Contract Negotiation and the Coase Conjecture*

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Abstract

This paper analyzes an explicit protocol of contract negotiation between a principal who has all the bargaining power and an agent with a privately known type, and provides a foundation for renegotiation-proof contracts in such environments. The model extends the framework of the Coase conjecture to situations in which the seller and buyer must determine the quantity or the quality of the good being sold. All equilibria converge to the same outcomes as renegotiation frictions become negligible. Those contracts are separating, efficient, and easily characterized.

1 Introduction

In the standard model of the durable-good monopolist, any sale is efficient and definitive: buyer and seller cannot both benefit from modifying the price of the sale. In richer contractual environments, however, a signed contract may be inefficient. For example, the parties may benefit from increasing the quantity of the good initially sold, or by agreeing on a different quality of that good.

This issue is particularly important when the buyer holds private information, because his willingness to sign some contract is informative of his type, and may thus reveal some inefficiency of the

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contract just signed. This, in turn, prompts the seller to propose a new contract, and may distort the buyer's ex ante incentives to accept any given contract.

Contract renegotiation with a privately informed agent has traditionally been studied from two different angles. The first approach is axiomatic, and focuses on "renegotiation-proof" contracts.[1] It essentially *assumes* that renegotiation leads to an efficient contract, even when one party holds private information. The second approach focuses on simple renegotiation protocols, in which the principal gets a single shot at renegotiating the contract, by making a take-it-or-leave-it offer. This approach typically results in *inefficient* contracts.[2]

The second approach seems incomplete: what, in reality, should prevent the principal from proposing a new contract after learning the inefficiency of the current contract? Such restriction amounts to a strong form of commitment power for the principal, and can even result in full commitment outcomes. For example, imposing any *finite* number k of negotiation opportunities results in the full commitment outcome as the friction parameter η goes to zero: the principal simply passes the first $k-1$ opportunities to negotiate the contract, and then proposes the full commitment allocation. Similarly, Wang (1998) has shown that the principal can implement the full commitment allocation if renegotiation stops as soon as the agent accepts an offer.[3]

This paper studies, instead, a more flexible negotiation protocol in which the principal can propose a new contract whenever he desires to do so, and in particular after learning any new information regarding the agent's type. Put differently, the principal cannot commit *not* to renegotiate a contract. While such flexibility seems necessary, as argued above, to guarantee efficient outcomes, establishing this result raises complex issues. To appreciate the difficulty, consider again the standard durable-good monopolist. Efficiency, in that context, means that the good is sold without delay, and was established by Gul, Sonnenschein, and Wilson (1986) as the discount rate,

[1] See Dewatripont (1989), Maskin and Tirole (1992), Battaglini (2007), Maestri (2012), and Strulovici (2011, 2013). A similar approach has been used to study renegotiation in repeated games with complete information by Bernheim and Ray (1989) and Farrell and Maskin (1989).

[2] See Hart and Tirole (1988) and Fudenberg and Tirole (1990). Wang (1998) considers a more flexible protocol, in which the principal proposes contracts until an agreement is reached. Such protocol leaves a high commitment power to the principal, since he cannot renegotiate any agreement. Indeed, Wang's main result is that, with this protocol, the principal achieves the full commitment allocation, which is also ex post inefficient.

[3] In Wang's model, the principal can repeatedly propose menus over quality–wage contracts but renegotiation stops as soon as the buyer has accepted a contract. Beaudry and Poitevin (1993) obtain a similar result if renegotiations break down as soon as a new proposal is *rejected*. In Beaudry and Poitevin, the informed party has the bargaining power and proposes a single new contract at each round. Renegotiation stops in the first round at which the other party refuses the new contract. In a different setting with moral hazard, Matthews (1995) considers one-shot renegotiation by the informed party and obtains ex post efficiency.

or breakdown probability, goes to zero.[4] The proof is sophisticated even in this simple contractual environment, where each contract amounts to a single posted price. The key question is to determine whether the seller can benefit from distorting the allocation of the low-valuation buyer by inefficiently delaying the sale, in order to extract some rent from the high-valuation buyer. In richer environments, the question is more complex because i) the signature of any contract may be followed by further negotiations (e.g., contractual covenants, increases in quantities or qualities), ii) the principal may benefit from proposing multiple new contracts at each round instead of single one,[5] iii) each type of the agent can randomize over all such contracts, and iv) in many contracting problems, the utility of the agent need not be linear or even separable in the contract components.

This paper analyzes a flexible negotiation protocol with the following properties. At each round, the principal can propose a menu of contracts (round zero starts with a default contract, which may correspond to the absence of a prior relationship, some status quo, or some unmodeled previous play). The agent then chooses a contract from that menu, or holds on to the last accepted contract. At the end of each round, negotiations exogenously break down with a fixed probability η, in which case the last accepted contract is implemented. The breakdown probability captures negotiation frictions: when it is equal to 1, the protocol reduces to full commitment, and the principal typically distorts the allocation of some type of the agent, creating some ex post inefficiency. The model focuses on a binary type structure, which satisfies a standard single-crossing condition. As a result, there is common knowledge of gains from renegotiation: as long as the types of the agent have not been fully separated, there is a strictly positive surplus to be extracted.

The main theorem of the paper is that, as η goes to zero, all PBE outcomes converge to the same separating and efficient contracts. The outcome of negotiation is therefore renegotiation-proof in the sense that P could not benefit from renegotiating the contract even if he were given an extra opportunity to do so after negotiations have broken down. As a result, flexible renegotiation provides a dynamic implementation, without commitment, of efficient allocations. The type-specific contracts to which all PBE outcomes converge are straightforward to characterize and determine

[4] The result is shown for the "gap" case and the "no gap" case under some Lipschitz condition on the distribution of types, for weak Markov equilibria (see also Sobel and Takahashi (1983) and Fudenberg, Levine, and Tirole (1985)). Ausubel and Deneckere (1989) show that the conjecture can fail when more general equilibria are allowed. The analysis of the Coase conjecture has been extended to various environments: interdependent values (Deneckere and Liang (2006)), incoming flow of new buyers (Fuchs and Skrzypacz (2010)), and outside options for the buyer (Board and Pycia (2013)). Skreta (2006) takes a mechanism design approach and shows the optimality price posting. All these models focus on the case in which the buyer can only buy one unit of the good, and a single quality of the good is available.

[5] For example, the principal may propose one contract for each type of the agent, or propose multiple almost identical contracts as a communication device to emulate cheap talk.

graphically. Unlike the full commitment case, these contracts are independent of the initial belief that the principal holds about the type of the agent. The contract space can be divided in three regions. In the "No-Rent" region, the principal extracts all surplus of renegotiation. Intuitively, this happens because no type of the agent cannot gain anything from mimicking the other type, and private information provides no leverage in the negotiation. While intuitive, even that case is nontrivial as there could a priori be equilibria where both types of the agent delay revealing their information until getting some rent. In the other two regions, there is a region-specific type that gains nothing from negotiation, leaving all the surplus to the principal. The other type, say H, gets a positive rent that depends on how inefficient the initial contract is for the other type, L. The more inefficient the initial contract is for L, and the more rent H can extract. Even in that case, though, the principal extracts positive surplus from negotiating with H *above and beyond* the surplus that he extracts from L. This stands in stark contrast with the standard Coase conjecture where the contracts that are efficient for one type (namely, immediate sale) are also the contracts that are efficient for the other type.

The fact that almost efficient contracts are proposed immediately implies that, *in equilibrium*, renegotiation plays a relatively minor role, even though the *possibility* of renegotiation has a major impact on the the outcome. This suggests that, empirically, one should not infer that renegotiation is impossible or difficult in practice just because the observed renegotiation activity seems negligible. Instead, it may well be that negotiation is feasible and cheap, but finds its expression in the very first contracts that are proposed.

