Universal Gravity*

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Abstract

Gravity trade models are the most important empirical tool in international trade. However, characterization of their theoretical and empirical properties has thus far been primarily focused on specific parametric examples. In this paper, we show there exist a number of theoretical and empirical properties that are universal to all gravity trade models, regardless of their microeconomic foundations. In particular, in any gravity trade model where goods and factor markets clear and trade is balanced, we (1) prove the existence of an equilibrium and provide sufficient conditions for its uniqueness; (2) provide sufficient conditions under which the equilibrium maximizes world income and is Pareto optimal; (3) derive an analytical expression for the elasticity of all trade flows and incomes to all bilateral trade frictions; (4) provide a general method to identify model fundamentals from observed trade flows; and (5) develop two easily implementable estimators of the gravity equation that respect the general equilibrium conditions.

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1 Introduction

The gravity relationship – where trade flows increase with the origin and destination countries incomes and decrease with the distance between the two countries – is one of the most robust empirical results in economics.[1] As such, economists have offered many alternative theoretical foundations for this relationship, see e.g. Anderson (1979); Eaton and Kortum (2002); Chaney (2008). However, much remains unknown about how assumptions regarding the source of the gravity relationship affect the implications of the model.

In this paper, we show that standard equilibrium conditions along with the gravity structure of trade flows are sufficient to characterize a number of properties that hold regardless of the particular microeconomic foundations of the model. In particular, we derive a number of theoretical results that hold universally for all gravity trade models where good and factor markets clear and trade is balanced, a class of models that we term general equilibrium gravity models. We classify the set of results into four groups: (1) existence and uniqueness; (2) efficiency; (3) comparative statics; and (4) identification and estimation.

First, we examine the existence and uniqueness properties common to all general equilibrium gravity trade models. We show that their solution can be represented by a nonlinear operator operating on a compact set, which allows us to prove existence of a trade equilibrium and provide conditions for which uniqueness is guaranteed. These sufficient conditions depend only on two parameters – α and β – which govern the relationship between factor market clearing and the gravity equation. In turn, this setup can be mapped to a wide range of parameterized gravity setups depending on the specification of the parameters α and β making it simple to check whether the equilibrium in any particular gravity trade model is unique. The set of gravity models for which uniqueness can be proven using this method is strictly larger than the set for which the gross substitutes property used in Alvarez and Lucas (2007) can be applied, and include a wider range of trade models with intermediate inputs and economic geography models where labor is mobile.

In addition, for the set of trade frictions that are the focus of much of the empirical gravity literature where frictions are symmetric up to an origin-specific and destination-specific shifter (which we call "quasi-symmetric"), we prove that balanced trade implies that the origin and destination fixed effects of the gravity equation are equal up to scale. This allows us to further extend the range of model parameters for which uniqueness can be guaranteed. It also provides a general theoretical underpinning for a result that has actually already been used in the literature (albeit implicitly); for example, it was this result that

[1] The literature on the gravity equation in trade is vast; an excellent starting place are the recent review articles by Baldwin and Taglioni (2006) and Head and Mayer (2013).

allowed Anderson and Van Wincoop (2003) to show that the "multilateral resistance" term was equal to the price index and allowed Allen and Arkolakis (2013) to simplify a set of non-linear integral equations into a single integral equation.

Second, we examine the efficiency properties of gravity trade models when trade costs are quasi-symmetric. We show that if the uniqueness conditions are satisfied, then the trade equilibrium can be equivalently expressed as the optimization problem of maximizing world trade flows subject to trade remaining balanced and an arbitrary normalization on the aggregate factor market clearing condition, i.e. general equilibrium gravity models maximize world income. Furthermore, if welfare in a location is increasing in its openness (as in the class of trade models considered by Arkolakis, Costinot, and Rodríguez-Clare (2012)), we prove that general equilibrium gravity models are Pareto optimal.

Third, we examine how changes in bilateral trade frictions affect equilibrium trade flows and incomes. We first derive an analytical expression for the (large) matrix of elasticities of all bilateral trade flows and incomes to all bilateral trade frictions. Somewhat surprisingly, this expression depends only on observed trade flows and α and β, indicating that apart from these two model parameters, there is no need to specify (let alone solve) a gravity trade model in order to determine how a change in any bilateral trade friction would affect all bilateral trade flows and incomes in all countries. We then derive a system of equations that show how arbitrary (possibly non-infinitesimal) changes to the trade friction matrix affect trade flows and incomes. The former result, to the best of our knowledge, is novel in the literature; the latter latter result generalizes the results of Dekle, Eaton, and Kortum (2008) to all general equilibrium gravity models; as with the elasticities, this system of equations depends only on observed trade flows and α and β.

Fourth, we examine the empirical properties of general equilibrium gravity models. We first show under what conditions trade frictions and the equilibrium origin and destination fixed effects can be recovered from observed trade flows. We then provide two new methods of estimating the gravity equation that combine the gravity structure of trade flows with the general equilibrium conditions of the model. First, we show how including factor market clearing conditions in the estimation allows one to recover the parameters α and β from a gravity equation as long as one can control for the exogenous income shifter (e.g. by including proxies for the effective units of labor). Second, we develop a simple method of estimating the gravity equation when trade costs are quasi-symmetric that respects the balanced trade condition that is twice as efficient (i.e. has half the asymptotic variance) as the widely used fixed effects gravity estimator (see e.g. Eaton and Kortum (2002); Waugh (2010)).

The paper is organized as follows. The next section defines the set of general equilibrium gravity models we are considering. Sections 3-6 present the theoretical results for existence and uniqueness, efficiency, comparative statics and counterfactuals, and identification and estimation, respectively. Throughout these sections, we present a number of theorems and propositions (the two being distinguished solely by the fact that we found the proofs for the former more mathematically interesting); for readability, all proofs are relegated to the appendix. Because of the rather abstract nature of the exercise, in Section 7 we show how our framework can be applied to a number of seminal gravity trade models and economic geography models. The paper finishes with a brief conclusion.

2 The General Equilibrium Gravity Model

Consider a world comprised of a set $S \equiv \{1, ..., N\}$ of locations.[2] We define a *gravity trade model* as any model which yields an equation of the following type for the value of bilateral trade flows from $i \in S$ to $j \in S$:

$$X_{ij} = K_{ij}\gamma_i\delta_j, \tag{1}$$

where X_{ij} is the value of bilateral trade flows, γ_i is an origin fixed effect, δ_j is a destination fixed effect, and K_{ij} is bilateral trade friction. The two fixed effects are endogenous model outcomes; depending on the specification of the model, they may include wages or the measure of producing firms. The bilateral trade frictions are exogenous and capture the effects of bilateral trade costs; they could be inverse functions of bilateral distance, various exporting barriers faced by exporting countries, etc. Whereas we do not take a particular stand on the model that yields the gravity specification (1), we explain how different models map to this specification and to our subsequent results below.

We proceed by defining three equilibrium conditions that are sine qua non for modern general equilibrium gravity models: goods market clearing, trade balance, and factor market clearing. Let Y_i be the total income derived from trade in a location $i \in S$. We say that *goods markets clear* if the income for all $i \in S$ is equal to the value of the good traded to other destinations:

$$Y_i = \sum_{j \in S} X_{ij}. \tag{2}$$

The goods market clearing condition is little more than an accounting identity.

We say that *trade is balanced* if the income for all $i \in S$ is equal to the amount spent on

[2]The choice of a finite number of locations is not necessary for the the results that follow, but it saves on notation, avoids several thorny technical issues, and is consistent with the majority of the trade literature. However, it does come at a cost: when there are a continuum of locations, Theorem 3 can be shown to hold for any set of trade frictions, not just quasi-symmetric trade frictions.

good purchased from all other destinations:

$$Y_i = \sum_{j \in S} X_{ji}. \tag{3}$$

While balanced trade is a standard equilibrium condition, it is important to note that trade is not balanced empirically. This empirical discrepancy is an inherent limitation arising from the use of a static model to explain an empirical phenomenon with dynamic aspects. However, given both its ubiquity in the literature and the necessarily ad hoc nature of any alternative assumption (e.g. exogenously trade deficits), balanced trade seems the natural assumption on which to focus.

We say that *factor markets clear* if for all $i \in S$ the income in the region is equal to a log-linear function of the origin and destination fixed effects:

$$Y_i = B_i \gamma_i^{\alpha} \delta_i^{\beta}, \tag{4}$$

where α and β are (exogenous) model parameters and $B_i > 0$ is an (exogenous) country specific shifter. The factor market clearing condition is analogous to the standard condition that the income in a location is equal to the income earned by the factors of production in that location but reformulated in terms of the origin and destination fixed effects of the gravity equation. This formulation is general enough to incorporate a number of seminal gravity trade models, e.g. Armington (1969); Anderson (1979); Krugman (1980); Eaton and Kortum (2002); Melitz (2003). In addition, with a slight modification,[3] this formulation also applies to many prominent economic geography models, e.g. Helpman (1998); Redding and Sturm (2008); Redding (2012); Allen and Arkolakis (2013). Table 1 shows how to write the factor market clearing condition in these models (and several others) in terms of equation (4); in Section 7 we consider several of these models in more detail.

The elasticity of income to the origin and destination fixed effects – which are captured by parameters α and β, respectively – turn out to be very important in characterizing the equilibrium properties of a gravity model. As we will see below, they are key determinants of whether or not an equilibrium is unique and determine how changes to model parameters affect trade flows and incomes. Figure 1 shows the range of α and β where trade and economic geography models may lie depending on their own parameter values (e.g. the elasticity of substitution and the importance of intermediaries). In trade models, when

[3]In economic geography models equation (4) must be modified to also include an unknown constant $\lambda > 0$. This constant is a a monotonic transformation of welfare which is pinned down in equilibrium by the total population of the world. The existence and uniqueness results that follow are unaffected by the inclusion of λ (although additional care must be taken in the proofs); see Section 7.2 for details.

goods are substitutes, α and β are either both negative (and $\alpha \leq \beta$) or both greater than one (and $\alpha \geq \beta$). In trade models when goods are complements, $\alpha \geq 1$ and $\beta \in [0,1]$. These regions are also inhabited by economic geography models (where the exact location depends on the preferences and the strength of spillovers). Economic geography models, however, can also reside in the space where $\alpha, \beta \in [0,1]$ and $\beta \geq \alpha$ (when goods are substitutes) and $\alpha \in [0,1]$, $\beta \geq 1$ (when goods are complements).

Finally, to choose the numeraire, we normalize world income equal to an arbitrary constant Y^W:

$$\sum_i Y_i = Y^W. \tag{5}$$

In what follows, we define a *general equilibrium gravity model* to be any gravity trade model such that goods market clears, trade is balanced, factor markets clear, and the normalization (5) is satisfied.

3 Existence and Uniqueness

In this section, we prove that there exists an equilibrium of any general equilibrium gravity model and provide conditions for its uniqueness.

Combining gravity (1) with goods market clearing (2) and the generalized labor marking clearing condition (4) yields:

$$B_i \gamma_i^{\alpha-1} \delta_i^{\beta} = \sum_j K_{ij} \delta_j \tag{6}$$

Combining gravity (1) with balanced trade (3) and the generalized labor marking clearing condition (4) yields:

$$B_i \gamma_i^{\alpha} \delta_i^{\beta-1} = \sum_j K_{ji} \gamma_j \tag{7}$$

Define $x_i \equiv B_i \gamma_i^{\alpha-1} \delta_i^{\beta}$ and $y_i \equiv B_i \gamma_i^{\alpha} \delta_i^{\beta-1}$. Then it can be shown that $\delta_i = x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}$ and $\gamma_i = x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}$ so that for any set of $\{B_i\} \in \mathbb{R}_{++}^N$, $\{K_{ij}\} \in \mathbb{R}_{++}^{N\times N}$, $\{\alpha, \beta\} \in \{\mathbb{R}^2 | \alpha + \beta \neq 1\}$, the equilibrium of a general equilibrium gravity model described by Equations (6) and (7) can be written using the equations

$$x_i = \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}, \tag{8}$$

and

$$y_i = \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}. \tag{9}$$

At this point, for given parameters K_{ij}, B_j, β, and α, the system takes the form of a standard system of non-linear equations. It turns out that this reformulation of the problem provides a method of solving for the trade equilibrium system using functions that map a compact space onto itself. This has two advantages over the standard formulation given in equations Equations (6) and (7): first, by restricting the potential solution space, it facilitates the calculation of the equilibrium; second, it allows us to generalize results used in the study of integral equations to prove the following theorem regarding the existence and uniqueness of general equilibrium gravity models:

Theorem 1. *Consider a general equilibrium gravity model. Then:*

i) As long as $\alpha + \beta \neq 1$, the model has a positive solution and all possible solutions are positive; and

ii) If α and β are both (weakly) negative or α and β are both (weakly) greater than 1, then the system has a unique solution.

Proof. See Appendix A.1. □

Note that condition (ii) of Theorem 1 provides sufficient conditions for uniqueness; for certain parameter constellations (e.g. particular geographies of trade costs), equilibria may be unique even if the conditions are not satisfied. In practice, however, we have found that there exist multiple equilibrium for particular geographies when condition (ii) is not satisfied; section 7.3 provides two examples.

It turns out that we can extend the range in which uniqueness is guaranteed if we constrain our analysis to a particular class of trade frictions which are the focus of a large empirical literature on estimating gravity trade models. We call these trade frictions quasi-symmetric.

Definition 1. Quasi Symmetry: We say the trade frictions matrix K is *quasi-symmetric* if there exists a symmetric $N \times N$ matrix $\tilde{K}$ (i.e. for all $i, j \in S$ we have $\tilde{K}_{ij} = \tilde{K}_{ji}$) and $N \times 1$ vectors K^A and K^B such that for all $i, j \in S$ we have:

$$K_{ij} = \tilde{K}_{ij} K_i^A K_j^B.$$

Loosely speaking, quasi-symmetric trade frictions are those that are reducible to a symmetric component and an origin- and destination-specific component. While restrictive, it is important to note that the vast majority of papers which estimate gravity equations assume that trade frictions are quasi-symmetric; for example Eaton and Kortum (2002) and Waugh

(2010) assume that trade costs are composed by a symmetric component that depends on bilateral distance and on a destination or origin fixed effect.

When trade frictions are quasi-symmetric we can show that the system of equations (8) and (9) can be dramatically simplified, and the uniqueness more sharply characterized.

Theorem 2. *Consider any general equilibrium gravity model with quasi-symmetric trade costs. Then:*

i) The balanced trade condition alone implies that the fixed effects that are equal up to scale, i.e.

$$\gamma_i K_i^A = \kappa \delta_i K_i^B$$

for some $\kappa > 0$ that is part of the solution of the equilibrium.

ii) If α and β satisfy

$$\alpha + \beta \leq 0 \text{ or } \alpha + \beta \geq 2 \tag{10}$$

the model has a unique positive solution.

Proof. See Appendix A.2. □

Part i) of the Theorem 2 is particularly useful since it allows to simplify the equilibrium system (8)-(9) into a single non-linear equation:

$$x_i = \kappa^{\frac{1-\alpha}{\alpha+\beta-1}} \sum_j \widetilde{K}_{i,j} K_i^A K_j^B B_j^{\frac{1}{1-\alpha-\beta}} \left(\frac{K_i^B}{K_i^A}\right)^{\frac{1-\alpha}{\alpha+\beta-1}} x_j^{\frac{1}{\alpha+\beta-1}} \tag{11}$$

In addition, because the origin and destination fixed effects in gravity models will (generally) be composites of exogenous and endogenous variables, by showing that the two fixed effects are equal up to scale, Theorem 2 provides a more precise analytical characterization of the equilibrium. We should note that the results of Theorem 2 has already been used in the literature for particular models, albeit implicitly. The most prominent example is Anderson and Van Wincoop (2003), who use the result to show the bilateral resistance is equal to the price index. The result is also used by Allen and Arkolakis (2013) to simplify a set on non-linear integral equations into a single integral equation. To our knowledge, Head and Mayer (2013) are the first to recognize the importance of balanced trade and market clearing in generating the result for the Armington model; however, Theorem 2 shows that the result applies more generally to any gravity equation with quasi-symmetrical trade costs.

One might wonder if the conditions for uniqueness given in Theorem 2 also hold for non-quasi symmetric trade costs. The answer is no: for certain parameter constellations, there exist multiple equilibria when trade costs are sufficiently asymmetric but only a single equilibrium if trade costs are quasi-symmetric. Section 7.3.2 provides an example.

Figure 2 illustrates the range of α and β for which uniqueness of model can be guaranteed. It should be noted that while most of the examination of existence and uniqueness of trade equilibria has proceeded on a model-by-model case, the gross substitute methodology used by Alvarez and Lucas (2007) has proven enormously helpful in establishing conditions for existence and uniqueness. It can be shown (see Online Appendix B.3) that the gross-substitutes methodology works only when $\alpha \leq 0$ and $\beta \leq 0$; hence, the tools used in Theorems 1 and 2 extend the range of trade models for which uniqueness can be proven. As we discuss below in Section 7, examples of trade models which Theorems 1 and 2 guarantee existence and uniqueness (that cannot be addressed using the gross substitute methodology) include trade models with intermediate inputs where the share of labor in the production function is less than $\frac{1}{\sigma}$, trade models with elasticities of substitution $\sigma < 1$, and trade models with labor mobility.

