Liquidity Trap and Stability of Taylor Rules[*]

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Abstract

We study a productive economy with fractional cash-in-advance constraint on consumption expenditures. Government issues safe bonds and levies taxes to finance public expenditures, while the Central Bank follows a feedback Taylor rules by pegging the nominal interest rate. We show that when the nominal interest rate is bound to be non-negative, under active policy rules a Liquidity Trap steady state does emerge besides the Leeper (1991) equilibrium. The stability of the two steady states depends, in turn, upon the amplitude of the liquidity constraint. When the share of consumption to be paid cash is set lower than one half, the Liquidity Trap equilibrium is indeterminate. The stability of the Leeper equilibrium too depends dramatically upon the amplitude of the liquidity constraint: for low amplitudes of the latter, the Leeper equilibrium, can be indeed stable. Policy and Taylor rules are thus theoretically rehabilitated since their targets, by contrast with a vast literature, may be reached for infinitely many agents' beliefs. We also show that a relaxation of the liquidity constraint is Pareto-improving and that the Liquidity Trap equilibrium Pareto-dominates the Leeper one, in view of the zero cost of money.

Keywords: Cash-in-Advance; Liquidity Trap; Monetary Policy; Multiple Equilibria.

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1   Introduction

Policy makers since a long time, when conducting the monetary policy, follow active Taylor rules consisting in changing nominal interest rates more than one-for-one when inflation deviates from a given target. This policy aims at keeping inflation anchored to its long run target and thus to stabilize the monetary and financial sector of the economy. The traditional approach is based upon the analysis of the linearized versions of the truly non-linear models and focuses on a local analysis of the Leeper (1991) equilibrium: under active Taylor rules, such an equilibrium is locally unstable and thus unique. This approach, actually, may be misleading since it rules out the study of all the trajectories, as the Liquidity Trap, other than those consistent with the targets. Benhabib *et al.* (2001), indeed, show that the combination of active Taylor rules and a zero bound on the nominal interest rate creates a new steady state for the economy, a steady state that the authors call "unintended". Moreover, this steady state is stable and thus the economy converges toward it giving raise to a deflationary path with very low levels for the nominal interest rate. As Bullard and Russell (1999) argue, such a feature is consistent with the deflation and nominal interest rate regime observed in Japan in the recent years (see Krugman (1998); Bernanke (1999); Meltzer (1999)). Moreover, the issue of Liquidity Trap during the 1929 crisis has been a source of intense debate in the cliometric literature (see among others, Damette and Parent (2016), Damette and Parent (2017); Basile *et al.* (2010); Romer (1992); Romer (2009); Friedman and Schwartz (1963); James (2001); Hanes (2006) and Gandolfi (1974)). However, Benhabib *et al.* (2001) theoretical findings have not received great attention in the policy debate which appears to be deaft to the perils underlying the conduct of standard and celebrated Taylor rules.

Besides early contributions as that of Brunner and Meltzer (1968), other recent theoretical developments suggest new avenues for research. Orphanides and Wieland (1998) are among the first to have re-addressed the issue in the last twenty years. Schmitt-Grohé and Uribe (2009) call into question the zero interest rate policy as an appropriate strategy to escape the Liquidity Trap occurrence. They demonstrate that pursuing a zero interest rate policy is not a way to escape Liquidity Trap but, on the contrary, leads to maintain the economy in a stable Liquidity Trap equilibrium. Airaudo and Zanna (2012) show that Taylor rules generate, besides liquidity traps, also aggregate instability, endogenous cycles and chaotic dynamics in open economies. Svensson (2000) too studies the liquidity trap in an open economy. Kudoh and Nguyen (2010) analyze the effects of fiscal policy in an economy in which the Central Bank pursue Taylor rules. They find that they depend dramatically upon the conduct of monetary policy. Schmitt-Grohé and Uribe (2013) observe that the great contraction of 2008 pushed the US economy into a lasting Liquidity

Trap characterized by zero nominal interest rates and inflation expectations below the target. At the same time, they find that output growth is recovered but nevertheless unemployment is increased. They refer to such a configuration as to "jobless recovery".

In our paper, following Benhabib *et al.* (2001), we confirm the utility of carrying out a global analysis rather than a pure local one, in order to unveil the existence of equilibrium outcomes other than the Leeper (1991) one, as Liquidity Traps, expectations-driven fluctuations and deterministic cycles. In our study, government issues bonds and levies taxes to finance public expenditures, while the Central Bank follows a feedback Taylor rules by pegging the nominal interest rate. The model is in infinite-horizon with endogenous labor supply and fractional cash-in-advance constraint on consumption expenditures as in Bosi *et al.* (2005). Within such a framework, we characterize the existence of stationary solutions and establish the conditions under which the different equilibria are or are not stable. All these things taken together lead to threshold phenomena in terms of the degree of liquidity of the economy such that, once one passes through them, some relevant change of instability does occur.

We consider both the case of passive and active Taylor rules following the definition of Benhabib *et al.* (2001) and of Schmitt-Grohé and Uribe (2000). In our framework we refer to active interest rate feedback rules in the case of rules that respond to increases in inflation with a more than one-for-one increase in the nominal interest rate. As a consequence, a passive interest rate feedback rule is such that the nominal interest rate reacts in a less than one-for-one increase in inflation. Since we refer to active or passive interest rate feedback rules in this spirit, the elasticity of the nominal interest rate with respect to the inflation rate falls within the above definition according to its magnitude which can be larger than one (active Taylor rules) or lower than one (passive Taylor rules). We find that under "passive" Taylor rules there is always an unique steady state, which can correspond either to the Leeper equilibrium or to the Liquidity Trap one. As is the case in Benhabib *et al.* (2001), under active Taylor rules, there may appear two stationary solutions simultaneously, one corresponding to the long-run Taylor target, the other sticking to a zero nominal interest rate.

We first Pareto-rank all the stationary equilibrium candidates and determine the GDP associated to each of them. Actually, we find that the Liquidity Trap equilibrium Pareto-dominates the Taylor target, since the former entails a zero cost of money holding. However, in correspondence to the Liquidity Trap equilibrium, the Central Bank is not more able to further revive economic activity by an additional cut in the interest rate, since the latter is already sticked to its minimum level. We simultaneously show that as soon as the share of consumption to be paid cash decreases, the welfare associated to both the Taylor equilibrium and the liquidity one increases. This result is easily interpretable, once one keeps in mind

that such a share represents the degree of financial market imperfection. By relaxing it, agents can thus buy more and more avoid the transaction cost represented by the nominal interest rate. Of course, when the latter is zero, such a cost vanishes and Pareto optimality is restored, as it is case at the Liquidity Trap equilibrium.

The stability of each steady state arising in our economy depends dramatically upon the amplitude of the liquidity constraint and changes as the latter is made to vary continuously. When the share of the consumption good which must be bought cash is included between zero and one-half, the Liquidity Trap equilibrium is stable and thus it is reached for infinitely many initial conditions for the control variables: it follows that the equilibrium is entirely driven by agents' "state" of expectations. On the other hand, when the degree of financial imperfection is above one-half, the Liquidity Trap equilibrium becomes unstable and thus the conclusions of Benhabib *et al.* (2001) are reversed: it is now the Taylor target to be indeterminate and thus compatible with infinitely many agents' self-fulfilling beliefs. We also prove the existence of deterministic cycles around the Taylor target, meanwhile the dynamics characterizing the Liquidity Trap equilibrium is shown to be always oscillatory.

Our results confirm the perils of analyzing uniquely the linearized versions of the truly non-linear models, since this rules out the study of all the trajectories, as the Liquidity Trap, other than those consistent with the targets. In addition, since the Liquidity Trap equilibrium is unstable for realistic low degrees of financial market imperfection, policy and Taylor rules are within our economy theoretically rehabilitated, not just as a tool for economic recovery but also as coordination devices as soon as the Liquidity Trap equilibrium represents a "pathological" and difficult to reach phenomenon, as claimed by the original Keynesian tradition (see Keynes (1936) and Orphanides and Wieland (1998)). As our paper shows, accounting for a partial cash-in-advance constraint allows to better appreciate the dynamic feature of the model and suggests that the Benhabib *et al.* (2001) results are non completely robust to some not negligible perturbation of the money demand.

The remainder of the paper is organized as follows. In Section 2 we present the economy; we describe the fiscal policy carried out by the government, the monetary policy pursued by the Central Bank, the households behavior and we derive the intertemporal equilibrium. Section 3 is devoted to the analysis of the stationary solutions and deals also with their welfare properties. The stability analysis represents the content of Section 4 meanwhile Section 5 concludes.

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2 The Economy

We consider an infinite horizon discrete time economy populated by the government, the Central Bank, a large number of infinitely lived households and a representative firm. In the sequel, we will describe the government and the Central Bank goals, the household behavior, the technology of the firm and the intertemporal equilibrium of the economy.

2.1 The Government and the Fiscal Policy

Let g_t denote the government public spending in real terms in period t and τ_t the tax revenue still in real terms relative to the same period. Let in addition $I_t \equiv (1 + i_t)$ be the nominal interest factor in period t, i_t being the nominal interest rate relative to the same period, and B_{t+1}^g the nominal amount of safe government bonds issued in period t. Setting p_t the price of the (unique) consumption good produced in the economy in period t, the government budget constraint relative to period t is therefore given by

$$B_{t+1}^g = p_t g_t - p_t \tau_t + I_t B_t^g. \tag{1}$$

Let us assume in addition that the initial amount of nominal government bonds issued in period zero is $B_0^g > 0$. Let us observe that the real interest factor R_t satisfies

$$R_t = I_t \frac{p_t}{p_{t+1}}.$$

The government budget constraint (1) expressed in real terms gives

$$\frac{p_{t+1}}{p_t} b_{t+1}^g = g_t - \tau_t + I_t b_t^g \tag{2}$$

where $b_t^g \equiv B_t^g / p_t$ denotes the real amount of government bonds issued in period $t - 1$.