Another contribution of the paper is to establish the existence of a PBE for a negotiation game with a (relatively) rich contract space. In the present setting, backward induction techniques cannot be applied. Instead, the proof takes a two-step approach: first, prove the existence of an equilibrium in an auxiliary game of perfect information between the principal and the high type of the agent, based on Harris (1985). Second, use that equilibrium to construct an equilibrium of the negotiation game with private information.

Finally, the paper is related to the literature on reputation, in which some players are trying to determine whether other players have a "commitment" type.[6] Compared to this literature, the present analysis differs in several ways: i) the "actions" of the players (the types) are endogenous, because the principal chooses which contracts the agent chooses from in each round, ii) the state space is large, because it includes the last accepted contract, in addition to the principal's belief, and iii) all types of the agent are strategic. The richer state space, in particular, requires specific

[6]See in particular Fudenberg and Levine (1989), Schmidt (1993), Abreu and Gul (2000), Cripps et al. (2005), and Atakan and Ekmekci (2012).

tools, and the use of a number of inequalities which combine the nonlinear geometry of the problem (as captured by the agent's and the principal's utility over the contract space) with the incentives of the players.[7]

2 Setting and Overview of the Results

There are two players, a principal (P) and an agent (A) who negotiate a contract lying in some compact and convex subset $\mathcal{C}$ of $\mathbb{R}^2$, whose components are denoted x_1 and x_2.

The agent has a utility function $u_\theta : \mathcal{C} \to \mathbb{R}$ where $\theta \in \{L, H\}$ denotes his type, and P has a cost function $Q : \mathcal{C} \to \mathbb{R}$. It is assumed throughout that u_L, u_H and Q are twice continuously differentiable and have strictly positive derivatives with respect to x_1 and x_2, that the functions u_L and u_H are concave, and that Q is convex.

A contract $C = (x_1, x_2) \in \mathcal{C}$ is θ-*efficient* if it is the cheapest contract in $\mathcal{C}$ providing θ with some given utility level. For each θ, let $\mathcal{E}_\theta$ denote the set θ-efficient contracts in the interior of $\mathcal{C}$; θ's iso-utility curve and P's iso-cost curve are tangent at any such contract. To rule out pathological cases, it is assumed that the efficiency curves $\mathcal{E}_L, \mathcal{E}_H$ are smooth and upward sloping and that, given any contract C on $\mathcal{E}_\theta$, P's isocost curve going through C and θ's isoutility curve going through C do not both have a zero curvature at C.[8]

The functions u_L and u_H are required to satisfy a standard single-crossing condition: iso-utility curves of L are steeper than those of H at their intersection point. This implies that the efficiency curve $\mathcal{E}_L$ lies to the lower right of $\mathcal{E}_H$. $\mathcal{C}$ can therefore be partitioned into three regions separated by $\mathcal{E}_L$ and $\mathcal{E}_H$. Contracts in the inner region are said to be in the 'No Rent' configuration, while contracts below $\mathcal{E}_L$ (above $\mathcal{E}_H$) are in the 'H-Rent' ('L-Rent') configuration. The set of contracts in the H-Rent configuration will be denoted by $\mathcal{H}$. This setting is represented on Figure 1 in the context of a trade application (other applications are described later in this section). $\mathcal{C}$ represents an Edgeworth box, delimited by the sum of endowments of the agent and the principal. Each contract C represents the a final allocation for the agent, the efficiency curve $\mathcal{E}_\theta$ is the 'contract curve' corresponding to type θ, and the status quo R_0 represents the endowment of the agent before any trade.

[7]Some of these differences can formally be incorporated into the standard reputation framework. For example, the action space of the agent can be assumed to be fixed by setting a default value if the agent chooses anything outside of the principal's proposed set. Such formal similarity does not affect the substantive differences between the settings.

[8]This guarantees that the two curves take off cleanly from each other as one moves away from C. That property is used in Lemma 12 in order to compute a lower bound on the inefficiency of some contracts.

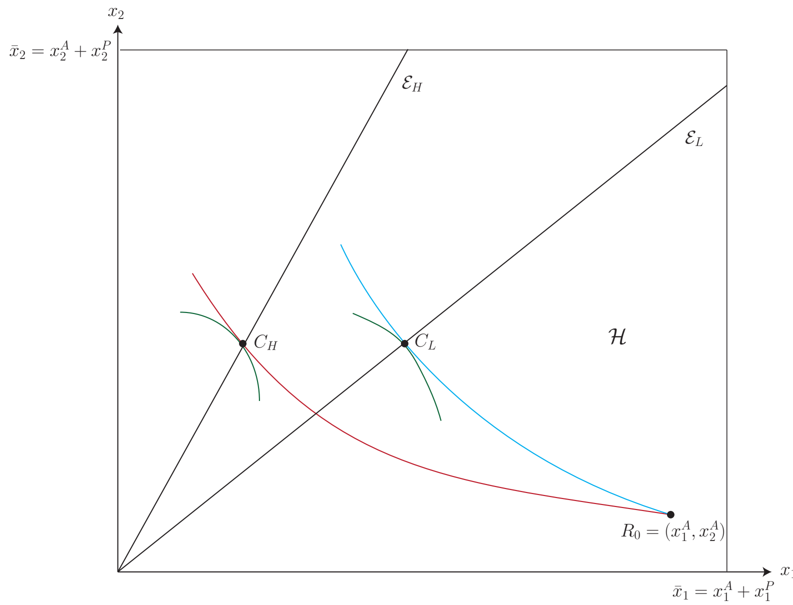

Figure 1: Setting (trade interpretation)

The Negotiation Game

The game unfolds as follows. First, the agent privately observes his type θ; P has a prior characterized by the probability $\beta_0 = Pr(\theta = H)$. The game starts with an initial contract $R_0 \in \mathcal{C}$, which may represent some status quo, the absence of a prior relationship, or the result of some unmodeled earlier play. There are countably many potential rounds, indexed by $n \in \mathbb{N}$. At each round n, P can propose a menu M_n of contracts in $\mathcal{C}$. The number of contracts in M_n is bounded by some constant $G \geq 2$ that is arbitrary but fixed throughout the game.[9] The agent chooses a contract in M_n or holds on to the last accepted contract, R_n. Any mixed strategy over the choice set $M_n \cup \{R_n\}$ is allowed; as in the standard Coase conjecture and in reputation models, mixing plays a key role in the analysis. The contract R_{n+1} that is selected by the agent becomes the new reference. At the end of each round, renegotiation breaks down with probability $\eta \in (0, 1]$ and the last accepted contract, R_{n+1}, is implemented. Otherwise, negotiations move on to the next round.

[9]There is no guarantee that proposing only two contracts at each round is without loss of generality. As Bester and Strausz (2001) have shown, the set of implementable outcomes can require more "messages" (or contracts) than the number of types of the agent, even in a two-period setting. While their modified revelation principle without commitment implies that *incentive efficient* contracts may require only 2 messages, here one must consider all possible continuation equilibria, including incentive inefficient ones (indeed, an inefficient continuation equilibrium may provide incentives at earlier stages of the game). Since $\mathcal{C}$ is a continuum, one could also imagine that P exploits a higher G by proposing several almost identical contracts as a 'cheap' way of communicating with the agent.

Letting $\{R_n\}$ denote the stochastic process of contracts entering each round n, the agent's expected utility is equal to

$$\mathcal{V}_\theta = E\left[\sum_{n\geq 0}(1-\eta)^n \eta u_\theta(R_{n+1})\right],$$

while P's expected cost is

$$\mathcal{Q} = E\left[\sum_{n\geq 0}(1-\eta)^n \eta Q(R_{n+1})\right].$$

The parameter η represents the *negotiation friction* of the game.[10] The objective of this paper is to characterize the PBEs of the game as the friction η goes to zero. Existence of a PBE is guaranteed by Theorem 1, whose proof is in Appendix A.