4 Efficiency

In this section, we examine the efficiency properties of gravity trade equilibria. There has been much work on the efficiency of trade equilibria (e.g. Dixit and Norman (1980); Helpman and Krugman (1985); Dhingra and Morrow (2012)); to our knowledge, however, there does not exist an efficiency proof that is sufficiently general to include any general equilibrium gravity model. In this section, we provide such a proof. To do so, we borrow a key insight from the study of integral equations: oftentimes integral equations can be equivalently considered as the solutions to "dual" maximization problems.[4] In our particular system of integral equations defined by equations (8) and (9), when trade costs are quasi-symmetric and the sufficient conditions for uniqueness of the equilibria are satisfied, we show the equilibrium of the gravity model can be equivalently interpreted as the set of origin and destination fixed effects that maximizes world income subject to a factor market clearing constraint, i.e. the equilibrium of the gravity model maximizes world income. Furthermore, if welfare is increasing in trade openness, maximizing world income is equivalent to maximizing a weighted average of a positive monotonic transform of country's welfare, i.e. the equilibrium of the gravity model is Pareto efficient.[5] We formalize these results in the following theorem:

Theorem 3. *Consider any general equilibrium gravity model with quasi-symmetric trade costs. If condition (ii) of Theorem 2 is satisfied (which guarantees uniqueness), then:*

[4] A simple example is that the eigenvalues of the system $\lambda x = Ax$ solve the maximization problem $\max_{x \in \{\mathbb{R}^N | x^T x = 1\}} x^T A x$. This dual approach has also been used previously in the international trade literature, see e.g. Dixit and Norman (1980); Costinot (2009) .

[5] This is not necessarily true for economic geography models in which labor is mobile, a result which we are exploring in ongoing research.

(1) The general equilibrium gravity model maximizes world income subject to trade being balanced and the aggregate factor markets clearing; and

(2) If for all $i \in S$, welfare can be written as a log linear function of the share of income:

$$W_i = C_i^W \lambda_{ii}^{-\rho}, \tag{12}$$

where $C_i^W > 0$ and $\rho > 0$ are constants, then there exists a set of weights $\omega_i > 0$, $\sum_{i \in S} \omega_i = 1$ and a constant $\eta > 0$ such that the trade equilibrium maximizes a weighted average of the following positive monotonic transform of welfare:

$$W = \left(\sum_{i \in S} \omega_i W_i^{\frac{\eta}{\rho}} \right)^{\frac{1}{\eta}},$$

i.e. the general equilibrium trade model is Pareto efficient.

Proof. See Appendix A.3. □

Note that Arkolakis, Costinot, and Rodríguez-Clare (2012) show that for a large class of trade models, the welfare of a country can be written solely as an increasing function of its openness to trade and an exogenous parameter, i.e. the equation (12) holds.

5 Comparative Statics

In this section, we consider how changes in model fundamentals affect trade flows and income. We first consider infinitesimal changes and derive an expression that yields the elasticities of all origin and destination fixed effects to all bilateral trade frictions that depends only on observed trade flows and the elasticities α and β. We then derive a system of equations that show how arbitrary changes to the trade friction matrix affect trade flows that also depend only on observed trade flows and the elasticities α and β.

5.1 Local Comparative Statics

Consider an infinitesimal change in any bilateral trade friction K_{ij}; how does this affect equilibrium trade flows and incomes? The following proposition provides a simple analytical expression for the elasticity of any origin or destination fixed effects to any change in bilateral trade frictions:

Proposition 1. *Consider any general equilibrium gravity model where condition (ii) of Theorem 1 is satisfied. Define the $2N \times 2N$ matrix $A \equiv \begin{pmatrix} (\alpha - 1) Y & \beta Y - X \\ \alpha Y - X^T & (\beta - 1) Y \end{pmatrix}^+$, where the "+" denotes the Moore-Penrose pseudo-inverse, Y is an $N \times N$ diagonal income matrix whose i^{th} diagonal element is Y_i and X is the $N \times N$ trade flow matrix whose $\langle i, j \rangle^{th}$ element is X_{ij}. Then:*

$$\frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} = X_{ij} \times (A_{l,i} + A_{N+l,j}) - c \text{ and } \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} = X_{ij} \times (A_{N+l,i} + A_{l,j}) - c, \tag{13}$$

where A_{kl} is the $\langle k, l \rangle^{th}$ element of A and c is a scalar[6] *that ensures the normalization $\sum_i B_i \gamma_i^\alpha \delta_i^\beta = Y^W$ holds.*

Proof. See Appendix A.4. □

We should note that the choice of the constant c (and hence the elasticities) will depend on the normalization chosen: for example, the alternative normalization that $\gamma_1 = 1$ implies $\frac{\partial \ln \gamma_1}{\partial \ln K_{ij}} = 0$, so that $c = X_{ij} \times (A_{1,i} + A_{N+1,j})$.

Because trade flows and country income are functions of the origin and destination fixed effects, Proposition 1 can be applied to determine how changing the trade costs from i to j affects trade flows between any other bilateral trade pair k and l:

$$\frac{\partial \ln X_{kl}}{\partial \ln K_{ij}} = \frac{\partial \ln \gamma_k}{\partial \ln K_{ij}} + \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} = X_{ij} \times (A_{k,i} + A_{N+k,j} + A_{N+l,i} + A_{l,j}) - 2c.$$

Similarly, Proposition 1 can be applied to determine how changing the trade costs from i to j affects income in any country l:

$$\frac{\partial \ln Y_l}{\partial \ln K_{ij}} = \alpha \frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} + \beta \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} = X_{ij} \times (\alpha (A_{l,i} + A_{N+l,j}) + \beta (A_{N+l,i} + A_{l,j})) - (\alpha + \beta) c.$$

If welfare can be written as in equation (12), then we can also determine the elasticity of welfare in any country l to any change in trade costs from i to j:

$$\frac{\partial \ln W_l}{\partial \ln K_{ij}} = X_{ij} \times \rho ((\alpha - 1) (A_{l,i} + A_{N+l,j}) + (\beta - 1) (A_{N+l,i} + A_{l,j})) - \rho (\alpha + \beta - 2) c$$

Hence, apart from a choice of α and β, there is no need to specify (let alone solve) a gravity model in order to calculate how, for example, a small reduction in trade costs between the U.S. and China would affect income in Vietnam or trade flows between Chile and Germany:

[6] In particular, $c \equiv \frac{1}{(\alpha+\beta) Y^W} X_{ij} \sum_l Y_l (\alpha (A_{l,i} + A_{N+l,j}) + \beta (A_{N+l,i} + A_{l,j}))$.

all that one needs to observe is trade flows.

5.2 Global Comparative Statics

Now consider how an arbitrary change in the trade friction matrix K affects bilateral trade flows. The following proposition, which generalizes the results of Dekle, Eaton, and Kortum (2008) for all general equilibrium gravity trade models, provides an analytical expression relating the change in the origin and destination fixed effects to the change in trade frictions and the initial exporting and importing shares:

Proposition 2. *Consider any general equilibrium gravity model where condition (ii) of Theorem 1 is satisfied. Consider any change in the trade friction $\hat{K}_{ij}$. Then the percentage change in the fixed effects, γ_i, δ_i can be computed as the unique solution of the following system:*

$$\hat{\gamma}_i^{\alpha-1}\hat{\delta}_i^{\beta} = \sum_j \pi_{ij}\hat{K}_{ij}\hat{\delta}_j \text{ and } \hat{\gamma}_i^{\alpha}\hat{\delta}_i^{\beta-1} = \sum_j \lambda_{ij}\hat{K}_{ji}\hat{\gamma}_j \quad (14)$$

where $\pi_{ij} = X_{ij}/\sum_j X_{ij}$ represents the exporting shares and $\lambda_{ij} = X_{ij}/\sum_i X_{ij}$ represents the import shares.

Proof. See Appendix A.6. □

Since equation (14) only depend on trade data and parameters α and β the proposition tells us that for any given change in variable trade costs, all the gravity trade models with the same α and β must imply the same change in the fixed effects γ_i and δ_i. This in turn implies that for any change in trade costs, all gravity models sharing the same α and β will imply the same change in trade flows (and hence trade shares) when calibrated to the same initial trade shares. If welfare can be written as in equation (12), the change in country and global welfare will also be the same.

This proposition characterizes the comparative statics for a wide class of gravity trade models. In the case where $\beta = 0$, it can be shown (see Online Appendix B.4) that the comparative statics can be characterized using import shares alone. This special case (and its welfare implications) is discussed in Proposition 2 of Arkolakis, Costinot, and Rodríguez-Clare (2012).

6 Identification and Estimation

Our final contribution is to examine how the general equilibrium conditions of gravity trade models facilitate the empirical analysis of trade flows. We first show the extent to which trade frictions can be recovered from observed trade flows. We then show how to use the factor market clearing condition to identify the trade elasticities α and β if proxies for the trade frictions and exogenous income shifters are observed. We finally present a simple variant of the standard fixed effects method of estimating the gravity equation which both improves the asymptotic efficiency of the estimator and ensures the balanced trade condition is satisfied.

6.1 Identification of trade frictions and origin and destination fixed effects

Suppose that we observe trade flows $\{X_{ij}\}$ and model parameters $\{B_i\}$, α, and β. Can we identify the trade frictions $\{K_{ij}\}$ and origin and destination fixed effects $\{\gamma_i\}$ and $\{\delta_i\}$?

It is important to note that the model can only rationalize observed trade flows where goods market clearing and balanced trade holds; that is, by definition, there does not exist a set of trade frictions for which there exists a trade equilibrium that generates trade flows where these two equilibrium conditions do not hold. Define the set of feasible trade flows that satisfy balanced trade and goods market clearing to be Ξ.[7] The following proposition summarizes the extent to which trade frictions and origin and destination fixed effects can be identified from observed trade flows:

Proposition 3. *For any set of observed trade flows* $\{X_{ij}\} \in \Xi$ *and parameters* $\{B_i\}$, α *and* β, *there exists a unique set of relative trade frictions* $\left\{\frac{K_{ij}}{\left(K_{ii}^{\beta}/K_{jj}^{\alpha}\right)^{\frac{1}{\alpha-\beta}}}\right\}$ *and origin and destination fixed effects* $\left\{\frac{\gamma_i}{K_{ii}^{\frac{\beta}{\alpha-\beta}}}\right\}$ *and* $\left\{\frac{\delta_i}{K_{ii}^{\frac{\alpha}{\beta-\alpha}}}\right\}$ *that are consistent with a trade equilibrium, which can be written solely as a function of observables:*

$$\frac{\gamma_i}{K_{ii}^{\frac{\beta}{\alpha-\beta}}} = \left(\frac{\sum_j X_{ij}}{B_i}\right)^{\frac{1}{\alpha-\beta}} X_{ii}^{\frac{\beta}{\beta-\alpha}},$$

$$\frac{\delta_i}{K_{ii}^{\frac{\alpha}{\beta-\alpha}}} = X_{ii}^{\frac{\alpha}{\alpha-\beta}} \left(\frac{\sum_j X_{ij}}{B_i}\right)^{\frac{1}{\beta-\alpha}}, \text{ and}$$

$$\frac{K_{ij}}{\left(K_{ii}^{\beta}/K_{jj}^{\alpha}\right)^{\frac{1}{\alpha-\beta}}} = X_{ij} \times \left(\frac{\sum_k X_{jk}}{\sum_k X_{ik}} \times \frac{B_i}{B_j} \times \frac{X_{ii}^{\beta}}{X_{jj}^{\alpha}}\right)^{\frac{1}{\alpha-\beta}}.$$

Proof. See Appendix A.5. □

[7] Specifically, $\Xi \equiv \left\{\{X_{ij}\} \in \mathbb{R}_{++}^{N\times N} | \sum_{j\in S} X_{ij} = \sum_{j\in S} X_{ji} \forall i \in S\right\}$.

Note that Proposition 3 implies that if we assume trade with ones own country is costless, i.e. $K_{ii} = 1$ for all $i \in S$, then $\{K_{ij}\}$, $\{\gamma_i\}$ and $\{\delta_i\}$ are all uniquely identified. If $\alpha = 0$, then $\{K_{ij}K_{ii}\}$, $\{\gamma_i K_{ii}\}$, and $\{\delta_i\}$ are uniquely identified. Conversely, if $\beta = 0$, then $\{K_{ij}K_{jj}\}$, $\{\gamma_i\}$, and $\{\delta_i K_{ii}\}$ are uniquely is identified. Intuitively, the reason that we can only identify K_{ij} when $K_{ii} = 1$ for all $i \in S$ is that it is only possible to identify the cost of trading with other countries *relative* to the cost of trading at home. Loosely speaking, this is because changes in the overall level of trade flows can be captured equally well by a change in the overall level of trade costs or the overall level of the origin and destination fixed effects.

A related procedure for identification of bilateral trade frictions from trade flows using the full structure of a general equilibrium trade model has been described by Burstein and Vogel (2012) and Arkolakis, Ramondo, Rodríguez-Clare, and Yeaple (2013), in parameterized setups that are closely related to the class of general equilibrium gravity trade model that we discuss. Our derivations show that there is formal mapping between bilateral data on trade flows and trade frictions that holds for any gravity model.[8]

6.2 Identification of trade elasticities α and β

In this subsection, we show that one can use the generalized labor market clearing condition in conjunction with the gravity structure to identify the trade elasticity parameters α and β.

Consider any gravity trade model where the generalized labor market clearing condition (4) holds, i.e.:

$$X_{ij} = K_{ij}\gamma_i\delta_j \text{ and } Y_i = B_i\gamma_i^{\alpha}\delta_i^{\beta},$$

where we assume that for all $i \in \{1, ..., N\}$, $K_{ii} = 1$. Suppose that the econometrician observes bilateral trade flows $\{X_{ij}\}$, income $\{Y_i\}$, and proxies for the bilateral trade frictions and the exogenous income shifters $\{B_i\}$, but does not observe the trade elasticities α and β. Is it possible for the econometrician to identify α and β? As is well known (see e.g. Anderson and Van Wincoop (2004)), the trade elasticities cannot be identified solely from the gravity structure of trade flows: for example, the coefficient of distance in a gravity equation captures both the effect of distance on trade costs and the elasticity of trade to the trade cost. However, it turns out that incorporating the generalized labor market clearing

[8] These results also provide a formal underpinning to the results of Eaton, Kortum, Neiman, and Romalis (2011). They describe a calibration procedure using the gravity setup of Eaton and Kortum (2002) whereas changes in the trade frictions can be inverted from the data using data on bilateral trade flows. We formally show that this inversion is unique not only for changes but also for levels of bilateral trade flows and holds for any gravity trade model. Note that Eaton, Kortum, Neiman, and Romalis (2011) use also price data to identify the changes in the productivity parameters that correspond to B_i in our notation.

condition allows for the identification of α and β.

To see this, note that rearranging the generalized labor market clearing condition implies the following relationship between the origin and destination fixed effects:

$$\gamma_i = \left(\frac{Y_i}{B_i}\right)^{\frac{1}{\alpha}} \delta_i^{-\frac{\beta}{\alpha}}. \tag{15}$$

From the gravity structure of trade flows and using the fact that $K_{ii} = 1$, we can write the destination fixed effect solely as a function of own trade flows, income, and the income shifter B_i, all of which are observed (although possibly with error):

$$\delta_i = X_{ii}^{\frac{\alpha}{\alpha-\beta}} \left(\frac{B_i}{Y_i}\right)^{\frac{1}{\alpha-\beta}}. \tag{16}$$

Combining equations (15) and (16), we can write the origin fixed effect solely as a function of observables as well:

$$\gamma_i = \left(\frac{Y_i}{B_i}\right)^{\frac{1}{\alpha-\beta}} X_{ii}^{\frac{\beta}{\beta-\alpha}} \tag{17}$$

Finally, substituting equations (16) and (17) back into the gravity equation lets us write the following log linear relationship between bilateral trade flows and observables:

$$\ln \frac{X_{ij}}{X_{ii}} = \ln K_{ij} + \left(\frac{1}{\alpha-\beta}\right) \ln \left(\frac{Y_i}{B_i} / \frac{Y_j}{B_j}\right) + \left(\frac{\alpha}{\alpha-\beta}\right) \ln \frac{X_{jj}}{X_{ii}}. \tag{18}$$

Recall that we assumed the econometrician observes proxies for bilateral trade frictions K_{ij} and the exogenous income shifter B_i; for simplicity, let us assume these proxies are related log linearly to the model parameters:

$$\ln K_{ij} = \mu^K \ln Z_{ij}^K + \varepsilon_{ij}^K$$

$$\ln B_i = \mu^B \ln Z_i^B + \varepsilon_i^B$$

so that equation (18) becomes:

$$\ln \frac{X_{ij}}{X_{ii}} = \mu^K \ln Z_{ij}^K + \left(\frac{1}{\alpha-\beta}\right) \ln \frac{Y_i}{Y_j} + \left(\frac{\mu^B}{\alpha-\beta}\right) \ln \frac{Z_j^B}{Z_i^B} + \left(\frac{\alpha}{\alpha-\beta}\right) \ln \frac{X_{jj}}{X_{ii}} + \varepsilon_{ij}, \tag{19}$$

where $\varepsilon_{ij} \equiv \varepsilon_{ij}^K + \left(\frac{1}{\alpha-\beta}\right)\left(\varepsilon_j^B - \varepsilon_i^B\right)$. Hence, α and β can be identified by comparing the estimated coefficients of $\ln \frac{Y_i}{Y_j}$ and $\ln \frac{X_{jj}}{X_{ii}}$.

