Optimal Financial Regulation and the Concentration of Aggregate Risk

Sebastian Di Tella
Stanford GSB

PRELIMINARY AND INCOMPLETE

July 2014

Abstract

Excessive concentration of aggregate risk can lead to financial fragility and may create the need for financial regulation. This paper studies the optimal financial regulation policy in a standard model where financial frictions are derived from a moral hazard problem, with a focus on the allocation of aggregate risk. First, I study the competitive equilibrium where agents can write complete long-term contracts, and I derive a simple formula for the allocation of aggregate risk. I then consider the optimal allocation that can be achieved by a social planner who faces the same informational asymmetries as the market, and show the competitive equilibrium is not constrained efficient. I identify a "moral hazard externality" that appears in a wide class of models, and show how it can create incentives for an inefficient allocation of aggregate risk. Finally, I show that although the competitive equilibrium may feature excessive concentration of aggregate risk, the optimal allocation can be implemented with a tax on capital.

1 Introduction

The concentration of aggregate risk on the balance sheets of highly leveraged agents can lead to financial fragility. Macroeconomic shocks will be amplified and propagated throughout the economy when these agents face large loses and reduce their demand for productive assets, depressing growth and leading to balance sheet recessions and financial crises. As a result, the aftermath of the last financial crisis has seen a renewed interest in financial regulation. This concentration of aggregate risk, however, might be an efficient feature of a market economy. In this paper I study the optimal financial regulation policy, with a focus on the allocation of aggregate risk.

There are three main contributions. First, I provide a tractable framework to study the role of financial frictions derived from a moral hazard problem when agents are able to write complete long-term contracts in a competitive environment. I show that concentration of aggregate risk can arise naturally even though the moral hazard problem does not limit agents' ability to share aggregate risk, and I derive a simple formula that characterizes the allocation of aggregate risk in the competitive equilibrium, in both financial and utility terms. I then turn to the problem of optimal financial regulation. Instead of committing to a set of policy instruments, I characterize the best allocation that can be achieved by a social planner who faces the same informational asymmetries as the market, and then ask how it can be implemented in a competitive equilibrium. The focus in this paper is on understanding under what conditions, and why, the allocation of aggregate risk in a competitive market will be inefficient, and what can be done about it.

I show that even though agents can write optimal long-term contracts, the competitive equilibrium is constrained inefficient. I identify a "moral hazard externality" that appears in a wide class[1] of models: agents don't internalize that by demanding capital and bidding up its price, they create a moral hazard problem for everyone else. While the moral hazard problem does not directly limit agents' ability to share aggregate risk, it may create incentives for an excessive concentration of aggregate risk, and I derive a simple formula that characterizes this inefficiency. Remarkably, even though the allocation of aggregate risk is inefficient, the socially optimal allocation can be implemented with a simple tax on capital, without the need to directly regulate agents' exposure to risk.

I then use these general results to study two applications. First, when the economy is hit only by Brownian TFP shocks, the social planner implements lower asset prices and growth than the unregulated competitive equilibrium, in order to reduce the cost of providing incentives. Aggregate risk sharing, however, is efficient. Neither the unregulated competitive equilibrium nor the optimal allocation feature concentration of aggregate risk. In contrast, when the economy is hit by uncertainty shocks that raise idiosyncratic risk, the allocation of aggregate risk is inefficient. After a bad shock, financial loses are excessively concentrated on financial intermediaries, while utility loses are excessively concentrated on households.

I use a standard continuous-time growth model as in the Di Tella (2013) and Brunnermeier and

[1] Such as, for example, Di Tella (2013), Brunnermeier and Sannikov (2012), or He and Krishnamurthy (2011)

Sannikov (2012) models of financial crises.[2] There are two goods: consumption and capital, and two types of agents: experts and households. Both have the same preferences, but only experts can trade and use capital to produce consumption goods. Capital is exposed to both aggregate and expert-specific idiosyncratic shocks. In addition, there is an investment goods sector with constant returns to scale that uses consumption goods and capital to produce new capital.[3]

Experts would like to raise funds and share risk with households, but they face a moral hazard problem. They can secretly divert (steal) capital and immediately sell it for consumption goods which they can add to their consumption (no hidden savings). To deal with this moral hazard problem they sign long-term contracts contingent on all observable variables. In order to provide incentives to not steal, contracts must expose experts' continuation utility to the idiosyncratic risk in their capital. The larger the value of capital under management, the higher the idiosyncratic risk the expert must be exposed to, relative to his continuation utility.[4] Experts' continuation utility therefore becomes an important state variable that can amplify and propagate aggregate shocks.

While moral hazard limits agent's ability to share idiosyncratic risk, it places no constraints on aggregate risk sharing, which can be characterized in terms of the marginal cost of providing utility to experts in terms of foregone utility for households. The competitive equilibrium allocates aggregate risk so that experts face large loses in continuation utility after an aggregate shock that raises the cost of providing utility to them. In terms of financial wealth, however, there is an *income* and a *substitution* effect. The substitution effect says experts should have relatively *more* net worth after an aggregate shock that reduces the cost of providing utility to them, in order to get more "bang for the buck". The income effect, on the other hand, says that they should have relatively *less* net worth after an aggregate shock that reduces the cost of providing utility to them, since in those states of the world they need less net worth to achieve any given level of utility. The income effect dominates in the empirically relevant case with relative risk aversion greater than one.

I then turn my attention to the problem of optimal financial regulation. I study the optimal allocation that can be achieved by a social planner who faces the same technological and informational environment. To this end, I consider a social planner who can a) determine consumption streams for households; b) determine consumption and capital under management for each expert; and c) provide consumption goods and capital to the investment sector and demand a flow of new capital from them. The planner faces the same informational problem as the market: experts can secretly divert capital. In the decentralized market problem they could immediately sell it in a competitive market for consumption goods. In the planner's problem, they can strike a side deal with an investment sector firm in exchange for consumption goods. With less consumption goods to invest, output of new capital will be lower, but the investment firm can use the stolen capital to make up the difference and avoid being detected (and punished) by the planner. The "black market" price of capital in this setting will then be equal to the marginal cost of producing new capital (which was equal to the price of capital in the competitive equilibrium). Thus, this setup

[2] A similar setup is also used in He and Krishnamurthy (2011).

[3] This sector doesn't have any frictions, and investment firms don't make any profits in equilibrium.

[4] Here is where the externality appears, as the private benefit of the hidden action depends on the price of capital.

captures the same contractual problem as in the decentralized market economy.

The competitive equilibrium is constrained inefficient. Agents don't internalize that by demanding capital and bidding up its price they worsen the moral hazard problem for others, limiting idiosyncratic risk sharing and increasing the cost of providing incentives to experts. The planner internalizes the resulting tradeoff between growth and idiosyncratic risk sharing. This moral hazard externality creates a wedge between the private and social marginal cost of providing utility to experts in terms of forgone utility for households. In addition, although the externality does not affect aggregate risk sharing directly, it can create incentives for an inefficient allocation of aggregate risk. The allocation of aggregate risk in the competitive equilibrium will be inefficient when the wedge between the private and social cost of experts' utility is correlated with aggregate shocks. Utility loses will be excessively concentrated on experts after an aggregate shock that increases the private cost of experts' utility relative to the social cost.[5]

I then show how the socially optimal allocation can be implemented in a competitive market with a simple tax on capital.[6] One advantage of the approach to optimal policy taken here is that we don't need to commit ex-ante to any given set of policy instruments. Instead, we let the model tell us what instruments are appropriate for the problem at hand.[7] In particular, although the allocation of aggregate risk may be inefficient, there is no need to directly regulate aggregate risk sharing. Once the externality is internalized with a tax on capital, agents can be allowed to share aggregate risk freely. In addition, the implementation of the socially optimal allocation as a competitive equilibrium allows us to study the allocation of aggregate risk in terms of financial wealth. As in the case of continuation utility, the allocation of aggregate financial risk is inefficient when the wedge between private and social cost of experts' utility is correlated with aggregate shocks. When the income effect dominates, financial loses are excessively concentrated on experts after an aggregate shock that reduces the private cost of experts' utility relative to the social cost.

Finally, I apply these general results to two settings. I first consider an economy that is hit only by Brownian TFP shocks. In line with Di Tella (2013), experts and households share aggregate risk proportionally, and the economy is deterministic (up to the level of capital): a negative TFP shock will reduce output in the economy, but will have no effect on asset prices, growth, interest rates, or the price of aggregate risk. The distribution of wealth (or continuation utility) matters for the equilibrium because of the financial friction created by the moral hazard, but they play no role in the propagation or amplification of aggregate shocks. In terms of efficiency, the unregulated competitive equilibrium is inefficient because of the moral hazard externality. The social planner implements lower asset prices and growth in order to reduce the cost of providing incentives to

[5]This is a statement about the ratio of experts' to households' utility. It might be the case that both gain utility after such a shock, but experts less so that households.

[6]The implementation of the optimal allocation is not unique. This implementation has the advantage of highlighting the margins the planner would like to manipulate.

[7]Sometimes we may be interested in what is the best we can do with a given policy instrument. Both approaches can yield valuable insights. In particular, the optimal allocation considered here provides an upper bound on what can be achieved with a given set of policy instruments, and is therefore the natural benchmark to evaluate those policies.

experts. Nevertheless, aggregate risk sharing is efficient because these TFP shocks don't affect the wedge between the private and social cost of experts' utility. As a result, the social planner also does not concentrate aggregate risk.

I then consider an economy hit by uncertainty shocks that raise idiosyncratic risk. In contrast to the TFP case, the competitive equilibrium concentrates aggregate financial risk on experts. After an uncertainty shock asset prices and growth falls, and financial loses are concentrated on experts because the private cost of experts' utility is lower when idiosyncratic risk, and therefore excess returns, are high. However, this concentration of aggregate risk is inefficient, because after a bad uncertainty shock the wedge between private and social cost is bigger (the private cost is smaller relative to the social cost). The social planner not only implements lower asset prices and growth, as in the TFP case, but also allocates financial risk more evenly. In utility terms the situation is reversed: in the competitive equilibrium utility loses are concentrated on households, while the social planner allocates utility risk more evenly. In other words, after a bad uncertainty shock, financial loses are too concentrated on experts, but utility loses are too concentrated on households.

Literature review. This paper fits in the literature on balance sheet recessions starting with Kiyotaki and Moore (1997) and Bernanke et al. (1999). More recently, Brunnermeier and Sannikov (2012) and He and Krishnamurthy (2011) use a similar setting, and restrict agents to short-term contracts that cannot be written on the aggregate state of world. Di Tella (2013) shows that when agents can write short-term contracts on all observable variables, the concentration of aggregate risk disappears in models driven by Brownian TFP shocks, but can arise naturally in an economy hit by uncertainty shocks. This paper studies the optimal financial regulation policy in a very similar environment. The main difference in the setting is that here I allow agents to write complete long-term contracts with full commitment. This is a simpler contractual structure that makes the comparison with the social planner's allocation cleaner. It is therefore a more natural starting point for the study of optimal financial regulation. Hidden savings are ruled out for the same reason, but both hidden savings and limited commitment seem like a natural direction for future research.

The private contractual environment in the competitive market is that same as in Di Tella and Sannikov (2014), where we explore it in more detail in a partial equilibrium setting, as well as the case with hidden savings. It is also related to the partial equilibrium settings in Sannikov (2008), DeMarzo and Sannikov (2006), and DeMarzo et al. (2012). The setting here has two features that make it particularly well suited to financial and macroeconomic applications. First, preferences with risk aversion and elasticity of intertemporal substitution are important in financial or macro settings. Second, the scale of the project can be continuously adjusted and is unbounded, allowing both growth and the hedging and replication strategies common in finance.

The approach to optimal policy is in the tradition of Mirrlees (1971) and the large literature on optimal taxation. Davila et al. (2012) study the optimal allocation in a model with exogenously uninsurable idiosyncratic risk, and no aggregate risk. Here, in contrast, agents face a moral hazard problem that endogenously limits idiosyncratic risk sharing, and the focus is on the allocation of aggregate risk.

Layout. The rest of the paper is organized as follows. In Section 2 I present the model and define the competitive equilibrium. In Section 3 I provide a recursive characterization of the competitive equilibrium and study aggregate risk-sharing. In Section 4 I introduce the planner's problem and characterize it recursively. I also show how to implement the optimal allocation as a competitive equilibrium. In Section 5 I apply the general results to two settings: I provide numerical solutions of the competitive equilibrium and the planner's allocation for TFP shocks and uncertainty shocks. Section 6 concludes.

2 The model

Technology. The economy is populated by two types of agents: "experts" $i \in \mathbb{I} = [0,1]$ and "households" $j \in \mathbb{J} = (1,2]$, identical in every respect except that experts are able to use capital. There are two goods, consumption and capital. Denote by k_t the aggregate "efficiency units" of capital in the economy, and by $k_{i,t}$ the individual holdings on expert i, where $t \in [0,\infty)$ is time. An expert can use capital to produce a flow of consumption goods

$$y_{i,t} = ak_{i,t}$$

There's a competitive investment sector that uses capital and consumption goods to produce new capital. The cost of producing a flow of new capital $k_t g_t$ is $\iota(g_t)k_t$, where the function $\iota' > 0$, $\iota'' > 0$ is a standard investment technology with convex adjustment costs. The competitive investment sector sets the growth rate to satisfy a Tobin's q FOC:

$$\iota'(g_t) = p_t$$

Since this technology has constant returns to scale, the investment sector has zero profits and experts who own capital add to their return from holding capital $(p_t g_t - \iota(g_t))k_{i,t}$.[8] In addition, capital is exposed to both aggregate and expert-specific idiosyncratic risk. If an expert i holds $k_{i,t}$ units of capital over a short period of time, the change in his capital stock is[9]

$$\sigma_t k_{i,t} dZ_t + \nu_t k_{i,t} dW_{i,t}$$

where $Z = \{Z_t \in \mathbb{R}^d; \mathcal{F}_t, t \geq 0\}$ is an aggregate brownian motion, and $W_i = \{W_{i,t} \in \mathbb{R}; \mathcal{F}_t, t \geq 0\}$ is an idiosyncratic brownian motion for each expert i, in a probability space $(\Omega, P, \mathcal{F})$ equipped with

[8] Formally, investment firms are owned by households or experts, but since they will have zero profits in equilibrium, this can be ignored in agents' problems. A firm in the investment sector chooses g and k to maximize profits $p_t gk - \iota(g)k - q_t k$, where q_t is rental price for the capital used in the production of new capital (or the difference between the price of old and new capital). The FOC yields $\iota'(g_t) = p_t$ and free entry implies zero profits, so $q_t = p_t g_t - \iota(g_t)$. Experts therefore receive a rental income for the capital they hold $q_t k_{i,t} = (p_t g_t - \iota(g_t))k_{i,t}$.

[9] The expert will be allowed to continuously trade capital, so $k_{i,t}$ will be a choice variable.

a filtration $\mathbb{F} = \{\mathcal{F}_t; t \geq 0\}$ generated by Z and $\{W_i\}_{i\in\mathbb{I}}$, with the usual conditions. The law of motion of the aggregate capital stock is not affected by idiosyncratic shocks[10]

$$\frac{dk_t}{k_t} = g_t dt + \sigma_t dZ_t$$

We can let several features of the environment depend on the history of aggregate shocks (in one of the examples, it will be idiosyncratic volatility ν_t). To this end, I introduce an exogenous aggregate state of the economy $Y_t \in \mathbb{R}^d$ following a Markov process

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dZ_t$$

We can later specify how this aggregate state affects the economy, e.g. $\nu_t = \nu(Y_t)$. Notice that the aggregate shocks and state can be multidimensional $d > 1$ so that different aggregate shocks can have different effects on the economy.

Preferences. Both experts and households have Epstein-Zin preferences with the same discount factor ρ, risk aversion γ, and elasticity of intertemporal substitution ψ^{-1}. For a consumption stream $c = \{c_t; t \geq 0\}$ in an appropriate set $\mathbb{C}$, the utility process U satisfies

$$U_t = \mathbb{E}_t\left[\int_t^\infty f(c_u, U_u)du\right] \tag{1}$$

where

$$f(c, U) = \frac{1}{1-\psi}\left\{\frac{c^{1-\psi}}{[(1-\gamma)U]^{\frac{\gamma-\psi}{1-\gamma}}} - \rho(1-\gamma)U\right\}$$

I will focus on the case where relative risk aversion is larger than log: $\gamma > 1$, and elasticity of intertemporal substitution is larger than 2: $\psi < \frac{1}{2}$.[11]

Markets. Experts can trade capital continuously at a competitive price $p_t > 0$, which we conjecture follows an Ito process

$$\frac{dp_t}{p_t} = \mu_{p,t}dt + \sigma_{p,t}dZ_t$$

The price of capital depends on the history of aggregate shocks Z and is determined endogenously in equilibrium. The total value of the capital stock in the economy is $p_t k_t$.