In the remainder of the paper we will focus on a fiscal policy ensuring a balanced government budget constraint, i.e. $p_t g_t = \tau_w w_t l_t$ for every $t \geq 1$, where $\tau_w \in [0, 1]$ is a flat tax on labor income and w_t is the nominal wage. Our specification consists in a government balanced-budget rule according to the specification used in Schmitt-Grohé and Uribe (2000); in their terminology, the fiscal policy is said to be active when the primary government surplus is exogenous, meanwhile if the primary surplus is increasing in and sensitive enough to the public debt, the fiscal policy is said to be passive. In our framework, the fiscal policy falls rather within the category of an active one, since we assume a zero primary surplus in

each period. Such a zero surplus is actually guaranteed by the fact that government spending is financed exactly out of labor income. Finally, we assume a Ricardian framework, so that the government budget constraint (1) must be respected for all possible sequence of prices $\{p_t\}_{t=0}^{+\infty}$. In order to complete the description of the government behavior we must introduce the transversality condition ensuring that the present value of national debt, for t going to infinite, is finite:

$$\lim_{T \to \infty} \frac{\Pi_{t=1}^{T} R_t b_0^G}{\Pi_{t=1}^{T} R_t} = b_0^G < +\infty,$$

where b_0^G is the real debt of the government at period 0. The transversality condition is thus always satisfied.

2.2 The Central Bank and the Monetary Policy

The Central Bank issues money against the purchase of government bonds through open market operations. Denoting B_{t+1}^{CB} the amount of nominal government bonds purchased by the Central Bank in period t and M_{t+1} the stock of nominal balances available in the economy at the outset of period t, the budget constraint of the Central Bank is

$$B_{t+1}^{CB} = I_t B_t^{CB} + M_{t+1} - M_t \tag{3}$$

which, setting $m_t \equiv M_t/p_t$ the real balances available at the beginning of period t and $b_t^{BC} = B_t^{BC}/p_t$ the real amount of the bonds held by the Central Bank issued in period $t - 1$, in real terms can be written as

$$\frac{p_{t+1}}{p_t} b_{t+1}^{BC} = I_t b_t^{BC} + m_{t+1} \frac{p_{t+1}}{p_t} - m_t. \tag{4}$$

Following Leeper (1991) and Kudoh and Nguyen (2010), we assume that the Central Bank follows a Taylor (1993) feedback rule

$$I_t = I(\pi_t) = \max \left\{ 1, I^* \left(\frac{\pi_t}{\pi^*} \right)^{\gamma} \right\}, \tag{5}$$

where $\pi_t \equiv \frac{p_t}{p_{t-1}}$ is the inflation factor between periods $t - 1$ and t, I^* and π^* are the implicit targets for, respectively, the nominal interest factor and for the inflation factor and $\gamma > 0$ is the elasticity of the nominal interest rate with respect to inflation satisfying

$$\frac{dI}{d\pi} \frac{\pi}{I} = \gamma. \tag{6}$$

Following Benhabib *et al.* (2001) and Schmitt-Grohé and Uribe (2000) in our framework we refer to active interest rate feedback rules in the case of rules that respond to increases in inflation with a more

than one-for-one increase in the nominal interest rate. As a consequence, a passive interest rate feedback rules is such that the nominal interest rate reacts in a less than one-for-one increase in inflation. Since we refer to active or passive interest rate feedback rules in this spirit, the elasticity of the nominal interest rate with respect to the inflation rate falls within one of the above definitions according to its magnitude which can be larger than one (active Taylor rules) or lower than one (passive Taylor rules).

However, the distinction between active monetary policy and passive monetary one is sometime different. For instance, Schmitt-Grohé and Uribe (2009) when the zero lower bound for the nominal interest rate is reached (with low inflation), they call it a passive monetary policy, a case in our paper obtained in the Liquidity Trap equilibrium (independetely of whether $\gamma < 1$ or $\gamma > 1$). In our paper, in any case, we refer to active or passive monetary rules according to the magnitude of the elasticity of the nominal interest with respect to the inflation rate. Conversely, when the economy is at the Liquidity Trap equilibrium, the interest rate does no more react to increases in inflation but sticks to zero.

Notice that in the Taylor rules we do not include the output gap since in our model is by construction zero (see, e.g., Woodford (1993)) and, in addition, according to several empirical estimates (see, e.g., Clarida *et al.* (1998)), its coefficient falls within a range including very small values for many Central Banks. Since the Central Bank pegs the nominal interest rate, it must supply as much money as the household do demand in correspondence to the chosen interest rate.

2.3 Households

We consider an infinite horizon discrete time economy populated by a constant mass of agents whose size is normalized to one. The preferences of the representative agent are described by the following intertemporal utility function:

$$\sum_{t=0}^{+\infty} \beta^t \left[u\left(c_t\right) - v\left(l_t\right) \right] \tag{7}$$

where c_t is the unique consumption good, l_t the labor supply, p_t the price of consumption good and $\beta \in (0, 1)$ the discount factor. The instantaneous utility function $u\left(c\right) - v\left(l\right)$ satisfies the following standard Assumption.

Assumption 1. *$u\left(c\right)$ is C^2 over $\mathbb{R}_+$, increasing and concave over $\mathbb{R}_+$. Moreover, $v\left(l\right)$ is C^2 over $\mathbb{R}_+$, strictly increasing and weakly convex.*

When maximizing (7) agents must respect the dynamic budget constraint

$$p_t c_t + M_{t+1} + B_{t+1} = M_t + \left(1 + i_t\right) B_t + \left(1 - \tau_w\right) w_t l_t \tag{8}$$

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where B_t denotes the safe nominal bonds issued by the government and held by the representative household. Following analogous lines as in Hahn and Solow (1995), we assume in addition that agents must pay cash at least a share $q \in (0, 1]$ of their consumption purchases

$$q p_t c_t \leq M_t. \tag{9}$$

Denoting $b_t = B_t / p_t$ the real governments bonds held by the representative consumer at the outset of period $t - 1$ and $\omega_t = w_t / p_t$ the real wage earned in period t, we get the intertemporal maximization problem of the representative agent:

$$\max_{(c_t, m_{t+1}, l_t, b_{t+1})_{t=0}^{\infty}} \sum_{t=0}^{+\infty} \beta^t \left[u(c_t) - v(l_t) \right] \tag{10}$$

subject to the dynamic budget constraint

$$c_t + \frac{p_{t+1}}{p_t} m_{t+1} + \frac{p_{t+1}}{p_t} b_{t+1} = m_t + (1 + i_t) b_t + (1 - \tau_w) \omega_t l_t \tag{11}$$

and to the cash-in-advance constraint

$$q c_t \leq m_t. \tag{12}$$

From now on, we will focus on the case where the cash-in-advance constraint is binding. This requires the nominal interest rate to be positive. We will show in the sequel that the intertemporal equilibrium evaluated at the steady states is consistent with such an assumption. As a consequence, the cash-in-advance constraint will be binding also in a neighborhood of each stationary solution. Notice that when $i = 0$, agents are indifferent between investing in money or in bonds. However, we can easily assume that in such a circumstance money is only used to purchase the consumption good meanwhile agents transfer wealth across time by exclusively investing in bonds. Equivalently, we can suppose that the lower bound for the nominal interest rate is not exactly zero but it is set arbitrarily close to zero from above. In such a case, of course, the cash-in-advance constraint will be binding. Actually, the bindness of the liquidity constraint is ensured by the same definition of the Liquidity Trap equilibrium here studied, that prevents the nominal interest rate from being negative.

Let denote λ and μ the Lagrangian multiplier associated to the dynamic budget constraint and the cash-in-advance constraint. Under a binding liquidity constraint, the first-order conditions are[1]

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[1] u' and v' denote respectively $\partial u(c) / \partial c$ and $\partial v(l) / \partial l$.

$$\beta^t u'(c_t) = \lambda_t + q\mu_t, \;\; \beta^t v'(l_t) = (1 - \tau_w)\,\omega_t\lambda_t, \;\; \lambda_t - \frac{p_t}{p_{t-1}}\lambda_{t-1} + \mu_t = 0, \tag{13}$$

and the Fisher equation is

$$\lambda_t(1 + i_t) - \frac{p_t}{p_{t-1}}\lambda_{t-1} = 0. \tag{14}$$

2.4 Intertemporal Equilibrium

We assume a linear technology such that one unit of labor can be used to produce one unit of output y according to the linear production function

$$y_t = l_t. \tag{15}$$

Equilibrium in the good market requires therefore $y_t = l_t$ in each period t meanwhile, in the labor market, firms, in view of the constant returns to scale technology, employ as much labor as it is supplied by the households. Since the technology is linear in labor, one has that the real wage is constant and equal to one, i.e. $\omega_t = 1$, for every t. In the good market, total government expenditures g_t and total households consumption c_t must, in each period, equalize total production y_t, i.e.

$$g_t + c_t = y_t = l_t \text{ for each } t \geq 1. \tag{16}$$

Notice that (16) is satisfied by the Walras' Law, once one takes into account the bonds market and the money market. As a consequence one has $c_t = (1 - \tau_w)y_t$. By manipulating appropriately (13)-(14), we obtain the following equations describing intertemporal equilibrium in terms of $(y_t, y_{t+1}, \pi_t, \pi_{t+1})$:

$$u'((1 - \tau_w)y_t) = \frac{\beta u'((1 - \tau_w)y_{t+1})}{\pi_{t+1}} \left[\frac{qI(\pi_t) + 1 - q}{q + (1 - q)\,[I(\pi_{t+1})]^{-1}} \right], \tag{17}$$

$$(1 - \tau_w)\beta u'((1 - \tau_w)y_{t+1}) = (1 - q)\beta v'(y_{t+1}) + qv'(y_t)\pi_{t+1}, \tag{18}$$

where $I(\pi)$ is the Taylor rule defined in (5). Notice that y and π are variables which are not predetermined and therefore equilibrium is locally unique if and only if both roots of the Jacobian evaluated at the steady state under study lay outside the unit circle. In the opposite case where the steady state is a sink (both stable roots) or a saddle (one stable root), the system will be locally indeterminate and there will be infinitely many choices for the initial conditions ensuring the convergence toward the stationary solution. Once a particular trajectory $\{y_t, \pi_t\}_{t=0}^{\infty}$ has been selected, the price level is nevertheless still

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indeterminate: however, from the cash-in-advance constraint and the Taylor rule, all the nominal variables are immediately derived. Of course (17)-(18) describe the intertemporal equilibrium under the hypothesis of a binding cash-in-advance constraint. In the next Section, we will prove that this is actually true in correspondence for each stationary solution and thus in a small neighborhood of them.