We illustrate the estimation procedure for the Armington model (both with and without intermediates) in Section 7.1.

6.3 Estimation with balanced trade

We now consider how imposing the balanced trade condition affects the estimation of the gravity equation when trade costs are quasi-symmetric. Suppose that the econometrician observes bilateral trade flows X_{ij} with log-additive measurement error ε_{ij} and is interested in estimating the origin and destination fixed effects $\{\gamma_i\}$ and $\{\delta_i\}$. For simplicity, suppose too that the econometrician observes trade frictions K_{ij}. As a result, trade flows can be written as:

$$\ln X_{ij} = \ln K_{ij} + \ln \gamma_i + \ln \delta_j + \varepsilon_{ij}. \tag{20}$$

Since Anderson and Van Wincoop (2003), the most common way to estimate $\ln \gamma_i$ and $\ln \delta_j$ is via ordinary least squares with origin and destination fixed effects (see e.g. Head and Mayer (2013)). We refer to this as the "traditional" estimator. However, this procedure relies only on the gravity structure of the trade model without imposing the general equilibrium conditions, in particular that trade is balanced. From part (i) of Theorem 2, however, we know that when trade costs are quasi-symmetric, we have that $\gamma_i K_i^A = \kappa \delta_i K_i^B$. This implies that equation (20) can be written as:

$$\ln X_{ij} = \ln K_{ij} + \ln K_i^A \gamma_i + \ln K_j^A \gamma_j - \ln \kappa + \varepsilon_{ij}. \tag{21}$$

Equation (21) says that the gravity regression respecting balanced trade should include a *single fixed effect for each country*:

$$\ln X_{ij} = \ln K_{ij} + z_i + z_j + \varepsilon_{ij}, \tag{22}$$

which we refer to as the "balanced trade gravity estimator." The origin and destination fixed effects can be identified (up to scale) from regression (22) as follows:

$$\hat{\gamma}_i = \frac{\exp\left(\hat{z}_i\right)}{K_i^A} \text{ and } \hat{\delta}_i = \frac{\exp\left(\hat{z}_i\right)}{K_i^B}.$$

Furthermore, if we suppose that for all $i, j \in S$, the variance of the idiosyncratic measurement error is constant, i.e. $E\left[\varepsilon_{ij}^2\right] = \sigma$, we can compare the asymptotic variance of the two estimators. A straightforward application of the central limit theorem yields an asymptotic variance of σ^2 for the traditional estimator, whereas the balanced trade gravity estimator has an asymptotic variance of $\frac{1}{2}\sigma^2$, i.e. the general equilibrium consistent gravity estimator is twice as precise (simply because there are only half the number of fixed effects to estimate).

7 Examples

Below, we study parametrized variations of the main gravity setup. In particular, we characterize the equilibrium in the perfect competition Armington (1969) setup considered by Anderson (1979), with and without intermediate inputs, and with labor mobility and spillovers in the setup considered by Allen and Arkolakis (2013). Table 1 summarizes how the universal gravity framework can be applied to many additional models as well. Finally, we provide examples of multiple equilibria for two simple geographies.

7.1 The Armington Model

In the Armington model, each location produces a differentiated variety (which is sold at marginal cost) and consumers have CES preferences with elasticity of substitution σ and where we denote by $P(i)$ the CES price index across all varieties. It is easy to solve for bilateral trade flows in this model and the value of trade between $i \in S$ and $j \in S$ is:

$$X_{ij} = \tau_{ij}^{1-\sigma} \left(\frac{w_i}{A_i} \right)^{1-\sigma} P_j^{\sigma-1} Y_j \tag{23}$$

where w_i is location's i wage, A_i is the location's productivity and the marginal production cost is $\frac{w_i}{A_i}$, τ_{ij} is the iceberg cost of delivering i's good in destination j, and Y_i is again its income.

Income is determined by labor market clearing:

$$Y_i = w_i L_i \tag{24}$$

where L_i is the population in location i. Note that the labor market clearing condition can be written as:

$$Y_i = \gamma_i^{\frac{1}{1-\sigma}} A_i L_i, \tag{25}$$

which is the factor market clearing condition (4) where $\alpha = \frac{1}{1-\sigma}$, $\beta = 0$, and $B_i = A_i L_i$.

In general, Theorem 1 implies that there is a unique solution of the system as long as $\sigma > 1$. From part (ii) of Theorem 2, in the case of quasi-symmetry the uniqueness region expands to $\sigma \geq 1/2$. Thus, it is possible that we have multiplicity when $\sigma < 1$ and trade costs are not quasi-symmetric, or in general, for $\sigma < 1/2$.

In addition, because wages are a function solely of the origin fixed effect and the productivity, i.e. $w_i = \gamma_i^{\frac{1}{1-\sigma}} A_i$, the existence of a unique set of origin and destination fixed effects implies the existence and uniqueness of the set of wages. Note that this also explains why we normalize $\sum_i w_i L_i = Y^W$, as it is straightforward to show that the equilibrium is homogeneous of degree zero in wages. (In contrast, since factor market clearing and balanced trade imply $\delta_i = \frac{w_i L_i}{\sum_j \tau_{ji}^{1-\sigma} A_j^{\sigma-1} w_j^{1-\sigma}}$, note that we cannot normalize δ_i, since its scale is pinned down by the normalization of γ_i).

7.1.1 Quasi-symmetric trade costs

When trade costs are quasi-symmetric, the equilibrium of the model can be further characterized. From part (i) of Theorem 2, we have $\gamma_i = \kappa\delta_i$, which implies:

$$\kappa K_i^A \left(\frac{w_i}{A_i}\right)^{1-\sigma} = K_i^B P_i^{\sigma-1} Y_i.$$

This can be rewritten to express welfare as a function of wages and model parameters, using the balanced budget condition $Y_i = w_i L_i$:

$$\kappa W_i^{\sigma-1} = \frac{K_i^B}{K_i^A} w_i^{2\sigma-1} L_i A_i^{1-\sigma}. \tag{26}$$

We can also provide a characterization of the welfare in the Armington model using the results of Theorem 1. In particular, we can re-write equation substitute equation (26) into the trade balance equation to derive a set of equations that characterize welfare across locations:

$$\kappa W_i^{\sigma\tilde{\sigma}} L_i^{\tilde{\sigma}} = \sum_j \tau_{ij}^{1-\sigma} A_i^{(\sigma-1)\tilde{\sigma}} A_j^{\sigma\tilde{\sigma}} L_j^{\tilde{\sigma}} W_j^{-(\sigma-1)\tilde{\sigma}}, \tag{27}$$

where $\tilde{\sigma} \equiv \frac{\sigma-1}{2\sigma-1}$. This equation reveals the fundamental forces acting upon the welfare of each country: As long as $\sigma \geq 1/2$ increased access into foreign markets (lower τ_{ij}) tends to increase the welfare of a worker i as is larger productivities anywhere in the world, either domestic because they increase the ability of i to export or for any destination country because they increase the demand for $i's$ goods. An increase in the domestic population in principle may decrease welfare but an increase in the foreign population typically expands domestic demand. Note too that equation (27) holds for both trade models (where labor is fixed) and economic geography models (where labor is mobile); in the former case, L_i is treated as exogenous parameter and W_i solved for; in the latter case L_i is treated as endogenous and W_i is assumed to be constant across locations.

7.1.2 Estimation of the trade elasticity in an Armington model

This section shows how one would estimate the trade elasticity in an Armington model using the methodology of Section 6.2. By combining the gravity structure of the Armington model with the factor market clearing condition $w_i L_i = Y_i$, it is straightforward to show that:

$$\frac{X_{ij}}{X_{ii}} = \tau_{ij}^{1-\sigma} \times \left(\left(\frac{Y_i}{A_i L_i} \right) / \left(\frac{Y_j}{A_j L_j} \right) \right)^{1-\sigma} \times \left(\frac{X_{jj}}{X_{ii}} \right),$$

which if proxies for A_i, L_i, and τ_{ij} are observed (i.e. $\ln A_i = \mu^A \ln Z_i^A + \varepsilon_i^A$, $\ln L_i = \mu^L \ln Z_i^L + \varepsilon_i^L$ and $\ln \tau_{ij} = \mu^\tau \ln Z_{ij}^\tau + \varepsilon_{ij}^\tau$, where the Z's are observables) yields the following estimating equation:

$$\ln \left(\frac{X_{ij}}{X_{jj}} \right) = (1-\sigma) \mu^\tau \ln Z_{ij}^\tau + (1-\sigma) \ln \left(\frac{Y_i}{Y_j} \right) + (1-\sigma) \mu^A \ln \left(\frac{Z_j^A}{Z_i^A} \right) + (1-\sigma) \mu^L \ln \left(\frac{Z_j^L}{Z_i^L} \right) + \varepsilon_{ij},$$

where $\varepsilon_{ij} \equiv (1-\sigma) \left(\varepsilon_{ij}^\tau + \varepsilon_j^L + \varepsilon_j^A - \varepsilon_i^L - \varepsilon_i^A \right)$. Hence, the elasticity of trade can be identified from the coefficient on $\ln \left(\frac{Y_i}{Y_j} \right)$, which then in turn allows us to identify $\left\{ \mu^\tau, \mu^A, \mu^L \right\}$.

7.1.3 Intermediate Inputs

Suppose now that we introduce intermediate inputs a la Eaton and Kortum (2002), so that the marginal production cost of is $\frac{w_i^\gamma P_i^{1-\gamma}}{A_i}$. Then the origin and destination fixed effects become:

$$\gamma_i \equiv \left(\frac{w_i^\gamma P_i^{1-\gamma}}{A_i} \right)^{1-\sigma}$$

$$\delta_i \equiv P_i^{\sigma-1} Y_i,$$

which allows us to write the factor market clearing condition in the form of equation (4):

$$Y_i = \gamma_i^{\frac{1}{1-\sigma\gamma}} \delta_i^{\frac{1-\gamma}{1-\sigma\gamma}} A_i^{\frac{\sigma-1}{\sigma\gamma-1}} L_i^{\frac{\gamma(\sigma-1)}{\sigma\gamma-1}},$$

so that $\alpha = \frac{1}{1-\sigma\gamma}$, $\beta = \frac{1-\gamma}{1-\sigma\gamma}$, and $B_i = A_i^{\frac{\sigma-1}{\sigma\gamma-1}} L_i^{\frac{\gamma(\sigma-1)}{\sigma\gamma-1}}$. From Theorem 1, there exists a unique origin and destination fixed effect if $\sigma \geq 1$. When trade costs are quasi-symmetric, from Theorem 2, there exists a unique origin and destination fixed effects if $\sigma \geq 1/2$.

With intermediate inputs, we can write the gravity equation as:

$$\frac{X_{ij}}{X_{ii}} = \tau_{ij}^{1-\sigma} \times \left(\frac{Y_i}{A_i^{\frac{1-\sigma}{1-\sigma\gamma}} L_i^{\frac{\gamma(1-\sigma)}{1-\gamma\sigma}}} / \frac{Y_j}{A_j^{\frac{1-\sigma}{1-\sigma\gamma}} L_j^{\frac{\gamma(1-\sigma)}{1-\gamma\sigma}}} \right)^{\frac{1-\sigma\gamma}{\gamma}} \times \left(\frac{X_{jj}}{X_{ii}} \right)^{\frac{1}{\gamma}},$$

which if $\ln A_i = \mu^A \ln Z_i^A + \varepsilon_i^A$, $\ln L_i = \mu^L \ln Z_i^L + \varepsilon_i^L$ and $\ln \tau_{ij} = \mu^\tau \ln Z_{ij}^\tau + \varepsilon_{ij}^\tau$, yields the following estimating equation:

$$\ln\left(\frac{X_{ij}}{X_{ii}}\right) = (1-\sigma)\,\mu^{\tau}\ln Z_{ij}^{\tau} + \left(\frac{1-\sigma\gamma}{\gamma}\right)\ln\left(\frac{Y_i}{Y_j}\right) + \frac{1}{\gamma}\ln\left(\frac{X_{jj}}{X_{ii}}\right) + \left(\frac{1-\sigma}{1-\sigma\gamma}\right)\mu^{A}\ln\frac{Z_j^A}{Z_i^A} + \left(\frac{\gamma(1-\sigma)}{1-\gamma\sigma}\right)\mu^{L}\ln\frac{Z_j^L}{Z_i^L} + \varepsilon_{ij},$$

where $\varepsilon_{ij} \equiv (1-\sigma)\,\varepsilon_{ij}^{\tau} + \left(\frac{1-\sigma}{1-\sigma\gamma}\right)\left(\varepsilon_j^A - \varepsilon_i^A\right) + \left(\frac{\gamma(1-\sigma)}{1-\gamma\sigma}\right)\left(\varepsilon_j^L - \varepsilon_i^L\right)$. Notice that the coefficients on $\ln\left(\frac{Y_i}{Y_j}\right)$ and $\ln\left(\frac{X_{jj}}{X_{ii}}\right)$ depend only on γ and σ, so that the two can be separately identified.

One can show directly that the Eaton and Kortum (2002) model of Ricardian comparative advantage maps to the Armington framework, and the two models generate the same predictions for trade flows as long as we set $\sigma - 1$ in the Armington model equal to the elasticity of the Frechet distribution, θ, in the Eaton and Kortum framework. The Frechet elasticity is restricted to be positive, which directly implies that there is always a unique equilibrium in the Eaton and Kortum framework.

7.2 Economic Geography Models

In this subsection, we examine the economic geography model considered by Allen and Arkolakis (2013), which is itself isomorphic to a number of seminal economic geography models (e.g. Roback (1982); Helpman (1998); Redding and Sturm (2008); Redding (2012)).

7.2.1 Setup

The model is based on an Armington model, labor mobility, and both productivity and amenity spillovers, and yields trade flows:

$$X_{ij} = \tau_{ij}^{1-\sigma} A_i^{\sigma-1} L_i^{a(\sigma-1)} w_i^{1-\sigma} P_j^{\sigma-1} Y_j,$$

where a is a parameter governing the strength of productivity spillovers. Labor is the only factor of production and markets are perfectly competitive, so income in location i can be written as $w_i L_i = Y_i$. Because workers are perfectly mobile, welfare is equalized across locations, which implies $\frac{w_i}{P_i} L_i^b = W$, where b is a parameter governing the strength of amenity spillovers. Finally, the total population in the world is set exogenously to $\bar{L}$, so that $\sum_i L_i = \bar{L}$.

Combining the gravity formulation with the factor market clearing condition and the welfare equalization condition yields the following expression for trade flows in the universal gravity notation:

$$X_{ij} = \lambda K_{ij} \gamma_i \delta_j,$$

where $K_{ij} \equiv \tau_{ij}^{1-\sigma}$, $\gamma_i \equiv A_i^{\sigma-1} w_i^{1-\sigma} L_i^{a(\sigma-1)}$, $\delta_j \equiv w_j^{\sigma} L_j^{1+(\sigma-1)b}$, and $\lambda = W^{1-\sigma}$. Substituting the expressions for γ_i and δ_i into the factor market clearing condition allows us to write income in the form of equation (4):

$$Y_i = \lambda B_i \gamma_i^{\alpha} \delta_i^{\beta},$$

where $B_i = A_i^{\frac{1+(\sigma-1)b-\sigma}{1+a\sigma+(\sigma-1)b}}$, $\alpha = \frac{1-b}{1+a\sigma+(\sigma-1)b}$ and $\beta = \frac{1+a}{1+a\sigma+(\sigma-1)b}$. Note that this framework is identical to the framework considered above, apart from the inclusion of an (endongeous) constant $\lambda > 0$, which is a monotonic transformation of welfare.

