume, and Adam Smith. As a famous example, the "invisible

hand" of the market is a metaphor conceived by Adam Smith to describe the self-regulating behaviour of the marketplace:

> *By preferring the support of domestic to that of foreign industry, he intends only his own security; and by directing that industry in such a manner as its produce may be of the greatest value, he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention. Nor is it always the worse for the society that it was not part of it.*

Emergent self-organisation appears frequently in cities where no planning predetermines the layout of the city, see [136]. Some urban planners believe that city planning is a problem of organised complexity that should be approached by observing the behaviour of individuals, instead of trying to control a single variable useful in mathematical formulae, see http://emergenturbanism.com/. Urban emergence has also been linked to theories of urban complexity, see [17] and urban evolution, see [150]. The last part of this HDR is dedicated to the idea that the individual minimisation of a utility by each agent can actually be seen as the optimisation of a functional at the population level. So that the Nash equilibrium associated with this game is actually seen at the level of the population, see below.

This *Habilitation à diriger des recherches* thesis is devoted to variational methods applied to models exhibiting emerging properties coming from biology and economics. Using theories from parabolic PDE, functional inequalities, optimal transport, kinetics, homogenisation, etc. we address diverse questions raised by the fields of application. Such an analysis requires to extend and develop new tools from nonlinear analysis in these different theories. Where possible, these results are published in journals of other disciplines, and constant dialogue between the mathematics community and the fields of application ensures the pertinence of the models studied. We give here a summary of the models which will be detailed in the forthcoming parts and chapters.

Applications to biology

Part I, dedicated to the applications in biology, is made up of two chapters.

The Keller-Segel systems

Chemo-taxis is defined as a move of an organism along a chemical concentration gradient. Bacterias can produce this chemo-attractant themselves, creating thus a long-range non-local interaction between them. We are interested in a very simplified model of aggregation at the scale of cells by chemo-taxis: myxamoebaes or bacterias experience a random walk to spread in the space and find food. But in starvation conditions, they emit a chemical signal: the cyclic adenosine monophosphate (cAMP). They move towards a higher concentration of cAMP. Their behaviour is thus the result of a competition between a random walk-based diffusion process and a chemo-taxis-based attraction. It was noticed experimentally that if there

are enough bacteria they aggregate whereas if they are not enough they go on spreading in a chemo-tactically inert environment, *e.g.* [51]. The typical time scale for the spreading of bacteria on a Petri dish is around one day, and a few minutes for the concentration. This concentration phenomenon is the first step for uni-cellular organisms to come together with others and form a multi-cellular organism. It can be seen as a hint on how, during the evolution of species, the passage from uni-cellular organisms to more complex structure was achieved. It is also a paradigm model for pattern formation of cells for meiose, embryo-genesis or angio-genesis.

In nature the dictyostelium discoideum spread on the soil and then come together to form a motile pseudoplasmodium. This slug creeps to a few centimetres below the soil surface where it forms a fruiting body with spores and a stalk. The spores are then blown away by the wind to colonise a new place. Around 20% of the cells which are in the stalk altruistically sacrifice themselves to allow the species to survive. They are an excellent example of social behaviour with outstanding coordination and sense of sacrifice for the benefit of the species. See Figure 1.

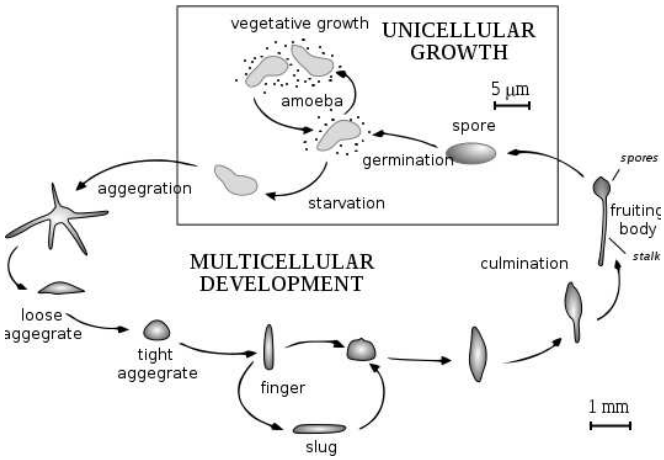

Figure 1 – *Dictyostelium discoideum cycle (source: Wikipedia).*

The general form of such models is a drift-diffusion equation given by

$$\begin{cases} \partial_t \rho = \operatorname{div}\left[\nabla \rho^m - \chi \rho \nabla \phi\right], \\ \tau \partial_t \phi = \Delta \phi - \alpha\, \phi + \rho, \end{cases} \qquad (t,x) \in (0,\infty) \times \mathbb{R}^d,$$

where $m \in (0,1)$ and $d \geq 2$. They exhibit a critical mass phenomenon in the sense that for diverse choices of m, τ and α, we are able to prove that there exists a critical mass under which all the solution are global-in-time.

The diffusion and the potential drift in general do not scale the same in the mass-invariant scaling. They actually exactly balance in the case where $m = m_d$ where

$$m = m_d =: 2\left(1 - \frac{1}{d}\right) \in (1,2).$$

In the case $m > m_d$ the diffusion is stronger and one can prove that all the solutions are global-in-time whereas when $m < m_d$, the drift term is predominant. These two cases are not studied in this thesis. The work presented here is devoted to the critical case $m = m_d$.

The most studied model in the case $m = m_d$ and $\tau = \alpha = 0$ when $d = 2$ and called the *(classical) parabolic-elliptic Keller-Segel system in dimension* 2. Such a model has attracted a lot of attention since [39]. The behaviour of the solutions is now better understood at least in the sub-critical regime. There actually exists a critical mass such that all the solutions are global-in-time if the mass is below this critical mass and above which all the solutions blowup in finite time. The convergence toward a self-similar profile was initiated in [39] and it was proved recently that such a convergence holds with rate for any mass below the critical mass [60]. Above the critical mass the situation is less clear, for a more detailed display see [96]. Let us however mention that researchers from the nonlinear Schödinger community have started to tackle such a problem with apparent success [173]. In the case of critical mass, the solution converges to the unique profile which is at finite Wasserstein distance. In particular, if the 2-moment is bounded then the solutions converge to a Dirac mass in infinite time whereas they do not blowup is they have a fat tail of the form

$$\frac{M}{\pi}\frac{\lambda}{(\lambda+|x|^2)^2}\,.$$

If we consider the case *parabolic-elliptic Keller-Segel system in highers dimensions* corresponding to $m = m_d$ and $\tau = \alpha = 0$ when $d \geq 3$, the situation is less clear. Indeed in such a model there also exists a critical mass under which all the solutions exist globally-in-time but we cannot prove that all the solutions with super-critical mass blowup in finite time. The main tractable questions are opened in this model.

The situation is even less clear for the cases when $\tau \neq 0$, *i.e* for the *parabolic-parabolic Keller-Segel system in highers dimensions*. In the case of nonlinear diffusion in higher dimensions we prove that all the solutions are global-in-time below a sub-critical case. The situation of super-critical case is far from being understood and it was even proved that there exists global-in-time solution with super-critical mass, see [26].

All these results are proved using entropy methods in connection with functional inequalities and the theory of optimal transport.

Stochastic Stokes' drift

Molecular motors are biological molecular machines that are the essential agents of movement in living organisms. In general terms, a motor may be defined as a device that consumes energy in one form and converts it into motion or mechanical work; for example, many protein-based molecular motors harness the chemical free energy released by the hydrolysis of ATP in order to perform mechanical work. In terms of energy efficiency, this type of motor can be superior to currently available man-made motors. One important difference between molecular motors and macroscopic motors is that molecular motors operate in the thermal bath, an environment in which the fluctuations due to thermal noise are significant. It first appeared to allow a mecanism which would create mechanical work without consuming energy by M. Smoluchowski: The simple machine, consisting of a tiny paddle

wheel and a ratchet, appears to be an example of a Maxwell's demon, able to extract useful work from random fluctuations (heat) in a system at thermal equilibrium in violation of the second law of thermodynamics, see Figure 2. Detailed analysis by Feynman and others showed why it cannot actually do this.

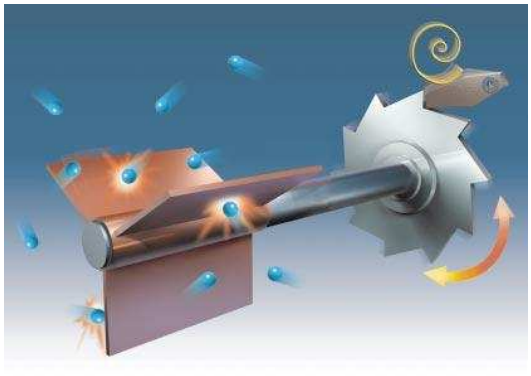

Figure 2 – *Ratchet mechanisms proposed by M. Smoluchowski. Source: D. Astumian.*

The *stochastic Stokes' drift*, see [23], is a simple model describing the diffusion of particles in the presence of a periodic, wave-like potential. Particles suspended in a liquid and subject to diffusion experience a net drift due to the wave travelling through the liquid. See Figure 3.

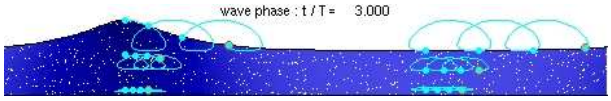

Figure 3 – *Stokes' drift. Source: Wikipedia*

It can also be seen as a simple model of *Brownian ratchet*. When there is no diffusion, the net drift of particles is proportional to ω when ω is small, but decays to 0 when ω is large. In the presence of a diffusion the situation is different since, due to the Brownian motion, some particles will move in the direction opposite to the wave train. Our goal is to study the net drift, or to be precise, the speed of the centre of mass, the formation of the front and its diffusion, when there are no spatial limitation for the solution, and to measure the *efficiency* in terms of *coherent transport*.

In Chapter 2, we study the simplest version of the stochastic Stokes' drift model, describing a density $f(t,x)$ of particles obeying the equation

$$f_t = \Delta f + \nabla \cdot \left[\nabla \psi(x - \omega\, t\, \mathsf{e})\, f\right], \quad x \in \mathbb{R}^d\,,\ t > 0\,.$$

where $\psi(y+\mathsf{k}) = \psi(y)$ for any $(y,\mathsf{k}) \in \mathbb{R}^d \times \mathbb{Z}^d$, and we will simply write ψ as a function of $y \in \mathbb{T}^d \approx [0,1)^d$. Furthermore, $\omega \in \mathbb{R}$ is a constant and $\mathsf{e} \in \mathbb{R}^d$ is a fixed vector, such that $|\mathsf{e}| = 1$. With these notations, $\psi(x - \omega\, t\, \mathsf{e})$ represents a periodic potential in $\mathbb{R}^d$ moving with a constant speed ω in the direction of the vector e, that is a *travelling potential*.

The asymptotic speed of the centre of mass is decreased by the diffusion. The effective diffusion of the particles is also changed by the travelling wave.

Surprisingly, it can be decreased or increased, depending on ω, an effect which is apparently not mentioned in physics literature. This last statement is perhaps less obvious although similar effects are already known in the context of homogenisation theory, see *e.g.*, [106, 203]. To address the mutual influence of transport and diffusion in the stochastic Stokes' drift, we will analyse the large time asymptotic profiles of solutions of (2.1.1). A first step will be to characterise the speed of the travelling front and to show that it is asymptotically the same as the speed of the centre of mass of the solution. Then, in the reference frame attached to the centre of mass, a time rescaling transforms the travelling potential into an oscillating term whose influence on the large time behaviour can be understood using the tools of homogenisation theory. Moreover, several length scales have to be taken into account. The position of the centre of mass is of the order of t, while the typical size of the front grows like $\sqrt{t}$. Typical relaxation rates are exponential at small scale, but of the order of $1/\sqrt{t}$ when measured globally in L^1.

A key tool for the understanding of the stochastic Stokes' drift rewritten in self-similar variables attached to the centre of mass, is the *logarithmic Sobolev inequality* for a Gaussian measure perturbed by a bounded oscillating potential, namely $d\mu_\varepsilon(x) := Z_\varepsilon^{-1} \mathrm{e}^{-\phi(x/\varepsilon)-|x|^2/2}\,\mathrm{d}x$. The main effort in this chapter is directed towards the study of the homogenised limit of a family of functional inequalities, which interpolate between Poincaré and logarithmic Sobolev inequalities, and govern the rate of convergence to equilibrium and the variance of the solution for large time.

Entropy methods and functional inequalities

Part II is dedicated to the development of entropy methods for standard linear equations: heat and fast-diffusion equations.

Asymptotics of the fast diffusion equation

In Chapter 3, we consider non-negative solutions of the fast diffusion equation

$$\begin{cases} \partial_\tau u = \Delta u^m \\ u(0,\cdot) = u_0\,, \end{cases}$$

with $m \in (0,1)$, in the Euclidean space $\mathbb{R}^d$, $d \ge 3$.

During the last few years, asymptotic rates of convergence for the solutions of nonlinear diffusion equations have attracted lots of attention, usually in connection with time-dependent scalings and entropy methods. This has been first done in the range of exponents corresponding to the porous medium equation, with $1 < m < 2$, and in the range where standard Gagliardo-Nirenberg inequalities apply, $1 - 1/d =: m_1 \le m < 1$, see [87, 66, 69]. The class of non-negative, finite mass solutions has to be narrowed to the smaller set of functions with finite free energy, or to be precise, with finite entropy and finite potential energy. In the rescaled variables, asymptotic stabilisation to the Barenblatt profiles holds at an exponential rate, while in the original time variable τ, the convergence of the difference

with the Barenblatt solutions holds at a power-law rate, which is shown to be optimal.

The next question was to understand what happens for $m < m_1$. After the linearised analysis of [68], the proof of convergence with rates was done in [72] in the range $1 - 2/d := m_c < m < m_1$ for which global existence of finite mass solutions still holds. The basin of attraction is narrowed to the class of solutions with finite relative entropy with respect to some Barenblatt solution.

A dramatic change occurs for $m < m_c$, since a large class of solutions vanish in finite time. As a consequence, mass is not conserved, and a key estimate for higher values of m is lost. It is however natural to investigate the basin of attraction of the pseudo-Barenblatt solutions for $m \le m_c$ using relative entropy techniques and to study the convergence rates. This can be done in a weighted space using functional inequalities, which can still be related to some spectral properties of a differential operator obtained by an appropriate linearisation.

To capture the asymptotic profile of extensions it is convenient to rescale the solutions and replace the study of intermediate asymptotics by the study of the convergence to stationary solutions in *rescaled variables*,

$$t := \log\left(\frac{R(\tau)}{R(0)}\right) \quad \text{and} \quad x := \frac{y}{R(\tau)}\,,$$

with

$$R(\tau) := \left[d\,(m_c - m)\,(T - \tau)\right]^{-\frac{1}{d\,(m_c - m)}}.$$

where T denotes the extinction time. In these new variables, if u is a solution to the fast-diffusion equation, the function

$$v(t,x) := R(\tau)^d\, u(\tau, y)$$

solves a nonlinear *Fokker-Planck type equation*,

$$\begin{cases} \partial_t v(t,x) = \Delta v^m(t,x) + \nabla\cdot(x\,v(t,x)) & (t,x) \in (0,+\infty)\times\mathbb{R}^d\,, \\ v(0,x) = v_0(x) & x \in \mathbb{R}^d\,. \end{cases}$$

We study the asymptotic stabilisation towards *self-similar asymptotic solutions* known as Barenblatt:

$$U_{D,T}(\tau,y) := \frac{1}{R(\tau)^d}\left(D + \frac{1-m}{2\,m}\left|\frac{y}{R(\tau)}\right|^2\right)^{-\frac{1}{1-m}}$$

which are transformed in the self-similar variable into stationary solutions given by

$$V_D(x) := \left(D + \frac{1-m}{2\,m}\,|x|^2\right)^{-\frac{1}{1-m}}$$

where $0 < m < 1$ and $D > 0$ is a free parameter.

Using the *free energy functional* defined by

$$\mathcal{E}[v] := \int_{\mathbb{R}^d}\left[\varphi(v) + \frac{1}{2}\,|x|^2\,v\right]\mathrm{d}x \quad \text{where} \quad \varphi(v) := \frac{v^m}{m-1}\,,$$

the entropy method allows to determine the large time asymptotics of the solution in the range $1 - 1/d =: m_1 \le m < 1$ and $1 < m < 2$. Below this exponent m_1 the functional fails to be displacement convex and the standard methods cannot be extended. Linearisation techniques allows to go beyond this exponent but not beyond m_c.

We push-forward this analysis by considering the *relative entropy* $\mathcal{E}[v|V_D] := \mathcal{E}[v] - \mathcal{E}[V_D]$. Such a quantity actually makes sense for all ranges of $0 < m < 1$ at least for a large class of initial data. It allows to determine the asymptotics of the fast-diffusion equation for any $m \in (0, m_1)$. We also prove that asymptotically the solutions converge with a rate.

The main idea of the method is to linearise the free energy and prove an inequality linking the linearised relative entropy to its derivative. Such an inequality is of Hardy-Poincaré type and was not available in the literature. For time large enough it is possible to relate the linear analysis to the analysis of the non-linear fast-diffusion equation and obtain the desired result.

Similar methods can be used for the doubly nonlinear equation.

Improved asymptotics for the heat equation

In Chapter 4, we determine suitable assumptions on the initial data in order to improve the rate of convergence of the solution to the standard heat equation:

$$\frac{\partial u}{\partial t} = \Delta u \quad t > 0\,,\ x \in \mathbb{R}^d$$

Various attempts have been made by fixing the centre of mass. In this chapter we prove that

Such an orthogonality condition actually allows to improve on various Poincaré and logarithmic Sobolev inequalities in the spirit of the previous chapter.

Applications to economics

The last part is consecrated to a series of work on models in economics. The first is a social interaction model in the literature of urban region, the second is in economics theory.

Since [150], it is known that both market and non-market forces play an important role in shaping the distribution of economic activities across space. The new economic geography literature has reemphasised the role of localised pecuniary externalities mediated by the market in a general equilibrium framework, see [136]. Social interactions through face-to-face contacts also contribute to the gathering of individuals in villages, agglomerations, or cities, see [105]. In [18], the urban structure emerges from the interplay between a spatial communication externality and the land market.

When studying the role of agglomeration forces on the urban structure, the existing literature traditionally relies on specific functional forms regarding utility functions or transportation costs. New economic geography models make a wide use of Dixit-Stiglitz or quadratic preferences over manufacturing varieties and of 'icerberg' transport costs, see [103, 166]. In Beckmann's spatial model of social interactions, the preference for land is logarithmic and the cost of accessing agents is linear, see [104].

Despite these various efforts in extending models addressing agglomeration forces mediated by the market mechanism, little effort has been made to extend further spatial models where agglomeration externalities are driven by non-market forces. The aim of this paper is to fill up this gap by addressing the existence and uniqueness of equilibrium for general spatial economies involving social interactions. In order to do so, we generalise Beckmann's spatial model of social interactions to the case of a two dimensional spatial economy, a large class of preferences for land, a general accessing cost, and space-dependent amenities.

Mathematically, we consider non-cooperative, anonymous non-atomic games with a continuum of players. Given a space of players types X endowed with a probability measure $\mu \in \mathcal{M}(X)$ which gives the exogenous distribution of the type of the agents, an action space Y and a cost Γ: $X \times Y \times \mathcal{M}(Y) \to \mathbb{R}$. The θ-type agents taking action x pay the cost $\Gamma(\theta, x, \lambda)$ where λ is the distribution of the players' actions. Such a problem seen from an individualistic point of view is very difficult to tackle. We are interested in the Nash equilibrium which is the situation in which no player has any incentive to relocate. We restrict ourselves to the case of potential games in which the externalities derive from a potential. In both these situations we prove that the equilibrium can be seen as the minimiser of a functional in the usual sense and in the sense of optimal transport. Actually we prove that a distribution is an equilibrium if and only if it is a minimiser of

$$\mathcal{J}_\mu[\nu] := \alpha \mathcal{W}_c(\mu, \nu) + \mathcal{E}[\nu]$$

where $\mathcal{W}_c(\mu, \nu)$ is the Wasserstein distance between μ and ν for the cost c and $\mathcal{E}$ has the typical form

$$\mathcal{E}[\nu] := \int_{\mathcal{K}} V[\nu(x)] \, \mathrm{d}y + \int_{\mathcal{K}} A(y)\lambda(y) \, \mathrm{d}y + \frac{1}{2} \iint_{\mathcal{K}\times\mathcal{K}} \phi(y-z)\lambda(y)\lambda(z) \, \mathrm{d}y \, \mathrm{d}z$$

and α is zero when we consider that the utility of the agents does not depend on their type, whereas $\alpha = 1$ when c measures the cost for an agent of type x to take action y. The study of the decentralised equilibrium emerges from the analysis of the minimiser of a functional. Standard optimal transport results can be applied and allow to deduce existence and uniqueness results together with a characterisation of the equilibrium in terms of a Monge-Ampère equation. We can even determine and numerically compute the taxes to restore the efficiency.

List of the presented articles

Applications to biology

On the Keller-Segel systems

- A. Blanchet, *On the parabolic-elliptic Patlak-Keller-Segel system in dimension 2 and higher*, To appear in Sémin. Équ. Dériv. Partielles.
- A. Blanchet, V. Calvez, and J. A. Carrillo, *Convergence of the mass-transport steepest descent scheme for the subcritical Patlak-Keller-Segel model*, SIAM J. Numer. Anal., 46 (2008),
- A. Blanchet, E. Carlen, and J. A. Carrillo, *Functional inequalities, thick tails and asymptotics for the critical mass Patlak-Keller-Segel model*, J. Funct. Anal., 262 (2012), pp. 2142–2230.
- A. Blanchet, J. A. Carrillo, and P. Laurençot, *Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimensions*, Calc. Var. Partial Differential Equations, 35 (2009),
- A. Blanchet, J. A. Carrillo, and N. Masmoudi, *Infinite time aggregation for the critical Patlak-Keller-Segel model in* $\mathbb{R}^2$, Comm. Pure Appl. Math., 61 (2008), pp. 1449–1481.
- A. Blanchet, J. Dolbeault, M. Escobedo, and J. Fernandez, *Asymptotic behaviour for small mass in the two-dimensional parabolic-elliptic Keller-Segel model*, Journal of Mathematical Analysis and Applications, 361 (2010), pp. 533–542.
- A. Blanchet and P. Laurençot, *Finite mass self-similar blowing-up solutions of a chemotaxis system with non-linear diffusion*, Communications on Pure and Applied Analysis, 11 (2011), pp. 47–,60.
- ——, *The parabolic-parabolic Keller-Segel system with critical diffusion as a gradient flow in* ^{d}, $d \geq 3$, To appear in Com. Par. Dif. Eq., 2012.

Stochastic Stokes' drift

- A. Blanchet, J. Dolbeault, and M. Kowalczyk, *Travelling fronts in stochastic Stokes' drifts*, Physica A: Statistical Mechanics and its Applications, 387 (2008), pp. 5741–5751.

- ——, *Stochastic Stokes' drift, homogenized functional inequalities, and large time behaviour of Brownian ratchets*, SIAM Journal of Mathematical Analysis, 41 (2009), pp. 46–76.

Entropy methods and functional inequalities

On the fast-diffusion equation

- M. Agueh, A. Blanchet, and J. A. Carrillo, *Large time asymptotics of the doubly nonlinear equation in the non-displacement convexity regime*, Journal of Evolution Equations, 10 (2010), pp. 59–84.
- A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, and J.-L. Vázquez, *Hardy-Poincaré inequalities and applications to nonlinear diffusions*, C. R. Math. Acad. Sci. Paris, 344 (2007), pp. 431–436.
- ——, *Asymptotics of the fast diffusion equation via entropy estimates*, Archive for Rational Mechanics and Analysis, 191 (2009), pp. 347–385.

On the heat equation

- J.-P. Bartier, A. Blanchet, J. Dolbeault, and M. Escobedo, *Improved intermediate asymptotics for the heat equation*, Applied Mathematics Letters, 24 (2011), pp. 76–81.

Applications to economics

- A. Blanchet and G. Carlier, *Optimal transport and Cournot-Nash equilibria*. Pre-print http://arxiv.org/abs/1206.6571, 2012.
- A. Blanchet, P. Mossay, and F. Santambrogio, *Existence and uniqueness of equilibrium for a spatial model of social interactions*. Pre-print, http://w3-gremaq.univ-tlse1.fr/blanchet/publication/BMS.pdf, 2012.

Part I

Applications to biology

1 On the Keller-Segel system in dimension 2 and higher

This chapter is dedicated to recent results on the parabolic-elliptic Keller-Segel model in dimension 2, and its generalisation with nonlinear diffusion in higher dimensions. These models have a critical mass M_c such that the solutions exist globally in time if the mass is less than M_c and above which there are solutions which blowup in finite time. The main tools, in particular the free energy, and the idea of the methods are set out. A number of open questions are also stated. Part of this chapter was published in [29].

The works presented here are

- A. Blanchet, *On the parabolic-elliptic Patlak-Keller-Segel system in dimension 2 and higher*, To appear in Sémin. Équ. Dériv. Partielles.
- A. Blanchet, V. Calvez, and J. A. Carrillo, *Convergence of the mass-transport steepest descent scheme for the subcritical Patlak-Keller-Segel model*, SIAM J. Numer. Anal., 46 (2008),
- A. Blanchet, E. Carlen, and J. A. Carrillo, *Functional inequalities, thick tails and asymptotics for the critical mass Patlak-Keller-Segel model*, J. Funct. Anal., 262 (2012), pp. 2142–2230.
- A. Blanchet, J. A. Carrillo, and P. Laurençot, *Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimensions*, Calc. Var. Partial Differential Equations, 35 (2009),
- A. Blanchet, J. A. Carrillo, and N. Masmoudi, *Infinite time aggregation for the critical Patlak-Keller-Segel model in* $\mathbb{R}^2$, Comm. Pure Appl. Math., 61 (2008), pp. 1449–1481.
- A. Blanchet, J. Dolbeault, M. Escobedo, and J. Fernandez, *Asymptotic behaviour for small mass in the two-dimensional parabolic-elliptic Keller-Segel model*, Journal of Mathematical Analysis and Applications, 361 (2010), pp. 533–542.
- A. Blanchet and P. Laurençot, *Finite mass self-similar blowing-up solutions of a chemotaxis system with non-linear diffusion*, Communications on Pure and Applied Analysis, 11 (2011), pp. 47–,60.

- ——, *The parabolic-parabolic Keller-Segel system with critical diffusion as a gradient flow in ^{d}, $d \geq 3$*, To appear in Com. Par. Dif. Eq., 2012.

1.1 The 2d parabolic-elliptic Keller-Segel system

1.1.1 The model

This chapter is dedicated to the Keller-Segel type model. Such a model describes a chemo-taxis phenomenon in which organisms emit a chemical signal which attracts all its fellow organisms. It was seen experimentally that if the cells are numerous enough they aggregate to form a multi-cellular body, whereas if they are too few they spread in their environment. It is precisely the evolution of such models, as the result of a competition between diffusion and drift driven by the chemical signal, that we propose to study. Such a model serves as a test to understand the migration of cells, and is used as a building block for more complex biological phenomenon such as polarisation, Balo disease or angio-genesis, etc. Such models are also known in physics as Smulochowski-Poisson systems.

The first mathematical attempt to model this aggregation phenomenon is often granted to E. F. Keller and L. A. Segel in [127] but this model was earlier described by C. S. Patlak in [169]. We consider the following simplified version given by [121]:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \Delta \rho - \nabla \cdot (\rho \nabla c) & x \in \mathbb{R}^2,\ t > 0\,, \\ -\Delta c = \rho & x \in \mathbb{R}^2,\ t > 0\,, \\ \rho(\cdot, t = 0) = \rho_0 \geq 0 & x \in \mathbb{R}^2\,. \end{cases} \tag{1.1.1}$$

Here ρ represents the cell density and c the concentration of chemo-attractant.

As the solution to the Poisson equation $-\Delta c = \rho$ is given up to a harmonic function, we choose the one given by $c = G * \rho$ where

$$G(|x|) := -\frac{1}{2\pi} \log |x|\,.$$

The Patlak-Keller-Segel system (1.1.1) can thus be written as a non-local parabolic equation:

$$\frac{\partial \rho}{\partial t} = \Delta \rho - \operatorname{div}(\rho \nabla G * \rho) \quad \text{in } (0, +\infty) \times \mathbb{R}^2\,.$$

Also note that the mass is conserved

$$\int_{\mathbb{R}^2} \rho(t, x)\, \mathrm{d}x = \int_{\mathbb{R}^2} \rho_0(x)\, \mathrm{d}x =: M\,.$$

S. Childress, J. Percus and V. Nanjundiah conjectured in [78, 164] that this system displays the existence of a critical mass above which the cells aggregate and below which they do not. For a complete review of the early literature, the interested reader could beneficially consult [117, 116]. For a more recent references see [171, 115, 75].

Except when it is clearly indicated, in this article we will assume that the initial condition ρ_0 satisfies:

$$(1 + |x|^2)\, \rho_0 \in \mathrm{L}^1_+(\mathbb{R}^2) \quad \text{and} \quad \rho_0 \log \rho_0 \in \mathrm{L}^1(\mathbb{R}^2)\,.$$

1.1.2 Blowup

Consider a smooth solution to the Patlak-Keller-Segel system (1.1.1), we formally compute a virial identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}|x|^2\rho(t,x)\,\mathrm{d}x = 4M\big(1-\frac{M}{8\pi}\big)\,. \tag{1.1.2}$$

So that all the solutions with mass bigger than 8π and finite 2-moment cannot be global in time:

Theorem 1.1.1 (Blowup) *Let ρ be a solution to the Patlak-Keller-Segel system (1.1.1) and $[0,T^*)$ its maximal interval of existence. If $M>8\pi$, then*

$$T^* \le \frac{2\pi}{M(M-8\pi)}\int_{\mathbb{R}^2}|x|^2\rho_0(x)\,\mathrm{d}x\,,$$

and $\rho(t,\cdot)$ converges, up to extraction of a sub-sequence, as $t\to T^*$ to a measure which is not in $L^1(\mathbb{R}^2)$.

This method, known as the virial method, was already used in the dispersive equations community. This kind of method is non-constructive and gives no hint on the nature of the blowup. Another non-constructive proof of this non global-in-time existence will be given in Remark 1.1.10 and will give similar results even in the case when the solutions are of 2-moment infinite.

The first proof that the solutions of super-critical mass develop singularities are due to [27, 162, 77, 121]. Such a solution develops Dirac peaks of mass exactly equal to 8π, see [163, 113]. Very recently, [173] proves the universality and the stability for small perturbations of the blowup around the minimiser Q of the HLS inequality: the solution admits for all times $t\in[0,T)$ a decomposition

$$u(t,x)=\frac{1}{\lambda^2(t)}(Q+\varepsilon)\left(t,\frac{x}{\lambda(t)}\right)$$

and the universal blowup speed:

$$\lambda(t)=\sqrt{T-t}\exp\left(\log(T-t)/2+O(1)\right)$$

as t goes to the blowup time T. Such a method was developed for different dispersive partial differential equations in [157].

Open question: *We can hope that a similar method should also gives such blowup results for the parabolic-parabolic Keller-Segel system.*

Concerning the continuation of the solution after blowup, the usual idea is to define a sequence of approximate problems containing a small parameter $\varepsilon>0$ which have global-in-time solutions and approach the original problem when ε goes to 0. The behaviour of the solution to the approximate model has to be close to the one of the Patlak-Keller-Segel system (1.1.1), except close to the singularities. The blowup indicates that the approximate problem is no longer valid close to the singularity. In [202, 201], using matched asymptotic expansions J. Velázquez describes in a rather detailed manner the formation and motion of some regions where the mass concentrates. He also proves local existence of the solution as long as there is no formation

of another Dirac mass and no collision of Dirac masses. The approximate problem is

$$\frac{\partial \rho}{\partial t} = \Delta \rho - \nabla \cdot (\Phi_\varepsilon(\rho) \nabla G * \rho) \quad x \in \mathbb{R}^2,\ t > 0\,,$$

where $\Phi_\varepsilon(\rho) := \varepsilon^{-1}\Phi(\varepsilon\rho)$ with ε a small parameter and Φ an increasing bounded function satisfying

$$\Phi(s) = s - \alpha s^2 + \cdots \text{ as } s \to 0 \text{ and } \Phi(s) \sim A \text{ as } s \to \infty\,,$$

where $A > 0$ is a given number. This model prevents overcrowding as the chemo-tactic function $\Phi_\varepsilon(\rho)$ saturates at the constant value A.

In [96], J. Dolbeault and C. Schmeiser define measure valued densities to give a sense to generalised global-in-time solutions for any mass. This extends the solution concept after blowup. They also show that the choice of a solution concept after blowup is not unique and depends on the type of regularisation. The regularised problem they consider is different from the one chosen above:

$$\frac{\partial \rho}{\partial t} = \Delta \rho - \nabla \cdot (\rho \nabla G_\varepsilon * \rho) \quad x \in \mathbb{R}^2,\ t > 0\,, \tag{1.1.3}$$

where $G_\varepsilon(x) := -\log(|x| + \varepsilon)/2\pi$.

Theorem 1.1.2 (Generalised solution, [96]) *For every $T > 0$, as $\varepsilon \to 0$, a sub-sequence of solutions ρ_ε to (1.1.3) converges tightly and uniformly in time to a time dependent measure $\rho(t)$. There exists $\nu(t)$ such that (ν, ρ) is a generalised solution in the distributional sense of*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(j[\rho, \nu] - \nabla \rho) = 0 \tag{1.1.4}$$

where the convective flux $j[\rho, \nu]$ is supported in the support of ν and is given by

$$\int_0^T \int_{\mathbb{R}^2} \varphi(t,x) j[\rho,\nu](t,x)\, \mathrm{d}x\, \mathrm{d}t = -\frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^2} \nu(t,x) \nabla \varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t$$
$$- \frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^4} (\varphi(t,x) - \varphi(t,y))\, K(x-y) \rho(t,x) \rho(t,y)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t$$

for any $\varphi \in \mathcal{C}_b^1((0,T) \times \mathbb{R}^2)$ with

$$\begin{cases} \dfrac{x}{|x|^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0\,. \end{cases}$$

If ρ does not charge points then the additional default measure ν vanishes and $j[\rho, 0] = \rho \nabla G * \rho$, so that (1.1.4) is generalisation of the Patlak-Keller-Segel system (1.1.1). They also obtain a strong formulation when the generalised solution is assumed to be the sum of a regular part and of Dirac masses:

Theorem 1.1.3 (Strong formulation, [96]) *Assume that the generalised solution to* (1.1.4) *has the form*

$$\rho(t,x) = \sum_{j\in N} M_j(t)\delta(x - x_j(t)) + \rho_{\text{reg}}(t,x)\ . \tag{1.1.5}$$

Then

$$\frac{\partial \rho_{\text{reg}}}{\partial t} = \Delta\rho_{\text{reg}} - \nabla\left(\rho_{\text{reg}}\nabla G * \rho_{\text{reg}} - \frac{1}{2\pi}\nabla\rho_{\text{reg}}\sum_{j_i n N} M_j(t)\frac{x - x_j(t)}{|x - x_j(t)|^2}\nabla\rho_{\text{reg}}\right)$$

with

$$\dot{M}_i(t) = M_i(t)\rho_{\text{reg}}(t, x_i(t))$$

and

$$\dot{x}_i(t) = \nabla G * \rho_{\text{reg}}(t, x_i(t)) - \frac{1}{2\pi}\sum_{j\in N, i\neq j} M_j(t)\frac{x - x_j(t)}{|x - x_j(t)|^2}\ .$$

A similar result was formally obtained in [202], with the last equation replaced by

$$\dot{x}_i(t) = \Gamma(M_i(t))\left(-\frac{1}{2\pi}\sum_{j\in N, i\neq j} M_j(t)\frac{x - x_j(t)}{|x - x_j(t)|^2} + \nabla G * \rho_{\text{reg}}(t, x_i(t))\right)$$

where $\Gamma(M)$ is a mean value of the derivative Φ is such that $0 < \Gamma(M) < 1$, $\Gamma(8\pi) = 1$ and $\Gamma(\infty) = 0$ and can be explicitly described, see [202, Equation (3.45)].

In [74], the authors prove that when t is large enough the solution is made of a Dirac peak of mass $M_0(t)$ surrounded by a dilute halo containing the remaining mass whose dynamical evolution is described by a Fokker-Planck equation. Therefore, they neglect the self-gravity of the halo and prove that the mass of the Dirac peak saturates to M algebraically rapidly as

$$1 - \frac{M_0(t)}{M} \sim t^{-a} \quad \text{with } a = \frac{M}{4\pi}\ .$$

Open question: *Actually the assumption* (1.1.5) *is valid only between two blowup events or between two collisions of Dirac masses. We expect that in the end the solution is made of one Dirac mass with all the mass but we are still missing such a rigorous theory.*

1.1.3 Global existence

A priori **estimates**

The natural idea is to regularise the Green kernel and to pass to the limit. The solutions have mass M so that the loss of compactness can come either from concentration or vanishing. By (1.1.2), the second moment remains bounded so that the main problem is to control the concentration of mass. W. Jäger and S. Luckhaus tried to obtain a bound on the entropy $\int_{\mathbb{R}^2}\rho\log\rho\ \mathrm{d}x$ by differentiating it and using an integration by parts and the equation for c, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\rho\log\rho\ \mathrm{d}x = -4\int_{\mathbb{R}^2}|\nabla\sqrt{\rho}|^2\ \mathrm{d}x + \int_{\mathbb{R}^2}\rho^2\ \mathrm{d}x\,.$$

Applying the Gagliardo-Nirenberg-Sobolev inequality:

$$\int_{\mathbb{R}^2} |u|^4 \, \mathrm{d}x \leq C_{\mathrm{GNS}} \int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x \int_{\mathbb{R}^2} |u|^2 \, \mathrm{d}x \quad \forall \, u \in H^1(\mathbb{R}^2) \,, \tag{1.1.6}$$

to $u = \sqrt{\rho}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} \rho \log \rho \, \mathrm{d}x \leq [-4 + MC_{\mathrm{GNS}}] \int_{\mathbb{R}^2} |\nabla \sqrt{\rho}|^2 \, \mathrm{d}x \,.$$

So that the entropy is non-increasing if $M \leq 4/C_{\mathrm{GNS}} \approx 1.862... \times 4\pi < 8\pi$. They hence obtained global-in-time existence in this case together with important propagation of the L^p-estimates.

We can indeed improve this result by using the following free energy:

$$\mathcal{F}_{\mathrm{PKS}}[\rho] := \int_{\mathbb{R}^2} \rho \log \rho \, \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^2} \rho c \, \mathrm{d}x \,.$$

A simple formal calculation shows that for all $u \in C_c^\infty(\mathbb{R}^2)$ with zero mean,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\mathcal{F}_{\mathrm{PKS}}[\rho + \epsilon u] - \mathcal{F}_{\mathrm{PKS}}[\rho] \right) = \int_{\mathbb{R}^2} \frac{\delta \mathcal{F}_{\mathrm{PKS}}[\rho]}{\delta \rho}(x) \, u(x) \, \mathrm{d}x$$

where

$$\frac{\delta \mathcal{F}_{\mathrm{PKS}}[\rho]}{\delta \rho}(x) := \log \rho(x) - G * \rho(x) \,.$$

It is then easy to see that the Patlak-Keller-Segel system (1.1.1) can be rewritten as

$$\frac{\partial \rho}{\partial t}(t,x) = \mathrm{div} \left(\rho(t,x) \nabla \left[\frac{\delta \mathcal{F}_{\mathrm{PKS}}[\rho(t)]}{\delta \rho}(x) \right] \right) \,. \tag{1.1.7}$$

It follows that at least along well-behaved solutions to the Patlak-Keller-Segel system (1.1.1),

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] = - \int_{\mathbb{R}^2} \rho(t,x) \left| \nabla \left[\frac{\delta \mathcal{F}_{\mathrm{PKS}}[\rho(t)]}{\delta \rho}(x) \right] \right|^2 \mathrm{d}x \,.$$

Or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] = - \int_{\mathbb{R}^2} \rho(t,x) \left| \nabla \left(\log \rho(t,x) - c(t,x) \right) \right|^2 \mathrm{d}x \,.$$

In particular, along such solutions, $t \mapsto \mathcal{F}_{\mathrm{PKS}}[\rho(t)]$ is monotone non-increasing.

The gap between the $4/C_{GNS}$ and 8π was not filled before [95] when the link with the *logarithmic Hardy-Littlewood-Sobolev inequality* was made: Let f be a non-negative function in $L^1(\mathbb{R}^2)$ such that $f \log f$ and $f \log(1 + |x|^2)$ belong to $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f \, \mathrm{d}x = M$, then

$$\int_{\mathbb{R}^2} f \log f \, \mathrm{d}x + \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x - y| \, \mathrm{d}x \, \mathrm{d}y \geq - C(M) \,, \tag{1.1.8}$$

with $C(M) := M(1+\log\pi-\log M)$. Moreover the minimisers of the Logarithmic Hardy-Littlewood-Sobolev inequality (1.1.8) are the translations of

$$\bar{\varrho}_\lambda(x) := \frac{M}{\pi}\frac{\lambda}{(\lambda+|x|^2)^2}\,.$$

Using the monotonicity of $\mathcal{F}_{\mathrm{PKS}}[\rho]$ and the Logarithmic Hardy-Littlewood-Sobolev inequality (1.1.8) it is easy to see that

$$\begin{aligned}
&\mathcal{F}_{\mathrm{PKS}}[\rho] = \\
&\frac{M}{8\pi}\left(\int_{\mathbb{R}^2}\rho(x)\,\log\rho(x)\,\mathrm{d}x+\frac{2}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\log|x-y|\rho(y)\,\mathrm{d}x\,\mathrm{d}y\right)\\
&\quad+\left(1-\frac{M}{8\pi}\right)\int_{\mathbb{R}^2}\rho(x)\,\log\rho(x)\,\mathrm{d}x\\
&\geq -\frac{M}{8\pi}C(M)+\left(1-\frac{M}{8\pi}\right)\int_{\mathbb{R}^2}\rho(x)\,\log\rho(x)\,\mathrm{d}x\,. \qquad (1.1.9)
\end{aligned}$$

It follows that for solutions ρ of the Patlak-Keller-Segel system (1.1.1),

$$\int_{\mathbb{R}^2}\rho(t,x)\log\rho(t,x)\,\mathrm{d}x\leq\frac{8\pi\mathcal{F}_{\mathrm{PKS}}[\rho_0]-M\,C(M)}{8\pi-M}\,. \qquad (1.1.10)$$

Therefore, for $M<8\pi$, the entropy stays bounded uniformly in time. This precludes the collapse of mass into a point mass for such initial data.

Coming back to the super-critical mass case, it is worth noticing that for a given ρ, if we set $\rho_\lambda(x)=\lambda^{-2}\rho(\lambda^{-1}x)$ then

$$\mathcal{F}_{\mathrm{PKS}}[\rho_\lambda]=\mathcal{F}_{\mathrm{PKS}}[\rho]-2M\left(1-\frac{M}{8\pi}\right)\log\lambda\,. \qquad (1.1.11)$$

So that as a function of λ, $\mathcal{F}_{\mathrm{PKS}}[\rho_\lambda]$ is bounded from below if $M<8\pi$, and not bounded from below if $M>8\pi$.

As an alternative to the regularisation/passing to the limit procedure, another conceited but smart way to prove the global existence is to use the gradient flow interpretation in the Wasserstein metric. For this purpose we need to introduce a few elements of optimal transport for more details see the Annexe.

Using this metric, we can see the Patlak-Keller-Segel system (1.1.1) as a gradient flow of the free energy in the Wasserstein metric:

$$\rho_t=-\nabla_W\mathcal{F}_{\mathrm{PKS}}[\rho(t)]\,.$$

In the sense that we can construct a solution using the minimising scheme, often known as the minimising Jordan-Kinderlehrer-Otto (JKO) scheme: given a time step τ, we define the solution by

$$\rho_\tau^{k+1}\in\operatorname{argmin}_{\rho\in\mathcal{K}}\left[\frac{\mathcal{W}_2^2(\rho,\rho_\tau^k)}{2\tau}+\mathcal{F}_{\mathrm{PKS}}[\rho]\right]\,,$$

where

$$\mathcal{K}:=\{\rho\,:\,\int_{\mathbb{R}^2}\rho=M,\ \int_{\mathbb{R}^2}\rho(x)\log\rho(x)\,\mathrm{d}x<\infty\text{ and }\int_{\mathbb{R}^2}|x|^2\rho(x)\,\mathrm{d}x<\infty\}\,.$$

For the analogy, note that if the metric was Euclidean, the Euler-Lagrange equation associated to

$$\rho_\tau^{k+1} \in \operatorname{argmin}\left[\frac{|\rho-\rho_\tau^k|^2}{2\tau} + \mathcal{F}_{\mathrm{PKS}}[\rho]\right] , \tag{1.1.12}$$

would be

$$\frac{\rho_\tau^{k+1}-\rho_\tau^k}{\tau} + \mathcal{F}_{\mathrm{PKS}}[\rho_\tau^{k+1}] = 0 ,$$

which is nothing but the implicit Euler scheme associated to

$$\rho_t = -\nabla \mathcal{F}_{\mathrm{PKS}}[\rho(t)] .$$

At this point it is convenient to emphasise that the functional $\mathcal{F}_{\mathrm{PKS}}$ is not convex, so even the existence of a minimiser is not clear. When the functional is convex, or even displacement convex, general results from [204, 5] can be applied.