There is also a complete financial market with a SDF $\eta = \{\eta_t; t \geq 0\}$ that follows

$$\frac{d\eta_t}{\eta_t} = -r_t dt - \pi_t dZ_t$$

[10] I am using an exact law of large numbers, assuming the $\{W_i\}$ and Z are essentially pairwise independent. This requires working with an extension of the Lebesgue interval and a Fubini extension of the product space. See Sun and Zhang (2009) for details. Abusing notation, I still write, for example, $\int_{\mathbb{I}} k_{i,t} di$.

[11] It is natural to focus on the case with elasticity of intertemporal substitution greater than 1, especially when studying economies with stochastic volatility. The further restriction to EIS > 2 is required to guarantee existence of the competitive equilibrium and is related to the assumption of no hidden savings.

Here r_t is the risk-free interest rate and π_t the price of aggregate risk Z. Both depend on the history of aggregate shocks, and are determined endogenously in equilibrium. When I write the law of motion of the SDF this way I am already using the fact that the price of idiosyncratic risks $\{W_i\}_{i\in[0,1]}$ is zero in equilibrium, since they can be aggregated away. Later I will use the Equivalent Martingale Measure Q, defined from the SDF η.

Households' problem. Households are all the identical and have homothetic preferences, so we may consider the problem faced by a representative household. It starts with some wealth w_0 (derived from its initial ownership of capital and government transfers) and choses consumption $c = \{c_t > 0; t \geq 0\} \in \mathbb{C}$ to maximize utility subject to the budget constraint

$$\max_c V_0(c)$$

$$st: \quad \mathbb{E}\left[\int_0^\infty \frac{\eta_t}{\eta_0} c_t dt\right] \leq w_0$$

This is equivalent to choosing c and the exposure of wealth to aggregate risk $\sigma_w = \{\sigma_{w,t}; t \geq 0\}$ to maximize utility subject to a dynamic budget constraint

$$st: \quad \frac{dw_t}{w_t} = (r_t + \sigma_{w,t}\pi_t - \hat{c}_t)dt + \sigma_{w,t}dZ_t$$

and a solvency constraint $w_t \geq 0$, where the hat on $\hat{c}_t$ denotes the variable is divided by wealth.[12] Implicit in the second formulation is the fact that since idiosyncratic risks $\{W_i\}_{i\in[0,1]}$ have price zero in equilibrium, it is wlog that they will never chose to be exposed to them. We refer to (c, w) as the representative household's portfolio plan: it is *optimal* if it solves the household's problem.

Experts' contracts. Experts take aggregate conditions as given, and can continuously trade and use capital to produce consumption goods. Each expert would like to borrow from and share risk with the market, but he faces a moral hazard problem: he can secretly steal capital at rate s_t and keep a fraction $\phi \in (0, 1)$.[13] He must immediately sell it at price p_t and consume the proceeds (no hidden savings). This partial equilibrium contractual setting is studied in more detail in Di Tella and Sannikov (2014).

Formally, the expert starts with net worth $n_{i,0}$ which he gives to a principal in exchange for a contract $\mathcal{C} = (e_i, k_i)$ that specifies his consumption $e_i = \{e_{i,t} > 0; t \geq 0\} \in \mathbb{C}$ and the capital he will manage $k_i = \{k_{i,t} \geq 0; t \geq 0\}$, both adapted to $\mathbb{F}$.[14] There is full commitment on both sides.

---

[12] The link is $w_t = \mathbb{E}_t\left[\int_t^\infty \frac{\eta_u}{\eta_t} c_u du\right]$ and $\sigma_{w,t}w_t$ is the loading on Z of w_t thus defined.

[13] The parameter ϕ captures the severity of the moral hazard problem. With $\phi = 0$ there is no moral hazard.

[14] To prevent arbitrage, we must restrict k_i so that pk_i is *martingale-generating*, and $\lim_{T\to\infty} \mathbb{E}_t^Q\left[\frac{p_T k_{i,T}}{B_T}\right] = 0$ *a.e.* A portfolio process is martingale-generating if the discounted gain process is a martingale under the equivalent martingale measure Q (see Karatzas and Shreve (1998), Definition 5.9), and is required in order to rule out, e.g. doubling strategies. In this setting we say pk_i is martingale generating if the local martingales $\int_0^t \frac{p_u k_{i,u}}{B_u}(\sigma + \sigma_{p,u})dZ_u^Q$ and $\int_0^t \frac{p_u k_{i,u}}{B_u}\nu_u dW_u$ are both martingales under Q, where $Z_t^Q = Z_t + \int_0^t \pi_u du$ is a Q-Brownian Motion. See Di Tella

Faced with a contract $\mathcal{C}$, the expert privately chooses a stealing plan $s_i = \{s_{i,t} \geq 0; t \geq 0\}$ adapted to $\mathbb{F}$, and obtains utility process $U_i^{s_i} = \{U_{i,t}^{s_i}; t \geq 0\}$ defined recursively

$$U_{i,t}^{s_i} = \mathbb{E}_t^{s_i}\left[\int_t^\infty f(e_{i,u} + \phi p_u k_{i,u} s_{i,u}, U_{i,u}^{s_i})du\right] \tag{2}$$

where the expectation is taken under the probability measure P^{s_i} induced by his stealing plan s_i. We restrict the expert to stealing plans that induce a probability measure P^{s_i} equivalent to P, since otherwise he would be detected and punished.[15] The observable return of capital $R_i = \{R_{i,t}; t \geq 0\}$ satisfies for any valid stealing plan s

$$dR_{i,t} = \left(\frac{a - \iota(g_t)}{p_t} + g_t + \mu_{p,t} + \sigma_t \sigma_{p,t}' - s_{i,t}\right)dt + (\sigma_t + \sigma_{p,t})dZ_t + \nu_t dW_{i,t}^s$$

where $W_{i,t}^{s_i} = W_t + \int_0^t \frac{s_{i,u}}{\nu_u}du$ is a Brownian Motion under P^{s_i}.[16] The informational asymmetry comes from the principal's inability to observe or contract on the stealing process s_i and therefore P^{s_i}, so he must provide incentives to the expert so he chooses the right s_i. It is always optimal to implement no stealing $s_i = 0$.[17,18]

A contract $\mathcal{C}$ is *incentive compatible* if and only if never stealing ($s_i = 0$) is optimal:

$$s_i = 0 \in \arg\max_{s_i} U_{i,0}^{s_i}$$

Let IC denote the set of incentive compatible contracts (for given aggregate conditions). Notice that due to the homotheticity of preferences and the linearity of stealing technology, if a contract is incentive compatible, then so is a scaled up version of the contract $\mathcal{C}' = (\alpha e, \alpha k)$ for $\alpha > 0$.

The principal represents the market and just wants to maximize the present value of profits

$$\mathbb{E}_0^{s_i=0,Q}\left[\int_0^\infty \frac{1}{B_t}\Big(p_t k_{i,t}(dR_{i,t} - (r_t + \tau_t^k)dt) - e_{i,t}dt\Big)\right]$$

where the expectation is taken under no stealing and the equivalent martingale measure Q, and $B_t = \exp(\int_0^t r_u du)$ is the value of a risk-free bond. τ_t^k is a tax on capital investment that the government may impose, as will be explained below (this will be used for the implementation of the socially optimal allocation; in the unregulated economy, $\tau_t^k = 0$). In equilibrium the principal's value must always be negative: it is costly to provide continuation utility to the expert.[19]

and Sannikov 2014.

[15] Technically, we also need to restrict the expert to stealing plans such that the resulting consumption stream has a defined utility. Let $\mathbb{C}(s_i)$ be the set of consumption streams for which the expert's utility is defined, under probability measure P^{s_i}. We require that the stream $e_i + \phi pk s_i \in \mathbb{C}(s_i)$. Of course, $s_i = 0 \in \mathbb{C}(0) = \mathbb{C}$.

[16] Stealing does not affect the probability over Z (the aggregate shocks), so Z is also a P^{s_i}-Brownian Motion.

[17] The standard argument applies: if the agent is stealing in equilibrium it's better to just give him what he steals and have him not steal instead. See DeMarzo and Sannikov (2006) for example.

[18] Notice that we allow the contract to vary the scale of the project continuously, this is a major difference with many contractual settings where the scale is either fixed, or can only be adjusted "slowly".

[19] If this is not the case, by scaling up the program $(\alpha e_i, \alpha k_i)$ the principal can achieve unbounded profits.

An incentive compatible contract $\mathcal{C}$ is *optimal* if it maximizes the present value of profits subject to providing continuation utility $U_{i,0}^{s=0} \geq u_{i,0}$ to the expert, and its value to the principal is $J_{i,0}$:

$$J_{i,0}(u_{i,0}) = \max_{e_i,k_i} \mathbb{E}_0^{s_i=0,Q}\left[\int_0^\infty \frac{1}{B_t}\Big(p_t k_{i,t}(dR_{i,t} - (r_t + \tau_t^k)dt) - e_{i,t}dt\Big)\right]$$

$$st: \quad U_{i,0}^{s=0} \geq u_{i,0}$$

$$(e_{i,}, k_i) \in IC$$

We set the expert's outside option $u_{i,0}$ at the level such that the principal breaks even $J_{i,0} + n_{i,0} = 0$ (this captures competition among principals).

Define the continuation value of the optimal contract for agent i at time t

$$J_{i,t} = \mathbb{E}_t^{s=0,Q}\left[\int_t^\infty \frac{B_t}{B_u}\Big(p_u k_{i,u}(dR_{i,u} - (r_u + \tau_u^k)du) - e_{i,u}du\Big)\right]$$

Government. In Section II, I study the unregulated economy where the government is inactive. I then consider in Section III the optimal allocation that can be achieved by a social planner who faces the same informational frictions as private agents, and show how it can be implemented as a competitive equilibrium with a simple tax on capital. To facilitate the exposition I introduce here the policy instruments that will implement the optimal allocation. For the unregulated economy, the role of government can be ignored. It should be stressed that I am not restricting the planner to use this instrument to control the economy, but instead finding that the optimal allocation can be implemented in this way.[20]

The government taxes investment in capital with a history-dependent tax $\tau^k = \{\tau_t^k; t \geq 0\}$ adapted to $\mathbb{F}$. An expert who has holdings in capital worth $p_t k_{i,t}$ must pay a tax flow $\tau_t^k p_t k_{i,t}$. As a result the government raises a total flow $\tau_t^k p_t k_t$. To balance the budget the government will distribute back the proceeds as lump-sum transfers to agents. Since there is a complete financial market agents can just sell these transfers, so the market value of the transfers they will receive is part of their initial wealth. The aggregate value of transfers is $T = \{T_t; t \geq 0\}$ per unit of capital

$$T_t = \frac{1}{k_t}\mathbb{E}_t\left[\int_t^\infty \frac{\eta_u}{\eta_t}\tau_u^k p_u k_u du\right]$$

Total wealth in the economy is therefore $k_t(p_t + T_t)$. In the unregulated economy, we simply take $\tau^k = 0$ and therefore $T = 0$.

Competitive Equilibrium. Take as given the initial capital stock k_0, and a tax τ^k. In addition, take as given the initial distribution of wealth for experts $\{\theta_i > 0\}_{i\in\mathbb{I}}$, such that $\int_{\mathbb{I}} \theta_i di < 1$ (the rest belongs to the representative household).

[20]Of course, the implementation of the optimal allocation is not unique. This is one possible way to implement it, that has the advantage of being simple and highlighting how the planner wants to distort agents' incentives.

Definition 1. *A competitive equilibrium is a set of $\mathbb{F}$-adapted stochastic processes: the price of capital p, value of taxes T, the state price density η, growth rate g, and the aggregate capital stock k; an optimal contract $\mathcal{C}_i = (e_i, k_i)$ with associated value J_i and continuation utility U_i for each expert $i \in \mathbb{I}$; and a portfolio plan (c, w) for the representative household, such that*

i. *The representative household's plan and experts contracts are optimal, with initial wealth $n_{i,0} = \theta_i(p_0 + T_0)k_0$ and $w_0 = (p_0 + T_0)k_0(1 - \int_{\mathbb{I}} \theta_i di)$.*

ii. *Investment is optimal $\iota'(g_t) = p_t$.*

iii. *The value of taxes is $T_t = \frac{1}{k_t}\mathbb{E}_t\left[\int_t^\infty \frac{\eta_u}{\eta_t}\tau_u^k p_u k_u du\right]$.*

iv. *Market clearing*

$$\int_{\mathbb{I}} e_{i,t} di + c_t = (a - \iota(g_t))k_t$$

$$\int_{\mathbb{I}} k_{i,t} di = k_t$$

v. *Aggregate capital satisfies the law of motion*

$$\frac{dk_t}{k_t} = g_t dt + \sigma_t dZ_t$$

We can take $n_{i,t} = -J_{i,t} > 0$ to be the net worth of expert i at time t. In equilibrium it will be the case that $w_t + \int_{\mathbb{I}}(-J_{i,t})di = k_t\big(p_t + T_t\big)$: the representative household and experts hold the total wealth in the economy, made up of the value of capital $p_t k_t$, plus the value of taxes $T_t k_t$. Recall, however, that in the unregulated economy we have $\tau^k = 0$ and therefore $T = 0$.[21]

Concentration of financial risk. We can think of the market value of experts' contracts as the *aggregate net worth of experts,* $n_t = -\int_{\mathbb{I}} J_{i,t} di$. Then the ratio of experts' net worth to households' wealth is

$$\omega_t = \frac{n_t}{w_t}$$

with law of motion

$$\frac{d\omega_t}{\omega_t} = \mu_{\omega,t} dt + \sigma_{\omega,t} dZ_t$$

We say *aggregate financial risk is concentrated on experts*, if after a bad shock their net worth n_t falls proportionally more than households' wealth w_t, so the ratio ω_t falls. To the extent that the distribution of wealth matters for the economy, the aggregate shock will be amplified and propagated through its effect on ω_t.

[21] We can alternatively define initial utility for each expert $\{u_{i,0}\}_{i\in\mathbb{I}}$ and define the equilibrium requiring contracts to deliver utility $u_{i,0}$ to each expert, and the household's portfolio plan (c, w) to be optimal for w_0. There then exist $\{\theta_i > 0\}_{i\in\mathbb{I}}$ such that $n_{i,0} = \theta_i(p_0 + T_0)k_0 = -J_{i,0} > 0$ and$w_0 = (p_0 + T_0)k_0(1 - \int_{\mathbb{I}} \theta_i di) > 0$. Both definitions are equivalent, but this one has the advantage of paralleling the social planner's problem.

3 Solving the competitive equilibrium

In this section I characterize the competitive equilibrium. Both experts and households face a dynamic problem where their optimal policies depend on endogenously stochastic investment possibility sets. The approach I take is to obtain a recursive characterization of their problems, and look for a Markov equilibrium in the exogenous state variable Y_t and an endogenous state variable X_t that captures the continuation utility of experts relative to the total capital stock in the economy k_t. This is the same endogenous state variable that a social planner would use, which makes comparisons between the competitive equilibrium and the social optimum clear.

The layout of this section is as follows: I first characterize a first best benchmark without moral hazard. I then provide a recursive characterization of agents' problems and define a Markov equilibrium which can be solved as a system of PDEs.

3.1 First Best without moral hazard

Consider the first best without moral hazard, and without taxes τ^k=0. Without moral hazard, optimal contracts will provide full insurance against idiosyncratic shocks and there is no financial friction. Capital is then priced by arbitrage, and the distribution of wealth between experts and households doesn't matter. If there are only TFP shocks, the competitive equilibrium is a balanced growth path with constant asset prices $p_t = p^*$ and growth $g_t = g^*$, given by

$$p^* = \frac{a - \iota(g^*)}{\rho - (1-\psi)g + (1-\psi)\frac{\gamma}{2}\sigma^2}$$

$$\iota'(g^*) = p^*$$

The interest rate and price of risk are then also constant, $r_t = r^* = \rho + g\psi - \frac{\gamma}{2}\sigma^2(1+\psi)$ and $\pi_t = \pi^* = \gamma\sigma$.