THEOREM 1 *For each $\eta \in (0,1]$, there exists a PBE of the negotiation game.*

For any contract $R \in \mathcal{C}$, let $E_H(R)$ and $E_L(R)$ denote the cheapest pair of H- and L-efficient contracts such that each type $\theta \neq \theta'$ weakly prefers $E_\theta(R)$ to $E_{\theta'}(R)$ and to R. That pair is well defined for each possible configuration of R.[11] Figure 2 represents these concepts for the case of CRRA utility functions and a linear cost function, and where $\mathcal{C}$ is the Cartesian product $[0,\bar{x}_1] \times [0,\bar{x}_2]$.

The proof of Theorem 2, below, exploits the inefficiency stemming from both types of the agent getting the same contracts at the time of a breakdown. To guarantee that such inefficiency actually does arise, one needs to rule out situations in which contracts lying at the boundary at the contract space arise in equilibrium, since such contracts may be efficient for both types.[12]

Accordingly, say that a contract R_0 is ***regular*** if it is in the No-Rent configuration or if it satisfies the following condition, stated when R_0 lies in $\mathcal{H}$ (the L-Rent case is defined analogously): for any $R' \in \mathcal{H}$,

$$u_H(E_L(R')) \geq u_H(R_0) \Rightarrow E_L(R') \neq E_H(R') \tag{1}$$

[10] There is another interpretation of the setting where η is the discount rate and the parties receive payoffs at each period of the on going relationship. This interpretation is discussed at the end of Section 5.

[11] If R is in the No-Rent configuration, $E_\theta(R)$ is simply the θ-efficient contract that gives θ the same utility as R. If R is in the H-Rent configuration, then $E_L(R)$ is similarly defined, while $E_H(R)$ is the H-efficient contract that gives H the same utility as $E_L(R)$. Because that contract gives a strictly higher utility to H than the initial contract R, H must be getting a positive rent in any equilibrium, hence the name of that configuration. A symmetric construction obtains if R is instead in the L-Rent configuration.

[12] At the boundary, even the strict single crossing property is ineffectual in separating efficient contracts: even if the isoutility curves for the low and high types have different slopes for a contract lying at the boundary, that contract may be efficient for both types.

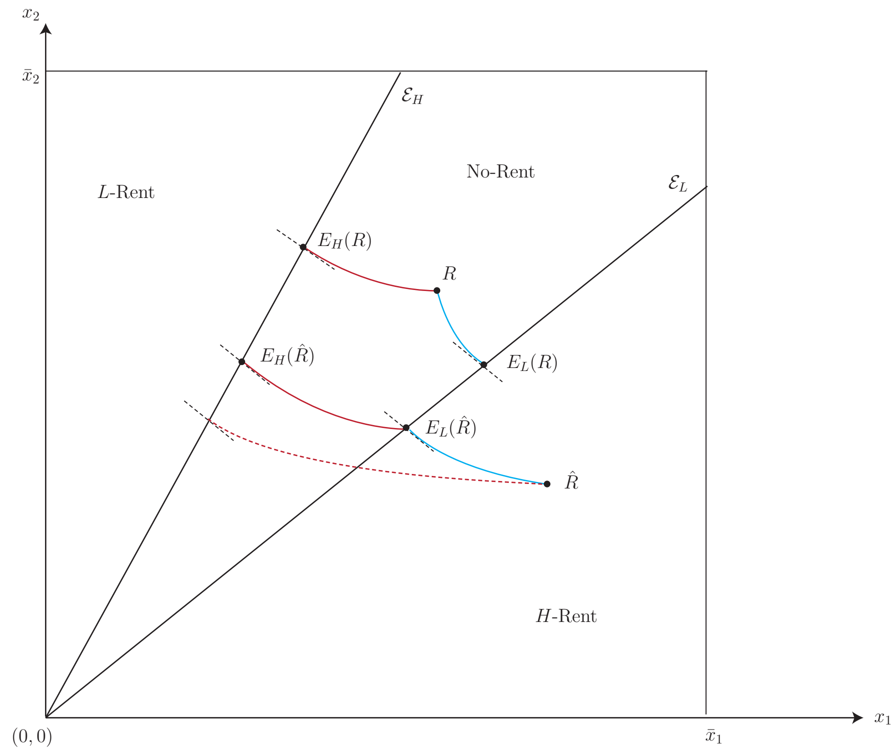

Figure 2: Renegotiation outcomes

This condition works because starting from R_0, any contract $R' \in \mathcal{H}$ which may arise in equilibrium must satisfy $u_H(E_L(R')) \geq u_H(R_0)$, as shown by Proposition 1, part iv) and thus satisfies the premise of Condition 1.

Even in situations where regularity does not hold, it is easy to slightly perturb the agent's utility function so as to satisfy it. This technique is illustrated in Section 4, which recovers the standard Coase conjecture as a limit case of the present setting. Another way to guarantee regularity is to expand the contract space.[13] The regularity condition then rules out only those initial contracts giving such a low utility to the agent that one may reach along the equilibrium path some efficient contracts hitting the lower boundaries of $\mathcal{C}$. In Figure 2 all contracts are regular except for the origin.

Theorem 2 *Consider any regular contract R_0 and belief β_0, and fix any $\varepsilon > 0$. There exists $\bar{\eta}(\varepsilon) > 0$ such that the following statements hold for any $\eta \leq \bar{\eta}(\varepsilon)$ and corresponding PBE:*
A: The expected utility of each type θ is bounded below by $u_\theta(E_\theta(R_0)) - \varepsilon$.

[13] If the utility and cost functions are defined on some upper orthant $\mathcal{O} = [\ell_1, +\infty) \times [\ell_2, +\infty)$ containing $\mathcal{C}$ (regardless of its shape), one can always expand $\mathcal{C}$ to the smallest rectangle $\mathcal{C}'$ containing $\mathcal{C}$ and such that $\mathcal{E}_L$ (resp. $\mathcal{E}_H$) hits the boundary of $\mathcal{C}'$ on its right (resp. upper) edge.

B: The probability that each type θ gets a contract within a distance[14] *ε of $E_\theta(R_0)$ when renegotiation breaks down is greater than $1-\varepsilon$.*

Statement B implies that the outcomes of renegotiation must get arbitrarily close to ex-post efficiency as the renegotiation friction η goes to zero, since each contract $E_\theta(R_0)$ is θ-efficient. That statement is a relatively straightforward consequence of Statement A, to which the quasi-totality of the proof is devoted.

Theorem 2 implies that P always gets some surplus from negotiation. When the contract is in the No-Rent configuration, P extracts, in fact, all the surplus regardless of the agent's type. When R_0 is in the H-Rent configuration, P extracts all the surplus from negotiating with L, and extracts some additional surplus in case he is facing H (the surplus obtained by moving from $E_L(R_0)$ to $E_H(R_0)$).

Applications

1. **Durable Good Monopolist.** The agent, A, is a buyer with quasi-linear utility $u_\theta(C) = \theta\bar{u}(x_2) + x_1$, where x_2 is the quantity of the good sold by P, x_1 is A's wealth, and u is A's concave utility function.[15] The initial contract, R_0, is equal to $(\bar{x}_1, 0)$ where $\bar{x}_1$ is A's initial wealth. P's cost is $Q(x_1, x_2) = cx_2 + x_1$, where $c > 0$ is the marginal cost for producing the good and x_2 captures how much wealth "P leaves to A".[16]

2. **Labor Contract.** P is a potential employer and A is a worker. $-x_2$ represents A's effort and x_1 is his wage. A gets utility $u_\theta(C) = \theta\psi(-x_2) + x_1$, where ψ is a common component of the agent's cost of effort, increasing in its argument, and θ is a worker-specific factor entering the cost of effort. The status quo $R_0 = (0, 0)$ represents unemployment, while P's profit is $\Pi(x_1, x_2) = -Q(x_1, x_2) = -x_2p - x_1$, where $p > 0$ is the unit price of the good.