7.2.2 Existence and Uniqueness

Even though the geography models contain an unknown constant determined in equilibrium, we can obtain similar results to one we get in gravity models. As we did in trade models, the system is written as follows:

$$\begin{aligned} x_i &= \lambda^{1-\beta} \sum_j K_{i,j} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \\ y_i &= \lambda^{1-\beta} \sum_j K_{j,i} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}. \end{aligned}$$

where the welfare level can be pinned down by the aggregate labor market clearing condition:

$$\lambda = W^{1-\sigma} = \frac{\sum_i A_i^{-\sigma} B_i^{\frac{1}{1-\alpha-\beta}\frac{2\sigma-1}{\sigma-1}} x_i^{\frac{1}{\alpha+\beta-1}\left(\alpha+(1-\beta)\frac{\sigma}{\sigma-1}\right)} y_i^{\frac{1}{\alpha+\beta-1}\left(1-\alpha+\beta\frac{\sigma}{\sigma-1}\right)}}{\left(\bar{L}\right)^{1+a\sigma+(\sigma-1)b}}.$$

Proposition 4. *Consider a general equilibrium geography model. Then:*

i) The model has a positive solution and all possible solutions are positive.

ii) If α and β are both negative or α and β are both greater than 1, then the system has an unique solution up to scale.

Proof. The proof is very similar to that of Theorem 1, with some additional work to account for the presence of λ. See Appendix A.7. □

Applying Proposition 4, it can be shown that if $\sigma \geq 1$ and $\alpha, \beta \in [-1, 1]$, there exists a unique set of equilibrium origin and destination fixed effects if $a + \beta \leq 0$. This extends the range of uniqueness beyond those proven by Allen and Arkolakis (2013), who only consider the cases where there are no spillovers $a = b = 0$ (so that $\alpha = \beta = 1$) or where trade costs are symmetric.

Given the origin and destination fixed effects, we can solve for wages and population:

$$L_i = \left(W^{\sigma-1} A_i^{-\sigma} \delta_i \gamma_i^{\frac{\sigma}{\sigma-1}} \right)^{\frac{1}{1+a\sigma+(\sigma-1)b}}$$

$$w_i = \left(W^{\frac{a(\sigma-1)}{1+(\sigma-1)b}} A_i \gamma_i^{\frac{1}{1-\sigma}} \delta_i^{\frac{a}{1+(\sigma-1)b}} \right)^{\frac{1+(\sigma-1)b}{1+a\sigma+(\sigma-1)b}}$$

Hence, Proposition 4 shows there exists a unique set of wages, population, and welfare consistent with the economic geography model.

7.3 Examples of multiple equilibria

We now consider two trade models with simple geographies to provide examples of the possibility of multiple equilibria.

7.3.1 Armington Model with two countries

Consider an Armington model with two countries. Note that with two countries, all trade costs are quasi-symmetric. We first provide an analytical characterization of the relative welfare in the two regions and then provide an example of multiple equilibria.

To study the two countries case we define the Kernel $M_{ij} = \tau_{ij}^{1-\sigma} L_i^{-\tilde{\sigma}} A_i^{(\sigma-1)\tilde{\sigma}} A_j^{\sigma\tilde{\sigma}} L_j^{\tilde{\sigma}}$. Our analysis will proceed by discussing the equilibrium levels of welfare but wages can be computed using equation (26). Solving for the trade balance of the two countries and dividing we directly obtain a non-linear equation in the relative welfares of the two countries

$$M_{22} \left(\frac{W_1}{W_2} \right)^{\sigma\tilde{\sigma}} - M_{11} \left(\frac{W_1}{W_2} \right)^{(1-\sigma)\tilde{\sigma}} + M_{21} \left(\frac{W_1}{W_2} \right)^{\tilde{\sigma}} = M_{12} \tag{28}$$

It is easy to show that there is a unique solution of this nonlinear equation if $\sigma > 1/2$, which is consistent with Theorem 2. In addition, we can directly show using the implicit function theorem that the relative welfare of country 1 to country 2 increases with M_{11}, M_{12}, and decreases with M_{21}, M_{22}, i.e. country 1 becomes relatively richer if either its productivity increases favorably compared to country's 2 productivity or it faces a relative reduction in its trade costs to selling to 2 versus the corresponding trade costs of country 2 selling to country 1.

A particularly interesting region of parameters to study is that of low elasticities of substitution, $\sigma < 1/2$, a case where multiple equilibria may arise as we discussed in Section 3.

In our numerical solutions of the equilibrium we find that multiple equilibria arise for reasonable parameter configurations. In particular, with symmetric countries and $M_{21} = M_{22} = 0.5$ when we set $\sigma = .25$ we obtain three possible equilibria. The first equilibrium is the symmetric, where wages and welfare are equal across countries. The other two equilibria are asymmetric where, despite the ex-ante symmetry in fundamental parameters, wages and welfare are higher in one country to the expense of the other, the more so the lower the elasticity of substitution. The intuition for this result is that with goods that are strong complements a country might be able to have relatively higher wage and still be very successful in exporting because a large share of the other country's spending is allocated to imported goods. Thus, with strong complementarities and under certain parameter configurations there exists two more equilibria where both terms of trade -relative wages- and exporting is in favor of one country resulting in large welfare differentials.

7.3.2 Multiple equilibria arising from trade cost asymmetry

We now provide an example of multiple equilibria that arises from the asymmetry of trade costs, i.e. an equilibria that would be unique if trade costs were quasi-symmetric. Because trade costs are always quasi-symmetric when there are two countries, we consider an Armington model with three countries. Suppose that the elasticity of substitution is one half, i.e. $\sigma = \frac{1}{2}$ and the share of labor and intermediates are both $\frac{1}{2}$, i.e. $\gamma = \frac{1}{2}$. In the universal gravity framework (see Section 7.1.3) this implies that $\alpha = \frac{4}{3}$ and $\beta = \frac{2}{3}$. Because $\alpha + \beta = 2$, Theorem 2 implies that if K is quasi-symmetric, then the equilibria is unique.

Consider instead the following matrix of trade frictions, which is not quasi symmetric:

$$K = \begin{bmatrix} 1 & .0191 & .0116 \\ .1 & 1 & .1 \\ .1 & .1 & 1 \end{bmatrix}.$$

With these trade frictions, countries 2 and 3 have symmetric trade costs, but country 1 faces much lower costs exporting to countries 2 and 3 than importing from 2 and 3. In this case, it can be shown that there exist multiple equilibria: equilibria exist in which country 2 and country 3 have the same price index, country 2 has a higher price index than country 3, and vice versa. Loosely speaking, since goods are complements and there are intermediate inputs, it is possible for either country 2 or country 3 to be the larger producer, despite the small differences in trade frictions from country 1 to both countries. This example shows that sufficiently asymmetric trade costs may result in multiple equilibria when uniqueness is guaranteed for quasi-symmetric trade costs.

8 Conclusion

Despite the empirical importance of gravity trade models, little is known about their theoretical and empirical properties which hold universally, i.e. regardless of the micro-economic foundation of the model. In this paper, we have established a number of properties that hold for any gravity trade model where goods and factor markets clear and trade is balanced. In particular, we have shown that the equilibrium exists and provided conditions for when it is unique and efficient. We have also derived analytical expressions that allow one to determine how changing any bilateral trade friction affects trade flows and incomes worldwide. Finally, we have developed new methods of bringing the gravity model to the data which are more consistent with the general equilibrium conditions.

These universal properties of gravity trade models have highlighted the importance of the relationship between the gravity structure and the factor market clearing condition. Indeed, once this relationship is known – and we provide a method of estimating it – much of the insight yielded by gravity trade models (e.g. for determining counterfactual incomes and trade flows) can be accomplished without specifying any particular trade model. This paper hence contributes to a growing literature emphasizing that the micro-economic foundations are not particularly important for determining a trade model's macro-economic implications.

The major limitation with the set of gravity models considered in this paper is that in reality trade is not balanced. We see this limitation arising as a result of the static nature of such gravity models, and look forward to future research incorporating the gravity structure into dynamic models of trade.

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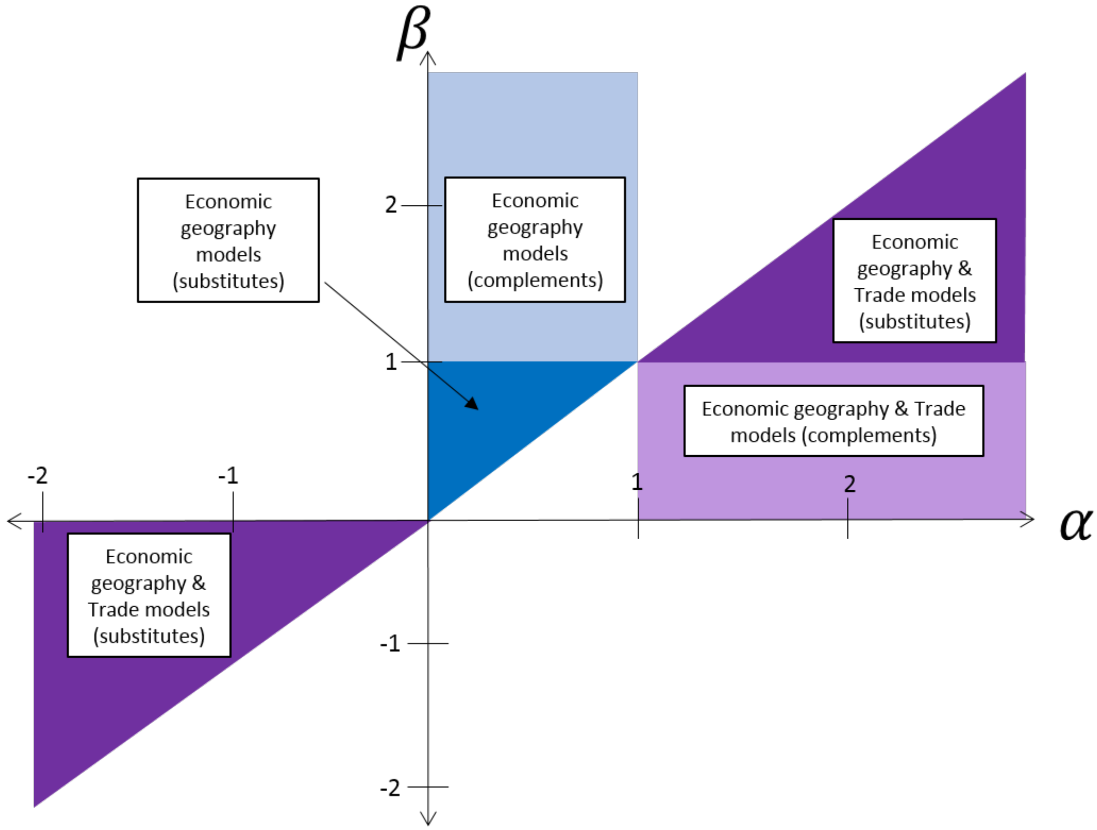
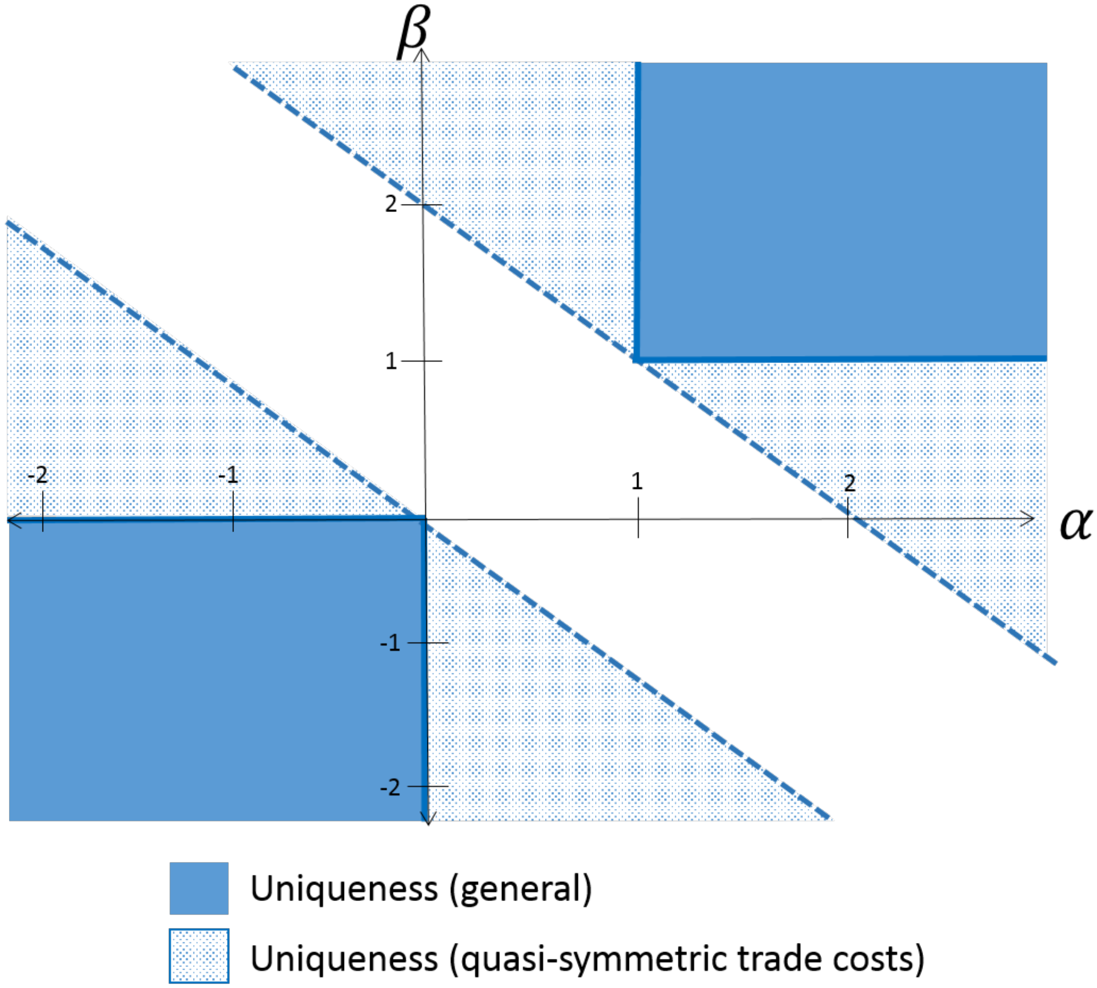
Tables and Figures

Figure 1: Gravity models and their locations in (α, β) space



Notes: This figure shows the regions in (α, β) space which correspond to different types of gravity trade models. Economic geography models where labor is mobile are represented in blue; regions which correspond to economic geography models and trade models (where labor is fixed) are in purple. Models in which goods are substitutes (e.g. the elasticity of substitution is greater than one) are represented by dark colors; models in which goods are complements (e.g. the elasticity of substitution is between zero and one) are represented by light colors.

Figure 2: Existence and Uniqueness



Notes: This figure shows the regions in (α, β) space for which the gravity equilibrium is unique generally and the when trade frictions are quasi-symmetric.

Table 1: Trade and Economic Geography Models

Model	Citation	Parameter	Symbol	Additional parameters	Symbol	Mapping to α	Mapping to β	Condition for uniqueness (general)	Condition for uniqueness (quasi-symmetry)
Trade Models (i.e. labor fixed)									
Armington	Armington (1969); Anderson (1979); Anderson and Van Wincoop (2003)	Elasticity of substitution	σ	N/A	N/A	$\alpha = \frac{1}{1-\sigma}$	$\beta = 0$	$\sigma \geq 1$	$\sigma \geq \frac{1}{2}$
Monopolistic competition, homogeneous firms	Krugman (1980)	Elasticity of substitution	σ	N/A	N/A	$\alpha = \frac{1}{1-\sigma}$	$\beta = 0$	$\sigma \geq 1$	$\sigma \geq \frac{1}{2}$
Perfect competition	Eaton and Kortum (2002)	Frechet shape parameter	θ	N/A	N/A	$\alpha = -\frac{1}{\theta}$	$\beta = 0$	$\theta \geq 0$	$\theta \geq 0$
Monopolistic competition, heterogeneous firms, exporting fixed costs in destination	Melitz (2003); Arkolakis, Demidova, Klenow, and Rodríguez-Clare (2008); Chaney (2008)	Pareto shape parameter	θ	N/A	N/A	$\alpha = -\frac{1}{\theta}$	$\beta = 0$	$\theta \geq 0$	$\theta \geq 0$
Monopolistic competition, heterogeneous firms, exporting fixed costs in origin	Melitz (2003); Di Giovanni and Levchenko (2009)	Elasticity of substitution	σ	Pareto shape parameter	θ	$\alpha = -\frac{\sigma-1}{(\theta-1)\sigma+1}$	$\beta = 0$	$\sigma \geq 1$, $\theta \geq \frac{\sigma-1}{\sigma}$	$\sigma \geq 1$, $\theta \geq \frac{1}{2}\frac{\sigma-1}{\sigma}$
Perfect competition with intermediate inputs	Eaton and Kortum (2002); Dekle, Eaton, and Kortum (2008); Caliendo and Parro (2010)	Elasticity of substitution	σ	Share of labor in production	γ	$\alpha = \frac{1}{1-\sigma\gamma}$	$\beta = \frac{1-\gamma}{1-\sigma\gamma}$	$\sigma \geq 1$	$\sigma \geq \frac{1}{2}$
Economic Geography Models (i.e. labor mobile)									
Armington (no spillovers)	Allen and Arkolakis (2013)	Elasticity of substitution	σ	N/A	N/A	$\alpha = 1$	$\beta = 1$	$\sigma \geq 0$	$\sigma \geq 0$
Free entry and non-tradable sector	Helpman (1998); Redding and Sturm (2008); Redding (2012)	Elasticity of substitution	σ	Share spent on non-tradable	γ	$\alpha = \frac{\sigma-1}{2\sigma-\sigma^2+\sigma^2\delta-1}$	$\beta = \frac{\sigma\delta}{2\sigma-\sigma^2+\sigma^2\delta-1}$	for $\sigma \geq 1$: $\frac{1+(\delta-1)\sigma}{\delta(\sigma-1)} \leq 0$	for $\sigma \geq 1$: $\frac{1+(\delta-1)\sigma}{\delta(\sigma-1)} \leq 0$
Armington (with spillovers)	Allen and Arkolakis (2013)	Elasticity of substitution	σ	Productivity spillover; amenity spillover	a; b	$\alpha = \frac{1-b}{1+a\sigma+(\sigma-1)b}$	$\beta = \frac{1+a}{1+a\sigma+(\sigma-1)b}$	for $\sigma \geq 1$ and $a, b \in [-1, 1]$: $\alpha + \beta \leq 0$	for $\sigma \geq 1$ and $a, b \in [-1, 1]$: $\alpha + \beta \leq 0$

Notes: This table includes a (non-exhaustive) list of trade and economic geography models that can be examined within the universal gravity framework. For brevity, the (sufficient) conditions for uniqueness in the economic geography models when goods are complements are not reported.