3   Steady State Analysis

In this Section, we provide conditions for the existence, the uniqueness and the multiplicity of the stationary solutions of the dynamic system defined by equations (17)-(18).

3.1   Existence and Multiplicity of Stationary Solutions

Our first task consists in studying the existence and the number of stationary solutions of the dynamic system defined by equations (17) and (18). For sake of precision, a steady state is a pair $(\pi, y) > (0,0)$ satisfying the following planar system of equations:

$$\pi \left[q + (1-q)I\left[(\pi)\right]^{-1} \right] = \beta \left[qI(\pi) + 1 - q \right] \tag{19}$$

and

$$(1 - \tau_w)\beta u'((1 - \tau_w)y) = (1 - q)\beta v'(y) + qv'(y)\pi \tag{20}$$

where the function $I(\pi)$ is defined in (5). Since $I(\pi) = \pi/\beta$ (as it is immediate to verify from (19)) can not be lower than one, the Taylor rule actually puts a lower bound on π, which we will refer as to $\pi_{\min}$. From (5), one has immediately that $\pi_{min} \equiv \pi^*/(I^*)^{\frac{1}{\gamma}}$. Since, according to the Taylor rule, the gross nominal interest rate is increasing in the inflation rate, one has for $\pi > \pi_{min}$ that the gross nominal interest is positive and thus the cash-in-advance constraint is binding. Our task consists thus in finding (π, y) solution of (19)-(20) for π larger or equal than π_{min}. As a matter of fact, for $\pi > \pi_{min}$, the gross nominal interest rate is positive and thus it is fixed on the basis of the Taylor rule. On the other hand, for $\pi \leq \pi_{\min}$, the economy boils down to the Liquidity Trap regime and the gross nominal interest rate sticks to one. Since (19) includes uniquely the inflation rate, we can derive the existence of a stationary π compatible with the inequality $I(\pi) \geq 1$ under the case, respectively, of a passive Taylor rule, i.e. $\gamma < 1$, and under the case of an active Taylor rule, i.e. $\gamma > 1$. Once a stationary value for π has been found, from (20) one immediately derives the corresponding (and unique) stationary value for the output y, provided the

following boundary conditions are satisfied:

$$\lim_{y \to 0} \frac{u'((1 - \tau_w)y)}{v'(y)} > \frac{(1 - q)\beta + q\pi}{\beta(1 - \tau_w)} > \lim_{y \to +\infty} \frac{u'((1 - \tau_w)y)}{v'(y)}.$$

In order to find the stationary values for the inflation rate π, let us define the two functions $G_0 \equiv \pi \left[q + (1 - q) [I(\pi)]^{-1} \right]$ and $G_1 \equiv \beta [qI(\pi) + 1 - q]$. Each π solving equation $G = G_0/G_1 = 1$ corresponds thus to a steady states we are looking for. Notice that, when $\pi < \pi_{min}$, one has $I(\pi) = 1$ and thus $G(\pi) = \pi/\beta$, meanwhile for $\pi > \pi_{min}$ the function G can be written as

$$G(\pi) = \frac{q\pi + (1 - q)(I^*)^{-1}(\pi^*)^{\gamma}(\pi)^{1-\gamma}}{q\beta I^*(\pi^*)^{-\gamma}\pi^{\gamma} + \beta(1 - q)}. \tag{21}$$

(a) $\gamma < 1$ and $\pi_{min} < \beta$.

(b) $\gamma < 1$ and $\pi_{min} > \beta$.

(c) $\gamma > 1$ and $\pi_{min} < \beta$.
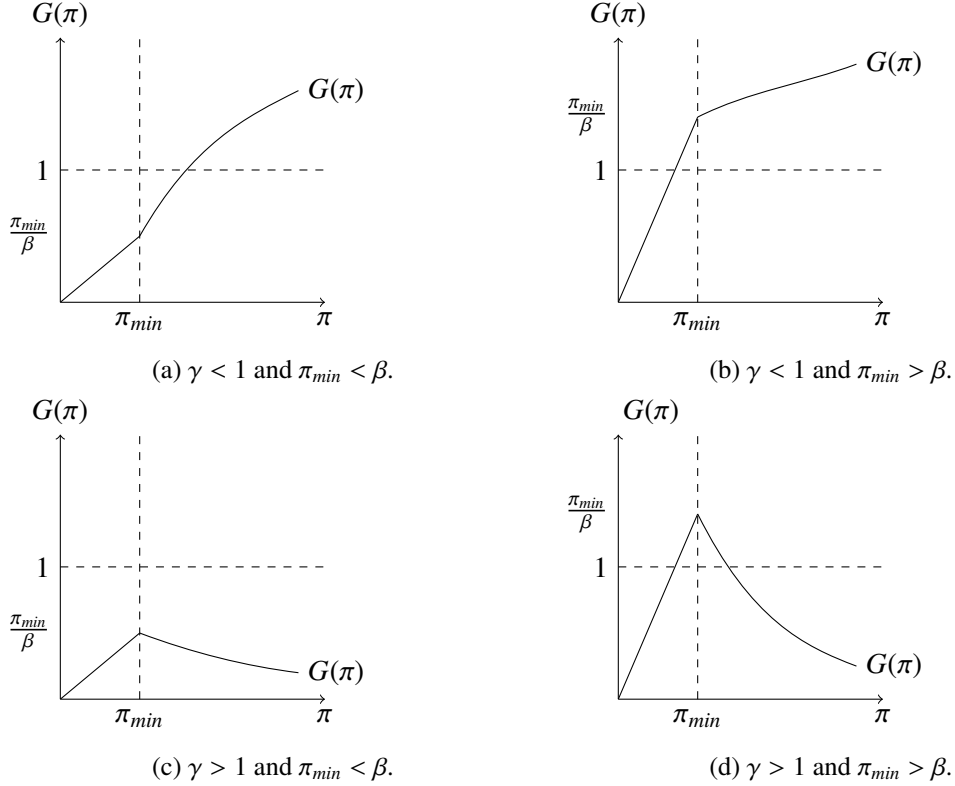
(d) $\gamma > 1$ and $\pi_{min} > \beta$.

Figure 1: Steady states.

One immediately verifies that for $\gamma < 1$, the function $G(\pi)$ is increasing in π. First, for $\pi < \pi_{min}$, it describes a line with slope β^{-1} and then, for $\pi > \pi_{min}$, it describes a convex curve strictly increasing and converging to infinite. It follows that there will exist exactly one stationary solution: if $\pi_{min} < \beta$, the unique stationary equilibrium will correspond to the Leeper case, meanwhile, for $\pi_{min} > \beta$, the unique

equilibrium will be the Liquidity Trap one. The case $\gamma < 1$ is depicted in Figure 1a and Figure 1b which refer, respectively, to the cases $\pi_{min} < \beta$ and $\pi_{min} > \beta$.

The case $\gamma > 1$ is also easy to study. Actually, the function $G(\pi)$ defined in (21), when $\pi > \pi_{min}$, is now monotonically decreasing and converges to zero when π tends to infinite. As it is depicted in Figure 1c, one thus has that, for $\pi_{min} < \beta$, there are no stationary solutions for the dynamic system defined by equations (17) and (18), meanwhile, for $\pi_{min} > \beta$, the function $G(\pi)$ intersects the horizontal line 1 twice, once in correspondence to some $\pi = \beta$ lower than π_{min}, giving thus raise to a Liquidity Trap equilibrium, and once for some π larger than π_{min} to which it is associated a gross nominal interest rate larger than one. This case is depicted in Figure 1d.

The following Proposition is thus immediately proved.

Proposition 1. *Under Assumption 1, let $q \in [0, 1]$. Then the following results hold:*

i⌋ Let $\gamma < 1$ and $\pi_{min} < \beta$. Then the unique (Leeper) steady state is such that $I(\pi) > 1$;

ii⌋ Let $\gamma < 1$ and $\pi_{min} > \beta$. Then the unique (Liquidity Trap) steady state is such that $I(\pi) = 1$;

iii⌋ Let $\gamma > 1$ and $\pi_{min} < \beta$. Then there is no steady state;

iv⌋ Let $\gamma > 1$ and $\pi_{min} > \beta$. Then there exists a (Liquidity Trap) steady state such that $I(\pi) = 1$ and a (Leeper) steady state such that $I(\pi) > 1$.