We interpolate between the terms of the sequence $\{\rho_\tau^k\}_{k\in\mathbb{N}}$ to produce a function from $[0,\infty)$ to $L^1(\mathbb{R}^2)$: For each positive integer k, let $\nabla\varphi^k$ be the optimal transportation plan with $\nabla\varphi^k \# \rho_\tau^k = \rho_\tau^{k-1}$. Then for $(k-1)\tau \le t \le k\tau$ we define

$$\rho_\tau(t) = \left(\frac{t-(k-1)\tau}{\tau}\mathrm{id} + \frac{k\tau - t}{\tau}\nabla\varphi^k\right)\#\rho_\tau^k .$$

Theorem 1.1.4 (Convergence of the scheme as $\tau \to 0$, [32]) *If $M < 8\pi$ then the family $(\rho_\tau)_{\tau>0}$ admits a sub-sequence converging weakly in $L^1(\mathbb{R}^2)$ to a weak solution to the Patlak-Keller-Segel system.*

In this proof the perturbation of the minimiser has to be done in the optimal transport way: Let ζ be a smooth vector field with compact support, we introduce $T_\varepsilon := \mathrm{id} + \varepsilon\zeta$. We define $\overline{\rho_\varepsilon}$ the push-forward perturbation of ρ_τ^{n+1} by T_ε:

$$\overline{\rho_\varepsilon} = T_\varepsilon \# \rho_\tau^{n+1} .$$

Let $\nabla\varphi^n$ be the unique transport map such that $\nabla\varphi^n \# \rho_\tau^{n+1} = \rho_\tau^n$. Standard computations give

$$\begin{aligned}&\int_{\mathbb{R}^2} \zeta(x)\frac{x-\nabla\varphi^n(x)}{\tau}\rho_\tau^{n+1}(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2}\left[\mathrm{div}\zeta(x) - \frac{1}{4\pi}\int_{\mathbb{R}^2}\frac{[\zeta(x)-\zeta(y)]\cdot(x-y)}{|x-y|^2}\rho_\tau^{n+1}(y)\,\mathrm{d}y\right]\rho_\tau^{n+1}(x)\,\mathrm{d}x ,\end{aligned}$$

which is the weak form of the Euler-Lagrange equation:

$$\frac{\mathrm{id}-\nabla\varphi^n}{\tau}\rho_\tau^{n+1} = -\nabla\rho_\tau^{n+1} + \rho_\tau^{n+1}\nabla c_\tau^{n+1} . \tag{1.1.13}$$

Using the Taylor's expansion $\zeta(x) - \zeta\left[\nabla\varphi^n(x)\right] = [x-\nabla\varphi^n(x)]\cdot\nabla\zeta(x) + O\left[|x-\nabla\varphi^n(x)|^2\right]$, we obtain for all $t_2 > t_1 \ge 0$,

$$\begin{aligned}\int_{\mathbb{R}^2}\zeta(x)\left[\rho_\tau(t_2,x)-\rho_\tau(t_1,x)\right]\,\mathrm{d}x &= \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\Delta\zeta(x)\,\rho_\tau(s,x)\,\mathrm{d}x\,\mathrm{d}s + O(\tau^{1/2}) \\ -\frac{1}{4\pi}\int_{t_1}^{t_2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho_\tau(s,x)\,\rho_\tau(s,y)\,&\frac{(x-y)\cdot(\nabla\zeta(x)-\nabla\zeta(y))}{|x-y|^2}\,\mathrm{d}y\,\mathrm{d}x .\end{aligned} \tag{1.1.14}$$

To pass to the limit, the scheme provides some *a priori* bounds: Taking ρ_τ^{n+1} as a test function in (1.1.12) we have:

$$\mathcal{F}_{\mathrm{PKS}}[\rho_\tau^{n+1}] + \frac{1}{2\tau}\mathcal{W}_2^2(\rho_\tau^n, \rho_\tau^{n+1}) \leq \mathcal{F}_{\mathrm{PKS}}[\rho_\tau^n]\,. \tag{1.1.15}$$

As a consequence we obtain an *energy estimate*

$$\sup_{n\in\mathbb{N}} \mathcal{F}_{\mathrm{PKS}}[\rho_\tau^n] \leq \mathcal{F}_{\mathrm{PKS}}[\rho_\tau^0],$$

which together with (1.1.9) forbids the concentration, and a *total square estimate*

$$\frac{1}{2\tau}\sum_{n\in\mathbb{N}} \mathcal{W}_2^2(\rho_\tau^n, \rho_\tau^{n+1}) \leq \mathcal{F}_{\mathrm{PKS}}[\rho_\tau^0] - \inf_{n\in\mathbb{N}} \mathcal{F}_{\mathrm{PKS}}[\rho_\tau^n]\,,$$

which rules out vanishing. These two estimates allow to pass to the limit in τ in (1.1.14), to obtain:

$$\int_{\mathbb{R}^2} \zeta(x)\,[\rho(t_2,x) - \rho(t_1,x)]\;\mathrm{d}x = \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \Delta\zeta(x)\,\rho(s,x)\;\mathrm{d}x\,\mathrm{d}s$$
$$- \frac{1}{4\pi}\int_{t_1}^{t_2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho_\tau(s,x)\,\rho(s,y)\,\frac{(x-y)\cdot(\nabla\zeta(x)-\nabla\zeta(y))}{|x-y|^2}\;\mathrm{d}y\,\mathrm{d}x\,.$$

Which is the definition of a weak solution. Note that the last term of (1.1.14) converges because $\rho_\tau(s)$ converges weakly in $\mathrm{L}^1(\mathbb{R}^2)$ and the other term is bounded in $\mathrm{L}^\infty(\mathbb{R}^2)$.

By proving the hyper-contractivity of the Patlak-Keller-Segel system (1.1.1), we can actually obtain:

Theorem 1.1.5 (Existence of solution is the subcritical case, [39]) *If $M < 8\pi$, then the Patlak-Keller-Segel system (1.1.1) has a global weak non-negative solution ρ with initial data ρ_0 such that*

$$(1+|x|^2+|\log\rho|)\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+, L^1(\mathbb{R}^2))\,,$$
$$\int_0^t\int_{\mathbb{R}^2} \rho|\nabla\log\rho - \nabla c|^2\;\mathrm{d}x\,\mathrm{d}t < \infty\,,$$
$$\int_{\mathbb{R}^2}|x|^2\rho(t,x)\;\mathrm{d}x = \int_{\mathbb{R}^2}|x|^2\rho_0(x)\;\mathrm{d}x + 4M\big(1-\frac{M}{8\pi}\big)t$$

for $t>0$. Moreover $\rho \in L^\infty_{\mathrm{loc}}((\varepsilon,\infty), L^p(\mathbb{R}^2))$ for any $p\in(1,\infty)$ and any $\varepsilon>0$, and the following inequality holds for any $t>0$:

$$\mathcal{F}_{\mathrm{PKS}}[\rho(\cdot,t)] + \int_0^t\int_{\mathbb{R}^2} \rho\,|\nabla(\log\rho - c)|^2\;\mathrm{d}x\,\mathrm{d}s \leq \mathcal{F}_{\mathrm{PKS}}[\rho_0]\,.$$

Similar results were first proved in [162] for radially symmetric solutions in a bounded domain with Neumann boundary conditions.

This notion of free energy solution allows to study the large time behaviour, intermediate asymptotics and convergence to asymptotically self-similar profiles: let (u_∞, v_∞) be the unique solution to the Gelfand equation

$$u_\infty = M\frac{e^{v_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{v_\infty-|x|^2/2}\,\mathrm{d}x} = -\Delta v_\infty\,, \quad \text{with} \quad v_\infty = G * u_\infty\,. \tag{1.1.16}$$

Using the comparison principle in the radial variable, it has been proven in [28] that the radial, non-negative smooth solution to this problem is unique. In the original variables, the self-similar solutions of (1.1.1) take the expression:

$$\rho_\infty(t,x) := \frac{1}{1+2t} u_\infty\left(\log(\sqrt{1+2t}), x/\sqrt{1+2t}\right), \tag{1.1.17}$$

$$c_\infty(t,x) := v_\infty\left(\log(\sqrt{1+2t}), x/\sqrt{1+2t}\right). \tag{1.1.18}$$

Theorem 1.1.6 (Large time behaviour, [39]) *Under the assumptions in Theorem 1.1.5,*

$$\lim_{t\to\infty} \|\rho(\cdot,t) - \rho_\infty(\cdot,t)\|_{L^1(\mathbb{R}^2)} = 0 \quad \textit{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot,t) - \nabla c_\infty(\cdot,t)\|_{L^2(\mathbb{R}^2)} = 0\,.$$

The proof follows the usual entropy/entropy production method in self-similar variables: We define the rescaled functions u and v by

$$\rho(t,x) = \frac{1}{R^2(t)} u\left(\frac{x}{R(t)}, \tau(t)\right) \quad \text{and} \quad c(t,x) = v\left(\frac{x}{R(t)}, \tau(t)\right)$$

with $R(t) = \sqrt{1+2t}$ and $\tau(t) = \log R(t)$. The rescaled system is

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla\cdot(u(x+\nabla v)) & x\in\mathbb{R}^2,\ t>0, \\ v = G * u & x\in\mathbb{R}^2,\ t>0, \\ u(\cdot, t=0) = \rho_0 & x\in\mathbb{R}^2, \end{cases} \tag{1.1.19}$$

and the associated free energy takes the form

$$\mathcal{F}^R_{\mathrm{PKS}}[u] := \int_{\mathbb{R}^2} u\log u\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^2} uv\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 u\,\mathrm{d}x\,. \tag{1.1.20}$$

If (u,v) is a smooth solution of the rescaled Patlak-Keller-Segel system (1.1.19) which decays sufficiently at infinity, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}^R_{\mathrm{PKS}}[u(t,\cdot)] = -\int_{\mathbb{R}^2} u\left|\nabla\left(\log u - v + \frac{|x|^2}{2}\right)\right|^2 \mathrm{d}x\,.$$

If we keep in mind the gradient flow interpretation (which is true also for the rescaled equation), we can imagine that the limit when t goes to infinity of a solution to the rescaled Patlak-Keller-Segel system (1.1.19) cancels the free energy dissipation (1.1.20). So that the limit solution satisfies the Gelfand equation (1.1.16).

The question of the speed of convergence has been very recently understood

Theorem 1.1.7 (Rate of convergence, [60]) *Assume that $n_0 \in \mathrm{L}^2_+(n_\infty^{-1}\,\mathrm{d}x)$, $M < 8\pi$ and there exists $\varepsilon \in (0, 8\pi - M)$ such that*

$$\int_0^s n_{0,*}(\sigma)\,\mathrm{d}\sigma \le \int_{B(0,\sqrt{s/\pi})} n_{\infty,M+\varepsilon}(x)\,\mathrm{d}x \quad \forall s\ge 0$$

then any solution to the Patlak-Keller-Segel system (1.1.1) with initial datum n_0 is such that

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 n_\infty^{-1}\,\mathrm{d}x \le C\mathrm{e}^{-2t} \quad \forall t\ge 0$$

for some positive constant C.

The proof relies on the linearisation of the problem around n_∞ which was already performed in [37]: Consider indeed f and g defined for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^2$ by

$$n(t,x) = n_\infty(x)(1+f(t,x)) \quad \text{and} \quad c(t,x) = c_\infty(x)(1+g(t,x)) \,.$$

Then (f,g) is a solution to the nonlinear problem

$$\begin{cases} f_t - \mathcal{L}f = -\dfrac{1}{n_\infty}\nabla\cdot[fn_\infty\nabla(gc_\infty)] \\ -\Delta(gc_\infty) = fn_\infty \end{cases}$$

where the linearised operator is

$$\mathcal{L}f = \frac{1}{n_\infty}\nabla\cdot[n_\infty\nabla(f-gc_\infty)]$$

Such an analysis allowed to prove convergence with rate for solution with small initial data, see [37]: there are two positive constants, C and δ, such that

$$\int_{\mathbb{R}^2} |u(t,x)-u_\infty(x)|^2 \frac{\mathrm{d}x}{u_\infty(x)} \le C\,e^{-\delta t} \quad \forall\, t>0\,.$$

As a function of M, δ is such that $\lim_{M\to 0_+}\delta(M) = 1$.

Using a decreasing rearrangement method due to J.I. Diaz, T. Nagai, and J.-M. Rakotoson, [60] improves this result by proving that such a result is valid in any L^p for n and any L^q for v where $p \in [1,\infty]$ and $q \in [2,\infty]$.

The authors of [60] then perform a precise study of the spectral gap of the linearised operator in an appropriate functional setting to prove the stated result.

1.1.4 Critical case

In the case $M = 8\pi$, the free energy $\mathcal{F}_{\mathrm{PKS}}$ is the same as the functional which appears in the logarithmic Hardy-Littlewood-Sobolev inequality (1.1.8). The remainder entropy which was controlled in (1.1.9) is thus entirely "eaten" by the logarithmic Hardy-Littlewood-Sobolev inequality (1.1.8). In [36], we use a three-steps procedure:

How would it blowup the space is split into balls and annulus. Using the diffusion it is possible to prove that in a ball the mass is less than 8π and to control the influence of the interactions outside the ball. So that only when all the mass is concentrated in a point, we cannot extend the solution to a bigger interval. If the solution blows up then it blows up as a Dirac mass concentrated in the centre of mass.

When would it blowup in this case $M = 8\pi$, by the virial computation (1.1.2), the 2-moment remains constant. A De la Vallée-Poussin's type argument, shows that the concentration cannot occur in finite time. If the solution blows up then it blows up as a Dirac peak concentrated in the centre of mass at infinite time.

Does it blowup Keeping in mind the gradient flow structure described in the previous section, we can imagine that the solutions converge to

the minimisers of the logarithmic Hardy-Littlewood-Sobolev inequality (1.1.8). But as the second-moment is constant thanks to the virial computation (1.1.2), the solutions converge to the only minimiser $\bar{\varrho}_\lambda$ of the logarithmic Hardy-Littlewood-Sobolev inequality (1.1.8) which is of finite moment: the Dirac mass.

As a consequence, we prove

Theorem 1.1.8 (Infinite Time Aggregation, [36]) *If the 2-moment is bounded, there is a global in time non-negative free-energy solution of the Patlak-Keller-Segel system (1.1.1) with initial data ρ_0. Moreover if $\{t_p\}_{p\in\mathbb{N}} \to \infty$ as $p \to \infty$, then $t_p \mapsto \rho(t_p, x)$ converges to a Dirac peak of mass 8π concentrated at the centre of mass of the initial data weakly-* in the sense of measure as $p \to \infty$.*

In the radial and bounded case with a Dirichlet boundary conditions, the blowup rate and refined asymptotics estimates are given in the following

Theorem 1.1.9 (Blowup profile, [126]) *In the radial case, in the ball, consider a solution to the Patlak-Keller-Segel system (1.1.1) with $\partial\rho/\partial\nu = \rho\partial c/\partial\nu$ and $c = 0$ on the boundary. Then when t goes to ∞,*

$$\rho(t,0) = 8e^{5/2+2\sqrt{2}t}\left(1 + O\left(t^{-1/2}\log(4t)\right)\right) .$$

In [186], C. Sire and P.-H. Chavanis predicted this result by a formal argument considering only the first order correction terms. In [126], N. Kavallaris and P. Souplet study the Patlak-Keller-Segel system (1.1.1). They make successive appropriate change of variables and of functions to reduce the system in radial coordinates to a degenerate parabolic problem. The precise results are difficult to translate back to the original Patlak-Keller-Segel system (1.1.1). In particular, they prove that the solution is the sum of a quasi-stationary profile and of a correction term which is significant only for x bounded away from 0.

The extension of Theorem 1.1.8 to the case when the second moment is not finite allows the solution to converge to the other minimisers of the logarithmic Hardy-Littlewood-Sobolev inequality (1.1.8). For this purpose we need to introduce another free energy functional which still has to be fully understood. Let us first recall the Fokker-Planck version of the fast diffusion equation corresponding to the fast diffusion equation

$$\frac{\partial u}{\partial t} = \Delta\sqrt{u}$$

by a self-similar change of variable:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta\sqrt{u(t,x)} + 2\sqrt{\dfrac{\pi}{\lambda M}}\,\mathrm{div}(x\,u(t,x)) & t > 0\,,\ x \in \mathbb{R}^2\,, \\ u(0,x) = u_0(x) \geq 0 & x \in \mathbb{R}^2\,. \end{cases} \tag{1.1.21}$$

This equation can also be written in a form analogous to (1.1.7): following [141] for $\lambda > 0$, define the relative entropy of the fast diffusion equation

with respect to the stationary solution $\bar{\varrho}_\lambda$ by

$$\mathcal{H}_\lambda[u] := \int_{\mathbb{R}^2} \frac{\left|\sqrt{u(x)} - \sqrt{\bar{\varrho}_\lambda(x)}\right|^2}{\sqrt{\bar{\varrho}_\lambda(x)}}\,\mathrm{d}x\,.$$

Equation (1.1.21) can be rewritten as

$$\frac{\partial u}{\partial t}(t,x) = \operatorname{div}\left(u(t,x)\nabla\frac{\delta\mathcal{H}_\lambda[u(t)]}{\delta u}(x)\right)\,,$$

with

$$\frac{\delta\mathcal{H}_\lambda[u]}{\delta u} = \frac{1}{\sqrt{\bar{\varrho}_\lambda}} - \frac{1}{\sqrt{u}}\,.$$

The connection with the Patlak-Keller-Segel system (1.1.1) can be seen through the minimisers of $\mathcal{H}_\lambda$ which are the same as those of the logarithmic Hardy-Littlewood-Sobolev inequality (1.1.8). The functional $\mathcal{H}_\lambda$ is a weighted distance between the solution and its unique minimiser $\bar{\varrho}_\lambda$. It is tempting to compute the dissipation of $\mathcal{H}_\lambda$ along the flow of solutions to the Patlak-Keller-Segel system (1.1.1): Let ρ be a sufficiently smooth solution of the Patlak-Keller-Segel system (1.1.1). Then we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[\rho(t)] = -\frac{1}{2}\int_{\mathbb{R}^2}\frac{|\nabla\rho(t)|^2}{\rho(t)^{3/2}}\,\mathrm{d}x + \int_{\mathbb{R}^2}\rho(t)^{3/2}\,\mathrm{d}x + 4\sqrt{\frac{M\,\pi}{\lambda}}\left(1-\frac{M}{8\pi}\right)\,. \tag{1.1.22}$$

In the critical case $M = 8\pi$ the dissipation of the $\mathcal{H}_\lambda$ free energy along the flow of the Patlak-Keller-Segel system (1.1.1) is

$$\mathcal{D}[\rho] := \frac{1}{2}\int_{\mathbb{R}^2}\frac{|\nabla\rho|^2}{\rho^{3/2}}\,\mathrm{d}x - \int_{\mathbb{R}^2}\rho^{3/2}\,\mathrm{d}x\,.$$

We use the following Gagliardo-Nirenberg-Sobolev inequality due to J. Dolbeault and M. Del Pino, see [87]: For all functions f in $\mathbb{R}^2$ with a square integrable distributional gradient ∇f,

$$\pi\int_{\mathbb{R}^2}|f|^6\,\mathrm{d}x \le \int_{\mathbb{R}^2}|\nabla f|^2\,\mathrm{d}x\int_{\mathbb{R}^2}|f|^4\,\mathrm{d}x\,,$$

and there is equality if and only if f is a multiple of a translate of $\bar{\varrho}_\lambda^{1/4}$ for some $\lambda > 0$.

As a consequence, taking $f = \rho^{1/4}$ so that $\int_{\mathbb{R}^2} f^4(x)\,\mathrm{d}x = 8\pi$, we obtain $\mathcal{D}[\rho] \ge 0$, and moreover, $\mathcal{D}[\rho] = 0$ if and only ρ is a translate of $\bar{\varrho}_\lambda$ for some $\lambda > 0$.

Remark 1.1.10. This free energy $\mathcal{H}_\lambda[\rho]$ gives another proof of non existence of global-in-time solutions in the super-critical case $M > 8\pi$. Indeed, by (1.1.22) and as $\mathcal{D}[\rho]$ is non-negative,

$$0 \le \mathcal{H}_\lambda[\rho(t)] \le 4\sqrt{\frac{M\,\pi}{\lambda}}\left(1-\frac{M}{8\pi}\right)t\,.$$

So that in the case $M > 8\pi$, there cannot be global-in-time solutions even with infinite 2-moment as long as $\int\sqrt{\rho_0}$ is bounded.

Based on this free energy $\mathcal{H}_\lambda[\rho]$, the main results of [33] in the critical case can be summarised in the following:

Theorem 1.1.11 (Existence of global solutions, [33]) *Let ρ_0 be any density in $\mathbb{R}^2$ with mass 8π, such that $\mathcal{F}_{\mathrm{PKS}}[\rho_0] < \infty$, and for some $\lambda > 0$, $\mathcal{H}_\lambda[\rho_0] < \infty$. Then there exists a global free energy solution of the Patlak-Keller-Segel equation (1.2.1) with initial data ρ_0. Moreover,*

$$\lim_{t\to\infty} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] = \mathcal{F}_{\mathrm{PKS}}[\bar\varrho_\lambda] \qquad \text{and} \qquad \lim_{t\to\infty} \|\rho(t) - \bar\varrho_\lambda\|_1 = 0\,.$$

We even prove further regularity using the propagation of the L^p-estimates and the hypercontractivity property of the equation. This theorem can be translated in terms of the Wasserstein distance of the initial data to $\bar\varrho_\lambda$ thanks to the Talagrand inequality:

$$\mathcal{W}_2(\rho, \bar\varrho_\lambda) \le \sqrt{\frac{2\mathcal{H}_\lambda[\rho]}{2\sqrt{\frac{\pi}{M\lambda}}}}.$$

Remember that the minimisers $\bar\varrho_\lambda$ of the logarithmic Hardy-Littlewood-Sobolev inequality (1.1.8) are of infinite 2-moment so that the condition $\mathcal{H}_\lambda[\rho_0] < \infty$ implies that ρ_0 is of infinite 2-moment. If we keep in mind that the 2-moment can be seen as the Wasserstein distance between the solution and the Dirac mass, we see that Theorem 1.1.11 completes the picture which emerged from Theorem 1.1.8 which states the convergence to the Dirac mass if the solution is initially at finite Wasserstein distance of the Dirac mass. As soon as we start at a finite distance from one of the minimisers $\bar\varrho_\lambda$ we can construct a solution which converges towards it. Note that this result is true for the solution that we construct as we do not have uniqueness of the solution to the Patlak-Keller-Segel system, even if we strongly believe that this is the case. Also observe that the equilibrium solutions $\bar\varrho_\lambda$ are infinitely far apart: let $\varphi(x) = \sqrt{\lambda/\mu}|x|^2/2$, one has $\nabla\varphi\#\varrho_\mu = \bar\varrho_\lambda$. Thus,

$$\mathcal{W}_2^2(\varrho_\mu, \bar\varrho_\lambda) = \frac{1}{2}\int_{\mathbb{R}^2} \left|\sqrt{\frac{\lambda}{\mu}}x - x\right|^2 \varrho_\mu(x)\,\mathrm{d}x = +\infty$$

since the equilibrium densities $\bar\varrho_\lambda$ all have infinite second moments. In particular, $\mathcal{H}_\lambda[\varrho_\mu] = +\infty$ for $\mu \neq \lambda$. There may still be initial data out of these basins of attraction.

Concerning the proof of Theorem 1.1.11, we expect the propagation of the bounds on $\mathcal{F}_{\mathrm{PKS}}[\rho]$ and $\mathcal{D}[\rho]$ to give compactness. Unfortunately, $\mathcal{D}[\rho]$ is a difference of two functionals of ρ that can each be arbitrarily large even when $\mathcal{D}[\rho]$ is very close to zero. Indeed, for $M = 8\pi$ and each $\lambda > 0$, $\mathcal{D}[\bar\varrho_\lambda] = 0$ while

$$\lim_{\lambda\to 0} \|\bar\varrho_\lambda\|_{3/2} = \infty\,, \quad \lim_{\lambda\to 0} \|\nabla\bar\varrho_\lambda^{1/4}\|_2 = \infty \quad \text{and} \quad \lim_{\lambda\to 0} \bar\varrho_\lambda = 8\pi\delta_0\,.$$

Likewise, an upper bound on $\mathcal{F}_{\mathrm{PKS}}[\rho]$ provides no upper bound on the entropy $\int_{\mathbb{R}^2}\rho\log\rho$. Indeed, $\mathcal{F}_{\mathrm{PKS}}[\rho]$ takes its minimum value for $\rho = \bar\varrho_\lambda$ for each $\lambda > 0$, while

$$\lim_{\lambda\to 0} \int \bar\varrho_\lambda \log\bar\varrho_\lambda = \infty\,.$$

Fortunately, an upper bound on both $\mathcal{H}_\lambda[\rho]$ and $\mathcal{F}_{\rm PKS}[\rho]$ *does* provide an upper bound on $\int \rho \log \rho$:

Theorem 1.1.12 (Concentration control for $\mathcal{F}_{\rm PKS}$, [33]) *Let ρ be any density with mass $M = 8\pi$ such that $\mathcal{H}_\lambda[\rho] < \infty$ for some $\lambda > 0$. Then there exist $\gamma_1 > 0$ and an explicit $C > 0$ depending only on λ and $\mathcal{H}_\lambda[\rho]$ such that*

$$\gamma_1 \int_{\mathbb{R}^2} \rho \log \rho \, dx \leq \mathcal{F}_{\rm PKS}[\rho] + C \, .$$

Here we also prove that since $\mathcal{H}_\lambda$ controls concentration, a uniform bound on both $\mathcal{H}_\lambda$ and $\mathcal{D}$ does indeed provide compactness:

Theorem 1.1.13 (Concentration control for $\mathcal{D}$, [33]) *Let ρ be any density in $L^{3/2}(\mathbb{R}^2)$ with mass 8π such that $\mathcal{F}_{\rm PKS}[\rho]$ is finite, and $\mathcal{H}_\lambda[\rho]$ is finite for some $\lambda > 0$. Then there exist constants $\gamma_1 > 0$ and an explicit $C > 0$ depending only on λ, $\mathcal{H}_\lambda[\rho]$ and $\mathcal{F}_{\rm PKS}[\rho]$ such that*

$$\gamma_2 \int_{\mathbb{R}^2} |\nabla \rho^{1/4}|^2 \, dx \leq \pi \mathcal{D}[\rho] + C \ .$$

The proofs of this two theorems lead on the following lemma:

$$\int_{\mathbb{R}^2} \sqrt{\lambda + |x|^2} \rho(x) \, dx \leq 2\sqrt{\lambda} \, M + 2M^{3/4} (\lambda/\pi)^{1/4} \sqrt{\mathcal{H}_\lambda[\rho]} \, . \quad (1.1.23)$$

As explained at the beginning of the section, in [36] we managed to find a ball in which the mass was smaller than 8π. Here, (1.1.23) gives a vertical cut to prove Theorem 1.1.12. Indeed, we split the function ρ in two parts: given $\beta > 0$, define $\rho_\beta(x) = \min\{\rho(x) \ , \ \beta\}$. By (1.1.23), for β large enough, $\rho - \rho_\beta$ is such that:

$$\int_{\mathbb{R}^2} (\rho - \rho_\beta) \leq \frac{C_1}{\beta} + C_2 \sqrt{\mathcal{H}_\lambda[\rho]} \leq \frac{C_1}{\beta} + \frac{8\pi - \varepsilon_0}{2} < 8\,\pi - \varepsilon_0 \, .$$

We then apply the logarithmic Hardy-Littlewood-Sobolev inequality method as in (1.1.10) to the function $\rho - \rho_\beta$ whose mass is less than 8π.

The same idea works for the Gagliardo-Nirenberg-Sobolev inequality to prove Theorem 1.1.13: Let $f := \rho^{1/4}$, we split f in two parts by defining $f_\beta := \min\{f, \beta^{1/4}\}$ and $h_\beta := f - f_\beta$. We use (1.1.23) and apply the Gagliardo-Nirenberg-Sobolev inequality to control h_β.

Idea of the proof of Theorem 1.1.11: It follows the line of the convergence of the JKO minimising scheme (1.1.12) exposed in the previous section to obtain the Euler-Lagrange equation (1.1.13). Dividing the Euler-Lagrange equation (1.1.13) by $\sqrt{\rho_\tau^{n+1}}$ we obtain:

$$2\nabla \sqrt{\rho_\tau^{n+1}} = \left(\nabla c_\tau^{n+1} - \frac{x - \nabla \varphi_\tau^n}{\tau} \right) \sqrt{\rho_\tau^{n+1}} \, , \quad (1.1.24)$$

where $\nabla \varphi_\tau^n \# \rho_\tau^{n+1} = \rho_\tau^n$. Integrating (1.1.24) we obtain

$$\int_{\mathbb{R}^2} \left| \sqrt{\rho_\tau^{n+1}} \nabla c_\tau^{n+1} - 2\nabla \sqrt{\rho_\tau^{n+1}} \right|^2 \, dx = \int_{\mathbb{R}^2} \left| \frac{x - \nabla \varphi_\tau^n}{\tau} \right|^2 \rho_\tau^{n+1} \, dx$$
$$= \mathcal{W}_2(\rho_\tau^n, \rho_\tau^{n+1})$$

which is bounded thanks to (1.1.15). But the left hand side is a sum of two terms so that we cannot conclude any compactness on each of them.

A powerful technique is developed in [152]: the idea is to perturb the minimiser by moving it along the gradient flow $S^{\mathcal{V}}$ generated by another functional $\mathcal{V}$ which is displacement convex. To present the idea let us first consider the two ordinary differential equations describing gradient flow:

$$\dot{x}(t) = -\nabla\Phi[x(t)] \quad \text{and} \quad \dot{y}(t) = -\nabla\Psi[y(t)]$$

Then of course $\Phi[x(t)]$ and $\Psi[y(t)]$ are monotone decreasing. Differentiate each function along the other's flow:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Phi[y(t)] &= -\langle\nabla\Phi[y(t)], \nabla\Psi[y(t)]\rangle \\ \frac{\mathrm{d}}{\mathrm{d}t}\Psi[x(t)] &= -\langle\nabla\Psi[x(t)], \nabla\Phi[x(t)]\rangle \end{aligned}$$

Thus, Φ is decreasing along the gradient flow of Ψ for any initial data if and only if Ψ is decreasing along the gradient flow of Φ for any initial data.

Let us consider the following variational problem:

$$\text{Find } u_{h,n} \text{ which minimises} \quad u \mapsto \frac{1}{2h}\mathcal{W}_2^2(u, u_{h,n-1}) + \mathcal{G}[u].$$

Imagine now that we can find a displacement convex functional $\mathcal{V}$ such that the dissipation of $\mathcal{G}$ along the flow $S^{\mathcal{V}}$:

$$D^{\mathcal{V}}\mathcal{G}[\mu] := \limsup_{t\to 0} \frac{\mathcal{G}[\mu] - \mathcal{G}[S_t^{\mathcal{V}}\mu]}{t}.$$

is non-negative. By the previous remark

$$D^{\mathcal{V}}\mathcal{G}[u_{h,n}] = \limsup_{t\to 0} \frac{\mathcal{V}[u_{h,n-1}] - \mathcal{V}[u_{h,n}]}{t}$$

And as $\mathcal{V}$ is displacement convex, the above the tangent formulation gives:

$$D^{\mathcal{V}}\mathcal{G}[u_{h,n}] \le \frac{\mathcal{V}[u_{h,n-1}] - \mathcal{V}[u_{h,n}]}{h}.$$

So that the differential estimate of $\mathcal{G}$ is converted into a discrete estimate for the approximation scheme.

Indeed, by definition of the minimising scheme, for any $u \in \mathcal{K}$

$$\frac{1}{2h}\mathcal{W}_2^2(u_{h,n}, u_{h,n-1}) + \mathcal{G}[u_{h,n}] \le \frac{1}{2h}\mathcal{W}_2^2(u, u_{h,n-1}) + \mathcal{G}[u] \tag{1.1.25}$$

Choosing $u = S_t^{\mathcal{V}}(u_{h,n})$ in (1.1.25) we obtain

$$\mathcal{G}[u_{h,n}] - \mathcal{G}[S_t^{\mathcal{V}}u_{h,n}] \le \frac{1}{2h}\left(\mathcal{W}_2^2(S_t^{\mathcal{V}}u_{h,n}, u_{h,n-1}) - \mathcal{W}_2^2(u_{h,n}, u_{h,n-1})\right)$$

Dividing by t and letting $t \to 0$, (1.1.26) with $u = u_{h,n}$ and $v = u_{h,n-1}$ yields

$$D^{\mathcal{V}}\mathcal{G}[u_{h,n}] \le \frac{\mathcal{V}[u_{h,n-1}] - \mathcal{V}[u_{h,n}]}{h}$$

Because $\mathcal{V}$ is displacement convex and $S^{\mathcal{V}}$ is the associated semi-group means

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}\mathcal{W}_2^2(S_t^{\mathcal{V}}u, v) \leq \mathcal{V}[v] - \mathcal{V}[S_t^{\mathcal{V}}u] \tag{1.1.26}$$

And as $\mathcal{V}$ is displacement convex: So that the differential estimate of $\mathcal{G}$ is converted into a discrete estimate for the approximation scheme.

Here, as already discussed the functional $\mathcal{F}_{\mathrm{PKS}}$ is not displacement convex but the flow constructed from this functional is also non-increasing along the flow of $\mathcal{H}_\lambda$. Moreover, the displacement convexity of $\mathcal{H}_\lambda$ is formally obvious from the fact that

$$\mathcal{H}_\lambda[u] = \int_{\mathbb{R}^2}\left(-2\sqrt{u(x)} + \sqrt{\frac{1}{2\lambda}}\frac{|x|^2}{2}u(x)\right)\,\mathrm{d}x + C\,.$$

where $-\sqrt{u(x)}$ and $|x|^2u(x)$ are displacement convex. So that at each step, we can use the convexity estimate of the type

$$\mathcal{H}_\lambda[\rho_0] - \mathcal{H}_\lambda[\rho_1] \geq \limsup_{t\to 0}\frac{\mathcal{H}_\lambda[\rho_t] - \mathcal{H}_\lambda[\rho_0]}{t}\,,$$

which gives in this optimal transport framework the *above the tangent formulation*:

$$\mathcal{H}_\lambda[\rho_\tau^{n+1}] - \mathcal{H}_\lambda[\rho_\tau^n] \geq \frac{1}{2}\int_{\mathbb{R}^2}\left[\sqrt{\frac{1}{2\lambda}}\,x + \frac{\nabla\rho_\tau^n}{(\rho_\tau^n)^{3/2}}\right]\cdot(\nabla\varphi(x) - x)\,\rho_\tau^n\,\mathrm{d}x\,.$$

where $\nabla\varphi$ is such that $\nabla\varphi\#\rho_\tau^{n+1} = \rho_\tau^n$. Inferring the Euler-Lagrange equation (1.1.13): $-\nabla\rho_\tau^{n+1} + \rho_\tau^{n+1}\nabla c_\tau^{n+1} = (\mathrm{id} - \nabla\varphi)\,\rho_\tau^{n+1}/\tau$, we obtain a discrete version of the entropy/entropy dissipation inequality

$$\mathcal{H}_\lambda[\rho_\tau^{n+1}] - \mathcal{H}_\lambda[\rho_\tau^n] \leq -\tau\,\mathcal{D}[\rho_\tau^n]\,. \tag{1.1.27}$$

This inequality is a skeleton version of the crucial estimate which allows to apply the standard entropy/entropy dissipation method to study the asymptotics. There are main technical difficulties and the methods to turn around them are interesting by themselves but we do not present them in details here.

Very recently, in [62], E. Carlen and A. Figalli use a argument of Bianchi-Egnell's type to obtain a quantitative stability for the logarithmic Hardy-Littlewood-Sobolev inequality (1.1.8) and prove:

$$\|\rho(t) - \bar{\varrho}_\lambda\|_{L^1(\mathbb{R}^2)} \leq \frac{C}{\sqrt{\log(e+t)}}\,.$$

Before closing this section let us mention that in the original system as studied by Nagai, the Poisson equation is replaced by a Bessel equation:

$$\begin{cases} \dfrac{\partial\rho}{\partial t}(t,x) &= \mathrm{div}\,[\nabla\rho(t,x) - \rho(t,x)\nabla\phi(t,x)] \qquad t>0\,,\ x\in\mathbb{R}^2\,,\\ -\Delta\phi(t,x) + \phi(t,x) &= \rho(t,x)\,, \qquad\qquad\qquad\qquad\qquad\qquad\quad t>0\,,\ x\in\mathbb{R}^2\,, \end{cases}$$

Open question: *For such a system, the kernel is weaker than the Poisson kernel and the existence results should still be valid. The blowup solution with super-critical mass which are initially concentrated enough comes from a argument similar to the one presented in the next section. The main open question is the behaviour of the solution at infinite time for a solution of critical mass. Indeed, in such a case there is no stationary solution and it is not clear to determine if the solution spreads or concentrates. The existence of solution for super-critical mass cannot be excluded and the method of [26] could probably be adapted here.*

1.2 The non-linear parabolic-elliptic Keller-Segel system

1.2.1 The model

In higher dimensions the critical quantity is no longer the mass but the $\mathcal{L}^{d/2}$-norm, see [81]. We can however replace the linear diffusion with a homogeneous non-linear diffusion:

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(t,x) &= \operatorname{div}\left[\nabla \rho^m(t,x) - \rho(t,x)\nabla\phi(t,x)\right] \qquad t>0\,,\ x\in\mathbb{R}^d\,,\\ -\Delta\phi(t,x) &= \rho(t,x)\,, \qquad\qquad\qquad\qquad\qquad\qquad\quad t>0\,,\ x\in\mathbb{R}^d\,, \end{cases} \tag{1.2.1}$$

where $m\in(0,1)$ and $d\geq 3$. In physics, this system models the motion of the mean field of many self-gravitating Brownian particles. This system is then known as the generalised Smulochowski-Poisson system, see [75, 74].

Define

$$\mathcal{K}(x) = c_d\,\frac{1}{|x|^{d-2}} \quad \text{and} \quad c_d := \frac{1}{(d-2)\sigma_d}$$

where $\sigma_d := 2\,\pi^{d/2}/\Gamma(d/2)$ is the surface area of the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. Up to a harmonic function $\phi = \mathcal{K}*\rho$, so that the system (1.2.1) can be rewritten as a non-local parabolic equation, for all $t>0$ and $x\in\mathbb{R}^d$:

$$\frac{\partial \rho}{\partial t}(t,x) = \operatorname{div}\left[\nabla\rho^m(t,x) - \rho(t,x)\nabla(\mathcal{K}*\rho)(t,x)\right]\,. \tag{1.2.2}$$

In this case too the mass is preserved and will be denoted M.

Let $\rho_\lambda(x) := \lambda^d\rho(\lambda\,x)$ with $\lambda>0$, the diffusion term scales like $\lambda^{d\,m+2}\Delta(\rho_\lambda^m)(\lambda\,x)$ whereas the interaction term scales like $\lambda^{2\,d}\operatorname{div}\left(\rho_\lambda\nabla(\mathcal{K}*\rho_\lambda)\right)(\lambda\,x)$. Hence the mass-invariant scaling of the diffusion term balances the potential drift in (1.2.2) if

$$m = m_d =: 2\left(1-\frac{1}{d}\right)\in(1,2)\,. \tag{1.2.3}$$

This difference of balance was studied to obtain

Theorem 1.2.1 (First criticality, [189, 190]) *Let m_d be as defined in (1.2.3).*

- *if $m>m_d$ then the solutions to (1.2.1) exist globally in time,*
- *if $m<m_d$ then solutions to (1.2.1) with sufficiently large initial data blowup in finite time,*

- *if $m = m_d$ there exist two constants $M_1 > 0$ and $M_2 > M_1$ such that*
  - *if $M < M_1$ then the solutions to (1.2.1) exist globally in time,*
  - *if $M > M_2$ there exist initial conditions such that the corresponding solution blows up in finite time.*

In dimension 2, these results were obtained by [134, 57]. We will concentrate on the critical diffusion case $m = m_d$. When no confusion is possible, the index d in m_d will be omitted and the critical exponent will be denoted m in the sequel of this article. The proof of the last point of Theorem 1.2.1 relies on the Gagliardo-Nirenberg-Sobolev inequality and is not sharp as we will see in Theorem 1.2.2 below.

The analogous of the free energy used in the previous section is:

$$\mathcal{G}[\rho(t)] := \int_{\mathbb{R}^d} \frac{\rho^m(t,x)}{m-1} - \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{K}(x-y)\, \rho(t,x)\, \rho(t,y)\, \mathrm{d}x\, \mathrm{d}y$$

which is related to its time derivative along the flow of (1.2.2) by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}[\rho(t)] = -\int_{\mathbb{R}^d} \rho(t,x) \left| \nabla \left(\frac{m}{m-1} \rho^{m-1}(t,x) - \phi(t,x) \right) \right|^2 \mathrm{d}x\,.$$

1.2.2 The sub-critical case

In [35], the functional inequality used is a *variant to the Hardy-Littlewood-Sobolev (VHLS) inequality*: for all $h \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$, there exists an optimal constant C_* such that

$$C_* = \sup_{h \neq 0} \left\{ \|h\|_m^{-m}\, \|h\|_1^{-2/d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{h(x)\, h(y)}{|x-y|^{d-2}}\, \mathrm{d}x\, \mathrm{d}y \right\}\,. \tag{1.2.4}$$

We define the critical mass by

$$M_c := \left[\frac{2}{(m-1) C_* c_d} \right]\,.$$

Theorem 1.2.2 (Global-in-time existence, [35, 193]) *If u_0 is of mass $M < M_c$ then there exists a global weak solution with initial condition u_0. Moreover, this solution satisfies the free energy/free energy dissipation inequality.*

The proof of existence follows the lines of the 2d Patlak-Keller-Segel system. Indeed, as a direct consequence of the VHLS inequality, for any solution ρ to the nonlinear Patlak-Keller-Segel system (1.2.2)

$$\frac{C_*\, c_d}{2} \left(M_c^{2/d} - M^{2/d} \right) \|\rho(t)\|_m^m \leq \mathcal{G}[\rho(t)] \leq \mathcal{G}[\rho_0] < \infty\,.$$

In the case $M < M_c$, it gives the concentration controlled analogous to the entropy *a priori* estimate (1.1.10) of the previous section. It should not be difficult to prove the existence of global-in-time solutions using the JKO minimising scheme.

Open question: *The convergence to the self-similar solution in the sub-critical case has been analysed in [187] but a mathematical study is still*

missing. By doing the porous medium scaling, we can prove, see [35, Theorem 5.2], that for any given mass $M < M_c$ there exists a unique minimiser W_M to the rescaled free energy. Moreover this minimiser is non-negative, radially symmetric and compactly supported. We expect this minimiser to attract all the solutions but have not been able to prove it. Such a result has been recently proved for radially symmetric solutions in [210].

1.2.3 The critical case

The balance in the mass-invariant scaling of diffusion and potential drift can also be seen in the free energy: If $h_\lambda(x) := \lambda^d h(\lambda x)$ then

$$\mathcal{G}[h_\lambda] = \lambda^{(m-1)d} \int_{\mathbb{R}^d} \frac{h^m(x)}{m-1} - \lambda^{d-2} \frac{c_d}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^{d-2}} h(x)\, h(y)\, \mathrm{d}x\, \mathrm{d}y\,.$$

The diffusion and interaction term balance if $m = m_d$. And in this case

$$\mathcal{G}[h_\lambda] = \lambda^{d-2} \mathcal{G}[h]\,. \tag{1.2.5}$$

This scaling has to be compared to the case $d = 2$, see (1.1.11). In the case of the 2d Patlak-Keller-Segel model (1.1.1), the authors were not able to apply a concentration compactness argument due to the rigidity in the scaling of the free energy. Indeed, the scaling which preserves the mass also preserves the free energy. Here, as will be shown below, it is possible to follow the line of P.-L. Lions's concentration-compactness.