3. **Consumption Smoothing and Insurance.** There are two periods and a single good. The dimensions of $\mathcal{C}$ represent A's consumption in each period. P is a social planner or a bank who can help the agent smooth his consumption. The type θ may be a privately known patience/discount factor, or a distribution parameter that describes how likely the agent is to value the good in the second period. For example $u(x_1, x_2) = v(x_1) + \theta v(x_2)$ or $u(x_1, x_2) = v(x_1) + E[w(x_2, \tilde{\rho})|\theta]$ where $\tilde{\rho}$ is a taste shock whose distribution FOSD increases in θ, and w is supermodular, so that

[14] The statement holds for any norm on $\mathbb{R}^2$.

[15] The iso-level curves of u_θ have positive curvature as long as the second derivative $\bar{u}$ is strictly negative, as is easily checked. A similar observation applies to ψ in the labor contract application.

[16] P's profit is $\Pi(t, x_2) = t - cx_2$, where t is how much the agent pays P. Letting $t = \bar{x}_1 - x_1$, we obtain the formulation in terms of the cost function Q.

$E[\partial w/\partial x(x_2, \tilde{\rho})|\theta]$ is increasing in θ.[17] R_0 is A's autarkic income stream. $Q(x_1, x_2) = p_1 x_1 + p_2 x_2$, where p_t is the market price for the good in period t.

4. **Trade.** More generally, the model describes a trade environment in which the dimensions of $\mathcal{C}$ represent distinct goods, with x_i denoting the quantity of good i consumed by A. Type L cares more about the first good than the second, relative to H. P (like A) has convex preferences, and Q is the negative of a utility function representing his preferences. R_0 denotes the agent's initial holdings of the goods.

3 Overview of the Arguments

This section provides a roadmap of the key steps and ideas used to establish Theorem 2. Most arguments and concepts are purposely simplified; all proofs are in the appendix. When the presentation of these steps and arguments could not be self-contained, precise pointers to the actual proofs have been included.

Preliminaries

When P assigns probability 1 to either type of the agent, or when the last accepted contract is in the No-Rent configuration, it is comparatively easy to show that there is a unique continuation PBE: P immediately extracts all the rent from negotiation and efficiency obtains (Proposition 1). The most challenging part of the analysis is to prove the theorem when R_0 is in the H-Rent or L-Rent configuration; this section focuses on the former case, without loss of generality.

With R_0 in the H-Rent configuration, one may show that, along any PBE, L accepts only contracts in the H-Rent configuration (Lemma 3). Without loss, a PBE looks like this: P proposes at each round an H-efficient contract, C_n, and some contracts in the H-Rent configuration. H mixes over all contracts, while L mixes over all contracts *but* C_n.

"Jump" deviation: P can, at any round n, propose the efficient contracts $E_H(R_n)$ and $E_L(R_n)$ and have them accepted by their respective types (Lemma 2). This deviation is feasible in any equilibrium and thus provides a key upper bound on P's expected cost.

Main question The objective of the proof is to show that P cannot do any better than the jump deviation as η goes to zero. It suffices to show that P must leave to H a utility arbitrarily close to $u_H(E_H(R_0))$; the other claims of Theorem 2 follow relatively easily from that statement.

[17] This application is explored in detailed by Strulovici (2013).

Contradiction hypothesis and block construction

The proof proceeds by contradiction. Fixing some $\varepsilon > 0$, suppose one may find some arbitrarily small η and associated PBE for which P extracts a rent of at least ε from H, compared to $u_H(E_H(R_0))$. To fix ideas, let $u_H(0)$ denote H's expected utility at the beginning of round 0, and suppose that $u_H(0) = u_H(E_H(R_0)) - \varepsilon$.[18]

The contradiction argument starts by constructing *blocks* of rounds delimited by continuation-utility thresholds for H, denoted $\hat{u}_0 = u_H(0) < \hat{u}_1 < \hat{u}_2 < \cdots$, and an equilibrium sequence of contract offers and choices along which P's assigned probability of facing H is guaranteed to drops by at least some factor $q < 1$ across each block.

The thresholds are defined inductively by the equation

$$\frac{\hat{u}_{k+1} - \hat{u}_k}{\hat{e}_k - \hat{u}_k} = \frac{1}{t} < 1 \tag{2}$$

for block $k+1$ (the first block has index 1), where $\hat{e}_0 = u_H(E_H(R_0))$ and, for $k \geq 1$, we have[19]

- $\hat{e}_k = u_H(E_H(R_{n_k}))$ where n_k is the last[20] round of block k, and
- $t > 1$ is a factor that will be defined shortly.

These thresholds are chosen so as to generate a geometric sequence for $\{\hat{e}_k - \hat{u}_k\}$. Proceeding by backward induction on k, as explained below, that geometric sequence is used to compute an upper bound on $\hat{e}_0 - \hat{u}_0$.[21] This bound is of obvious interest because $\hat{e}_0 - \hat{u}_0 = u_H(E_H(R_0)) - u_H(0)$ is exactly the quantity which must be shown to be less than ε in the contradiction hypothesis above.

It is easily shown that H's continuation utility at the beginning of round n, $u_H(n)$, increases by steps of order at most η between consecutive rounds. In fact, H's Bellman equation implies that

$$u_H(n) = \eta u_H(R_{n+1}) + (1-\eta) u_H(n+1) \tag{3}$$

for any contract R_{n+1} chosen by H with positive probability, which implies the η bound on $u_H(n+1) - u_H(n)$. Therefore, each block k contains at least in the order of $(\hat{u}_k - \hat{u}_{k-1})/\eta$ rounds. Since each round is followed by a breakdown with probability η, this puts a lower bound, of order $\hat{u}_k - \hat{u}_{k-1}$ on the probability of a breakdown within block k. Moreover, each breakdown creates an inefficiency,

[18] In general, $u_H(0)$ is strictly less than $u_H(E_H(R_0)) - \varepsilon$. However, this distinction can be easily addressed by an initialization phase. See Lemma 4.

[19] The actual definition of $\hat{e}_k$ is more complicated and involves the entire history of play.

[20] Lemma 14 shows that H's continuation utility must eventually exceed $\hat{u}_k$ in finite time, so that the final round n_k is well-defined.

[21] Precisely, one can show that $\hat{e}_0 - \hat{u}_0 \leq \frac{t}{t-1}^k (\hat{e}_k - \hat{u}_k)$ for any relevant k. See the argument preceding (22).

conditional on facing H, that is bounded below by some constant $D > 0$.[22] Given this inefficiency, the only reason for P to proceed with the equilibrium instead of directly jumping to the efficient contracts $E_H(R_n)$ and $E_L(R_n)$ is to extract some rent from H. That rent is bounded above by $\hat{e}_k - \hat{u}_k$, because $\hat{e}_k$ is what P would leave to H if he gave up on rent extraction, while $\hat{u}_k$ is H's continuation utility if P goes on with the equilibrium. This puts an upper bound $a(\hat{e}_k - \hat{u}_k)$ on the cost reduction that P can hope to get through rent extraction, where a is a constant "translating" any upper bound on utility increments for H into an upper bound on the corresponding cost increment for P.

Taken together, these observations imply an upper bound on the probability μ_k that H rejects the rent-extracting contracts at block k:

$$\mu_k \leq \frac{a}{a+D} \frac{\hat{e}_k - \hat{u}_k}{\hat{u}_{k+1} - \hat{u}_k} = t^{-1} < 1. \tag{4}$$

where the equality comes from (2) and from choosing the factor $t = \sqrt{\frac{a+D}{a}}$. Since H's probability of accepting only contracts in $\mathcal{H}$ is bounded above by (4), while L accepts contracts in $\mathcal{H}$ with probability 1, P's posterior probability of facing H conditional of facing these contracts must go down by at least some factor related the bound provided by (4), at least for some choice sequences of contracts in $\mathcal{H}$. Letting $\hat{\beta}_k$ denote P's assigned probability of facing H at the end of block k, this implies that

$$\hat{\beta}_k \leq g\hat{\beta}_{k-1}$$

for some factor $g < 1$ independent of η.