3.2   Welfare Analysis

Once established the existence of at most two stationary solutions of the dynamic system described by equations (17)-(18), one may wonder at this point whether one of them Pareto-dominates the other one. To this end, let us totally differentiate (20) with respect to y and π in order to obtain

$$\frac{dy}{d\pi} = \frac{(1 - \tau_w)\beta u'' ((1 - \tau_w)y) - [(1 - q)\beta + q\pi] v'' (y)}{q v' (y)} < 0. \tag{22}$$

The other piece of information needed is provided by the differentiation of the stationary utility of the representative household, (7) which gives $u'((1 - \tau_w)y) - v'(y)$ and by the stationary relationship

$$\frac{u'((1 - \tau_w)y)}{v'(y)} = \frac{q\pi + (1 - q)\beta}{(1 - \tau_w)\beta} > 1. \tag{23}$$

Taking into account simultaneously (22) and (23) and by recalling to mind that the inflation rate corresponding to the liquidity trap equilibrium is lower than the one corresponding to the Leeper stationary solution, one has that households are better off in the former one, as stated in the following Proposition.

Proposition 2. *Under Assumption 1, the utility of the representative household, evaluated at the liquidity trap equilibrium, is larger than that corresponding to the Leeper equilibrium.*

Proposition 2 opens the door for some important considerations. The fact the Liquidity Trap dominates the Leeper equilibrium is indeed easily interpretable in the light of the fact that a lower inflation reduces the burden of the inflationary tax. This seems to suggest that a further decline in inflation below the Liquidity Trap equilibrium could entail a Pareto-improvement. However, the same definition of the liquidity trap puts a lower bound on the nominal interest rate (which cannot be negative) and thus, in view of the Taylor rules (5), on the inflation rate, which cannot be lower than $\pi_{\min}$. Were it not be the case, money would dominate government bonds in terms of returns and the liquidity constraint would not more be binding, the households investing wealth exclusively in money balances. This confirms the policy implications of the Liquidity Trap: in correspondence to the latter, it is not more possible to stimulate economic activity by a further cut in the interest rate. It follows that monetary policy in this configuration is completely impotent.

Another interesting question is concerned with the effect of an increase of q on the welfare, evaluated at the steady state under study, of the representative agent. Since the amplitude q of the liquidity constraint can be viewed as a measure of the degree of the capital market imperfection, one is tempted to guess that an increase in the latter is Pareto-worsening. This is actually true, in respect to the Leeper equilibrium. To prove this, let us first look at equation (19): as we have already seen, it is independent upon the output y and therefore it allows to find the stationary solutions in terms uniquely of π. At this point, we can totally differentiate (19) with respect to π and q in order to obtain

$$\frac{d\pi}{dq} = -\frac{(I(\pi) - 1)\,(\pi - \beta I(\pi))}{I(\pi)q\,(1 - \beta I'(\pi)) + (1 - q)\,(1 - \gamma)}. \tag{24}$$

By a direct inspection of (19), one immediately verifies that at the Leeper equilibrium it is $\pi = \beta I(\pi)$ and at the stationary solution corresponding to the Liquidity Trap it is by definition $I(\pi) = 1$. It follows that under both cases under consideration, $d\pi/dq = 0$ and therefore the stationary inflation rate is independent upon the amplitude q of the liquidity constraint. On the other hand, by differentiating (20) with respect to y and q, one easily derives:

$$\frac{dy}{dq} = -\frac{v'\,(y)\,(\beta - \pi)}{\beta(1 - \tau_w)u''\,((1 - \tau_w)y) - [(1 - q)\beta + q\pi]\,v''\,(y)}. \tag{25}$$

Notice now that at the Leeper equilibrium one has $\pi > \beta$ and therefore (25) is strictly negative. On

the other hand, at the Liquidity Trap equilibrium it is $\pi = \beta$ and therefore (25) is equal to zero. These informations, gathered together with (23), establish that

$$\frac{d}{dq}\left(u\left((1 - \tau_w)y\left(q\right)\right) - v\left(y\left(q\right)\right)\right) \tag{26}$$

is lower than zero at the Leeper equilibrium and zero at the Liquidity trap equilibrium and therefore proves that an increase in the amplitude q of the liquidity constraint is Pareto-worsening at the Leeper equilibrium, meanwhile does not entail any effect on household welfare at the Liquidity Trap equilibrium. This result is immediately interpretable once one recalls to mind at the Liquidity Trap equilibrium the cost of money, i.e. the nominal interest, is zero. These results are gathered in the following Proposition which can be immediately proved.

Proposition 3. *Under Assumption 1, the utility of the representative household, evaluated at the Leeper equilibrium, is decreasing in $q \in (0, 1]$, meanwhile it is not affected by q at the Liquidity Trap equilibrium.*

4   Stability Analysis

In the Section, we analyze the stability of the Liquidity Trap steady state and of the Leeper one. We will show that it depends dramatically upon the amplitude of liquidity constraint q. We will first analyze the case of a complete cash-in-advance constraint, i.e. $q = 1$, and then the case of fractional liquidity constraint, i.e. $q < 1$. This will allow us to appreciate how the local dynamic features do change as soon as q is relaxed from one to zero. As a matter of fact, we will show that the stability properties do change dramatically as soon as q is progressively relaxed.

4.1   The benchmark case: $q = 1$

When $q = 1$, the steady state analysis of Section 3 does not undergoes any modification as it is possible to verify by directly inspecting equations (19)-(20). Therefore, we can refer again to Figure 1a to Figure 1d and to Proposition 1. In order to study the dynamics of the economy under the hypothesis $q = 1$ we must inspect the the following two equations:

$$\pi_{t+1}u'((1 - \tau_w)y_t) = \beta u'((1 - \tau_w)y_{t+1})I(\pi_t), \tag{27}$$

$$(1 - \tau_w)\beta u'((1 - \tau_w)y_{t+1}) = \pi_{t+1}v'(y_t). \tag{28}$$

We first study the stability of the Liquidity Trap steady state. Such an analysis turns out to be straight-forward. In fact, by replacing π_{t+1}, obtained in (28), into (27), and by exploiting the fact that $I(\pi_t) = 1$, one has

$$(1 - \tau_w)u'((1 - \tau_w)y_t) = v'(y_t). \tag{29}$$

In view of Assumption 1, there will be a unique solution y_t for (29) which is time invariant. It follows that the Liquidity Trap equilibrium does not generate any dynamics.

Let us now consider the Leeper Equilibrium, i.e. the stability of the steady state where $I(\pi) > 1$. From (28) it is possible to express π_{t+1} in terms of y_t and y_{t+1}. By plugging such an expression into (27), we obtain the following expression:

$$I(\pi_t) = \frac{(1 - \tau_w)u'((1 - \tau_w)y_t)}{v'(y_t)}. \tag{30}$$

By taking into account the Taylor rule (5), it is possible to obtain the smooth function

$$\pi_t = \left[\frac{(\pi^*)^\gamma (1 - \tau_w)u'((1 - \tau_w)y_t)}{I^* v'(y_t)} \right]^{\frac{1}{\gamma}} \equiv \pi(y_t). \tag{31}$$

The derivative $d\pi_t/dy_t$ is easily derived as

$$\frac{d\pi_t}{dy_t} = \frac{\pi_t}{\gamma} \left[\frac{(1 - \tau w)u''((1 - \tau_w)y_t)}{u'((1 - \tau_w)y_t)} - \frac{v''(y_t)}{v'(y_t)} \right] < 0. \tag{32}$$

Therefore intertemporal equilibrium is defined by a sequence $\{y_t\}_{t=0}^{\infty}$ satisfying the following first-order difference equation

$$\pi(y_{t+1})u'((1 - \tau_w)y_t) = \beta u'((1 - \tau_w)y_{t+1})I(\pi(y_t)). \tag{33}$$

Let us define, at this stage, $\varepsilon_{ll} = v''(l)l/v'(l) \in (0, +\infty)$ and $\varepsilon_{cc} = -u''(c)c/u'(c) \in (0, +\infty)$, respectively, the inverse of the elasticity of the labor supply and the elasticity of the marginal utility of consumption. In order to simplify our analysis, it is useful to introduce here the elasticity ε_{cl} that we will refer to as the elasticity of the offer curve which includes both the previous elasticities:

$$\varepsilon_{cl} = \frac{\varepsilon_{ll}}{\varepsilon_{cc}}. \tag{34}$$

Notice that ε_{cl} too belongs to $(0, +\infty)$. By taking into account (32) evaluated at the steady state and (34), we obtain the following Jacobian:

$$\frac{dy_{t+1}}{dy_t} = \frac{\gamma}{1 + (1 - \gamma)\varepsilon_{cl}}. \tag{35}$$

By inspecting (35), one immediately verifies than when $\gamma < 1$ the Jacobian is positive and lower than one, and thus the steady state is stable (locally indeterminate). On the other hand, under the hypothesis $\gamma > 1$, one easily sees that the Jacobian is larger than one for $\varepsilon_{cl} < 1/(\gamma - 1) \equiv \hat{\varepsilon}_{cl}$, and thus the steady state is unstable (locally determinate). In addition, when ε_{cl} goes throught $\hat{\varepsilon}_{cl}$ from below the Jacobian tends to $+\infty$; conversely, when ε_{cl} goes throught $\hat{\varepsilon}_{cl}$ from above, the Jacobian tends to $-\infty$. In addition, when ε_{cl} goes through $\varepsilon_{cl}^{\mathcal{F}} = (1 + \gamma)/(\gamma - 1)$, then the Jacobian is -1. Eventually, when ε_{cl} tends to $+\infty$, the Jacobian converges monotonically to zero. Then, the following Proposition is immediately verified

Proposition 4. *Under Assumption 1, the following results hold:*

i] Let $\gamma < 1$. Then $dy_{t+1}/dy_t \in (0, 1)$ and the steady state is stable (locally indeterminate);

ii] Let $\gamma > 1$. If $\varepsilon_{cl} < \varepsilon_{cl}^F$, then $dy_{t+1}/dy_t \in (1, +\infty) \cup (-\infty, -1)$ and the steady state is unstable (locally determinate). If $\varepsilon_{cl} > \varepsilon_{cl}^F$, then $dy_{t+1}/dy_t \in (-1, 0)$ and the steady state is stable (locally indeterminate).