The minimisers of $\mathcal{G}$ of mass M_c are such that there are $R > 0$ and $z \in \mathbb{R}^d$ with

$$V(x) = \begin{cases} \dfrac{1}{R^d} \left[\zeta\left(\dfrac{x-z}{R} \right) \right]^{d/(d-2)} & \text{if } x \in B(z,R), \\ 0 & \text{if } x \in \mathbb{R}^d \setminus B(z,R) \end{cases} \tag{1.2.6}$$

where ζ is the unique positive radial classical solution to

$$\Delta \zeta + \frac{m-1}{m} \zeta^{1/(m-1)} = 0 \quad \text{in} \quad B(0,1) \quad \text{with} \quad \zeta = 0 \quad \text{on} \quad \partial B(0,1)\,.$$

Open question: *On the contrary to the 2d Patlak-Keller-Segel system, this proves that there exist compactly supported stationary solutions not blowing-up at infinite time. We were however not able to prove that they attract some solutions. Such a result was recently announced in [20].*

Here, thanks to (1.2.5) we can adapt the concentration-compactness method to prove

Proposition 1.2.3 (How would it blowup, [35]) *Let $T \in (0, \infty]$ and a sequence $(t_k)_k$ converging to T. If*

$$\lim_{k\to\infty} \|\rho(t_k)\|_m = \infty\,.$$

then there are a sub-sequence $(t_{k_j})_j$ and a sequence $(x_j)_j$ in $\mathbb{R}^d$ such that

$$\lim_{j\to\infty} \left\| \rho(t_{k_j}, x + x_j) - \frac{1}{\lambda_{k_j}^d} V\left(\frac{x}{\lambda_{k_j}} \right) \right\|_{L^1} = 0\,,$$

where $\lambda_k := \|\rho(t_k)\|_m^{-m/(d-2)}$ and V is the minimiser of $\mathcal{G}$ of the form (1.2.6) with $\|V\|_m = 1$.

The main ingredient of the proof is the following: we set $v_k(x) := \lambda_k^d \rho(t_k, \lambda_k x)$ so that $\|v_k\|_m = 1$. By the concentration compactness principle there exists a sub-sequence satisfying compactness, vanishing or dichotomy. As already discussed, contrary to the 2d Keller-Segel system, here

$$\lim_{k\to\infty} \mathcal{G}[v_k] = \lim_{k\to\infty} \|\rho(t_k)\|_m^{-m}\, \mathcal{G}[\rho(t_k)] = 0\,.$$

As a consequence

$$\lim_{k\to\infty} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{v_k(t,x)\, v_k(t,y)}{|x-y|^{d-2}} = \lim_{k\to\infty} \frac{2}{c_d}\left(\frac{1}{m-1}\|v_k\|_m^m - \mathcal{G}[v_k]\right) > 0\,.$$

Whereas this quantity goes to zero if vanishing or dichotomy should occur.
Open question: *Except in the radially symmetric case, we are not able to say if the blowup occurs at the centre of mass or if the blowup escapes at infinity. We could not even rule out the possibility that a sub-sequence diverges whereas the other does not. In the radially symmetric case, a comparison principle is available and [21] prove that all the solutions are global-in-time and converge to the stationary solution.*

1.2.4 The super-critical case

The answer was clear for the super-critical case for the 2d Patlak-Keller-Segel thanks to the constant sign of the derivative of the 2-moment. Here for any ρ solution to (1.2.1) the virial identity is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |x|^2\, \rho(t,x)\, \mathrm{d}x = 2\,(d-2)\,\mathcal{G}[\rho(t)]\,. \tag{1.2.7}$$

Theorem 1.2.4 (Blowup, [189, 190, 35, 193]) *If $M > M_c$, there exist initial data of mass M such that the L^m-norm of the corresponding solution blows up in finite time.*

In [35], the proof relies on a procedure directly adapted from [209]: let $\tilde\rho$ be a minimiser of the form (1.2.6) and consider

$$\rho_0 = \frac{M}{M_c}\tilde\rho\,.$$

Then,

$$\mathcal{G}[\rho_0] = \frac{1}{m-1}\left(\frac{M}{M_c}\right)^m \left[1 - \left(\frac{M}{M_c}\right)^{2-m}\right] \|\tilde\rho\|_m^m$$

is negative if $M \ge M_c$. This result combined with (1.2.7) gives the expected result.
Open question: *We cannot exclude the possibility that solutions with positive free energy exist globally in time. In [21, Corollary 1], the authors prove, using the comparison principle in radial coordinates, that there are radially symmetric blowing-up solution of positive energy.*

In [40] a more precise answer is proven: the blowup time T being given, we can look for solution to (1.2.1) of the form

$$\rho(t,x) = \frac{1}{s(t)^d}\Psi\left(\frac{x}{s(t)}\right) \quad \text{and} \quad c(t,x) = \frac{1}{s(t)^{d-2}}\Phi\left(\frac{x}{s(t)}\right)$$

where $s(t) := [d(T-t)]^{1/d}$.

Theorem 1.2.5 (Self-similar blowing-up solutions, [40]) *There exists $\tilde{M}_c \in (M_c, \infty)$ such that for any $M \in (M_c, \tilde{M}_c]$, there exists a self-similar blowing-up solution with a radially symmetric, compactly supported and non-increasing profile Ψ, satisfying $\|\rho(t)\|_1 = \|\Psi\|_1 = M$ for $t \in [0, T)$ and $\|\rho(t)\|_\infty$ going to ∞ as t goes to T.*

The method relies on the study of a boundary value problem for the following non-linear ordinary differential equation:

$$\begin{cases} u''(r,a) + \dfrac{d-1}{r}\, u'(r,a) + |u(r,a)|^{p-1}\, u(r,a) - 1 = 0\,, \quad r \in [0, r_{\max}(a))\,, \\ u(0,a) = a\,, \quad u'(0,a) = 0\,, \end{cases}$$

with $r_{\max}(a) \in (0, \infty]$ and $p = d/(d-2)$. We prove that there are global solutions to this problem and that the solutions oscillate around the stationary solution 1, see Figure 1.1. The solution of Theorem 1.2.5 corresponds to that which vanishes and the support corresponds to its first zero. The proofs rely on ordinary differential equation tools. For recent results in this direction, where asymptotic expansions are performed for $a \to \infty$, see [191].

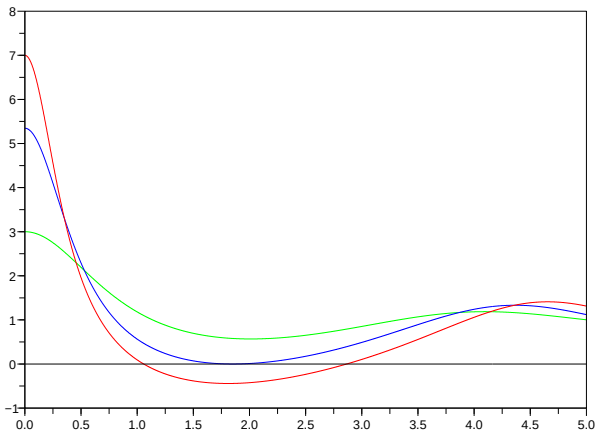

Figure 1.1 – *Behaviour of $u(.,a)$ for $a > a_c$, $a = a_c$ and $a < a_c$, where here $a_c \sim 5.4$.*

Open question: *If $a > a_c$ is large enough, $u(a,\cdot)$ may have several zeros, see Figures 1.2 and each hump corresponds to a solution. It is possible to construct self-similar blowing-up solutions of any mass?*
Open question: *The stability of blowing-up solutions is also of interest but seems yet unclear according to numerical simulations performed in [187].*

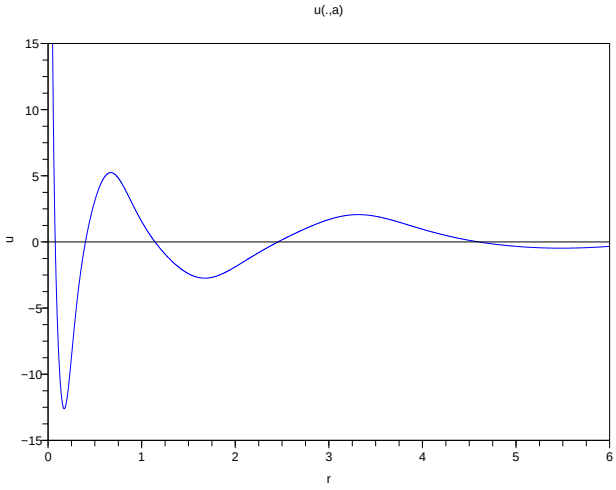

Figure 1.2 – *Positivity set of $u(.,a)$ with three connected components when $a = 90$ ($d = 3$).*

1.3 The non-linear parabolic-parabolic Keller-Segel system

1.3.1 The model

We consider now the following parabolic-parabolic version of the Keller-Segel system:

$$\begin{cases} \partial_t \rho = \operatorname{div}\left[\nabla \rho^m - \chi \rho \nabla \phi\right], \\ \tau \partial_t \phi = \Delta \phi - \alpha\, \phi + \rho, \end{cases} \qquad (t,x) \in (0,\infty) \times \mathbb{R}^d, \qquad (1.3.1)$$

where $m \in (0,1)$ and $d \geq 3$.

For the case $d = 2$, global-in-time existence for a mass less that M_c was proved in [58]. But there are also global-in-time self-similar solutions for larger masses [26]. The question of the eventuality of blowups of solutions to this system remains opened. We were however able to extend the global-in-time existence results of dimension 2 to higher dimensions in the following:

Theorem 1.3.1 (Global existence, [41]) *Let $\tau > 0$, $\alpha \geq 0$, u_0 be a non-negative function in $\mathrm{L}^1(\mathbb{R}^d, (1+|x|^2)\,\mathrm{d}x) \cap \mathrm{L}^m(\mathbb{R}^d)$ satisfying $\|u_0\|_1 = 1$ and $v_0 \in \mathcal{H}^1(\mathbb{R}^d)$. If $\chi < \chi_c$ then there exists a weak solution (u,v) to the parabolic-parabolic Keller-Segel system (1.3.1), that is, for all $t > 0$ and $\xi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,*

- $u \in \mathrm{L}^\infty(0,t;\mathrm{L}^1(\mathbb{R}^d,(1+|x|^2)\,\mathrm{d}x) \cap \mathrm{L}^m(\mathbb{R}^d))$, $u^{m/2} \in \mathrm{L}^2(0,t;\mathcal{H}^1(\mathbb{R}^d))$,
- $u(t) \geq 0$, $\|u(t)\|_1 = 1$,
- $v \in \mathrm{L}^\infty(0,t;\mathcal{H}^1(\mathbb{R}^d)) \cap \mathrm{L}^2(0,t;\mathcal{H}^2(\mathbb{R}^d)) \cap \mathrm{W}^{1,2}(0,t;\mathrm{L}^2(\mathbb{R}^d))$, $v(0) = v_0$,

and

$$\begin{aligned} \int_{\mathbb{R}^d} \xi\, (u(t) - u_0)\,\mathrm{d}x \;&+ \int_0^t \int_{\mathbb{R}^d} (\nabla(u^m) - \chi\, u\, \nabla v) \cdot \nabla \xi \,\mathrm{d}x\,\mathrm{d}s = 0, \\ \tau\, \partial_t v - \Delta v + \alpha\, v \;&= u \quad \text{a.e. in} \quad (0,t) \times \mathbb{R}^d. \end{aligned}$$

1.3.2 Idea of the proof

The main difficulty stems from the fact that, unlike the parabolic-elliptic Keller-Segel system, the Cauchy problem (1.3.1) cannot be reduced to a single equation with a nonlinear term involving a convolution with a time independent kernel. The main difficulty here is that the system cannot easily be reduced to a single non-local parabolic equation. Actually the corresponding free energy has the two quantities ρ and ϕ:

$$\mathcal{K}_\alpha[\rho,\phi] := \int_{\mathbb{R}^d} \left\{ \frac{|\rho(x)|^m}{\chi(m-1)} - \rho(x)\,\phi(x) + \frac{1}{2}\,|\nabla\phi(x)|^2 + \frac{\alpha}{2}\,\phi(x)^2 \right\} \mathrm{d}x\,. \tag{1.3.2}$$

The minimising scheme has thus to be replaced by a gradient flow of this energy in $\mathcal{P}_2(\mathbb{R}^d)\times \mathrm{L}^2(\mathbb{R}^d)$ endowed with the Monge-Kantorovich metric for the first component and the usual L^2-norm for the second component. Such a strategy has already been developed to prove existence of the thin film approximation of the Muskat problem [139].

The minimising scheme is as follows: given an initial condition $(\rho_0,\phi_0)\in \mathcal{K}$ and a time step $h>0$, we define a sequence $(\rho_{h,n},\phi_{h,n})_{n\ge 0}$ in $\mathcal{K}$ by

$$\begin{cases} (\rho_{h,0},\phi_{h,0}) = (\rho_0,\phi_0)\,, \\ (\rho_{h,n+1},\phi_{h,n+1}) \in \mathrm{Argmin}_{(\rho,\phi)\in\mathcal{K}} \mathcal{F}_{h,n}[\rho,\phi]\,, \qquad n\ge 0\,, \end{cases} \tag{1.3.3}$$

where

$$\mathcal{F}_{h,n}[\rho,\phi] := \frac{1}{2h}\,\left[\frac{\mathcal{W}_2^2(\rho,\rho_{h,n})}{\chi} + \tau\,\|\phi-\phi_{h,n}\|_2^2\right] + \mathcal{K}_\alpha[\rho,\phi]\,,$$

and $\mathcal{W}_2$ is the Monge-Kantorovich distance on $\mathcal{P}_2(\mathbb{R}^d)$.

Several difficulties arise in the proof of the well-posedness and convergence of the previous minimising scheme. First, as the energy $\mathcal{K}_\alpha$ is not displacement convex, standard results from [204, 5] do not apply and even the existence of a minimiser is not clear. Nevertheless, the assumption $\chi<\chi_c$ and a further development of the modified Hardy-Littlewood-Sobolev inequality (1.2.4) allow us to obtain an $(\mathrm{L}^1\cap\mathrm{L}^m)(\mathbb{R}^d)\times\mathcal{H}^1(\mathbb{R}^d)$ bound on minimising sequences which permits in particular to pass to the limit in the term in $\mathcal{K}_\alpha[\rho,\phi]$ involving the product $\rho\phi$, and prove the existence of a minimiser. To obtain the Euler-Lagrange equation satisfied by a minimiser $(\bar\rho,\bar\phi)$ of $\mathcal{F}_{h,n}$ in $\mathcal{K}$, the parameters h and n being fixed, we consider an "optimal transport" perturbation for $\bar\rho$ and a L^2-perturbation for $\bar\phi$ defined for $\delta\in(0,1)$ by

$$\rho_\delta = (\mathrm{id}+\delta\,\zeta)\#\bar\rho\,, \quad \phi_\delta := \bar\phi + \delta\,w\,,$$

where $\zeta\in\mathcal{K}_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and $w\in\mathcal{K}_0^\infty(\mathbb{R}^d)$ are two smooth test functions. Identifying the Euler-Lagrange equation requires to pass to the limit as $\delta\to 0$ in

$$\frac{\mathcal{W}_2^2(u_\delta,u_{h,n}) - \mathcal{W}_2^2(\bar\rho,\rho_{h,n})}{2\delta} \quad \text{and} \quad \frac{\|\rho_\delta\|_m^m - \|\bar\rho\|_m^m}{\delta}\,,$$

which can be performed by standard arguments [204, 5], but also in

$$\frac{1}{\delta}\int_{\mathbb{R}^d}(\bar\rho\,\bar\phi-u_\delta\,\phi_\delta)(x)\,\mathrm{d}x \\ =\int_{\mathbb{R}^d}\bar\rho(x)\left[\frac{\bar\phi(x)-\bar\phi(x+\delta\zeta(x))}{\delta}-w(x+\delta\zeta(x))\right]\,\mathrm{d}x\,.$$

This is where the main difficulty lies: indeed, since $\bar v\in\mathcal{H}^1(\mathbb{R}^d)$, we only have

$$\frac{\bar v\circ(\mathrm{id}+\delta\zeta)-\bar v}{\delta}\rightharpoonup\zeta\cdot\nabla\bar v\quad\text{in }\mathrm{L}^2(\mathbb{R}^d),$$

while $\bar u$ is only in $(\mathrm{L}^1\cap\mathrm{L}^m)(\mathbb{R}^d)$ with $m<2$. So even the product $\bar u\zeta\cdot\nabla\bar v$ which is the candidate for the limit is not well defined and the regularity of $(\bar u,\bar v)$ has to be improved. To this end, a powerful technique is developed in [152]. The main idea was already presented above in Section 1.1.4: it is to find a functional $\mathcal{G}$ (different from the energy $\mathcal{K}_\alpha$) with the following properties: it is displacement convex and the energy $\mathcal{K}_\alpha$ is a Liapunov functional (up to lower order terms) for the gradient flow associated to $\mathcal{G}$. If such a functional $\mathcal{G}$ exists, the associated displacement convexity inequality can be converted into additional estimates on the minimisers of $\mathcal{K}_\alpha$. Of course, the cornerstone of this method is the availability of the functional $\mathcal{G}$ and the simplest situation is the case where the flow has a displacement convex Liapunov functional which is different from the energy. Unfortunately, there does not seem to be a natural choice of such a functional $\mathcal{G}$ here. A first try is to choose $\mathcal{G}$ as the displacement convex part of $\mathcal{K}_\alpha$, that is,

$$\mathcal{G}[u,v]:=\int_{\mathbb{R}^d}\left(\frac{|u(x)|^m}{\chi(m-1)}+\frac12\,|\nabla v(x)|^2+\frac{\alpha}{2}\,|v(x)|^2\right)\,\mathrm{d}x\,.$$

The associated gradient flow is the solution (U,V) to

$$\partial_s U-\Delta U^m=0\quad\text{in }(0,\infty)\times\mathbb{R}^d,\qquad U(0)=\bar u\,,$$

and

$$\partial_s V-\Delta V+\alpha V=0\quad\text{in }(0,\infty)\times\mathbb{R}^d,\qquad V(0)=\bar v\,.$$

Computing $\mathrm{d}\mathcal{K}_\alpha[U(s),V(s)]/\,\mathrm{d}s$ leads to the sum of a negative term and a remainder but the remainder terms cannot be controlled. Despite this failed attempt, it turns out that, somehow unexpectedly, the following functional

$$\mathcal{G}[u,v]:=\int_{\mathbb{R}^d}\left(u(x)\log\left(u(x)\right)+\frac12\,|\nabla v(x)|^2+\frac{\alpha}{2}\,|v(x)|^2\right)\,\mathrm{d}x$$

provide the right information. Indeed, its associated gradient flow is the solutions U and V to the initial value problems

$$\partial_s U-\Delta U=0\quad\text{in }(0,\infty)\times\mathbb{R}^d,\qquad U(0)=\bar u\,,$$

and

$$\partial_s V-\Delta V+\alpha V=0\quad\text{in }(0,\infty)\times\mathbb{R}^d,\qquad V(0)=\bar v\,,$$

and, as we shall see below, $\mathrm{d}\mathcal{K}_\alpha[U(s),V(s)]/\,\mathrm{d}s$ is in that case the sum of a negative term and a remainder which we are able to control. This regularity allows us to pass to the limit in the Euler-Lagrange equation.

The kernel which appears in $\mathcal{K}_\alpha$, $\alpha \geq 0$ is the Bessel kernel, $\mathcal{Y}_\alpha$, defined for $\alpha \geq 0$ by:

$$\mathcal{Y}_\alpha(x) := \int_0^\infty \frac{1}{(4\pi s)^{d/2}} \exp\left(-\frac{|x|^2}{4s} - \alpha s\right) \, \mathrm{d}s \,, \quad x \in \mathbb{R}^d \,,$$

the case $\alpha = 0$ corresponding to the already defined Poisson kernel. For $u \in \mathrm{L}^1(\mathbb{R}^d)$, $S_\alpha(u) := \mathcal{Y}_\alpha * u$ solves

$$-\Delta S_\alpha(u) + \alpha S_\alpha(u) = u \quad \text{in } \mathbb{R}^d \tag{1.3.4}$$

in the sense of distributions, see [143, Theorem 6.23]. The Bessel kernel is also referred to as the screened Poisson or Yukawa potential in the literature. The crucial inequality is thus a modified Hardy-Littlewood-Sobolev inequality is valid for the Bessel kernel $\mathcal{Y}_\alpha$ for $\alpha > 0$:

Lemma 1.3.2 (Hardy-Littlewood-Sobolev inequality for the Bessel kernel) *For $\alpha > 0$,*

$$\sup\left\{ \frac{\int_{\mathbb{R}^d} h(x)\,(\mathcal{Y}_\alpha * h)(x)\,\mathrm{d}x}{\|h\|_m^m \, \|h\|_1^{2/d}} \,:\, h \in (\mathrm{L}^1 \cap \mathrm{L}^m)(\mathbb{R}^d), h \neq 0 \right\} = C_{\mathrm{HLS}} \,, \tag{1.3.5}$$

where C_{HLS} is defined in (1.2.4).

Note that the constant is the exact same as for the case $\alpha = 0$ so that the critical mass below which all the solutions exist globally-in-time is the same as for the parabolic-elliptic version.

1.4 Concluding remarks

The Keller-Segel models have attracted much attention these last years. The literature is vast and drastically increasing. There exist many variants of the presented models with prevention of overcrowding [114, 53] or with non-linear chemo-sensitivity [22, 145], etc. The Keller-Segel model has recently been used as a basis for more complete models [59]. This chapter is dedicated to the Keller-Segel models and tries to describe the progress made through energy and functional inequalities methods in the idea of [204, 5]. For more complete reviews see [117, 116, 171, 115].

From the author's point of view, the most challenging question is the understanding of the blowup. And in this direction progress is still to be made. We are now at a point where we need to develop new methods to address those questions. The answer could come from interaction with the non-linear Schrödinger equation (NLS) and the unstable thin-film equation (UTF). Indeed, the Patlak-Keller-Segel, the NLS and the UTF equation have two levels of criticality. The first level is given by the homogeneity of the "attractive" and "repulsive" terms in each problem. In our particular case, this refers to the aggregation versus diffusion mechanisms. In NLS it is the balance between dispersion and nonlinear attraction. As seen above, the balance happens precisely for our chosen exponent $m = m_d$. In the NLS equation this happens for the so-called *pseudo-conformal* non-linearity, see [192] or [73, Chapter 6]. In the UTF equation this happens in the so-called *marginal* case, see [24, 25]. In theses three equations, a second

level of critically occurs when the attractive and repulsive are balanced. In that particular case, and for the three models, there exists a critical value M_c of the mass which is the maximum value of the mass below which the solutions exist globally in time, see [73, 209, 156] for the pseudo-conformal NLS equation and [25, 188] for the marginal UTF equation. Note that mass refers to the total number of particles for the NLS equation and the L^1-norm for (1.2.2) and the UTF equation.

Let us also point out that in all these three problems, the virial method is an elegant way to prove that there are solutions which blowup above the critical mass, but it does not give any hint on the mechanism of the blowup. However, in [156], for the NLS equation, the result goes further and clarifies the blowup for super-critical masses close to critical. The techniques used by [156, 157] describes the manifold where the blowup occurs for the Schrodinger operator. For the Keller-Segel system, we can expect that such a study would lead to a set defined by $M \geq 8\pi$. There is here a whole programme to extend these results to non-radial solutions, to others Keller-Segel models -and in particular to the parabolic-parabolic models where even the occurence of blowup is not proved yet.

On the Stochastic Stokes' drifts

2

We study the large time properties of the solution of a simple model of stochastic Stokes' drift. The analysis also applies to *Brownian ratchets* and molecular motors in biology. We first establish a transport phenomenon. Asymptotically, the center of mass of the solution moves with a constant velocity, which is determined by a doubly periodic problem. In the traveling frame, the macroscopic profile obeys to an isotropic diffusion. Compared with the original diffusion, diffusion is enhanced or reduced, depending on the regime. At least in the limit cases, the rate of convergence to the effective profile is always decreased. All these considerations allow us to define a notion of efficiency for coherent transport, characterized by a dimensionless number, which is illustrated on two simple examples of traveling potentials with a sinusoidal shape in the first case, and a sawtooth shape in the second case.

The proof relies on an entropy estimate based on homogenized logarithmic Sobolev inequalities. A periodic perturbation of a Gaussian measure modifies the sharp constants in Poincaré and logarithmic Sobolev inequalities in the homogenization limit, that is, when the period of a periodic perturbation converges to zero. We use variational techniques to determine the homogenized constants and get optimal convergence rates towards equilibrium of the solutions of the perturbed diffusion equations.

Most of this chapter in inspired form [38] and presents the works:

- A. Blanchet, J. Dolbeault, and M. Kowalczyk, *Travelling fronts in stochastic Stokes' drifts*, Physica A: Statistical Mechanics and its Applications, 387 (2008), pp. 5741–5751.

- ——, *Stochastic Stokes' drift, homogenized functional inequalities, and large time behaviour of Brownian ratchets*, SIAM Journal of Mathematical Analysis, 41 (2009), pp. 46–76.

2.1 Model

The literature on the *stochastic Stokes' drift* and *Brownian ratchets* is huge. We first refer to [122]: the drift velocity is computed in the case of a sinusoidal traveling potential (also see [149, 91]) and the diffuse traveling front is exhibited on the basis of numerical results. *Brownian ratchets* generically refer to drift-diffusion models in which a time periodic forcing coupled to some asymmetry induces a transport at large scale. The notion of traveling potential is explored in [48, 109]. We refer to [177] and references therein for the notion of *tilted Smoluchowski-Feynman ratchet*, which makes an explicit connection between the stochastic Stokes' drift and ratchet mechanisms. As we shall see later, a change of variables indeed reduces the model to a simple tilted Brownian ratchet, with no more explicit time-dependence. An historical perspective of the physics of ratchets and useful definitions are given in [109, 178, 148]. Many important issues, like effects due to the asymmetry of the potential, the geometry of the domain in higher dimensional models, see for instance [176], or applications of molecular motors in biology, see for instance [4, 123, 168], will not be addressed here.

In [179, 180], the *effective diffusion constant* is computed by a method which differs from ours, based on statistical fluctuations. Also see [82] for some earlier considerations. The ratio $\mathsf{k_B}\,\Theta$ of the diffusion constant κ_ω to the differential mobility μ_ω is in general not equal to the temperature of the environment (multiplied by Boltzmann's constant $\mathsf{k_B}$). The physical meaning of Θ far from equilibrium is analyzed in [111] and interpreted as an effective temperature in the large scale description of the system so that the relation $\kappa_\omega = \mu_\omega\,\mathsf{k_B}\,\Theta$ can be interpreted as an extension of Einstein's relation. An interesting experiment for measuring the violation of Einstein's relation can be found in [110].

Experimental measurements of the drift velocity corresponding to the diffusion of colloidal particles in presence of optical traps and a detailed explanation of the method can be found in [99], with abundant theoretical justifications. Some of the qualitative features were at least partially known before, see, *e.g.*, [144]. In [142, 43, 98], the analysis of [99] is refined and emphasis is put not only on the computation of the drift velocity, but also on the effective diffusion constant. Interestingly, the authors of [175] favorably evaluate the possibility of using tilted ratchet mechanisms to implement separation of two types of filaments of DNA.

This phenomenon is also related to Parrondo's paradox in game theory. Let us consider the following coin-tossing example: Let C_t be the player's capital at time t and $\epsilon > 0$.

- Winning a game earns us 1€ and losing requires us to surrender 1€.
- In Game A, we toss a biased coin, Coin 1, with a probability of winning $P_1 = 1/2 - \epsilon$.
- In Game B, we first determine if C_t is a multiple of 3.
  - If it is, we toss a biased coin, Coin 2, with a probability of winning $P_2 = 1/10 - \epsilon$.

– If it is not, we toss another biased coin, Coin 3, with a probability of winning $P_3 = 3/4 - \epsilon$.

As can be seen in Figure 2.1 both games are losing games. However, counter to common intuition, it is possible to mix two losing games into a winning combination, see Figure 2.1.

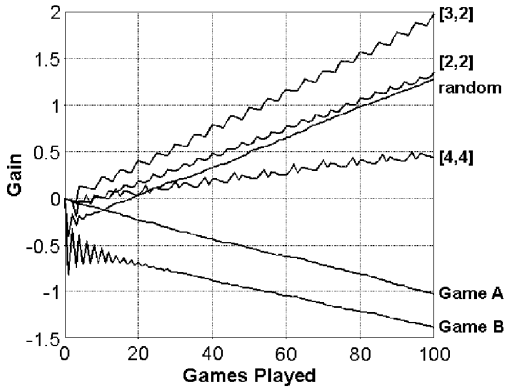

Figure 2.1 – *Average over 50.000 runs of switching between games A and B. The simulation was performed by playing game A a times, then game B b times, game A a times, and so on with* $e = 0.005$ *and averaged over 50,000 runs. The notation* $v = [a, b]$ *is used for each combination. Source: S. Sinha*

The simplest version of the stochastic Stokes' drift model describes a density $f(t,x)$ of particles obeying to the equation

$$f_t = \Delta f + \nabla \cdot \left[\nabla \psi(x - \omega\, t\, \mathsf{e})\, f\right], \quad x \in \mathbb{R}^d\,,\ t > 0\,. \tag{2.1.1}$$

where $\psi(y + \mathsf{k}) = \psi(y)$ for any $(y, \mathsf{k}) \in \mathbb{R}^d \times \mathbb{Z}^d$, and will simply write ψ as a function of $y \in \mathbb{T}^d \approx [0,1)^d$. Furthermore $\omega \in \mathbb{R}$ is a constant and $\mathsf{e} \in \mathbb{R}^d$ is a fixed vector, such that $|\mathsf{e}| = 1$. With these notations, $\psi(x - \omega\, t\, \mathsf{e})$ represents a periodic potential in $\mathbb{R}^d$ moving with a constant speed ω in the direction of the vector e, that is a *traveling potential*. At $t = 0$, $f(0,\cdot) = f_0$ is a given smooth probability distribution, so that, by conservation of mass, $\int_{\mathbb{R}} f(t,x)\,dx = 1$ for any $t \geq 0$. The question we investigate in this paper is the behavior of f for large values of t.

2.2 Transport phenomenon

A first case, which is particularly simple, is the case $\omega = 0$. Let $R(t) := \sqrt{1 + 2\,t}$. The function u defined by the change of coordinates

$$f(t,x) = \frac{1}{R^d(t)}\, u\left(\log R(t), \frac{x}{R(t)}\right),$$

is a solution of

$$\begin{cases} u_t = \Delta u + \nabla \cdot (x\, u) + R\, \nabla \cdot (u\, \nabla \psi(R\, x))\,, & x \in \mathbb{R}^d\,,\ t > 0\,, \\ u(t = 0, x) = f_0(x)\,, & x \in \mathbb{R}^d\,, \end{cases} \tag{2.2.1}$$

where, in the new variables,

$$R(t) = e^t \quad \forall\, t > 0\,.$$

For large values of t, we can formally regard $\mathsf{e} = 1/R(t)$ as a small parameter and it is reasonable to expect that the behavior of the solution is well described by (2.2.1) with $\phi = \psi$ in the limit $\mathsf{e} \to 0_+$.

When $\omega > 0$, Equation (2.2.1) is also going to play a role in the large time behavior of the solutions of (2.1.1), but the description is not as simple as above. The combination of the drift, which is time-periodic, and of the diffusion induces a motion of the center of mass. The speed of displacement is known as the *ballistic velocity*, or *drift velocity*,. On large time scales, the constant drift term, $\omega\,\tilde{f}_x$, is responsible for a displacement of the center of mass, but the solution is also spreading on a large number of periods of ψ. It is therefore natural to expect that the speed of the center of mass is determined by the flux of mass through one cell of period $\ell = 1$, supplemented with periodic conditions. This can be made rigorous by the following *folding* transformation as follows.

$$\begin{cases} g_t = \Delta g + \nabla\cdot\big(g\,\nabla\psi(x-\omega\,t\,\mathsf{e})\big), & x\in\mathbb{T}^d\,,\ t>0\,,\\ g(t=0,x) = g_0(x) = \displaystyle\sum_{k\in\mathbb{Z}} f_0(x+k), & x\in\mathbb{T}^d\,, \end{cases} \tag{2.2.2}$$

for which, by linearity of the equations, we obtain

$$g(t,x) = \sum_{k\in\mathbb{Z}^d} f(t,x+k) \quad \forall\,(t,x)\in\mathbb{R}^+\times\mathbb{T}^d\,.$$

Consider now a solution f of (2.1.1). Assume for simplicity that $\int_{\mathbb{R}^d} f_0\,\mathrm{d}x = 1$. Then $\int_{\mathbb{R}^d} f(t,\cdot)\,\mathrm{d}x = 1$ for any $t \ge 0$ and we can define the position of the center of mass by

$$\bar{x}(t) := \int_{\mathbb{R}^d} x\, f(t,x)\,\mathrm{d}x\,.$$

An integration by parts shows that

$$\begin{aligned} \frac{d\bar{x}}{dt} = \int_{\mathbb{R}^d} x\, f_t\,\mathrm{d}x \quad &= \quad -d\int_{\mathbb{R}^d} \nabla\psi(x-\omega\,t\,\mathsf{e})\, f(t,x)\,\mathrm{d}x\\ &= \quad -d\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}^d} \nabla\psi(x-\omega\,t\,\mathsf{e})\, f(t,x+k)\,dx\\ &= \quad -d\int_{\mathbb{T}^d} \nabla\psi(x-\omega\,t\,\mathsf{e})\, g(t,x)\,dx\\ &\underset{t\to\infty}{\sim} \quad -d\int_{\mathbb{T}^d} \nabla\psi(x-\omega\,t\,\mathsf{e})\, g_\infty(t,x)\,dx\,. \end{aligned}$$

If we define

$$c_\omega = -d\int_0^1 dt\int_{\mathbb{T}^d} \nabla\psi(x-\omega\,t\,\mathsf{e})\, g_\infty(t,x)\,dx\,, \tag{2.2.3}$$

then a more careful analysis of (2.2.2) shows that $\frac{d\bar{x}}{dt} - c_\omega$ converges to 0 at an exponential rate. Hence

$$\bar{x}(t) \sim c_\omega\, t \quad \text{as} \quad t\to\infty\,,$$

and it makes sense to introduce the change of coordinates

$$f(t,x) = \frac{1}{R^d}\, u\left(\log R, \frac{x - c_\omega\, t\, \mathsf{e}}{R}\right), \tag{2.2.4}$$

with $R(t) = \sqrt{1+2t}$ as above, in order to understand the large time behavior of f. In the new variables, the equation is

$$u_t = \Delta u + \nabla\,(x\,u) + R\,\nabla\cdot\left[\,u\,\left(c_\omega\,\mathsf{e} + \nabla\psi\left(R\,x + \tfrac{1}{2}\,(R^2-1)\,(c_\omega-\omega)\,\mathsf{e}\right)\right)\right]\,.$$

At this point, we shall assume that $d = 1$ to simplify the discussion. The higher dimensional case is similar. The time-periodic solution g_∞ can also be written as a function of $x - \omega\, t$ (here $\mathsf{e} = 1$) since the solution is unique and can be obtained as follows. The function $g_\infty(t,x) = g_\omega(x-\omega\,t)$ solves the equation

$$(g_\omega)_{xx} + \left((\omega+\psi')\,g_\omega\right)_x = 0\,, \tag{2.2.5}$$

with periodic boundary conditions. If we take a primitive of (2.2.5), we get that

$$x \mapsto (g_\omega)_x + (\omega+\psi')\,g_\omega =: A(\omega) \tag{2.2.6}$$

is constant. By taking one more integral of (2.2.6), using the normalization condition $\int_0^1 g_\omega(x)\,dx = 1$ and the definition of $c_\omega = c_\omega(\omega)$ given by (2.2.3), we get that

$$\omega - c_\omega(\omega) = \omega\int_0^1 g_\omega\,dx + \int_0^1 \psi'\,g_\omega\,dx = A(\omega)\,.$$

Some elementary but tedious computations show that $c_\omega(\omega) < \omega$, $\lim_{\omega\to 0_+} c_\omega(\omega)/\omega > 0$, $c_\omega(\omega)$ is positive for large values of ω, and $\lim_{\omega\to\infty} c_\omega(\omega) = 0$. We will now illustrate our results in the case of $\psi(x) = \sin(2\pi x)$ (sinusoidal case) and of an asymmetric smooth sawtooth potential, see Fig. 2.2. Notice that in the tilted ratchet point of view, the current is $A(\omega)$. It is actually very interesting to compare c_ω with the asymp-

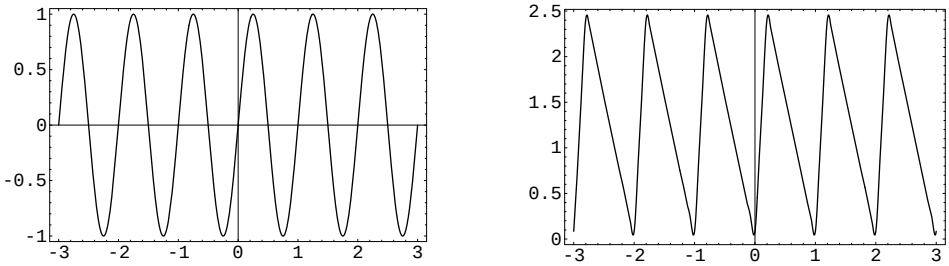

Figure 2.2 – *Plots of the potential ψ in the sinusoidal case (left) and in the asymmetric smooth sawtooth potential (right), which is computed here as a truncated Fourier series of $x \mapsto x/x_0$ if $x < x_0$ and $x \mapsto (1-x)/(1-x_0)$ if $x > x_0$, with $x_0 = 0.2$.*

totic drift velocity c_ω^0 when there is no diffusion. See [132] for similar considerations. The solutions of $f_t = \left(\psi'(x-\omega\,t)\,f\right)_x$ are easily solved by considering the equations of the characteristics, $\frac{dx}{dt} = -\psi'(x(t)-\omega\,t)$. Let $y(t) := x(t) - \omega\,t$ and consider the corresponding equation $\frac{dy}{dt} = -\psi'(y) - \omega$. For $\omega > 0$, there are two main regimes:

(i) Case $0 < \omega < \max_{[0,1]} \psi'$: any solution $t \mapsto y(t)$ converges to a local minimum of the function $y \mapsto \omega\, y - \psi(y)$, and so

$$c^0_\omega := \lim_{t\to\infty} \frac{x(t)}{t} = \omega\,.$$

(ii) Case $\omega > \max_{[0,1]} \psi'$: we observe that $\tau(\omega) := \int_{y(t)}^{y(t)+1} \frac{dx}{\omega+\psi'(x)}$ does not depend on t, and so $y(t) \sim -t/\tau(\omega)$ as $t \to \infty$. It follows that

$$\tau(\omega) = \int_0^1 \frac{dx}{\omega + \psi'(x)} \quad \text{and} \quad c^0_\omega := \lim_{t\to\infty} \frac{x(t)}{t} = \omega - \frac{1}{\tau(\omega)}\,.$$

A characteristic property of the curve $\omega \mapsto c^0_\omega$ is the *critical tilt:* the discontinuity of the derivative separates the two regimes. The curve $\omega \mapsto c_\omega$ is a smoothed version of $\omega \mapsto c^0_\omega$. When ψ is not symmetric, asymmetry effects are present when ω is replaced by $-\omega$, as shown in the case of the asymmetric smooth sawtooth potential. See Fig. 2.3.

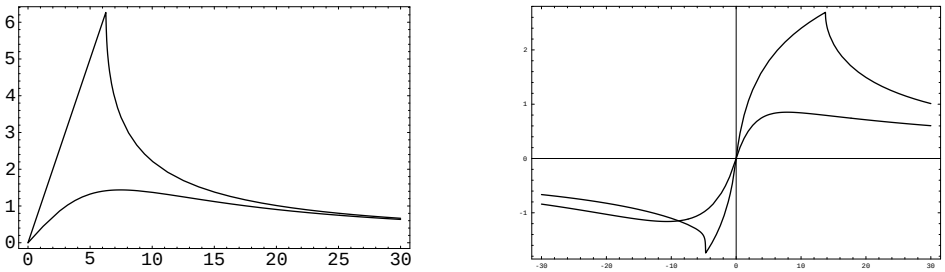

Figure 2.3 – *Plots of c_ω and c^0_ω as functions of ω in the sinusoidal case (left) and in the case of the asymmetric smooth sawtooth potential (right, in logarithmic coordinates: $\omega \mapsto \log(1 + c_\omega)$ for $\omega > 0$). In the sinusoidal case, the symmetry is reflected by the fact that $c_{-\omega} = -c_\omega$ (values corresponding to $\omega < 0$ are not represented). This is not true in the sawtooth case.*

Theorem 2.2.1 (Main results on c_ω) *For any $\omega > 0$, we have $c_\omega < \omega$,*

$$\lim_{\omega\searrow 0} \frac{c_\omega}{\omega} = 1 - \frac{1}{\int_0^1 e^{\psi}\,\mathrm{d}z \int_0^1 e^{-\psi}\,\mathrm{d}z} \quad \textit{and} \quad \lim_{\omega\nearrow\infty} c_\omega = 0\,.$$

2.3 The diffusive traveling front

After rescaling, the equation for u is

$$u_t = u_{xx} + (x\,u)_x + R\left[\left(\psi'\left(R\,x - \tfrac{1}{2}\,(R^2-1)\,A(\omega)\right) + c_\omega(\omega)\right)\,u\right]_x \quad (2.3.1)$$

with $R(t) = e^t$.

Let us continue our heuristic approach by introducing a two-scale function U. Since f in (2.2.4) only depends on $x - c_\omega\, t\, \mathsf{e}$, only one variable, z, is needed at small scale. Let

$$u(t,x) = \mathsf{U}(t,x;z) \quad \text{with} \quad z := R\,x - \tfrac{1}{2}\,(R^2-1)\,A(\omega)\,,$$

in the large $R = e^t$ limit. Using the chain rule, we see that U should be solution of

$$\mathsf{U}_t = R^2\Big(\mathsf{U}_{zz} + A\,\mathsf{U}_z + \big[(\psi'(z)+c_\omega)\,\mathsf{U}\big]_z\Big) + R\Big(2\,\mathsf{U}_z + (\psi'(z)+c_\omega)\,\mathsf{U}\Big)_x + \Big(\mathsf{U}_{xx} + (x\,\mathsf{U})_x\Big)\,.$$

Introduce the two-scale function U such that

$$u(t,x) = \mathsf{U}(t,x;z)$$

and make the ansatz

$$\mathsf{U} = \mathsf{U}_0 + R^{-1}\,\mathsf{U}_1 + R^{-2}\,\mathsf{U}_2$$

The equation of U can be written $R^2\,\mathsf{L}_0\,\mathsf{U} + R\,\mathsf{L}_1\,\mathsf{U} + \mathsf{L}_2\,\mathsf{U} = 0$ where

$$\begin{aligned}\mathsf{L}_0\,\mathsf{U} &:= \mathsf{U}_{zz} + \big((\omega+\psi'(z))\,\mathsf{U}\big)_z\\ \mathsf{L}_1\,\mathsf{U} &:= \Big(2\,\mathsf{U}_z + (\psi'(z)+c_\omega)\,\mathsf{U}\Big)_x\\ \mathsf{L}_2\,\mathsf{U} &:= \mathsf{U}_{xx} + (x\,\mathsf{U})_x - \mathsf{U}_t\end{aligned}$$

• At order R^2: we find $\mathsf{L}_0\,\mathsf{U}_0 = 0$ that is solved by $\mathsf{U}_0(t,x;z) = g_\omega(z)\,h(t,x)$, where recall that g_ω is the unique 1-mean valued, 1-periodic solution of

$$(g_\omega)_{xx} + \big((\omega+\psi')\,g_\omega\big)_x = 0\,.$$

• At order R^1: $\mathsf{L}_0\,\mathsf{U}_1 + \mathsf{L}_1\,\mathsf{U}_0 = 0$ gives $\mathsf{U}_1(t,x;z) = g^{(1)}_\omega(z)\,h_x(t,x)$, where $g^{(1)}_\omega$ is a solution of

$$(g^{(1)}_\omega)_{zz} + \Big((\omega+\psi')\,g^{(1)}_\omega\Big)_z = -2\,(g_\omega)_z - (\psi'+c_\omega)\,g_\omega\,.$$

For the solvability condition we recover the definition of c_ω.