Adapting backward induction to an infinite horizon

The construction is interrupted at the first block K for which the difference $\hat{e}_K - \hat{u}_K$ is less than $\bar{W}\eta$, where $\bar{W}$ is chosen high enough to guarantee that $\hat{e}_K - \hat{u}_K$ lies *above* some threshold $\underline{W}\eta$, for some $\underline{W} \in (0, \bar{W})$. This lower bound of order η is guaranteed to exist and plays an important role in establishing the contradiction, as explained below.

The remainder of the analysis hinges on the value of the posterior $\hat{\beta}_K$ at the end of block K. To see this, consider the standard Coase conjecture with 2 types. In that simpler setting, there exists a belief threshold $\hat{\beta}$ below which the seller immediately sets the price at the low-buyer valuation, completely giving up on rent extraction. If such a threshold existed in our problem, then because $\hat{\beta}_k \leq g^k \beta_0$, this would put a *fixed* bound on the number of blocks until $\hat{\beta}$ is reached. Proceeding

[22] The existence of such a constant hinges on the geometric assumption of regularity, as explained before stating Theorem 2. See also Lemma 16.

backward on the utility thresholds, this would imply that the initial difference $u_H(E_H(R_0)) - u_H(0) = \hat{e}_0 - \hat{u}_0$ is also of order η. This, for η small enough, would yield the desired contradiction.

Unfortunately, such belief threshold does not exist here. Unlike the gap case of the Coase conjecture, where the "no sale" outcome generates first-order inefficiency, here the default contract R_n varies over time and asymptotically becomes L-efficient. Instead of jumping to an L-efficient contract when β_n becomes small enough, the principal makes active proposals yielding ever smaller improvements of the contract until negotiations exogenously break down. As a result, the relatively simple backward induction argument in the standard Coase conjecture must be modified. The key is to establish the next proposition.

Proposition (Belief Bound) *For any $d > 0$, there exists an upper threshold $\bar{\eta}(d) > 0$ such that $\hat{\beta}_K > \eta^d$ for all PBEs corresponding to any $\eta < \bar{\eta}(d)$.*

Define $\rho > 0$ by $g^{-\rho} = \frac{t}{t-1}$ and set $d = \frac{1}{2}\min\{\frac{1}{\rho}, 1\} \in (0, 1/2]$. Proceeding by backward induction on the constructed blocks and applying the above proposition to that value of d, one may easily show (see (22)) that $\hat{e}_0 - \hat{u}_0$ is of order $\sqrt{\eta}$, which yields the desired contradiction. Proving this proposition is challenging, however, and takes up most of the proof.

The basic intuition for the proposition is – deceivingly – simple: if $\hat{\beta}_K$ were smaller than η^d, for η arbitrarily small, then P would be willing to sacrifice almost all rent extraction from type H, who is extremely unlikely anyway, in order to avoid any inefficiency on type L. This, it turns out, would imply that the rent extraction $\hat{e}_K - \hat{u}_K$ is of order η^{1+d} and contradict, for η small enough, its lower bound of $\underline{W}\eta$, which was mentioned above. The actual argument is complex because we are comparing arbitrarily small rent extraction gains and inefficiency losses. Making things worse, as R_n gets arbitrarily close to the efficiency curve $\mathcal{E}_L$, the inefficiency loss on L is *one order of magnitude smaller* than the rent extraction on H.

Deriving the belief bound for $\hat{\beta}_K$

The idea for proving the proposition is to transform (beyond recognition!) H's Bellman equation (3) into the following dynamic belief equation:

$$\frac{\beta_{n+2}}{\beta_{n+1}} \geq \frac{\beta_{n+1}}{\beta_n} - c\sqrt{\beta_{n+1}}. \tag{5}$$

This transformation requires a number of conceptual and technical steps, outlined below, and involves the observation that, for n large and η small, one can set without loss of generality

$$\frac{aw_n}{\eta D} = \frac{\beta_{n+1}}{\beta_n}, \tag{6}$$

where $w_n = u_H(E_H(R_n)) - u_H(n)$ is the *rent extraction index*. As shown by Proposition 2, w_n must converge to zero as n goes to infinity. Intuitively, the probability β_n of facing H converges to zero and R_n converges to an L-efficient contract as n goes to infinity. This implies that $E_H(R_n)$ gives roughly the same utility to H as R_n does for n large (see Figure 2) or, in other words, that w_n converges to zero. It turns out however (Lemma 9) that if $\hat{\beta}_K = \beta_{n_K}$ is too small (in particular, below the belief bound that we are trying to establish), then no solution to the dynamic equation (5) is such that β_{n+1}/β_n converges to zero. From (6), this implies that w_n cannot converge to zero as n goes to infinity, yielding the desired contradiction.

Transforming H's Bellman equation into the dynamic belief equation (5)

The transformation of H's Bellman equation is based on P's IC constraint. One may consider, at each round n, P's incentive to jump to $E_H(R_n), E_L(R_n)$ instead of pursuing active negotiations (this jump deviation is always available to P, as mentioned earlier). This IC condition yields the equation

$$w_n a \beta_n \geq \sum_{R_{n+1} \in (M_n \cup \{R_n\}) \cap \mathcal{H}} \left[\beta_n \mu_n^H(R_{n+1}) \eta D + (1-\beta_n)\mu_n^L(R_{n+1})\eta(Q(R_{n+1}) - Q(E_L(R_n)))\right], \quad (7)$$

where $\mu_n^\theta(R_{n+1})$ is type θ's probability of accepting any given contract R_{n+1} of $\mathcal{H}$ that lies in the menu $M_n \cup R_n$ of available contracts. The left-hand side is an upper bound on P's gain, as in the block construction described earlier. The right-hand side consists of the loss on the high and low types, respectively, for each possible chosen contract R_{n+1}. Rewriting (7) as

$$w_n a \beta_n \geq \sum_{R_{n+1} \in (M_n \cup \{R_n\}) \cap \mathcal{H}} \mu_n^L(R_{n+1}) \left[\beta_n \frac{\mu_n^H(R_{n+1})}{\mu_n^L(R_{n+1})} \eta D + (1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(R_n)))\right], \quad (8)$$

the right-hand side may now be seen as a convex combination of these losses, weighted by L's probability $\mu_n^L(R_{n+1})$ of choosing each contract R_{n+1}. Therefore, there must exist a contract R_{n+1}, chosen with positive probability, for which[23]

$$w_n a \beta_n \geq \mu_n \beta_n \eta D + (1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(R_n))), \quad (9)$$

where $\mu_n = \frac{\mu_n^H(R_{n+1})}{\mu_n^L(R_{n+1})}$. That condition is then broken up into two weaker conditions[24]

$$\frac{w_n a}{\eta D} \geq \mu_n, \qquad (IC_n^{LL})$$

[23]Some "loss" terms may be nonpositive, but very slightly so. The argument and the breakup into the two weaker conditions IC^{LL} and IC^{LH} can be adapted to account for this, as shown in the appendix.

[24]Some of the terms can be negative, but only very slightly so. This complication does not change the gist of the analysis, as shown in the appendix.

and

$$\beta_n w_n a \geq (1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(R_n))) \qquad (IC_n^{LH})$$

This establishes the existence of a sequence $\{R_n\}$ of on-equilibrium-path contracts such that the coupled inequalities IC_n^{LL} and IC^{LH} are satisfied for all n large enough. The posterior β_n associated with that sequence satisfies in good approximation $\beta_{n+1} = \beta_n \mu_n$. It turns out that, starting from some given round n, μ_n can be increased so as to satisfy IC_n^{LL} as an equality, yielding $\frac{\beta_{n+1}}{\beta_n} \sim \mu_n = \frac{w_n a}{\eta D}$, without violating any other relevant inequality for indices $m \geq n$. This change makes the analysis more tractable. Moreover, it has no impact on the *contracts* involved in these inequalities. Therefore, if one can show that these contracts violate some equilibrium property (in particular, asymptotic convergence to efficiency), this will establish a contradiction, irrespective of the fact that the beliefs were modified to establish that violation.