This is also the case when the exogenous state Y_t affects only idiosyncratic volatility $\nu_t = \nu(Y_t)$, because with full idiosyncratic risk-sharing this volatility is irrelevant, and so are shocks to it. Instead if aggregate shocks affect other features of the environment, the equilibrium will not be a balanced growth path in general, but it will still be true that the distribution of wealth doesn't matter. Experts and households are equivalent, and they share aggregate risk proportionally.

3.2 Back to moral hazard

Optimal contracts. It's a standard result that experts' optimal contracts are recursive in their continuation utility $U_{i,t} = U_{i,t}^{s=0}$. In this section I drop the i subscript to simplify notation. First note that for an agent i, we can wlog restrict attention to contracts $\mathcal{C} = (e, k)$ adapted to the filtration generated by Z and W_i, $\mathbb{F}^i \subset \mathbb{F}$. There is no point is exposing agent i to other agents' idiosyncratic risk. With this in mind, we obtain a representation of the expert's continuation utility.

Lemma 1. *For any contract $\mathcal{C} = (e, k)$ and no stealing $s = 0$, the expert's continuation utility process $U^{s=0} = \{U_t^{s=0}; t \geq 0\}$ satisfies the following BSDE*

$$dU_t^{s=0} = -f(e_t, U_t^{s=0})dt + \sigma_{U,t}dZ_t + \tilde{\sigma}_{U,t}dW_t \tag{3}$$

for some $\mathbb{F}^i$-adapted processes $\sigma_U = \{\sigma_{U,t}; t \geq 0\} \in \mathcal{L}$ and $\tilde{\sigma}_U = \{\tilde{\sigma}_{U,t}; t \geq 0\} \in \mathcal{L}$, with boundary condition $\lim_{t\to\infty} \mathbb{E}^{s=0}[U_t^{s=0}] = 0$.[22]

Under a stealing plan $s \neq 0$ the probability measure over observed returns changes to P^s equivalent[23] to P. The utility the expert gets from stealing is given

$$U_t^s = \mathbb{E}_t^s \left[\int_t^\infty f(e_u + \phi p_u k_u s_u, U_u^s) du \right]$$

We can use this representation to obtain an IC constraint for the case of interest where the EIS and the risk aversion are both larger than 1.[24]

Lemma 2. *If the EIS $\psi^{-1} > 1$ and the risk aversion $\gamma > 1$, an incentive compatible contract $\mathcal{C} = (e, k)$ must satisfy the following condition (ω, t)-almost everywhere:*

$$0 \in \arg\max_{s\geq 0} f(e_t + \phi p_t k_t s, U_t^{s=0}) - \tilde{\sigma}_{U,t} \frac{s}{\nu_t} - f(e_t, U_t^{s=0}) \tag{4}$$

Taking FOC in (4) we obtain

$$\tilde{\sigma}_{U,t} \geq f_1(e_t, U_t^{s=0})\phi p_t k_t \nu_t = \frac{e_t^{-\psi}}{((1-\gamma)U_t^{s=0})^{\frac{\gamma-\psi}{1-\gamma}}} \phi p_t k_t \nu_t \geq 0 \tag{5}$$

From now on, let U_t denote the continuation utility under no stealing ($U_t^{s=0}$). It's easy to see that because the expert is risk averse, while the principal is risk-neutral with respect to idiosyncratic risk W, the local IC will always be binding in the optimal contract. We can also verify that if contract $\mathcal{C} = (e, k)$ is IC, then so is a scaled up version of the contract $\mathcal{C}' = (\alpha e, \alpha k)$ for any $\alpha > 0$. In consequence, because expert's preferences are homothetic and the principal's objective function is linear, the value of the contract for the principal will take the following form

$$J_t = -\xi_t((1-\gamma)U_t)^{\frac{1}{1-\gamma}}$$

for some process $\xi = \{\xi_t > 0; t \geq 0\}$ that depends only on the history of aggregate shocks Z and follows

$$\frac{d\xi_t}{\xi_t} = \mu_{\xi,t}dt + \sigma_{\xi,t}dZ_t$$

[22]A stochastic process $y = \{y_t; t \geq 0\}$ is in $\mathcal{L}$ if $\mathbb{E}\left[\int_0^T y_t^2 dt\right] < \infty$ for all T.

[23]Recall agents must choose a stealing plan s that induces an absolutely continuous change of measure, since otherwise they would be detected and punished.

[24]The proof works for other parameters values as well.

for some processes μ_ξ and σ_ξ that must be determined in equilibrium. The HJB equation associated with the optimal contract is

$$r_t J_t dt = \max_{e,k,\sigma_U} p_t k_t \mathbb{E}_t^Q \left[dR_t - (r_t + \tau_t^k) dt \right] - e_t + \mathbb{E}_t^Q \left[dJ_t \right]$$

Because expectations are taken under the equivalent martingale measure Q, it is useful to write $Z_t = Z_t^Q - \int_0^t \pi_u du$, where Z^Q is a Brownian motion under Q to obtain a representation for ξ and for U_t under Q:

$$\frac{d\xi_t}{\xi_t} = (\mu_{\xi,t} - \pi_t \sigma_{\xi,t}) dt + \sigma_{\xi,t} dZ_t^Q$$

$$dU_t = (-f(e_t, U_t) - \sigma_{U,t}\pi_t) dt + \sigma_{U,t} dZ_t^Q + \tilde{\sigma}_{U,t} dW_t$$

Finally, let us normalize the controls $e_t = \hat{e}_t((1-\gamma)U_t)^{\frac{1}{1-\gamma}}$, $k_t = \hat{k}_t((1-\gamma)U_t)^{\frac{1}{1-\gamma}}$, and $\sigma_{U,t} = \hat{\sigma}_{U,t}(1-\gamma)U_t$. The HJB equation then takes the following form[25]

$$0 = \max_{\hat{e},\hat{k},\hat{\sigma}_U} p_t \hat{k} \left(\frac{a - \iota(g_t)}{p_t} + g_t + \mu_{p,t} + \sigma_t \sigma'_{p,t} - (r_t + \tau_t^k) - (\sigma_t + \sigma_{p,t})\pi_t \right) - \hat{e}$$

$$+ \xi_t \left\{ r_t + \frac{1}{1-\psi} \left\{ \hat{e}^{1-\psi} - \rho \right\} + \hat{\sigma}_U \pi_t - \mu_{\xi,t} + \sigma_{\xi,t}\pi_t \right.$$

$$\left. - \frac{1}{2}\gamma \hat{\sigma}_U^2 - \frac{1}{2}\gamma (\hat{e}_t^{-\psi} \phi p_t \hat{k} \nu_t)^2 - \sigma_{\xi,t} \hat{\sigma}_U \right\}$$

Households' value function. Likewise, households have a value function

$$V_t(w_t) = \frac{(\zeta_t w_t)^{1-\gamma}}{1-\gamma}$$

where $\zeta = \{\zeta_t > 0; t \geq 0\}$ depends only on the history of aggregate shocks Z with law of motion

$$\frac{d\zeta_t}{\zeta_t} = \mu_{\zeta,t} dt + \sigma_{\zeta,t} dZ_t$$

for some μ_ζ and σ_ζ which we must find in equilibrium. ζ captures the endogenously stochastic investment possibilities for households, and must satisfy the following HJB equation

$$\frac{\rho}{1-\psi} = \max_{\hat{c},\sigma_w} \frac{\hat{c}^{1-\psi}}{1-\psi} \zeta_t^{\psi-1} + r_t + \sigma_w \pi_t - \hat{c} + \mu_{\zeta,t} - \frac{\gamma}{2}\sigma_{\zeta,t}^2 - \frac{\gamma}{2}\sigma_w^2 + (1-\gamma)\sigma_{\zeta,t}\sigma_w$$

[25] Existence of the optimal contract requires $\psi < \frac{1}{2}$. This is related to the assumption of no hidden savings, as will be explained below. Once we assume this, the objective function in the HJB is concave, and the FOC are sufficient for optimality.

The value of taxes. Finally, since T_t only depends on the history of aggregate shocks Z, we can write

$$\frac{dT_t}{T_t} = \mu_{T,t}dt + \sigma_{T,t}dZ_t$$

The value of taxes T_t must satisfy a no-arbitrage pricing equation

$$\frac{p_t \tau_t^k}{T_t} + \mu_{T,t} + g_t + \sigma_t \sigma_{T,t}' - r_t = \pi_t(\sigma_{T,t} + \sigma_t) \tag{6}$$

Markov equilibrium. Experts' continuation utilities $\{U_{i,t}\}_{i\in[0,1]}$ are a state variable for the equilibrium. In fact, contracts are linear in $x_{i,t} = ((1-\gamma)U_{i,t})^{\frac{1}{1-\gamma}}$. This suggests using

$$X_t = \frac{\int_{[0,1]} x_{i,t} di}{k_t}$$

as an endogenous aggregate state variable, with law of motion

$$\frac{dX_t}{X_t} = \mu_{X,t}dt + \sigma_{X,t}dZ_t$$

We therefore look for a Markov equilibrium in X_t and Y_t, where, e.g. $\xi_t = \xi(X_t, Y_t)$ for some function ξ twice continuously differentiable. We can then use Ito's lemma to obtain expressions for e.g.

$$\mu_\xi = \frac{\xi_X}{\xi}\mu_X X + \frac{\xi_Y}{\xi}\mu_Y + \frac{1}{2}\left(\frac{\xi_{XX}}{\xi}(\sigma_X X)^2 + \frac{\xi_{YY}}{\xi}\sigma_Y^2 + 2\frac{\xi_{XY}}{\xi}\sigma_X X \sigma_Y\right)$$

$$\sigma_\xi = \frac{\xi_X}{\xi}\sigma_X X + \frac{\xi_Y}{\xi}\sigma_Y$$

where the functions are evaluated at (X, Y) (and analogous expressions for μ_ζ, σ_ζ, μ_p and σ_p). For this to work, it must be the case that taxes are also Markov, $\tau_t^k = \tau^k(X_t, Y_t)$, which will be the case for the implementation of the optimal allocation.

We can now provide a definition of a Markov equilibrium for a given tax policy $\tau^k(X, Y)$.

Definition 2. *A Markov equilibrium in (X, Y) is a set of functions of (X, Y) for prices p, T, r, and π and growth g; value functions ξ, ζ and policy functions $\hat{e}$,$\hat{k}$, $\hat{\sigma}_U$ for experts and $\hat{c}$, σ_w for households, and a law of motion for the endogenous state variable X given by $\frac{dX_t}{X_t} = \mu_X(X, Y)dt + \sigma_X(X, Y)dZ_t$ such that:*

i. *The value functions ξ and ζ and the associated policy functions solve experts' and households HJB equations, taking prices p, r, and π as given.*

ii. *The investment sector optimizes $\iota'(g) = p$.*

iii. *The value of taxes T satisfies pricing equation (6).*

iv. *Market clearing*

$$\hat{c}(p + T - \xi X) + \hat{e}X = a - \iota(g) \quad \textit{[consumption goods]}$$

$$\hat{k} = \frac{1}{X} \quad \textit{[capital]}$$

v. *The aggregate law of motion for X is derived from the policy functions*

$$\mu_X(X, Y) = \frac{\rho}{1-\psi} - \frac{\hat{e}^{1-\psi}}{1-\psi} + \frac{\gamma}{2}\hat{\sigma}_U^2 + \frac{\gamma}{2}(\hat{e}^{-\psi}\frac{\phi p \nu}{X})^2 - g - \sigma\hat{\sigma}_U + \sigma^2$$

$$\sigma_X(X, Y) = \hat{\sigma}_U - \sigma$$

To understand the market clearing conditions, notice that the aggregate value of experts' contracts is $-\xi_t X_t k_t$, so household wealth $w_t = p_t k_t + T_t k_t - \xi_t X_t k_t$ in equilibrium. We use this for the market clearing condition for consumption goods (and divide by k_t). Market clearing for capital goods requires total demand for capital $\hat{k}Xk$ be equal to total supply k. Finally, the law of motion of the endogenous state variable X is derived using Ito's lemma.

Demand for capital and growth. Taking FOC in the HJB, and using the equilibrium condition $\hat{k} = X^{-1}$ we obtain a pricing equation for capital

$$\underbrace{\frac{a - \iota(g_t)}{p_t} + g_t + \mu_p + \sigma\sigma_p' - r_t - \tau_t^k}_{\text{excess return}} = \underbrace{(\sigma_t + \sigma_{p,t})\pi_t}_{\text{agg. risk premium}} + \underbrace{\gamma\xi_t(\hat{e}_t^{-\psi}\phi\nu_t)^2\frac{p_t}{X_t}}_{\text{id. risk premium}}$$

Notice that although the market price of idiosyncratic risk W_i is zero, capital must pay a premium for its exposure to this risk. The reason is as follows: because of the moral hazard, the contract must expose the expert's continuation utility to his idiosyncratic risk in order to provide incentives to not steal, and this exposure is proportional to the amount of capital the expert manages. However, exposing the expert to risk is costly because he is risk averse. The principal knows that if he wants to have the expert manage a large amount of capital, he must be willing to bear the extra cost associated with the volatility in the expert's continuation utility, so he will only demand capital if it pays a premium for this risk.

Naturally, this premium is higher the higher idiosyncratic risk is (higher ν_t). In addition, the premium for idiosyncratic risk is higher when experts' continuation utility is low relative to the capital stock (small X_t) because the idiosyncratic risk contained in capital is large relative to the continuation utility of experts. It is through this pricing equation for capital that both idiosyncratic risk ν_t and the endogenous state variable X_t affect the equilibrium. Notice how this premium would disappear if we had no moral hazard ($\phi = 0$) or no idiosyncratic risk ($\nu_t = 0$).

The price of capital determines growth through Tobin's q:

$$\iota'(g_t) = p_t$$

so any shocks that depress the price of capital will result in lower growth.

Consumption. The FOC for experts' consumption $\hat{e}$ also takes a nonstandard form

$$\xi_t \hat{e}_t^{-\psi} + \underbrace{\xi_t \gamma \psi (\phi p_t \frac{\nu_t}{X_t})^2 \hat{e}_t^{-2\psi-1}}_{\text{front-loading}} = 1$$

The first term on the left hand side is standard: by giving the expert consumption today, we can promise lower utility in the future, so the principal trades off the cost of providing consumption today (on the RHS) against the reduction in cost of providing utility in the future. The higher the cost of providing utility ξ_t, the more attractive it is to give consumption to the expert today. However, there is another term on the LHS. The principal realizes that by front-loading consumption he can reduce the expert's marginal utility of consumption, and therefore render stealing (and immediately consuming) less attractive. To see this, use (5) and the market clearing condition for capital to write the exposure of the expert's utility to his idiosyncratic risk W as

$$\tilde{\sigma}_{U,t} = (1-\gamma) U_t \hat{e}_t^{-\psi} \phi p_t \frac{\nu_t}{X_t}$$

The optimal contract will distort the optimal intertemporal profile of consumption in order to relax the constraint on idiosyncratic risk sharing. In fact, this tool is so powerful that if experts' preferences where very curved (EIS < 2) the principal would be able to obtain unbounded profits by scaling up the contract, and the optimal contract would not exist.[26] This is the reason we need to assume $\psi < \frac{1}{2}$ (EIS > 2). The principal can do this because the expert doesn't have hidden savings, so introducing hidden savings seems like an important direction for future research.

Households have a more standard FOC for consumption

$$\hat{c}_t^{-\psi} \zeta_t^{\psi-1} = 1$$

Aggregate risk-sharing. The FOC for $\hat{\sigma}_U$ that controls the exposure of the expert's continuation utility to aggregate risk Z is

$$\pi_t - \sigma_{\xi,t} = \gamma \hat{\sigma}_{U,t}$$

$$\Longrightarrow \hat{\sigma}_{U,t} = \frac{\pi_t - \sigma_{\xi,t}}{\gamma} \tag{7}$$

[26] As long as capital pays a premium for idiosyncratic risk. If it doesn't then the optimal contract has $\hat{k} = 0$ and this can't be an equilibrium.

The cost of the contract to the principal is higher when the expert's continuation utility is larger, so the principal prefers to give him more utility when a) the value of money for the principal, captured by the SDF η_t, is lower (the term π_t); and b) when it is relatively cheaper to provide utility to the expert, as captured by ξ_t (the term $\sigma_{\xi,t}$).