• At order R^0: we obtain $\mathsf{L}_0\,\mathsf{U}_2 + \mathsf{L}_1\,\mathsf{U}_1 + \mathsf{L}_2\,\mathsf{U}_0 + R^{-1}\,(\mathsf{L}_1\,\mathsf{U}_2 + \mathsf{L}_2\,\mathsf{U}_1) + R^{-2}\,\mathsf{L}_2\,\mathsf{U}_2 = 0$.
And the solvability condition is: $\int_0^1 (\mathsf{L}_1\,\mathsf{U}_1 + \mathsf{L}_2\,\mathsf{U}_0)\,dz = 0$ which leads to

$$h_t = \kappa_\omega\,h_{xx} + (x\,h)_x \quad \text{where} \quad \kappa_\omega := 1 + \int_0^1 \psi'(z)\,g^{(1)}_\omega(z)\,dz > 0\,.$$

Remind that by classic entropy method, if $\int_{\mathbb{R}} h(0,x)\,dx = 1$, then h converges exponentially fast to

$$h_\infty(x) := \frac{e^{-|x|^2/(2\kappa_\omega)}}{\sqrt{2\pi\kappa_\omega}}$$

Summarising, we have formally found that

$$u(t,x) = \mathsf{U}(t,x;z) = \left(g_\omega(z) - \frac{x}{\kappa_\omega\,R}\,g^{(1)}_\omega(z)\right)h_\infty(x)\,\Big(1+o(1)\Big)\,.$$

By undoing the change of variables, we obtain

$$f(t,x) = \left[g_\omega(x-\omega\, t) - \frac{x-c(\omega)\, t}{\kappa_\omega\, \sqrt{1+2\, t}}\, g_\omega^{(1)}(x-\omega\, t) \right] \frac{h_\infty\left(\frac{x-c(\omega)\, t}{\sqrt{1+2\, t}}\right)}{\sqrt{2\,\pi\,(1+2\, t)}} \Big(1+o(1)\Big)$$

Define

$$u_\infty(x,t) := g_\omega(z)\, \frac{e^{-\frac{|x|^2}{2\kappa_\omega}}}{\sqrt{2\pi\kappa_\omega}}$$

where g_ω is the unique 1-mean valued, 1-periodic solution of

$$(g_\omega)_{zz} + \big((\omega+\psi')\, g_\omega\big)_z = 0\,.$$

The function u_∞ therefore describes the asymptotic regime of u, in self-similar, traveling variables. In the original variables, $f_\infty(t,x) = \frac{1}{R}\, u_\infty\left(\log R, \frac{x-c_\omega\, t}{R}\right)$ with $R(t) := \sqrt{1+2t}$ describes the *intermediate asymptotics* of the solution of (2.1.1). It is highly oscillatory, with an effective profile given by $F_\infty(t,x) := \frac{1}{R(t)}\, h_\infty\left(\frac{x-c_\omega\, t}{R(t)}\right)$, which is the diffuse, traveling front. See Fig.reffig3.

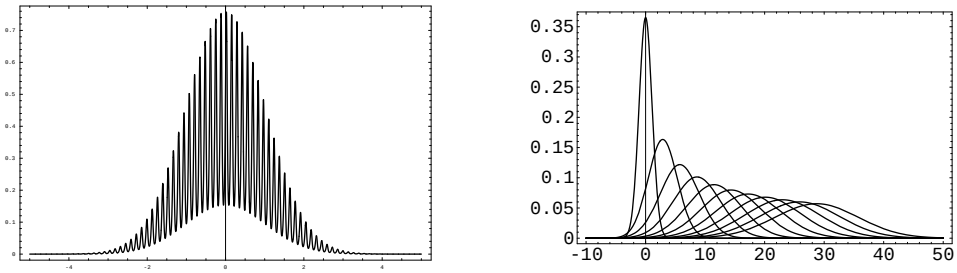

Figure 2.4 – *In the sinusoidal case, the limiting function u_∞ is shown on the left, in self-similar-variables, while on the right, the diffuse, traveling front F_∞ is plotted in the original variables for $t = 0,\, 1,\, \ldots 20$. Here we take $\omega = 5$ and (left) $u_\infty(t,x)$ is shown as a function of x for $t = 2$.*

The effective diffusion coefficient κ_ω is a global, macroscopic quantity, which should not be confused with the local effective diffusion constant which appears in some papers, see [137, 146]. It plays two roles:

(1) The effective diffusion coefficient κ_ω determines the variance of the Gaussian function h_∞ and therefore controls the size of the traveling font. A pure diffusion $f_t = f_{xx}$ would give rise to a self-similar Gaussian function $(4\pi t)^{-1/2}\, e^{-|x|^2/(4t)}$, and so κ_ω has to be compared with 1. When $\kappa_\omega < 1$, the front is more peaked than what we would get from a pure diffusion, while on the opposite, $\kappa_\omega > 1$ corresponds to a front which is diffusing faster.

(2) In (2.3.2), the functional inequality:

$$\int_{\mathbb{R}} v\, \log\left(\frac{v}{u_\infty}\right)\, dx \le C(t) \int_{\mathbb{R}} \left| \frac{v_x}{v} - \frac{(u_\infty)_x}{u_\infty} \right|^2 v\, dx\,,$$

holds for any function v, for some $C(t) > 0$ such that $\lim_{t\to\infty} C(t) = \tau/2$. We conjecture that the optimal possible value of τ is $\tau = 2\kappa_\omega/\kappa_0$, at least for a large class of potentials ψ, but the question is still mathematically

open. Now, if $\kappa_\omega/\kappa_0 \geq 1$, then $\tau = 2\kappa_\omega/\kappa_0$ governs the rate of convergence in (2.3.2). If $\kappa_\omega/\kappa_0 < 1$, other terms of order e^{-t}, which means $O(1/\sqrt{t})$ in the original variables, would eventually dominate the convergence process. This last case is never observed numerically.

Theorem 2.3.1 (Main results on κ_ω) *We have*

$$\lim_{\omega\to 0} \kappa_\omega = \kappa_0 = \left(\int_0^1 e^{\psi}\, \mathrm{d}z \int_0^1 e^{-\psi}\, \mathrm{d}z\right)^{-1} < 1$$

and

$$\lim_{\omega\to\infty} \kappa_\omega = 1^+ .$$

As a consequence, $\omega \mapsto \kappa_\omega$ has a maximum, which is strictly bigger than 1, see Fig. 2.5.

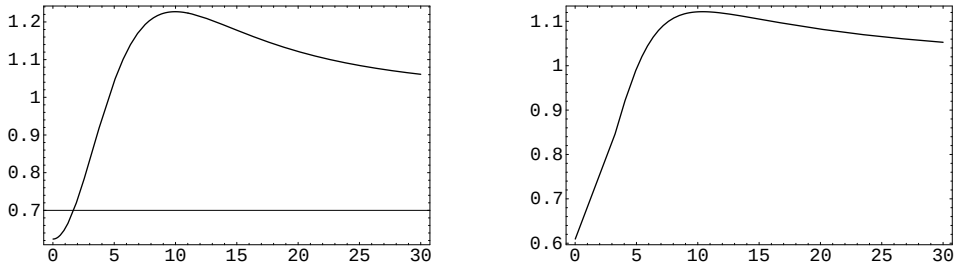

Figure 2.5 – *Plot of the diffusion coefficient κ_ω as a function of ω in the sinusoidal case (left) and in the smooth sawtooth potential case (right).*

We will able to prove by entropy methods that

Theorem 2.3.2 (Convergence with rate) *For any $\delta > 0$, there is $C_\delta = C_\delta(\psi, f_0) > 0$, such that*

$$\int_{\mathbb{R}} |u(t,x) - u_\infty(t,x)|\, \mathrm{d}x \leq C_\delta\, e^{-t/\tau}\,, \quad \forall t \geq 0$$

where

$$\tau \geq 2\kappa_\omega/\kappa_0 + \delta\,,$$

and $\kappa_0 = \left(\int_0^1 e^{\psi}\, dz \int_0^1 e^{-\psi}\, dz\right)^{-1}$.

Before discussing the entropy method and the associated homogeneised inequality let us discuss the physical parameters.

2.3.1 Measuring the efficiency of coherent transport

Measuring the efficiency of Brownian motors is a tricky issue. It requires specific tools. We may refer for instance to [133] for a recent reference in this direction, with some numerical simulations. Also see [122, Fig. 2] for an early result in the context of the stochastic Stokes' drift, [208] for recent simulations corresponding to a simple model, and [100] for detailed considerations on transport coherence and values of the Péclet number. In the very simple model considered in this paper, there are only few available parameters. As explained in [100, 208], the Péclet number Pe describes the

competition between the directional drift and the stochastic diffusion of the particle. It is defined, with our notations, by

$$\mathsf{Pe} := \frac{c_\omega\,\ell}{\kappa_\omega}$$

where ℓ is a typical length scale. One can easily check that this is a dimensionless number. Larger Pe number means that the drift predominates over diffusion and there is high transport coherence. In other words, the effective distribution is transported far away from the original data and stays peaked around its maximum value, at least when the variance is measured in the same units as the displacement of the center of mass.

In [100, 208], the typical length scale is the period of the potential ψ, that is $\ell = 1$ in our notations, and it is suggested that a criterion for efficient transport, preserving simultaneously the coherence of the distribution and optimizing its displacement, is $\mathsf{Pe} > 2$. This criterion does not make much sense for a study of the large time behavior, as the variance of the distribution, which is of diffusive nature and grows like $\sqrt{\kappa_\omega\,t}$, is always dominated by the displacement, which is linear and of the order of $c_\omega\,t$, when t is large. To remedy this, we can suggest the following analysis. We may first use the Péclet number to define a *characteristic length scale*

$$\mathrm{L} := \frac{\ell}{\mathsf{Pe}}\,.$$

Recall that with our notations, $\ell = 1$. The value of L corresponds to the displacement of the center of mass for which this displacement is equal to the variance of the effective distribution. If the effective distribution is originally centered at zero and evolves according to

$$h_t = \kappa_\omega\,h_{xx} + (x\,h)_x\,, \tag{2.3.2}$$

up to a translation at constant velocity c_ω, this occurs for $t = \mathrm{T}$ such that $\sqrt{\kappa_\omega\,\mathrm{T}} = c_\omega\,\mathrm{T} = \mathrm{L}$, and at that time, the percentage of the initial distribution which is still in the $x < 0$ region is given by $\int_{-\infty}^{0} \exp\big[-|x-\mathrm{L}|^2/(2\kappa_\omega \mathrm{T})\big]\,dx = \frac{1}{2}\mathrm{Erf}(1/\sqrt{2}) \approx 16\%$. See Fig 2.6.

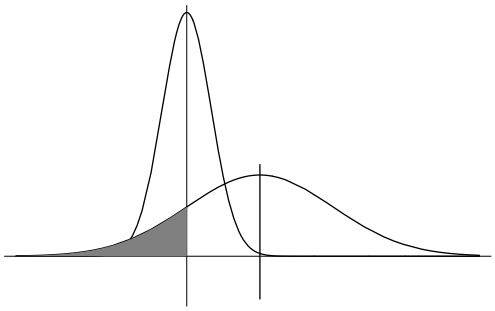

Figure 2.6 – *Definition of* L *and* T *can be understood as follow. If one starts with a Gaussian distribution centered at* $x = 0$ *and evolve it according to* (2.3.2), T *is the time for which the solution (centered at* L *in the above plot) has a variance equal to* L. *The grey area represents* 16% *of the area below the solution at time* $t = \mathrm{T}$.

Now we may observe that in the above discussion, we have also introduced a *characteristic time scale* $\mathrm{T} = \kappa_\omega/c_\omega^2$ which is related with the Péclet

number by the formula

$$\mathrm{T} = \frac{\ell}{c_\omega \,\mathsf{Pe}} \,.$$

It turns out that the stochastic Stokes' drift has a natural time scale, which is the time period of the potential $\mathrm{T}_0 := \ell/\omega$. Hence it is meaningful to consider

$$\mathrm{N} := \frac{\mathrm{T}}{\mathrm{T}_0} = \frac{\omega\,\kappa_\omega}{\ell\, c_\omega^2} = \frac{\omega}{c_\omega\,\mathsf{Pe}} \,,$$

which measures the time in takes to achieve the equality $\sqrt{\kappa_\omega\,\mathrm{T}} = c_\omega\,\mathrm{T}$ in natural units, and to define the efficiency of the transport by

$$\mathsf{E} := \frac{1}{\mathrm{N}} = \frac{\ell\, c_\omega^2}{\omega\,\kappa_\omega} = \mathsf{Pe}\,\frac{c_\omega}{\omega} \,.$$

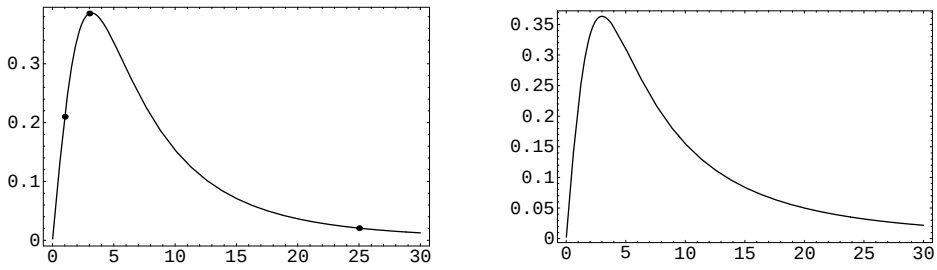

Figure 2.7 – *Plot of the efficiency* E *as a function of* ω *in the sinusoidal case (left) and in the smooth sawtooth potential case (right). We observe that in both cases, the maximum is extremely well defined. Dots (left) correspond* $(\omega, \mathsf{E}(\omega))$ *taking the values* $(1, 0.210)$, $(3, 0.385)$, $(25, 0.021)$ *and will be reused in Fig. 2.8.*

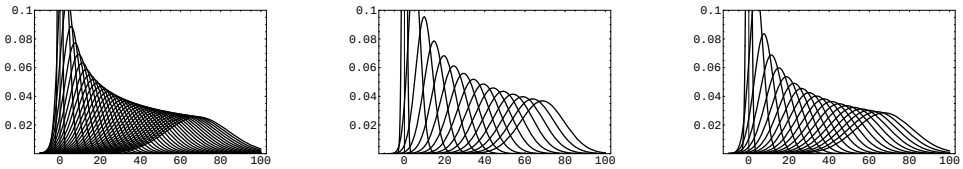

Figure 2.8 – *The effective profile* F_∞ *is represented for* ω *taking the values* 1, 3 *and* 25, *which correspond to the dots in Fig. 2.7 (left). Curves are plotted for* $\omega = 1$ *(left),* 3 *(center),* 25 *(right) for* $t = 0$, 5, 10, *etc, as long as* $c_\omega\, t \leq 70$. *The curve corresponding to* $\omega = 3$ *(center) is the most efficient, in the sense that* $c_\omega\, t \approx 70$ *is reached for a smaller value of* t *than for the other curves and the solution is kept more peaked. Computations are done in the case of the sinusoidal potential.*

The strength of our approach is that by our asymptotic expansion, we have been able to identify κ_ω and we have a formula which allows us to plot it precisely, see Fig. 2.5. The shapes of the curves $\omega \mapsto c_\omega$ and $\omega \mapsto \kappa_\omega$ combine well to define an optimum of the efficiency, characterized by the dimensionless number E, which reflects the idea of coherent transport.

2.3.2 Mobility and Einstein's relation

In the tilted ratchet picture, ω is proportional to the applied force F:

$$\omega = \frac{1}{\eta}\,\mathsf{F}$$

where η is the viscous friction coefficient, has the dimension of the inverse of a time, and takes value 1 in our units. The mobility is defined by

$$\mu_\omega := \frac{c_\omega}{\mathsf{F}} = \frac{c_\omega}{\eta\,\omega}$$

so we can write

$$\mathsf{E} = \mathsf{Pe}\,\frac{c_\omega}{\omega} = \eta\,\mu_\omega\,\mathsf{Pe}\,.$$

It has been argued that Einstein's relation

$$\frac{\kappa_\omega}{\mu_\omega} = \mathsf{k_B}\,\Theta$$

defines a notion of effective temperature Θ. Here $\mathsf{k_B}$ is Boltzmann's constant. As can be seen on Fig. 2.9, Θ is not constant in terms of ω.

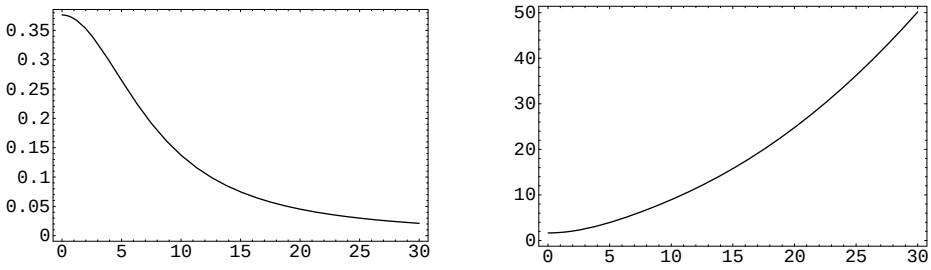

Figure 2.9 – *Plot of the mobility μ_ω (left) and the effective temperature Θ (measured in units of $1/\mathsf{k_B}$) of the tilted Brownian ratchet for values of ω ranging between 0 and 30 in the sinusoidal case.*

2.4 Homogeneisation of functional inequalities

In $\mathbb{R}^d \ni x$, define

$$\mu_0(x) := Z_0^{-1}\,e^{-|x|^2/2} \quad \text{where} \quad Z_0 = (2\pi)^{d/2}$$

and, for any $\varepsilon > 0$,

$$\mu_\varepsilon(x) := Z_\varepsilon^{-1}\,e^{-\psi(x/\varepsilon)}\,\mu_0(x) \quad \text{where} \quad Z_\varepsilon = \int_{\mathbb{R}^d} e^{-\psi(x/\varepsilon)}\,\mu_0(x)\,\mathrm{d}x\,.$$

For any $\varepsilon \geq 0$, to the measures $(\mu_\varepsilon)_{\varepsilon\geq 0}$, we associate the optimal Poincaré constant

$$\mathcal{C}_\varepsilon^{(2)} := \inf_{\substack{\int_{\mathbb{R}^d} u\,\mathrm{d}\mu_\varepsilon = 0 \\ 0 \neq u \in H^1(\,\mathrm{d}\mu_\varepsilon)}} \frac{\int_{\mathbb{R}^d} |\nabla u|^2\,\mathrm{d}\mu_\varepsilon}{\int_{\mathbb{R}^d} |u|^2\,\mathrm{d}\mu_\varepsilon}\,.$$

We can also define the optimal constant in the logarithmic Sobolev inequality by

$$\mathcal{C}_\varepsilon^{(1)} := \inf_{\substack{\nabla u \neq 0\ d\mu_\varepsilon \text{ a.e.} \\ u \in H^1(\,\mathrm{d}\mu_\varepsilon)}} \frac{\int_{\mathbb{R}^d} |\nabla u|^2\,\mathrm{d}\mu_\varepsilon}{\int_{\mathbb{R}^d} |u|^2 \log\left(\frac{|u|^2}{\int_{\mathbb{R}^d} |u|^2\,\mathrm{d}\mu_\varepsilon}\right)\,\mathrm{d}\mu_\varepsilon}$$

and the optimal constant in a family of generalised Poincaré inequalities

$$\mathcal{C}_\varepsilon^{(p)} := (p-1) \inf_{\substack{\nabla u \neq 0 \; d\mu_\varepsilon \text{ a.e.} \\ u \in H^1(\,\mathrm{d}\mu_\varepsilon)}} \frac{\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}\mu_\varepsilon}{\int_{\mathbb{R}^d} |u|^2 \,\mathrm{d}\mu_\varepsilon - \left(\int_{\mathbb{R}^d} |u|^{2/p} \,\mathrm{d}\mu_\varepsilon\right)^p}$$

where $p \in (1,2)$ is a parameter. Then we prove

Theorem 2.4.1 (Constants in the homogenisation limit) *Assume that ψ is a C^2 function in $\mathbb{T}^d := [0,1)^d$*

$$\forall p \in (1,2],\ \lim_{\varepsilon \to 0_+} \mathcal{C}_\varepsilon^{(p)} = \mathsf{K}\,\mathcal{C}_0^{(p)} \quad \textit{and} \quad \lim_{\varepsilon \to 0_+} \mathcal{C}_\varepsilon^{(1)} \in [\mathsf{k}\,\mathcal{C}_0^{(1)}, \mathsf{K}\,\mathcal{C}_0^{(1)}]$$

where

$$\mathsf{k} := \exp\left(-\|\psi\|_{L^{L^\infty(\mathbb{T}^d)}(\mathbb{R}^d)}\right)$$

and

$$\mathsf{K} := \frac{1}{\int_{\mathbb{T}^d} e^{\psi(y)} \,\mathrm{d}y \int_{\mathbb{T}^d} e^{-\psi(y)} \,\mathrm{d}y} \le 1\,.$$

2.5 Idea of the proof

The main tools of our approach are variational. We perform a detailed analysis of minimizing sequences. The difficulty comes from the fact that equality cases are sometimes achieved only by trivial functions, *e.g.* constant functions in the case of a Gaussian weight. E. Carlen and M. Loss proved in [63] that equality in the Euclidean logarithmic Sobolev inequalities, that is for Lebesgue's measure on $\mathbb{R}^d$, occurs for and only for Gaussian functions, which make simultaneously the entropy and the energy terms equal to zero. In some cases, this can also be seen as a consequence of the Bakry-Emery method, see [195], but this is not the case in the present framework. Hence, one has to carry a detailed analysis of the convergence and handle possible lacks of compactness.

Although not surprising from the point of view of homogenization theory, our estimates differ by several aspects of standard problems which have been abundantly treated in the literature. For instance, we deal with non compact domains, in functional spaces with oscillatory measures and determine sharp constants even in cases where there is no nontrivial solution of the Euler-Lagrange equations associated to the corresponding variational problem. As far as we know, tools of homogenization theory have not been used much in the framework of logarithmic Sobolev inequalities and semi-group theory. We think that this is an extremely interesting field with applications of large interest.

2.5.1 The entropy method

For any $p \in (1,2]$, introduce the entropy functional

$$\mathsf{E}_\varepsilon^{(p)}[u] := \frac{1}{p-1} \int_{\mathbb{R}^d} \left[\left(\frac{u}{u_\infty^\varepsilon}\right)^p - 1 - p\left(\frac{u}{u_\infty^\varepsilon} - 1\right)\right] u_\infty^\varepsilon \,\mathrm{d}x$$

and the entropy production:

$$\mathsf{I}_\varepsilon^{(p)}[u] := p \int_{\mathbb{R}^d} \left(\frac{u}{u_\infty^\varepsilon}\right)^{p-2} \left|\nabla\left(\frac{u}{u_\infty^\varepsilon}\right)\right|^2 u_\infty^\varepsilon \,\mathrm{d}x$$

If u^ε is a solution then

$$\frac{d}{dt}\,\mathsf{E}_{\mathsf{e}}^{(p)}[u^\varepsilon(t,\cdot)] = -\,\mathsf{I}_\varepsilon^{(p)}[u^\varepsilon(t,\cdot)]\;.$$

Recall the Csiszár-Kullback inequality: there exists a constant $C > 0$ such that

$$\|u^\varepsilon(t,\cdot) - u_\infty^\varepsilon\|_{L^p\left(\mathbb{R}^d,(u_\infty^\varepsilon)^{1-p}dx\right)} \le C\,\mathsf{E}_\varepsilon^{(p)}[u^\varepsilon(t,\cdot)]$$

Lemma 2.5.1 (Entropy/entropy production inequality) *If we assume that $M = 1$ then*

$$\frac{4}{p}\,\mathcal{C}_\varepsilon^{(p)}\,\mathsf{E}_\varepsilon^{(p)}[u] \le \mathsf{I}_\varepsilon^{(p)}[u]$$

Upper estimate Define $u_{\mathsf{e}}(x) = x\cdot\mathsf{e}$ with $\mathsf{e}\in S^{d-1}$.

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} |u_{\mathsf{e}}|^2\,\mathrm{d}\mu_\varepsilon = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} |\nabla u_{\mathsf{e}}|^2\,\mathrm{d}\mu_\varepsilon = 1\;,\\
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} |u_{\mathsf{e}}|^{2/p}\,\mathrm{d}\mu_\varepsilon = \frac{2^{1/p}}{\sqrt{\pi}}\,\Gamma\left(\frac12+\frac1p\right)\;,\\
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} |u_{\mathsf{e}}|^2\,\log|u_{\mathsf{e}}|^2\,\mathrm{d}\mu_\varepsilon = \log 2 - 2 + \gamma \approx -0.729637
\end{aligned}$$

where $\gamma\approx 0.577216$ is Euler's constant. The function

$$\kappa(p) := \frac{p-1}{1-\frac{2^{1/p}}{\sqrt{\pi}}\,\Gamma\left(\frac12+\frac1p\right)}\;,\quad p\in(1,2)\;,$$

is an increasing function on $(1,2)$ such that $\lim_{p\to 1_+} 1/\kappa(p) = -\log 2 + 2 - \gamma \approx 1.37054$ and $1/\kappa(2) = 1-\sqrt{2/\pi}\approx 4.94767$.

Lemma 2.5.2 (Upper estimate) *Assume that $\psi\in L^\infty(\mathbb{T}^d)$. For any $p\in[1,2]$*

$$\lim_{\varepsilon\to 0}\mathcal{C}_\varepsilon^{(p)} \le \kappa(p)\;.$$

Lower estimate Assume that for some probability measure $d\mu$, the following convex Sobolev inequality holds

$$\int \left[\varphi(u) - \varphi(\bar u) - \varphi'(\bar u)(u-\bar u)\right] d\mu \le \mathcal{C}_\varphi \int \varphi''(u)|\nabla u|^2\,d\mu \quad \forall\, u\in H^1(d\mu)\;.$$

Here $\bar u := \int u\,d\mu$. Assume that $d\tilde\mu$ there exist constants a, $b\in\mathbb{R}$ such that

$$e^{-b}\,d\mu \le d\tilde\mu \le e^{-a}\,d\mu \quad \mu\text{ a.e.}$$

Lemma 2.5.3 (Holley-Stroock, 1987) *If φ is a strictly convex C^3 function, then for all $u\in H^1(d\mu)$*

$$\int \left[\varphi(u) - \varphi(\tilde u) - \varphi'(\tilde u)(u-\tilde u)\right] d\tilde\mu \le e^{b-a}\,\mathcal{C}_\varphi \int \varphi''(u)\,|\nabla u|^2\,d\tilde\mu\;.$$

where $\tilde u := \int u\,d\tilde\mu / \int d\tilde\mu$.

Applied to the case $\varphi(u) = \frac{u^p - 1 - p(u-1)}{p-1}$.

Corollary 2.5.4 (Lower estimate) *With the above notations, if ψ is bounded on $\mathbb{T}^d$, then for any $p \in [1,2]$,*

$$\mathcal{C}_\varepsilon^{(p)} \geq \frac{p}{2}\, e^{-\|\psi\|_{L^\infty(\mathbb{R}^d)}}\,.$$

The Poincaré inequality We first need to remind two standard results of homogeneisation:

Proposition 2.5.5 (Two-scale convergence (L^2-case)) *Let Ω be an open set in $\mathbb{R}^d$. If $(u_\varepsilon)_{\varepsilon>0}$ is a bounded sequence in $L^2(\Omega)$, then there exists a sub-sequence of $(u_\varepsilon)_{\varepsilon>0}$, still denoted by $(u_\varepsilon)_{\varepsilon>0}$, and a function $u_0 \in L^2(\Omega \times \mathbb{T}^d)$ such that*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\, \varphi\left(x, \frac{x}{\varepsilon}\right)\, \mathrm{d}x = \int\int_{\Omega\times\mathbb{T}^d} u_0(x,y)\, \varphi(x,y)\, \mathrm{d}x\, \mathrm{d}y\,,$$

for all smooth φ which is periodic in y. We say that u_ε two-scale converges to u_0. Moreover, $(u_\varepsilon)_{\varepsilon>0}$ weakly converges in $L^2(\Omega)$ to

$$u_*(x) := \int_{\mathbb{T}^d} u_0(x,y)\, \mathrm{d}y\,.$$

Proposition 2.5.6 (Two-scale convergence (H^1-case)) *Let Ω be an open set in $\mathbb{R}^d$ and consider a sequence $(u_\varepsilon)_{\varepsilon>0}$ which weakly converges to u_* in $\mathcal{H}^1(\Omega)$. Then there exist a sub-sequence of $(u_\varepsilon)_{\varepsilon>0}$, still denoted $(u_\varepsilon)_{\varepsilon>0}$, which two-scale converges to u_*. Moreover, there exists a function $u_1 \in \mathrm{L}^2\left(\Omega, \mathcal{H}^1(\mathbb{T}^d)\right)$ such that $(\nabla u_\varepsilon)_{\varepsilon>0}$ two-scale converges to $(x,y) \mapsto \nabla_x u_*(x) + \nabla_y u_1(x,y)$.*

Now we can prove the Poincaré inequality.

For any $\varepsilon > 0$, let u_ε be a non-trivial minimiser to $\mathcal{C}_\varepsilon^{(2)}$ such that $\int_{\mathbb{R}^d} u_\varepsilon\, \mathrm{d}\mu_\varepsilon = 0$, $\int_{\mathbb{R}^d} |u_\varepsilon|^2\, \mathrm{d}\mu_\varepsilon = 1$ and

$$-\nabla \cdot \left(e^{-\frac{1}{2}|x|^2 - \psi(x/\varepsilon)}\, \nabla u_\varepsilon(x)\right) = \mathcal{C}_\varepsilon^{(2)}\, u_\varepsilon(x)\, e^{-\frac{1}{2}|x|^2 - \psi(x/\varepsilon)}\,.$$

Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi_1 \in \mathcal{D}(\mathbb{R}^d, C^\infty(\mathbb{T}^d))$. We have

$$\begin{aligned}\int_{\mathbb{R}^d} \nabla_x u_\varepsilon \left[\nabla_x \varphi(x) + \varepsilon\, \nabla_x \varphi_1\left(x, \frac{x}{\varepsilon}\right) + \nabla_y \varphi_1\left(x, \frac{x}{\varepsilon}\right)\right]\, \mathrm{d}\mu_\varepsilon \\ = \mathcal{C}_\varepsilon^{(2)} \int_{\mathbb{R}^d} u_\varepsilon \left[\varphi(x) + \varepsilon\, \varphi_1\left(x, \frac{x}{\varepsilon}\right)\right]\, \mathrm{d}\mu_\varepsilon\end{aligned}$$

Taking the limit $\varepsilon \to 0_+$, we obtain a two-scale homogenised equation:

$$\begin{aligned}\int\int_{\mathbb{R}^d\times\mathbb{T}^d} \left[\nabla_x u_*(x) + \nabla_y u_1(x,y)\right] \left[\nabla_x \varphi(x) + \nabla_y \varphi_1(x,y)\right] e^{-\frac{1}{2}|x|^2 - \psi(y)}\, \mathrm{d}x\, \mathrm{d}y \\ = \mathcal{K}_0^{(2)} \int\int_{\mathbb{R}^d\times\mathbb{T}^d} u_*(x)\, \varphi(x)\, e^{-\frac{1}{2}|x|^2 - \psi(y)}\, \mathrm{d}x\, \mathrm{d}y\,.\end{aligned}$$

where $\mathcal{K}_0^{(2)} := \lim_{\varepsilon\to 0_+} \mathcal{C}_\varepsilon^{(2)}$.

We test the above equation with $\varphi = 0$:

$$\nabla_y \cdot \left[e^{-\psi(y)} \left(\nabla_y u_1(x,y) + \nabla_x u_*(x)\right)\right] = 0\,,$$

Exactly as in the introduction, this amounts to write that

$$u_1(x,y) = \nabla_x u_*(x) \cdot w(y)$$

where $w = (w_j)_{j=1}^d$ is the solution to the *cell equation*, *i.e.*

$$\nabla_y u_1(x,y) = \left[\frac{e^\psi}{\int_{\mathbb{T}^d} e^{\psi(y)}\, dy} - 1 \right] \nabla_x u_*(x) \, .$$

We test now the above equation with $\varphi = u_*$:

$$\int_{\mathbb{R}^d} \frac{|\nabla_x u_*|^2}{\int_{\mathbb{T}^d} e^{\psi(y)}\, dy} \, \mathrm{d}\mu_0 = \mathcal{K}_0^{(2)} \int_{\mathbb{R}^d} |u_*(x)|^2 \, \mathrm{d}\mu_0 \, .$$

Observe that $\int_{\mathbb{R}^d} u_* \, \mathrm{d}\mu_0 = \lim_{\varepsilon \to 0_+} \int_{\mathbb{R}^d} u_\varepsilon \, \mathrm{d}\mu_\varepsilon = 0$.

Altogether this proves that

$$\mathcal{K}_0^{(2)} \geq \frac{\mathcal{C}_0^{(2)}}{\int_{\mathbb{T}^d} e^{\psi(y)}\, dy} = \mathsf{K}\, \mathcal{C}_0^{(2)} \, .$$

If we use $\tilde{u}_\varepsilon(x) := u_{\mathsf{e}}(x) + \varepsilon \, \nabla_x u_{\mathsf{e}}(x) \, w\left(\frac{x}{\varepsilon}\right)$ we find that

$$\mathcal{K}_0^{(2)} \leq \lim_{\varepsilon \to 0_+} \frac{\int_{\mathbb{R}^d} |\nabla \tilde{u}_\varepsilon|^2 \, \mathrm{d}\mu_\varepsilon}{\int_{\mathbb{R}^d} |\tilde{u}_\varepsilon|^2 \, \mathrm{d}\mu_\varepsilon - \left(\int_{\mathbb{R}^d} \tilde{u}_\varepsilon \, \mathrm{d}\mu_\varepsilon \right)^2} = \mathsf{K}\, \mathcal{C}_0^{(2)} \, .$$

2.5.2 Generalised Poincaré inequalities

Using similar arguments we can prove

Proposition 2.5.7 *Let ψ be a continuous function on $\mathbb{T}^d$ and take $p \in (1,2)$, $\varepsilon > 0$. Then, with the above notations, either $\mathcal{C}_\varepsilon^{(p)} \leq \frac{p}{2}\, \mathcal{C}_\varepsilon^{(2)}$ is achieved by some non trivial function, or $\mathcal{C}_\varepsilon^{(p)} = \frac{p}{2}\, \mathcal{C}_\varepsilon^{(2)}$ is not achieved by any non trivial function.*

2.6 Convergence with rate

Let us now come back to our analysis. Let

$$\mathsf{U}(t,x;z) := \\ \frac{1}{Z(t)} \left[g_\omega(z)\, h(t,x) + \frac{1}{R}\, g_\omega^{(1)}(z)\, h_x(t,x) + \frac{1}{R^2}\, \mathsf{V}\left(\tfrac{1}{2}\,(R^2-1), R\, x; z \right) \right]$$

where g_ω, $g_\omega^{(1)}$ and h are defined as above. The coefficient $Z(t)$ is determined in such a way that, for any $t > 0$,

$$\int_{\mathbb{R}} \mathsf{U}(t,x; R\, x - \tfrac{1}{2}\,(R^2-1)\, A(\omega))\, dx = 1$$

and V is a solution which cancels the R^2-term

$$\mathsf{V}_t = \mathsf{V}_{xx} + \mathsf{V}_{zz} + \omega\, \mathsf{V}_z + \left(\psi'\, \mathsf{V} \right)_z + \Big(2\, \mathsf{V}_z + \left(\psi' + c_\omega \right) \mathsf{V} \Big)_x \\ + \Big[(1 - \kappa_\omega)\, g_\omega + 2\, (g_\omega^{(1)})_z + \left(\psi' + c_\omega \right)\, g_\omega^{(1)} \Big]\, h_{xx} \, .$$

Denoting by F the error we find

$$-R^2\,L_0U - R\,L_1U - L_2U =: -\frac{\dot Z}{Z}\,\mathsf{U} + \frac{1}{Z}\,\mathsf{F}\,.$$

Let us define

$$u_\infty(t,x) := \mathsf{U}\left(t, x; e^t\,x - \tfrac12\,(e^{2t}-1)\,A(\omega)\right)$$
$$\text{and}\quad \mathsf{f}(t,x) := \mathsf{F}\left(t, x; e^t\,x - \tfrac12\,(e^{2t}-1)\,A(\omega)\right).$$

Let u be a solution of (2.3.1). Then

$$\frac{d}{dt}\int_{\mathbb{R}} u\,\log\left(\frac{u}{u_\infty}\right)\,dx = -\int_{\mathbb{R}}\left|\left(\log\left(\frac{u}{u_\infty}\right)\right)_x\right|^2 u\,dx + \frac{\dot Z}{Z} + e^{-t}\int_{\mathbb{R}}\mathsf{f}\,\frac{u}{u_\infty}\,dx$$

Proposition 2.6.1 (Logarithmic Sobolev inequality) *There exists a function $t\mapsto C(t)$, with $\lim_{t\to\infty} C(t) = \mathtt{k}/2$, which is positive, finite for any $t>0$, such that, for any $u\in L^1(\mathbb{R})$,*

$$\int_{\mathbb{R}} u\,\log\left(\frac{u}{u_\infty}\right)\,dx \le C(t)\int_{\mathbb{R}}\left|\left(\log\left(\frac{u}{u_\infty}\right)\right)_x\right|^2 u\,dx\,.$$

where $\kappa_\omega/\mathsf{K} \le \mathtt{k} \le \kappa_\omega\,\max_{[0,1]} g_\omega\cdot\left(\min_{[0,1]} g_\omega\right)^{-1}$ where $\mathsf{K}^{-1} = \int_0^1 g_\omega\,dz\int_0^1 g_\omega^{-1}\,dz$, and $\lim_{\omega\to0}\mathsf{K}/\kappa_\omega = 1$.

We compute

$$\limsup_{t\to+\infty}\frac{\dot Z(t)}{Z(t)} < C\,e^{-t}\,.$$

Next we obtain the result if we assume that u is a solution of (2.3.1)

$$\limsup_{t\to+\infty}\int_{\mathbb{R}}|x|^4 u(t,x)\,dx < \infty\,.$$

Such a restriction was shnown not to be necessary in a recent work [83].

2.7 Concluding remarks

Our results are based on a very simple model, but show how to compute analytically and numerically various quantities which are not easy to obtain by direct Monte-Carlo simulations. The main difficulty comes from the oscillatory behavior of the potential, which is very clear in self-similar variables, and results in highly non-trivial attractors. Mathematically, this can be handled with the tools of homogenization theory, which provide an equation for the macroscopic profile and formulae for the two main parameters, the speed c_ω of the center of mass (or drift velocity) and the effective diffusion coefficient κ_ω. This should not hide a major mathematical difficulty: the time t is not independent of the small parameter in the homogenization approach, namely $1/\sqrt{t}$, in the original variables. Moreover, several length scales have to be taken into account. The position of the center of mass is of the order of t, while the typical size of the front grows like $\sqrt{t}$. Typical relaxation rates are exponential at small scale, but of the order of $t^{-1/\tau}$ or

$1/\sqrt{t}$ when measured globally. Hence asymptotic expansions are not at all easy to handle even at a formal level and quite hard to justify. The way out for such difficulties is the homogenized logarithmic Sobolev inequality, with its own difficulties. The inequality anyway has the very nice feature of connecting the rates of convergence with κ_ω, something which definitely should be further investigated from a numerical point of view.

However, knowing c_ω and κ_ω accurately is a major step in the understanding of the asymptotic behavior of the solutions of the stochastic Stokes' drift. It gives solid grounds to a notion of effective diffusion. A striking consequence is that it gives a new criterion for measuring the efficiency of coherent transport using the number E.

We hope that our contribution will contribute to more realistic models from a theoretical point of view and will be used for benchmarking the numerous simulations that are being performed mostly with Monte-Carlo approaches.

Concerning the functional inequalities, the case $p = 1$ corresponding to the logarithmic Sobolev inequality is not completely understood. As far as we know, it is an open question to determine whether $\lim_{\mathrm{e}\to 0_+} \mathcal{C}_{\mathrm{e}}^{(1)} = \mathsf{K}\,\mathcal{C}_0^{(1)}$ or not.

Part II

Entropy methods and functional inequalities

Asymptotics of the fast diffusion equation

3

We consider non-negative solutions of the fast diffusion equation $u_t = \Delta u^m$ with $m \in (0,1)$, in the Euclidean space $\mathbb{R}^d$, $d \geq 3$, and study the asymptotic behaviour of a natural class of solutions, in the limit corresponding to $t \to \infty$ for $m \geq m_c = (d-2)/d$, or as t approaches the extinction time when $m < m_c$. For a class of initial data we prove that the solution converges with a polynomial rate to a self-similar solutions, for t large enough if $m \geq m_c$, or close enough to the extinction time if $m < m_c$. Such results are new in the range $m \leq m_c$ where previous approaches fail. In the range $m_c < m < 1$ we improve on known results.

The proof relies on a systematic study of weighted Poincaré type inequalities which are closely connected with Hardy type inequalities and establish the form of the optimal constants in some cases. Such inequalities are then used to relate entropy with entropy production.

This chapter is a summary of:

- M. Agueh, A. Blanchet, and J. A. Carrillo, *Large time asymptotics of the doubly nonlinear equation in the non-displacement convexity regime*, Journal of Evolution Equations, 10 (2010), pp. 59–84.
- A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, and J.-L. Vázquez, *Hardy-Poincaré inequalities and applications to nonlinear diffusions*, C. R. Math. Acad. Sci. Paris, 344 (2007), pp. 431–436.
- ——, *Asymptotics of the fast diffusion equation via entropy estimates*, Archive for Rational Mechanics and Analysis, 191 (2009), pp. 347–385.

3.1 Introduction

We study the Cauchy problem for the fast diffusion equation posed in the whole Euclidean space, that is, we consider the solutions $u(\tau, y)$ of

$$\begin{cases} \partial_\tau u = \Delta u^m \\ u(0,\cdot) = u_0\,, \end{cases} \tag{3.1.1}$$

where $m \in (0,1)$ (which means fast diffusion) and $(\tau, y) \in (0,T) \times \mathbb{R}^d$ for some $T > 0$. We consider non-negative initial data and solutions. Existence and uniqueness of weak solutions of this problem with initial data in $L^1_{\rm loc}(\mathbb{R}^d)$ was first proved by M.A. Herrero and M. Pierre in [115]. In the whole space, the behaviour of the solutions is quite different in the parameter ranges $m_c < m < 1$ and $0 < m < m_c$, the critical exponent being defined as

$$m_c := \frac{d-2}{d}\,.$$

Note that $m_c > 0$ only if $d \ge 3$, so that the lower range does not exist for $d = 1, 2$. For $m > m_c$ the mass $\int_{\mathbb{R}^d} u(y,t)\,dy$ is preserved in time if the initial datum u_0 is integrable in $\mathbb{R}^d$. Besides, non-negative solutions are positive and smooth for all $x \in \mathbb{R}^d$ and $t > 0$. On the contrary, solutions may extinguish in finite time in the lower range $m < m_c$, for instance when the initial data is in $L^{p_*}(\mathbb{R}^d)$ with $p_* = d\,(1-m)/2$: then there exists a time $T > 0$ such that

$$\lim_{\tau \nearrow T} u(\tau, y) = 0\,.$$

Many computations are however similar in both ranges, from an algebraic point of view. We refer to the monograph [198] for a detailed discussion of the existence theory and references to the subject. The extension to exponents $m \le 0$ is also treated, and it is natural but it will not be the focus of this paper.

In the last two decades, special attention has been given to the study of large time asymptotics of these equations, starting with the pioneering work of A. Friedman and S. Kamin [101] and completed in [197], when m is in the range (m_c, ∞). In those studies the class of non-negative, finite mass solutions are considered. Asymptotic stabilisation towards *self-similar asymptotic solutions* known as Barenblatt solutions is shown. For $m_c < m < 1$, such solutions take the form:

$$U_{D,T}(\tau, y) := \frac{1}{R(\tau)^d}\left(D + \frac{1-m}{2\,m}\left|\frac{y}{R(\tau)}\right|^2\right)^{-\frac{1}{1-m}} \tag{3.1.2}$$

with $R(\tau) := \left[d\,(m-m_c)\,(\tau+T)\right]^{\frac{1}{d\,(m-m_c)}}$. Here $D, T \ge 0$ are free parameters. While the second parameter means a time displacement and does not play much role in the asymptotic behaviour, the first does and can be computed from the mass of the solution. The value m_c is the critical exponent below which the Barenblatt solutions cease to exist in this standard form.

Here, we are mainly interested in addressing the question of the asymptotic behaviour of (3.1.1) when $0 < m < m_c$. We consider a wide class of

solutions which vanish in finite time T and describe their behaviour as τ goes to T. We point out that our methods allow to treat simultaneously the ranges $0 < m < m_c$ and $m_c \le m < 1$, in which one is interested in the behaviour of the solutions as τ goes to infinity. For this purpose, we extend the Barenblatt solutions to the range $0 < m < m_c$ with the same expression (3.1.2), but a different form for R, that is

$$R(\tau) := \left[d\,(m_c - m)\,(T - \tau) \right]^{-\frac{1}{d\,(m_c - m)}} .$$

The parameter T now denotes the extinction time. Following [198], we shall call such solutions the *pseudo-Barenblatt solutions*. Notice that Barenblatt and pseudo-Barenblatt solutions $U_{D,T}$, with D, $T > 0$, are such that $U_{D,T}^p$ is integrable if and only if $p > p_*$ (p_* is defined above, and $p_* > 1$ means $m < m_c$). Consistently with the above choices, for $m = m_c$, one has to choose $R(\tau) := e^{\tau + T}$ with free parameter T, see [198], in order to obtain pseudo-Barenblatt solutions; then, $p_* = 1$.