Subtracting $u_H(E_H(R_n))$ from both sides of H's Bellman equation (3) and rearranging (also recalling that $w_n = u_H(E_H(R_n)) - u_H(n))$ yields

$$w_{n+1} = w_n - \eta(u_H(E_H(R_n)) - u_H(R_{n+1})) + \eta w_{n+1} + (1-\eta)(u_H(E_H(R_{n+1})) - u_H(E_H(R_n))). \quad (10)$$

The last two terms can be shown to be negligible, resulting in the simpler equation

$$w_{n+1} = w_n - \eta(u_H(E_H(R_n)) - u_H(R_{n+1})). \quad (11)$$

One can further show, by exploiting IC_n^{LH} and some geometric inequalities (Lemma 12) that H's utility difference in (11) is of order $\sqrt{\beta_{n+1}}$ (see (39)). Since w_n is proportional to $\eta\frac{\beta_{n+1}}{\beta_n}$, as explained above, (11) can then be turned into (5).

Bridging blocks and the dynamic belief equation

The previous paragraphs have naturally made abstraction from a number of complications arising in the actual proof. One intermediary step must be mentioned here, however, as it takes up almost a third of the actual proof. The dynamic belief equation and the non-convergence to zero that it implies only work as long as β_n remains small during all the rounds following block K. Unfortunately, however, one cannot guarantee that β_n remains small: Even though, *on average*, L is more likely than H to choose contracts in $\mathcal{H}$ (since H can also choose the H-efficient contract C_n), there may be contracts in the menu $M_n \cap \mathcal{H}$ that are chosen with much higher probability by H than L, resulting in a spike up of the posterior β_n. And, at least in principle, one cannot rule out the possibility that those contracts creating the spike are precisely the contracts chosen in P's IC constraint (8) to get the simpler condition (9).

Fortunately, this issue does not arise if w_n is small enough (precisely, below $\frac{\eta D}{2a}$), because in that case μ_n can be shown to be smaller than 1, implying that β_n must be decreasing. But the last block K ended up with the condition $w_n \leq \eta \bar{W}$, and the constant $\bar{W}$ could a priori be well above $\frac{D}{2a}$. To address this difficulty, the idea is to insert a number of intermediary blocks, following block K, and build a hybrid argument drawing on the arguments used along the blocks constructed earlier and the arguments used to derive the dynamic belief equation. Along those new blocks, w_n is guaranteed to decrease until it drops below $\frac{\eta D}{2a}$, but not *too* fast, as it must remain exactly of order η: remember that, in the above contradiction argument, we used the fact $\hat{w}_K = \hat{e}_K - \hat{u}_K$ was *greater* than $\eta \underline{W}$ for some fixed $\underline{W} > 0$. Moreover, these intermediary blocks are built so that β_n remains of order η^d. The dynamic belief equation argument can then be applied at the end of those intermediary blocks.

4 Relation to the standard Coase conjecture

Theorem 2 does not exactly cover the standard Coase conjecture, in which a contract amounts to a unit sale. However, it is possible to recover the conjecture as a limit of Theorem 2. To see this, suppose that the first contractual dimension represents the agent's wealth, while the second dimension represents the probability that the agent gets the good (alternatively, it could represent the quantity, between 0 and 1, of a divisible good). The initial contract is $(W, 0)$, where W is the agent's initial wealth. The principal incurs a cost c for the good (or marginal cost c, in the divisible-good interpretation), resulting in linear isocost curves. To bridge the two settings, we fix an arbitrarily small constant δ and define the agent's utility as follows. The agent's utility is quasilinear[25] for $x_2 < 1 - 2\delta$, given by $u(x_1, x_2; \theta) = \theta x_2 + x_1$ with $\theta_H > \theta_L > c$; the iso-utility curves are then kinked for $x_2 > 1 - 2\delta$ as depicted on Figure 3, so as to guarantees that all efficient contracts for H (L) involve a fixed probability $x_2 = 1 - \delta$ $(1 - 2\delta)$ of getting the good. The red (blue) curves represent the iso-utility curve of the high (low) type. The boundary of regular contracts is shown on the left of the figure. All contracts of $\mathcal{H}$ lying to the right of that boundary, including R_0, are regular.

This situation approximates, for δ arbitrarily small, the setting of the Coase conjecture, in which it is always efficient to sell the good to the agent. For any $\delta > 0$, however, it also satisfies the

[25]The paper assume a nonzero curvature condition for isoutility curves. However, the curvature can be arbitrarily close to zero, so that the quasilinear case can be approximated arbitrarily well, and this can be done simultaneously as δ is taken to zero. Moreover, the Coase conjecture really concerns the value of the agent for zero and one unit of the good. It doesn't have to be linear for intermediary quantities of the good, though this restriction is natural if one think of the "quantity" as being the probability of that the agent gets the good.

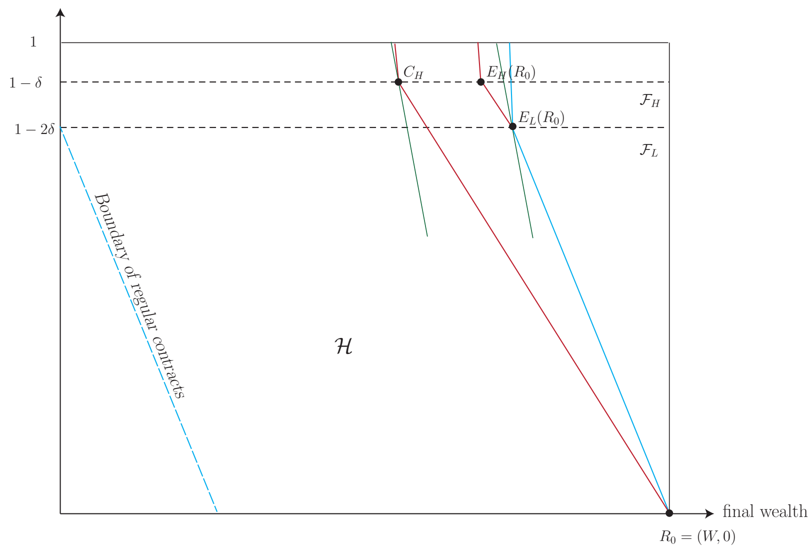

Figure 3: Recovering the standard Coase conjecture

assumptions of Theorem 2. In particular, the efficiency curves of the types are separated.

The Coase conjecture is then recovered as follows: if P were sure to face H, he would move to the contract C_H appearing on Figure 3. With uncertainty about the buyer's type, however, Theorem 2 implies that the outcome is given by the contracts $E_H(R_0), E_L(R_0)$, which converge to the same contract as δ goes to zero. Both types of the buyer obtain essentially the same outcome, which is the (almost) sure sale of the good at the same price. The high type gets a rent corresponding to the distance between $E_H(R_0)$ and C_H, while L gets no rent.

It must be emphasized that the standard Coase conjecture is not representative of the general situation. In particular, the similarity completely breaks down when the efficiency curves of H and L lie far away from each other. The principal then extracts some rent from H above and beyond the contract that L gets in equilibrium.

Moreover, the stark discontinuity arising in the standard Coase conjecture between the gap and no gap cases does not arise here. In that setting with two types, H's rent *increases* as L's valuation v_L (and hence the equilibrium price) becomes lower. However, when L's valuation reaches P's marginal cost c, turning into the "no gap" case, H's rent suddenly drops to zero and P's profit leaps up from zero to $\beta_0(v_H - v_L)$. Consider a similar exercise in the setting of Theorem 2, where L' efficiency curve is lowered until it goes through R_0. The surplus that P extracts from H then varies continuously, until $\mathcal{E}_L$ goes exactly through R_0 and P extracts all surplus from H.[26]

[26]The principal's rent extraction does not have to be monotonic through the change of $\mathcal{E}_L$, but it evolves continuously

5 Conclusion

Whether it concerns trade, production, employment, finance, or other economic activities, contract negotiation is central to economic analysis and has been widely studied from both pure and applied perspectives. However, much remains to understand when i) contracts are more complex than binary sales or divisions of a given "pie", ii) some parties hold private information, and iii) negotiation is not limited to a single take-it-or-leave-it-offer. In this common situation, private information is endogenously revealed through negotiation, even in the absence of any exogenous information arrival, and this information affects ongoing negotiations, providing a dynamic interaction between beliefs and contracts.