In addition, when ε_{cl} goes through $\varepsilon_{cl}^{\mathcal{F}}$, the steady state undergoes a flip bifurcation.

The Leeper steady state equilibrium, provided it exists, is thus stable for $\gamma < 1$ and can be stable or unstable for $\gamma > 1$ according to the magnitude of the elasticity ε_{cl} of the offer curve. These results seem to contradict the findings of Benhabib *et al.* (2001) within a money-in-the-utility framework. In fact, under active Taylor rules, they always find a locally determinate Leeper equilibrium and a stable Liquidity Trap one. But according to our results, this is not always true, since the stability of the Leeper equilibrium depends crucially upon the elasticity ε_{cl} of the offer curve. At a same time, we find that the Liquidity Trap steady state equilibrium is always determinate since it does not involve any dynamics. These results suggest the idea that modeling a cash-in-advance constraint implies a discontinuity with respect to money-in-the-utility function approach: it is not straightforward, in fact, to view the cash-in-advance constraint framework as a limit case of the money-in-the-utility function approach where consumption and real balances are perfect complements.

As a matter of fact, when $\gamma < 1$, the Leeper steady state equilibrium (provided it exists) is stable and thus there will be infinitely many initial conditions compatible with the transversality condition. On the other hand, when $\gamma > 1$ and for high enough values for the elasticity ε_{cl} of the offer curve, in corre-spondence to some initial condition located close to the Leeper steady state equilibrium, the system will converge toward the latter which therefore turns out to be indeterminate. At the same time, the unique way to attain the Liquidity Trap equilibrium will require agents to coordinate since the beginning on it.

The case $q = 1$ has been studied by Schmitt-Grohé and Uribe (2000). They find that the price level is indeterminate for both low and high values of the inflation elasticity of the feedback rule and determinate for intermediate values. More in details, they show that if the steady state leisure-consumption ratio is greater than one and the elasticity of the feedback rule is lower than one or larger than a given threshold, then there exists a continuum of perfect foresight equilibria in each of which the sequence of real balances is different. In correspondence to each equilibria, in view of the quantitative theory of money, corresponding to a binding liquidity constraint, the initial price level too is indeterminate. On the other hand, Schmitt-Grohé and Uribe (2000) show that for intermediate values of the inflation elasticity of the feedback rule the only perfect foresight equilibrium is the steady state equilibrium, and thus equilibrium price level is unique. If the steady state leisure-consumption ratio is less than one, they show that when the inflation elasticity of the feedback rule is lower than one, then there exist multiple perfect foresight equilibria in terms of the dynamics of real balances. As a consequence, there will correspond the indeterminacy of the price level. At a same time, if the inflation elasticity of the feedback rule is larger than one, the unique equilibrium will correspond to the steady state and thus the equilibrium price level is unique.

In our model, the households' objective function is defined over consumption utility and labor disutility. It follows that a direct comparison with the findings of Schmitt-Grohé and Uribe (2000) is hard to be carried out. In addition, they focus on a log utility function, meanwhile we consider a general specification for the utility function, although separable in consumption and labor. Were we focusing on a log specification, our elasticity of the offer curve would be equal to one and the Jacobian would boil down to $\gamma/(2 - \gamma)$. As it is immediate to verify, in such a case, the Jacobian would lie the unite circle if and only if $\gamma < 1$. Our conditions for indeterminacy seem thus to be similar to those provided by Schmitt-Grohé and Uribe (2000) in Propositions 3 and 4, at least for the part concerned with an inflation elasticity of the feedback rule lower than one.

To understand the mechanism leading to indeterminacy in our model under the hypothesis $q = 1$, let us suppose that the system is initially at the Leeper steady state equilibrium and assume that agents anticipate, say, a lower inflation rate. In such a case, next period consumption will be cheaper and so agent will invest more money balances by increasing present labor supply. It follows that, in view of (33), the present inflation rate will decreases. However, in order this expectation to be fulfilled, one needs that the output moves back to the steady state equilibrium and thus agents to react by increasing current output more than the next period expected one. This is of course true if the next period nominal interest rate will not decrease too much, i.e. if γ is set low enough below one. In the opposite case, the system will undergo an explosive dynamics.

Notice, on the other hand, that the Liquidity Trap equilibrium boils down to a static relationship. Indeed, in this case, the nominal interest and thus the inflation rate are constant. It follows that the arbitrage equation (33) is satisfied for a unique, and time invariant, value for the output.

In the following Subsection, we show that the features under the case of a complete cash-in-advance constraint are not necessarily preserved when the amplitude of the same liquidity constraint is relaxed. Actually, the two steady states corresponding to the Leeper equilibrium and to the Liquidity Trap may easily change their stability.

4.2   The Liquidity Trap for $q < 1$

Let consider now the case $q < 1$. When the economy is at the Liquidity Trap equilibrium and thus the gross nominal interest rate sticks to one, equation (17) boils down to

$$u'(y_t - g) = \frac{\beta u'((1 - \tau_w)y_{t+1})}{\pi_{t+1}}$$

and thus the inflation rate relative to period t does not appear anymore. It is thus possible to express the inflation rate relative to period $t + 1$ in terms of y_t and y_{t+1} as $\pi_{t+1} = \beta u'((1 - \tau_w)y_{t+1})/u'((1 - \tau_w)y_t)$. By plugging such an expression into (18), the dynamic system loses one dimension and boils down to a simple one-dimensional difference equation in terms of the input lagged once. It follows that an intertemporal equilibrium corresponding to a Liquidity Trap configuration is a sequence $\{y_t\}_{t=0}^{\infty}$ satisfying for each t the following first-order difference equation

$$(1 - \tau_w)u'((1 - \tau_w)y_{t+1}) = (1 - q)v'(y_{t+1}) + qv'(y_t)\frac{u'((1 - \tau_w)y_{t+1})}{u'((1 - \tau_w)y_t)}. \tag{36}$$

Using the fact that (36) evaluated at the steady state is $(1 - \tau_w)u'((1 - \tau_w)y) = v'(y)$, it follows that the Jacobian, evaluated at the steady state, is

$$\frac{dy_{t+1}}{dy_t} = -\frac{q}{1 - q}. \tag{37}$$

By inspecting (37), one immediately sees that the stability of the system depends dramatically of the amplitude of the liquidity constraint q. As a matter of fact the following Proposition is immediately proved.

Proposition 5. *Under Assumption 1, let $\pi_{min} > \beta$. Then the following results hold:*

i] if $q \in (0, 1/2)$, then $dy_{t+1}/dy_t \in (-1, 0)$. The steady state is thus stable (locally indeterminate);

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ii] *if $q \in (1/2, 1]$, then $dy_{t+1}/dy_t < -1$. The steady state is thus unstable (locally determinate).*

In addition when $q = 1/2$, the steady state undergoes a flip bifurcation.

As stated in Proposition 5, for q sufficiently high the Liquidity Trap steady state becomes unstable. This suggests that the findings of Benhabib *et al.* (2001) are not robust: the Liquidity Trap steady state is indeed stable only for low enough amplitude q of the liquidity constraint. When the latter is increased, conversely, the steady state from stable becomes unstable, and thus locally determinate.

In order to understand the mechanism leading to multiple self-fulfilling equilibria, let consider the following example. Let suppose that system (36) is at its steady state at the current period. Let us assume that agents anticipate, say, an increase in the next period output level. By looking at (36), one immediately verifies that its left-hand side decreases meanwhile its right-hand side increases. It follows that in order to reestablish the equality in (36), one needs a decrease in current output level. The required decreases will be the larger, the lower the corresponding value of q. If the latter is low enough, then the decrease of the output level in the current period will be in absolute value larger than the increase of the anticipated output level. It follows that the system will move back toward its steady state following an oscillatory path.

4.3   The Leeper Equilibrium for $q < 1$

When we are in the region in which the nominal interest rate is strictly positive, the intertemporal equilibrium of the economy is described, as we have already seen, by equations (17)-(18). Our goal now is to linearize such equation around the unique Leeper stationary equilibrium. By proceeding in this way, we obtain the following Proposition which defines the characteristic polynomial of the Jacobian evaluated at the steady state.