Households have a similar FOC for their exposure to aggregate risk σ_w:

$$\pi_t = \gamma\sigma_{w,t} - (1-\gamma)\sigma_{\zeta,t}$$

$$\implies \sigma_{w,t} = \frac{\pi_t}{\gamma} + \frac{1-\gamma}{\gamma}\sigma_{\zeta,t} \tag{8}$$

Households' hold aggregate risk because it carries a positive premium π_t (a myopic motive), and because it correlates with their investment possibility set, as captured by ζ_t (hedging motive).

To understand aggregate risk sharing, it's useful to consider households' utility. Let V_t be households continuation utility, and consider the increasing transformation

$$S_t = \frac{((1-\gamma)V_t)^{\frac{1}{1-\gamma}}}{k_t}$$

with law of motion $dS_t = \mu_{S,t}S_t dt + \sigma_{S,t}S_t dZ_t$, analogous to X_t for experts. It follows that $S_t = \zeta_t\frac{w_t}{k_t} = \zeta_t(p_t + T_t - \xi_t X_t)$. Define the (private) relative cost of providing utility to experts in terms of forgone utility for households

$$\Lambda_t = \xi_t\zeta_t$$

with law of motion $d\Lambda_t = \mu_{\Lambda,t}\Lambda_T dt + \sigma_{\Lambda,t}\Lambda_t dZ_t$. If we increase an expert's utility x_i by Δ, the cost of the contract increases by $\xi_t\Delta$, which translates into a decrease of $\zeta_t\xi_t\Delta = \Lambda_t\Delta$ in the (transformed) utility of the representative household. We can use Ito's lemma and the FOC for aggregate risk sharing (7) and (8), and the definition of $\sigma_X = \hat{\sigma}_U - \sigma$ to obtain an expression for the allocation of aggregate risk in utility terms

$$\sigma_{X,t} - \sigma_{S,t} = -\frac{1}{\gamma}(\sigma_{\zeta,t} + \sigma_{\xi,t}) = -\frac{1}{\gamma}\sigma_{\Lambda t} \tag{9}$$

Equation (9) says that *the market will allocate more continuation utility to experts, relative to households, when the cost of experts' utility Λ_t is low.*

Of course, the cost of experts' utility Λ_t is endogenous and must be determined in equilibrium. In particular, it depends on the endogenous state variable X, which creates a two-way feedback loop: a) aggregate risk sharing (captured by σ_X) affects how Λ_t and S respond to aggregate shocks

$$\sigma_\Lambda = \frac{\Lambda_X}{\Lambda}\sigma_X X + \frac{\Lambda_Y}{\Lambda}\sigma_Y \quad \sigma_S = \frac{S_X}{S}\sigma_X X + \frac{S_Y}{S}\sigma_Y$$

and b) these in turn provide incentives for the concentration of aggregate risk given by (9). We can then obtain an expression for σ_X in terms of functions of the Markov equilibrium objects:

$$\sigma_X = \frac{\frac{S_Y}{S} - \frac{1}{\gamma}\frac{\Lambda_Y}{\Lambda}}{1 - \left(\frac{S_X}{S} - \frac{1}{\gamma}\frac{\Lambda_X}{\Lambda}\right) X}\sigma_Y \tag{10}$$

Concentration of financial risk. While equation (9) determines the allocation of aggregate risk in utility terms, we might be more interested in the allocation of financial risk. Consider the ratio of experts' net worth to households' wealth

$$\omega_t = \frac{n_t}{w_t}$$

The proportional exposure of experts' net worth to aggregate risk is $\sigma_{n,t} = \sigma_{\xi,t} + \sigma_{X,t} + \sigma_t$, while for households' wealth we have $\sigma_{w,t}$. The volatility of the wealth ratio ω_t is then

$$\sigma_{\omega,t} = \sigma_{n,t} - \sigma_{w,t}$$

If aggregate risk was distributed proportionally, then $\sigma_{n,t} = \sigma_{w,t}$ and $\sigma_{\omega,t} = 0$. If instead aggregate risk is concentrated on experts, then $\sigma_{n,t} > \sigma_{w,t}$ and $\sigma_{\omega,t} > 0$. Using agents' FOC for aggregate risk sharing, and market clearing in the financial market, we can obtain an expression for $\sigma_{\omega,t}$

$$\sigma_{\omega,t} = \frac{\gamma - 1}{\gamma}\sigma_{\Lambda,t} \tag{11}$$

Here we have two opposing effects. On the one hand, there is a *substitution effect*: experts should have more net worth when it is cheap to provide utility to them relative to households (i.e. when the cost Λ_t is low) in order to get more "bang for the buck". But there is also an *income effect*: experts need more net worth in order to achieve any given utility level when it is expensive to provide utility to them relative to households (i.e. when Λ_t is high). If agents are very risk averse, $\gamma > 1$, they are more interested in stabilizing their utility across states of the world and the income effect dominates, so the market allocates more financial wealth to experts after aggregate shocks that increase the cost of providing utility to them (ω_t is positively correlated with Λ_t).

Solving the competitive equilibrium. The full competitive equilibrium can be characterized with a system of PDEs for $\xi(X,Y)$, $\zeta(X,Y)$, and $p(X,Y)$. We can use the FOC for $\hat{c}$ and $\hat{e}$, the optimal growth formula $\iota'(g) = p$, and formula (10) for σ_X, plus the market clearing conditions and formula for μ_X, and plug them into the HJB for experts, the HJB for consumers, and the FOC for capital. Appendix B describes the procedure in detail.

4 Planner's Problem

I now turn to the problem of optimal financial regulation. I take a mechanism design approach, and ask what is the best allocation that can be achieved by a social planner who faces the same informational frictions as the market. We can then ask how the optimal allocation can be implemented as a competitive equilibrium. The main advantage of this approach is that we are not pre-committing to a given set of policy instruments which might be inappropriate for the problem at hand. We are letting the model tell us what policy instruments we should use.

Even though we allow agents to write complete optimal contracts, the competitive equilibrium is inefficient because there is a "moral hazard externality". The private moral hazard problem arises because experts can divert capital and sell it for consumption goods: the private benefit of the hidden action depends on prices. Agents don't internalize that by competing for capital and bidding up its price, they are creating a moral hazard problem for everyone else. Indeed, if the equilibrium price of capital was $p_t = 0$, there wouldn't be any moral hazard problem: there's no point in stealing something that is worthless.

The price of capital, however, plays an allocative role in equilibrium inducing investment and growth. There is then a tradeoff between growth and idiosyncratic risk sharing that is not internalized by market participants. The focus here is not only on the tension between idiosyncratic risk-sharing and growth, but on the sharing of aggregate risk between households and experts. Since the market does not properly reflect the economic tradeoffs given the aggregate state of the world, it may also misallocate aggregate risk. We want to understand under what conditions the allocation of aggregate risk in the competitive equilibrium is inefficient, and what can be done about it.

To this end, consider a social planner who faces the same informational frictions as private agents in the market. He can a) control households' consumption; b) give capital and a flow of consumption goods to invest to the investment sector, and order them to deliver a flow of new capital; c) give consumption and capital to experts to manage, but they can secretly divert it. In the competitive equilibrium setup they could sell capital at price p_t and consume the proceeds right away. Here there is no market for capital, but experts can strike a side deal with the investment sector to obtain consumption goods from the capital they divert. An expert with a flow of stolen capital $k_{i,t}s_{i,t}$ can give it to a firm in the investment sector in exchange of $\iota'(g)$ units of consumption good. The investment firm will then be able to produce less new capital, but it will instead present the stolen capital to the planner, so that the side deal is not detected.[27]

A plan $\mathcal{P} = (c, \{e_i, k_i\}_{i\in[0,1]}, g, k\}$ is a consumption stream for households $c = \{c_t > 0; t \geq 0\}$ and each expert $e_i = \{e_{i,t} > 0; t \geq 0\}$; capital allocation for each expert $k_i = \{k_{i,t} \geq 0; t \geq 0\}$ and

[27] To formalize this, let $\tilde{p}_t$ be the black market price for stolen capital. The investment firm receives an order to use k_t units of capital and $\iota(g_t)k_t$ consumption goods to deliver a flow of new capital $g_t k_t$. However, it can instead chose to buy a flow of stolen capital $k_t s_t$ and do actual investment $k_t\iota(\tilde{g}_t)$ in order to maximize its surplus consumption (that it rebates to its owners)

$$\max_{s,\tilde{g}} k_t \left(\iota(g_t) - \iota(\tilde{g}) - \tilde{p}_t s\right)$$

$$st: \quad k_t\tilde{g} + k_t s = k_t g_t$$

Optimality implies

$$\tilde{p}_t = \iota'(\tilde{g}_t)$$

So if the planner is implementing his desired investment rate g_t and there is no stealing in equilibrium, the black market price of capital is

$$\tilde{p}_t = \iota'(g_t)$$

and investment firms have no surplus consumption.

an investment/growth rate $g = \{g_t; t \geq 0\}$ and aggregate capital $k = \{k_t; t \geq 0\}$, all $\mathbb{F}$-adapted. A plan $\mathcal{P}$ is *feasible* if it satisfies the aggregate consistency conditions

$$c_t + \int_{\mathbb{I}} e_{i,t} di = (a - \iota(g_t))k_t$$

$$\int_{\mathbb{I}} k_{i,t} = k_t$$

and aggregate capital follows the law of motion

$$dk_t = k_t g_t dt + \sigma_t k_t dZ_t$$

Let F be the set of feasible plans. Faced with a feasible plan $\mathcal{P}$, each expert choses a stealing plan s_i in order to maximize his utility defined recursively

$$U_{i,t}^{s_i} = \mathbb{E}_t^{s_i} \left[\int_t^{\infty} f(e_{i,u} + \phi \iota'(g_t) k_{i,u} s_{i,u}, U_{i,u}^{s_i}) du \right]$$

Notice that this is the same as expression (2) in the private contract, with the difference that instead of p_t we have $\iota'(g_t)$. A feasible plan is *incentive compatible* if every expert i chooses $s_i = 0$:[28]

$$s_i = 0 \in \arg\max_{s_i} U_{i,0}^{s_i} \quad \forall i \in \mathbb{I}$$

Let ICP be the set of incentive compatible plans. Given initial utility levels for each expert $\{u_i^0\}_{i \in [0,1]}$, an incentive compatible plan $\mathcal{P}$ is *optimal* if it maximizes households' utility subject to delivering utility u_i^0 to each expert:

$$\max_{\mathcal{P}} V_0(c)$$

$$st: \quad U_{i,0}(e_i) = u_i^0 \quad \forall i \in \mathbb{I}$$

$$\mathcal{P} \in ICP$$

4.1 A recursive formulation of the planner's problem

Just as in the private contract case, we look for an optimal mechanism that is recursive in the continuation utility of experts $\{U_i\}_{i \in \mathbb{I}}$ and the aggregate state variable. Experts' utility follows a law of motion under no stealing $s = 0$

$$dU_{i,t}^{s=0} = -f(e_{i,t}, U_{i,t}^{s=0})dt + \sigma_{U,i,t} dZ_t + \tilde{\sigma}_{U,i,t} dW_{i,t}$$

and we can get an expression for incentive compatibility

$$0 \in \arg\max_{s \geq 0} f(e_{i,t} + \phi \iota'(g_t) k_{i,t} s, U_{i,t}^{s=0}) - \tilde{\sigma}_{U,i,t} \frac{s}{\nu_t} - f(e_{i,t}, U_{i,t}^{s=0}) \tag{12}$$

[28] As in the CE, it is optimal to implement no stealing always.

Taking FOC with respect to s yields

$$\tilde{\sigma}_{U,i,t} \geq f_1(e_{i,t}, U_{i,t}^{s=0})\phi\iota'(g_t)k_{i,t}\nu_t = \frac{e_{i,t}^{-\psi}}{((1-\gamma)U_{i,t}^{s=0})^{\frac{\gamma-\psi}{1-\gamma}}}\phi\iota'(g_t)k_{i,t}\nu_t \geq 0 \tag{13}$$

which will be binding in the optimal plan because experts are risk averse and the planner can aggregate their idiosyncratic risks $\{W_i\}_{i\in\mathbb{I}}$ away.

Introduce $x_{i,t} = ((1-\gamma U_{i,t})^{\frac{1}{1-\gamma}}$ as in the private contract, and write $e_{i,t} = \tilde{e}_{i,t}x_{i,t}$ and $k_{i,t} = \tilde{k}_{i,t}x_{i,t}$, and $\sigma_{U,i,t} = \tilde{\sigma}_{U,i,t}(1-\gamma)U_{i,t}$. We can verify that, just as in the private contract, the planner will choose the same $\tilde{e}_{i,t} = \tilde{e}_t$, $\tilde{k}_{i,t} = \tilde{k}_t$, and $\tilde{\sigma}_{U,i,t} = \tilde{\sigma}_{U,t}$ for all experts. For consumers write $c_{i,t} = \tilde{c}_{i,t}k_t$. The planner's problem must be markov in the same endogenous state variable as the competitive equilibrium

$$X_t = \frac{\int_{\mathbb{I}} x_{i,t}di}{k_t}$$

in addition to the exogenous state variable Y_t. Thanks to homothetic preferences and the linear technology, the planner's value at time t then takes the following power form

$$\frac{(S_t k_t)^{1-\gamma}}{1-\gamma}$$

for some process S_t which depends only on the history of aggregate shocks Z. We look for a value function S and the policy functions $\tilde{e}$, $\tilde{k}$, g, and $\tilde{\sigma}_U$, all functions of (X, Y) (recall the economic environment is also a function of Y_t). We can then re-write the aggregate consistency conditions

$$\tilde{c} + \tilde{e}X = a - \iota(g)$$

$$\tilde{k}X = 1$$

and the law of motion of the endogenous state variable X is

$$\mu_X = \frac{\rho}{1-\psi} - \frac{\tilde{e}^{1-\psi}}{1-\psi} + \frac{\gamma}{2}\tilde{\sigma}_U^2 + \frac{\gamma}{2}(\tilde{e}^{-\psi}\phi\frac{\iota'(g)}{X}\nu)^2 - g - \sigma\tilde{\sigma}_U + \sigma^2 \tag{14}$$

$$\sigma_X = \tilde{\sigma}_U - \sigma \tag{15}$$

Using Ito's lemma we obtain

$$\mu_S = \frac{S_Y}{S}\mu_Y + \frac{S_X}{S}\mu_X X + \frac{1}{2}\frac{S_{YY}}{S}\sigma_Y^2 + \frac{1}{2}\frac{S_{XX}}{S}(\sigma_X X)^2 + \frac{S_{XY}}{S}\sigma_X X\sigma_Y \tag{16}$$

$$\sigma_S = \frac{S_X}{S}\sigma_X X + \frac{S_Y}{S}\sigma_Y \tag{17}$$

The associated HJB equation is

$$\frac{\rho}{1-\psi} = \max_{g,\tilde{e},\tilde{\sigma}_U} \frac{(a-\iota(g)-\tilde{e}X)^{1-\psi}}{1-\psi}S^{\psi-1} + (\mu_S + g - \frac{\gamma}{2}\sigma_S^2 - \frac{\gamma}{2}\sigma^2 + (1-\gamma)\sigma_S\sigma) \tag{18}$$

The (social) cost of providing utility to experts, in terms of forgone utility for households is

$$\Lambda = -S_X > 0$$

with law of motion $d\Lambda_t = \mu_{\Lambda,t}\Lambda_t dt + \sigma_{\Lambda,t}\Lambda_t dZ_t$.[29] Just as its private counterpart in the competitive equilibrium, this will play an important role in the socially optimal allocation.

Consumption. The FOC for $\tilde{e}$ takes the following form (dropping the time subscript)

$$\Lambda \left(\tilde{e}^{-\psi} + \gamma\psi(\phi\iota'(g)\frac{\nu}{X})^2 \tilde{e}^{-2\psi-1} \right) = \left(\frac{\tilde{c}}{S} \right)^{-\psi} \tag{19}$$

On the right hand side we have the cost of giving consumption to experts, in terms of the necessary reduction in households consumption. The benefit is the reduction is promised utility for experts. First, by giving them consumption now. Second, by front-loading consumption the planner can relax the idiosyncratic risk-sharing problem, which is costly because agents are risk averse. The planner is willing to distort the optimal intertemporal allocation of consumption in order to improve idiosyncratic risk sharing, just like the private contracts.