I have taken the view that parties should be able to react to this endogenous flow of information, particularly when it reveals some inefficiency of the current agreement, instead of being stuck with this agreement. This view captures the idea of unconstrained negotiation, and seems essential to generate a foundation for renegotiation-proof contracts.[27] While the analysis has focused on a specific negotiation protocol (just like foundations of the Nash bargaining solution and of the Coase conjecture), it would be useful to explore, in future research, more general protocols of negotiation. A natural conjecture is that contract negotiation should lead to efficient outcomes under these more general protocols as long as it is unconstrained in the above sense and does not entail additional frictions, such as explicit renegotiation costs.[28]

In the present setting, where the efficiency "gap" is endogenous, varies over time, and converges to zero, it is perhaps surprising that ex post efficiency should obtain for all equilibria – including non-stationary ones – of the negotiation game, as frictions vanish. It would be natural to study extensions of these results to more than two types or contract dimensions and to independent values imposed on the model.[29] While those extensions do not seem straightforward, the techniques developed here should prove helpful to analyze them. When the agent has more than two types, one may conjecture that renegotiation will similarly lead to ex post efficiency, and it is actually easy to guess the limit contracts in that case.[30]

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and reaches its maximum as $\mathcal{E}_L$ goes through R_0.

[27]For instance, Wang (1998) shows that when the first signed contract is always implemented, outcomes can be ex post efficient. In fact, those outcomes are, in his model, identical to the full commitment outcome.

[28]See, e.g., Brennan and Watson (2013).

[29]Deneckere and Liang (2006) provide an in-depth analysis of the interdependent value case, for a binary sale.

[30]With three types, H, M, L, for instance the contracts $\{E_\theta(R_0)\}_{\theta \in \{H,M,L\}}$ would the cheapest θ-efficient contracts that are incentive compatible. The conjecture extends trivially to finitely many types and can also be extended to a differential equation characterizing $\{E_\theta(R_0)\}_{\theta \in \Theta}$ for a continuum of types, although proving in that case seems particularly challenging.

Allowing shared bargaining power between the principal and the agent is also natural,[31] although this extension is not needed to provide a foundation for the renegotiation-proof contracts, most common in the literature,[32] that are *proposed by the principal*. In fact, any negotiation protocol in which the bargaining power is shared by the parties should typically *not* yield the renegotiation-proof contracts that are optimal for the principal. That point is straightforward to see in the case of symmetric information.

The analysis has focused on a single "delivery" time, at which the contract is implemented. To provide a foundation for multi-period renegotiation-proof contracts, one should consider a general model with multiple "physical" events, i.e., times at which payments, efforts are made, or exogenous information arrives (e.g., if the type of the agent is persistent but not constant). A renegotiation protocol like the one studied in this paper could be inserted between physical events, and negotiation would pertain to continuation contracts for the remaining horizon. For example, if good deliveries or monetary transfers occur at integer times $t = 1, 2, \ldots$, the renegotiation rounds between dates $t-1$ and t would occur at times $\tau_n^t = t - \frac{1}{2n}$, for $n \geq 1$. This double time scale is natural in many contexts where physical deliveries have a particular calendar structure (e.g., monthly wage, weekly delivery, quarterly report, etc.), but parties' ability to negotiate has no reason to be thus constrained.[33]

Finally, there is a formal equivalence between the model presented here and a model in which η is a discount rate, rather than a breakdown probability, and the contract chosen in period n is implemented in that period, generating payoffs for both players. With that interpretation, the principal and the agent are in an ongoing relationship that yields payoffs at *all* rounds, and the same contract is implemented over and over again unless players decide to renegotiate it. This interpretation is similar to an infinite horizon version of one of the models analyzed by Hart and Tirole (1988) where, also, players' value functions are not constrained to be quasilinear. Reinterpreted as such, the results say, first, that there exists an equilibrium of this infinite horizon game (Theorem 1) and, second, that the players converge almost immediately (relative to the discount factor) to separating and efficient contracts in all equilibria of the game (Theorem 2).[34]

[31] Ausubel and Deneckere (1992) analyze a model with shared bargaining power. In that model, the informed party (buyer) is getting all the surplus, even when the seller is making all the offers. When given the opportunity to make offers, therefore, the buyer does not lose anything by remaining silent. In the present setting however, the agent does not get all the surplus and it is clear that, even in the absence of private information, sharing bargaining power would affect the equilibrium allocation of surplus. The key question is whether shared bargaining power would create some efficiency loss.

[32] See, e.g., Dewatripont (1989), Maskin and Tirole (1992), and Battaglini (2007).

[33] The time structure described here allows unrestricted renegotiation between physical events, as there are infinitely many renegotiation rounds, and could in principle also be captured by a continuous-time model, in which the principal can propose new contracts at any instant between physical events occurring one a discrete time set.

[34] This result echoes Maestri (2013) who considers an infinite horizon version of Hart and Tirole (1988) and,

Appendices

A Proof of Theorem 1 (Existence of a PBE)

It suffices to prove the result for R_0 in the H-Rent configuration and $\beta_0 \in (0,1)$: the degenerate prior and No-Rent cases obtain as direct consequences of Proposition 1 below, which proves (independently of this section) the existence and uniqueness of an equilibrium in those cases, while the L-Rent case obtains by symmetry of the H-Rent case. The proof proceeds in two steps:

- Step 1 - Prove the existence of an equilibrium in an auxiliary game played between P and H.
- Step 2 - Construct a strategy profile of the original game based on the equilibrium established in Step 1, and verify that it defines a PBE of the original game.

Step 1: Auxiliary game

The game starts with a contract $R_0 \in \mathcal{H}$ in the H-Rent configuration and a parameter $\beta \in (0,1)$. For this auxiliary game, β is just a parameter affecting the payoff functions and is devoid of its interpretation as a belief.

The auxiliary game is a dynamic game with infinitely many rounds. At each round n, starting in state R_n, P proposes new contracts $R_{n+1} \in \mathcal{H}$ and $C_n \in \mathcal{E}_H$ subject to the constraints

$$u_L(R_{n+1}) \geq u_L(R_n) \tag{12}$$

$$u_L(R_{n+1}) \geq u_L(C_n) \tag{13}$$

$$u_H(C_n) \geq u_H(R_n). \tag{14}$$

H then chooses a number $\mu_n \in [0,1]$. The interpretation of this choice is that H accepts R_{n+1} with probability μ_n and C_n with probability $(1-\mu_n)$. For this auxiliary game, however, μ_n is simply an action deterministically affecting payoffs.

The principal's cost, for strategies $\{R_{n+1}, C_n\}$ and $\{\mu_n\}$, is given by

$$\begin{aligned} \mathcal{Q}(\{R_{n+1}, C_n\}, \{\mu_n\}) = \sum_{n\geq 0} Q(C_n)\beta(1-\eta)^n(1-\mu_n)\prod_{k=0}^{n-1}\mu_k \\ + \sum_{n\geq 0} Q(R_{n+1})\left(\beta(1-\eta)^n\eta\prod_{k=0}^{n}\mu_j + (1-\beta)(1-\eta)^n\eta\right), \end{aligned} \tag{15}$$

introducing a concept of renegotiation-proofness suitable for infinite-horizon models with private information, shows that all renegotiation proof equilibria must result in efficient contracts as the discount rate goes to zero. By contrast, one of the main motivations and contributions of the present paper, using the interpretation of the main text, is to show that contracts must be renegotiation-proof as the discount rate goes to zero.

while H's payoff is

$$\mathcal{V}(\{R_{n+1}, C_n\}, \{\mu_n\}) = \sum_{n\geq 0} u_H(C_n)(1-\eta)^n(1-\mu_n)\prod_{k=0}^{n-1}\mu_k + \sum_{n\geq 0} u_H(R_{n+1})(1-\eta)^n\eta\prod_{k=0}^{n}\mu_n. \tag{16}$$

These payoffs correspond to the expected cost and utility that P and H would obtain in an equilibrium of the *original* game in which P proposes two contracts at each round, the breakdown probability is η, $\{\mu_n\}$ is the mixing strategy of H, L always accepts R_{n+1}, and the initial probability of facing H is equal to β.