The family of Barenblatt (respectively pseudo-Barenblatt) solutions represents the asymptotic patterns to which many other solutions converge for large times if $m > m_c$ (respectively as t goes to T if $0 < m < m_c$). We are interested in the class of solutions for which such a convergence takes place and in the rates of convergence. Both questions strongly depend on m. Let us emphasise for instance that the Barenblatt solution $U_{D,T}$ is integrable in y for $m > m_c$, while the pseudo-Barenblatt solution corresponding to $m \le m_c$ is not integrable. Since much is known in the case $m > m_c$, see for instance [72, 87] and [47, 66, 68, 71, 94, 141, 197] for more complete results, the main novelty of our paper is concerned with the lower range $m \le m_c$, which has several interesting new features. For instance, in the analysis in high space dimensions, that is $d > 4$, another critical exponent appears,

$$m_* := \frac{d-4}{d-2} < m_c .$$

A key property of m_* is that the difference of two pseudo-Barenblatt solutions is integrable for $m \in (m_*, m_c)$, while it is not integrable for $m \in (0, m_*]$.

The convergence towards Barenblatt and pseudo-Barenblatt solutions is subtle since the solutions converge to zero everywhere. To capture the asymptotic profiles, it is therefore convenient to rescale the solutions and replace the study of intermediate asymptotics by the study of the convergence to stationary solutions in *rescaled variables*,

$$t := \log\left(\frac{R(\tau)}{R(0)}\right) \quad \text{and} \quad x := \frac{y}{R(\tau)} , \tag{3.1.3}$$

with R as above. In these new variables, if u is a solution to (3.1.1), the function

$$v(t,x) := R(\tau)^d\, u(\tau, y)$$

solves a nonlinear *Fokker-Planck type equation*,

$$\begin{cases} \partial_t v(t,x) = \Delta v^m(t,x) + \nabla \cdot (x\, v(t,x)) & (t,x) \in (0,+\infty) \times \mathbb{R}^d , \\ v(0,x) = v_0(x) & x \in \mathbb{R}^d . \end{cases} \tag{3.1.4}$$

The initial data for (3.1.1) and for the rescaled equation (3.1.4) are related by

$$u_0(y) = R(0)^{-d}\, v_0(y/R(0))\ ,$$

where $R(0) = \left[d\,|m-m_c|\,T\right]^{\frac{1}{d\,(m-m_c)}}$ only depends on T. In this formulation, the Barenblatt and pseudo-Barenblatt solutions are transformed into stationary solutions given by

$$V_D(x) := \left(D + \frac{1-m}{2\,m}\,|x|^2\right)^{-\frac{1}{1-m}} \tag{3.1.5}$$

where $0 < m < 1$ and $D > 0$ is a free parameter. With a straightforward abuse of language, we say that V_D is a *Barenblatt profile*, including the case $m \le m_c$. The value $D = 0$ can also be admitted as a limit case, but the corresponding solution is singular at $x = 0$. See [198] for more details. The parameter T has disappeared from the new problem, but it enters in the change of variables. Note that in all cases, t runs from 0 to infinity in these rescaled variables.

3.2 Motivation

The *generalised entropy functional*, or *free energy functional*, is defined as

$$\mathcal{E}[v] := \int_{\mathbb{R}^d} \left[\varphi(v) + \frac{1}{2}\,|x|^2\,v\right] \mathrm{d}x \quad \text{where} \quad \varphi(v) := \frac{v^m}{m-1}\ .$$

It is then observed that the free energy of the Barenblatt profiles, *cf.* [69, 87], becomes infinite if $m \le m_0$, where $m_0 := d/(d+2) \in (m_c, m_1)$. In order to avoid this difficulty, it is convenient to work with the *relative entropy* of v with respect to V_D defined as follows:

$$\mathcal{E}[v|V_D] := \int_{\mathbb{R}^d} \left[\varphi(v) - \varphi(V_D) - \varphi'(V_D)\,(v - V_D)\right] \mathrm{d}x\ .$$

The *relative entropy* is the key tool of our analysis. It is such that $\mathcal{E}[v|V_D] := \mathcal{E}[v] - \mathcal{E}[V_D]$ if $m \in (m_0, 1)$ and $\int_{\mathbb{R}^d} v\,\mathrm{d}x = \int_{\mathbb{R}^d} V_D\,\mathrm{d}x$, that is for $D = D_*$. The functional $\mathcal{E}[v|V_{D_*}]$ can also be defined for $m \le m_0$. By homogeneity of φ, we can indeed rewrite it as

$$\mathcal{E}[v|V_{D_*}] := \int_{\mathbb{R}^d} \left[\varphi(w) - \varphi(1) - \varphi'(1)\,(w-1)\right] V_{D_*}^m\,\mathrm{d}x \quad \text{with} \quad w = \frac{v}{V_{D_*}}\ .$$

This makes clear why it is well defined at least for w close enough to 1 as $|x| \to \infty$. The functional $v \mapsto \mathcal{E}[v|V_{D_*}]$ is convex and achieves its minimum, 0, for $v = V_{D_*}$. If v is a solution of (3.1.4), the *entropy production term* takes the form

$$-\frac{d}{dt}\mathcal{E}[v(t)|V_{D_*}] = \mathcal{I}[v(t)|V_{D_*}]\ ,$$

where the functional

$$v \mapsto \mathcal{I}[v|V_D] := \int_{\mathbb{R}^d} v\left|\nabla\varphi'(v) - \nabla\varphi'(V_D)\right|^2 \mathrm{d}x$$

will be called *the relative Fisher information*. For any $m \in [m_1, 1)$, $\mathcal{E}[v|V_{D_*}] \le \frac{1}{2}\,\mathcal{I}[v|V_{D_*}]$ holds for any smooth function v and the inequality

is nothing else than the optimal Gagliardo-Nirenberg inequality, for which equality is achieved precisely by the Barenblatt profiles, see [87]. In such a case,

$$\mathcal{E}[v(t)|V_{D_*}] \le \mathcal{E}[v_0|V_{D_*}]\, e^{-2t} \quad \forall\, t \ge 0\,.$$

The limit case $m = m_1$ corresponds to the critical Sobolev inequality whose optimal form was established by T. Aubin and G. Talenti in [10, 194], while in the limit $m \to 1$ one recovers Gross' logarithmic Sobolev inequality, see [108, 87]. For $m \in [m_1, 1)$, F. Otto in [167] noticed that (3.1.4) can be interpreted as the gradient flow of the free energy with respect to the Wasserstein distance. The exponent $m = m_1$ is the limit case for which the *displacement convexity* property holds true.

Pushing the method to the case $0 < m < m_1$ requires the use of the relative entropy in place of the free energy. The method applies only to a class of initial data which have a finite relative entropy with respect to some Barenblatt profile V_{D_*} and satisfy convenient bounds. Mass can be finite in the case $m \in (m_c, m_1)$, which was the framework of some earlier studies, see [68, 72], or infinite if $m \in (0, m_c)$. Two Barenblatt profiles V_{D_0} and V_{D_1} have finite relative entropy, *i.e.* $\mathcal{E}[V_{D_1}|V_{D_0}] < \infty$ if and only if either $d \le 4$, or $d \ge 5$ and $m > m_*$, $m_* = (d-4)/(d-2)$. Hence, for $d \ge 5$, $m = m_*$ is a threshold not only for defining the relative mass of two pseudo-Barenblatt solutions, but also for defining their relative entropies or for the integrability of $V_{D_*}^{2-m}$. Note that $m_* < m_c$ for all $d \ge 5$. The proof amounts to prove that the relative entropy $\mathcal{E}[v|V_{D_*}]$ decays in time and converges to 0 at an exponential rate when $t \to \infty$. For $m > \min\{0, m_*\}$, $\mathcal{E}[v|V_{D_*}]$ is well defined under condition (H1'). For $m < m_*$, an additional restriction is required, which is precisely the purpose of (H2').

Our approach of course covers the case $m \ge m_c$ and we recover some of the results found in [68, 72]. Some of our results can also be extended to the range $m < 0$, but additional technical complications arise, which are still to be studied. In this paper, we leave apart several interesting questions, like the precise study of the case of $m = m_*$ or the equation $u_t = \Delta \log u$ in dimension $d \ge 2$, see *e.g.* [84, 85, 182, 200], which is the natural limiting equation to study in the limit $m \to 0$. Also see [118, 119, 120] for results which seem closely related to ours, and [90] in the case $m = (d-2)/(d+2)$. In particular we do not use the Bakry-Emery method introduced in [12], on which the results of [69, 66, 141, 68, 72] are based. We prove a conservation of relative mass, which allows us to remove the limitation $m > m_c$. Neither mass transportation techniques nor Wasserstein distance are needed, although the approach of is not unrelated, see [44, 13, 14, 158].

3.3 Main results

For $0 < m < m_c$, assume that u extinguish in finite time T. The analysis of the large time behaviour is better seen in self-similar variables: Define the rescaled function v by

$$v(t,x) := R^d(\tau)\, u(\tau, y) \quad \text{where } t := \log\left(\frac{R(\tau)}{R(0)}\right) \text{ and } x := \frac{y}{R(\tau)}$$

with

$$\begin{cases} R(\tau) := \left[d\,(m-m_c)\,(T+\tau)\right]^{\frac{1}{d\,(m-m_c)}} & \text{if } m_c < m < 1, \\ R(\tau) := \mathrm{e}^{T+\tau} & \text{if } m_c = m\,, \\ R(\tau) := \left[d\,(m_c-m)\,(T-\tau)\right]^{-\frac{1}{d\,(m_c-m)}} & \text{if } 0 < m < m_c. \end{cases}$$

The function v is solution to the *non-linear Fokker-Planck equation*:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \nabla \cdot [\nabla(v^m) + x\,v] & \text{in } (0,+\infty)\times\mathbb{R}^d, \\ v(0,\cdot) = v_0 = R(0)^d u_0(\cdot\,R(0)) & \text{in } \mathbb{R}^d\,, \end{cases} \tag{NLFP}$$

Remark: T disappeared from the equation but is still in the change of variable.
Remark: The stationary solution is the (pseudo)-Barenblatt solution:

$$V_D(x) := \left(D + \frac{1-m}{2\,m}\,|x|^2\right)^{-\frac{1}{1-m}}$$

where D is a free parameter.

On these solutions we assume the following:

Assumption 3.3.1. (H1) v_0 is a non-negative function in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and there exist positive constants $D_0 > D_1$ such that

$$V_{D_0}(x) \le v_0(x) \le V_{D_1}(x) \quad \forall\, x \in \mathbb{R}^d\,.$$

(H2) Let $m_* := (d-4)/(d-2)$. If $m \in (0, m_*]$, there exist $D_* \in [D_1, D_0]$ and $f \in L^1(\mathbb{R}^d)$ such that

$$v_0(x) = V_{D_*}(x) + f(x) \quad \forall\, x \in \mathbb{R}^d\,.$$

We can now state the convergence of $v(t)$ towards a unique Barenblatt profile:

Theorem 3.3.2 (Convergence without rate) *Consider v the solution of (3.1.4) with initial data satisfying* (H1)-(H2).

$$\lim_{t\to\infty} \|v(t) - V_{D_*}\|_{\mathrm{L}^p(\mathbb{R}^d)} = 0 \quad \textit{for any } p \in (1,\infty]\,.$$

Moreover,

$$\lim_{t\to\infty} \left\| \frac{v(t)}{V_{D_*}} - 1 \right\|_{\mathrm{L}^q(\mathbb{R}^d)} = 0 \quad \textit{for any } q \in \left(\frac{d}{2}, \infty\right]\,.$$

This convergence occurs with a rate if t is large enough:

Theorem 3.3.3 (Convergence with rate) *Under the assumptions of Theorem 3.3.2, if $m \neq m_*$, there exists $t_0 \ge 0$ such that there exists a positive constant C such that*

$$\|v(t) - V_{D_*}\|_{\mathrm{L}^2(\mathbb{R}^d)} \le C\,\mathrm{e}^{-\lambda_{m,d}\,t} \quad \forall\, t \ge t_0\,.$$

where $\lambda_{m,d}$ is the eigen-value in the Hardy-Poincaré inequality.

These two results can be translated back to the original problem in the following:

Corollary 3.3.4 (Intermediate asymptotics) *Under the assumptions of Theorem 3.3.2, if $m \neq m_*$ and if $T - \tau > 0$ is small, when $m < m_c$ (resp. τ large if $m \geq m_c$), then there exists a positive constant C such that*

$$\|u(\tau) - U_{D_*,T}(\tau)\|_{\mathrm{L}^2(\mathbb{R}^d)} \leq C\, R(\tau)^{-\lambda_{m,d}-\frac{d}{2}}$$

where

$$U_{D,T}(\tau,y) := \frac{1}{R(\tau)^d}\left(D + \frac{1-m}{2\,m}\left|\frac{y}{R(\tau)}\right|^2\right)^{-\frac{1}{1-m}} .$$

As we will sketch below, these results are proved using entropy methods. The corresponding functional inequality is:

Theorem 3.3.5 (Hardy-Poincaré inequality) *Let $d \geq 1$ and $D > 0$. If $m \in (0,1)$ and $1 \leq d \leq 4$, or $m \in (m_*, 1)$ and $d \geq 5$. Define the measures*

$$\mathrm{d}\mu := V_D^{2-m}\,\mathrm{d}x \quad \text{and} \quad \mathrm{d}\nu := V_D\,\mathrm{d}x\,,$$

where $V_D(x) = \left(D + \frac{1-m}{2\,m}\,|x|^2\right)^{-1/(1-m)}$. To a function $g \in \mathrm{L}^1(\,\mathrm{d}\mu)$, we associate its average $\overline{g} = \int_{\mathbb{R}^d} g(x)\,\mathrm{d}\mu$. Then there exists a positive constant $\mathcal{C}_{m,d}$, which does not depend on D, such that

$$\int_{\mathbb{R}^d} |g-\overline{g}|^2\,\mathrm{d}\mu \leq \mathcal{C}_{m,d}\int_{\mathbb{R}^d} |\nabla g|^2\,\mathrm{d}\nu \quad \forall\, g \in \mathcal{D}(\mathbb{R}^d)\,, \quad \overline{g} = \int_{\mathbb{R}^d} g\,d\mu\,. \tag{3.3.1}$$

In case $d \geq 5$ and $m \in (0, m_*)$, we have

$$\int_{\mathbb{R}^d} g^2\,\mathrm{d}\mu \leq \mathcal{C}_{m,d}\int_{\mathbb{R}^d} |\nabla g|^2\,\mathrm{d}\nu \quad \forall\, g \in \mathcal{D}(\mathbb{R}^d) \tag{3.3.2}$$

and $\mathcal{C}_{m,d} = \frac{8\,m\,(1-m)}{[(d-2)\,(m-m_*)]^2}$ is optimal.

3.4 Idea of the proof

3.4.1 Entropy method

Let D_* be chosen to satisfy the relative conservation of mass. To study the convergence we introduce

$$w(t,x) := \frac{v(t,x)}{V_{D_*}(x)} \quad \forall\,(t,x) \in (0,\infty)\times\mathbb{R}^d\,. \tag{3.4.1}$$

Next, we rewrite Problem (3.1.4) in terms of w:

$$\begin{cases} w_t = \dfrac{1}{V_{D_*}}\,\nabla\cdot\left[w\,V_{D_*}\,\nabla\left(\dfrac{m}{m-1}\,(w^{m-1}-1)\,V_{D_*}^{m-1}\right)\right] & \text{in } (0,+\infty)\times\mathbb{R}^d\,,\\ w(0,\cdot) = w_0 := \dfrac{v_0}{V_{D_*}} & \text{in } \mathbb{R}^d\,. \end{cases} \tag{3.4.2}$$

Remark: Under assumption (H1)-(H2), the equation is non-singular and non-degenerate. Standard parabolic theory implies that w is bounded in $\mathrm{L}^\infty(0,T;\mathcal{C}^k(\mathbb{R}^d))$, for all k. In this framework the derivative of the relative entropy translates into:

Proposition 3.4.1 (Relative entropy/entropy production) *If v is solution to (3.1.4) and $w = v/v_{D_*}$ then*

$$\frac{d}{dt}\mathcal{F}[w] = -\mathcal{J}[w] \,.$$

Convergence to the Barenblatt The convergence is obtained by standard entropy methods that we sketch here for the convenience of the reader: Define $w_\tau(t,x) := w(t+\tau,x)$. By uniform $\mathcal{C}^k(\mathbb{R}^d)$-estimates and Ascoli-Arzelá's theorem, $(w_{\tau_n})_n$ converges to a function $w_\infty > 0$ locally uniformly in (t,x) as well as all the derivatives.

Under assumption (H1)-(H2), the relative entropy is bounded.

$$\infty > \mathcal{F}[w(\tau_n)] - \mathcal{F}[w(\tau_n+1)] = \int_{\tau_n}^{\tau_n+1} \mathcal{J}[w(s)]\,\mathrm{d}s = \int_0^1 \mathcal{J}[w(s+\tau_n)]\,\mathrm{d}s \,.$$

Hence $\mathcal{J}[w_{\tau_n}]$ is integrable on $[0,1]$ and converges to zero as $n \to \infty$:

$$\lim_{n\to\infty} \int_0^1 \int_{\mathbb{R}^d} \left| \nabla \left[\left(w_{\tau_n}^{m-1}(t,x) - 1 \right) V_{D_*}^{m-1}(x) \right] \right|^2 w_{\tau_n}(t,x)\, V_{D_*}(x)\,\mathrm{d}x\,\mathrm{d}t = 0 \,.$$

By Fatou's lemma, w_∞ satisfies

$$\nabla \left[\left(w_\infty^{m-1} - 1 \right) V_{D_*}^{m-1} \right] = 0 \quad \text{a.e. in } (0,1)\times\mathbb{R}^d$$

As a consequence of the conservation of relative mass $w_\infty = 1$. By uniqueness of the limit the whole sequence converges.

By the Lebesgue dominated convergence theorem and

$$|v(t) - V_{D^*}| \le \max\{|V_{D_0} - V_{D^*}|; |V_{D_1} - V_{D^*}|\}$$

the L^p-convergence holds for any p such that the difference between two Barenblatt solutions belongs to L^p.

Linearisation In order to better understand the asymptotic behaviour of the solutions of (3.4.2), we linearise the equation around the equilibrium, introducing a convenient weight. Let g be such that

$$w(t,x) = 1 + \varepsilon\, \frac{g(t,x)}{V_{D_*}^{m-1}(x)} \quad \forall\, t > 0 \,, \quad \forall\, x \in \mathbb{R}^d \,, \tag{3.4.3}$$

for some $\varepsilon > 0$, small. Substituting this expression in Equation (3.4.2) and letting $\varepsilon \to 0$, we formally obtain a linear equation for g,

$$g_t = A_m\, g \quad \text{where} \quad A_m\, g := m\, V_{D_*}^{m-2}(x)\, \nabla \cdot [V_{D_*}\, \nabla g] \,. \tag{3.4.4}$$

The linear operator $A_m : \mathrm{L}^2(\mathbb{R}^d, V_{D_*}^{2-m}\,\mathrm{d}x) \to \mathrm{L}^2(\mathbb{R}^d, V_{D_*}^{2-m}\,\mathrm{d}x)$ is the positive self-adjoint operator associated to the closure of the quadratic form defined for $\phi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ by

$$\mathsf{I}[\phi] := m \int_{\mathbb{R}^d} |\nabla g|^2\, V_{D_*}\,\mathrm{d}x \,. \tag{3.4.5}$$

See [86, Theorem 2.6] for more details.

With the same heuristics, we linearise the relative entropy $\mathcal{F}$ and the relative Fisher information $\mathcal{J}$, which provides the functionals F and I, where I is given by (3.4.5) and F is defined by

$$\mathsf{F}[g] := \frac{1}{2}\int_{\mathbb{R}^d} |g|^2\, V_{D_*}^{2-m}\,\mathrm{d}x\,. \tag{3.4.6}$$

Note that $\mathsf{F}[g]$ is the $\mathrm{L}^2(\mathbb{R}^d, V_{D_*}^{2-m}\,\mathrm{d}x)$-norm up to a factor $1/2$. If g is a solution of (3.4.4), then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{F}[g(t)] = -\,\mathsf{I}[g(t)]\,. \tag{3.4.7}$$

Still at a formal level, the conservation of relative mass amounts to require

$$\int_{\mathbb{R}^d}(v_0 - V_{D_*})\,\mathrm{d}x = \int_{\mathbb{R}^d}(w-1)\,V_{D_*}\,\mathrm{d}x = \varepsilon\int_{\mathbb{R}^d} g\,V_{D_*}^{2-m}\,\mathrm{d}x$$

in the limit $\varepsilon\to 0$. Hence, it makes sense to require that $\int_{\mathbb{R}^d} g\,V_{D_*}^{2-m}\,\mathrm{d}x = 0$ and use the spectral gap estimate, see Theorem 3.3.5. With $\mathcal{C}_{m,d} = m/\lambda_{m,d}$, we obtain

$$2\,\mathsf{F}[g] \le \frac{\mathcal{C}_{m,d}}{m}\,\mathsf{I}[g]\,, \tag{3.4.8}$$

which gives, for the solution of (3.4.3), an exponential decay of the relative entropy,

$$\mathsf{F}[g(t)] \le \mathrm{e}^{-2\,\lambda_{m,d}\,t}\,\mathsf{F}[g(0)] \quad \forall\, t\ge 0\,.$$

The connection with the Fokker-Planck equation is easy to understand at the level of the linearised problem. In the limit $m\to 1$, we observe that

$$\lim_{m\to 1^-} D_*^{1/(1-m)}\,V_{D_*} = (2\,\pi\,D_*)^{d/2}\,\mu \quad\text{with}\quad \mu(x) = \frac{\mathrm{e}^{-\frac{|x|^2}{2D_*}}}{(2\,\pi\,D_*)^{d/2}}\,.$$

Equation (3.4.4) formally converges to the Ornstein-Uhlenbeck equation,

$$g_t = \mu^{-1}\,\nabla\cdot\left(\,\mu\,\nabla g\,\right)\,.$$

The spectral gap inequality (3.4.8) corresponds in such a limit to the well-known Poincaré inequality with Gaussian weight,

$$\int_{\mathbb{R}^d}|\phi|^2\,\mathrm{d}\mu \le \int_{\mathbb{R}^d}|\nabla\phi|^2\,\mathrm{d}\mu \quad \forall\,\phi\in C^\infty(\mathbb{R}^d) \text{ such that } \int_{\mathbb{R}^d}\phi\,\mathrm{d}\mu = 0\,,$$

where $\mathrm{d}\mu := \mu\,\mathrm{d}x$. Note that in the Gaussian case, a logarithmic Sobolev inequality holds, see [108],

$$\int_{\mathbb{R}^d}|\phi|^2\,\log\left(\frac{|\phi|^2}{\int_{\mathbb{R}^d}|\phi|^2\,\mathrm{d}\mu}\right)\mathrm{d}\mu \;\le\; 2\int_{\mathbb{R}^d}|\nabla\phi|^2\,\mathrm{d}\mu\,,$$

which is stronger than the Gaussian Poincaré inequality. This is not the case with the measure $V_{D_*}\,\mathrm{d}x$. Although the spectral gap inequality (3.4.8) holds true, there is no corresponding logarithmic Sobolev inequality.

Comparison linear/non-linear Remains now to relate the results obtained at the linear level to the non-linear problem. The main tool is to notice that if w is close enough to 1, then there exists C_0, C_1, β_1 and β_2 such that

$$C_0\,\mathsf{F}[g] \le \mathcal{F}[w] \le C_1\,\mathsf{F}[g]$$

and

$$\mathsf{I}[g] \le \beta_1\,\mathcal{J}[w] + \beta_2\,\mathsf{F}[g]$$

with $g := (w-1)\,V_{D_*}^{m-1}$.

Using the Hardy-Poincaré inequality

$$2\,\mathsf{F}[g] \le \frac{1}{\lambda_{m,d}}\,\mathsf{I}[g] \le \frac{1}{\lambda_{m,d}}\,(\beta_1\,\mathcal{J}[w] + \beta_2\,\mathsf{F}[g])\ ,$$

from which we deduce that

$$\frac{2\,\lambda_{m,d}-\beta_2}{\beta_1}\mathsf{F}[g] \le \mathcal{J}[w]\ .$$

As $\mathcal{F}[w] \le C_1\,\mathsf{F}[g]$, there exists

$$\gamma := C_1\,\frac{2\,\lambda_{m,d}-\beta_2}{\beta_1}$$

such that

$$\gamma\,\mathcal{F}[w] \le \mathcal{J}[w]\ .$$

And so by Gronwall's estimate

$$\mathcal{F}[w(t)] \le \mathcal{F}[w_0]\ \mathrm{e}^{-\gamma\,t}\ .$$

Note that γ is positive for w close enough to 1.

Finally,

$$\|v-V_{D_*}\|_{\mathrm{L}^2}^2 \le \left\|V_{D_*}^{2-m}\right\|_{\mathrm{L}^\infty}\int |v-V_{D_*}|^2\,V_{D_*}^{m-2}\,\mathrm{d}x = C\,\mathsf{F}[v-V_{D_*}]$$
$$\le C\frac{1}{C_0}\mathcal{F}[w] \le \tilde{C}\,\mathrm{e}^{-\gamma\,t}\,.$$

We can improve this convergence by redoing the computations by replacing W_0 and W_1 by $1-\sigma\,\mathrm{e}^{\gamma t}$ and $1+\sigma\,\mathrm{e}^{\gamma t}$ in the Lemma "comparison linear/non-linear" we improve the rate on convergence up to $\gamma = \lambda_{m,d}$.

3.4.2 Hardy-Poincaré inequalities

The cornerstone of this proof is the Hardy-Poincaré inequality which we prove here. We observe that as $|x| \to \infty$, $\mathrm{d}\mu \sim \mathrm{d}\nu/|x|^2$. Hence, if $m \in (0, m_*)$, Inequality (3.3.2) is of Hardy type. Otherwise, if $m \in (m_*, 1)$, Inequality (3.3.1) involves an average and is rather of Poincaré type. In such a case, we shall also say that it is a weighted Poincaré inequality, or that there is a spectral gap, since for the associated operator, the lowest eigenvalue, 0, is achieved by the constant functions, and the second eigenvalue corresponds to $\lambda_{m,d} = m/\mathcal{C}_{m,d}$ where $\mathcal{C}_{m,d}$ is the best constant in the inequality.

For the case $m \in (0, m_*)$, "completing the square method" gives

Proposition 3.4.2 (Weighted Hardy inequality) *With the above notations, for any $\alpha \in \mathbb{R}$, $\alpha \neq \alpha_*$,*

$$\int_{\mathbb{R}^d} \frac{|g|^2}{|x|^2}\,|x|^{2\alpha}\,\mathrm{d}x \leq \kappa_\alpha \int_{\mathbb{R}^d} |\nabla g|^2\,|x|^{2\alpha}\,\mathrm{d}x \quad \forall\, g \in \mathcal{D}(\mathbb{R}^d)\,,$$

with the additional requirement that g is supported in $\mathbb{R}^d \setminus \{0\}$ if $\alpha < \alpha_*$, and κ_α is optimal.

For the other range, our result relies on an abstract spectral argument involving Persson's characterisation of the continuous spectrum of an operator, [170].

3.5 Asymptotics of the doubly nonlinear equation

Such a method is robust and can be extended for the doubly nonlinear diffusion equation $(t,x) \in (0,\infty)\times\mathbb{R}^n$ by

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \Delta_p(\rho^m) := \operatorname{div}\left[|\nabla \rho^m|^{p-2}\,\nabla\left(\rho^m\right)\right]\,, & (x\in\mathbb{R}^n, t>0) \\ \rho(t=0) = \rho_0 \geq 0\,, & (x\in\mathbb{R}^n) \end{cases} \tag{3.5.1}$$

with $1 < p < \infty$, $0 < m$ and $n \geq 3$. This class of equations contains the linear diffusion equation, $(p=2,\ m=1)$, commonly known as the heat equation, $\partial_t\rho = \Delta\rho$; the nonlinear diffusion equation $\partial_t\rho = \Delta\rho^m$, known as the porous medium equation $(p=2,\ m>1)$, or the fast diffusion equation $(p=2,\ m<1)$, and the gradient-dependent diffusion equation, $\partial_t\rho = \operatorname{div}(|\nabla\rho|^{p-2}\nabla\rho) := \Delta_p\rho$, that is, the p-Laplacian equation, $(p\neq 2,\ m=1)$. When $p\neq 2$ and $m\neq 1$, Eq. (3.5.1) is called the doubly nonlinear diffusion equation, due to the fact that its diffusion term depends non-linearly on both the unknown density ρ, and its gradient $\nabla\rho$.

Furthermore for $n \geq 3$, there exists a critical exponent,

$$m_c(p) := \frac{n-p}{n(p-1)},$$

such that if $m > m_c(p)$, then the mass of the solution is conserved, $\int_{\mathbb{R}^n}\rho(t)\ \mathrm{d}x = \int_{\mathbb{R}^n}\rho_0\ \mathrm{d}x$, while if $m < m_c(p)$, the solution vanishes in finite time, see [76, 199] and the references therein. In particular, for the p-Laplacian equation, this corresponds to the critical p-exponent,

$$p_c := \frac{2n}{n+1},$$

above which the mass of the solution is conserved, while the solution disappears in finite time if $p < p_c$. Therefore, up to renormalising the mass of ρ_0 to unity, we can assume without loss of generality that, under the condition $m > m_c(p)$, the solution $\rho(t)$ of (3.5.1) is a density in $\mathbb{R}^n$, for all times $t \geq 0$.

By similarity and scaling, it can be shown that, above the critical exponent $m_c(p)$, Eq.(3.5.1) has a unique self-similar solution ρ_{D_*}, whose initial value is the Dirac mass at the origin, that is, the fundamental solution of Eq.(3.5.1). In fact, among all the radially symmetric solutions of (3.5.1),

this solution is the most concentrated whose initial data have the same mass as ρ_0. It is called the Barenblatt solution, and it is precisely:

$$\rho_{D_*}(t,x) = \frac{1}{(\delta_p t)^{n/\delta_p}} u_{D_*}\left(\frac{x}{(\delta_p t)^{1/\delta_p}}\right), \tag{3.5.2}$$

where

$$\delta_p := n(p-1)(m-m_c(p)) > 0,$$

and

$$u_{D_*}(y) = \begin{cases} \dfrac{1}{\sigma}\exp\left(-\dfrac{|p-1|^2}{p}|y|^{p/(p-1)}\right) & \text{if} \quad m = \dfrac{1}{p-1} \\ \left(D_* - \dfrac{m(p-1)-1}{mp}|y|^{p/(p-1)}\right)_+^{\frac{p-1}{m(p-1)-1}} & \text{if} \quad m \neq \dfrac{1}{p-1}, \end{cases}$$

with σ and D_* are uniquely determined by the mass conservation: $\|u_{D_*}\|_{L^1(\mathbb{R}^n)} = \|\rho_{D_*}(t)\|_{L^1(\mathbb{R}^n)} = \|\rho_0\|_{L^1(\mathbb{R}^n)}$.

When $p = 2$ and $m > 1 - 2/n$, the existence and uniqueness of the Barenblatt solution was proved by Friedmann and Kamin in [101]. Moreover, they showed that the solution $\rho(t)$ of the Cauchy problem converges to $\rho_{D_*}(t)$ w.r.t. the $L^1(\mathbb{R}^n)$-norm, as $t \to \infty$, with no rates. Rates of convergence were computed by Carrillo and Toscani [70] if $m > 1$, independently by Del Pino and Dolbeault [87], and Otto [167] if $m \geq 1 - 1/n$. The rates found in this range were generically optimal. In the range $1 - 2/n < m < 1 - 1/n$, there were studies of the linearised problem by Carrillo, Lederman, Markowich and Toscani [67], and Denzler and McCann [93]. These linearisations were useful to obtain rates of decay for the nonlinear fast diffusion equation by Carrillo and Vázquez [72] and later Kim and McCann [130]. The decay rates obtained by using the linearisations are in general non optimal and is optimal in some sub-range, see [130].

When $p \neq 2$ and $m = 1$, Kamin and Vázquez [125] proved existence and uniqueness of the Barenblatt solution ρ_{D_*} for the p-Laplacian equation when $p > p_c$, along with an L^1-convergence of the solution $\rho(t)$ of the Cauchy problem to $\rho_{D_*}(t)$, with no rates. Their proof extends to the doubly nonlinear equation as long as $m > m_c(p)$, see [199]. Rates of convergence were computed by Del Pino and Dolbeault [88] when $p_c + 1/(n+1) \leq p < n$ for the p-Laplacian equation, but their rates are not optimal; see also a similar result for the doubly nonlinear equation in [89]. In [2], Agueh generalises previous results by deriving optimal rates for the convergence of the solution of the Cauchy problem (3.5.1) to $\rho_{D_*}(t)$, for all $m \geq m_c(p) + 1/(n(p-1)) = (n-p+1)/(n(p-1))$ and $p > 1$. For instance, when $p = 2$, this condition coincides with the case $m \geq 1 - 1/n$, while for the p-Laplacian equation ($p \neq 2$, $m = 1$), it corresponds to $p \geq p_c + 1/(n+1) = (2n+1)/(n+1)$, and therefore covers the range $p \geq n$ left in [88], but not the remaining exponent interval $2n/(n+1) < p < (2n+1)/(n+1)$. Similarly, for the doubly nonlinear diffusion equation, the rate of convergence remains unknown in the range

$$m_c(p) < m < m_c(p) + \frac{1}{n(p-1)} = \frac{n-p+1}{n(p-1)}. \tag{3.5.3}$$

Indeed, the proof of [1] is based on optimal transportation inequalities, which follows from the displacement convexity [154] of the energy functional associated with (3.5.1), that is, $H^F(\rho) = \int_{\mathbb{R}^n} F[\rho]\,\mathrm{d}x$, where

$$F(x) = \begin{cases} \dfrac{1}{p-1} x \ln x & \text{if} \quad m = \dfrac{1}{p-1} \\ \\ \dfrac{m x^\gamma}{\gamma(\gamma-1)} & \text{if} \quad m \neq \dfrac{1}{p-1}, \end{cases}$$

and

$$\gamma := m + \frac{p-2}{p-1}. \tag{3.5.4}$$

This energy functional is displacement convex if and only if $\gamma \geq 1 - \frac{1}{n}$, or equivalently $m \geq (n-p+1)/(n(p-1))$. This explains why the method of [2] does not extend to the interval (3.5.3).

The goal of this work is then precisely to derive a rate of convergence w.r.t the $L^1(\mathbb{R}^n)$-norm, of the non-negative solution ρ of the Cauchy problem (3.5.1), to the Barenblatt solution $\rho_{D_*}(t)$, as $t \to \infty$, provided that m belongs to the remaining exponent interval (3.5.3), that is,

$$\frac{n-p}{n(p-1)} < m < \frac{n-p+1}{n(p-1)} \quad \Leftrightarrow \quad 1 - \frac{q}{n} < \gamma < 1 - \frac{1}{n}\,. \tag{3.5.5}$$

For convenience we rewrite the Cauchy problem (3.5.1) as:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \mathrm{div}\Big\{ \rho \nabla c^* \left[\nabla \left(F' \circ \rho \right) \right] \Big\}, & (x \in \mathbb{R}^n,\ t > 0) \\ \\ \rho(t=0) = \rho_0, & (x \in \mathbb{R}^n), \end{cases} \tag{3.5.6}$$

where $c^*(x) = |x|^p/p$ is the Legendre transform of the convex function

$$c(x) = \frac{|x|^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

By rescaling in time and space ρ as follows:

$$\rho(t,x) = \frac{1}{R(t)^n} u\left(\tau, y\right), \tag{3.5.7}$$

where $\tau = \ln R(t)$, $y = x/R(t)$ with

$$R(t) = (1 + \delta_p t)^{1/\delta_p}, \quad \text{and} \quad \delta_p = (p-1)(nm+1) + 1 - n, \tag{3.5.8}$$

it is easy to show that ρ solves (3.5.6) if and only if u solves the rescaled convection-diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \mathrm{div}\Big\{ u \nabla c^* \left[\nabla \left(F' \circ u \right) \right] + u y \Big\} & (y \in \mathbb{R}^n,\ \tau > 0) \\ u(\tau = 0) = \rho_0 & (y \in \mathbb{R}^n). \end{cases} \tag{3.5.9}$$

Moreover, by conservation of mass there exists a unique D_* such that the Barenblatt profile u_{D_*} is the equilibrium solution of (3.5.9). Remark that in the considered range of exponents (3.5.5), $m(p-1) - 1 < 0$, and then

$$F(x) = \frac{m x^\gamma}{\gamma(\gamma-1)}, \quad \gamma := m + \frac{p-2}{p-1}. \tag{3.5.10}$$

Therefore, the Barenblatt profile is simply given by

$$u_{D_*}(y) = \left(D_* + \frac{1-\gamma}{m} c(y)\right)^{\frac{1}{\gamma-1}}. \tag{3.5.11}$$

In fact, u_{D_*} is the unique density function of same mass as u_0 which satisfies on its support,

$$\nabla\left(F'\circ u_{D_*}\right) = -\nabla c. \tag{3.5.12}$$

The main result is the following:

Theorem 3.5.1 (Rates of convergence, [3]) *Let m, p be in the range (3.5.5), and let u_0 be a density such that there exist positive constants $D_0 > D_1$ for which*

$$u_{D_0}(x) \le \rho_0(x) = u_0(x) \le u_{D_1}(x) \quad \forall\, x \in \mathbb{R}^n. \tag{H1}$$

Consider u a solution to (3.5.9) with initial data u_0, there exists a unique D_* such that $u(\tau)$ converges to the Barenblatt profile u_{D_*} in $L^1(\mathbb{R}^n)$. Moreover, there exist a time τ_0 and two positive constants λ and $M = M(m,n,p,u_0,\tau_0)$ such that, for any time $\tau > \tau_0$

$$\|u(\tau) - u_{D_*}\|_{L^1(\mathbb{R}^n)} \le M e^{-\frac{\lambda}{2}\tau}. \tag{3.5.13}$$

As a consequence, for a time large enough the corresponding solution $\rho(t)$ of (3.5.1) converges to the Barenblatt solution $\rho_{D_*}(t)$, algebraically fast in the L^1-norm, at the rate $\lambda/(2\delta_p)$: there exist a time t_0 and a constant $C = C(m,n,p,\rho_0,t_0)$ such that, for any time $t > t_0$

$$\|\rho(t) - \rho_{D_*}(t)\|_{L^1(\mathbb{R}^n)} \le C\, t^{-\lambda/(2\delta_p)}, \tag{3.5.14}$$

where $\delta_p = (p-1)(nm+1) + 1 - n$.

The main tool is the following *relative free energy* with respect to the Barenblatt solution u_{D_*} defined by

$$\mathcal{G}[u|u_{D_*}] := \int_{\mathbb{R}^n} \left[F\circ u(y) - F\circ u_{D_*}(y) - F'\circ u_{D_*}(y)(u(x) - u_{D_*}(y))\right]\, \mathrm{d}y \tag{3.5.15}$$

for any given $u \in \mathrm{L}^1_+(\mathbb{R}^n)$. Its derivative along the flow of (3.5.6) is formally given by

$$-\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{G}[u(\tau)|u_{D_*}] = \mathcal{I}[u(\tau)|u_{D_*}]$$

where

$$\mathcal{I}[u(\tau)|u_{D_*}] := \int_{\mathbb{R}^n} u(\tau,y)\,\nabla\left(F'\circ u(\tau,y) + c(y)\right)\cdot\left(\nabla c^*\circ\nabla F'\circ u(\tau,y) + y\right)\, \mathrm{d}y\,.$$

In this chapter, we prove that the relative entropy decays exponentially fast in the form

$$\mathcal{G}[u(\tau)|u_{D_*}] \le \mathrm{e}^{-\beta\tau}\mathcal{G}[u_0|u_{D_*}], \tag{3.5.16}$$

for some $\beta > 0$. This is obtained in two steps. First, we linearise (3.5.9) at the equilibrium solution u_{D_*} by using the linear perturbation $u(\tau) = u_{D_*} + \epsilon v(\tau)$, and we show that the linearised version of the relative energy

converges to 0 exponentially fast, as in [67]. For that, we use the Hardy-Poincaré inequality recently established by Blanchet, Bonforte, Dolbeault, Grillo and Vázquez in [30]. Next, following the strategy exposed above, we try to compare the relative energy and the dissipation of the relative energy –that is, the Fisher information– for both linearised and nonlinear equations, to deduce the exponential decay (3.5.16) for the nonlinear equation. The main differences with respect to the previous section lie in the fact that a direct relation between the linearised and the nonlinear Fisher information is not clear due to the singular characters at the origin of the weights when $1 < p < 2$. Therefore, we are forced to use a sort of regularised linearised Fisher information instead. Moreover, the control of the additional terms appearing in the regularised entropy dissipation of the linearised problem and in the relation between the entropy dissipations is more involved in our case.

We note that, based on our computations, the Bakry-Emery approach used in [67], which consists of differentiation twice the relative energy $\mathsf{F}[[]v(\tau)]$ to estimate the spectral gap at the eigenvalue 0, does not yield a positive result for our equation when $1 < p < 2$, and thus, a similar procedure to [72] for the doubly nonlinear equations is not feasible. Moreover, the Hardy-Poincaré inequality used here to establish the linear stability is actually valid on a larger interval, $m_*(p) < m < m_c(p) + \frac{1}{n(p-1)}$, which includes our interval $m_c(p) < m < m_c(p) + \frac{1}{n(p-1)}$, as $m_*(p) < m_c(p)$, where $m_*(p) := \frac{n-2q}{n-q} + \frac{2-p}{p-1}$. Therefore, our linearisation result extends naturally to the interval $m_* < m \leq m_c(p)$ where mass conservation for the nonlinear equation fails. In this range, one needs to carefully define the right class of initial data and a substitute of the Barenblatt solution, as done in the previous section when $p = 2$. Here, we will not follow this path and we will restrict ourselves to the case $m_c(p) < m < m_c(p) + \frac{1}{n(p-1)}$ where mass is conserved to concentrate in the main new difficulties.

3.6 Concluding remarks

The interplay between entropy methods and functional inequalities gives in these series of work pertinent result on the long-time asymptotics of the fast-diffusion equation for a large value of the diffusion.

The analysis of the spectrum of the linearised operator has been carried out in [46] and very recently in [92].

In connexion with [33, 61], the method was also used in [60] to obtain various connexion between functional inequalities.

Improved asymptotics for the heat equation

4

This chapter is devoted to results on intermediate asymptotics for the heat equation. We study the convergence towards a stationary solution in self-similar variables. By assuming the equality of some moments of the initial data and of the stationary solution, we get improved convergence rates using entropy / entropy-production methods. We establish the equivalence of the exponential decay of the entropies with new, improved functional inequalities in restricted classes of functions. This letter is the counterpart in a linear framework of a recent work on fast diffusion equations, see [46]. Results extend to the case of a Fokker-Planck equation with a general confining potential.

Most of this chapter in inspired from [15]:

- J.-P. Bartier, A. Blanchet, J. Dolbeault, and M. Escobedo, *Improved intermediate asymptotics for the heat equation*, Applied Mathematics Letters, 24 (2011), pp. 76–81.

4.1 The standard entropy method

Consider the *heat equation* in the euclidean space,

$$\frac{\partial u}{\partial t} = \Delta u \quad t > 0\,,\ x \in \mathbb{R}^d \tag{4.1.1}$$

with an initial condition $u_0 \in L^1(\mathbb{R}^d)$. By writing $u = u_+ - u_-$ where u_+ and u_- are respectively the positive and negative parts of u and solving (4.1.1) with initial data $(u_0)_+$ and $(u_0)_-$, we may reduce the problem to the case of a non-negative function, corresponding to a non-negative initial condition u_0, without restriction. The heat equation being linear, we can assume without loss of generality that u_0 is a probability measure so that in the sequel of this note $\int_{\mathbb{R}^d} u_0\ \mathrm{d}x = 1 = \int_{\mathbb{R}^d} u(t,x)\ \mathrm{d}x$ for any $t \ge 0$. Getting decay rates and even an asymptotic expansion for large values of t is completely standard, see for instance [97]. However, a few details and some notations will be useful for later purpose.

First of all, as a straightforward consequence of the expression of the Green function, $G(t,x,y) := (4\pi t)^{-d/2}\, e^{-\frac{|x-y|^2}{4t}}$, any solution u of (4.1.1) can be written as $u(t,x) = \int_{\mathbb{R}^d} u_0(y)\, G(t,x,y)\, dy$ and therefore uniformly decays like $O(t^{-d/2})$ since, as $t \to \infty$, $u(t,x) \sim G(t,x,0)$. It is also classical to estimate the decay of $u(t,\cdot) - G(t,\cdot,0)$ in various $L^p(\mathbb{R}^d)$ norms. Such estimates are called *intermediate asymptotics* estimates. The point is to determine the first term of an asymptotic expansion of the solution as $t \to \infty$. For instance, as we shall see below, it can be proved that $\|u(t,\cdot) - G(t,\cdot,0)\|_{L^1(\mathbb{R}^d)} = O(t^{-1/2})$ as $t \to \infty$.