Growth and idiosyncratic risk sharing. The FOC for investment/growth g is

$$\left(\frac{\tilde{c}}{S} \right)^{-\psi} \iota'(g) + \overbrace{\Lambda X \gamma (\tilde{e}^{-\psi}\phi\iota'(g)\frac{\nu}{X})^2 \frac{\iota''(g)}{\iota'(g)}}^{\text{externality}} = S + \Lambda X \tag{20}$$

On the right hand side we have the benefit of increasing the growth rate of the economy: first, we can deliver more utility to households with more capital, and second we reduce the utility promised to experts (per unit of capital), so we save on the cost Λ.

On the left hand side we have the cost of increasing growth: first, we must reduce households' consumption to allocate goods to investment. But in addition, the planner realizes that a higher growth rate will increase the marginal cost of capital, and therefore will require a larger exposure to idiosyncratic risk for experts. The planner internalizes the tradeoff between growth and idiosyncratic risk sharing which private agents in the competitive equilibrium don't. This is where we see the moral hazard externality. To see this, we can obtain a similar equation for the unregulated competitive equilibrium:

$$\left(\frac{\tilde{c}}{S} \right)^{-\psi} \iota'(g) = S + \Lambda X$$

Because of the moral hazard externality, the competitive equilibrium is not efficient.

Proposition 1. *The competitive equilibrium is constrained inefficient, and there is a wedge between*

[29] Recall in the competitive equilibrium, the private relative cost of experts utility in terms of forgone utility for households is $\Lambda^{CE} = \xi\zeta$.

the private and social cost of experts' utility

$$\Gamma = \frac{\Lambda^{CE}}{\Lambda^{SP}} \neq 1$$

The wedge Γ captures the inefficiency associated with the moral hazard externality and depends on the history of aggregate shocks, with law of motion

$$d\Gamma = \mu_\Gamma \Gamma dt + \sigma_\Gamma \Gamma dZ_t$$

Aggregate risk sharing. For experts' exposure to aggregate risk $\tilde{\sigma}_U$ we have the following FOC.

$$\sigma_X - \sigma_S = -\frac{1}{\gamma}\sigma_\Lambda \tag{21}$$

The planner wants to give more utility to experts when the social cost of experts' utility Λ is low. After some algebra, using Ito's lemma, we can obtain an expression for σ_X

$$\sigma_X = \frac{\frac{S_Y}{S} - \frac{1}{\gamma}\frac{\Lambda_Y}{\Lambda}}{1 - (\frac{S_X}{S} - \frac{1}{\gamma}\frac{\Lambda_X}{\Lambda})X}\sigma_Y \tag{22}$$

Notice how equations (21) and (22) parallel equations (9) and (10) for aggregate risk sharing in the competitive equilibrium. The only difference is that the competitive equilibrium uses the private cost of experts' utility $\Lambda^{CE} = \xi\zeta$ while the planner uses the social cost $\Lambda^{SP} = -S_X^{SP}$ (where CE and SP denote the competitive equilibrium and the planner's allocation). Although the externality only directly relates to the tradeoff between growth and idiosyncratic risk sharing, it will also distort the allocation of aggregate risk.

Proposition 2. *The allocation of aggregate risk in the unregulated competitive equilibrium will be inefficient when the wedge Γ between private and social cost of experts' utility is correlated with the aggregate shocks.*

$$\left(\sigma_X^{CE} - \sigma_S^{CE}\right) - \left(\sigma_X^{SP} - \sigma_S^{SP}\right) = -\frac{1}{\gamma}\left(\sigma_\Lambda^{CE} - \sigma_\Lambda^{SP}\right) = -\frac{1}{\gamma}\sigma_\Gamma \tag{23}$$

Experts will lose "too much" utility relative to households, after shocks that make the private cost Λ^{CE} of providing utility to them high relative to the social cost Λ^{SP}.

If the wedge Γ, which reflects the moral hazard externality, is not affected by aggregate shocks, although the unregulated competitive equilibrium may be constrained inefficient the allocation of aggregate risk will be efficient. If instead aggregate shocks affect the wedge Γ, the unregulated competitive equilibrium will overexpose experts' continuation utility to aggregate shocks that reduce the wedge Γ. Although Proposition 2 is useful to understand the efficiency of the equilibrium allocation, sometimes we may be more interested in the allocation of financial risk. Proposition 4 below deals with this.

Remark. The total loss or gain of utility for experts and households in response to an aggregate shock will in general be different in the unregulated competitive equilibrium and the optimal allocation. Equation (23) refers to how this loss or gain is allocated between experts and households. In particular, it may very well be the case that after a "bad" shock, both experts and households lose utility. Equation (23) tells us if the loss is shared correctly.

Solving the planner's problem. The planner's problem boils down to solving a PDE for $S(X, Y)$. We use the FOC for $\tilde{e}$, $\tilde{g}$ and σ_X and plug them into the HJB equation (18) to obtain a second order PDE. Appendix C describes the procedure in more detail.

4.2 Implementation of the optimal allocation

The planner's optimal allocation can be implemented as a competitive equilibrium with a tax on capital holdings $\tau^k = \{\tau_t^k; t \geq 0\}$. If an agent holds $p_t k_{i,t}$ in capital, he must pay a tax flow $\tau_t^k p_t k_t$ to the government. To balance the budget, the government distributes the proceeds from this tax via lump sum transfers to agents. The value of taxes is given by $T_t k_t$ with

$$T_t = \frac{1}{k_t}\mathbb{E}_t\left[\int_t^\infty \frac{\eta_u}{\eta_t}\tau_u^k p_u k_u du\right]$$

Remarkably, although the competitive equilibrium may feature an excessive concentration of aggregate risk on experts, it is not necessary to directly regulate the allocation of aggregate risk. Aggregate risk sharing will be inefficient in the unregulated competitive equilibrium if the wedge Γ between the social and private cost of experts' utility is correlated with aggregate shocks. But once taxes on capital internalize the moral hazard externality and align the private and social costs of providing utility to experts ($\Lambda^{CE} = \Lambda^{SP}$), aggregate risk sharing becomes optimal without the need for further regulation. The same reasoning applies to consumption and growth.

Consider an optimal plan $\mathcal{P}$ and the associated value function S and policy functions $\tilde{e}$, g, and the law of motion of the endogenous state, μ_X and σ_X, functions of (X, Y). We can build a Markov equilibrium using the same law of motion μ_X and σ_X, as well as the policy functions g and $\hat{e} = \tilde{e}$. From the FOC for growth we get

$$p = \iota'(g) \tag{24}$$

And we want to build the competitive equilibrium so that the private cost of experts' utility is aligned with the social cost[30]

$$\xi\zeta = -S_X$$

We build ξ and ζ as follows. Households' wealth per unit of capital is $p + T - \xi X > 0$, and their utility $S > 0$, so we set

$$S = \zeta(p + T - \xi X)$$

[30]This will only work if $S_X < 0$, i.e. if it is costly for households to provide utility to experts.

In addition, from the FOC for $\hat{c}$ we obtain

$$\left(\frac{\tilde{c}}{p+T-\xi X}\right)^{-\psi} = \zeta^{1-\psi}$$

The FOC for $\hat{e}$ will then be automatically satisfied. Putting these equations together we obtain

$$\zeta = \left(\frac{\tilde{c}}{S}\right)^{-\psi} > 0, \quad \xi = -\frac{S_X}{\zeta} > 0, \quad T = \frac{S}{\zeta} - p + \xi X \tag{25}$$

We can use the representative household's HJB equation to pin down r, and from his FOC for σ_w we pin down π (experts' FOC for $\hat{\sigma}_U$ will be automatically satisfied).[31] The optimal tax on capital τ^k that implements the optimal allocation as a competitive equilibrium is then obtained from the FOC for capital.

$$\frac{a-\iota(g)}{p} + \mu_p + g + \sigma'\sigma_p - (r+\tau^k) = \pi(\sigma+\sigma_p) + \gamma\xi(\hat{e}^{-\psi}\phi\nu)^2\frac{p}{X} \tag{26}$$

Appendix C shows the procedure in detail and proves the following result:

Proposition 3. *Given the planner's optimal allocation with value function S satisfying $S_X < 0$, and associated policy functions $\tilde{e}$, g and $\hat{\sigma}_U$, we can implement it as a markov competitive equilibrium with a tax on capital τ^k as in (26).*

Remark. Although the allocation of aggregate risk in the competitive equilibrium is inefficient, the optimal allocation can be implemented without directly regulating agents' aggregate risk sharing. This may not be the only way in which we can implement the optimal allocation, but it has the benefit that it highlights the source of the inefficiency and deals with it directly. In addition, taxes on capital, or capital income, are relatively easy to implement. On the other hand, the optimal tax τ^k is contingent on aggregate shocks, which may limit its feasibility. A complementary approach would study what is the best we can do with an ad-hoc simple instrument, and use the planner's optimal allocation as a benchmark for welfare, e.g. check if a simple instrument achieves most of the welfare gains from the optimal allocation.

An expression for the size of the externality. We can use the planner's FOC for g to obtain a useful expression for the wedge. The market value of capital in the optimal allocation would be $p+T$ in the absence of taxes τ^k. This is also the shadow value of an extra unit of capital, in terms of consumption goods, for the planner: $\partial_k Sk = \zeta(p+T)$, where $\zeta = \left(\frac{\tilde{c}}{S}\right)^{-\psi}$ captures the marginal value of an extra unit of consumption for households. It would seem that investment should be pinned down by $\iota'(g) = p+T$. That is, devote consumption goods to create new capital up to the point where the cost in consumption goods equals the shadow value, also in consumption goods. However, this is the shadow value of a unit of capital that falls from the sky. The planner knows

[31] Using the envelope theorem for the planner's HJB, we can show that experts' HJB is satisfied, and therefore also the pricing equation for taxes. See Appendix C.

that in order to create more capital using his investment technology, he must distort idiosyncratic risk sharing, so in the implementation of the optimal allocation we have

$$\iota'(g) = p = (p + T)(1 - \Omega)$$

The wedge Ω captures the externality as the difference between the shadow value of an extra unit of capital, and the value of creating an extra unit of capital through investment. From the FOC for g (20) we get an expression for this wedge

$$\Omega = \frac{T}{p+T} = \gamma \frac{\xi X}{p+T} \overbrace{(\tilde{e}^{-\psi}\phi\iota'(g)\frac{\nu}{X})^2}^{\tilde{\sigma}^2_{x,i}} \frac{\iota''(g)}{\iota'(g)}$$

The wedge Ω is larger when experts' exposure to idiosyncratic risk, $\tilde{\sigma}_{x,i}$ is higher, and when the investment function is very curved. If if was linear $\iota''(g) = 0$, the marginal cost of investment would be fixed and there wouldn't be any point is distorting the growth rate of the economy. Likewise, if experts' exposure to idiosyncratic risk is very small, the cost of the externality is very small, so it doesn't make sense to distort investment in order to improve idiosyncratic risk sharing.

Concentration of financial risk in the regulated competitive equilibrium. We can use expression (11) to obtain an expression for the concentration of aggregate financial risk $\sigma_\omega = \sigma_n - \sigma_w$. Use SP and CE to distinguish the optimal allocation from the unregulated competitive equilibrium. We have from equation (11)

$$\sigma_\omega^{SP} = \frac{\gamma - 1}{\gamma}\sigma_\Lambda^{SP}$$

$$\sigma_\omega^{CE} = \frac{\gamma - 1}{\gamma}\sigma_\Lambda^{CE}$$

We can then obtain the following equation comparing the concentration of financial risk in the unregulated competitive equilibrium and the implementation of the optimal allocation

Proposition 4. *The allocation of aggregate financial risk in the unregulated competitive equilibrium will be inefficient when the wedge between private and social cost of experts' utility* $\Gamma = \frac{\Lambda^{CE}}{\Lambda^{SP}}$ *is correlated with the aggregate shock:*

$$\sigma_\omega^{CE} - \sigma_\omega^{SP} = \frac{\gamma - 1}{\gamma}\left(\sigma_\Lambda^{CE} - \sigma_\Lambda^{SP}\right) = \frac{\gamma - 1}{\gamma}\sigma_\Gamma \tag{27}$$

When the income effect dominates ($\gamma > 1$*), experts will lose "too much" net worth after a shock that makes the private cost of providing them utility* Λ^{CE} *lower relative to the social cost* Λ^{SP}.

Remark. As in the case of utility in equation (23), the total loss or gain of aggregate wealth for experts and households in response to an aggregate shock will in general be different in the unregulated competitive equilibrium and the optimal allocation. Equation (27) refers to how this loss or gain is allocated between experts and households.

5 Two numerical examples: TFP and uncertainty shocks

In this section I provide numerical solutions for the competitive equilibrium and the social planner's allocation for two different setups that illustrate the general results. First I consider an economy driven purely by TFP shocks. In this case there is no concentration of aggregate risk in the unregulated competitive equilibrium. As a result, this case holds less interest as a positive theory of financial crises, but is very useful as a benchmark. The competitive equilibrium is inefficient because of the moral hazard externality: the private and social cost of experts' utility are not aligned. The regulated economy has lower asset prices and growth, in order to reduce the cost of providing incentives to experts. However, the wedge between the private and social cost of experts' utility is not correlated with TFP shocks, and therefore the allocation of aggregate risk is efficient.

I then consider an economy hit by uncertainty shocks that increase idiosyncratic risk ν_t, as in Di Tella (2013). Just as in in the TFP case, the unregulated competitive equilibrium is inefficient because of the moral hazard externality, and the private and social cost of experts' utility are not aligned. In contrast to the TFP case, the unregulated competitive equilibrium concentrates financial risk on experts. A "bad" uncertainty shock that increases idiosyncratic risk ν_t drives excess returns up, and this reduces the private cost of experts' utility. Because the income effect dominates (recall we assume throughout that $\gamma > 1$), privately optimal contracts concentrate financial loses on experts (they need less net worth to achieve a given utility level).

This concentration of aggregate risk is inefficient, however, because uncertainty shocks are correlated with the wedge Γ between private and social cost of experts' utility. Uncertainty shocks reduce the private cost Λ^{CE} relative to the social cost Λ^{SP} (excess returns on assets go up too much during periods of high idiosyncratic risk), so in the unregulated equilibrium aggregate financial risk is too concentrated on experts: after a bad uncertainty shock, experts lose too much net worth. Utility loses, however, are too concentrated on households (experts face proportionally larger financial loses, but they make up for it with high excess returns looking forward).

5.1 Brownian TFP Shocks

In this section I solve for a Markov equilibrium in an economy driven only by TFP shocks. There are no shocks Y to the economic environment ($\mu_Y = \sigma_Y = 0$) which is fixed: $\nu_t = \nu$ and $\sigma_t = \sigma$.

Parametrization: I use the following parametrization for this numerical example. *Preferences*: $\gamma = 5$, $\psi = \frac{1}{3}$, $\rho = 0.1$; *technology*: $a = 1$, $\sigma = 0.03$, $\nu = 0.25$, $\iota(g) = 200g^2$; *moral hazard*: $\phi = 0.2$.

Competitive equilibrium vs. Social planner. Figure 1 shows the welfare of households S (per unit of capital) for a given promised utility for experts. For households, in the competitive equilibrium we have $S^{CE} = \zeta^{CE}(p^{CE} - \xi^{CE}X)$, while in the social planners solution we simply have S^{SP}. The curve for the social planner can be interpreted as the constrained Pareto frontier because it is decreasing throughout. The unregulated competitive equilibrium is below the frontier,

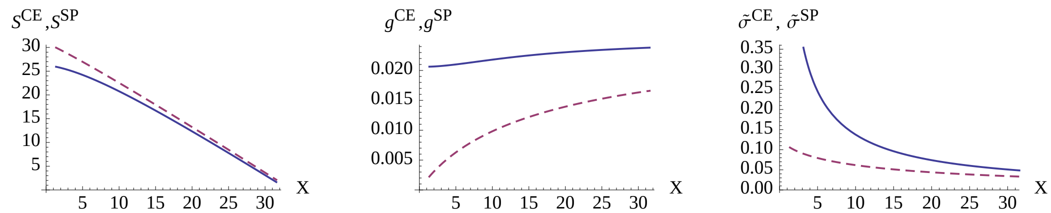

Figure 1: Welfare, growth, and exposure to idiosyncratic risk as a function of X, under the CE (solid) and the SP (dashed)

as expected since it is inefficient. A social planner can deliver the same continuation utility to experts X while providing more utility to households. This is achieved by reducing the growth rate in order to reduce the cost of providing incentives to experts, as Figure 1 shows.