A Proofs

A.1 Proof of Theorem 1

The proof of Theorem 1 proceeds in four parts. In the first part, we consider a general mathematical structure, for which the general equilibrium gravity model (defined by equations (8) and (9)) is a special case. In the second part, we prove a lemma that will allow us to convert the general mathematical result to the particular case of the gravity trade model. In the third and fourth parts, we show how the general mathematical result can be applied to the trade model to prove existence and uniqueness, respectively.

A.1.1 The general case

Lemma 1. *Consider the following system of non-linear equations; for all $i \in S$,*

$$x_i = \frac{\sum_j F_{i,j} x_j^a y_j^b}{\sum_{i,j} F_{i,j} x_j^a y_j^b} \tag{29}$$

$$y_i = \frac{\sum_j H_{ij} x_j^c y_j^d}{\sum_{i,j} H_{ij} x_j^c y_j^d}, \tag{30}$$

for some $a, b, c, d \in \mathbb{R}$, $C_x, C_y \in \mathbb{R}_{++}$ and matrices F, H with all elements non-negative and the diagonal strictly positive (i.e. for all $i \in \{1, ..., N\}$, $F_i > 0$ and $H_i > 0$). Then the system has a positive solution $x, y \in \mathbb{R}_+^S$ and all its possible solutions are positive.

Proof. We use the following fixed point theorem to show the existence.

Theorem [Schauder's fixed point] Suppose that $D \subset V$, where V is a topological vector space. If a continuous function $f : D \to D$ satisfies the condition that $f(D)$ is a compact subset of D, then there exists $x \in D$ such that $f(x) = x$.

To apply the theorem, we have to find a proper subset D of R^{2S} such that D satisfies the condition in theorem 1.

Now consider the system (29)(30). We define the set Γ as

$$\Gamma = \left\{(x, y) \in \Delta\left(R^S\right) \times \Delta\left(R^S\right); m_x \leq x_i \leq M_x,\ m_y \leq y_i \leq M_y \quad \text{for all } i\right\},$$

and the following constants

$$\begin{aligned}
M_x &\triangleq \max_{i,j} \frac{F_{i,j}}{\sum_i F_{i,j}} \\
m_x &\triangleq \min_{i,j} \frac{F_{i,j}}{\sum_i F_{i,j}} \\
M_y &\triangleq \max_{i,j} \frac{H_{i,j}}{\sum_i H_{i,j}} \\
m_y &\triangleq \min_{i,j} \frac{H_{i,j}}{\sum_i H_{i,j}}.
\end{aligned}$$

In addition, we define the following operator for $d = (x, y) \in \Gamma$.

$$\begin{aligned}
Td &= T(x, y) \\
&= \left(\left(T^x(x, y) \right), \left(T^y(x, y) \right) \right),
\end{aligned}$$

where

$$\begin{aligned}
T_i^x(x, y) &= \frac{\sum_j F_{i,j} x_j^a y_j^b}{\sum_i \sum_j F_{i,j} x_j^a y_j^b} \\
&= \sum_j \frac{F_{i,j}}{\sum_i F_{i,j}} \frac{\sum_i F_{i,j} x_j^a y_j^b}{\sum_j \sum_i F_{i,j} x_j^a y_j^b}. \\
T_i^y(x, y) &= \frac{\sum_j H_{i,j} x_j^c y_j^d}{\sum_i \sum_j H_{i,j} x_j^c y_j^d} \\
&= \sum_j \frac{H_{i,j}}{\sum_i H_{i,j}} \frac{\sum_i H_{i,j} x_j^c y_j^d}{\sum_j \sum_i H_{i,j} x_j^c y_j^d}.
\end{aligned}$$

It is easy to show that

$$m_x \leq T_i^x(x, y) \leq M_x, m_y \leq T_i^x(x, y) \leq M_y$$

so that the operator T is from Γ to Γ, where Γ is compact.

To show that T is continuous, it suffices to show that T_i^x and T_i^y are continuous for all i. Since the range is compact, these functions are trivially continuous.

Since Schauder's fixed point theorem is applied for T, then there exits an solution to the system. Also by construction, any fixed points satisfy

$$\begin{aligned}
0 &\leq m_x \leq x_i \\
0 &\leq m_y \leq y_i
\end{aligned}$$

for all i. $\square$

A.1.2 Preliminary mathematical result

Lemma 2. *Suppose that* (x, y) *satisfies*

$$\begin{aligned} x_i &= \frac{\sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}}{\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}} \\ y_i &= \frac{\sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}}{\sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}}. \end{aligned}$$

Then we have

$$\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} = \sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}.$$

Proof. Note that

$$x_i = \lambda_x \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}},$$

where

$$\lambda_x = \sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}$$

Multiply both sides by $x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}$, which yields:

$$\begin{aligned} x_i \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) = \lambda_x \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) \iff \\ x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} = \lambda_x \sum_j K_{ij} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) \end{aligned}$$

Now sum over all i and rearrange to solve for λ_x:

$$\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} = \lambda_x \sum_i \sum_j K_{ij} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) \iff$$

$$\lambda_x = \frac{\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}}{\sum_i \sum_j K_{ij} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right)}$$

$$= \frac{\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}}{\sum_i \left[\sum_j K_{ij} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right) \right] \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right)}.$$

Now let us consider the second equilibrium condition:

$$y_i = \lambda_y \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}$$

where

$$\lambda_y = \sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}.$$

Multiply both sides by $x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}$:

$$y_i \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) = \lambda_y \sum_j K_{ji} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) \iff$$

$$x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} = \lambda_y \sum_j K_{ji} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right)$$

Now sum over all i and rearrange to solve for λ_y:

$$\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} = \lambda_y \sum_i \sum_j K_{ji} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) \iff$$

$$\lambda_y = \frac{\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}}{\sum_i \sum_j K_{ji} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right)}$$

$$= \frac{\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}}{\sum_i \left[\sum_j K_{ij} \left(x_j^{\frac{\alpha}{\beta+\alpha-1}} y_j^{\frac{1-\alpha}{\beta+\alpha-1}} B_j^{\frac{1}{1-\alpha-\beta}} \right) \right] \times \left(B_i^{\frac{1}{1-\alpha-\beta}} x_i^{\frac{1-\beta}{\alpha+\beta-1}} y_i^{\frac{\beta}{\alpha+\beta-1}} \right)}.$$

Comparing the expressions for λ_x and λ_y, we immediately have $\lambda_x = \lambda_y \equiv \lambda$. $\square$

A.1.3 Existence for trade models

We now consider the existence of a strictly positive solution to the general equilibrium gravity model defined by equations (8) and (9).

Proof. We apply Lemma 1 with

$$a = \frac{\alpha}{1-\alpha-\beta}, \qquad b = \frac{1-\alpha}{\alpha+\beta-1}$$
$$c = \frac{1-\beta}{\alpha+\beta-1}, \qquad d = \frac{\beta}{\alpha+\beta-1}$$
$$F_{i,j} = K_{ij} B_j^{\frac{1}{1-\alpha-\beta}}, \qquad H_{i,j} = K_{ji} B_j^{\frac{1}{1-\alpha-\beta}}.$$

Then there exits a solution to the system. Define t as follows

$$t = \left(\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right)^{\frac{1}{1-\frac{\alpha}{\alpha+\beta-1}}}.$$

From the lemma, we have

$$\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} = \sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} = t^{1-\frac{\alpha}{\alpha+\beta-1}} = t^{\frac{\beta-1}{\alpha+\beta-1}}.$$

Then if we show that (tx, y) is a solution to

$$\begin{aligned} (tx_i) &= \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \\ y_i &= \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}, \end{aligned}$$

then (tx, y) is a solution to a general equilibrium trade model.

To see this, note that

$$tx_i = t^{1-\frac{\alpha}{\alpha+\beta-1}} \frac{\sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}}{\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (x_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}} = \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}.$$

The equality holds by construction of t. Thus first equation is satisfied. To show second equation, it suffices to show

$$\sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} = 1.$$

This holds since

$$\sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} = t^{\frac{1-\beta}{\alpha+\beta-1}} \sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}$$
$$= \frac{\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}}{\sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}} = 1,$$

where the last line followed from Lemma 2.[9] □

A.1.4 Uniqueness for trade models

We now consider the uniqueness of the general equilibrium gravity model. We prove uniqueness by contradiction.

Proof. For Part ii), uniqueness, we make use of the same Proposition. Gravity models imply the following restrictions to the coefficients of equations (29) and (30):

$$a = \frac{\alpha}{\alpha+\beta-1}, b = \frac{1-\alpha}{\alpha+\beta-1}$$
$$c = a-1 = \frac{1-\beta}{\alpha+\beta-1}$$
$$d = b+1 = \frac{\beta}{\alpha+\beta-1}.$$

Suppose that there are two solutions $(x,y), (\widetilde{x},\widetilde{y})$ for the system. Also assume that there are no constants t such that

$$x = t\widetilde{x}. \tag{31}$$

Without loss of generality, we can assume that for all i,

$$\sum_j F_{i,j} = \sum_j H_{i,j} = 1.$$

Also we can take $(\widetilde{x},\widetilde{y}) = (1,1)$ since

$$1 = \sum_j F_{i,j} 1^a 1^b$$
$$1 = \sum_j H_{i,j} 1^c 1^d.$$

[9] If $\beta = 1$, then this last line is not true, since the equation for y is no longer dependent on x. In this case, however, existence and uniqueness follows immediately from Theorem 1 of Karlin and Nirenberg (1967), as the two integral equations can be treated as distinct from each other.

Define

$$\begin{aligned} m_x &= \min_i x_i \\ M_x &= \max_i x_i \\ m_y &= \min_i y_i \\ M_y &= \max_i y_i. \end{aligned}$$

From (31), $m_x\,(m_y)$ is strictly less than $M_x\,(M_y)$ respectively.

Then we can show that;

$$\begin{aligned} \max x_i &= M_x = \max \sum_j F_{i,j} x_j^a y_j^b \leq M_x^a m_y^b \\ \max y_i &= M_y = \max \sum_j H_{i,j} x_j^c y_j^d \leq m_x^c M_y^d \\ m_x = \min x_i &= \min \sum_j F_{i,j} x_j^a y_j^b \geq m_x^a M_y^b \\ m_y = \min y_i &= \min \sum_j H_{i,j} x_j^c y_j^d \geq M_x^c m_y^d. \end{aligned}$$

It is easy to show[10]

$$\begin{aligned} \left(\frac{M_x}{m_x}\right)^{1-a} \left(\frac{M_y}{m_y}\right)^{b} &< 1 \\ \left(\frac{M_x}{m_x}\right)^{c} \left(\frac{M_y}{m_y}\right)^{1-d} &< 1. \end{aligned}$$

[10]To obtain first equation, multiply first and third equation.

$$M_x \left(m_x^b M_y^b\right) \leq m_x \left(M_x^a m_y^b\right),$$

which is equivalent to

$$\left(\frac{M_x}{m_x}\right)^{1-a} \left(\frac{M_y}{m_y}\right)^{b} < 1.$$

For second equation, multiply second and fourth equation.

$$(M_y)\, M_x^c m_y^d \leq \left(m_x^c M_y^d\right) m_y,$$

which implies

$$\left(\frac{M_x}{m_x}\right)^{c} \left(\frac{M_y}{m_y}\right)^{1-d} \leq 1$$

Since $c = a - 1$, and $d = b + 1$,

$$\left(\frac{M_x}{m_x}\right)^{1-a}\left(\frac{M_y}{m_y}\right)^{b} \;<\; 1$$
$$\left(\frac{M_x}{m_x}\right)^{a-1}\left(\frac{M_y}{m_y}\right)^{-b} \;<\; 1.$$

Therefore the following holds.

$$\left(\frac{M_x}{m_x}\right)^{1-a}\left(\frac{M_y}{m_y}\right)^{b} < 1 < \left(\frac{M_x}{m_x}\right)^{1-a}\left(\frac{M_y}{m_y}\right)^{b},$$

which is a contradiction. □

A.2 Proof of Theorem 2

Proof. Part i) This relation comes from the balanced trade conditions and labor market clearing conditions.

$$\sum_i X_{i,j} = \sum X_{j,i},$$

which is equivalent to

$$\begin{aligned}
\frac{K_i^A \gamma_i}{K_i^B \delta_i} &= \frac{\sum_j \widetilde{K}_{i,j} K_j^A \gamma_j}{\sum_j \widetilde{K}_{i,j} K_j^B \delta_j} \\
&= \sum_j \frac{\widetilde{K}_{i,j} K_j^B \delta_j}{\sum_j \left(\widetilde{K}_{i,j} K_j^B \delta_j\right)} \times \frac{K_j^A \gamma_j}{K_j^B \delta_j}.
\end{aligned}$$

It is easy to show that

$$\frac{K_i^A \gamma_i}{K_i^B \delta_i} = 1$$

is a solution to the problem. From the Perron-Frobenius theorem, this solution is unique up to scale. Therefore for some κ, we have

$$\gamma_i K_i^A = \kappa \delta_i K_i^B. \tag{32}$$

Part ii) The relation (32) implies

$$y_i = \frac{\gamma_i}{\delta_i} x_i = \kappa \frac{K_i^B}{K_i^A} x_i.$$

Substituting this expression into (8), we get

$$x_i = \kappa^{\frac{1-\alpha}{\alpha+\beta-1}} \sum_j \widetilde{K}_{i,j} K_i^A K_j^B B_j^{\frac{1}{1-\alpha-\beta}} \left(\frac{K_i^B}{K_i^A}\right)^{\frac{1-\alpha}{\alpha+\beta-1}} x_j^{\frac{1}{\alpha+\beta-1}}. \tag{33}$$

Also if we substitute the same expression into (9), we get the exact same expression. Therefore one of the two equations is trivially satisfied. From Theorem 1 of Karlin and Nirenberg (1967), the system has an unique solution if $\left|\frac{1}{\alpha+\beta-1}\right| \leq 1$, which is equivalent to (10). □

A.3 Proof of Theorem 3

A.3.1 Part (i): The trade equilibrium maximizes world income.

The proof of part (i) of Theorem 3 proceeds in three parts. First, we provide the necessary and sufficient conditions of a maximization problem that will turn out to be a positive monotonic transformation of world income. Second, we show that the general equilibrium gravity model satisfies these conditions. Finally, we show that the object being maximized is indeed a positive monotonic transformation of world income.

Step #1: Necessary and sufficient conditions for a maximization problem

Proof. Assume that $\alpha + \beta \geq 2$. The case where $\alpha + \beta \leq 0$ is similarly proved. We proceed as follows. We first derive the FONCs for a maximization problem, which implies that from Kuhn-Tucker theorem, the solution satisfies the associated FONCs. Second we show that the general equilibrium trade model solves the FONCs.

Consider the following maximization problem:[11]

$$\max_{\{\widetilde{\gamma}_i\}_{i\in S}} \sum_{i\in S} \sum_{j\in S} \widetilde{K}_{ij} \widetilde{\gamma}_i \widetilde{\gamma}_j \text{ s.t. } \sum_{i\in S} B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \widetilde{\gamma}_i^{\alpha+\beta} \leq 1, \tag{34}$$

where $\widetilde{\gamma}_i = K_i^A \gamma_i$. This part of the proof shows (1) that there exists a unique $\tilde{\gamma}$ that satisfies the first order conditions of this maximization problem; (2) that there exists a solution to the maximization problem; and (3) that the maximization must be reached at an interior point, thereby showing that the unique $\tilde{\gamma}$ that satisfies the first order conditions is also the unique $\tilde{\gamma}$ that solves the maximization problem.