The *entropy method* can be used among various other approaches to obtain such an estimate. It relies on the logarithmic Sobolev inequality and goes as follows. First consider the time-dependent rescaling

$$u(t,x) = R^{-d}\, v\,(\log R, x/R) \qquad \text{with} \quad R = R(t) := \sqrt{1+2\,t}\,, \quad t > 0\,,\ x \in \mathbb{R}^d\,. \tag{4.1.2}$$

If u is a solution of (4.1.1), then v solves the *Fokker-Planck equation*

$$\frac{\partial v}{\partial t} = \Delta v + \nabla \cdot (x\, v) \tag{4.1.3}$$

with same initial condition $v(t=0,\cdot) = u_0$. Let $v_\infty(x) := (2\pi)^{-d/2}\, e^{-|x|^2/2}$ be the unique stationary solution of (4.1.3) with mass 1, and define $d\mu := v_\infty\ \mathrm{d}x$ as the Gaussian measure. We denote by $L^p(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d, d\mu)$ the Lebesgue spaces corresponding respectively to Lebesgue's measure and to the Gaussian measure. Understanding the intermediate asymptotics for u amounts to study the convergence of v to vs_∞, as $t \to \infty$. Define the *entropy* by $\mathcal{E}_1[w] := \int_{\mathbb{R}^d} w \log w\, d\mu$. Let v be a solution of (4.1.3) and define $w(t,\cdot) := v(t,\cdot)/v_\infty$, $w_0 := w(t=0,\cdot)$. Then $\frac{d}{dt}\,\mathcal{E}_1[w(t,\cdot)] = -\mathcal{I}_1[w(t,\cdot)]$ where $\mathcal{I}_1$ is the *Fisher information* defined by $\mathcal{I}_1[w] := \int_{\mathbb{R}^d} w\,|\nabla \log w|^2\ \mathrm{d}\mu$. Gross' *logarithmic Sobolev inequality* exactly amounts to $\mathcal{E}_1[v/v_\infty] \le \frac{1}{2}\,\mathcal{I}_1[v/v_\infty]$ and so, it follows that

$$\mathcal{E}_1[w(t,\cdot)] \le \mathcal{E}_1[w_0]\, e^{-2\,t} \quad \forall\, t \ge 0\,.$$

By the *Csiszár-Kullback inequality*, see for instance [196], we get $\|v(t,\cdot)-v_\infty\|^2_{L^1(\mathbb{R}^d)} \le \frac{1}{4}\,\mathcal{E}_1[w(t,\cdot)]$ and deduce that

$$\|v(t,\cdot)-v_\infty\|_{L^1(\mathbb{R}^d)} \le \frac{1}{2}\,\sqrt{\mathcal{E}_1[w_0]}\,e^{-t} \quad \forall\, t\ge 0\,.$$

Undoing the change of variables (4.1.2) and observing that $u_\infty(t,x) := R(t)^{-d}\,v_\infty\left(x/R(t)\right) = G(t+1/2,\cdot,0)$, we finally get

$$\|u(t,\cdot)-u_\infty(t,\cdot)\|_{L^1(\mathbb{R}^d)} \le \frac{1}{2}\,\sqrt{\frac{\mathcal{E}_1[w_0]}{1+2\,t}} \quad \forall\, t\ge 0\,,$$

which establishes the claimed estimate, namely:

$$\|u(t,\cdot)-G(t,x,0)\|_{L^1(\mathbb{R}^d)} \le O\left(t^{-1/2}\right)$$

as $t\to\infty$. Such an estimate is quite classical. The above method is known as the *Bakry-Emery method* or *entropy / entropy-production method* and also provides a proof of the logarithmic Sobolev inequality. See [195, 9] for some references on this topic, in the context of partial differential equations.

By combining $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ estimates using Hölder's inequality, we get that

$$\|u(t,\cdot)-G(t,\cdot,0)\|_{L^p(\mathbb{R}^d)} \le O\Big(t^{-\frac{1}{2\,p}\,(1+(p-1)\,d)}\Big) \quad \text{as} \quad t\to\infty\,.$$

In a $L^2(\mathbb{R}^d)$ framework, a much more detailed description can be achieved using a spectral decomposition. If v is a solution of (4.1.3), then $w=v/v_\infty$ is a solution of the *Ornstein-Uhlenbeck equation*

$$\frac{\partial w}{\partial t} = \Delta w - x\cdot\nabla w \tag{4.1.4}$$

with initial data $w_0 = u_0/v_\infty$. Notice that $\int_{\mathbb{R}^d} w_0\;\mathrm{d}\mu{=}1$ and, as a consequence, $\int_{\mathbb{R}^d} w(t,\cdot)\;\mathrm{d}\mu{=}1$ for all $t\ge 0$.

4.2 Improved inequalities

Define by $(H_k)_{k\in\mathbb{N}^d}$ the sequence of Hermite type polynomials acting on $x=(x_1,x_2\ldots x_d)\in\mathbb{R}^d$, such that $H_k(x) := \prod_{j=1}^d h_{k_j}(x_j)$ where $h_n(y) := (-1)^n\,(n!)^{-1/2}\,e^{y^2/2}\frac{d^n}{dy^n}\big(e^{-y^2/2}\big)$, $y\in\mathbb{R}$ and $k=(k_1,...,k_d)\in\mathbb{N}^d$. These functions provide an orthonormal family of eigenfunctions in $L^2(\mathbb{R}^d,d\mu)$ which spans the eigenspaces of the Ornstein-Uhlenbeck operator, that is $-\left(\Delta H_k - x\cdot\nabla H_k\right) = |k|\,H_k$, where $|k| := \sum_{j=1}^d k_j$. Up to a scaling, $(h_n)_{n\in\mathbb{N}}$ is the usual family of Hermite polynomials on $\mathbb{R}$.

If w_0 satisfies the orthogonality condition

$$\int_{\mathbb{R}^d} w_0\,H_k\;\mathrm{d}\mu{=}0 \quad \forall\, k\in\mathbb{N}^d \text{ such that } 0<|k|<n\,, \tag{4.2.1}$$

then an improved rate of convergence follows, in the sense that

$$\|w(t,\cdot)-1\|_{L^2(\mathbb{R}^d,d\mu)} \le e^{-n\,t}\,\|w_0-1\|_{L^2(\mathbb{R}^d,d\mu)} \quad \forall\, t\ge 0\,.$$

If (4.2.1) initially holds, we indeed have $\int_{\mathbb{R}^d} w(t,\cdot)\, H_k \ \mathrm{d}\mu = 0$ for any $t \geq 0$ and any $k \in \mathbb{N}^d$such that $0 < |k| < n$. Then, since $\frac{d}{dt}\|w(t,\cdot)-1\|^2_{L^2(\mathbb{R}^d,\,\mathrm{d}\mu)} = -2\int_{\mathbb{R}^d} |\nabla w(t,\cdot)|^2 \ \mathrm{d}\mu$, the conclusion holds using the following result.

Proposition 4.2.1 (Improved Poincaré inequality) *Assume that $w \in L^2(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^d} w \ \mathrm{d}\mu{=}1$ and the condition $\int_{\mathbb{R}^d} w\, H_k \ \mathrm{d}\mu{=}0$ holds for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$. Then the following inequality holds, with optimal constant:*

$$\|w-1\|^2_{L^2(\mathbb{R}^d,\,\mathrm{d}\mu)} \leq \frac{1}{n}\,\|\nabla w\|^2_{L^2(\mathbb{R}^d,\,\mathrm{d}\mu)}\,.$$

The proof is no more than a straightforward rewriting of the Rayleigh quotient $\|\nabla w\|^2_{L^2(\mathbb{R}^d,\,\mathrm{d}\mu)}/\|w-1\|^2_{L^2(\mathbb{R}^d,\,\mathrm{d}\mu)}$ under the appropriate orthogonality condition. Notice that polynomials H_k are of degree $|k|$ so that the Condition (4.2.1) can be rephrased in terms of moment conditions. See [97, 129] for further results in this direction.

It is natural to search for improved estimates of convergence also in $L^p(\mathbb{R}^d)$ with $p \in [1,2)$ by looking for improved functional inequalities whenever condition (4.2.1) is fulfilled. We may for instance quote [8] in which improvements on the constant, but not on the rates, have been achieved for $p = 1$.

For any $p \in (1,2]$, consider the *generalised entropy*

$$\mathcal{E}_p[w] := \int_{\mathbb{R}^d} \frac{w^p-1}{p-1}\,\mathrm{d}\mu\,.$$

This definition is consistent with the definition of $\mathcal{E}_1$ because, under the condition $\int_{\mathbb{R}^d} w \ \mathrm{d}\mu{=}1$, $\mathcal{E}_p[w] = \int_{\mathbb{R}^d} \frac{w^p-w}{p-1} \ \mathrm{d}\mu \to \mathcal{E}_1[w]$ as $p \to 1$. The functional $\mathcal{E}_p$ controls the convergence in $L^p(\mathbb{R}^d,\,d\mu)$ using a generalised Csiszár-Kullback inequality. In [54, 16], it has been proved that $\|w-1\|^2_{L^p(\mathbb{R}^d,d\mu)} \leq \frac{1}{p}\,2^{2/p}\,\max\left\{\,\|w\|^{2-p}_{L^p(\mathbb{R}^d,d\mu)},1\right\}\mathcal{E}_p[w]$, for any $p \in [1,2]$. Since $\|w\|_{L^1(\mathbb{R}^d,\,\mathrm{d}\mu)} = 1$, we have $1 \leq \|w\|^p_{L^p(\mathbb{R}^d,\,\mathrm{d}\mu)} = 1 + (p-1)\,\mathcal{E}_p[w]$, and so

$$\|w-1\|_{L^p(\mathbb{R}^d,d\mu)} \leq \mathcal{A}_p\left(\mathcal{E}_p[w]\right)$$
$$\text{with}\quad \mathcal{A}_p(s) := \frac{2^{1/p}}{\sqrt{p}}\left[1+(p-1)\,s\right]^{1-p/2}\sqrt{s}\,. \tag{4.2.2}$$

Next, assume that $\int_{\mathbb{R}^d} w\, H_k \ \mathrm{d}\mu{=}0$ for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$ and consider the *generalised Poincaré inequalities,* with $p \in [1,2]$, namely

$$\mathcal{E}_p[w] \leq \mathcal{B}_{n,p}\int_{\mathbb{R}^d}\left|\nabla w^{p/2}\right|^2 \mathrm{d}\mu \quad \forall\, w \in H^1(\mathbb{R}^d,\,d\mu)\,. \tag{4.2.3}$$

Such inequalities have been established for $n = 1$ by W. Beckner in [19] with optimal constant $\mathcal{B}_{1,p} = 2/p$ for the Gaussian measure. By the same method, it has been shown in [7] that for a larger class of measures $d\mu$, if (4.2.3) holds for $p = 1$ and $p = 2$, for some positive constants $\mathcal{B}_{n,1}$ and $\mathcal{B}_{n,2}$ respectively, then it also holds for any $p \in (1,2)$ with

$$\mathcal{B}_{n,p} = \tfrac{1}{p-1}\left[1-\left((2-p)/p\right)^{\mathcal{B}_{n,1}/(2\,\mathcal{B}_{n,2})}\right]\mathcal{B}_{n,2}\,. \tag{4.2.4}$$

By the logarithmic Sobolev inequality and the improved Poincaré inequality, see Proposition 4.2.1, we know that $\mathcal{B}_{n,1} \le 2$ and $\mathcal{B}_{n,2} = 1/n$. Hence it follows that $\mathcal{B}_{n,p} \le \frac{1}{p-1}\left[1-\left((2-p)/p\right)^n\right]\frac{1}{n}$. On the other hand, as in [9], if w is a solution of (4.1.4), then

$$\frac{d}{dt}\,\mathcal{E}_p[w(t,\cdot)] = -\frac{4}{p}\int_{\mathbb{R}^d} \left|\,\nabla w^{\,p/2}\,\right|^2 \mathrm{d}\mu\,. \tag{4.2.5}$$

If (4.2.1) is satisfied, we conclude using (4.2.3) and (4.2.2) that any solution of (4.1.4) with initial data w_0 satisfies

$$\mathcal{E}_p[w(t,\cdot)] \le \mathcal{E}_p[w_0]\,e^{-2\,\lambda(n,p)\,t}$$
$$\text{and}\quad \|w(t,\cdot)-1\|_{L^p(\mathbb{R}^d,\,d\mu)} \le \mathcal{A}_p\left(\mathcal{E}_p[w_0]\right)\,e^{-\lambda(n,p)\,t}\quad\forall\,t\ge 0\,,$$

with $\lambda(n,p) := \frac{2}{p}\,n\,(p-1)\left[1-\left((2-p)/p\right)^n\right]^{-1}$. The last estimate holds because, for any $t \ge 0$,

$$\|w(t,\cdot)-1\|_{L^p(\mathbb{R}^d,\,d\mu)} \le \mathcal{A}_p\left(\mathcal{E}_p[w(t,\cdot)]\right) \le \mathcal{A}_p\left(\mathcal{E}_p[w_0]\,e^{-2\,\lambda(n,p)\,t}\right)$$
$$\le \mathcal{A}_p\left(\mathcal{E}_p[w_0]\right)e^{-\lambda(n,p)\,t}$$

Notice that $\lambda(1,p) = 1$ and $\lambda(n,2) = n$. Nothing is gained as $p \to 1$, since $\lim_{p\to 1}\lambda(n,p) = 1$ is independent of n.

On the other hand, by Hölder's inequality, we have for free that $\|w-1\|_{L^p(\mathbb{R}^d,\,\mathrm{d}\mu)} \le \|w-1\|_{L^2(\mathbb{R}^d,\,\mathrm{d}\mu)}$. Hence, if w is a solution of (4.1.4) with initial data w_0, we know that $\|w(t,\cdot)-1\|_{L^p(\mathbb{R}^d,\,\mathrm{d}\mu)} \le e^{-n\,t}\,\|w_0-1\|_{L^2(\mathbb{R}^d,\,\mathrm{d}\mu)}$ as $t\to\infty$, for any $p\in[1,2]$, if (4.2.1) is satisfied. By interpolation, we recover the rates of [97, 129]. However, this is not satisfactory since neither $\|w_0-1\|_{L^p(\mathbb{R}^d,\,\mathrm{d}\mu)}$ nor $\mathcal{E}_p[w_0]$ are involved in the right hand side of the above estimate.

Consider first the case $p = 1$. An alternative approach is suggested by the method of [30, 31], which applies to the fast diffusion equation $\frac{\partial u}{\partial t} = \Delta u^m$ for $m < 1$. By assuming some uniform bound on the initial data, which is preserved along the evolution, it is possible to relate the asymptotic rate for intermediate asymptotics with the spectrum of the linearised operator. We can indeed observe that $\|w_0-1\|^2_{L^2(\mathbb{R}^d,\,\mathrm{d}\mu)} \le \|w_0-1\|_{L^1(\mathbb{R}^d,\,\mathrm{d}\mu)}\,\|w_0-1\|_{L^\infty(\mathbb{R}^d,\,\mathrm{d}\mu)} \le \frac12\sqrt{\mathcal{E}_1[w_0]}\,\|w_0-1\|_{L^\infty(\mathbb{R}^d,\,\mathrm{d}\mu)}$ using Hölder's inequality and the Csiszár-Kullback inequality. This proves that

$$\|w(t,\cdot)-1\|^2_{L^1(\mathbb{R}^d,\,\mathrm{d}\mu)} \le \tfrac12\,\|w_0-1\|_{L^\infty(\mathbb{R}^d,\,\mathrm{d}\mu)}\,\sqrt{\mathcal{E}_1[w_0]}\,e^{-n\,t}\quad\text{as } t\to\infty$$

if (4.2.1) is satisfied initially. Still, this provides neither an estimate of $\int_{\mathbb{R}^d} w(t,\cdot)\,\log w(t,\cdot)\,\mathrm{d}\mu$ nor a functional inequality which improves upon the logarithmic Sobolev inequality. To prove such an inequality, we keep following the strategy of [31]. A simple but key idea is to observe that the functions defined for any $p\in[1,2]$ by $h_p(0) = 1$, $h_p(1) = p/2$ and, for any $s\in(0,1)\cup(1,\infty)$ by $h_p(s) := [s^p - 1 - p\,(s-1)]/[(p-1)\,|s-1|^2]$ if $p > 1$, $h_1(s) := [s\,\log s - (s-1)]/|s-1|^2$, are continuous, non-negative, decreasing on $\mathbb{R}^+$ and achieve their maximum at 0. Define on $L^\infty(\mathbb{R}^d)$ the functional

$$\mathcal{H}_p[w] := \|w\|^{2-p}_{L^\infty(\mathbb{R}^d)}\,\sup_{x\in\mathbb{R}^d} h_p(w(x)) = \|w\|^{2-p}_{L^\infty(\mathbb{R}^d)}\,h_p\left(\inf_{x\in\mathbb{R}^d} w(x)\right).$$

Theorem 4.2.2 (Improved logarithmic Sobolev inequality) *Assume that $w \in L^\infty_+(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^d} w \, \mathrm{d}\mu = 1$ and satisfies the condition $\int_{\mathbb{R}^d} w \, H_k \, \mathrm{d}\mu = 0$ for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$. Then the following inequality holds, with optimal constant:*

$$\int_{\mathbb{R}^d} w \log w \, \mathrm{d}\mu \le \frac{\mathcal{H}_1[w]}{n} \int_{\mathbb{R}^d} \frac{|\nabla w|^2}{w} \, \mathrm{d}\mu \,.$$

Proof. We may indeed observe that by the Poincaré inequality and using the definition of $\mathcal{H}_1$, we get

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{|\nabla w|^2}{w} \, \mathrm{d}\mu \ge \frac{1}{\|w\|_{L^\infty(\mathbb{R}^d)}} \int_{\mathbb{R}^d} |\nabla w|^2 \, \mathrm{d}\mu \ge \frac{n}{\|w\|_{L^\infty(\mathbb{R}^d)}} \int_{\mathbb{R}^d} |w-1|^2 \, \mathrm{d}\mu \\ \ge \frac{n}{\mathcal{H}_1[w]} \int_{\mathbb{R}^d} w \log w \, \mathrm{d}\mu \,. \end{aligned}$$

The optimality of the constant can be checked by a lengthy but elementary computation using the functions $w^k_\varepsilon := H_k(x)\,\chi\big(x\,\varepsilon^{1/(2n)}\big) + C^k_\varepsilon$ for some smooth truncation function χ such that $0 \le \chi \le 1$, $\chi \equiv 1$ on $B(0,1)$ and $\chi \equiv 0$ in $\mathbb{R}^d \setminus B(0,2)$. Here for $k \in \mathbb{N}^d$ is such that $|k| = n$ and the constant C^k_ε is chosen so that $\int_{\mathbb{R}^d} w^k_\varepsilon \, \mathrm{d}\mu = 1$. □

As a consequence of the Maximum Principle applied to the heat equation (4.1.1) and the fact that to $u_0 = v_\infty$ corresponds a self-similar solution of (4.1.1), namely $u(t,x) = G(t + \frac{1}{2}, x, 0)$, we have the estimate

$$\mathcal{H}_1[w(t,\cdot)] \le \mathcal{H}_1[w_0] \quad \forall\, t \ge 0 \,.$$

By applying Theorem 4.2.2, we obtain a new result of decay for $\mathcal{E}_1[w(t,\cdot)]$ with a constant which is exactly $\mathcal{E}_1[w_0]$, to the price of a rate which is less than $2\,n$.

Corollary 4.2.3 (Improved decay rate of the entropy) *Let w be a solution of (4.1.4) with a non-negative bounded initial data $w_0 \in L^1(\mathbb{R}^d, d\mu)$ such that $\int_{\mathbb{R}^d} w_0 \, \mathrm{d}\mu = 1$ and (4.2.1) is satisfied. Then*

$$\mathcal{E}_1[w(t,\cdot)] \le \mathcal{E}_1[w_0]\, e^{-n\,t/\,\mathcal{H}_1[w_0]} \quad \forall\, t \ge 0 \,.$$

This result is actually equivalent to Theorem 4.2.2, as follows by differentiating the above inequality at $t = 0$ (for which equality is trivially satisfied) and using the fact that $-\int_{\mathbb{R}^d} |\nabla w_0|^2/w_0 \, \mathrm{d}\mu = \frac{d}{dt}\,\mathcal{E}_1[w(t,\cdot)]_{|t=0} \le \mathcal{E}_1[w_0]\,\frac{d}{dt}\,e^{-n\,t/\,\mathcal{H}_1[w_0]}{}_{|t=0}$. What we have achieved is a global, improved exponential decay of the entropy $\mathcal{E}_1$ in a restricted class of functions. To simplify even further, for any $\varepsilon \in (0,1)$ and $n \in \mathbb{N}^*$, consider the set $\mathcal{X}^n_\varepsilon := \{ w \in L^1(\mathbb{R}^d, d\mu) : 1-\varepsilon \le w \le 1+\varepsilon$ *a.e. and* $\int_{\mathbb{R}^d} w\, H_k \, \mathrm{d}\mu = 0$ *holds for any* $k \in \mathbb{N}^d$ *such that* $0 < |k| < n \}$, which is appropriate to handle the optimality case corresponding to $\varepsilon \to 0_+$. The best constant in Theorem 4.2.2 is indeed asymptotically equivalent to the sharp rate of convergence in Corollary 4.2.3, in the sense that $\lim_{\varepsilon\to 0_+} \inf_{w\in\mathcal{X}^n_\varepsilon} n/\mathcal{H}_1[w] = \lim_{\varepsilon\to 0_+} n/[(1+\varepsilon)\,h(1-\varepsilon)] = 2\,n$.

For simplicity, we have considered only the case $p = 1$, but the method also applies to any $p \in (1,2)$. We obtain an improved version of (4.2.3) under the restriction that $w \in L^1(\mathbb{R}^d, d\mu)$ is bounded non-negative and the

condition $\int_{\mathbb{R}^d} w\, H_k \, \mathrm{d}\mu_{=}0$ holds for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$. With $\mathcal{B}_{n,1} = 4\,\mathcal{H}_1[w]/n$ and $\mathcal{B}_{n,2} = 1/n$, we get $\mathcal{B}_{n,p} \le \mathcal{K}[n,p,w] := (n\,(p-1))^{-1}\left[1-((2-p)/p)^{2\,\mathcal{H}_1[w]}\right]$ by (4.2.4). Using the entropy / entropy-production identity (4.2.5), the fact that $\mathcal{K}[n,p,w(t,\cdot)] \le \mathcal{K}[n,p,w_0]$ and the generalised Csiszár-Kullback inequality (4.2.2), we obtain

$$\mathcal{E}_p[w(t,\cdot)] \le \mathcal{E}_p[w_0]\, e^{-\frac{4\,t}{p\,\mathcal{K}[n,p,w_0]}}$$
$$\text{and} \quad \int_{\mathbb{R}^d} w - 1p \, \mathrm{d}\mu \le \mathcal{A}_p\left(\mathcal{E}_p[w_0]\right)\, e^{-\frac{2\,t}{p\,\mathcal{K}[n,p,w_0]}} \quad \forall\, t \ge 0\,. \quad (4.2.6)$$

Alternatively, an elementary computation as in the proof of Theorem 4.2.2 gives a similar result:

$$\frac{4}{p^2}\int_{\mathbb{R}^d} \left|\,\nabla w^{p/2}\,\right|^2 \mathrm{d}\mu = \int_{\mathbb{R}^d} w^{p-2}\,|\nabla w|^2\,\mathrm{d}\mu$$
$$\ge \frac{1}{\|w\|^{2-p}_{L^\infty(\mathbb{R}^d)}}\int_{\mathbb{R}^d}|\nabla w|^2\,\mathrm{d}\mu \ge \frac{n}{\|w\|^{2-p}_{L^\infty(\mathbb{R}^d)}}\int_{\mathbb{R}^d}|w-1|^2\,\mathrm{d}\mu \ge \frac{n}{\mathcal{H}_p[w]}\,\mathcal{E}_p[w]$$

if $\int_{\mathbb{R}^d} w\,\mathrm{d}\mu_{=}1$ and the condition $\int_{\mathbb{R}^d} w\, H_k\,\mathrm{d}\mu_{=}0$ holds for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$. This proves that

$$\mathcal{E}_p[w] \le \frac{4}{p^2}\,\frac{\mathcal{H}_p[w]}{n}\int_{\mathbb{R}^d}\left|\,\nabla w^{p/2}\,\right|^2\mathrm{d}\mu\,.$$

Using (4.2.5) and (4.2.2), this proves that any solution of (4.1.4) with initial data in $w_0 \in L^1\cap L^\infty(\mathbb{R}^d,\,d\mu)$ satisfies

$$\mathcal{E}_p[w(t,\cdot)] \le \mathcal{E}_p[w_0]\, e^{-n\,p\,t/\,\mathcal{H}_p[w_0]}$$
$$\text{and} \quad \|w-1\|_{L^p(\mathbb{R}^d,\,d\mu)} \le \mathcal{A}_p\left(\mathcal{E}_p[w_0]\right)\, e^{-n\,p\,t/(2\,\mathcal{H}_p[w_0])} \quad \forall\, t \ge 0\,. \quad (4.2.7)$$

Comparing the rates of (4.2.6) and (4.2.7) is a natural question. In the limit $\varepsilon \to 0$, $\inf_{w\in\mathcal{X}^n_\varepsilon}\mathcal{H}_p[w] \sim \sup_{w\in\mathcal{X}^n_\varepsilon}\mathcal{H}_p[w] \to p/2$ and it follows that $\lim_{\varepsilon\to 0}\frac{4}{p\,\mathcal{K}[n,p,w_0]} = \frac{4}{p}\,n\,(p-1)/[1-((2-p)/p)^p] < 2\,n = \lim_{\varepsilon\to 0}\frac{n\,p\,t}{\mathcal{H}_p[w_0]}$. Hence, at least in the regime $\varepsilon \to 0$, (4.2.7) is a better estimate in terms of rates than (4.2.6). Undoing the change of variables (4.1.2), we have achieved a detailed result on improved u_0.

Corollary 4.2.4 (Improved intermediate asymptotics for the heat equation) *Let $p \in [1,2]$ and assume that u_0 is a probability measure such that $w_0 = u_0/v_\infty$ is bounded and satisfies the condition $\int_{\mathbb{R}^d} u_0\, H_k\,\mathrm{d}x = 0$ for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$. If u is the solution of (4.1.1) with initial condition u_0, then*

$$\|u(t,\cdot) - u_\infty(t,\cdot)\|_{L^p(\mathbb{R}^d)}$$
$$\le (2\pi)^{-\frac{d}{2}\left(1-\frac{1}{p}\right)}\,\mathcal{A}_p\left(\mathcal{E}_p[w_0]\right)\,(1+2\,t)^{-\frac{n\,p}{4\,\mathcal{H}_p[w_0]}-\frac{d}{2}\left(1-\frac{1}{p}\right)} \quad \forall\, t \ge 0\,.$$

The proof relies on the remark that $\|u(t,\cdot)-u_\infty(t,\cdot)\|_{L^p(\mathbb{R}^d)} \le \|u_\infty(t,\cdot)\|^{1-\frac{1}{p}}_{L^\infty(\mathbb{R}^d)}\,\|w(t,\cdot)-1\|_{L^p(\mathbb{R}^d,\,\mathrm{d}\mu)}$ where $u_\infty(t,\cdot) := G(t+1/2,\cdot,0)$. The conclusion holds using $\|u_\infty(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} = (2\pi\,R^2)^{-d/2}$ with $R = \sqrt{1+2\,t}$.

Up to now, we have considered the simple case of the harmonic potential, $V(x) = \frac{1}{2}|x|^2$. As in [7], the previous results can be extended to more general potentials as follows. Consider $V \in W^{1,2}_{\rm loc} \cap W^{2,2}_{\rm loc}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} e^{-V(x)} dx = 1$, and define the probability measure $d\mu(x) := e^{-V(x)} dx$ in $\mathbb{R}^d$, which generalises the Gaussian measure. Under the above conditions on V, the logarithmic Sobolev inequality holds (resp. (4.2.3) for $p = 1$) for some positive constant (resp. for $\mathcal{B}_{1,1} > 0$). The Ornstein-Uhlenbeck operator $\mathsf{N} := -\Delta + \nabla V \cdot \nabla$ is essentially self-adjoint on $L^2(d\mu)$, has a non-degenerate eigenvalue $\lambda_0 = 0$ and a spectral gap $\lambda_1 > 0$. According to [207, Theorem 2.1], N has a pure point spectrum without accumulation points. Since $\lim_{k\to\infty} \lambda_k = \infty$, then by [174, Theorem XIII.64], the eigenfunctions of N form a complete basis of $L^2(\mathbb{R}^d, d\mu)$. We shall denote the eigenvalues by λ_k, $k \in \mathbb{N}$, and by E_k the corresponding eigenspaces.

Theorem 4.2.2 adapts without changes. Assume that $w \in L^\infty_+(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^d} w \, \mathrm{d}\mu = 1$. Then

$$\int_{\mathbb{R}^d} w \log w \, \mathrm{d}\mu =\leq \frac{\mathcal{H}_1[w]}{\lambda_n} \int_{\mathbb{R}^d} \frac{|\nabla w|^2}{w} \, \mathrm{d}\mu$$

under the orthogonality condition: $w \in \left(\bigcup_{k=1}^{n-1} E_k\right)^\perp$, that is $\int_{\mathbb{R}^d} w \, f_k \, \mathrm{d}\mu = 0$ for any $f_k \in E_k$, $k = 1, 2, \ldots n-1$. Next, consider the solution w of the Ornstein-Uhlenbeck equation

$$\frac{\partial w}{\partial t} = -\mathsf{N}\, w = \Delta w - \nabla V \cdot \nabla w \,, \tag{4.2.8}$$

with initial condition $w_0 \in \left(\bigcup_{k=1}^{n-1} E_k\right)^\perp \cap L^\infty(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^d} w_0 \, \mathrm{d}\mu = 1$. With the same definition as above for $\mathcal{E}_p$, for any solution of (4.2.8) with initial data w_0, (4.2.7) is now replaced by

$$\mathcal{E}_p[w(t,\cdot)] \le \mathcal{E}_p[w_0]\, e^{-\lambda_n\, p\, t/\, \mathcal{H}_p[w_0]}$$
$$\text{and} \quad \|w-1\|_{L^p(\mathbb{R}^d, d\mu)} \le \mathcal{A}_p\left(\mathcal{E}_p[w_0]\right)\; e^{-\lambda_n\, p\, t/\,(2\,\mathcal{H}_p[w_0])} \quad \forall\, t \ge 0\,.$$

4.3 Discussion

Let us conclude this part by some comments and open questions. It is standard in entropy / entropy-production methods that determining sharp rates of convergence in an evolution equation is equivalent to finding sharp constants in functional inequalities, as we have seen in the case of the heat equation: the rate of convergence in $L^2(\mathbb{R}^d, \mathrm{d}\mu)$ is given by the Poincaré inequality, while the rate of convergence in entropy, which controls the $L^1(\mathbb{R}^d, \mathrm{d}\mu)$ norm, is related with the logarithmic Sobolev inequality. This is also true for nonlinear diffusion equations, see for instance [87]. In this case, a breakthrough came from the observation that uniform norms can also be used, see [65, 30, 31], to the price of a restricted functional framework. This allows to relate nonlinear quantities of entropy type with spectral properties of the linearised problem, in an appropriate functional space and, again, to relate sharp rates with best constants, see [46]. As long as nonlinear evolution problems are concerned, only a few invariant quantities are usually available: the mass and the position of the centre of mass of the solution, for

instance. In linear evolution problems, we can impose an arbitrary number of orthogonality conditions, which are preserved along the evolution. Improved rates of convergence are then expected, even when measured with nonlinear quantities like the entropy. Various attempts have been done, see for instance [8], but the question has been left open for many years. Such ideas have been partially explored by R.J. McCann, including in the linear case (see [79]), based on considerations on an appropriate Hessian matrix. Our approach provides a simpler and elementary answer under restrictions which are natural in view of [31]. It also raises a number of questions concerning the optimality of the new functional inequalities from a variational point of view, the convergence of minimising sequences and the symmetry of the eventual minimisers.

Part III

Applications to economics

Optimal transport applied to game theory

5

We study a class of games with a continuum of players for which Nash equilibria can be obtained by the minimisation of some cost, related to optimal transport. This cost is not convex in the usual sense in general but it turns out to have hidden strict convexity properties in many relevant cases. This enables us to obtain new uniqueness results. In a first section, we apply this method to social interaction in the framework of urban equilibria. In a second section we push-forward this analysis for a general Cournot-Nash equilibria analysis and give a characterisation of equilibria in terms of some partial differential equations, a simple numerical scheme in dimension one as well as an analysis of the inefficiency of equilibria.

This chapter presents part of:

- A. Blanchet and G. Carlier, *Optimal transport and Cournot-Nash equilibria*. Pre-print http://arxiv.org/abs/1206.6571, 2012.

- A. Blanchet, P. Mossay, and F. Santambrogio, *Existence and uniqueness of equilibrium for a spatial model of social interactions*. Pre-print http://w3-gremaq.univ-tlse1.fr/blanchet/publication/BMS.pdf, 2012.

5.1 Introduction

This section does not aim to provide an introduction to game theory but to introduce lineage of the results of the next sections. Classical monographs on game theory includes [102, 206, 161, 181].

In decision making, game theory studies "mathematical models of conflict and cooperation between intelligent rational decision-makers". A game consists of

- a set of *players*,
- a set of *strategies* available to those players,
- a specification of *payoffs* for each combination of strategies.

Mathematically, we can define a game in the following

Definition 5.1.1 (Strategic game)**.** A strategic game is a triplet $G = (I, (A^t)_{t\in I}, (g^t)_{t\in I})$ where:

- I is a non-empty set called the set of players,
- For each player $t \in I$, A^t is a non-empty set called the *set of actions* (or strategies) for player t,
- For each player $t \in I$, g^t is a mapping from the distribution of the actions of all the players to $\mathbb{R}$ called the *payoff* function of the player t.

Game theory has been used to study a wide variety of human and animal behaviours. It was initially developed in economics to understand a large collection of economic behaviours, including behaviours of firms, markets, and consumers. The use of game theory in the social sciences has expanded, and game theory has been applied to political, sociological, and psychological behaviours as well. Eight game-theorists have won the Nobel Memorial Prize in Economic Sciences.

5.1.1 Finite games

In a game with a finite number of players we define

$$A = \Pi_{t\in I} A^t \, .$$

An element of A is called an *action profile*. For $a = (a^t)_{t\in I} \in A$, we denote the action profile of all the players except for player t: $a^{-t} := (a^s)_{s\in T\setminus t} \in A^{-t}$.

The easiest example of such games are non-cooperative games defined in [165] as:

> game based on the absence of coalitions in that it is assumed that each participant acts independently, without collaboration and communication from any of the others.

Such games can be described by using the normal forms: there are 2 players *i.e.* $I = \{1, 2\}$, player 1 chooses either top or bottom *i.e.* $A^1 = \{T, B\}$ and player 2 chooses either left or right *i.e.* $A^2 = \{L, R\}$. The payoff can be represented in a matrix in the following

$$\begin{bmatrix} (1,1) & (3,0) \\ (0,3) & (0,0) \end{bmatrix}$$

where the first component of each element of the matrix represents the player's 1 payoff and the second player's 2 payoff. Both players plays simultaneously.

Another interesting example of such game is the matching pennies where the matrix of payoff is given by

$$\begin{bmatrix} (1,-1) & (-1,1) \\ (-1,1) & (1,-1) \end{bmatrix} \tag{5.1.1}$$

Nash notion of equilibria describes a situation in which no player can benefit by changing her strategy while the other players keep theirs unchanged:

Definition 5.1.2 (Nash equilibrium)**.** Let $a \in A$. The action profile a is a *Nash equilibirum* if

$$\forall t \in I,\ \forall b^t \in A^t, \quad g^t(a^t, a^{-t}) \geq g^t(b^t, a^{-t})\ .$$

In the matching pennies example (5.1.1), no such equilibrium seems to exist. However, the notion of strategy profile can be divided in two main categories which will play a crucial role in the sequel:

- A *pure strategy* fully specifies all actions in a game. In particular, it determines the move a player will make for any situation she could face. A player's strategy set is the set of pure strategies available to that player.
- A *mixed strategy* is an assignment of a probability to each pure strategy. This allows for a player to randomly select a pure strategy.

Formally the notion of mixed strategies is the following

Definition 5.1.3 (Mixed strategies)**.** Let I be a finite set of cardinal $n \in \mathbb{N}$. Denote $\Delta(I)$ the set of probabilities over I and $x = (x(t))_{t\in I}$ a probability over I. The *mixed extension* of a finite game $G = (I, (A^t)_{t\in I}, (g^t)_{t\in I})$ is the strategic game $(I, \Delta((A^t)_{t\in I}), (\tilde{g}^t)_{t\in I})$ where for each player $t \in I$ the payoff function $\tilde{g}^t$ is defined by

$$\tilde{g}^t(x^1, \dots, x^n) = \sum_{(a^1,\dots,a^n)\in A} \Pi_{i=1}^n x^i(a^i) g^i(a^1, \dots, a^n)$$

Nash famous result reads as follows:

Theorem 5.1.4 (Nash's theorem) *In a finite game, there exists a Nash equilibrium in mixed strategies.*

5.1.2 Game with a continuum of players

There are two main type of non-cooperative games

- *conflicts* among a small group of agents each of whom can make unilateral decisions which may significantly affect the welfare of the others as well as his own welfare (ex.: Card games, battles between opposing generals, etc.).
- the *individualistic* but not deliberately adversary behaviour of a large number of agents, none of whom alone is able to affect the circumstances of anyone except himself but whose actions in the aggregate determine the environment in which all must live (ex.: competitive markets).

In individualistic game it is often relevant to consider that only the distribution of the actions of the players matters rather than specifying the actions of each single individual. Such games are called *anonymous*.

Nash's result provides an existence result for all these games but such equilibria are more and more difficult to describe when the number of agents increases, see [107]:

> Microeconomics is full of elegant and persuasive arguments about the behaviour of representative firms and representative consumers in competitive markets in general, but in contrast it requires a great deal of elaborate computation to show that even a simple model of non-cooperative exchange yields competitive outcomes when there are many traders.

However, as was noticed by [185] when we deal with individualistics-type non-cooperative games:

> Institutions having a large number of competing participants are common in political and economic life (...) game theory has not yet been able so far to produce much in the way of fundamental principles of mass competition that might help to explain how they operate in practice. (...) it might be worth while to spend a little effort looking at the behaviour of existing n-person solution concepts, as n becomes very large.

Such phenomenon is perfectly known in the physics literature and was already pointed out in [206] in very explicit terms:

> An almost exact theory of a gas, containing about 10^{25} freely moving particles, is incomparably easier than that of the solar system, made up of 9 major bodies. (...)
>
> It is a well known phenomenon in many branches of the exact and physical sciences that very great numbers are often easier to handle than those of medium size. This is of course due to the excellent possibility of applying the laws of statistics and probabilities in the first case. (...)
>
> When the number of participants becomes really great, some hope emerges that the influence of every particular participant will become negligible, and that the above difficulties may recede and a more conventional theory become possible

Such a notion of games with a *continuum of players* was formalised in [11] as

> The most natural model for this purpose contains a continuum of participants, similar to the continuum of points on a line or the continuum of particles in a fluid. (...)
>
> The continuum can be considered an approximation to the "true" situation in which there is a large but finite number of particles.(...)
>
> The purpose of adopting the continuous approximation is to make available the powerful and elegant methods of a branch of mathematics called 'analysis' in a situation where treatment by finite methods would be much more difficult or hopeless.

Aumann even points that

> The choice of the unit interval as a model for the set of traders is of no particular significance. In technical terms, T can be any measure space without atoms. The condition that T have no atoms is precisely what is needed to ensure that each individual trader have no influence.

First results for this kind of game were obtained for *non-atomic games* described in [184] by

> Non-atomic games enable us to analyse a conflict situation where the single player has no influence on the situation but the aggregative behaviour of "large" sets of players can change the payoffs. The examples are numerous: Elections, many small buyers from a few competing firms, drivers that can choose among several roads, and so on.

[184] proves the existence of an equilibria in a non-atomic game with an arbitrary finite number of pure strategies when the payoff of the player only depend on the mean of distribution of all the payers actions. Note that such a result is wrong for a finite number of players as can be seen in the matching penny example. See [128] for more references.

Mathematically, we use here the formalism of [151]: given a space of players types X endowed with a probability measure $\mu \in \mathcal{M}(X)$ (which gives the exogenous distribution of the type of the agents), an action space Y and a cost Γ: $X \times Y \times \mathcal{M}(Y) \to \mathbb{R}$. The θ-type agents taking action x pay the cost $\Gamma(\theta, x, \nu)$ where ν is the distribution of the players' actions. A *Cournot-Nash equilibrium* is a joint probability measure $\gamma \in \mathcal{M}(X \times Y)$ with first marginal μ such that

$$\gamma(\{(\theta, x) \in X \times Y \ : \ \Gamma(\theta, x, \nu) = \min_{z \in Y} \Gamma(\theta, z, \nu)\}) = 1$$

where ν represents γ's second marginal.

Concerning the externalities exerted by the action of all the players we will consider two types

- Rivalry/Congestion: *The utility of the agent decreases when the number of players who choose the same action increases.* Examples:

  - Consumption of the same public good (motorway game),
  - Food supply in an habitat decreases with the number of its users (ex. Sticklebacks),
  - More populated areas lead to higher *competition for land*.

- Social interactions *The utility of the agents increases because some other agents play a similar action.* Examples:

  - Location to go spend holidays,
  - Quality of a product in a differentiated industry (technological choice),
  - The agents benefit from *social interactions* but there is a cost to access to distant agents,

The results of [184, 151] and extensions see [124] assume that $\nu \mapsto \Gamma(\cdot, \cdot, \nu)$ is, in some sense, continuous. However this continuity assumption excludes the case of a purely local dependence which is relevant to capture congestion effects.

5.2 Equilibrium for a spatial model of social interactions

5.2.1 The equilibrium model

We consider a closed spatial economy $\mathcal{E}$ extending along the geographical space $\mathcal{K} \subset \mathbb{R}^d$. We consider a game in which a continuum of agents have to choose an allocation in the space $\mathcal{K}$. A feasible allocation consist in a distribution $\nu : \mathcal{K} \to \mathbb{R}_+$ of agents with $\int_{\mathcal{K}} \nu(x)\, \mathrm{d}x = 1$. We introduce the set $\mathcal{M}(\mathcal{K})$ of spatial absolutely continuous probabilities in $\mathcal{K}$ and such that $\int_{\mathcal{K}} |x|^2 \nu(x)\, \mathrm{d}x < \infty$ if $\mathcal{K}$ is unbounded.

Agents benefit from social contacts with other agents. The social utility $S(x)$ that an agent in location $x \in \mathcal{K}$ derives from interacting with other agents is given by

$$S(x) = B - \int_{\mathcal{K}} \phi(x - y)\nu(y)\, \mathrm{d}y \tag{5.2.1}$$

where the constant B denotes the total benefit from interacting with other agents and $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ the cost of accessing them.

Agents in location $x \in \mathcal{K}$ consume a composite good z and some land space s. As they also benefit from social contacts with other agents, their utility U is given by

$$U(s, z, x) \;=\; z + u(s) + S(x) + A(x)$$

where S is the social utility defined in Relation (5.2.1), $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ the utility of land consumption, and $A : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ the spatial distribution of amenities. The budget constraint faced by agents is

$$z + R(x)\, s \;=\; Y$$

where Y is the income of agents (e.g., the endowment of the composite good) and $R(x)$ the land rent in location x.

As is usual in the urban economics literature, we assume the presence of an absentee landlord who collects the rent paid by agents. Also, we assume that land has no alternate use other than residence. The agent's bid rent function in location x is defined as the maximum rent that an agent is willing to pay for residing in that location,

$$\psi(x, \bar{U}) = \max_{s,z} \frac{Y - z}{s} \quad \text{such that} \quad U(s, z, x) = \bar{U} \, .$$

Lemma 5.2.1 (Spatial indirect function) *Assume the utility of land consumption $u \in \mathcal{C}^1(\mathbb{R}_+)$ is concave. The spatial indirect function $\mathcal{V}$ is given by*

$$\mathcal{V}[\nu](y) := S - v\left[\nu(y)\right] - \int_{\mathcal{K}} \phi(|y - z|)\nu(z) \, \mathrm{d}z + A(y) \, . \tag{5.2.2}$$

where $S = Y + B$ and the residence cost v defined by

$$v(\nu) = \frac{1}{\nu} \, u'\left(\frac{1}{\nu}\right) - u\left(\frac{1}{\nu}\right)$$

is an increasing function of ν.