The intuition is as follows: with high growth, experts' hidden action has a high private benefit (asset prices are high, so diverting capital is very profitable). They must therefore receive very high powered incentives that expose them to high idiosyncratic risk, proportionally to their utility. Because they are risk averse, this is costly: to deliver the promised utility to the experts, the planner must give them a lot of consumption, either now or in the future; this comes at the cost of reduced consumption for households. So the planner prefers to reduce the growth rate of the economy and the idiosyncratic risk of experts as shown in Figure 1 ($\tilde{\sigma}_x = \frac{idvol(x_i)}{x_i} = \hat{e}^{-\psi}\phi\iota'(g)\frac{\nu}{X}$). In other words, the cost of high growth is that experts appropriate large rents to compensate for their large exposure to risk, which is necessary to sustain high growth: households are better off with a slower growing economy where experts get less rewards.[32]

In both the competitive equilibrium and the social planner's allocation, growth increases with X. When experts' utility is low, exposing them to idiosyncratic volatility is relatively more costly. Capital is therefore less attractive in the market, so its price falls, along with investment. The social planner faces the same logic but reacts more strongly, as Figure 1 shows. Growth falls much more as X falls in the social planner's solution, and the gap between growth in the competitive equilibrium and the social planner's solution widens when X is low. The intuition is that when X is low, a small increase in the price of capital has a bigger impact on experts' proportional exposure to idiosyncratic risk necessary to provide incentives to experts. The moral hazard externality is therefore larger, so the planner wants to reduce asset prices and growth to a larger extent. This can be seen in Figure 2. The private cost of experts' utility Λ^{CE} is always smaller than the social cost Λ^{SP}, and the wedge Γ is larger for small X. The tax τ^k that implements the optimal allocation tells the same story.

Figure 2 also shows the law of motion of the endogenous state variable X. In both the competitive equilibrium and the optimal allocation, X has a positive drift, and in the long-run experts have

[32]It may seem surprising that the social planner wants to reduce growth in the economy, since the first best level of growth is higher than in equilibrium. This makes sense, however, because the social planner also faces the same informational frictions. The first best is unattainable.

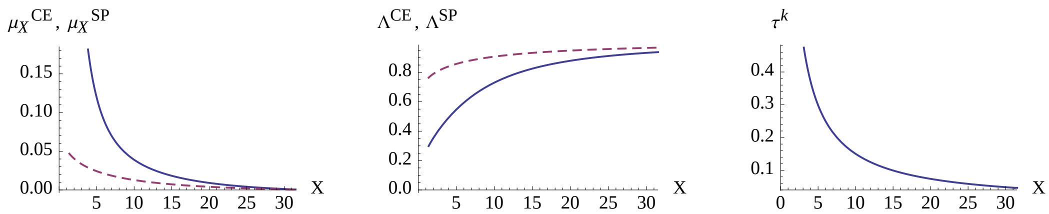

Figure 2: Drift of the endogenous state variable μ_X, the cost of experts' utility Λ, and the tax on capital τ, as a function of X, under the CE (solid) and the SP (dashed)

all the utility in the economy. This is a standard result in models of financial frictions.[33] In the unregulated competitive equilibrium experts' utility X grows faster, because they are exposed to higher idiosyncratic risk.

In both the unregulated competitive equilibrium and the optimal allocation aggregate risk is shared proportionally between experts and households, both in utility and financial terms. TFP shocks don't affect the private relative cost of experts' utility Λ^{CE}. As a result, from equation (9)

$$\sigma_X^{CE} - \sigma_S^{CE} = -\frac{1}{\gamma}\overbrace{\sigma_\Lambda^{CE}}^{=0} = 0$$

In fact, since $\sigma_Y = 0$, from equation (10) we know $\sigma_X^{CE} = 0$, which yields $\sigma_\Lambda^{CE} = \frac{\Lambda_X^{CE}}{\Lambda^{CE}}\sigma_X^{CE} X = 0$. Aggregate TFP shocks affect only the level of capital, but because they are not concentrated on experts, they don't affect growth g^{CE}, asset prices p^{CE}, interest rates r^{CE}, or the price of risk π^{CE}. Furthermore, this risk sharing is efficient since the ratio of private to social relative cost of experts' utility is not affected by TFP shocks: $\sigma_X^{SP} = 0$ from equation (22), and therefore $\sigma_\Lambda^{SP} = \frac{\Lambda_X^{SP}}{\Lambda^{SP}}\sigma_X^{SP} X = 0$. As a result, TFP shocks don't affect the wedge Γ between private and social cost of experts' utility. From equation (23) we obtain

$$\left(\sigma_X^{CE} - \sigma_S^{CE}\right) - \left(\sigma_X^{SP} - \sigma_S^{SP}\right) = -\frac{1}{\gamma}\overbrace{\sigma_\Gamma}^{=0} = 0$$

Although the unregulated competitive equilibrium is constrained inefficient due to the moral hazard externality, aggregate risk sharing is efficient. The same analysis can be done in terms of the allocation of financial risk. From equation (11) we get the concentration of financial risk in the unregulated competitive equilibrium:

$$\sigma_\omega^{CE} = \frac{\gamma - 1}{\gamma}\overbrace{\sigma_\Lambda^{CE}}^{=0} = 0$$

[33] This can be avoided by introducing retirement among experts, as in Di Tella (2013), or giving them higher discount rates, as in Brunnermeier and Sannikov (2012), for example. Here I present the "bare bones" model.

and from equation (27) we learn that this allocation of aggregate financial risk is efficient:

$$\sigma_\omega^{CE} - \sigma_\omega^{SP} = \frac{\gamma - 1}{\gamma} \overbrace{\sigma_\Gamma}^{=0} = 0$$

5.2 Uncertainty Shocks

Now consider an economy hit by uncertainty shocks. The idiosyncratic risk of capital, $\nu_t = Y_t$, now follows an autoregressive stochastic process[34]

$$\underbrace{d\nu_t}_{dY_t} = \underbrace{\beta(\bar{\nu} - \nu_t)}_{\mu_Y(Y_t)} dt + \underbrace{\sqrt{\nu_t}\sigma_\nu}_{\sigma_Y(Y_t)} dZ_t$$

By convention, we will take $\sigma_\nu < 0$, so that we may think of Z as a "good" shock that drives idiosyncratic risk ν_t down. The aggregate risk of capital is still fixed σ. For simplicity we consider only one aggregate shock (Z is unidimensional). If $\sigma > 0$ the good shock Z also improves TFP, but there is no loss in intuition from setting $\sigma = 0$.

Parametrization: I use the same parametrization as for the TFP case and add a stochastic process for ν_t. *Preferences*: $\gamma = 5$, $\psi = \frac{1}{3}$, $\rho = 0.1$; *technology*: $a = 1$, $\sigma = 0.03$, $\iota(g) = 200g^2$; *moral hazard*: $\phi = 0.2$; *uncertainty shock*: $\beta = 0.2$, $\bar{\nu} = 0.24$, $\sigma_\nu = -0.13$.

Competitive Equilibrium vs. Social Planner. Figure 3 shows households' welfare S as a function of experts utility X, for the competitive equilibrium and the social planner's allocation. The social planner can always deliver more utility to households, for a given promised utility to experts. The gap is larger when X is low and ν is large. The social planner reduces growth and asset prices, $p = \iota'(g)$, with respect to the unregulated competitive equilibrium in order to reduce the cost of providing incentives to experts. With lower asset prices the private benefit of experts' hidden action is lower, so idiosyncratic risk sharing is improved, as shown also in Figure 3. The planner does this to a larger extent when X is low and ν is large, when the moral hazard externality is large. A marginal increase in the price of capital increases the required exposure of experts to idiosyncratic risk more when X is low and ν is large, and therefore has a larger impact on the cost of providing incentives to experts.

In contrast to the TFP case, the unregulated competitive equilibrium concentrates aggregate financial risk on experts. Figure 4 shows the volatility of the wealth ratio $\omega = \frac{N}{W}$. In the competitive equilibrium after a bad uncertainty shock raises idiosyncratic risk ν, loses are concentrated on experts, so ω falls. We can understand this looking at the private cost of providing utility to experts Λ^{CE}:

$$\sigma_\omega^{CE} = \frac{\gamma - 1}{\gamma} \sigma_\Lambda^{CE}$$

[34] This is a CIR process. If $2\beta\bar{\nu} \geq \sigma_\nu^2$, then $\nu_t > 0$ always, and it has a long-run distribution with mean $\bar{\nu}$.

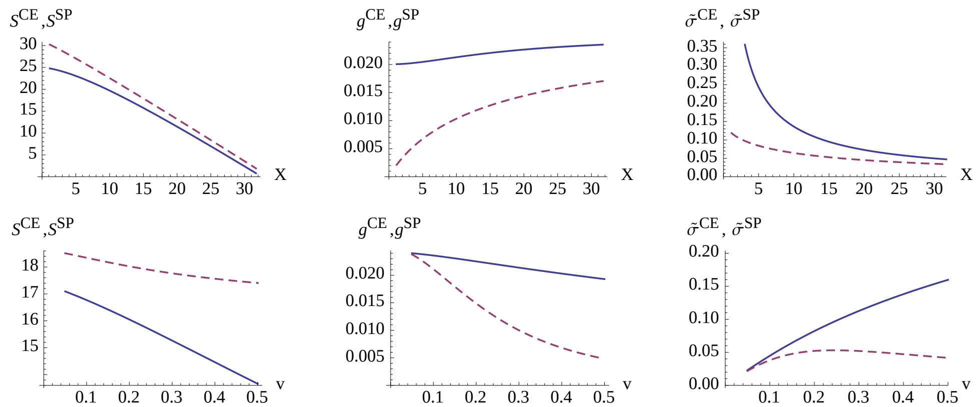

Figure 3: Experts and households welfare, the growth rate g, and experts' idiosyncratic risk $\tilde{\sigma}_U$, in the CE (solid) and SP (dashed), as functions of X for a fixed $\nu_0 = \bar{\nu} = 0.24$.(above) and as a function of ν for a fixed $X_0 = 15$ (below).

When the income effect dominates ($\gamma > 1$) optimal contracts devote less resources to experts when it is cheaper to provide them utility (they need less in order to achieve any given level of utility). Figure 4 shows Λ^{CE} is lower when X is low and ν is high. This is because capital pays a high premium for idiosyncratic risk to convince experts to hold it (the price of this idiosyncratic risk is proportional to $\tilde{\sigma}$ from Figure 3). Since they are compensated at the margin (for the last "unit" of idiosyncratic risk), the surplus from managing capital is larger, and therefore the private cost of experts' utility Λ^{CE} is lower. After a bad uncertainty shock ν rises and X falls (because $\sigma_X > 0$ as we will see below[35]), and therefore it becomes cheaper to provide utility to experts:

$$\sigma_\Lambda^{CE} = \overbrace{\frac{\Lambda_X^{CE}}{\Lambda^{CE}}}^{>0} \overbrace{\sigma_X^{CE} X}^{>0} + \overbrace{\frac{\Lambda_\nu^{CE}}{\Lambda^{CE}}}^{<0} \overbrace{\sqrt{\nu}\sigma_\nu}^{<0} > 0 \quad \Longrightarrow \quad \sigma_\omega^{CE} > 0$$

The resulting concentration of aggregate financial risk on experts is inefficient, however. Figure 4 shows the regulated competitive equilibrium is very close to allocating aggregate financial risk proportionally, and sometimes slightly concentrates financial risk on households. The social cost of providing utility to experts Λ^{SP} is less sensitive to increases in ν and reductions in X, and therefore less sensitive to uncertainty shocks. We can see this better looking at the wedge Γ between private and social cost of experts' utility in Figure 4. It is always below one: the social cost of experts' utility is always higher that the private cost because the private cost doesn't take into account the moral hazard externality. In addition, the wedge Γ is increasing in X and decreasing in ν: when idiosyncratic risk is very high relative to experts' continuation utility, the impact of a small increase in the price of capital on the moral hazard problem each expert faces is large (the externality is large, so the wedge is big). After a bad uncertainty shock X is low ν high, so the private cost of

[35] Recall that a bad uncertainty shock is a negative $dZ < 0$.

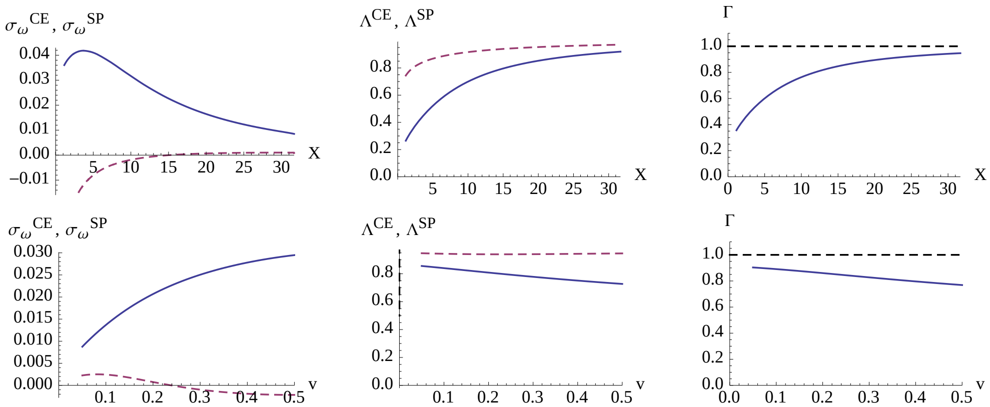

Figure 4: The volatility of the wealth ratio σ_ω, the marginal cost of experts' utility Λ (private and social) and the wedge Γ in the CE (solid) and SP (dashed), as functions of X for a fixed $\nu_0 = \bar{\nu} = 0.24$.(above) and as a function of ν for a fixed $X_0 = 15$ (below).

experts' utility is particularly low relative to the social cost

$$\sigma_\Gamma = \overbrace{\frac{\Gamma_X}{\Gamma}}^{>0} \overbrace{\sigma_X X}^{>0} + \overbrace{\frac{\Gamma_\nu}{\Gamma}}^{<0} \overbrace{\sigma_\nu \sqrt{\nu}}^{<0} > 0$$

The unregulated competitive equilibrium therefore has an excessive concentration of aggregate financial risk on experts. From equation (27)

$$\sigma_\omega^{CE} - \sigma_\omega^{SP} = \frac{\gamma - 1}{\gamma} \overbrace{\sigma_\Gamma}^{>0} > 0$$

Even though aggregate financial risk is too concentrated on experts, aggregate risk in utility terms is too concentrated on households. Figure 5 shows the law of motion of the endogenous state variable X. $\sigma_X^{CE} > \sigma_X^{SP} > 0$ so after a bad uncertainty shock experts lose continuation utility both in the unregulated and the regulated economy. The volatility of experts' utility in the regulated economy is smaller because the planner intervenes and dampens the impact of shock for both experts and households. In fact, in the unregulated competitive equilibrium the loss of utility is concentrated on households. This can be seen by looking at the volatility of the ratio X/S, in Figure 5. After a bad uncertainty shock, experts' utility falls proportionally less than households'. This is because although they lose more net worth (proportionally), they at least get very high excess returns, so the private cost of providing them utility falls, as we saw before ($\sigma_\Lambda^{CE} > 0$). Privately optimal contracts therefore give experts relatively more utility (compared to households) after an uncertainty shock:

$$\sigma_X^{CE} - \sigma_S^{CE} = -\frac{1}{\gamma} \overbrace{\sigma_\Lambda^{CE}}^{>0} < 0$$

This allocation of aggregate risk in utility terms is inefficient: even though experts' utility in the

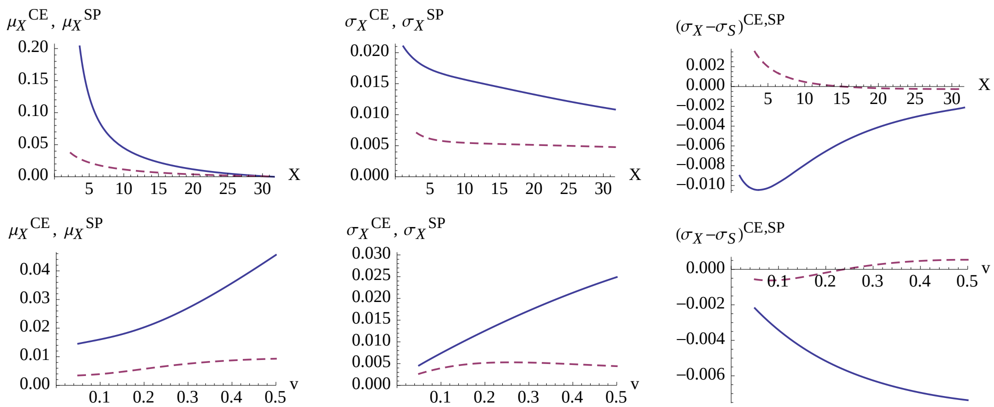

Figure 5: Drift μ_X and volatility σ_X of the endogenous state variable X, and the volatility of the utility ratio X/S, in the CE (solid) and SP (dashed), as functions of X for a fixed $\nu_0 = \bar{\nu} = 0.24$.(above) and as a function of ν for a fixed $X_0 = 15$ (below).

optimal allocation falls less after a bad uncertainty shock, their share of the total loss is higher. The distribution of aggregate risk in utility terms is closer to proportional, and sometimes concentrates this risk on experts. The inefficiency comes from the correlation of the wedge Γ with the uncertainty shock. From equation (23):

$$\overbrace{\left(\sigma_X^{CE} - \sigma_S^{CE}\right)}^{<0} - \overbrace{\left(\sigma_X^{SP} - \sigma_S^{SP}\right)}^{\approx 0} = -\frac{1}{\gamma} \overbrace{\sigma_\Gamma}^{>0} < 0$$

A bad uncertainty shock makes the private cost lower relative to the social cost, so utility loses are excessively concentrated on households in the unregulated competitive equilibrium. Figure 5 also shows the drift of X. Since the planner improves idiosyncratic risk sharing, experts continuation utility rises slower in the optimal allocation.