[11] For the case where $\alpha + \beta \leq 0$, the constraint becomes

$$\sum_{i\in S} B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \widetilde{\gamma}_i^{\alpha+\beta} \geq 1.$$

The associated Lagrangian is:

$$\mathcal{L} = \widetilde{\gamma}^T \widetilde{K} \widetilde{\gamma} + \lambda \left[1 - \sum_{i \in S} B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \widetilde{\gamma}_i^{\alpha+\beta} \right].$$

The first order conditions are:

$$2 \sum_j \widetilde{K}_{i,j} \widetilde{\gamma}_j = (\alpha+\beta) \lambda \left(B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \widetilde{\gamma}_i^{\alpha+\beta-1} \right).$$

The associated Lagrange multiplier λ is expressed as:

$$\lambda = \frac{2}{\alpha+\beta} \widetilde{\gamma}^T \widetilde{K} \widetilde{\gamma} > 0. \tag{35}$$

The strict inequality follows from the fact that $\sum_j \widetilde{K}_{i,j}$ is strictly positive and at the optimal γ^* the objective function is strictly positive.[12]

We now redefine the variables so as to allow us to apply Theorem 1 in Karlin and Nirenberg (1967):

$$\begin{aligned} F_{i,j} &= \frac{\widetilde{K}_{i,j}}{B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta}} \\ x_i &= \widetilde{\gamma}_i^{\alpha+\beta-1} \\ a &= \frac{1}{\alpha+\beta-1}. \end{aligned}$$

Then with these variables the FONCs are rewritten;

$$\sum_j F_{i,j} x_j^a = \lambda x_j.$$

As we did in the proof for Theorem 1, first we solve the following sub-problem.

$$x_j = \frac{\sum_j F_{i,j} x_j^a}{\sum_{i,j} F_{i,j} x_j^a}.$$

Theorem 1 in Karlin and Nirenberg (1967) assets that there exits a solution for any F and a. Furthermore if $|a| \leq 1$, which is assumed, a solution is unique. Consider $(tx_i)_i$ where $t = \left(\left(\sum_{i,j} \widetilde{K}_{i,j} x_i^a x_j^a \right)^{-1} \left(\sum_{i,j} F_{i,j} x_j^a \right) \right)^{\frac{1}{1+a}}$. It is easy to show that $(tx_i)_i$ satisfy;

[12] More formally we can choose a sufficiently small ε such that $\widetilde{\gamma}_i = \varepsilon$ satisfies the constraint and the attains a positive value for the objective function.

$$
\begin{aligned}
tx_j &= \frac{t^{1-a}}{\sum_{i,j} F_{i,j} x_j^a} \sum_j F_{i,j} (tx_j)^a \\
&= \frac{\left(t^{2a} \sum \widetilde{K}_{i,j} x_i^a x_j^a\right)^{-1} \sum_{i,j} F_{i,j} x_j^a}{\sum_{i,j} F_{i,j} x_j^a} \sum_j F_{i,j} (tx_j)^a \\
&= \left(\sum_{i,j} \widetilde{K}_{i,j} (tx_i)^a (tx_j)^a\right)^{-1} \sum_j F_{i,j} (tx_j)^a .
\end{aligned}
$$

Therefore $\widetilde{\gamma}_i = (tx_i)^a$ solves

$$
(\widetilde{\gamma}_i)^{\alpha+\beta-1} \left(\sum_{i,j} \widetilde{K}_{i,j} \widetilde{\gamma}_i \widetilde{\gamma}_j\right) = \sum_j F_{i,j} (\widetilde{\gamma}_j) .
$$

Substituting F into the equation, we get

$$
B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} (\widetilde{\gamma}_i)^{\alpha+\beta-1} \left(\sum_{i,j} \widetilde{K}_{i,j} \widetilde{\gamma}_i \widetilde{\gamma}_j\right) = \sum_j \widetilde{K}_{i,j} (\widetilde{\gamma}_j) .
$$

Set the Lagrange multiplier as follows.

$$
\lambda = \frac{2}{\alpha+\beta} \left(\sum_{i,j} \widetilde{K}_{i,j} \widetilde{\gamma}_i \widetilde{\gamma}_j\right) .
$$

These two equations tell us that $(\widetilde{\gamma}_i)_i$ solves the FONCs for the maximization problem.

It is easy to show that:

$$
\sum_i \left(B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \widetilde{\gamma}_i^{\alpha+\beta}\right) = 1.
$$

which is implies the constraint is satisfied. Therefore we have shown that if $\alpha + \beta \geq 2$ (or $\alpha + \beta \leq 0$), there exits a unique $\{\widetilde{\gamma}\}$ which solves the FONCs for the maximization problem (34).

It remains to show that solving the FONCs for the maximization problem are sufficient for finding the maximization. This requires (1) showing that there exists a maximum for the maximization problem (34); and (2) showing that the maximization does not occur at a boundary (where the FONCs may not hold).

That there exists a maximum for the maximization problem (34) follows from compactness of the constraints. Remember that the constraint set is given by

$$\sum_{i\in S} B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \widetilde{\gamma}_i^{\alpha+\beta} = 1$$

Then the constraint set is trivially closed, and bounded.[13]

We establish that the maximization does not occur at a boundary in the following lemma:

□

Lemma 3. *Any solution γ^* to the maximization problem (34) must be strictly positive, i.e. $\gamma_i^* > 0$ for all $i \in S$.*

Proof. Suppose not, i.e. there exists an $i \in S$ such that $\gamma_i^* = 0$. We show that this cannot be a maximum. Let $\varepsilon > 0$ and consider the alternative $\tilde{\gamma}$ where $\tilde{\gamma}_i = \varepsilon$ and $\tilde{\gamma}_j = \left(\left(\gamma_j^*\right)^{\alpha+\beta} - \frac{B_i\left(K_i^A\right)^{-\alpha}\left(K_i^B\right)^{-\beta}}{B_j\left(K_j^A\right)^{-\alpha}\left(K_j^B\right)^{-\beta}} \frac{1}{N-1}\varepsilon^{\alpha+\beta}\right)^{\frac{1}{\alpha+\beta}}$. Note that since the solution γ^* satisfies the constraint, i.e. $\sum_i \left(B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \left(\gamma_i^*\right)^{\alpha+\beta}\right) = 1$, so too does the alternative $\tilde{\gamma}$, i.e. $\sum_i \left(B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \left(\tilde{\gamma}_i\right)^{\alpha+\beta}\right) = 1$. Intuitively, we have chosen an alternative $\tilde{\gamma}$ where we increase γ_i^* by ε by taking an equal amount from all $j \neq i$. We can write the change on the objective function, $Z(\varepsilon)$ moving from γ^* to $\tilde{\gamma}$ as follows:

$$Z(\varepsilon) \equiv \sum_{i\in S}\sum_{j\in S} \widetilde{K}_{ij} K_i^A K_j^A \gamma_i^* \gamma_j^* - \sum_{i\in S}\sum_{j\in S} \widetilde{K}_{ij} K_i^A K_j^A \tilde{\gamma}_i \tilde{\gamma}_j \implies$$

$$\frac{Z(\varepsilon)}{2} = \varepsilon \left(\sum_{j\in S} \left(\widetilde{K}_{ij} K_i^A K_j^A \left(\left(\gamma_j^*\right)^{\alpha+\beta} - \frac{B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta}}{B_j \left(K_j^A\right)^{-\alpha} \left(K_j^B\right)^{-\beta}} \frac{1}{N-1} \varepsilon^{\alpha+\beta} \right)^{\frac{1}{\alpha+\beta}} \right) \right) -$$

$$\sum_{j\neq i}\sum_{k\neq i} (\widetilde{K}_{ij} K_i^A K_j^A \left(\gamma_j^* - \left(\left(\gamma_j^*\right)^{\alpha+\beta} - \frac{B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta}}{B_j \left(K_j^A\right)^{-\alpha} \left(K_j^B\right)^{-\beta}} \frac{1}{N-1} \varepsilon^{\alpha+\beta} \right)^{\frac{1}{\alpha+\beta}} \right)$$

$$\times \left(\gamma_k^* - \left(\left(\gamma_k^*\right)^{\alpha+\beta} - \frac{B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta}}{B_k \left(K_k^A\right)^{-\alpha} \left(K_k^B\right)^{-\beta}} \frac{1}{N-1} \varepsilon^{\alpha+\beta} \right)^{\frac{1}{\alpha+\beta}} \right)$$

[13] To get the bounds note that if $(\widetilde{\gamma}_i)$ satisfies the constraint, then

$$0 \leq \widetilde{\gamma}_i \leq \left\{ \sum_{i\in S} B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \right\}^{-1}.$$

Therefore the constraint set is included by

$$\left\{ (\widetilde{\gamma}_i) \in R^N; \sum_{i\in S} B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \widetilde{\gamma}_i^{\alpha+\beta} \leq 1 \right\} \subseteq \left[0, \left\{ \sum_{i\in S} B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \right\}\right]^N.$$

Taking the derivative of $Z(\varepsilon)$ evaluated at $\varepsilon = 0$ yields:

$$Z'(0) = \sum_{j \in S} \widetilde{K}_{ij} K_i^A K_j^A \gamma_j^* > 0,$$

i.e. the feasible deviation $\tilde{\gamma}$ increases the objective function. This is a contradiction since γ^* was supposed to maximize the objective function, thereby proving the lemma.[14]

The previous lemma shows that we can constrain our focus on interior solutions where the FONCs must be satisfied. We can also check the second order conditions locally to see that the maximization problem (34) is indeed a maximization. The Hessian of the Lagrangian is:

$$\begin{aligned} H(\widetilde{\gamma}) &= 2\widetilde{K} - \lambda(\alpha+\beta)(\alpha+\beta-1) \begin{bmatrix} C_1 \widetilde{\gamma}_1^{\alpha+\beta-2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_N \widetilde{\gamma}_N^{\alpha+\beta-2} \end{bmatrix} \\ &= 2\widetilde{K} - 2\widetilde{\gamma}^T \widetilde{K} \widetilde{\gamma} (\alpha+\beta-1) \begin{bmatrix} C_1 \widetilde{\gamma}_1^{\alpha+\beta-2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_N \widetilde{\gamma}_N^{\alpha+\beta-2} \end{bmatrix}. \end{aligned}$$

Note that:

$$\begin{aligned} &\widetilde{\gamma}^T H(\widetilde{\gamma}) \widetilde{\gamma} = 2\widetilde{\gamma}^T \widetilde{K} \widetilde{\gamma} - 2\widetilde{\gamma}^T \widetilde{K} \widetilde{\gamma} (\alpha+\beta-1) \widetilde{\gamma}^T \begin{bmatrix} C_1 \widetilde{\gamma}_1^{\alpha+\beta-2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_N \widetilde{\gamma}_N^{\alpha+\beta-2} \end{bmatrix} \widetilde{\gamma} \iff \\ &\widetilde{\gamma}^T H(\widetilde{\gamma}) \widetilde{\gamma} = 2\widetilde{\gamma}^T \widetilde{K} \widetilde{\gamma} (1 - (\alpha+\beta)) \leq 0, \end{aligned}$$

since $\alpha+\beta \geq 2$, i.e. the second order conditions in the direction of $\widetilde{\gamma}$ are negative, confirming that in this direction, the problem (34) is indeed a maximization problem.[15] When $\alpha + \beta \leq 0$, note that the second term of the Hessian is positive, which yields $\widetilde{\gamma}^T H(\widetilde{\gamma}) \widetilde{\gamma} = 2\widetilde{\gamma}^T \widetilde{K} \widetilde{\gamma} (\alpha+\beta)$, which is also negative.

□

[14]When $\alpha + \beta \leq 0$, this lemma is unnecessary since it is immediately obvious from the constraint $\sum_i \left(B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} (\gamma_i)^{\alpha+\beta} \right) = 1$ that $\gamma_i \neq 0$.

[15]If it was the case that the Hessian was negative definite, then we could have simply relied upon the second order sufficiency conditions. However, it turns out that the Hessian is indefinite, which is why we instead show (1) there is a unique solution satisfying the necessary first order conditions; and (2) the maximization cannot be achieved at a boundary where the first order conditions may not hold.

Step #2: The general equilibrium gravity model solves the maximization problem.

Proof. Now we show that a solution to a general equilibrium gravity model satisfies the FONCs for the maximization (and hence are the unique solution to the maximization problem (34)). If (γ_i, δ_i) is a solution to a general equilibrium gravity model, then (γ_i, δ_i) solves

$$\begin{aligned} K_i^B \delta_i &= \kappa K_i^A \gamma_i \\ \sum_i B_i (\gamma_i)^\alpha (\delta_i)^\beta &= Y^W \\ B_i (\gamma_i)^\alpha (\delta_i)^\beta &= \sum_j K_{i,j} \gamma_i \delta_j. \end{aligned}$$

Substituting first equation into the other conditions,

$$\kappa^\beta \sum_i B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \widetilde{\gamma}_i^{\alpha+\beta} = Y^W \tag{36}$$

$$\kappa^{\beta-1} \left(B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} (\widetilde{\gamma}_i)^{\alpha+\beta-1} \right)_i = (\widetilde{\gamma})^T \widetilde{K}. \tag{37}$$

Note here that κ is used for normalization, and take κ as follows:

$$\kappa = \left(Y^W\right)^{\frac{1}{\beta}}, \tag{38}$$

which is equivalent to

$$\sum_i B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \widetilde{\gamma}_i^{\alpha+\beta} = 1.$$

This equation is one of the FONCs for the maximization problem.

To obtain the other set of the conditions, multiply $\widetilde{\gamma}$ on (37) and sum over i, we get

$$\kappa^{\beta-1} \underbrace{\sum_i B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} (\widetilde{\gamma}_i)^{\alpha+\beta}}_{=1} = \kappa^{\beta-1} = (\widetilde{\gamma})^T \widetilde{K} (\widetilde{\gamma}). \tag{39}$$

Then (36) simplifies to:

$$\begin{aligned} (\widetilde{\gamma})^T \widetilde{K} &= \kappa^{\beta-1} \left(B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} (\widetilde{\gamma}_i)^{\alpha+\beta-1} \right)_i \\ &= (\widetilde{\gamma})^T \widetilde{K} (\widetilde{\gamma}) \left(B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} (\widetilde{\gamma}_i)^{\alpha+\beta-1} \right)_i, \end{aligned}$$

which is the other FONC of the maximization problem. We have hence shown that the general equilibrium gravity model is the unique solution to the maximization problem (34). ☐

Step #3: The maximization problem maximizes world income.

Proof. It remains to show that any solution to the maximization problem maximizes world income. Combining equations (38) and (39), we have that:

$$\left(\left(\widetilde{\gamma}\right)^T \widetilde{K}\left(\widetilde{\gamma}\right)\right)^{\frac{\beta}{\beta-1}} = Y^W,$$

i.e. the maximand of (34) is monotonically increasing in Y^W as long as $\frac{\beta}{\beta-1} > 0$. Note that we could have done the entire proof maximizing the δ instead of maximization γ; in this case, following the same methodology as above, it is straightforward to show that the maximand of (34) would be monotonically increasing in Y^W as long as $\frac{\alpha}{\alpha-1} > 0$. Note that the assumption $\alpha+\beta \geq 2$ (or $\alpha+\beta \leq 0$) implies either $\frac{\beta}{\beta-1} > 0$ or $\frac{\alpha}{\alpha-1} > 0$. Hence, since the solution of the general equilibrium gravity model, maximizes $\left(\widetilde{\gamma}\right)^T \widetilde{K}\left(\widetilde{\gamma}\right)$, it also maximizes Y^W, thereby completing the part (i) of the proof. ☐

A.3.2 Part (ii): The trade equilibrium maximizes a weighted average of world welfare.