Proposition 6. *Under Assumption 1, the characteristic polynomial is defined by $\mathscr{P}(\lambda) = \lambda^2 - \lambda \mathscr{T} + \mathscr{D}$ where*

$$\mathscr{T}(\varepsilon_{cl}) = -\frac{q\left\{(1-\gamma)\left(q\frac{\pi}{\beta} + 1 - q\right) + \varepsilon_{cl}\left[q\frac{\pi}{\beta} + (1-q)(1-2\gamma)\right]\right\}}{(1-q)\left\{(1-\gamma)\left(q\frac{\pi}{\beta} + 1 - q\right) + \varepsilon_{cl}\left[q\frac{\pi}{\beta} + (1-q)(1-\gamma)\right]\right\}} \tag{38}$$

$$\mathscr{D}(\varepsilon_{cl}) = -\frac{q^2\gamma\left(\frac{\pi}{\beta}\right)^2 \varepsilon_{cl}}{(1-q)\left\{(1-\gamma)\left(q\frac{\pi}{\beta} + 1 - q\right) + \varepsilon_{cl}\left[q\frac{\pi}{\beta} + (1-q)(1-\gamma)\right]\right\}} \tag{39}$$

Proof: See Appendix 6.1.   ■

In view of the complicated form of the above expressions, it may seem that the study of the local dynamics of system (17)-(18) requires long and tedious computations. However, by applying the geometrical method adopted in Grandmont *et al.* (1998) and Cazzavillan *et al.* (1998), it is possible to analyze

qualitatively the (in)stability of the characteristic roots of the Jacobian evaluated at the steady state of system defined by (17)-(18) and their bifurcations (changes in stability) by locating the point $(\mathscr{T}, \mathscr{D})$ in the plane and studying how $(\mathscr{T}, \mathscr{D})$ varies when the value of some parameter changes continuously. If $\mathscr{T}$ and $\mathscr{D}$ lie in the interior of the triangle $\mathscr{ABC}$ depicted in Figure 2, the stationary solution is a sink. In the opposite case, it is either a saddle, when $\mid \mathscr{T} \mid > \mid 1 + \mathscr{D} \mid$, or a source. If we fix all the parameters of the model with exception of ε_{cl} (which we let vary from zero to $+\infty$) we obtain a parametrized curve $\{\mathscr{T}(\varepsilon_{cl}), \mathscr{D}(\varepsilon_{cl})\}$ that describes a half-line $\Delta(\mathscr{T})$ starting from the point $(\mathscr{T}_0, \mathscr{D}_0)$ when ε_{cl} is close to zero. The linearity of such locus can be verified by direct inspection of the expressions for $\mathscr{T}$ and $\mathscr{D}$ and from the fact they share the same denominator. This geometrical method makes it possible also to characterize the different bifurcations that may arise when ε_{cl} moves from zero to $+\infty$. In particular, as shown in Figure 5, when the half-line $\Delta(\mathscr{T})$ intersects the line $\mathscr{D} = \mathscr{T} - 1$ (at $\varepsilon_{cl} = \varepsilon_{cl}^{\mathscr{T}}$), one eigenvalue goes through unity and a saddle-node bifurcation generically occurs; accordingly, we should expect a change in the number and in the stability of the steady states. When $\Delta(\mathscr{T})$ goes through the line $\mathscr{D} = -\mathscr{T} - 1$ (at $\varepsilon_{cl} = \varepsilon_{cl}^{\mathscr{F}}$), one eigenvalue is equal to -1 and we expect a flip bifurcation: it follows that there will arise nearby two-period cycles, stable or unstable, according to the direction of the bifurcation. Eventually, when $\Delta(\mathscr{T})$ intersects the interior of the segment $\mathscr{BC}$ (at $\varepsilon_{cl} = \varepsilon_{cl}^{\mathscr{H}}$), the modulus of the complex conjugate eigenvalues is one and the system undergoes, generically, a Hopf bifurcation. Therefore, around the stationary solution, there will emerge a family of closed orbits, stable or unstable, depending on the nature of the bifurcation (supercritical or subcritical).

Following Grandmont *et al.* (1998) and Cazzavillan *et al.* (1998), this analysis is also powerful enough to characterize the occurrence of sunspot equilibria around an indeterminate stationary solution of system (17)-(18) as well as along flip and Hopf bifurcations[2]. Actually, system defined by (17)-(18) has at each period *t* two non-predetermined variables, the output and the inflation factor. In such a configuration, the existence of local indeterminacy requires that at least one of the two characteristic roots associated to the linearization of the dynamic system (17)-(18) around the steady state has modulus less than one.

The bifurcation parameter we will adopt through our analysis is the elasticity of the offer curve ε_{cl}. Then the variation of the Trace $\mathscr{T}$ and of the Determinant $\mathscr{D}$ in the $(\mathscr{T}, \mathscr{D})$ plane will be studied as ε_{cl} is made to vary continuously within the $(0, +\infty)$ interval. The relationship between $\mathscr{T}$ and $\mathscr{D}$ is given by a

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[2]In the case of supercritical flip bifurcation and of supercritical Hopf bifurcation, sunspot remain in a compact set containing in its interior, respectively, the stable two-period cycle and the stable closed orbit. Unstable cycles and closed orbits emerge in the opposite case of subcritical bifurcations.
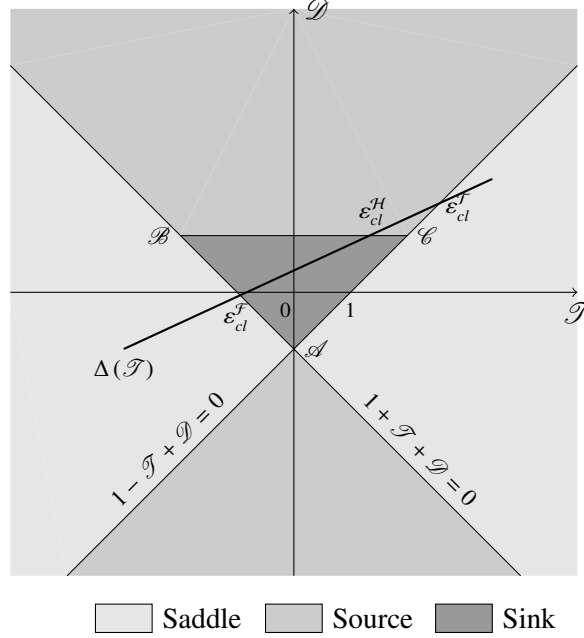
Figure 2: Stability triangle and $\Delta(\mathscr{T})$ segment.

half-line $\Delta(\mathscr{T})$. $\Delta(\mathscr{T})$ is obtained from (38)-(39) and yields to the following linear relationship

$$\mathscr{D} = \Delta(\mathscr{T}) = \mathscr{S}\mathscr{T} + \mathscr{Z} \tag{40}$$

where $\mathscr{Z}$ is a constant term, whose expression is

$$\mathscr{Z} = -\frac{q^2\left(\frac{\pi}{\beta}\right)^2}{(1-q)^2} \tag{41}$$

The slope of $\Delta(\mathscr{T})$ is given by

$$\mathscr{S} = -\frac{q\pi}{\beta(1-q)}. \tag{42}$$

When ε_{cl} is made to vary in the interval $(0, +\infty)$, $\mathscr{T}(\varepsilon_{cl})$ and $\mathscr{D}(\varepsilon_{cl})$ move linearly along the line $\Delta(\mathscr{T})$. As $\varepsilon_{cl} \in (0, +\infty)$, the properties of the line $\Delta(\mathscr{T})$ are derived from the consideration of its extremities. Actually, the starting point is the couple $(\lim_{\varepsilon_{cl}\to+\infty} \mathscr{T} \equiv \mathscr{T}_\infty, \lim_{\varepsilon_{cl}\to+\infty} \mathscr{D} \equiv \mathscr{D}_\infty)$. The corresponding expressions are given by

$$\mathscr{T}_\infty = -\frac{q\frac{\pi}{\beta}\left[q\frac{\pi}{\beta} + (1-q)(1-2\gamma)\right]}{(1-q)\left[q\frac{\pi}{\beta} + (1-q)(1-\gamma)\right]}, \mathscr{D}_\infty = -\frac{q^2\gamma\left(\frac{\pi}{\beta}\right)^2}{(1-q)\left[q\frac{\pi}{\beta} + (1-q)(1-\gamma)\right]}. \tag{43}$$

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The coordinates of the origin are easily obtained and write

$$\mathscr{T}_0 = -\frac{q\pi}{\beta(1-q)}, \mathscr{D}_0 = 0. \tag{44}$$

The half-line $\Delta(\mathscr{T})$ is pointing upward or downward according to the sign of $\mathscr{D}'(\varepsilon_{cl})$:

$$\mathscr{D}'(\varepsilon_{cl}) = -\frac{(1-\gamma)q^2\gamma\left(\frac{\pi}{\beta}\right)^2\left(q\frac{\pi}{\beta}+1-q\right)}{(1-q)\left\{(1-\gamma)\left(q\frac{\pi}{\beta}+1-q\right)+\varepsilon_{cl}\left[q\frac{\pi}{\beta}+(1-q)(1-\gamma)\right]\right\}^2}. \tag{45}$$

Finally, we consider the location of the end point $(\mathscr{T}_\infty, \mathscr{D}_\infty)$. In order to proceed in this way, we analyze the sign of $1 - \mathscr{T}_\infty + \mathscr{D}_\infty$ and of $1 + \mathscr{T}_\infty + \mathscr{D}_\infty$. It follows from (43) that

$$1 - \mathscr{T}_\infty + \mathscr{D}_\infty = \frac{(1-\gamma)\left[\frac{\pi}{\beta}q+1-q\right]^2}{(1-q)\left[q\frac{\pi}{\beta}+(1-q)(1-\gamma)\right]}, \quad 1 + \mathscr{T}_\infty + \mathscr{D}_\infty = \frac{(1-q)^2 - \frac{\pi}{\beta}q^2 - \gamma\left[1-q\left(1+\frac{\pi}{\beta}\right)\right]^2}{(1-q)\left[q\frac{\pi}{\beta}+(1-q)(1-\gamma)\right]}. \tag{46}$$

We now study the general case corresponding to q belonging to $(0, 1)$. We will consider first the case $\gamma < 1$ and then the case of $\gamma > 1$.

4.3.1 $\gamma < 1$

Under the hypothesis $\gamma < 1$, we have seen in the steady state analysis that it must be $\pi_{min} < \beta$ in order the Leeper steady state equilibrium to exist. When $\pi_{min} > \beta$, conversely, no stationary solution does exist.