State-contingent taxes. Figure 6 shows the tax on capital holdings that implements the optimal allocation as a competitive equilibrium. As expected, the tax is larger when X is low and ν high, reflecting the larger externality. This may seem counterintuitive if we think that low asset prices and growth are part of the problem after bad shocks, and this is true when comparing the equilibrium with the first best without moral hazard. But the social planner cannot get rid of moral hazard, it can only deal with the resulting externality, which is larger after a bad uncertainty shock (with low X and high ν). If we could prevent the increase in idiosyncratic risk, we may well want to do so, but once the shock occurs it is optimal to also reduce asset prices and growth to reduce the cost of providing incentives.

A related puzzle is that asset values are more volatile in the regulated equilibrium, i.e. $\sigma + \sigma_p$ is larger as Figure 6 shows. Notice, however, that in the regulated equilibrium the total wealth is not only capital, but also the present value of government transfers, and total wealth $(p + T)k$ is less volatile in the regulated equilibrium.

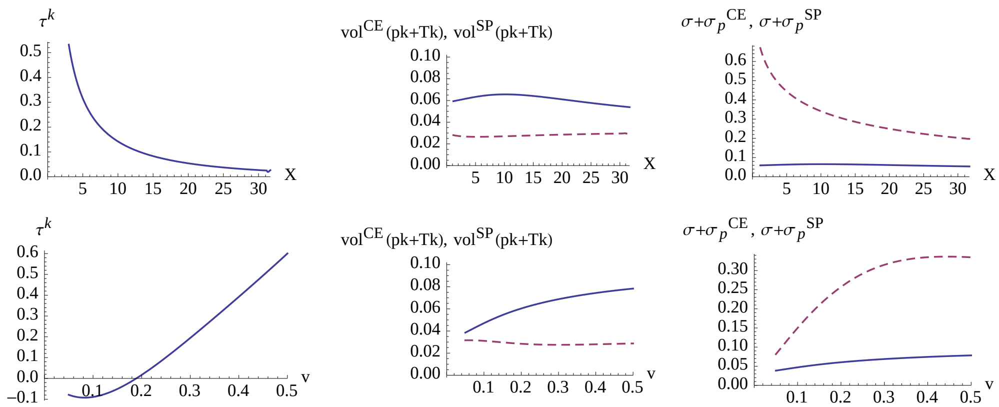

Figure 6: Optimal tax on capital τ^k, the volatility of aggregate wealth $vol(pk + Tk)$ and the volatility of the value of capital $\sigma + \sigma_p$, in the CE (solid) and SP (dashed), as functions of X for a fixed $\nu_0 = \bar{\nu} = 0.24$.(above) and as a function of ν for a fixed $X_0 = 15$ (below).

6 Conclusion

This paper studies the optimal financial regulation policy in a setting where financial frictions are derived from a moral hazard problem, focusing on the allocation of aggregate risk. Even if agents can write complete long-term contracts, the competitive environment is constrained inefficient. Agents don't internalize that by competing for capital and bidding up its price they create a moral hazard problem for everyone else. A social planner who faces the same informational asymmetries will internalize the resulting tradeoff between growth and idiosyncratic risk sharing. As a result, there is a wedge between the private and social costs of providing utility to experts. Even though the moral hazard problem does not prevent agents from sharing aggregate risk, and even though the externality does not involve aggregate risk sharing directly, it can create incentives for an excessive concentration of aggregate risk. The equilibrium allocation of aggregate risk will be inefficient when aggregate shocks are correlated with the wedge. However, even though the competitive equilibrium may face an excessive concentration of aggregate risk, the socially optimal allocation can be implemented as a competitive equilibrium with a tax on capital, without the need to directly regulate aggregate risk sharing.

I then study two specific settings that illustrate the general results. In the first, the economy is hit only by TFP shocks. Asset prices and growth are too high in the unregulated competitive equilibrium, but the allocation of aggregate risk is efficient: experts and households share it proportionally, both in utility and financial terms. In contrast, in the second setting the economy is hit by uncertainty shocks that raise idiosyncratic risk. Now the competitive equilibrium concentrates aggregate financial risk on experts, while concentrating aggregate utility risk on households. Furthermore, this allocation of aggregate risk is inefficient, because uncertainty shocks are correlated with the wedge between the private and social cost of experts' utility. After a bad uncertainty shock, financial loses are too concentrated on experts, and utility loses too concentrated on households.

There are three broad avenues for future research. First, this paper uses a simple and tractable contractual setting. Contracts with full long-term commitment and no hidden savings make clear the comparison with the socially optimal allocation, and are a natural starting point to study optimal financial regulation. The case with hidden savings or with limited commitment seem like natural next steps that can provide new insights and make the setting more realistic. Second, the general results show that the type of aggregate shock can play an important role explaining not only the concentration of risk in the competitive equilibrium, but also its efficiency. In particular, monetary shocks play an important role in business cycle theory, as well as in theories of financial crises.[36] Finally, an advantage of the approach to optimal policy taken here is that we are not pre-committing to a set of policy instruments, but rather letting the economic environment tells us what are the tools appropriate for the job. In some circumstances, however, we might be more interested in the optimal use of a given policy instrument. Both approaches can yield valuable insights, especially when combined, since the socially optimal allocation studied here provides a natural benchmark to evaluate different policy instruments.

References

Bernanke, B., M. Gertler, and S. Gilchrist. 1999. "The Financial Accelerator in a Quantitative Business Cycle Framework." In *Handbook of Macroeconomics*, edited by J.B. Taylor and M. Woodford, Vol. 1, chap. 21, 1341–1393. Elsevier.

Brunnermeier, M. and Y. Sannikov. 2012. "A Macroeconomic Model with a Financial Sector." Working paper, Princeton University.

Davila, J., J. Rios-Rull, J. Hong, and P. Krusell. 2012. "Constrained efficiency in the neoclassical growth model with uninsurable idiosyncratic shocks." *Econometrica* 80(6): 2431–2467.

DeMarzo, P., Z. He, M. Fishman, and N. Wang. 2012. "Dynamic Agency Theory meets the Q-Theory of Investment." Forthcoming.

DeMarzo, P. and Y. Sannikov. 2006. "Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model." *The Journal of Finance* 61 (6): pp. 2681–2724.

Di Tella, S. 2013. "Uncertainty Shocks and Balance Sheet Recessions." Stanford GSB working paper.

Di Tella, S. and Y. Sannikov. 2014. "Optimal incentive scheme for delegated asset management." Work in progress, Stanford GSB and Princeton University.

Duffie, D. and L. Epstein. 1992. "Stochastic Differential Utility." *Econometrica* 60 (2): pp. 353–394.

Fisher, I. 1933. "The Debt-Deflation Theory of of Great Depressions." *Econometrica* 1 (7): 337–357.

[36] Going all the way back to Fisher's debt-deflation theory Fisher (1933), for example.

He, Z. and A. Krishnamurthy. 2011. "A Model of Capital and Crises." Forthcoming, Review of Economic Studies.

Karatzas, I. and S. Shreve. 1998. *Methods of Mathematical Finance.* Springer.

Kiyotaki, N. and J. Moore. 1997. "Credit Cycles." *Journal of Political Economy* 105: 211–248.

Kraft, Holger, Frank Seifried, and Mogens Steffensen. 2011. "Consumption-Portfolio Optimization with Recursive Utility in Incomplete Markets."

Mirrlees, J.A. 1971. "An Exploration in the Theory of Optimum Income Taxation." *Review of Economic Studies* 38: 175–208.

Sannikov, Y. 2008. "A Continuous- Time Version of the Principal: Agent Problem." *The Review of Economic Studies* 75 (3): pp. 957–984.

Sun, Y. and Y. Zhang. 2009. "Individual risk and Lebesgue extension without aggregate uncertainty." *Journal of Economic Theory* (144): 432–443.

7 Appendix

7.1 Appendix A: Omitted Proofs

Proof of Lemma 1. Consider the process $Y = \{Y_t; 0 < t < T\}$ for some arbitrary T

$$Y_t = \mathbb{E}_t^{s=0}\left[\int_0^T f(e_u, U_u^{s=0})du + U_T^{s=0}\right] = \int_0^t f(e_u, U_u^{s=0})du + U_t^{s=0}$$

It is a P-square integrable martingale adapted to $\mathbb{F}^i$, so we can write

$$dY_t = f(e_t, U_t^{s=0})dt + dU_t^{s=0} = \sigma_{U,t}dZ_t + \tilde{\sigma}_{U,t}dW_t$$

$$dU_t^{s=0} = -f(e_t, U_t^{s=0})dt + \sigma_{U,t}dZ_t + \tilde{\sigma}_{U,t}dW_t$$

for some $\mathbb{F}^i$-adapted processes $\sigma_U = \{\sigma_{U,t}; t \geq 0\}$ and $\tilde{\sigma}_U = \{\tilde{\sigma}_{U,t}; t \geq 0\}$ in $\mathcal{L}$. The transversality condition comes from taking limits as $T \to \infty$.

Proof of Lemma 2. Given contract $\mathcal{C} = (e, k)$, the utility the expert obtains from following stealing plan s is U^s defined recursively so that for any $t < T$ it holds that

$$U_t^s = \mathbb{E}_t^s\left[\int_t^T f(e_u + \phi p_u k_u s_u, U_u^s)du + U_T^s\right]$$

Meanwhile, the utility of not stealing is $U^{s=0}$ and satisfies (3). We can then write

$$dU_t^{s=0} = \left[-f(e_t, U_t^{s=0}) - \frac{s_t}{\nu_t}\tilde{\sigma}_{U,t}\right] dt + \sigma_{U,t} dZ_t + \tilde{\sigma}_{U,t} dW_t^s$$

and integrating and taking expectation under P^s we obtain for any $t < T$

$$U_t^{s=0} = \mathbb{E}_t^s \left[\int_t^T f(e_u, U_u^{s=0}) + \frac{s_t}{\nu_t}\tilde{\sigma}_{U,t} du + U_T^{s=0}\right]$$

Now suppose the IC condition (4) fails, and pick a bounded stealing strategy s such that $f(e_t + \phi p_t k_t s, U_t^{s=0}) - \tilde{\sigma}_{U,t} s - f(e_t, U_t^{s=0}) > 0$ on a set A of positive measure in $[0, T] \times \Omega$, for some large enough T, and zero outside of A. After this T, the expert doesn't steal anymore, so $U_{T+u}^s = U_{T+u}^{s=0}$, $u \geq 0$. We can compare the utility from this strategy with the utility from not stealing for $t < T$. We will prove that $U_0^s > U_0^{s=0}$ which contradicts incentive compatibility. To this end, write:

$$U_t^s - U_t^{s=0} = \mathbb{E}_t^s \left[\int_t^T \left[f(e_u + \phi p_u k_u s_u, U_u^s) - f(e_u, U_u^{s=0}) - \frac{s_u}{\nu_u}\tilde{\sigma}_{U,u}\right] du\right]$$

Now look at the integrand, and obtain the following inequality

$$\begin{aligned}
f(e_u + \phi p_u k_u s_u, U_u^s) - f(e_u, U_u^{s=0}) - \tilde{\sigma}_{U,u}\frac{s_u}{\nu_u} &= f(e_u + \phi p_u k_u s_u, U_u^s) - f(e_u + \phi p_u k_u s_u, U_u^{s=0}) \\
&\quad + f(e_u + \phi p_u k_u s_u, U_u^{s=0}) - f(e_u, U_u^{s=0}) - \tilde{\sigma}_{U,u}\frac{s_u}{\nu_u} \\
&\geq f(e_u + \phi p_u k_u s_u, U_u^s) - f(e_u + \phi p_u k_u s_u, U_u^{s=0})
\end{aligned}$$

with strict inequality on A. Now we use an interesting fact about the EZ aggregator $f(c, U)$: if $\gamma > 1$ and $\psi < 1$, then there is a constant $\lambda > 0$ such that $f(c, y) - f(c, x) \leq \lambda(y - x)$ for $y \geq x$, and any c.[37] We can then write

$$\underbrace{f(e_u + \phi p_u k_u s_u, U_u^s) - f(e_u, U_u^{s=0}) - \tilde{\sigma}_{U,u}\frac{s_u}{\nu_u}}_{H_u} \geq \lambda \underbrace{(U_u^s - U_u^{s=0})}_{M_u} \quad \text{when } \underbrace{U_u^s - U_u^{s=0}}_{M_u} \leq 0$$

and the inequality is strict on A. Now define the process $M_t = U_t^s - U_t^{s=0}$ and write

$$M_t = U_t^s - U_t^{s=0} = \mathbb{E}_t^s \left[\int_t^T H_u du\right] \quad \text{with } H_t \geq \lambda M_t \text{ whenever } M_t \leq 0$$

We can now use a generalized version of Skiadas' Lemma[38] to obtain that $M_t = U_t^s - U_t^{s=0} \geq 0$ for all $t \in [0, T]$, a.s. as follows. Let $\tau = \inf\{t : U_t^{s=0} \leq U_t^s\}$ and write

$$M_t \mathbb{1}_{\{\tau > t\}} \geq \mathbb{E}_t^s \left[\int_t^{\tau \wedge T} \lambda M_u \mathbb{1}_{\{\tau > t\}} du + M_{\tau \wedge T} \mathbb{1}_{\{\tau > t\}}\right]$$

[37] see Proposition 3.2 in Kraft et al. (2011)

[38] The strategy is similar to Theorem A.2 in Kraft et al. (2011).

$$M_t \mathbb{1}_{\{\tau>t\}} \geq \mathbb{E}_t^s \left[\int_t^T \lambda M_u \mathbb{1}_{\{\tau>u\}} du + M_T \mathbb{1}_{\{\tau>T\}} \right] = \mathbb{E}_t^s \left[\int_t^T \lambda M_u \mathbb{1}_{\{\tau>u\}} du \right]$$

Applying the stochastic Gronwall-Bellman inequality,[39] we get that $M_t \mathbb{1}_{\{\tau>t\}} \geq 0$ for $0 \leq t \leq T$. Since $M_0 \mathbb{1}_{\{\tau=0\}} \geq 0$, we conclude $M_0 \geq 0$. We can apply a similar argument for any $u < T$ (redefining the stopping time $\tau^u = \inf\{t \geq u : U_t^{s=0} \leq U_t^s\}$) and get $M_u \geq 0$ for all $0 < u < T$.