Proof. From part (i) of Proposition 3, we know that if part (ii) of Theorem 1 is satisfied, the equilibrium of the general equilibrium gravity model maximizes world income subject to trade being balanced in all regions and a normalization on the factor market clearing condition. As a result, it is sufficient to show that world income can be written as a weighted average of welfare in each country, i.e.:

$$\sum_i \sum_j X_{ij} = c\left(\sum_i \omega_i \left(W_i^{\rho}\right)^{\eta}\right)^{\frac{1}{\eta}},$$

where $\rho > 0$, some constant η, weights $\omega_i > 0$ such that $\sum_i \omega_i = 1$ are functions solely of exogenous model parameters, and c is a constant common to all countries (which hence does not affect the maximization). Recall that $\lambda_{ii} \equiv \frac{X_{ii}}{Y_i}$, which combining the gravity structure of trade flows from equation (1) and the factor market clearing condition from equation (4) can be written as:

$$\lambda_{ii} = \frac{K_{ii}\gamma_i\delta_i}{B_i\gamma_i^{\alpha}\delta_i^{\beta}} \iff$$
$$\gamma_i = \left(\frac{\lambda_{ii}B_i}{K_{ii}}\right)^{\frac{1}{2-(\alpha+\beta)}} \left(\frac{\delta_i}{\gamma_i}\right)^{\frac{\beta-1}{2-(\alpha+\beta)}}.$$

Substituting this expression into the factor market clearing condition allows us to write the income in region i solely as a function of λ_{ii}, the ratio of the destination to origin fixed effect, and exogenous model parameters:

$$Y_i = B_i\gamma_i^{\alpha}\delta_i^{\beta} \iff$$
$$Y_i = B_i^{\frac{2}{2-(\alpha+\beta)}} \left(\frac{\lambda_{ii}}{K_{ii}}\right)^{\frac{\alpha+\beta}{2-(\alpha+\beta)}} \left(\frac{\delta_i}{\gamma_i}\right)^{\frac{\alpha-\beta}{\alpha+\beta-2}}$$
$$= \kappa^{\beta} B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \widetilde{\gamma}_i^{\alpha+\beta}.$$

Then the trade openness is expressed by

$$\begin{aligned}\lambda_{ii} &= \frac{K_{ii}\gamma_i\delta_i}{B_i\gamma_i^{\alpha}\delta_i^{\beta}} \\ &= \kappa \frac{\widetilde{\gamma}_i\widetilde{\gamma}_i}{\kappa^{\beta} B_i \left(K_i^A\right)^{\alpha} \left(K_i^B\right)^{\beta} \widetilde{\gamma}_i^{\alpha+\beta}} = \frac{\kappa^{1-\beta}}{B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \widetilde{\gamma}_i^{\alpha+\beta-2}}.\end{aligned}$$

Then the adjusted origin effect $\widetilde{\gamma}_i$ is a function of the trade openness for country i:

$$\widetilde{\gamma}_i = \left[\lambda_{ii}^{-1} \frac{\kappa^{1-\beta}}{B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta}}\right]^{\frac{1}{\alpha+\beta-2}}.$$

Substituting this expression into the constraint considered in the maximization problem,

$$\sum_i B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \widetilde{\gamma}_i^{\alpha+\beta} = 1.$$

allows us to write the auxiliary variableκ as:

$$\kappa^{\beta-1} = \left\{\sum_i \left[B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta}\right]^{1-\frac{\alpha+\beta}{\alpha+\beta-2}} \lambda_{ii}^{-\frac{\alpha+\beta}{\alpha+\beta-2}}\right\}^{\frac{\alpha+\beta-2}{\alpha+\beta}}.$$

From equation (35), we know κ is related with the maximization problem

$$\kappa^{\beta-1} = \widetilde{\gamma}\widetilde{K}\widetilde{\gamma} = \left\{ \sum_i \left[B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \right]^{1-\frac{\alpha+\beta}{\alpha+\beta-2}} \lambda_{ii}^{-\frac{\alpha+\beta}{\alpha+\beta-2}} \right\}^{\frac{\alpha+\beta-2}{\alpha+\beta}}.$$

Therefore the maximization problem attains the weighted welfare maximization since trade openness λ_{ii} is inversely related with nominal income

$$W_i = C_i^W \lambda_{ii}^{-\rho}.$$

To see this, we substitute welfare equation (12) into the trade openness for country i, λ_{ii}, yielding:

$$\begin{aligned} \widetilde{\gamma}\widetilde{K}\widetilde{\gamma} &= \left\{ \sum_i \left[B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \right]^{1-\frac{\alpha+\beta}{\alpha+\beta-2}} \left(\frac{C_i^W}{W_i} \right)^{-\frac{1}{\rho}\frac{\alpha+\beta}{\alpha+\beta-2}} \right\}^{\frac{\alpha+\beta-2}{\alpha+\beta}} \\ &= \left\{ \sum_i \omega_i \left(W_i^{\frac{1}{\rho}} \right)^{\eta} \right\}^{\frac{1}{\eta}}, \end{aligned}$$

where the weights are given by $\omega_i = \left[B_i \left(K_i^A\right)^{-\alpha} \left(K_i^B\right)^{-\beta} \right]^{\frac{-2}{\alpha+\beta-2}} \left(C_i^W\right)^{-\frac{1}{\rho}\frac{\alpha+\beta}{\alpha+\beta-2}}$, and $\eta = \frac{\alpha+\beta}{\alpha+\beta-2}$, which completes the proof. □

A.4 Proof of Proposition 1

Proof. First some notation is necessary. Define $y_i \equiv \ln \gamma_i$, $z_i \equiv \ln \delta_i$, $k_{ij} \equiv \ln K_{ij}$. Let $\vec{y} \equiv \{y_i\}$ and $\vec{z} \equiv \{z_i\}$ both be $N \times 1$ vectors and let $\vec{x} \equiv \{\vec{y}; \vec{z}\}$ be a $2N \times 1$ vector. Let $\vec{k} \equiv \{k_{ij}\}$ be a $N^2 \times 1$ vector. Now consider the function $f\left(\vec{x}; \vec{k}\right) : R^{2N} \times R^{N^2} \to R^{2N}$ given by:

$$f\left(\vec{x}; \vec{k}\right) = \begin{bmatrix} \left[B_i \left(\exp\{y_i\}\right)^{\alpha} \left(\exp\{z_i\}\right)^{\beta} - \sum_j \exp\{k_{i,j}\} \left(\exp\{y_i\}\right) \left(\exp\{z_j\}\right) \right]_i \\ \vdots \\ \left[B_i \left(\exp\{y_i\}\right)^{\alpha} \left(\exp\{z_i\}\right)^{\beta} - \sum_j \exp\{k_{j,i}\} \left(\exp\{y_j\}\right) \left(\exp\{z_i\}\right) \right]_i \end{bmatrix}.$$

In the general equilibrium trade model, we have:

$$f\left(\vec{x}; \vec{k}\right) = 0.$$

Full differentiation of the function hence yields:

$$f_{\vec{x}} D_{\vec{k}} \vec{x} + f_{\vec{k}} = 0, \tag{40}$$

where $f_{\vec{x}}$ is the $2N \times 2N$ matrix:

$$f_{\vec{x}}\left(\vec{x}; \vec{k}\right) = \begin{pmatrix} (\alpha - 1) Y & \beta Y - X \\ \alpha Y - X^T & (\beta - 1) Y \end{pmatrix},$$

where Y is a $N \times N$ diagonal matrix whose i^{th} diagonal is equal to Y_i and X is the $N \times N$ trade matrix.

Similarly, $f_{\vec{k}}$ is a $2N \times N^2$ matrix that depends only on trade flows:

$$f_{\vec{k}}\left(\vec{x}, \vec{k}\right) = - \begin{pmatrix} X_{11} & \cdots & X_{1N} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & X_{21} & \cdots & X_{2N} & \cdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & X_{N1} & \cdots & X_{NN} \\ X_{11} & \cdots & 0 & X_{21} & \cdots & 0 & \cdots & X_{N1} & \cdots & 0 \\ 0 & \ddots & \vdots & 0 & \ddots & \vdots & \cdots & 0 & \ddots & \vdots \\ 0 & \cdots & X_{1N} & 0 & \cdots & X_{2N} & \cdots & 0 & \cdots & X_{NN} \end{pmatrix}$$

If $f_{\vec{x}}$ was of full rank, we could immediately invert equation (40) (i.e. apply the implicit function theorem) to immediately yield:

$$D_{\vec{k}} \vec{x} = - \left(f_{\vec{x}}\right)^{-1} f_{\vec{k}}.$$

However, because Walras Law holds and we can without loss of generality apply a normalization to $\{\gamma_i\}$ and $\{\delta_i\}$ (see Online Appendix B.1 for details), we effectively have $N - 1$ equations and $N - 1$ unknowns, i.e. matrix $f_{\vec{x}}$ is of rank $2N - 1$. Hence, there exists an infinite number of solutions to equation (40), each corresponding to a different normalization. To find the solution that corresponds to our choice of world income as the numeraire, note that from equation (5):

$$\sum_l B_l \gamma_l^{\alpha} \delta_l^{\beta} = Y^W \implies$$

$$\sum_l Y_l \left(\alpha \frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} + \beta \frac{\partial \ln \delta_l}{\partial \ln K_{ij}}\right) = 0. \tag{41}$$

We claim that if $\frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} = X_{ij} \times \left(A_{l,i} + A_{N+l,j}\right) - c$ and $\frac{\partial \ln \delta_l}{\partial \ln K_{ij}} = X_{ij} \times \left(A_{N+l,i} + A_{l,j}\right) - c$, where $c \equiv \frac{1}{Y^W(\alpha+\beta)} X_{ij} \sum_l Y_l \left(\alpha \left(A_{l,i} + A_{N+l,j}\right) + \beta \left(A_{N+l,i} + A_{l,j}\right)\right)$, then $\frac{\partial \ln \gamma_l}{\partial \ln K_{ij}}$ and $\frac{\partial \ln \delta_l}{\partial \ln K_{ij}}$

solve equations (40) and (41). It is straightforward to see that our assumed solution ensures equation (40) holds, as the generalized inverse is a means of choosing from one of the infinitely many solutions; see James (1978). It remains to scale the set of elasticities appropriately to ensure that our normalization holds as well. Given our definition of the scalar c, it is straightforward to verify that equation (41) holds:

$$\begin{aligned}
\sum_l Y_l \left(\alpha \frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} + \beta \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} \right) &= \sum_l Y_l (\alpha \left(X_{ij} \times \left(A_{l,i} + A_{N+l,j} \right) - c \right) + \\
&\quad + \beta \left(X_{ij} \times \left(A_{N+l,i} + A_{l,j} \right) - c \right)) \\
&= X_{ij} \sum_l Y_l \left(\alpha \left(X_{ij} \times \left(A_{l,i} + A_{N+l,j} \right) \right) + \beta \left(X_{ij} \times \left(A_{N+l,i} + A_{l,j} \right) \right) \right) \\
&\quad - c \left(\alpha + \beta \right) \sum_l Y_l \\
&= X_{ij} \sum_l Y_l \left(\alpha \left(X_{ij} \times \left(A_{l,i} + A_{N+l,j} \right) \right) + \beta \left(X_{ij} \times \left(A_{N+l,i} + A_{l,j} \right) \right) \right) - \\
&\quad \left(\frac{1}{Y^W (\alpha + \beta)} X_{ij} \sum_l Y_l \left(\alpha \left(A_{l,i} + A_{N+l,j} \right) + \beta \left(A_{N+l,i} + A_{l,j} \right) \right) \right) \left(\alpha + \beta \right) Y^W \\
&= 0,
\end{aligned}$$

i.e. equation (41) also holds. More generally, different choices of c correspond to different normalizations. A particularly simple example is if we choose the normalization $\gamma_1 = 1$. Since this implies that $\frac{\partial \ln \gamma_1}{\partial \ln K_{ij}} = 0$, $c = X_{ij} \times (A_{1,i} + A_{N+1,j})$. In this case, however, an alternative procedure is even simpler: the elasticities for all $i > 1$ can be calculated directly by inverting the $(2N-1) \times (2N-1)$ matrix generated by removing the first row and first column of $f_{\vec{x}}$. □

A.5 Proof of Proposition 3

Proof. From the gravity equation (1) we have:

$$\begin{aligned}
X_{ij} &= K_{ij} \gamma_i \delta_j \iff \\
K_{ij} &= \frac{X_{ij}}{\gamma_i \delta_j}
\end{aligned} \tag{42}$$

Combining factor market clearing (4) with goods market clearing yields:

$$\begin{aligned}
B_i \gamma_i^\alpha \delta_i^\beta &= \sum_j X_{ij} \iff \\
\gamma_i^\alpha \delta_i^\beta &= \frac{\sum_j X_{ij}}{B_i}.
\end{aligned} \tag{43}$$

The gravity equation (1) yields the following relationship between origin and destination fixed effects:

$$\begin{aligned} X_{ii} &= K_{ii}\gamma_i\delta_i \iff \\ \delta_i &= \frac{X_{ii}}{K_{ii}\gamma_i}. \end{aligned} \tag{44}$$

Combining equations (43) and (44) to solve for γ_i and δ_i yields:

$$\gamma_i = \left(\frac{\sum_j X_{ij}}{B_i}\right)^{\frac{1}{\alpha-\beta}} \left(\frac{X_{ii}}{K_{ii}}\right)^{\frac{\beta}{\beta-\alpha}} \text{ and } \delta_i = \left(\frac{X_{ii}}{K_{ii}}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{\sum_j X_{ij}}{B_i}\right)^{\frac{1}{\beta-\alpha}},$$

which substituting into equation (42) yields an expression for trade frictions K_{ij} that depends only on observed model parameters and trade flows

$$K_{ij} = X_{ij} \times \left(\frac{\sum_k X_{jk}}{\sum_k X_{ik}} \times \frac{B_i}{B_j} \times \frac{X_{ii}^{\beta}}{X_{jj}^{\alpha}} \times \frac{K_{jj}^{\alpha}}{K_{ii}^{\beta}}\right)^{\frac{1}{\alpha-\beta}},$$

thereby proving the claim. □

A.6 Proof of Proposition 2

Proof. We want to rewrite the equilibrium conditions in changes by defining $(\hat{x}_i) = x_i'/x_i$. Starting from (6) we have

$$\begin{aligned} \hat{\gamma}_i^{\alpha}\hat{\delta}_i^{\beta} &= \sum_j \frac{K_{ij}'\gamma_i'\delta_j'}{\sum_j K_{ij}\gamma_i\delta_j} \implies \\ \hat{\gamma}_i^{\alpha}\hat{\delta}_i^{\beta} &= \sum_j \pi_{ij}\hat{K}_{ij}\hat{\gamma}_i\hat{\delta}_j \implies \\ \hat{\gamma}_i^{\alpha-1}\hat{\delta}_i^{\beta} &= \sum_j \pi_{ij}\hat{K}_{ij}\hat{\delta}_j \end{aligned}$$

where $\pi_{ij} = X_{ij}/\sum_j X_{ij}$ represents the exporting shares. Similarly we can rewrite the second equilibrium condition, Equation (7), in changes as

$$\begin{aligned}
\hat{\gamma}_i^{\alpha}\hat{\delta}_i^{\beta} &= \frac{\sum_j K'_{ji}\gamma'_j\delta'_i}{\sum_j K_{ji}\gamma_j\delta_i} \implies \\
\hat{\gamma}_i^{\alpha}\hat{\delta}_i^{\beta} &= \sum_j \lambda_{ij}\hat{K}_{ji}\hat{\gamma}_j\hat{\delta}_i \implies \\
\hat{\gamma}_i^{\alpha}\hat{\delta}_i^{\beta-1} &= \sum_j \lambda_{ij}\hat{K}_{ji}\hat{\gamma}_j
\end{aligned}$$

where $\lambda_{ij} = X_{ij}/\sum_i X_{ij}$ represents the import shares. This system of equations in changes is the same as the system of equations in levels. As long as λ_{ij}, π_{ij} are the same and α,β are the same all the gravity models give the same changes in γ_i, δ_j for a given change in K_{ij}.

□

A.7 Proof of Proposition 4

The basic idea is the same as the proof of Theorem 1; see Appendix A.1.