By inspecting (42), we have in such case that the slope $\mathscr{S}$ of the half-line $\Delta(\mathscr{T})$ is always negative. As a matter of fact, it is zero when $q = 0$, then decreases monotonically with q: it is -1 when $q = \beta/(\beta + \pi) \equiv \tilde{q}$ (which corresponds also to $\mathscr{T}_0 = -1$) and when q tends to 1 it converges to $-\infty$. These pieces of information, gathered together with the fact that, in view of (45), the Determinant $\mathscr{D}$ is decreasing in ε_{cl}, is sufficient to rule out the flip bifurcation. In addition, one has from (46) that $1 - T_\infty + \mathscr{D}_\infty > 0$: it follows that the half-line $\Delta(\mathscr{T})$ does not cross the line $\mathscr{D} = \mathscr{T} - 1$, and thus there is no room for the saddle-node bifurcation. This should not be surprising, since we have seen that in our model each one of the steady state is locally unique. All these observations taken together show that the Trace $\mathscr{T}$ and the Determinant $\mathscr{D}$ lie in the interior the triangle $\mathscr{A}\mathscr{B}\mathscr{C}$ for $q < \tilde{q}$ and, for $q > \tilde{q}$, in the saddle region delimitated by the negative axis of the abscissas, the line $\mathscr{D} = \mathscr{T} - 1$ and the line $\mathscr{D} = -\mathscr{T} - 1$. These cases are depicted in Figure 3a and Figure 3b.

The above results are summarized in the following Proposition which is immediately proved.

Proposition 7. *Under Assumption 1, let $\pi_{min} < \beta$ and $\gamma < 1$. Then there exist $\tilde{q} = \beta/(\beta + \pi)$ such that the following results hold:*

  i] *if $q \in (0, \tilde{q})$, then the steady state is a sink, i.e. locally indeterminate;*

  ii] *if $q \in (\tilde{q}, 1)$, then the steady state is a saddle, i.e. locally indeterminate.*

(a) $\gamma < 1$ and $q \in (0, \tilde{q})$.   (b) $\gamma < 1$ and $q \in (\tilde{q}, 1]$.

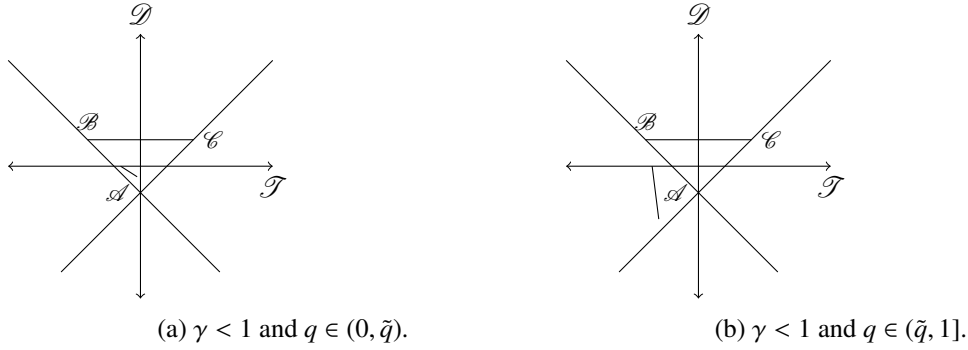Figure 3: Case $\gamma < 1$

4.3.2 $\gamma > 1$

When $\gamma > 1$, we need $\pi_{min} > \beta$ in order to ensure that both the Leeper steady state equilibrium and the Liquidity Trap one do exist. In the opposite case, there will not be any steady state. The case $\gamma > 1$, actually, presents some more complications with respect to the configuration $\gamma < 1$ previously analyzed. Indeed, by looking at the expressions (43) of the end points ($\mathscr{T}_\infty, \mathscr{D}_\infty$), one immediately sees that their common denominator can be either positive or negative according to the value of γ (obviously higher than one). Notice, indeed, that when $\gamma > \tilde{\gamma} = 1 + q\pi/[\beta(1 - q)]$, the common denominator of the end points ($\mathscr{T}_\infty, \mathscr{D}_\infty$) is negative. Moreover, in view of expression (45), $\mathscr{D}'$ is always positive. Therefore, by relaxing ε_{cl} from zero to $+\infty$, one describes a half-line $\Delta(\mathscr{T})$ negatively sloped, starting from ($\mathscr{T}_0, \mathscr{D}_0$) and with a determinant $\mathscr{D}$ increasing in ε_{cl}. The slope $\mathscr{S}$ of the half-line $\Delta(\mathscr{T})$, as one can immediately verify from (42), is larger than -1 when $q < \tilde{q}$, it is -1 when $q = \tilde{q}$ and it is lower than -1 when $q > \tilde{q}$. These informations are sufficient to rule out the occurrence of a saddle-node bifurcation and, on the other hand, to prove the existence of a flip bifurcation at $\varepsilon_{cl}^{\mathscr{F}}$ whose expression is given by

$$\varepsilon_{cl}^{\mathscr{F}} = \frac{(1 - \gamma)\left[q\frac{\pi}{\beta} + 1 - q\right](1 - 2q)}{q\gamma\left[q\left(\left(\frac{\pi}{\beta}\right)^2 + 1\right) - 1\right] - \left[q\frac{\pi}{\beta} + 1 - q\right](1 - 2q)} \tag{47}$$

At a same time, one can immediately show that when $\mathscr{D} = 1$ we have

$$\varepsilon_{cl}^{\mathscr{D}=1} = -\frac{(1-q)(1-\gamma)\left(q\frac{\pi}{\beta}+1-q\right)}{(1-\tau_w)\left[q\frac{\pi}{\beta}\left(q\gamma\frac{\pi}{\beta}+1\right)+(1-q)(1-\gamma)\right]}$$

By replacing $\varepsilon_{cl}^{\mathscr{D}=1}$ in $\mathscr{T}$ we obtain

$$\mathscr{T}^{\mathscr{D}=1} = -\frac{q^2\left(\frac{\pi}{\beta}\right)^2+(1-q)^2}{(1-q)q\left(\frac{\pi}{\beta}\right)^2}.$$

One can immediately verify that $\mathscr{T}^{\mathscr{D}=1}$ is always lower than -2. Such findings ensure that the steady state never undergoes a Hopf bifurcation. Gathering together all the previous considerations, one can reach the conclusion that, for $q < \tilde{q}$, the steady state is first a sink for low values of ε_{cl}, it undergoes a flip bifurcation at $\varepsilon_{cl} = \varepsilon_{cl}^F$ and for $\varepsilon_{cl} > \varepsilon_{cl}^F$ it becomes a saddle. On the other hand, for $q > \tilde{q}$, the steady is first a saddle for low values of ε_{cl}, and it undergoes a flip bifurcation at $\varepsilon_{cl} = \varepsilon_{cl}^F$. Finally, for $\varepsilon_{cl} > \varepsilon_{cl}^F$ it becomes a source. In Figure 4a and Figure 4b we have depicted the case $\gamma > \tilde{\gamma}$.

In the following Proposition we summarize the above results.

Proposition 8. *Under Assumption 1, let $\pi_{min} > \beta$ and $\gamma > 1$. Then there exist $\tilde{q} = \beta/(\beta+\pi)$ and $\tilde{\gamma} = 1 + q\pi/[\beta(1-q)]$ such that when $\gamma > \tilde{\gamma}$ the following results hold:*

i] Let $q \in (0,\tilde{q})$. If $\varepsilon_{cl} \in (0,\varepsilon_{cl}^F)$, the steady state is a sink, i.e. locally indeterminate, and, if $\varepsilon_{cl} \in (\varepsilon_{cl}^F,+\infty)$, the steady state is a saddle, i.e. locally indeterminate.

ii] Let $q \in (\tilde{q},1)$. If $\varepsilon_{cl} \in (0,\varepsilon_{cl}^F)$, the steady state is a saddle, i.e. locally indeterminate, and, if $\varepsilon_{cl} \in (\varepsilon_{cl}^F,+\infty)$, the steady state is a source, i.e. locally determinate.

In addition, when ε_{cl} goes through $\varepsilon_{cl}^{\mathscr{F}}$, the steady state undergoes a flip bifurcation.

Let us now consider the case $\gamma \in (1,\tilde{\gamma})$. By a direct inspection of (38)-(39), one now can immediately verify that there exists a ε_{cl}^0 such that the common denominator of the trace $\mathscr{T}$ and the determinant $\mathscr{D}$ is equal to zero. Since the above mentioned results in terms of the non existence of the Hopf bifurcation and of the saddle-node one still hold, and taking into account that the determinant $\mathscr{D}$ is increasing in ε_{cl}, one has that for $q < \tilde{q}$, the trace $\mathscr{T}$ and the determinant $\mathscr{D}$ will lie in the stability region $\mathscr{ABC}$ for $\varepsilon < \varepsilon_{cl}^F$, and in the saddle region for $\varepsilon \in (\varepsilon_{cl}^F,\varepsilon_{cl}^0)$. When ε_{cl} converges from below to ε_{cl}^0, the trace $\mathscr{T}$ is $-\infty$ and the determinant $\mathscr{D}$ is $+\infty$, meanwhile, when ε_{cl} converges to ε_{cl}^0 from above, the trace $\mathscr{T}$ is $+\infty$ and the determinant $\mathscr{D}$ is $-\infty$. Eventually, for $\varepsilon_{cl} > \varepsilon_{cl}^0$, the trace $\mathscr{T}$ and the determinant $\mathscr{D}$ lie in the saddle region located below the line $\mathscr{D} = \mathscr{T}-1$: this is the consequence of the fact that, in view of (46), $1-T_\infty+\mathscr{D}_\infty < 0$. Figure 4c corresponds to the case $q < \tilde{q}$.