Following [165], a distribution of agents ν constitutes a *spatial equilibrium* of the economy $\mathcal{E}$ if no agent has an incentive to relocate:

Definition 5.2.2 (Spatial equilibrium)**.** A feasible allocation ν constitutes a *spatial equilibrium* of the economy $\mathcal{E}$ if there exists $\bar{V}$ such that

$$\begin{cases} \mathcal{V}[\nu](y) \leq \bar{V} & \text{for almost every } y \in \mathcal{K}, \\ \mathcal{V}[\nu](y) = \bar{V} & \text{for almost every } y \in \mathcal{K} \text{ such that } \nu(x) > 0. \end{cases} \tag{5.2.3}$$

5.2.2 Existence of equilibria

Let $V : \mathbb{R}_+ \to \mathbb{R}_+$ be a primitive of v. We build the following functional $\mathcal{F} : \mathcal{M}(\mathcal{K}) \to \mathbb{R} \cup \{+\infty\}$

$$\mathcal{F}[\nu] = \mathcal{V}[\nu] + \mathcal{W}[\nu] + \mathcal{A}[\nu] \tag{5.2.4}$$

where ν denotes a spatial probability density in $\mathcal{M}(\mathcal{K})$ and the terms $\mathcal{V}$, $\mathcal{A}$ and $\mathcal{W}$ are defined by

$$\mathcal{V}[\nu] := \int_{\mathcal{K}} V[\nu(x)] \, \mathrm{d}x \, , \quad \mathcal{A}[\nu] := -\int_{\mathcal{K}} A(x)\nu(x) \, \mathrm{d}x$$
$$\text{and} \quad \mathcal{W}[\nu] := \frac{1}{2} \iint_{\mathcal{K}\times\mathcal{K}} \phi(x - y)\nu(x)\nu(y) \, \mathrm{d}x \, \mathrm{d}y$$

Assumption 5.2.3 (Spatial symmetry, [42])**.** Assume that

- $\mathcal{K}$ is symmetric *i.e.* for all $z \in \mathbb{R}^2$, $(z \in \mathcal{K} \Rightarrow -z \in \mathcal{K})$,
- ϕ is even *i.e.* for all $z \in \mathcal{K}$, $\phi(z) = \phi(-z)$.

We now consider the minimisation of $\mathcal{F}$ on $\mathcal{M}(\mathcal{K})$.

Lemma 5.2.4 (Necessary condition of existence, [42]) *Under Assumption 5.2.3. If the agent spatial distribution ν minimises the potential functional $\mathcal{F}$ in the set $\mathcal{M}(\mathcal{K})$, then it is a spatial equilibrium of $\mathcal{E}$.*

The proof of this result consists in deriving the optimality condition for the minimisation problem of functional $\mathcal{F}$. The main idea is that the spatial indirect function U is a differential of $\mathcal{F}$ in the sense that for every (ρ, ν) admissible densities on $\mathcal{K}$, we have

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{F}[\nu + \varepsilon(\rho - \nu)] - \mathcal{F}[\nu]}{\varepsilon} = \int_Y U(x, \nu)(\rho(x) - \nu(x)) \, \mathrm{d}x.$$

It is clear for the first two term whereas the last term gives

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{\mathcal{E}[\nu + \varepsilon(\rho - \nu)] - \mathcal{E}[\nu]}{\varepsilon} \\ &\qquad = \frac{1}{2} \iint \phi(y, z)[\mathrm{d}\nu(y) \, \mathrm{d}(\rho - \nu)(z) + \mathrm{d}\nu(z) \, \mathrm{d}(\rho - \nu)(y)] \\ &\qquad\qquad = \frac{1}{2} \iint [\phi(y, z) + \phi(z, y)] \, \mathrm{d}\nu(y) \, \mathrm{d}(\rho - \nu)(z) \, . \end{aligned}$$

So that

$$\frac{\delta \mathcal{E}}{\delta \nu}(y) = \int_Y \phi^{\mathrm{sym}}(y, z) \, \mathrm{d}\nu(z) \ : \ \phi^{\mathrm{sym}}(y, z) = \frac{\phi(y, z) + \phi(z, y)}{2}$$

Hence $\mathcal{V}$ is the differential of $\mathcal{E}$ on $\mathcal{P}(Y)$ as soon as ϕ is symmetric *i.e.* $\phi(y, z) = \phi(z, y)$

Once we have realised that the first-order condition to the optimisation problem $\min_\nu \mathcal{F}[\nu]$ corresponds to the spatial equilibrium condition (5.2.3). Very mild assumptions are needed to guarantee the existence of minimisers and so of equilibrium:

Assumption 5.2.5. Assume

- The utility of land consumption $u \in \mathcal{C}^1(\mathbb{R}_+)$, is concave, and $\lim_{s \to 0^+} u(s) = -\infty$,
- The accessing cost ϕ is continuous on $\mathcal{K}$,
- The amenity function A is continuous on $\mathcal{K}$ and bounded from above,
- If $\mathcal{K}$ is unbounded, and either $\lim_{|x| \to \infty} A(x) = -\infty$ or A is constant and $\lim_{|z| \to \infty} \phi(z) = +\infty$.

Theorem 5.2.6 (Existence of equilibrium, [42]) *Under Assumptions 5.2.3 and 5.2.5, the spatial economy $\mathcal{E}$ admits a spatial equilibrium.*

The convexity of the potential functional $\mathcal{F}$ would ensure the critical points of $\mathcal{F}$ to be minimisers of $\mathcal{F}$, and therefore spatial equilibria of $\mathcal{E}$. Moreover, if the potential functional $\mathcal{F}$ were strictly convex, then it would not have more than one minimiser. This would provide the uniqueness of spatial equilibrium. Unfortunately, the potential functional $\mathcal{F}$ fails to be convex because of the bi-linear form of the aggregate accessing cost $\mathcal{W}$. This term corresponds to the spatial externality associated with the social interactions between agents located at different locations.

5.2.3 Uniqueness of equilibrium

In this Section, in order to overcome the lack of convexity of the potential $\mathcal{F}$, we rely on the notion of *displacement convexity* introduced by [154]. We will show that the functional $\mathcal{F}$ is displacement convex under mild assumptions on the primitives of the spatial economy $\mathcal{E}$ (*i.e.*, the spatial domain $\mathcal{K}$, the utility function u, the accessing cost ϕ, and the space-dependent amenities A). As a consequence, for a wide class of spatial economies, there is an equivalence between the critical points of $\mathcal{F}$ and the minimisers of $\mathcal{F}$. This will provide a variational characterisation of the spatial equilibria of $\mathcal{E}$. Moreover, if $\mathcal{F}$ is strictly displacement convex, the uniqueness of minimiser is ensured, and therefore that of equilibrium as well. Furthermore, spatial symmetry of the equilibrium will also be obtained depending on the geometry of the spatial domain $\mathcal{K}$.

In all the sequel, we assume that $\mathcal{K} = \overline{\Omega}$ where Ω is some open bounded convex subset of $\mathbb{R}^2$. We first introduce some basic concepts of the theory of optimal transportation, for a detailed exposition of this subject, we refer the interested reader to [204], [5, 205], or [172].

McCann provided some assumptions ensuring the displacement convexity of the functional $\mathcal{F}$ defined in (5.2.4).

Assumption 5.2.7 (Displacement convexity)**.** Let $\mathcal{K} = \overline{\Omega}$ where Ω is some open bounded convex subset of $\mathbb{R}^2$ and assume

- $V(0) = 0$ and $r \mapsto r^2 V(r^{-2})$ is convex non-increasing in $(0, +\infty)$,
- The accessing cost ϕ is convex,
- The amenity function A is concave.

Theorem 5.2.8 (Variational characterization, [42]) *Under Assumption 5.2.7, the agent distribution ν is a spatial equilibrium of the economy $\mathcal{E}$ if and only if it is a minimiser of $\mathcal{F}$ in the set $\mathcal{M}(\mathcal{K})$.*

To ensure the uniqueness of minimiser, [154] provides criteria so as to obtain the strict displacement convexity of $\mathcal{F}$: under Assumption 5.2.7, if ϕ is strictly convex or if A is strictly concave, then the functional $\mathcal{F}$ is strictly displacement convex.

Theorem 5.2.9 (Uniqueness of the equilibrium, [42]) *Under Assumption 5.2.7, if A is strictly concave (resp. ϕ is strictly convex), then any spatial equilibrium ν of the spatial economy $\mathcal{E}$ is unique (resp. unique up to translation).*

And as a direct consequence:

Assumption 5.2.10 (Even symmetry)**.** Suppose that Assumption 5.2.3 holds. Moreover, the spatial distribution of amenities A is also even: for all $x \in \mathcal{K}$, $A(x) = A(-x)$.

Assumption 5.2.11 (Radial symmetry)**.** Let $\mathcal{K}$ be $\mathbb{R}^2$ or a centred ball in $\mathbb{R}^2$. The function W and the spatial distribution of amenities A are radially symmetric: $A(x) = A(x')$ and $W(x) = W(x')$ for all $x, x' \in \mathcal{K}$ with $|x| = |x'|$.

Corollary 5.2.12 (Spatial symmetry of equilibria, [42]) *Under Assumption 5.2.10 (resp. Assumption 5.2.11), any spatial equilibrium ν is even (resp. radially symmetric).*

5.2.4 A circular economy: a non-convex example

In this Section, we revisit the model by [160] along the unit circle $\mathcal{K} = \mathcal{C} = [0, 2\pi]$. In the light of Assumption 5.2.7 and Theorem 5.2.9, the emergence of multiple spatial equilibria can be explained by a lack of convexity of the spatial domain. As the problem along the circle is not convex, Theorem 5.2.8 does not apply. This is the reason why the model exhibits multiple equilibria along the circle while it admits a unique spatial equilibrium along the real line.

Studying spatial economies extending along a circle has a long tradition in economics, ranging from the circular Hotelling model in the industrial organisation literature to the more recent racetrack economy used in the New Economic Geography literature. However, here, the circular model of spatial interactions cannot be interpreted as a simple variant of the corresponding model along the real line. As the spatial equilibria arising along the circle may involve disconnected cities, we find it useful to introduce the following Definition.

Definition 5.2.13 (City, city-centre and multiple cities)**.** Let λ be a spatial density of agents. A *city* is defined as a connected component of the support of λ, and a *city-centre* (or centre) of a city as any point x which is a strict local maximum of λ. The spatial economy is said to be a *multiple-city* economy if it consists of disjoint cities.

Following [160], we consider a linear utility function, $u(r) = \beta r$ where β denotes the preference for land, and a linear accessing cost $W(z)$ equal to τz, for $z \in [0, \pi]$, and to $\tau(2\pi - z)$, for $z \in [\pi, 2\pi]$, where τ is the accessing cost.

Mossay and Picard used a constructive method to solve the model, making conjectures about candidates for equilibrium and, only then, determining which of these candidates do actually satisfy the equilibrium condition (5.2.3). In contrast to their, we propose a direct method which allows to determine all the spatial equilibria of the economy as solutions to a differential equation.

By spatial periodicity, we impose that $\lambda(x + 2\pi) = \lambda(x)$. Also, the point opposite to x along $\mathcal{C}$ is denoted by $\bar{x}$. Any spatial equilibrium λ satisfies

$$\lambda(x) = \frac{1}{\beta}\left(Y + B - \overline{U} - \int_0^{2\pi} W(x-y)\lambda(y)\,\mathrm{d}y\right)_+ .$$

We make the following change of functions by defining the auxiliary function ϕ

$$\phi(x) := \frac{1}{\tau}\int_0^{2\pi} W(x-y)\lambda(y)\,\mathrm{d}y - \frac{\pi}{2}\,. \tag{5.2.5}$$

This allows to rewrite the spatial distribution λ as

$$\lambda(x) = \frac{1}{2}\left(C - \delta^2\phi(x)\right)_+ \tag{5.2.6}$$

where $\delta^2 = 2\tau/\beta$ and $C = 2[Y + B - \overline{U} - \tau\pi/2]/\beta$.

We now derive an equation for function ϕ.

Proposition 5.2.14 (Differential equation for ϕ) *If λ is a spatial equilibrium along the geographical space $\mathcal{C}$, then the function ϕ defined in Expression (5.2.5) belongs to $C^2(\mathcal{C})$ and satisfies the following ordinary differential equation*

$$\phi'' = (C - \delta^2\phi)_+ - (C + \delta^2\phi)_+ \tag{5.2.7}$$

with the periodic condition

$$\phi(x) = -\phi(x \pm \pi), \quad \forall x \in [0, \pi) \tag{5.2.8}$$

Proof. By using relation (5.2.5), function ϕ can be rewritten as

$$\phi(x) := \int_{x-\pi}^{x} (x - y)\lambda(y)\, \mathrm{d}y + \int_{x}^{x+\pi} (2\pi - x + y)\lambda(y)\, \mathrm{d}y - \pi\,.$$

By inspection of this expression, ϕ is differentiable. Its derivative is given by

$$\phi'(x) = \int_{x-\pi}^{x} \lambda(y)\, \mathrm{d}y - \int_{x}^{x+\pi} \lambda(y)\, \mathrm{d}y.$$

As ϕ is differentiable and thus continuous, λ is also continuous given Relation (5.2.6). The fundamental theorem of calculus allows to differentiate ϕ'. This leads to

$$\phi''(x) = \lambda(x) - \lambda(x - \pi) - \lambda(x + \pi) + \lambda(x) = 2[\lambda(x) - \lambda(\bar{x})]\,.$$

This implies that function $\phi \in C^2(\mathcal{C})$. By using Relation (5.2.6), we get $\phi''(x) = (C - \delta^2\phi(x))_+ - (C - \delta^2\phi(\bar{x}))_+$. We also have

$$\begin{aligned}\phi(x) + \phi(\overline{x}) &= 1/\tau \int W(x - y)\lambda(y)\, \mathrm{d}y - \frac{\pi}{2} + 1/\tau \int W(\overline{x} - y)\lambda(y)\, \mathrm{d}y - \frac{\pi}{2} \\ &= 1/\tau \int [W(x - y) + W(\overline{x} - y)]\, \lambda(y)\, \mathrm{d}y - \pi = 0\end{aligned}$$

given the relation $W(x - y) + W(\overline{x} - y) = \tau\pi$ and the total population constraint $\int_{\mathcal{C}} \lambda(y)\, \mathrm{d}y = 1$. Finally, we get $\phi''(x) = (C - \delta^2\phi(x))_+ - (C + \delta^2\phi(x))_+$. □

Our resolution method consists in determining the solutions ϕ to Equation (5.2.7) with the periodic condition (5.2.8). Only then, the spatial equilibria λ will be obtained by Relation (5.2.6). Mossay and Picard identified spatial equilibria involving cities distributed according to a cosine function given by $\cos(\delta x)$. In what follows, these equilibria are referred to as one-frequency (δ) equilibria, as opposed to other solutions derived in this paper involving two frequencies (δ and $\sqrt{2}\delta$). We summarize them in the following Proposition.

Proposition 5.2.15 (Spatial equilibria along the circle) *The spatial equilibria arising in the circular economy $\mathcal{C}$ can be described as follows. Of course, the uniform spatial distribution is always an equilibrium. If $\sqrt{2}\delta$ happens to be an odd number, there exists a spatial equilibrium with full support exhibiting $\sqrt{2}\delta$ centres, see the illustration in Figure 5.1. When $\sqrt{2}\delta$ is not an odd number, for any odd number J such that $J \leq \delta$ (resp. such that $\delta < J \leq \sqrt{2}\delta$), there is a one-frequency (resp. two-frequency) spatial equilibrium with J identical and evenly spaced cities, see the illustration in Figure 5.2 (resp. Figure 5.3).*

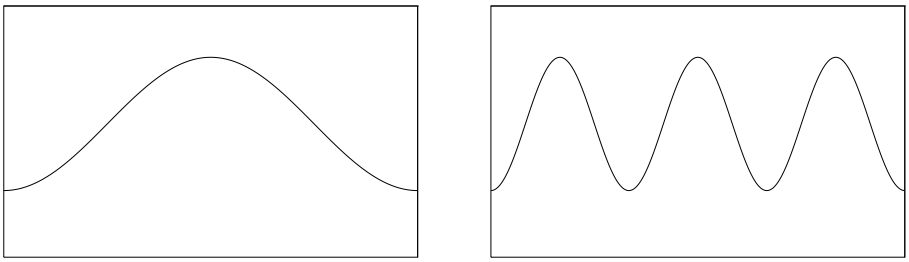

Figure 5.1 – *Spatial equilibria with full support involving an odd number of centres. In the left panel, the spatial economy displays one centre for* $\delta = \sqrt{2}/2$. *In the right panel, the spatial economy displays three centres for* $\delta = 3\sqrt{2}/2$.

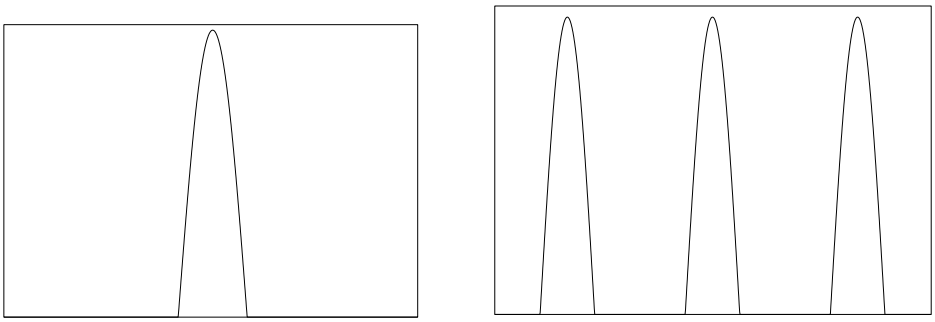

Figure 5.2 – *One-frequency spatial equilibria involving an odd number of cities. In the left panel, the spatial economy displays* $J = 1$ *city for* $\delta = 3$. *In the right panel, the spatial economy displays* $J = 3$ *cities for* $\delta = 4$.

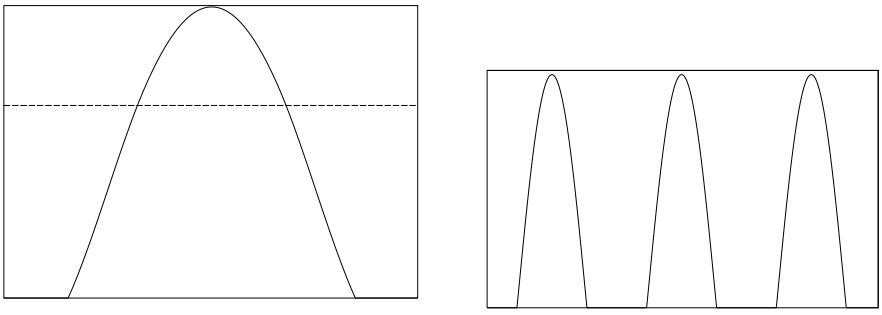

Figure 5.3 – *Two-frequency spatial equilibria involving an odd number of cities. In the left panel, for* $\delta = 3/4$, *the equilibrium displays* $J = 1$ *city where the frequency is* $\sqrt{2}\delta$ *for the portion of the curve above the line and* δ *for the portion of the curve below that line. In the right panel, for* $\delta = 2.8$, *the equilibrium displays* $J = 3$ *cities.*

Our direct resolution method has allowed us to determine all the spatial equilibria of the circular economy. This completes the analysis initiated by Mossay and Picard and reemphasizes the emergence of multiple equilibria, which has been interpreted here as a lack of convexity arising in the circular model.

5.3 Cournot-Nash equilibria

In the previous problem the utility function $\mathcal{V}$ given by

$$\begin{array}{rcl} \mathcal{V} : Y \times \mathcal{P}(Y) & \to & \mathbb{R} \\ (y, \nu) & \mapsto & \mathcal{V}(y, \nu) \end{array}$$

is *the same for all the agents*. We now consider that agents have a type $x \in X$ and that the cost depend on the type:

$$\begin{array}{rcl} \mathcal{V} : X \times Y \times \mathcal{P}(Y) & \to & \mathbb{R} \\ (x, y, \nu) & \mapsto & \mathcal{V}(x, y, \nu) \end{array}$$

However in order to develop our optimal transport approach it seems necessary to assume that the costs $\mathcal{V}$ is on the additive separable form

$$\mathcal{V}(x, y, \nu) := c(x, y) + \mathcal{V}[\nu](y) \ .$$

5.3.1 The equilibrium model

For simplicity we consider X and Y to be compact metric spaces but unbounded domains could be considered too. To give a sense to the local congestion term we take m_0 be a reference probability measure. The definition of equilibrium reads:

Definition 5.3.1 (Cournot-Nash equilibrium)**.** Let μ be a Borel probability measure on X giving the exogenous distribution of the type of the agents. A *Cournot-Nash equilibrium* is a joint probability measure $\gamma \in \mathcal{P}(X \times Y)$ such that

- $\gamma \in \Pi(\mu, \nu) := \{\gamma \in \mathcal{P}(X \times Y) \ : \ \pi_{X\#}\gamma = \mu, \ \pi_{Y\#}\gamma = \nu\}$
- and there exists $\varphi \in \mathcal{K}(X)$ such that for all $x \in X$ and a.e. y

$$\begin{cases} c(x, y) + \mathcal{V}[\nu](y) \geq \varphi(x) \\ c(x, y) + \mathcal{V}[\nu](y) = \varphi(x) \qquad \text{for } \gamma\text{-a.e. } (x, y). \end{cases} \tag{5.3.1}$$

To fix the idea let us give two examples where such equilibria arise:

Holiday choice

Let us consider a population of agents whose location is distributed according to some probability distribution $\mu \in \mathcal{P}(X)$ where X is some compact subset of $\mathbb{R}^2$ (say). These agents have to choose their holidays destination (possibly in mixed strategy). The set of possible holiday destinations is some compact subset of the plane Y (it can be X, a finite set, ...). The commuting cost from x to y is $c(x, y)$. In addition to the commuting cost, agents incur costs resulting from interactions with other agents, this is captured by a map $\nu \mapsto \mathcal{V}[\nu]$ that can be modelled as follows. A natural effect that has to be taken into account is congestion, *i.e.* the fact that more crowded location results in more disutility for the agents. Congestion thus requires to consider local effects and actually imposes that ν is not too concentrated; a way to capture this is to impose that ν is absolutely continuous with respect to some

reference probability measure m_0. Still denoting by ν the Radon-Nikodym derivative of ν, a natural congestion cost is of the form $y \mapsto f(\nu(y))$ with f non-decreasing. In addition to the negative externality due to congestion effect, there may be a positive externality effect due to the positive social interactions between agents which can be captured through a non-local term of the form $y \mapsto \int_Y \phi(y,z)\ \mathrm{d}\nu(z)$ where for instance $\phi(y,.)$ is minimal for $z = y$ so that the previous term represents a cost for being far from the rest of the population. Finally, the presence of purely geographical factors (e.g. distance to the sea) can be reflected by a term of the form $y \mapsto v(y)$. The total externality cost generated by the distribution ν combines the three effects of congestion, positive interactions and geographical factors and can then be taken of the form

$$\mathcal{V}[\nu](y) = f(\nu(y)) + \int_Y \phi(y,z)\ \mathrm{d}\nu(z) + v(y).$$

Technological choice

Consider now a simple model of technological choice in the presence of externalities. There is a set of consumers indexed by a type $x \in X$ drawn according to the probability μ, and a set of technologies Y for a certain good (cell-phone, computer, tablet...). On the supply side, assume there is a single profit maximising profit firm with convex production cost $F(y,.)$ producing technology y, the supply (equals demand at equilibrium) of this firm is thus determined by the marginal pricing rule $p(y) = \partial_\nu F(y, \nu(y))$. Agents aim to minimise with respect to y a total cost which is the sum of their individual purchasing cost $c(x,y) + p(y) = c(x,y) + \partial_\nu F(y,\nu(y))$ and an additional usage/maintenance or accessibility cost which is positively affected by the number of consumers having purchased similar technologies *i.e.* a term of the form $\int_Y \phi(y,z)\ \mathrm{d}\nu(z)$ where ϕ is increasing in the distance between technologies y and z.

5.3.2 Connexion with optimal transport

Let $\mathcal{W}_c$ be the value of the Monge-Kantorovich optimal transport problem:

$$\mathcal{W}_c(\mu,\nu) := \inf_{\gamma \in \Pi(\mu,\nu)} \iint_{X\times Y} c(x,y)\ \mathrm{d}\gamma(x,y)$$

and let us also denote by $\Pi_o(\mu,\nu)$ the set of optimal transport plans *i.e.*

$$\Pi_o(\mu,\nu) := \{\gamma \in \Pi(\mu,\nu)\ :\ \iint_{X\times Y} c(x,y)\ \mathrm{d}\gamma(x,y) = \mathcal{W}_c(\mu,\nu)\}.$$

Proposition 5.3.2 (Connexion with optimal transport) *If γ is a Cournot-Nash equilibrium and ν denotes its second marginal then $\gamma \in \Pi_o(\mu,\nu)$.*

Proof. Indeed, let $\varphi \in \mathcal{K}(X)$ be such that (5.3.1) holds and let $\eta \in \Pi(\mu,\nu)$ then we have

$$\begin{aligned}\iint_{X\times Y} c(x,y)\ \mathrm{d}\eta(x,y) &\geq \iint_{X\times Y} (\varphi(x) - V[\nu](y))\ \mathrm{d}\eta(x,y)\\ &= \int_X \varphi(x)\ \mathrm{d}\mu(x) - \int_Y V[\nu](y)\ \mathrm{d}\nu(y) = \iint_{X\times Y} c(x,y)\ \mathrm{d}\gamma(x,y)\end{aligned}$$

so that $\gamma \in \Pi_o(\mu,\nu)$. □

In an euclidean setting, there are well-known conditions on c and μ which guarantee that such an optimal γ necessarily is pure whatever ν is. It is the case for instance if μ is absolutely continuous with respect to the Lebesgue measure, $c(x,y)$ is a smooth and strictly convex function of $x-y$ (see [155] who extended the seminal results of [49] in the quadratic cost case), or more generally, when it satisfies a generalised Spence-Mirrlees condition (see [64] for details):

Proposition 5.3.3 (Purification of equilibria) *Assume that $X = \overline{\Omega}$ where Ω is some open connected bounded subset of $\mathbb{R}^d$ with negligible boundary, that μ is absolutely continuous with respect to the Lebesgue measure, that c is differentiable with respect to its first argument, that $\nabla_x c$ is continuous on $\mathbb{R}^d \times Y$ and that it satisfies the generalised Spence-Mirrlees condition:*

$$\text{for every } x \in X, \text{ the map } y \in Y \mapsto \nabla_x c(x,y) \text{ is injective,}$$

then for every $\nu \in \mathcal{P}(Y)$, $\Pi_o(\mu,\nu)$ consists of a single element and the latter is of the form $\gamma = (\mathrm{id}, T)_{\#}\mu$ hence every Cournot-Nash equilibrium is pure.

5.3.3 Variational approach

In this section, we will see that in many relevant cases, one may obtain equilibria by the minimisation of some functional over a set of probability measures[1]. The main assumption for this variational approach to be valid is that the interaction map $\mathcal{V}[\nu]$ has the structure of a differential *i.e.* that $\mathcal{V}[\nu]$ can be seen as the first variation of some function $\nu \mapsto \mathcal{E}[\nu]$. In this case, the variational approach is based on the observation that the equilibrium condition is the first-order optimality condition for the minimisation of $\mathcal{W}_c(\mu,\nu) + \mathcal{E}[\nu]$.

Let $\mathcal{D}$ be defined by

$$\mathcal{D} := \{\nu \in \mathcal{L}^1(m_0) \ : \ V[\nu] \in \mathcal{L}^1(\nu)\} = \{\nu \in \mathcal{L}^1(m_0) \ : \ \int_Y |V[\nu]| \, \mathrm{d}\nu < +\infty\}.$$

Assume $a(\nu^\alpha - 1) \leq f(y,\nu) \leq b(\nu^\alpha + 1)$, for $\alpha \geq 0$ and that ϕ is symmetric.

$$\mathcal{V}[\nu](y) := f(\nu(y)) + \int_Y \phi(y,z) \, \mathrm{d}\nu(z) + v(y) \ .$$

is the differential of $\mathcal{E}$ in $\mathcal{D} := \mathcal{P}(Y) \cap \mathcal{L}^{\alpha+1}$ where

$$\mathcal{E}[\nu] = \int_Y F(\nu(y)) \, \mathrm{d}y + \frac{1}{2} \iint_{Y\times Y} \phi(y,z) \, \mathrm{d}\nu(y) \, \mathrm{d}\nu(z) + \int_Y v(y) \, \mathrm{d}\nu(y) \ ,$$

with $F' = f$.

Define the variational problem:

$$\inf_{\nu \in \mathcal{D}} \mathcal{J}_\mu[\nu] \quad \text{where} \quad \mathcal{J}_\mu[\nu] := \mathcal{W}_c(\mu,\nu) + \mathcal{E}[\nu]. \tag{5.3.2}$$

Proposition 5.3.4 (Existence of equilibria, [34]) *If*

- *ν solves (5.3.2),*

[1] Note the analogy with the variational approach of [159] for potential games, *i.e.* games for which theequilibria can be obtained by minimising some potential function.

- *and γ is an optimal transport between μ and ν*

then γ is a Cournot-Nash equilibrium.

Moreover, (5.3.2) admits minimisers in $\mathcal{P}(Y) \cap \mathcal{L}^{\alpha+1}$ so that there exists Cournot-Nash equilibria.

The proof of this result uses a usual vertical perturbation of the functional. Let us mention that the optimality condition for (5.3.2) is the following: there is a constant M such that

$$\begin{cases} \varphi^c + \mathcal{V}[\nu] \geq M \\ \varphi^c + \mathcal{V}[\nu] = M \qquad \nu\text{-a.e. ,} \end{cases} \tag{5.3.3}$$

where φ^c is the c-transform of φ.

5.3.4 Hidden convexity and further uniqueness results

So far, our variational approach has enabled us to prove the existence of equilibria by the minimisation problem (5.3.2). However, the previous results are not totally satisfying since in general there might exist equilibria that are not minimisers and even if we are only interested in the special equilibria obtained by minimisation, optimality conditions:

$$\nu(y) = f^{-1}\left(M - \varphi^c(y) - \int_Y \phi(y,z)\, \mathrm{d}\nu(z)\right). \tag{5.3.4}$$

are not tractable enough to provide a full characterisation. In the case where

$$\mathcal{E}[\nu] = \int_Y F(y,\nu(y))\, \mathrm{d}m_0(y) + \frac{1}{2}\iint_{Y\times Y} \phi(y,z)\, \mathrm{d}\nu(y)\, \mathrm{d}\nu(z)$$

there is a competition between the convexity of the congestion term that favours dispersion and the non-convexity of the interaction term so that in general nothing can be said about the convexity of $\mathcal{E}$ in the usual sense. However, for people familiar with optimal transport there is an hidden convexity due to R. McCann [153] which would restore the equivalence between being a minimiser and being an equilibrium. It would also give new uniqueness and characterisation results.

Throughout this section, we will assume the following:

- $X = Y = \overline{\Omega}$ where Ω is some open bounded convex subset of $\mathbb{R}^d$,
- μ is absolutely continuous with respect to the Lebesgue measure (that will be the reference measure m_0 from now on) and has a positive density on Ω,
- c is quadratic *i.e.*

$$c(x,y) := \frac{1}{2}|x-y|^2,\ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d,$$

- $\mathcal{V}$ again takes the form

$$\mathcal{V}[\nu](y) = f(\nu(y)) + v(y) + \int_Y \phi(y,z)\, \mathrm{d}\nu(z)$$

where v is convex, f satisfies the Inada condition and $\phi \in \mathcal{C}(\mathbb{R}^d \times \mathbb{R}^d)$ is symmetric and $\mathcal{K}^{1,1}_{\mathrm{loc}}$ (*i.e.* $\mathcal{K}^1$ with a locally Lipschitz gradient).

The variational problem (5.3.2) then takes the form

$$\inf_{\nu\in\mathcal{P}(\overline{\Omega})} \mathcal{J}_\mu[\nu] \quad \text{where} \quad \mathcal{J}_\mu[\nu] := \frac{1}{2}\mathcal{W}_2^2(\mu,\nu) + \mathcal{E}[\nu] \tag{5.3.5}$$

with $\mathcal{W}_2^2(\mu,\nu)$ is the squared-2-Wasserstein distance between μ and ν i.e.:

$$\mathcal{W}_2^2(\mu,\nu) := \inf_{\gamma\in\Pi(\mu,\nu)} \int_{X^2} |x-y|^2 \,\mathrm{d}\gamma(x,y).$$

Two more structural assumptions are needed to guarantee the strict convexity of $\mathcal{J}_\mu$ along generalised geodesics with base μ, namely McCann's condition:

$$\nu \mapsto \nu^d F(\nu^{-d}) \text{ is convex non-increasing on } (0,+\infty) \tag{5.3.6}$$

and that ϕ is convex. Note that McCann's condition is satisfied for standard utility function as the power functions ν^m with an exponent larger than 1 as well as by the entropy $\log(\nu)$.

Proposition 5.3.5 (Uniqueness of the equilibrium, [34]) *Under the above assumptions there is a unique equilibrium (which is actually pure).*

A partial differential equation

For computational simplicity, we take $v = 0$ and $f(\nu) = \log(\nu)$ but any convex, $\mathcal{K}^{1,1}_{\mathrm{loc}}$, symmetric v and any increasing f satisfying McCann's condition would lead to a similar partial differential equation. The optimality condition (5.3.3) reads

$$\nu(y) = f^{-1}\left(M - \varphi^c(y) - \int_Y \phi(y,z)\,\mathrm{d}\nu(z)\right).$$

It can be rephrased using the Brenier map ∇u which satisfies the Monge-Ampère equation:

$$\mu(x) = \det(D^2u(x))\,\nu(\nabla u(x)), \quad \forall x\in\Omega$$

supplemented with the natural sort of boundary condition $\nabla u(\Omega) = \Omega$. As

$$\varphi(x) = \frac{1}{2}|x|^2 - u(x), \quad \varphi^c(y) = \frac{1}{2}|y|^2 - v(y), \quad \forall (x,y)\in\Omega\times\Omega\,.$$

The equilibrium problem is therefore equivalent to a non-local and nonlinear partial differential equation:

$$\mu(x) = \det(D^2u(x)) \exp\left(-\frac{1}{2}|\nabla u(x)|^2 + x\cdot\nabla u(x) - u(x)\right) \times \exp\left(-\int_\Omega \phi(\nabla u(y),\nabla u(z))\,\mathrm{d}\mu(z)\right). \tag{5.3.7}$$

This kind of partial differential equation is rather complicated. However, in dimension 1, *i.e.* when Ω is an open interval, which we can assume to be $(0,1)$, the boundary condition is $u'(0) = 0$, $u'(1) = 1$ and the Monge-Ampère equation (5.3.7) simplifies to

$$\mu(x) = u''(x)\exp\left(-\frac{1}{2}u'(x)^2 + x\cdot u'(x) - u(x) - \int_{(0,1)} \phi(u'(x),u'(z))\,\mathrm{d}\mu(z)\right).$$

Numerical simulation in dimension 1

Looking for ν amounts to look for its rearrangement:

$$G(x) := \inf\{\nu \ : \ \nu([0,\nu]) \geq x\}, \ \forall x \in (0,1).$$

We also denote by H the quantile of μ.

Then

$$\mathcal{W}_2^2(\mu,\nu) = \int_0^1 |G(x) - H(x)|^2 \ \mathrm{d}x.$$

So that the variational problem reduces to minimise the strictly convex functional

Proposition 5.3.6 *One dimensional scheme*

$$\frac{1}{2}\int_0^1 |G-H|^2 - \int_0^1 \log(G'(x)) \ \mathrm{d}x + \frac{1}{m+1}\int_{[0,1]^{m+1}} \phi(G(x_0),G(x_1)) \ \mathrm{d}x_0 \ \mathrm{d}x_1$$

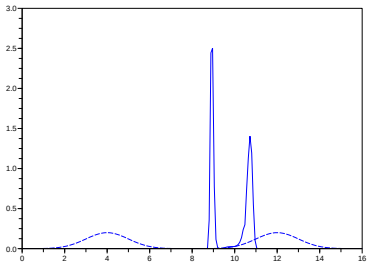

Figure 5.4 – *The distribution μ of the agents is dash line and the solution ν in the case $f(x) = x^8$, $\phi(z) = 10^{-4}|z|^2$ and $v = (x-10)^4$.*

5.4 Extension to utility which are not differentials

The methods used previously need the utility function to be a differential. When ϕ is not symmetric, such a property is no longer satisfied.

As is customary in economics, one may look for equilibria by a fixed-point argument. This can be done where $\mathcal{V}[\nu]$ is a continuous function for every ν and $\nu \mapsto \mathcal{V}[\nu]$ is a continuous map from $(\mathcal{P}(Y), \mathrm{w}-*)$ to $(\mathcal{C}(Y), \|\cdot\|_\infty)$. This is an extremely strong assumption that rules out the case of a local dependence of the type

$$\mathcal{V}[\nu](y) := f(y,\nu(y)), \tag{5.4.1}$$

But it enables us to treat a non-local dependence of the form

$$\mathcal{V}[\nu](y) := \int_{Y^m} \phi(y,z_1,\dots,z_m) \ \mathrm{d}\nu(z_1)\cdots \ \mathrm{d}\nu(z_m) \tag{5.4.2}$$

where ϕ is a continuous function on Y^{m+1} but is not assumed to be symmetric. The following existence result is a simple corollary of [151, Theorem 1], and can be easily proved by using Kakutani's fixed point theorem

Theorem 5.4.1 (Existence of equilibria in the regular case) *Let $\mu \in \mathcal{P}(Y)$ be given. Assume that X and Y are compact metric spaces, c is continuous on $X \times Y$ and $\nu \mapsto \mathcal{V}[\nu]$ is a continuous map from $(\mathcal{P}(Y), w-*)$ to $(\mathcal{C}(Y), \|\cdot\|_\infty)$, then there exists an equilibrium.*

5.4.1 Uniqueness under monotonicity

In the framework of mean-field games, [138] established that some simple monotonicity property of $\nu \mapsto \mathcal{V}[\nu]$ is enough to guarantee uniqueness of the equilibrium. A simple adaptation of their argument gives the elementary uniqueness result:

Theorem 5.4.2 (Uniqueness of the equilibrium) *Let $\mu \in \mathcal{P}(X)$ be given. If $\nu \mapsto \mathcal{V}[\nu]$ is strictly monotone in the sense that for every ν_1 and ν_2 in $\mathcal{P}(Y)$*

$$\int_Y (\mathcal{V}[\nu_1] - \mathcal{V}[\nu_2])\, \mathrm{d}(\nu_1 - \nu_2) \geq 0$$

and the inequality is strict whenever $\nu_1 \neq \nu_2$ then all equilibria have the same second marginal.

5.4.2 Equilibria by best-reply iteration

In this section, we adopt a more direct approach when c is quadratic and $\mathcal{V}[\nu]$ satisfies some suitable convexity condition which makes solving type x agents program $\min_{z\in Y}\{c(x,z) + \mathcal{V}[\nu](z)\}$, given ν, more explicit by a first-order condition.

Throughout this section, we will assume that

- $X = \overline{\Omega}$, $Y = \overline{U}$, where Ω and U are some open bounded convex subsets of $\mathbb{R}^d$,
- the cost is quadratic:

$$c(x,y) := \frac{1}{2}|x-y|^2,\ \forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d\,,$$

- μ is absolutely continuous with respect to the Lebesgue measure on X and has a bounded density still denoted μ,
- $\mathcal{V}[\nu]$ is a smooth and convex function for every $\nu \in \mathcal{P}(Y)$ (which is the case if $\mathcal{V}[\nu]$ has the form (5.4.2) with ϕ smooth and convex with respect to its first argument),
- for every $\nu \in \mathcal{P}(Y)$ and every $x \in X$, the solution of

$$\inf_{y\in Y} \left\{ \frac{1}{2}|x-y|^2 + \mathcal{V}[\nu](y) \right\} \tag{5.4.3}$$

  belongs to U (which is the case as soon as $\mathcal{V}[\nu]$ fulfils some coercivity assumption and U is chosen large enough).

Convergence to the unique equilibrium

The solution of (5.4.3) is obtained by a first-order condition which gives

$$y = (\mathrm{id} + \nabla \mathcal{V}[\nu])^{-1}(x).$$

The resolvent operator $(\mathrm{id} + \nabla \mathcal{V}[\nu])^{-1}$ is a very natural operator in convex analysis where it is known as the *proximal operator* of $\mathcal{V}[\nu]$. If agents have

a prior ν on the other agents actions, their cost-minimising behaviour leads to another *a posteriori* measure on the action space Y, namely

$$T\nu := (\mathrm{id} + \nabla\mathcal{V}[\nu])^{-1}_{\#}\mu. \tag{5.4.4}$$

One easily checks that (γ, ν) is an equilibrium if and only if $\nu = T\nu$ and $\gamma = (\mathrm{id}, (\mathrm{id} + \nabla\mathcal{V}[\nu])^{-1})_{\#}\mu$ is *the* optimal transport plan between μ and ν for the quadratic cost. Finding an equilibrium thus amounts to finding a fixed point of T and we shall see some additional conditions that ensure that T is a contraction of $\mathcal{P}(Y)$ endowed with the 1-Wasserstein distance $\mathcal{W}_1$:

$$\mathcal{W}_1(\nu_1, \nu_2) := \inf_{\eta \in \Pi(\nu_1, \nu_2)} \iint_{Y\times Y} |y_1 - y_2| \, \mathrm{d}\eta(y_1, y_2) \, .$$

Since $(\mathcal{P}(Y), \mathcal{W}_1)$ is a complete metric space, these conditions will therefore imply the existence and the uniqueness of an equilibrium (and more importantly, from a numerical point, this equilibrium can be approximated by the iterates of T applied to any $\nu_0 \in \mathcal{P}(Y)$). Our additional assumptions read as : there exists $\lambda > 0$, $C \geq 0$ and $M > 0$ such that for every $(\nu_1, \nu_2) \in \mathcal{P}(Y) \times \mathcal{P}(Y)$ the following inequalities hold

$$D^2\mathcal{V}[\nu_1] \geq \lambda \mathrm{I}_d \text{ on } X \tag{5.4.5}$$

$$\det(\mathrm{I}_d + D^2\mathcal{V}[\nu_1]) \leq M \text{ on } X \tag{5.4.6}$$

$$\int_Y |\nabla\mathcal{V}[\nu_1](y) - \nabla\mathcal{V}[\nu_2](y)| \, \mathrm{d}y \leq C\mathcal{W}_1(\nu_1, \nu_2) \tag{5.4.7}$$

Using a series of estimates we prove

Theorem 5.4.3 (Convergence of the best-reply iteration scheme) *Under the assumptions of this section, if* (5.4.5), (5.4.6) *and* (5.4.7) *hold and if*

$$MC\|\mu\|_{\mathcal{L}^\infty} < 1 + \lambda \tag{5.4.8}$$

then the map T defined by (5.4.4) *is contraction of $(\mathcal{P}(Y), \mathcal{W}_1)$. Therefore there exists a unique equilibrium (γ, ν) where*

$$\gamma = (\mathrm{id}, (\mathrm{id} + \nabla\mathcal{V}[\nu])^{-1})_{\#}\mu \quad \textit{and} \quad \nu = (\mathrm{id} + \nabla\mathcal{V}[\nu])^{-1}_{\#}\mu \, .$$

Moreover, for every $\nu_0 \in \mathcal{P}(Y)$, the sequence $T^n\nu_0$ converges to ν in the distance $\mathcal{W}_1$ and hence for the weak-$*$ topology.

It may seem difficult at first glance to check the assumptions of Theorem 5.4.3 but we give now a class of examples. Namely, we consider the case where

$$\mathcal{V}[\nu](y) = \mathcal{V}_0(y) + \varepsilon \int_Y \phi(y, z) \, \mathrm{d}\nu(z) \tag{5.4.9}$$

where $\varepsilon > 0$ is a scalar parameter capturing the strength of interaction, $\mathcal{V}_0$ is a smooth and convex function such that $D^2\mathcal{V}_0 \geq \lambda_0 \mathrm{I}_d$ on Y with $\lambda_0 > 0$ and ϕ is $\mathcal{K}^2(\mathbb{R}^d \times \mathbb{R}^d)$.

Corollary 5.4.4 *Assume that $\nu \mapsto \mathcal{V}[\nu]$ has the form* (5.4.9) *and that the previous assumptions are satisfied, then for ε small enough, the map T defined by* (5.4.4) *satisfies* (5.4.5)- (5.4.6)- (5.4.7)- (5.4.8) *and so there is a unique equilibrium.*

Best-reply numerical scheme

Theorem 5.4.3 gives a natural numerical scheme which converges to the equilibrium. Let μ be the distribution of agents and ν_0 be their prior. The best-reply iteration consist, at each step, starting from a distribution of action ν_k in determining

$$\nu_{k+1} = (\mathrm{id} + \nabla \mathcal{V}[\nu_k])_{\#}^{-1} \mu \ .$$

By definition of the push-forward and the change of variable formula, it can be rewritten for all $\phi \in \mathcal{K}(Y)$

$$\begin{aligned} &\int_X \phi((\mathrm{id} + \nabla \mathcal{V}[\nu_k])^{-1}(x)) \, \mathrm{d}\mu(x) = \int_Y \phi(y) \, \mathrm{d}\nu_{k+1}(y) \\ &\Leftrightarrow \int_Y \phi(y) \det(\mathrm{I}_d + D^2 \mathcal{V}[\nu_k])(y) \, \mathrm{d}\mu(\mathrm{id} + \nabla \mathcal{V}[\nu_k])(y) = \int_Y \phi(y) \, \mathrm{d}\nu_{k+1}(y) \ . \end{aligned}$$

So that at each step we have to solve

$$\nu_{k+1} = \mu(\mathrm{id} + \nabla \mathcal{V}[\nu_k]) \det(\mathrm{I}_d + D^2 \mathcal{V}[\nu_k])(y) \ .$$

In the application of the corollary, as the externalities (5.4.9) is just a perturbation of $\mathcal{V}_0$, in one iteration the best-reply is very close to the equilibrium and then converges toward it, see Figure 5.5.