Proof. Consider the following subproblem.

$$\begin{aligned}
x_i &= \frac{\sum_j K_{ij}B_j^{\frac{1}{1-\alpha-\beta}}x_j^{\frac{\alpha}{\alpha+\beta-1}}y_j^{\frac{1-\alpha}{\alpha+\beta-1}}}{\sum_{i,j} K_{ij}B_j^{\frac{1}{1-\alpha-\beta}}x_j^{\frac{\alpha}{\alpha+\beta-1}}y_j^{\frac{1-\alpha}{\alpha+\beta-1}}} \\
y_i &= \frac{\sum_j K_{ji}B_j^{\frac{1}{1-\alpha-\beta}}x_j^{\frac{1-\beta}{\alpha+\beta-1}}y_j^{\frac{\beta}{\alpha+\beta-1}}}{\sum_{i,j} K_{ji}B_j^{\frac{1}{1-\alpha-\beta}}x_j^{\frac{1-\beta}{\alpha+\beta-1}}y_j^{\frac{\beta}{\alpha+\beta-1}}}.
\end{aligned}$$

From the proof of Theorem 1, there exists (x, y) satisfies these two equations. Now consider (tx, y).

$$\begin{aligned}
tx_i &= t\frac{\sum_j K_{ij}B_j^{\frac{1}{1-\alpha-\beta}}x_j^{\frac{\alpha}{\alpha+\beta-1}}y_j^{\frac{1-\alpha}{\alpha+\beta-1}}}{\sum_{i,j} K_{ij}B_j^{\frac{1}{1-\alpha-\beta}}x_j^{\frac{\alpha}{\alpha+\beta-1}}y_j^{\frac{1-\alpha}{\alpha+\beta-1}}} \\
&= \frac{t^{1-\frac{\alpha}{\alpha+\beta-1}}}{\sum_{i,j} K_{ij}B_j^{\frac{1}{1-\alpha-\beta}}x_j^{\frac{\alpha}{\alpha+\beta-1}}y_j^{\frac{1-\alpha}{\alpha+\beta-1}}}\sum_j K_{ij}B_j^{\frac{1}{1-\alpha-\beta}}\left(tx_j\right)^{\frac{\alpha}{\alpha+\beta-1}}y_j^{\frac{1-\alpha}{\alpha+\beta-1}}.
\end{aligned}$$

Geography models require that

$$\frac{t^{1-\frac{\alpha}{\alpha+\beta-1}}}{\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}} = \lambda^{1-\beta} = \left(\frac{\sum_i A_i^{-\sigma} B_i^{\frac{1}{1-\alpha-\beta}\frac{2\sigma-1}{\sigma-1}} (tx_i)^{\frac{1}{\alpha+\beta-1}\left(\alpha+(1-\beta)\frac{\sigma}{\sigma-1}\right)} y_i^{\frac{1}{\alpha+\beta-1}\left(1-\alpha+\beta\frac{\sigma}{\sigma-1}\right)}}{\left(\bar{L}\right)^{1+a\sigma+(\sigma-1)b}} \right)^{1-\beta}$$

which implies

$$t = \left[\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \left(\frac{\sum_i A_i^{-\sigma} B_i^{\frac{1}{1-\alpha-\beta}\frac{2\sigma-1}{\sigma-1}} x_i^{\frac{1}{\alpha+\beta-1}\left(\alpha+(1-\beta)\frac{\sigma}{\sigma-1}\right)} y_i^{\frac{1}{\alpha+\beta-1}\left(1-\alpha+\beta\frac{\sigma}{\sigma-1}\right)}}{\left(\bar{L}\right)^{1+a\sigma+(\sigma-1)b}} \right)^{1-\beta} \right]^{\frac{\alpha+\beta-1}{(\beta-1)\left(\frac{2\sigma-1}{\sigma-1}\right.}}$$

Then the associated welfare under (tx, y) satisfies

$$W^{1-\sigma} = \frac{\sum_i A_i^{-\sigma} B_i^{\frac{1}{1-\alpha-\beta}\frac{2\sigma-1}{\sigma-1}} x_i^{\frac{1}{\alpha+\beta-1}\left(\alpha+(1-\beta)\frac{\sigma}{\sigma-1}\right)} y_i^{\frac{1}{\alpha+\beta-1}\left(1-\alpha+\beta\frac{\sigma}{\sigma-1}\right)}}{\left(\bar{L}\right)^{1+a\sigma+(\sigma-1)b}},$$

and

$$\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} = \lambda^{1-\beta}.$$

Therefore

$$tx_i = \lambda^{1-\beta} \sum_j K_{i,j} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}.$$

From Lemma 2, we get

$$y_i = \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}},$$

which implies that (tx, y) solves the geography model.

Repeating the exact same argument, we can show that under the same conditions, the solutions are unique up to scale. □

B Online Appendix (not for publication)

This Online Appendix provides some additional theoretical results referenced in the paper.

B.1 Normalization

Without loss of generality we can normalize the world income.

Proposition 5. *Suppose that* (γ, δ) *solves the non-linear system. Denote the associated* (x, y). *Then* $\left(t\gamma, t^{-\frac{1-\alpha}{1-\beta}}\delta\right)$ *induces* $\left(t^{-\frac{1-\alpha}{1-\beta}}x, ty\right)$*, which again solves the non-linear equation. The world income* Y^W *under* $\left(t\gamma, t^{-\frac{1-\alpha}{1-\beta}}\delta\right)$ *is* $t^{\frac{\alpha-\beta}{1-\beta}}$. *In particular if* $t = (Y^w)^{-\frac{1-\beta}{\alpha-\beta}}$, *then* $Y^W = 1$.

Proof. Take $(t\gamma, s\delta)$, where

$$s = t^{-\frac{1-\alpha}{1-\beta}}.$$

Denote the associated $(x(t,s), y(t,s))$ .Then

$$\begin{aligned}
x(t,s) &= t^{\alpha-1}s^{\beta}x \\
&= t^{\alpha-1}t^{-\beta\frac{1-\alpha}{1-\beta}}x \\
&= t^{\alpha-1-\beta\frac{1-\alpha}{1-\beta}}x \\
&= t^{-\frac{1-\alpha}{1-\beta}}x \\
y(t,s) &= t^{\alpha}s^{\beta-1}y. \\
&= ty.
\end{aligned}$$

It is easy to show

$$\begin{aligned}
x_i(t,s) &= t^{\alpha-1}s^{\beta}x_i = \sum_j K_{ij}B_j^{\frac{1}{1-\alpha-\beta}}\left(t^{\alpha-1}s^{\beta}x_j\right)^{\frac{\alpha}{\alpha+\beta-1}}\left(t^{\alpha}s^{\beta-1}y_j\right)^{\frac{1-\alpha}{\alpha+\beta-1}} \\
&= \sum_j K_{ij}B_j^{\frac{1}{1-\alpha-\beta}}\left(x_j(t,s)\right)^{\frac{\alpha}{\alpha+\beta-1}}\left(x_j(t,s)\right)^{\frac{1-\alpha}{\alpha+\beta-1}} \\
y_i(t,s) &= \sum_j K_{ji}B_j^{\frac{1}{1-\alpha-\beta}}\left(t^{\alpha-1}s^{\beta}x_j\right)^{\frac{1-\beta}{\alpha+\beta-1}}\left(t^{\alpha}s^{\beta-1}y_j\right)^{\frac{\beta}{\alpha+\beta-1}} \\
&= \sum_j K_{ji}B_j^{\frac{1}{1-\alpha-\beta}}\left(x_j(t,s)\right)^{\frac{1-\beta}{\alpha+\beta-1}}\left(y_j(t,s)\right)^{\frac{\beta}{\alpha+\beta-1}}.
\end{aligned}$$

Thus a solution to the The world income induced by $\left(t\gamma, t^{-\frac{1-\alpha}{1-\beta}}\delta\right)$ is

$$\begin{aligned}\sum_i B_i \left(t\gamma_i\right)^{\alpha} \left(t^{-\frac{1-\alpha}{1-\beta}}\delta_i\right)^{\beta} &= t^{\alpha-\frac{1-\alpha}{1-\beta}\beta} \sum_i B_i \gamma_i^{\alpha} \delta_i^{\beta} \\ &= t^{\frac{\alpha-\beta}{1-\beta}} Y^w.\end{aligned}$$

In particular if we take $t^{-\frac{\alpha-\beta}{1-\beta}} = Y^w$, then the world income is normalized to 1. □

B.2 Walras law

In the previous section, we showed that without loss of generality, we can normalize the system of equations so that world income is equal to an arbitrary constant. In this section, we show that Walras law holds, i.e. if all equilibrium equations but one hold with equality, then the remaining one holds with equality as well. The two facts together imply that the equilibrium is really defined by $2N-1$ equations and $2N-1$ unknowns.

To see this, define $\gamma \equiv \{\gamma_i\}$, $\delta \equiv \{\delta_i\}$ and $x \equiv \{\gamma; \delta\}$, where x is a $2N \times 1$ vector. Consider the function $f(x) : R^{2N} \to R^{2N}$ given by:

$$f(x) = \begin{bmatrix} \left[B_i \gamma_i^{\alpha-1} \delta_i^{\beta} - \sum_j K_{ij} \delta_j\right]_i \\ \vdots \\ \left[\sum_j K_{ji} \gamma_j - B_i \gamma_i^{\alpha} \delta_i^{\beta-1}\right]_i \end{bmatrix}.$$

Note that the general equilibrium trade model is in equilibrium if $f(x) = 0$. Walras law can be written as:

$$f(x) \cdot x = 0.$$

To see this is the case, note that:

$$\begin{aligned} f(x) \cdot x = 0 &\iff \\ \sum_i \left(B_i \gamma_i^{\alpha-1} \delta_i^{\beta} - \sum_j K_{ij} \delta_i\right) \times \gamma_i + \sum_i \left(\sum_j K_{ij} \gamma_i - B_i \gamma_i^{\alpha} \delta_i^{\beta-1}\right) \times \delta_i = 0 &\iff \\ \sum_i B_i \gamma_i^{\alpha} \delta_i^{\beta} - \sum_i \sum_j K_{ij} \gamma_i \delta_i + \sum_i \sum_j K_{ji} \gamma_j \delta_i - \sum_i B_i \gamma_i^{\alpha} \delta_i^{\beta} = 0 &\iff \\ 0 = 0. & \end{aligned}$$

Hence, Walras law holds.

B.3 Existence and Uniqueness using Gross Substitutes Methodology (a la Alvarez and Lucas (2007))

We will illustrate the application of the gross-substitute property to prove uniqueness equilibrium in an excess demand system. This is a necessary step in the proof of Alvarez and Lucas (2007) but it is not sufficient, as a number of other properties need to be proved for an equation to be an excess demand system, as we discuss below.

Because of the complexity of the system that we analyze we cannot apply the gross-substitutes property directly to equations (6) and (7).

$$B_i \gamma_i^{\alpha-1} \delta_i^{\beta} = \sum_j K_{ij} \delta_j \tag{45}$$

Combining gravity (1) with balanced trade (3) and the generalized labor marking clearing condition (4) yields:

$$B_i \gamma_i^{\alpha} \delta_i^{\beta-1} = \sum_j K_{ji} \gamma_j \tag{46}$$

In order to find the equation that can be used to prove, we need to eliminate one variable. Use (7) to express δ_i as

$$\delta_i = \left(\frac{\sum_{s \in S} \gamma_s K_{si}}{B_i \gamma_i^{\alpha}} \right)^{\frac{1}{\beta-1}} \tag{47}$$

into equation (6), we obtain

$$\begin{aligned} B_i \gamma_i^{\alpha} \left(\frac{\sum_{s \in S} \gamma_s K_{si}}{B_i \gamma_i^{\alpha}} \right)^{\frac{\beta}{\beta-1}} &= \sum_{j \in S} \gamma_i \left(\frac{\sum_{s \in S} \gamma_s K_{sj}}{B_j \gamma_j^{\alpha}} \right)^{\frac{1}{\beta-1}} K_{ij} \iff \\ B_i^{\frac{1}{1-\beta}} \gamma_i^{\frac{\alpha}{1-\beta}-1} \left(\sum_{s \in S} \gamma_s K_{si} \right)^{\frac{\beta}{\beta-1}} &= \sum_{j \in S} \left(\frac{\sum_{s \in S} \gamma_s K_{sj}}{B_j \gamma_j^{\alpha}} \right)^{\frac{1}{\beta-1}} K_{ij} \end{aligned} \tag{48}$$

We define the corresponding excess demand function might be

$$Z_i(\gamma) = \frac{1}{\gamma_i} \left[B_i^{\frac{1}{\beta-1}} \gamma_i^{\frac{\alpha+\beta-1}{1-\beta}} \left(\sum_{s \in S} \gamma_s K_{si} \right)^{\frac{\beta}{\beta-1}} - \sum_{j' \in S} \left(\frac{\sum_{s \in S} \gamma_s K_{sj'}}{B_{j'} \gamma_{j'}^{\alpha}} \right)^{\frac{1}{\beta-1}} K_{ij'} \right]$$

This system written as such needs to satisfy 5 properties to be an excess demand system and the gross substitute property to establish existence and uniqueness (see Propositions 17.B.2, 17.C.1 and 17.F.3 of Mas-Colell, Whinston, and Green (1995)). The six conditions are:

1. $Z(\gamma)$ is continuous for $\gamma \in \left(\Delta\left(R_{+}^{N}\right)\right)^{o}$
2. $Z(\gamma)$ is homogenous of degree zero.
3. $Z(\gamma) \cdot \gamma = 0$ (Walras' Law).
4. There exists a $k > 0$ such that $Z_j(\gamma) > -k$ for all j.
5. If there exists a sequence $w^m \rightarrow w^0$, where $w^0 \neq 0$ and $w_i^0 = 0$ for some i, then it must be that:

$$max_j\{Z_j(w^m)\} \rightarrow \infty \tag{49}$$

and the gross-substitute property:

6. Gross substitutes property: $\frac{\partial Z(w_j)}{\partial w_k} > 0$ for all $j \neq k$.

Properties 1-3 are trivial by the way we define the system. Properties 4 and 5 are challenging and may require an analysis case-by-case which restrict further the set of parameters that uniqueness applies. We thus only discuss the region where gross-substitutes applies. To consider this system as an excess demand system and apply the tools originally developed in Alvarez and Lucas (2007), we need to differentiate the expression above. We only use the bracketed term without loss of generality. We have:

$$\begin{aligned}
\frac{\partial Z_i(\gamma)}{\partial \gamma_j} &= \frac{\beta}{\beta-1} K_{ji} B_i^{\frac{1}{1-\beta}} \gamma_i^{\frac{\alpha+\beta-1}{1-\beta}} \left(\sum_{s\in S} \gamma_s K_{si}\right)^{\frac{1}{\beta-1}} - \frac{1}{\beta-1} \sum_{j'\in S, j'\neq j} \left[K_{ij'} \left(\frac{\sum_{s\in S}\gamma_s K_{sj'}}{B_{j'}\gamma_{j'}^{\alpha}}\right)^{\frac{-\beta+2}{\beta-1}} K_{jj'}\right] - \\
&- \frac{1}{\beta-1} K_{ij} \left(\frac{\sum_{s\in S}\gamma_s K_{sj}}{B_j \gamma_j^{\alpha}}\right)^{\frac{-\beta+2}{\beta-1}} \left[\frac{K_{jj} B_j \gamma_j^{\alpha} - \alpha B_j \gamma_j^{\alpha-1} \sum_{s\in S}\gamma_s K_{sj'}}{B_j \gamma_j^{\alpha}}\right] \\
&= \frac{\beta}{\beta-1} K_{ji} B_i^{\frac{1}{1-\beta}} \gamma_i^{\frac{\alpha+\beta-1}{1-\beta}} \left(\sum_{s\in S} \gamma_s K_{si}\right)^{\frac{1}{\beta-1}} - \frac{1}{\beta-1} \sum_{j'\in S, j'\neq j} \left[K_{ij'} \left(\frac{\sum_{s\in S}\gamma_s K_{sj'}}{B_{j'}\gamma_{j'}^{\alpha}}\right)^{\frac{-\beta+2}{\beta-1}} K_{jj'}\right] - \\
&- \frac{1}{\beta-1} K_{ij} \left(\frac{\sum_{s\in S}\gamma_s K_{sj}}{B_j \gamma_j^{\alpha}}\right)^{\frac{-\beta+2}{\beta-1}} \left[\frac{\gamma_j K_{jj} - \alpha \sum_{s\in S}\gamma_s K_{sj'}}{\gamma_j}\right]
\end{aligned}$$

Let $\beta < 0$ and $\alpha < 0$ then the expression is positive and the gross-substitute property holds. Similar results can be easily established for $\beta = 0$, $\alpha < 0$ and $\beta < 0$, $\alpha = 0$. The same cannot be, in generally, established if $\beta > 1$ or $\alpha > 1$ since the expression cannot be signed in that case, and in particular we have found parametric specifications where the gross-substitutes property may fail.[16] Thus, the region that uniqueness applies with this

[16] In particular, we analyzed the Armington case with intermediate inputs as in Section 7.1.3. We can show that this model for $\sigma = 3$ and $\gamma = 1/4$ corresponds to the case $\alpha, \beta > 1$ but the gross-substitute condition does not obtain in the case of many symmetric regions with symmetric trade costs or even two regions with no trade costs.

approach is $\alpha \leq 0, \beta \leq 0$.

B.4 Comparative Statics when $\beta = 0$

Let us consider a particularly interesting special case, $\beta = 0$. We have in this case that the equilibrium is characterized by

$$B_i \gamma_i^{\alpha-1} = \sum_{j \in S} \left(\frac{\sum_{s \in S} \gamma_s K_{sj}}{B_j \gamma_j^{\alpha}} \right)^{-1} K_{ij} \implies$$

$$\gamma_i^{\alpha-1} = \sum_{j \in S} \left(\frac{B_j \gamma_j^{\alpha}}{\sum_{s \in S} \gamma_s K_{sj}} \right) B_i K_{ij},$$

which is the standard single-equation gravity model that we find in papers such as Anderson (1979); Eaton and Kortum (2002); Chaney (2008). We can rewrite this system re-written using 4 as

$$Y_i = \sum_{j \in S} \left(\frac{\gamma_i K_{ij}}{\sum_{s \in S} \gamma_s K_{sj}} \right) Y_j$$

In this last equation the technique developed by Dekle, Eaton, and Kortum (2008) can be applied (see details in Arkolakis, Costinot, and Rodríguez-Clare (2012)) so that computing the changes in γ_i require only knowledge of changes in K_{ij} and initial trade and output levels across all the models that can be captured by this formulation.

Notice that given equation 47 and the above equation we have for $\beta = 0$ that we can express the origin fixed effects as a function of the destination fixed effects and parameters

$$\gamma_i = \left(\frac{\sum_j K_{ij} \delta_j}{B_i} \right)^{\frac{1}{\alpha-1}}. \tag{50}$$