Now to make the inequality strict, if $M_t = 0$ a.e. on $[0,T] \times \Omega$, then $H_t \geq \lambda M_t = 0$, and the inequality is strict on positive measure subset A, and therefore $M_0 > 0$. If $M_t > 0$ for at least some (ω, t) with $t < T$, with positive probability, we do the following. For some small $\epsilon > 0$, let $\tau^\epsilon = \inf\{t : M_t \geq \epsilon\}$. If we take ϵ small enough, the probability that we get to such a point before T is positive: $P^s(\{\tau^\epsilon \wedge T < T\}) > 0$ (for any P^s since they are equivalent). It must be that there is some stealing going on after this, since otherwise M_{τ^ϵ} would be zero. Now consider the alternative stealing plan s' that steals only until τ^ϵ and then stops, that is $s'_t = s_t$ for $t < \tau^\epsilon$ and $s' = 0$ after this. By a similar argument as before, $U_0^{s=0} \leq U_0^{s'}$. Utility under this plan satisfies $U_{\tau^\epsilon \wedge T}^{s'} = U_{\tau^\epsilon \wedge T}^{s=0} < U_{\tau^\epsilon \wedge T}^{s}$ if $\tau^\epsilon \wedge T < T$, and equal otherwise. Now if we compare s and s', both plans induce the same probability measure until $\tau^\epsilon \wedge T$, and the same consumption stream, but the payoff at $\tau^\epsilon \wedge T$ is larger for s (strictly so with positive probability):

$$U_t^{s'} = \mathbb{E}_t^s \left[\int_t^{\tau^\epsilon \wedge T} f(e_u + \phi p_u k_u s_u, U_u^{s'}) + U_{\tau^\epsilon \wedge T}^{s=0} \right]$$

$$U_t^{s} = \mathbb{E}_t^s \left[\int_t^{\tau^\epsilon \wedge T} f(e_u + \phi p_u k_u s_u, U_u^{s}) + U_{\tau^\epsilon \wedge T}^{s} \right]$$

By strict monotonicity of EZ preferences with respect to terminal value, we get $U_0^s > U_0^{s'} \geq U_0^{s=0}$. This proves stealing is attractive if (4) fails.

Proof of Proposition 1. Using the FOCs for consumption we obtain in the CE:

$$\Lambda \left(\hat{e}^{-\psi} + \gamma\psi(\phi\iota'(g)\frac{\nu}{X})^2 \tilde{e}^{-2\psi-1} \right) = \left(\frac{\tilde{c}}{S} \right)^{-\psi}$$

where $\tilde{c} = \frac{c}{k}$. This is the same as equation (19) in SP. It follows that if the CE is efficient, then $\Lambda^{CE} = \Lambda^{SP}$, since all other objects in the equation must coincide by definition, including S.

Now use the FOC for g, $\iota'(g) = p$, and for consumption to obtain in the CE

$$\left(\frac{\tilde{c}}{S} \right)^{-\psi} \iota'(g) = S + \Lambda X$$

[39] See Duffie and Epstein (1992)

while in the SP we have

$$\left(\frac{\tilde{c}}{S}\right)^{-\psi} \iota'(g) + \overbrace{\Lambda X \gamma (\tilde{e}^{-\psi} \phi \iota'(g) \frac{\nu}{X})^2 \frac{\iota''(g)}{\iota'(g)}}^{\text{externality}} = S + \Lambda X$$

This shows the CE is not constrained efficient.

Proof of Proposition 2. Straightforward from equilibrium and optimal allocation conditions.

Proof of Proposition 4. Straightforward from equilibrium and optimal allocation conditions.

7.2 Appendix B: Solving the Competitive Equilibrium

The strategy to solve the competitive equilibrium is to transform the problem into a system of PDEs for p, ξ, and ζ. Suppose we are given these functions. We can build $S = \zeta(p - \xi X)$ and $\Lambda = \xi\zeta$, and use Ito's lemma to compute the drift and volatility of all these objects, in terms of μ_X and σ_X, which we still don't know, and μ_Y and σ_Y, which are exogenously given. We use equation (10) to obtain σ_X. Then use the definition of σ_X to write $\hat{\sigma}_U = \sigma_X + \sigma$, and use the FOC for $\hat{\sigma}_U$, equation (7), to write $\pi = \gamma\hat{\sigma}_U + \sigma_\xi$. Then using the FOC for σ_w, equation (8), we get $\sigma_w = \frac{\pi}{\gamma} + \frac{1-\gamma}{\gamma}\sigma_\zeta$. Now we need to compute the drifts. First, use the FOCs to obtain $\hat{e}$ and $\hat{c}$, and we can use the definition of μ_X to compute it. Now use households' HJB to compute r. We now have experts' HJB (with $\hat{k} = X^{-1}$ from market clearing for capital), the FOC for capital, and the market clearing condition for consumption goods (this is an algebraic constraint), and we need to find ξ, ζ and p.

The system of equations can be solved by adding a fictitious finite time horizon T, with some terminal values for these functions. A time derivative must be added to the computation of all drifts, and we can then solve backwards in time. In this respect we have a system of first order ODEs with respect to time, which can be solved with a standard integrator, such as Runge-Kutta 4 for example. If the time derivatives vanish as we solve backwards, we have a solution to the system of PDEs we were interested in (infinite horizon). Terminal conditions are not important as long as the time derivatives vanish in the limit. Since the market clearing condition for consumption is an algebraic constraint, it is easier to differentiate it with respect to time to obtain a differential equation. We just need to make sure that terminal conditions are consistent with market clearing for consumption goods, and the algorithm will preserve this as we solve backwards. We can also verify ex-post that this condition is satisfied by the solution.

There are two complications. The first is that the FOC for $\hat{e}$ cannot be solved analytically, and solving it numerically at each step would make the algorithm much slower. What we can do is add $\hat{e}$ as a function to be solved for, and differentiate the FOC for $\hat{e}$ with respect to time, like we did for market clearing for consumption. We get an extra unknown but also an extra differential equation, and terminal conditions must be chosen so that the FOC for $\hat{e}$ is satisfied. This can also be verified ex-post (the benefit is we only solve the FOC for $\hat{e}$ once at the beginning). The second complication is that the domain of the system $(X, Y) \in D \subset \mathbb{R}^2_+$ is unknown. Basically, for a given Y we know that $X \in (0, \bar{X}(Y))$, but we don't know what is the maximum utility that can be delivered to experts for each exogenous state Y. To deal with this we can do a change of variables, such as $\tilde{X} = \frac{X}{X+\zeta(p-\xi X)} \in (0,1)$, and solve the resulting system.

7.3 Appendix C: Solving the Planner's Problem

The planner's HJB is a PDE for $S(X,Y)$. As in the competitive equilibrium, we can solve it by adding a fictitious finite horizon T. This requires us to add a time derivative when computing μ_S. We can then solve backward for arbitrary terminal conditions. If the time derivative vanishes as we solve back, we found the original PDE.

Just like in the competitive equilibrium case, we need to deal with two complications. The first is that now the FOC for both $\tilde{e}$ and g are difficult to solve analytically, so we add both as functions of (X,Y) and differentiate the FOCs with respect to time to obtain two more PDEs. We just need to ensure that terminal conditions satisfy the FOCs (the benefit, as before, is that we only solve them numerically once). We can check at the end that the FOC are satisfied. The second problem is that as before we don't know the domain, so we need to do a change of variables as in the competitive equilibrium, such as $\tilde{X} = \frac{X}{X+\zeta(p-\xi X)} \in (0,1)$, and solve the resulting system.

Implementation of the planner's allocation (and proof of Proposition 3). Using the construction described in Section 4.2, we only need to check that experts' and the representative household's HJB equations are satisfied, and the pricing equation for taxes holds. Start with households. By construction, their FOC for consumption is satisfied. Using $w = (p+T-\xi X)k > 0$ we derive an expression for σ_w and then set $\pi = \gamma\sigma_w - (1-\gamma)\sigma_\zeta$. We then set r so that their HJB equation is satisfied. We are in effect choosing r and π so that $(a - \iota(g) - \hat{e}X)k = \tilde{c}k$ is the optimal choice of consumption for the household, and their wealth $(p+T-\xi X)k$. Notice that the FOC for experts' $\hat{e}$ and $\hat{\sigma}_U$ will be satisfied automatically. To see this, first use the planner's FOC for $\tilde{e}$

$$\Lambda \left(\tilde{e}^{-\psi} + \gamma\psi(\phi\iota'(g)\frac{\nu}{X})^2 \tilde{e}^{-2\psi-1} \right) = \left(\frac{\tilde{c}}{S} \right)^{-\psi}$$

and using $\left(\frac{\tilde{c}}{S}\right)^{-\psi} = \zeta$ and $\Lambda = \xi\zeta$ and multiplying by $\hat{e}$ on both sides (recall $\tilde{e} = \hat{e}$) we get experts' FOC for $\hat{e}$ in the private contract

$$\xi \left(\hat{e}^{-\psi} + \gamma\psi(\hat{e}^{-\psi}\phi\iota'(g)\frac{\nu}{X})^2 \right) = \hat{e}$$

From planner's optimal aggregate risk sharing we get

$$\sigma_X = \sigma_S - \frac{1}{\gamma}\sigma_\Lambda = \sigma_\zeta + \sigma_w - \sigma - \frac{1}{\gamma}(\sigma_\zeta + \sigma_\xi)$$

and using $\pi = \gamma\sigma_w - (1-\gamma)\sigma_\zeta$ and $\sigma_X = \hat{\sigma}_U - \sigma$ we get experts' FOC for $\hat{\sigma}_U$ in the private contract:

$$\hat{\sigma}_U = \frac{\pi - \sigma_\xi}{\gamma}$$

Now we want to prove that $\xi = -\frac{S_X}{\zeta}$ satisfies experts' HJB equation. For this we will use the planner's HJB equation (18). Multiply by S on both sides, take the derivative with respect to X using the envelop theorem, and divide throughout by S_X to obtain

$$\begin{aligned}\frac{\rho}{1-\psi} = \frac{(a-\iota(g)-\tilde{e}X)^{1-\psi}}{1-\psi}\psi S^{\psi-1} - \frac{(a-\iota(g)-\tilde{e}X)^{-\psi}S^{\psi}}{S_X}\hat{e} + (g - \frac{\gamma}{2}\sigma^2) + \frac{S_{XY}}{S_X}\mu_Y + \frac{S_{XX}}{S_X}\mu_X X \\ + \mu_X - \gamma(\tilde{e}^{-\psi}\phi\frac{\iota'(g)}{X}\nu)^2 + \frac{1}{2}\frac{S_{XYY}}{S_X}\sigma_Y^2 + \frac{1}{2}\frac{S_{XXX}}{S_X}(\sigma_X X)^2 + \frac{S_{XX}}{S_X}\sigma_X^2 X + \frac{S_{XXY}}{S_X}\sigma_X X\sigma_Y \\ + \frac{S_{XY}}{S_X}\sigma_X\sigma_Y + (1-\gamma)\sigma\left(\frac{S_{XX}}{S_X}\sigma_X X + \frac{S_{XY}}{S_X}\sigma_Y\right) + (1-\gamma)\sigma\sigma_X \\ -\gamma\left(\frac{S_X}{S}\sigma_X X + \frac{S_Y}{S}\sigma_Y\right)\left(\frac{S_{XX}}{S_X}\sigma_X X + \frac{S_{XY}}{S_X}\sigma_Y + \sigma_X\right) + \frac{\gamma}{2}\left(\frac{S_X}{S}\sigma_X X + \frac{S_Y}{S}\sigma_Y\right)^2\end{aligned} \tag{28}$$

Now use $-\xi\zeta = S_X$ to obtain

$$\frac{S_{XX}}{S_X} = \frac{\xi_X}{\xi} + \frac{\zeta_X}{\zeta}, \quad \frac{S_{XY}}{S_X} = \frac{\xi_Y}{\xi} + \frac{\zeta_Y}{\zeta}, \quad \frac{S_{XXX}}{S_X} = \frac{\zeta_{XX}}{\zeta} + 2\frac{\xi_X}{\xi}\frac{\zeta_X}{\zeta} + \frac{\xi_{XX}}{\xi},$$

$$\frac{S_{XYY}}{S_X} = \frac{\zeta_{YY}}{\zeta} + 2\frac{\zeta_Y}{\zeta}\frac{\xi_Y}{\xi} + \frac{\xi_{YY}}{\xi}, \quad \frac{S_{XXY}}{S_X} = \frac{\zeta_{XY}}{\zeta} + \frac{\xi_Y}{\xi}\frac{\zeta_X}{\zeta} + \frac{\xi_X}{\xi}\frac{\zeta_Y}{\zeta} + \frac{\xi_{XY}}{\xi}$$

Now plug this into (28), use the definition of μ_X and the FOC for $\hat{e}$ in the private contract (which we already know holds), and simplify to obtain

$$0 = \frac{\psi}{1-\psi}\hat{c}^{1-\psi}\zeta^{\psi-1} + \mu_\xi + \mu_\zeta - \psi\frac{\tilde{e}^{1-\psi}}{1-\psi} + \frac{\gamma}{2}\sigma_X^2 - \gamma(\frac{1}{2}-\psi)(\tilde{e}^{-\psi}\phi\frac{\iota'(g)}{X}\nu)^2$$

$$\sigma_\xi\sigma_\zeta + \sigma_X(\sigma_\xi + \sigma_\zeta) + (1-\gamma)\sigma(\sigma_\xi+\sigma_\zeta)$$

$$-\gamma\sigma_S(\sigma_\xi + \sigma_\zeta + \sigma_X) + \frac{\gamma}{2}\sigma_S^2$$

Now from household's HJB and using $\sigma_w = \sigma_S - \sigma_\zeta + \sigma$ we get

$$\frac{\rho}{1-\psi} = \psi\frac{\hat{c}^{1-\psi}}{1-\psi}\zeta^{\psi-1} + r + \frac{\gamma}{2}(\sigma_S - \sigma_\zeta + \sigma)^2 + \mu_\zeta - \frac{\gamma}{2}\sigma_\zeta^2$$

which we plug into our expression. After some algebra using $\sigma_S = \sigma_X + \frac{1}{\gamma}(\sigma_\xi + \sigma_\zeta)$ and $\hat{\sigma}_U = \sigma_X + \sigma$, as well as the FOC for $\hat{e}$ and $\hat{\sigma}_U$, we get

$$\gamma(\tilde{e}^{-\psi}\phi\frac{\iota'(g)}{X}\nu)^2 - \hat{e} + \xi\left\{r + \frac{\hat{e}^{1-\psi}}{1-\psi} - \frac{\rho}{1-\psi} - \frac{\gamma}{2}(\tilde{e}^{-\psi}\phi\frac{\iota'(g)}{X}\nu)^2 - \mu_\xi + \hat{\sigma}_U\pi + \sigma_\xi\pi - \frac{\gamma}{2}\hat{\sigma}_U^2 - \sigma_\xi\hat{\sigma}_U\right\} = 0$$

Because τ is chosen so that the pricing equation for capital holds we get

$$\gamma(\tilde{e}^{-\psi}\phi\frac{\iota'(g)}{X}\nu)^2 = \frac{p}{X}\left(\frac{a-\iota(g)}{p} + \mu_p + g + \sigma'\sigma_p - (r+\tau^k) - \pi(\sigma+\sigma_p)\right)$$

and plugging this in, we obtain experts' HJB.

Finally, we just need to check that the pricing equation for taxes is satisfied. First, use the planner's HJB and households' HJB to obtain a version of the dynamic budget constraint of the household

$$p\mu_p + T\mu_T - \xi X(\mu_\xi + \mu_X + \sigma_\xi\sigma_X) + g(p+T-\xi X) + \sigma(p\sigma_p + T\sigma_T - \xi X(\sigma_\xi + \sigma_X))$$

$$= r(p+T-\xi X) + \pi(p\sigma_p + T\sigma_T + \xi X(\sigma_\xi+\sigma_X) + \sigma(p+T-\xi X)) - \hat{c}(p+T-\xi X)$$

Multiply experts' HJB by X to obtain

$$a - \iota(g) - \hat{e}X + p(g + \mu_p + \sigma\sigma_p - (r+\tau^k) - (\sigma+\sigma_p)\pi$$

$$+\xi X\left\{r + \frac{1}{1-\psi}(\hat{e}^{1-\psi} - \rho) - \frac{\gamma}{2}\hat{\sigma}_U^2 - \frac{\gamma}{2}(\hat{e}^{-\psi}\phi\frac{\iota(g)}{X}v)^2 + \hat{\sigma}_U(\pi - \sigma_\xi) - \mu_\xi + \sigma_\xi\pi\right\} = 0$$

Combining these two expressions, and using the definition of μ_X, σ_X, and $\hat{c} = \frac{a-\iota(g)-\hat{e}X}{p+T-\xi X}$ we get the pricing equation for capital. This completes the proof of Proposition 3.