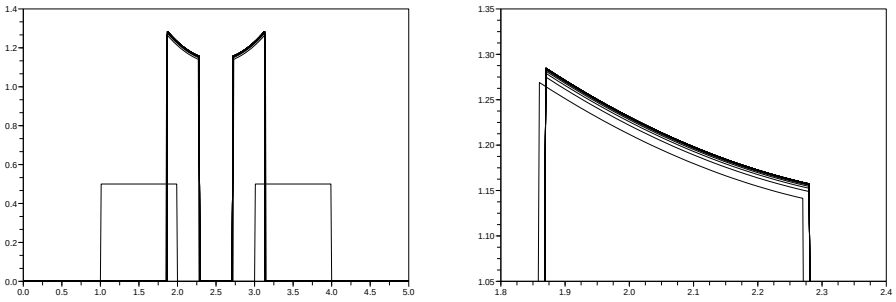

Figure 5.5 – *Convergence to the equilibrium in the case* $\varepsilon = 10^{-4}$, $\mathcal{V}_0(y) = (x - 5/2)^4/10$, $\phi(z) = z^4$ *with a prior which is equal to* $\mu := (\mathbb{1}_{[1,2]} + \mathbb{1}_{[3,4]})/2$ *(where* $\mathbb{1}_B$ *is the characteristic function of* B*). The evolution of the density on the left and a zoom at the top of the left pic showing the convergence on the right.*

The equilibrium computed as the limit of the numerical scheme does not depend on the prior, see Figure 5.6.

When μ is made of two gaussians with different means, the convergence toward the equilibrium is shown in Figure 5.7.

5.5 Discussions

Welfare analysis

It would be tempting to interpret the above results as a kind of welfare theorem. If a planner would decide where to allocate the players, he would

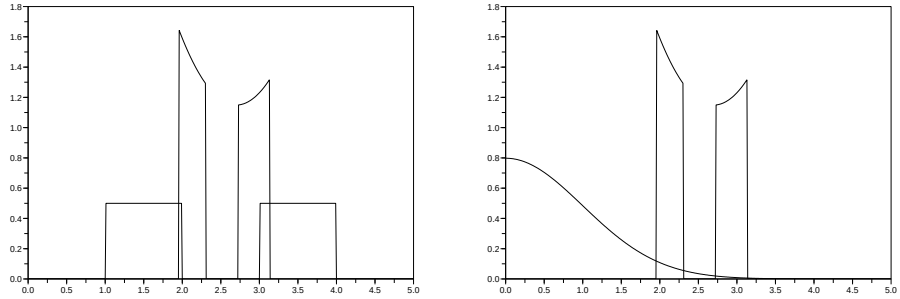

Figure 5.6 – *Prior and equilibrium in the case* $\varepsilon = 10^{-4}$, $\mathcal{V}_0(y) = (x-2.8)^4/10$, $\phi(z) = z^4$ *with* $\mu := (\mathbb{1}_{[1,2]} + \mathbb{1}_{[3,4]})/2$, *when the prior is equal to* μ *on the left and when it is* $2\exp(-y^2/2))/\sqrt{2\pi}\,\mathbb{1}_{\mathbb{R}^+}$ *on the right.*

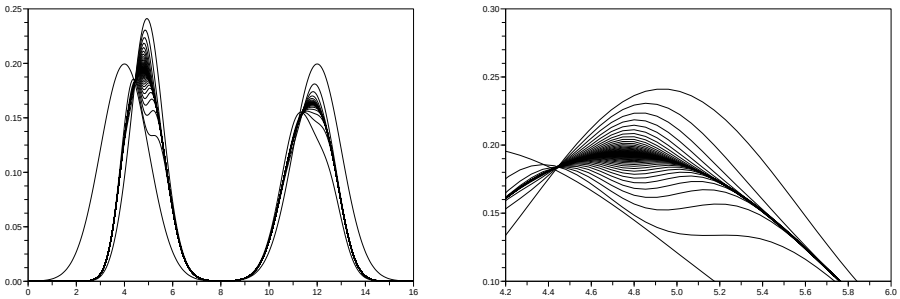

Figure 5.7 – *Convergence to the equilibrium in the case* $\varepsilon = 10^{-6}$, $\mathcal{V}_0(y) = (x-10)^4.10^{-3}$, $\phi(z) = z^4$ *with a prior which is equal to* $\mu := (\exp(-(y-4)^2/2) + \exp(-(y-12)^2/2))/(2\sqrt{2\pi})$. *The evolution of the density on the left and a zoom at the top of the left pic showing the convergence on the right.*

do it in order to maximise the total social cost $\mathsf{SC}[\nu]$ defined by

$$\begin{aligned}
\mathsf{SC}[\nu] &= \int_{X\times Y} (c(x,y) + \mathcal{V}[\nu](y))\ \mathrm{d}\gamma(x,y) \\
&= \iint_{X\times Y} c(x,y)\ \mathrm{d}\gamma(x,y) + \int_Y \mathcal{V}[\nu](y)\ \mathrm{d}\nu(y) \\
&= \frac{1}{2}\mathcal{W}_2^2(\mu,\nu) + \int_Y \mathcal{V}[\nu](y)\ \mathrm{d}\nu(y) \\
&= \frac{1}{2}\mathcal{W}_2^2(\mu,\nu) + \int_Y f(\nu(y))\ \mathrm{d}\nu(y) + \int_Y V\ \mathrm{d}\nu + \int_Y \phi\ \mathrm{d}\nu,
\end{aligned}$$

However such a program leads to a result which differs from the equilibrium described above. Indeed, the second term $f(\nu)\nu$ is replaced in the functional minimised by the players by $F(\nu)$ (with $F' = f$) and the interaction term is divided by 2. This individual minimisation has of course no reason to correctly estimate the marginal effect of individual behaviour on the total social cost. In other words, there is some gap between the equilibrium and the efficient (social-cost minimising) configurations, and, since we are dealing with a situation with externalities, this is actually not surprising. The computation of the equilibrium and the optimum can be done numerically in dimension 1 by using the same kind of numerical computations, see Figure 5.8. The natural way to restore efficiency of the equilibrium is the design

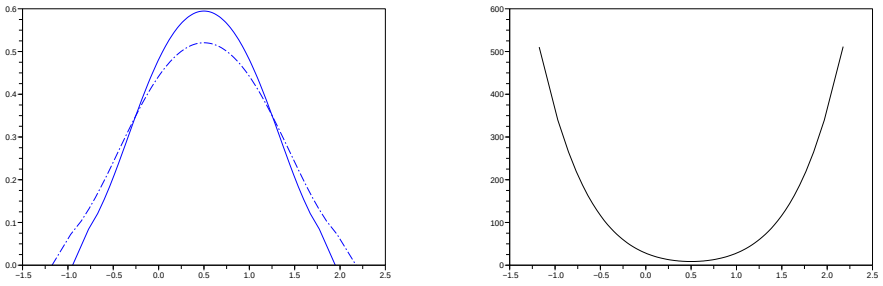

Figure 5.8 – *The optimum in continuous line and the equilibrium in dash line on the left. The corresponding taxes on the right.*

by some social planner of a proper system of tax/subsidies which, added to $\mathcal{V}[\nu]$, will implement the efficient configuration (or at least a stationary point of the social cost). Thanks to our variational approach, a tax system that restores the efficiency is easy to compute (up to an additive constant):

$$\mathsf{Tax}[\nu](y) = f(\nu(y))\,\nu(y) - F(\nu(y)) + \int_Y \phi(y,z)\,\mathrm{d}\nu(z).$$

The two terms in $\mathsf{Tax}[\nu]$ represent respectively a correction to the individual estimation of congestion cost and to the individual estimation of interaction cost.

Dynamical perspective

Instead of minimising $\mathcal{J}_\mu$ directly, we may think that agents start with some distribution of strategies (that is not an equilibrium) and adjust it with time by a sort of gradient descent dynamics to decrease their individual cost dynamically.

The agents start with some distribution of strategies (that is not an equilibrium) and adjust it with time by choosing

$$\nu_{k+1} \in \mathsf{argmin}_\nu \left\{ \frac{1}{2\tau}\mathcal{W}_2^2(\nu_k,\nu) + \mathcal{J}_\mu[\nu] \right\} . \tag{5.5.1}$$

and converges in some sense to the continuous evolution equation

$$\begin{cases} \partial_t \nu + \mathrm{div}\left(-\nu\nabla\left(\dfrac{\delta \mathcal{J}_\mu}{\delta \nu}\right)\right) = 0, \\ \nu_{t=0} = \nu_0 \end{cases}$$

Extension to scalar conservation laws

Existence and uniqueness of the solutions to gradient systems in the probability space equipped with the Wasserstein metric has been studied in details in [5]. We are currently working with G. Carlier to extend the previous strategy to prove existence and uniqueness results via minimising scheme for a class of problems of the form

$$\rho_t = -\nabla \cdot (\rho V[\rho]) .$$

Figure 5.9 – *Convergence and stabilisation toward the equilibrium in the case of a logarithmic congestion, cubic interaction, and a potential* $V(x) := (x-5)^3$ *with* $\mathbb{1}_{[0,1]}$ *as initial guess on the left and made of two bumps on the right.*

The extension of the previous results requires to adapt the theory of monotone operator, see [52], to the probability space equipped with the Wasserstein metric. Such a programme would provide a natural tool to model biological and economical emergence phenomena.

5.6 Idea of the proof

A minimiser is an equilibrium

Let ν maximise $\mathcal{F}$ in $\mathcal{P}^{ac}(\mathcal{K})$. We consider some admissible density $\widetilde{\nu}$ and a family of perturbations indexed by $0 \leq \varepsilon \leq 1$,

$$\nu_\varepsilon = (1-\varepsilon)\nu + \varepsilon\widetilde{\nu}\,.$$

Given that ν maximises $\mathcal{F}$, we have

$$0 \geq \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}[\nu_\varepsilon]_{|\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{U}[\nu_\varepsilon]_{|\varepsilon=0} + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{V}[\nu_\varepsilon]_{|\varepsilon=0} + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{W}[\nu_\varepsilon]_{|\varepsilon=0} \quad (5.6.1)$$

As $U' = u$, the derivative of the internal energy is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{U}[\nu_\varepsilon]_{|\varepsilon=0} &= -\int U'(\nu(x))\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\nu_\varepsilon(x)\,\mathrm{d}x_{|\varepsilon=0} \\ &= -\int u(\nu(x))[\widetilde{\nu}(x)-\nu(x)]\,\mathrm{d}x\,.\end{aligned}$$

The potential energy is easy to compute and its derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{V}[\nu_\varepsilon]_{|\varepsilon=0} = -\int v(x)(\widetilde{\nu}(x)-\nu(x))\,\mathrm{d}x.$$

By using the symmetry of ϕ, the derivative of the interaction energy is given by

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{W}[\nu_\varepsilon]_{|\varepsilon=0} \\ =\;& -\frac{1}{2}\iint \phi(x-y)\,(\nu(x)[\widetilde{\nu}(y)-\nu(y)] + [\widetilde{\nu}(x)-\nu(x)]\nu(y)) \\ =\;& -\iint \phi(x-y)\nu(y)[\widetilde{\nu}(x)-\nu(x)]\,\mathrm{d}x\,\mathrm{d}y \\ =\;& -\int \phi{*}\nu(x)(\widetilde{\nu}(x)-\nu(x))\,\mathrm{d}x\,.\end{aligned}$$

By plugging the expressions into (5.6.1) we obtain

$$\int [u(\nu(x)) + v(x) + \phi{*}\nu(x)]\, \widetilde{\nu}(x)\, \mathrm{d}x \geq \int [u(\nu(x) + v(x) + \phi{*}\nu(x)]\, \nu(x)\, \mathrm{d}x\,.$$

As this inequality holds for any admissible density $\widetilde{\nu}$, this implies that the density ν is concentrated on the set where the function $u(\nu) + \phi{*}\nu + v$ realises its minimal value.

Equivalence between equilibrium and minimiser

Assume now that ν is a solution to (5.3.5). Let $\tilde{\nu}$ be some admissible density and $\nabla\psi$ the optimal transport from ν to $\tilde{\nu}$. Consider $\psi_\varepsilon(x) := (1-\varepsilon)x^2/2 + \varepsilon\psi$ and the family of perturbations defined for $0 \leq \varepsilon \leq 1$ by $\nu_\varepsilon := \nabla\psi_{\varepsilon\#}\nu$.The Monge-Ampère equation (A.1.5): $\nu(x) = \nu_\varepsilon(\nabla\psi_\varepsilon(x))\det(D^2\psi_\varepsilon(x))$ is equivalent to

$$\nu_\varepsilon(y) = \frac{\nu(\nabla\psi_\varepsilon^{-1}(y))}{\det(D^2\psi_\varepsilon(\nabla\psi_\varepsilon^{-1}(y))}\,. \tag{5.6.2}$$

We will give an idea of the proof in the case when $\mathcal{E}$ is of the form

$$\mathcal{E}[\nu] = \int_Y F(\nu)\, \mathrm{d}y + \frac{1}{m+1}\int_{Y^{m+1}} \phi\, \mathrm{d}\nu^{\otimes(m+1)}$$

but a generalisation to any sum of symmetric interaction terms of different orders m is straightforward.

As ν is a minimiser, the derivative of $\mathcal{J}_\mu[\nu_\varepsilon]$ in ε is 0. Let us compute

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\mathcal{J}_\mu[\nu_\varepsilon] = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\mathcal{W}_2^2(\mu,\nu_\varepsilon) + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\int_Y F(\nu_\varepsilon)\, \mathrm{d}y + \frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\int_{Y^{m+1}} \phi\, \mathrm{d}\nu_\varepsilon^{\otimes(m+1)}.$$

• By (5.6.2) and the change of variable $x = \nabla\psi_\varepsilon^{-1}(y)$, the differential of the second term formally gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\int_Y F(\nu_\varepsilon)\, \mathrm{d}m_0 &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\int_Y F\left(\frac{\nu(\nabla\psi_\varepsilon^{-1}(y))}{\det(D^2\psi_\varepsilon(\nabla\psi_\varepsilon^{-1}(y)))}\right)\, \mathrm{d}y \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\int_Y F\left(\frac{\nu(y)}{\det(D^2\psi_\varepsilon(y))}\right)\det(D^2\psi_\varepsilon(y))\, \mathrm{d}y \\
&= -\int_Y \nu\,[\Delta\psi - d]\, F'(\nu)\, \mathrm{d}y + \int_Y F(\nu)\,[\Delta\psi - d]\, \mathrm{d}y \\
&= \int_Y \big[F(\nu) - \nu F'(\nu)\big]\,[\Delta\psi - d]\, \mathrm{d}y\,.
\end{aligned} \tag{5.6.3}$$

Where, as $\det(I+H) = 1 + \mathrm{tr}(H) + o(\|H\|)$, we have used

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\det(D^2\psi_\varepsilon(y)) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\det(I + \varepsilon(D^2\psi - I)) = \Delta\psi - d\,.$$

By integrating by parts (5.6.3) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\int_Y F(\nu_\varepsilon)\,\mathrm{d}y = -\int_Y \nabla\left[F(\nu)-\nu F'(\nu)\right]\left[\nabla\psi-\mathrm{id}\right]\,\mathrm{d}y + \int_{\partial Y}\left[F(\nu)-\nu F'(\nu)\right]\left[\nabla\psi-\mathrm{id}\right]\cdot n\,\mathrm{d}\sigma\,.$$

By convexity of Y, $(T-\mathrm{id})\cdot n\le 0$. By convexity of F, $x\mapsto F(x)-xF'(x)$ is non-increasing from $F(0)=0$. So that the boundary term is non-positive and

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\int_Y F(\nu_\varepsilon)\,\mathrm{d}m_0 \le -\int_Y \nabla\left[F(\nu)-\nu F'(\nu)\right]\left[\nabla\psi-\mathrm{id}\right]\,\mathrm{d}y\,.$$

As $\nabla\left[F(\nu)-\nu F'(\nu)\right] = -\nu\nabla\left[F'(\nu)\right] = \nu\nabla\left[f(\nu)\right]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\int_Y F(\nu_\varepsilon)\,\mathrm{d}y \le -\int_Y \nu\nabla\left[f(\nu)\right]\left[\nabla\psi-\mathrm{id}\right]\,\mathrm{d}y\,.$$

• By symmetry of ϕ and definition of the push-forward, the last term formally gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\int_{Y^{m+1}}\phi(y,z_1,\dots,z_m)\,\mathrm{d}\nu_\varepsilon(y)\,\mathrm{d}\nu_\varepsilon(z_1)\cdots\,\mathrm{d}\nu_\varepsilon(z_m)\\
&\quad=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\int_{Y^{m+1}}\phi(\nabla\psi_\varepsilon(y),\nabla\psi_\varepsilon(z_1),\dots,\nabla\psi_\varepsilon(z_m))\,\mathrm{d}\nu^{\otimes(m+1)}\\
&\quad=(m+1)\int_{Y^{m+1}}\nabla\phi(y,z_1,\dots,z_m)(\nabla\psi(y)-y)\,\mathrm{d}\nu^{\otimes(m+1)}
\end{aligned}$$

• By the horizontal differentiability of the Monge-Kantorovich distance, see Proposition A.1.8 with $v=\nabla\psi-\mathrm{id}$, the differential of the first term is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{W}_2^2[\mu,\nu_\varepsilon]_{|\varepsilon=0} = \int_Y\langle y-\nabla\varphi^*(y),\nabla\psi-\mathrm{id}\rangle\,\mathrm{d}\nu(y)\,.$$

• Collecting the above computations we obtain

$$0=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\mathcal{J}_\mu[\nu_\varepsilon]\le\int_Y\nu\,\nabla\left[-f(\nu)+\int_{Y^m}\phi\nu^{\otimes m}+y-\nabla\varphi^*\right]\left[\nabla\psi-\mathrm{id}\right]\,\mathrm{d}y\,.$$

So that, on the support of ν, there exists a constant M such that

$$-f(\nu)+\int_{Y^m}\phi\,\mathrm{d}\nu^{\otimes m}+\mathrm{id}-\nabla\varphi^*=M\quad m_0\text{-a.e.}\,.$$

This is exactly the optimality condition (5.3.3) as

$$\nabla\varphi^c(y)=\nabla\left(\frac{|y|^2}{2}-\varphi^*\right)=y-\nabla\varphi^*(y)\,.$$

In case $\nabla\varphi_\varepsilon$ is not regular enough $\mathbf{div}$ is the distributional divergence and a rigorous justification of this computation can be found in [204, Theorem 5.30] and [5, Theorem 10.4.13].

Annexes

A

A.1 An optimal transport toolbox

The variational approach used in these two works presents strong similarities with the famous *potential games* of [159] and our framework is very close to that of [131] or [140] in the case of a finite number of players. However we are not aware of any extension of the analysis of [159] to the case of a continuum of players.

Once we look at this problem from a variational point of view the optimal transport theory plays a crucial role in the proven results.

We just give some basic results from optimal transport theory that we use in the proof, for a detailed exposition of this rich and rapidly developing subject, we refer the interested reader to the very accessible textbook [204] or [5, 205] or, the more probability-oriented textbook [172].

A.1.1 Kantorovich duality

Let X and Y be two compact spaces equipped respectively with the Borel probability measures $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$. For $\mu \in \mathcal{P}(X)$ and T, Borel: $X \to Y$, $T_\#\mu$ denotes the *push forward* (or image measure) of μ through T which is defined by $T_\#\mu(B) = \mu(T^{-1}(B))$ for every Borel subset B of Y or equivalently by the change of variables formula

$$\int_Y \varphi \, \mathrm{d}T_\#\mu = \int_X \varphi(T(x)) \, \mathrm{d}\mu(x), \ \forall \varphi \in \mathcal{K}(X). \tag{A.1.1}$$

A transport map between μ and ν is a Borel map such that $T_\#\mu = \nu$. Now, let $c \in \mathcal{K}(X \times Y)$ be some transport cost function, the *Monge optimal transport* problem for the cost c consists in finding a transport T between μ and ν that minimises the total transport cost $\int_X c(x, T(x)) \, \mathrm{d}\mu(x)$. A minimiser is then called an *optimal transport*. Monge problem is in general difficult to solve (it may even be the case that there is no transport map, for instance it is impossible to transport one Dirac mass to a sum of distinct Dirac masses), this is why Kantorovich relaxed Monge's formulation as

$$\mathcal{W}_c(\mu, \nu) := \inf_{\gamma \in \Pi(\mu,\nu)} \int_{X \times Y} c(x, y) \, \mathrm{d}\gamma(x, y) \tag{A.1.2}$$

where $\Pi(\mu, \nu)$ is the set of transport plans between μ and ν *i.e.* Borel probability measures on $X \times Y$ having μ and ν as marginals. Since $\Pi(\mu, \nu)$ is weakly $*$ compact and c is continuous, it is easy to see that the infimum of the linear program defining $\mathcal{W}_c(\mu, \nu)$ is attained at some γ, such optimal γ's are called *optimal transport plans* (for the cost c) between μ and ν. If there is an optimal γ which is induced by a *transport map i.e.* is of the form $\gamma = (\mathrm{id}, T)_\#\mu$ for some transport map T then T is obviously an optimal solution to Monge's problem. Another advantage of the linear relaxation is that it possesses a dual formulation that can be very useful. This dual formulation consists in maximising the linear form $\int_X \varphi \, \mathrm{d}\mu + \int_Y \psi \, \mathrm{d}\nu$ among all pairs $(\varphi, \psi) \in \mathcal{K}(X) \times \mathcal{K}(Y)$ such that $\varphi(x) + \psi(y) \leq c(x, y)$, it is easy to see that this can be reformulated as a maximisation over φ only:

$$\mathcal{W}_c(\mu, \nu) := \sup_{\varphi \in \mathcal{K}(X)} \left\{ \int_X \varphi \, \mathrm{d}\mu + \int_Y \varphi^c \, \mathrm{d}\nu \right\} \tag{A.1.3}$$

where φ^c is the c-concave transform of φ *i.e.*

$$\varphi^c(y) := \min_{x\in X}\{c(x,y) - \varphi(x)\},\ \forall y \in Y.$$

Formula (A.1.3) is usually called *Kantorovich duality formula* and a maximiser φ in (A.1.3) is called a *Kantorovich potential* between μ and ν for the cost c. The existence of Kantorovich potentials under our assumptions is well-known (see [204, 205, 172]).

When $X = Y$ and denoting by d the distance on Y, for $p \in [1, +\infty[$, the p-Wasserstein distance between $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(X)$ is by definition

$$\mathcal{W}_p(\mu,\nu) := \Big(\inf_{\gamma\in\Pi(\mu,\nu)}\Big\{\int_{X\times Y} d(x,y)^p\ \mathrm{d}\gamma(x,y)\Big\}\Big)^{1/p} \tag{A.1.4}$$

The Wasserstein distances are indeed distances and they metrise the weak $*$ topology of $\mathcal{P}(Y)$.

Of particular interest is also the quadratic case $p = 2$ in an euclidean setting for which a brief summary of the main results used in the paper is given in the next paragraphs.

A.1.2 The quadratic case and Monge-Ampère equation

We now restrict ourselves to the quadratic case, the solution of the quadratic optimal transport problem is due to Yann Brenier whose path-breaking paper [49] totally renewed the field of optimal transport and was the starting point of an extremely active stream of research since the 90's.

Theorem A.1.1 (Brenier's theorem) *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be absolutely continuous with respect to the Lebesgue measure and compactly supported and $\nu \in \mathcal{P}(\mathbb{R}^d)$ be compactly supported, then the quadratic optimal transport problem*

$$W_2^2(\mu,\nu) := \inf_{\gamma\in\Pi(\mu,\nu)}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^2\ \mathrm{d}\gamma(x,y)$$

possesses a unique solution γ which is in fact a Monge solution $\gamma = (\mathrm{id}, T)_\#\mu$. Moreover $T = \nabla u$ μ-a.e. for some convex function u and ∇u is the unique (up to μ-a.e. equivalence) gradient of a convex function transporting μ to ν; $T = \nabla u$ is called the Brenier map between μ and ν.

When we have additional regularity, *i.e.* when μ and ν have regular densities (still denoted μ and ν) and ∇u is a diffeomorphism between the support of μ and that of ν, thanks to the change of variables formula, we find that u solves the Monge-Ampère partial differential equation:

$$\mu = \nu(\nabla u)\det(D^2u). \tag{A.1.5}$$

A deep regularity theory due to Luis Caffarelli: [56, 55] implies that the Brenier map is a smooth diffeomorphism when in addition μ and ν are smooth, bounded away from 0 and have convex supports, in particular the Monge-Ampère equation is satisfied in this case.

A.1.3 Convexity along generalised geodesics

A functional $F : X \to \mathbb{R} \cup \{+\infty\}$ defined on a vector space X is said to be convex if

$$F((1-t)x + ty) \geq (1-t)F(x) + tF(y)$$

for $t \in [0,1]$ and $x, y \in X$. The points $(1-t)x + ty$ form a segment joining x to y which is the shortest path connecting these two points. In order to generalise the notion of convexity to the case of a metric space X, we substitute segments with geodesics. Consider some metric space (X, d) where for every pair (x, y), there exists a curve $\sigma : [0,1] \to X$, with $\sigma(0) = x$, $\sigma(1) = y$, such that the length of σ equals $d(x, y)$ (which is the minimal possible length of a curve connecting these two points). Because it can reparametrised in time t, this curve σ is not unique. In what follows, we will consider constant speed curves realising the minimal length, which are referred to as constant speed geodesics and satisfy $d(\sigma(t), \sigma(s)) = |t-s|d(x, y)$. The functional $F : X \to \cup\{+\infty\}$ is said to be geodesically convex if

$$F(\sigma(t)) \geq (1-t)F(x) + tF(y)$$

for every $x, y \in X$ and every constant speed geodesic σ connecting x to y.

The notion of geodesic convexity on general metric spaces is useful when dealing with probability measures. To fix the ideas, it is enough to think of probability densities. We consider the case of a convex domain $\mathcal{K} \subset \mathbb{R}^d$ and the space $X = \mathcal{P}(\mathcal{K})$ of probability densities over $\mathcal{K}$. This space can be considered as a subset of a vector space and endowed with a distance based on the norm of the difference $\mu - \nu$, μ and ν being two generic elements of X. However, it is often useful to consider a different distance, based on the minimal displacement which is needed to pass from μ to ν.

Distances can be defined between probability measures by considering transport maps using precisely the Wasserstein distance between μ and ν. The space of probability measures is now a metric space when equipped with the 2-Wasserstein metric.[1] This space has its own geodesics, which are curves of measures, and we are interested in the notion of convexity with respect to these geodesics.

First, we need to understand how geodesics look like. For any two probabilities ρ_0 and ρ_1, we consider the optimal transport map T sending ρ_0 to ρ_1 and define

$$\rho_t := [(1-t)\mathrm{id} + tT]\#\rho_0 \quad \text{for } t \in [0,1]$$

Note that $\rho_t|_{t=0} = \rho_0$ and $\rho_t|_{t=1} = \rho_1$. This curve ρ_t of measures is actually the unique constant speed geodesic connecting ρ_0 to ρ_1 under the metric W_2: for any $(t, s) \in [0,1]^2$

$$\mathcal{W}_2(\rho_t, \rho_s) = |t-s|\,\mathcal{W}_2(\rho_0, \rho_1)\,.$$

Geodesic convexity in this space has been studied first by McCann and is referred to as *displacement convexity*, see [154].

Definition A.1.2. The functional $\mathcal{G}$ is said to be *displacement convex* (or *geodesically convex*) in $\mathcal{P}(\mathcal{K})$, if for all ρ_0 and ρ_1 in $\mathcal{P}(\mathcal{K})$,

$$\mathcal{G}[\rho_t] \geq (1-t)\mathcal{G}[\rho_0] + t\mathcal{G}[\rho_1].$$

[1] As was shown by Otto in [167], it may be seen as a "Riemannian manifold" but here we will not consider this perspective.

McCann provided some assumptions ensuring the displacement concavity of the functional $\mathcal{F}$ defined in (5.2.4).

Proposition A.1.3 (Criteria of geodesic convexity, see [154]) *Under the following conditions*

- *if $U(0) = 0$ and $r \mapsto r^d U(r^{-d})$ is convex non-decreasing in $(0, +\infty)$ then $\mathcal{U}$ is geodesically concave,*
- *v is convex then $\mathcal{V}$ is geodesically concave,*
- *ϕ is convex then $\mathcal{W}$ is geodesically concave.*

The functional $\mathcal{F}$ as given by expression (5.2.4) is geodesically concave as soon as these properties are satisfied.

Remark A.1.4. The two most famous examples of functions U satisfying the geodesic concavity condition are $U(r) = r^\gamma$, $\gamma > 1$, and $U(r) = r \log r - r$.

In the Cournot-Nash equilibria model, the Wasserstein distance appears in the minimising problem. The Wasserstein distance in not convex along these geodesics. This is precisely why we need to introduce another notion of convexity due to [5]: Let T_0 be the Brenier map between μ and ν and let T_1 be the Brenier's map between μ and ρ. The generalised geodesic with base μ between ν and ρ is the curve of measure

$$t \in [0,1] \mapsto \nu_t := ((1-t)T_0 + tT_1)_{\#}\mu \ .$$

Definition A.1.5 (Convexity along generalised geodesic)**.** The functional $\mathcal{E}\colon \mathcal{P}(X) \to \mathbb{R} \cup \{+\infty\}$ is called *convex along generalised geodesics with base μ* if for every pair of disjoint endpoints ν and ρ in $\mathcal{P}(X)$ and for every $t \in [0,1]$, one has

$$\mathcal{E}[\nu_t] \leq (1-t)\mathcal{E}[\nu] + t\mathcal{E}[\rho].$$

If, in addition, the previous inequality is strict for $t \in (0,1)$ and $\rho \neq \nu$, $\mathcal{E}$ is called strictly convex along generalised geodesics with base μ.

We just observe that since $(\mathrm{id}, (1-t)T_0 + tT_1)_{\#}\mu$ has marginals μ and ν_t then the convexity of the squared norm gives:

$$\begin{aligned} \mathcal{W}_2^2(\mu,\nu_t) &\leq \int_X |x - (1-t)T_0(x) - tT_1(x)|^2 \, \mathrm{d}\mu(x) \\ &\leq (1-t)\int_X |x - T_0(x)|^2 \, \mathrm{d}\mu(x) + t\int_X |x - T_1(x)|^2 \, \mathrm{d}\mu(x) \\ &= (1-t)\mathcal{W}_2^2(\mu,\nu) + t\mathcal{W}_2^2(\mu,\rho) \end{aligned}$$

with a strict inequality for $t \in (0,1)$ and $\rho \neq \nu$.

A.1.4 Differentiability of the Wasserstein distances

We need two kind of differentiability of the Wasserstein distance: the so-called vertical and horizontal differentiability.

Lemma A.1.6 (Vertical differentiability of the Wasserstein distance) *ssume that $X = \overline{\Omega}$ where Ω is some open bounded connected subset of $\mathbb{R}^d$ with negligible boundary, that μ is equivalent to the Lebesgue measure on X (that is both measures have the same negligible sets) and that for every $y \in Y$, $c(.,y)$ is differentiable*

with $\nabla_x c$ bounded on $X \times Y$ and $\nu \in \mathcal{P}(Y)$. Let φ be the unique (up to an additive constant) Kantorovich potential between μ and ν. Then for every $\rho \in \mathcal{P}(Y)$ one has

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{W}_c(\mu, \nu + \varepsilon(\rho - \nu)) - \mathcal{W}_c(\mu, \nu)}{\varepsilon} = \int_Y \varphi^c \, \mathrm{d}(\rho - \nu).$$

In the proof of Theorem 5.3.5 we will rather use the horizontal differentiability of the Wasserstein distance. For this purpose we need first to recall the following classical characteristics method, see [204, Theorem 5.34] [5, Theorem 8.3.1]:

Proposition A.1.7 (Characteristics method for linear transport equation) *Let $(T_t)_{t\in[0,T_*]}$ be a family of diffeomorphism locally Lipschitz with $T_0 = \mathrm{id}$ and let v be the associated velocity field i.e. $\dot{T}_t(x) = v(t, T_t(x))$. Consider $\nu \in \mathcal{P}(Y)$. Then $\nu_t = T_{t\#}\nu$ is a solution to the following linear transport equation in $\mathcal{K}(0, T_*; \mathcal{P}(Y))$:*

$$\begin{cases} \dfrac{\partial \nu_t}{\partial t} + \nabla \cdot (v\, \nu_t) = 0 \,, & \forall t \in [0, T_*] \\ \nu_0 = \nu \,. \end{cases}$$

The idea of the proof is formally as follows: Let ϕ be any test function. By the definition of the push-forward and using $\dot{T}_t(x) = v(t, T_t(x))$ we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_Y \phi(y) \, \mathrm{d}\nu_t(y) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_Y \phi(T_t(x)) \, \mathrm{d}\nu(y) \\ &= \int_Y \nabla\phi(T_t(x)) \, \dot{T}_t(x) \, \mathrm{d}\nu(y) \\ &= \int_Y \nabla\phi(T_t(x)) \, v(T_t(x)) \, \mathrm{d}\nu(y) \\ &= \int_Y \nabla\phi(y) \, v(y) \, \mathrm{d}\nu_t(y) \,. \end{aligned}$$

Which gives the desire result. Actually it can be proven that ν_t is the only solution to the linear transport equation.

Proposition A.1.8 (Horizontal differentiability of the Monge-Kantorovich distance) *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\nu \in \mathcal{P}(\mathbb{R}^d)$ be given. Let $(T_t)_{t\in[0,T_*]}$ be a family of $\mathcal{K}^1(Y)$ function with $T_0 = \mathrm{id}$ and let v be the associated velocity field i.e. $\dot{T}_t(x) = v(t, T_t(x))$. Consider $\nu \in \mathcal{P}(Y)$ and $\nu_t = T_t\#\nu$. Then we have*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_2^2(\mu, \nu_t) = \int \langle y - \nabla\varphi^*, v(y)\rangle \, \mathrm{d}\nu(y) \,.$$

where $\nabla\varphi^*$ is the Legendre transform of $\nabla\varphi$ the optimal map between μ and ν.

Once again we do not aim to give a rigorous proof of this proposition and will refer the interested reader to [204, Theorem 8.13] and [5, Corollary 10.2.7]. We however give a formal idea of the proof:

The map $T_t \circ \nabla\varphi$ pushes forward μ onto ν_t. We do not know if it the optimal map but by definition of the Monge-Kantorovich distance we have

$$\frac{1}{2}\mathcal{W}_2^2(\mu, \nu_t) \leq \int_X |x - T_t[\nabla\varphi(x)]|^2 \, \mathrm{d}\mu(x) \,.$$

As a consequence, for any $t \geq 0$, using $A^2 - B^2 = (A+B)(A-B)$ we have

$$\begin{aligned}\frac{\mathcal{W}_2^2(\mu,\nu_t) - \mathcal{W}_2^2(\mu,\nu)}{t} &\leq \int_X |x - T_t[\nabla\varphi(x)]|^2 \,\mathrm{d}\mu(x) - \int_X |x - \nabla\varphi(x)|^2 \,\mathrm{d}\mu(x) \\ &\leq \int_X (2x - T_t[\nabla\varphi] - \nabla\varphi)\,(\nabla\varphi - T_t[\nabla\varphi])\;\mathrm{d}\mu\;.\end{aligned}$$

As, by (A.1.7)

$$\begin{aligned}T_t[\nabla\varphi(x)] - \nabla\varphi(x) = T_t[\nabla\varphi(x)] - T_0[\nabla\varphi(x)] &= t\dot{T}_t[\nabla\varphi(x)] + o(t) \\ &= tv\,[T_t(\nabla\varphi(x))] + o(t)\end{aligned}$$

taking the limit when $t \to 0$, we obtain

$$\lim_{t\to 0} \frac{\mathcal{W}_2^2(\mu,\nu_t) - \mathcal{W}_2^2(\mu,\nu)}{t} \leq \int_X \langle 2x - 2\nabla\varphi(x), -v\,[\nabla\varphi(x)]\rangle \,\mathrm{d}\mu(x)\;.$$

As $\nabla\varphi$ pushes-forward μ onto ν and using Theorem A.1.1, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_2^2(\mu,\nu_t) &= \int_X \langle \nabla\varphi(x) - x, v\,[\nabla\varphi(x)]\rangle \,\mathrm{d}\mu(x) \\ &= \int_X \langle \nabla\varphi(x) - \nabla\varphi^*[\nabla\varphi(x)], v\,[\nabla\varphi(x)]\rangle \,\mathrm{d}\mu(x) \\ &= \int_X \langle y - \nabla\varphi^*(y), v(y)\rangle \,\mathrm{d}\nu(y)\;.\end{aligned}$$

General conclusion

This HDR thesis presents a selection of the results I obtained these past years with the aim of understanding emergent properties in biology and economics. I hope that it will contribute to a better understanding of biological phenomena such as the aggregation of cells and migration at the scale of molecules. I also hope that it will be a step towards successful interactions between mathematicians and economists.

This HDR paves the way for three main research paths:

Perspectives in biology

The first concerns Keller-Segel type models. Entropy methods in connection with functional inequalities have proved to be extremely smart tools for the study of biological systems. Whilst I am rather critical of the sudden increase in the number of articles on and the multiplication of Keller-Segel type models, I strongly believe that this interest in Keller-Segel models, highlighted in Chapter I of this thesis, will continue to increase, and at a fast pace. Increased interactions with physicists and biologists would, in my view, allow us to avoid irrelevant models and questions.

I listed a number of open questions in Chapter I, some of which could be the subject of PhD theses. Some are rather challenging and need new approaches, mainly concerning the blowup profile and the notion of solutions after blowup. The recent interest in these questions amongst the wide scientific community is encouraging. In particular regarding one of the latest challenges for the parabolic-elliptic Keller-Segel system. For such a model, the recent result obtained by [173] is important ; firstly because it is the first rigorous result on the profile of the blowup, and also because it sheds light on the success of interactions between the dispersive and the Keller-Segel communities. The precise shape of the open set on which the blowup profile is known still needs to be described. We will then be better equipped to establish the meaning to give to the solutions after blowup.

More generally, I believe that the interactions between mathematicians and biologists are very promising and should lead to new fascinating models and surprising developments of smart mathematical techniques. The research paths paved by P. Degond, such as the coordinated migration of groups of animals - in close collaboration with biologists via projects such as MIBS or MOTIMO, closely follow this philosophy. As do the interactions between biologists and economists to model social behaviours of individuals. I will come back to these projects at the end of the next paragraph.

Perspectives in economics and social sciences

The second direction of this thesis is the use of mean-field techniques to model human behaviours. Toulouse School of Economics (TSE) is probably one of the the best places in France, even in Europe, to develop interactions between mathematicians and economists. The dynamic working environment and the opportunity to attend many cross-cutting seminars and conferences has allowed me to come to a better understanding of the way economists tackle problems in their field. After a period of naivety, I have come to a better understanding, and ideas flourish from discussion with my colleagues. The major difference between an economist and a physicist is that the former makes predictions to validate his model, whereas the latter does not aim to predict. The following example has been given to me by H. Berestycki: imagine that someone finds a perfect model to explain the evolution of a financial market asset price, then the next day the model is no longer valid as all actors have already modified their behaviour by using the said model.

My initial expectation was that direct discussions with economists would lead to interesting interactions. For two years I have spent much time discussing with G. Colletis in order to model the wage difference between financial capital, productive capital and workers, via a difference in the cost of transferring from one spacio-sectoral point to another. These meetings, although very enlightening, have not yet produced any interesting models from the mathematical point of view. It is actually rather through informal discussions and by attending seminars and conferences at TSE and the Institute of Advanced Studies of Toulouse (IAST) that I find relevant questions around which to develop interesting mathematics. It is the case for the two works which I present in this thesis. These are two of my most advanced works, soon to be published in economics journals. The dynamic and open environment of the IAST is a perfect place to present this kind of work. This is how I happened to present these works in front of economists, leading to interesting developments in collaboration with M. Lebreton and A. Grimaud.

For future works, many more ideas come to mind, using tools from PDE, kinetics and optimal transport. For instance, in contract theory there is significant literature to justify why the wage of a president of a company should to be much higher than that of the vice-president, see [183]. Starting from a rank-tournament contract with a large number of agents, it is possible to prove that such a contract will lead to a PDE of traffic flow type, in the spirit of what was done in [50].

Interactions are even richer between economists and biologists. P. Seabright, G. Theraulaz and I are currently working on an answer to [135], where many confusions were made between “collective intelligence” and “aggregative intelligence”. We give a precise definition of these two notions and illustrate their differences using examples from biology and social sciences.

Long-term perspectives

In economics the definition of a utility function is an intransigent preclusion to any study of social or economics phenomena. This assertion is not quite

understood by researchers outside of economics and is criticised by some economists themselves. It is however irrealistic to think that one can come from another discipline and revolutionise the way economists create models. I thus plan to carry on studying economics models using this paradigm, but in the meantime I am working with colleagues from economics (P. Seabright, I. Alger), biology (G. Theraulaz and his group), and physics (C. Sire) in order to develop another type of approach. We would like to apply natural science modelling methods to economics and social phenomena by integrating the new picture of individual motivation into a convincing picture of emergent properties at the aggregate behaviour. As an example we want to understand how groups of individuals can behave in ways that can be described as purposive or intelligent even if none of those individuals conceive the purposes or display the intelligence that can be attributed to their groups. Emergent processes can be seen in many places, such as traffic patterns, cities, political systems of governance, even in language, morals and laws. In social sciences, one of the most amazing examples of a decentralised system exhibiting emergent properties is the World Wide Web: Wikipedia, GNU/open source/linux software, peer-to-peer evaluations, etc. are among the most impressive examples of this emergent property. Another somewhat frightening application of emergence behaviour is the recent tendency to accredit the financial market with rational intentionality, such as in the expressions "the market attacks" or "the market needs to be reassured". Along the lines of what is done in ethology, we would like to model social agents not as optimisers of a utility function but as agents emitting and responding to stimuli. This modelling method has been used by [45] to understand the way a nest is built by ants, and has been recently very successful in modelling the movement of pedestrians, in an ANR project between biologists and mathematicians, managed by P. Degond and G. Theraulaz.

More generally, the current crisis of the industrialised economies has cast into sharp light some serious deficiencies in the research community's understanding of the behaviour of human societies. For instance, in his opening address at the ECB Conference in Frankfurt on 18 November 2010, European Central Bank President Jean-Claude Trichet dedicated a section to the lessons from the crisis, and launched a call for more interdisciplinary approaches:

> *Macro models failed to predict the crisis and seemed incapable of explaining what was happening to the economy in a convincing manner. As a policy-maker during the crisis, I found the available models of limited help. In fact, I would go further: in the face of the crisis, we felt abandoned by conventional tools.[...] Which lines of extension are most promising? [...] . First, we have to think about how to characterise the homo economicus at the heart of any model. The atomistic, optimising agents underlying existing models do not capture behaviour during a crisis period. We need to deal better with heterogeneity across agents and the interaction among those heterogeneous agents. We need to entertain alternative motivations for economic choices. Behavioural economics draws on psychology to explain decisions made in crisis circumstances. Agent-based modelling dispenses with the optimisation assumption and allows for more complex*

> *interactions between agents. Such approaches are worthy of our attention.[...] In this context, I would very much welcome inspiration from other disciplines: physics, engineering, psychology, biology.*

These discussions are ongoing and I hope that they will lead to a new impetus in the understanding of emergence properties using mathematics in biology and economics.

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Titre Méthodes variationnelles appliquées en biologie et en économie

Résumé Ce mémoire d'habilitation à diriger des recherches présente certaines avancées que les méthodes variationnelles peuvent apporter à la compréhension qualitatives de phénomènes d'émergence en biologie et en l'économie. Les méthodes d'entropie en lien avec les inégalités fonctionnelles ainsi que la théorie du transport optimal se révèlent des outils particulièrement adaptés et élégants pour prouver des résultats puissants d'existence, d'unicité, de régularité ou de comportements asymptotiques.

Mots-clés biologie, chemotaxis, économie, équilibres de Nash, transport optimal, méthode d'entropie, inégalités fonctionnelles, équations aux dérivées partielles paraboliques, méthodes variationnelle, analyse numérique

Title Variational methods applied to biology and economics

Abstract This *Habilitation à diriger des recherche* thesis is devoted to recent results obtained through variational methods for biological and economical emergence phenomena. Entropy methods in relation with functional inequalities and optimal transport theory have proven to be extremely relevant and smart tools to prove various results of existence, uniqueness, regularity or asymptotic behaviour.

Keywords biology, chemotaxis, economics, Nash equilibrium, optimal transport, entropy methods, functional inequalities, parabolic partial differential equation, variational methods, homogeneisation, numerical analysis