(a) $\gamma > \tilde{\gamma}$ and $q \in (0, \tilde{q})$.

(b) $\gamma > \tilde{\gamma}$ and $q \in (\tilde{q}, 1)$.

(c) $\gamma \in (1, \tilde{\gamma})$ and $q \in (0, \tilde{q})$.

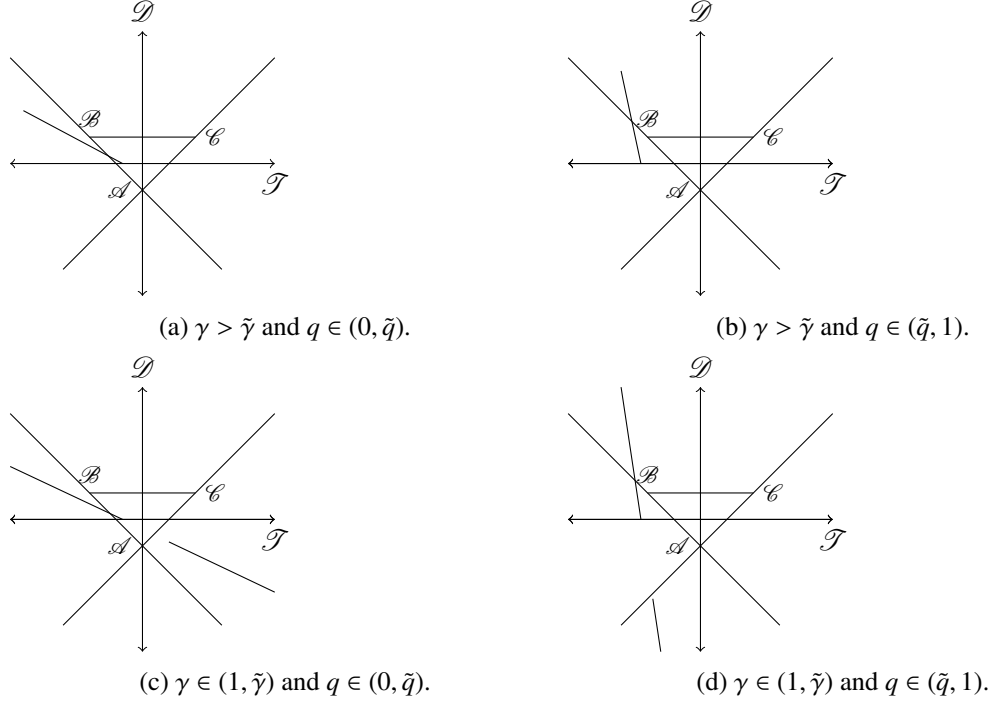(d) $\gamma \in (1, \tilde{\gamma})$ and $q \in (\tilde{q}, 1)$.

Figure 4: Case $\gamma > 1$

For $q > \tilde{q}$, the picture is different since the starting point $(\mathscr{T}_0, \mathscr{D}_0)$ now lies outside the $\mathscr{A}\mathscr{B}\mathscr{C}$ triangle. As a matter of fact, when $\varepsilon_{cl} < \varepsilon_{cl}^F$, the steady state is a saddle; then it undergoes a flip bifurcation at $\varepsilon_{cl} = \varepsilon_{cl}^F$, and, for $\varepsilon_{cl} \in (\varepsilon_{cl}^F, \varepsilon_{cl}^0)$, it becomes a source. The source configuration prevails also for $\varepsilon_{cl} > \varepsilon_{cl}^0$ since now trace $\mathscr{T}$ and the determinant $\mathscr{D}$ lie in the region delimitated from above by the line $\mathscr{D} = \mathscr{T} - 1$ and the line $\mathscr{D} = -\mathscr{T} - 1$. Such a case is depicted in Figure 4d.

The next Proposition is thus immediately proved.

Proposition 9. *Under Assumption 1, let $\pi_{min} > \beta$ and $\gamma > 1$. Then there exist $\tilde{q} = \beta/(\beta + \pi)$ and $\tilde{\gamma} = 1 + q\pi/[\beta(1 - q)]$ such that, when $\gamma \in (1, \tilde{\gamma})$, the following results hold:*

i] Let $q \in (0, \tilde{q})$. If $\varepsilon_{cl} \in (0, \varepsilon_{cl}^F)$, the steady state is a sink, i.e. locally indeterminate, and, if $\varepsilon_{cl} \in (\varepsilon_{cl}^F, +\infty)$, the steady state is a saddle, i.e. locally indeterminate.

ii] Let $q \in (\tilde{q}, 1)$. If $\varepsilon_{cl} \in (0, \varepsilon_{cl}^F)$, the steady state is a saddle, i.e. locally indeterminate, and, if $\varepsilon_{cl} \in (\varepsilon_{cl}^F, +\infty)$, the steady state is a source, i.e. locally determinate.

In addition, when ε_{cl} goes through $\varepsilon_{cl}^{\mathscr{F}}$, the steady state undergoes a flip bifurcation.

As we seen in the above Propositions, the stability of the Leeper equilibrium under the hypothesis $q < 1$ is quite different with respect to the case of a full cash-in-advance constraint ($q = 1$) studied in Schmitt-Grohé and Uribe (2000). As a matter of fact, we have shown that when q is set very close

to zero, the Leeper equilibrium is locally stable whatever the monetary policy is (active or passive). It follows that the Leeper steady state will be locally indeterminate and the degree of indeterminacy will be two. Indeed, there are two degrees of freedom in terms of the initial level of output and inflation rate that turn out to be compatible with the convergence toward the steady state. In order to provide an intuition of the mechanism leading to indeterminacy for q small enough, we will assume for sake of simplicity the case of an active Taylor rules, i.e. $\gamma > 1$. Furthermore, this case has more direct implications since usually Central Banks pursue active monetary rules.

Suppose that the system is at the steady state equilibrium and suppose that agents anticipate, say, a lower inflation rate and a higher output in the next period. Since under the assumption of a q very close to zero, the return on investment is very close to the real interest rate, and the latter is given by the nominal interest/inflation rate ratio, in view of the Euler equation (17) one has that the output in the current period must increase less than that of the next one. At a same time, in view of the arbitrage equation (18) relative to the current period, one has that the inflation rate must be lower than its steady state value although its gap from the latter, in absolute value, is lower than the corresponding increase in the subsequent period. Therefore, the expected decrease in the inflation rate is self-fulfilling and the system will move back toward the steady state following a monotonically decreasing sequence of inflation rates and an oscillatory but shrinking sequence of output levels.

5 Conclusion

In this paper we have provided a theoretical contribution to the debate running around the plausibility of the emergence of the Liquidity Trap equilibrium as well as its stability features. In order to motivate a positive money demand, we have assumed that agents must pay cash a given share of the value of consumption expenditures. We have shown that the Liquidity Trap is not bound to be a stable equilibrium but that, instead, its stability depends dramatically upon the degree of liquidity of the economy. By showing that the Liquidity Trap is not necessarily the unique stable stationary solution of the economy, the original intuition of Keynes is henceforth consolidated on a theoretical point of view: in contrast with Benhabib *et al.* (2001), the Liquidity Trap represents again a limit case that can be avoided by means of an appropriate public policy aiming at coordinating agents toward the Taylor target. In contrast to a money-in-the utility approach, as in Benhabib *et al.* (2001), and to a full cash-in-advance specification, as in Schmitt-Grohé and Uribe (2000), we are able, in fact, to capture the degree of market imperfection by simply letting the amplitude of the liquidity constraint on consumption expenditures to vary. However,

we have proved that there is not always a rationale for escaping from the Liquidity Trap equilibrium, since it Pareto-dominates the Taylor target, in view of the zero interest rate associated to it and the consequent zero cost of money holding. In addition, the dynamics around the Liquidity Trap equilibrium is always oscillatory; similarly, also the behavior characterizing the neighborhood of the Taylor equilibrium can be easily cyclical.

A natural extension of the economy here treated, is to account for physical capital accumulation: such an asset will be indeed held by agents in order to carry over wealth from one period to another, beside government bonds and money. Also accounting for international trade would be an interesting issue to explore, in view of the close ties that would arise between the monetary policies implemented in each country involved.

6 Appendix

6.1 Proof of Proposition 6

Letting $u' = u'((1-\tau_w)y)$, $u'' = u''((1-\tau_w)y)$, $v' = v'(y)$ and $v'' = v''(y)$, using (6), and taking into account that at steady state one has $(1 - \tau_w)u' = v'(qI + 1 - q)$ and $I = \pi/\beta$, we obtain from the linearization of (17)-(18):

$$
\underbrace{\begin{pmatrix} -\beta(1-\tau_w)u''\left(q\frac{\pi}{\beta} + 1 - q\right) & u'\frac{\beta}{\pi}\left[q\frac{\pi}{\beta} + (1-q)(1-\gamma)\right] \\ \beta\left[(1-\tau_w)^2 u'' - (1-q)v''\right] & -qv' \end{pmatrix}}_{A}\begin{pmatrix} dy_{t+1} \\ d\pi_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} -\beta(1-\tau_w)u''\left(q\frac{\pi}{\beta} + 1 - q\right) & u'q\gamma \\ q\pi v'' & 0 \end{pmatrix}}_{B}\begin{pmatrix} dy_t \\ d\pi_t \end{pmatrix}
$$

$$(48)$$

The Jacobian matrix J is given by $J = A^{-1}B$ and its characteristic polynomial is defined by $P(\lambda) = \lambda^2 - \lambda\mathcal{T} + \mathcal{D}$ where $\mathcal{T}$ and $\mathcal{D}$ are given by, respectively, (38) and (39).

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