LEMMA 1 *For any initial R_0 and $\beta \in (0, 1)$, there exists a perfect equilibrium of the auxiliary game*

Proof. The result is direct consequence of Theorem 1 in Harris (1985). We check Assumptions 1–5 of that theorem. The payoff function of the principal is simply the negative of his cost, $\mathcal{Q}$. P's (unconstrained) action set in round n is $S_{P_n} = \mathcal{H} \times \mathcal{E}_H$, while H's action space is $S_{H_n} = [0, 1]$ which are both compact and Hausdorff spaces. Hence, Assumptions 1 and 2 are satisfied. P's feasible set at each round n, as defined by the constraints (12) and (14), is closed and depends continuously on the current state. Therefore, the set $\mathcal{S}_f$ of feasible sequences is closed in $\mathcal{S} = \times_n(S_{P_n} \times S_{H_n})$ endowed with the product topology, and the set of feasible actions in round n depends continuously on past play. Thus, Assumptions 3 and 4 are satisfied. Finally, the payoffs $-\mathcal{Q}$ and $\mathcal{V}$ are clearly continuous on their domain $\mathcal{S}_f$, so Assumption 5 is satisfied as well. The result follows. ■

REMARK 1 *We can similarly define an auxiliary game and equilibrium when instead R_0 is in the L-Rent configuration. This equilibrium yields an expected utility for H, as a passive player of the auxiliary game, given, by*

$$V_H(\beta) = \sum_{n\geq 0}(1-\eta)^n\eta u_H(R_{n+1}). \tag{17}$$

This equilibrium and payoff is used to define H's strategy, off the equilibrium path, in the PBE construction for the original game.

Step 2: Equilibrium of the original game

Starting from $R_0 \in \mathcal{H}$ and a belief $\beta_0 \in (0, 1)$, the equilibrium strategies are defined as follows:

At each round n:

- P proposes the sequence of contracts $\{C_n, R_{n+1}\}$ corresponding to the auxiliary game started at (R_0, β_0)
- L accepts R_{n+1} with probability 1, while H accepts R_{n+1} with probability μ_n and C_n with probability $(1-\mu_n)$, where μ_n is H's equilibrium choice in the auxiliary game. If $R_{n+1} \neq R_n$ and the agent accepts R_n, P assigns probability 1 to H, so the continuation play is trivially defined in that case, by Proposition 1 (whose proof is independent of Theorem 1).
- If P proposes, at some round n, a menu M_n that does *not* correspond to the pair of contracts defined by the auxiliary game, let $\bar{R}_{n+1}$ denote the contract of $M_n \cup \{R_n\}$ that maximizes L's utility and

$\bar{C}_n$ denote the contract of $M_n \cup \{R_n\}$ that maximizes H's utility.[35] By construction, $\bar{R}_{n+1}$ and $\bar{C}_n$ satisfy (12)–(14). Let $\hat{R}_{n+1}$ denote the L-efficient contract that gives L the same utility as $\bar{R}_{n+1}$ and $\hat{C}_n$ denote the H-efficient contract that gives H the same utility as $\bar{C}_n$. There are three cases to consider: a) $u_H(\hat{C}_n) \geq u_H(\hat{R}_{n+1})$ and $u_L(\hat{R}_{n+1}) \geq u_L(\hat{C}_n)$, b) $u_H(\hat{C}_n) < u_H(\hat{R}_{n+1})$ and $u_L(\hat{R}_{n+1}) \geq u_L(\hat{C}_n)$, and c) $u_H(\hat{C}_n) \geq u_H(\hat{R}_{n+1})$ and $u_L(\hat{R}_{n+1}) < u_L(\hat{C}_n)$. Because of the single-crossing property, the fourth and last logical case cannot occur, as is easily checked.

Continuation play is then defined as follows, according to each case.

a) L chooses $\bar{R}_{n+1}$ with probability 1, H chooses $\bar{C}_n$ with probability 1. If the agent chooses any other contract R in $M_n \cup \{R_n\}$, then the principal assigns probability 1 to a type θ of the agent such that the other type $\theta' \neq \theta$ cannot benefit from choosing that contract if the principal put probability 1 on θ.[36] There always exists at least one such type, as is easily checked.

b) Case b) can occur only if $\bar{R}_{n+1}$ is in the H-Rent configuration. Continuation play is defined by the continuation equilibrium of the auxiliary game in which, following R_n, P proposes $\bar{R}_{n+1}$ and $\hat{C}_n$, but replacing $\hat{C}_n$ by $\bar{C}_n$.[37] In particular, L accepts $\bar{R}_{n+1}$ with probability 1, and H randomizes between the contracts $\bar{C}_n$ and $\bar{R}_{n+1}$ according to the probability μ_n of coming from the auxiliary equilibrium if $\bar{C}_n$ is replaced by $\hat{C}_n$. If the agent picks any contract other than $\bar{R}_{n+1}$ and $\bar{C}_n$, P assigns probability 1 to one type according to the same rule as in Case a). Continuation for rounds $m \geq n+1$ is also determined by the equilibrium of the auxiliary game.

c) Case c) is symmetric to Case b), and can only occur if $\bar{C}_n$ is in the L-Rent configuration. The continuation equilibrium is defined by the continuation equilibrium, from period 1 onwards (see Remark 1) of the auxiliary game starting in period 0 with belief $\tilde{\beta}_0 = 1 - \beta_n$ (since L now plays the role of H and vice versa) and at some fictitious contract $\tilde{R}$ in the L-Rent configuration such that $\hat{R}_{n+1}$ and $\bar{C}_n$ satisfy equations (14) and (12), respectively (the inequalities are reversed, because the equilibrium is in the L-Rent configuration).

This construction defines the continuation strategies after any possible history. We now verify that the strategy profile forms an equilibrium, using the one-shot deviation principle, which applies since the breakdown probability η has the effect of discounting the utility of future rounds at a geometric rate.

Consider first L's strategy, assuming that P follows the prescribed sequence of contracts. From (13), L cannot benefit from picking C_n: indeed, doing so causes β_n jumps to 1, and L to be stuck with utility $u_L(C_n)$, which is (weakly) lower than $u_L(R_{n+1})$ and hence lower than his continuation utility if he chooses R_{n+1}.[38] Similarly, if $R_{n+1} \neq R_n$ and L chooses R_n, then β_n jumps to 1, and L's continuation utility is bounded above[39] by $u_L(R_n)$, which is weakly dominated by accepting R_{n+1} by (12) (guaranteeing that L's

[35] If there are several maximizers, the equilibrium selects any of them. There must be at least one maximizer, because the menu is finite.

[36] That is, θ' prefers the contract that he is supposed to take with probability 1 in equilibrium (e.g., $\bar{C}_n$ if $\theta' = H$) to the θ-efficient contract that gives θ the same utility as R.

[37] By construction, $\bar{R}_{n+1} \in \mathcal{H}$, $\hat{C}_n$ is H-efficient, and the contracts satisfy conditions (12), (13), and (14), so the contract pair is feasible for the auxiliary games.

[38] That utility is always weakly higher than $u_L(R_{n+1})$, since L can always hold on to R_{n+1}.

[39] Indeed, P then proposes the H-efficient contract R that gives H utility $u_H(R_n)$, and $u_L(R) \leq u_L(R_n)$ by the single-crossing property and the fact that R_n is in the H-Rent configuration.

continuation utility is bounded below by $u_L(R_{n+1})$.

Let us now consider the optimality of H's strategy. From (14), $u_H(C_n) \geq u_H(R_n)$. Therefore, if H holds on to R_n, his continuation utility is equal to $u_H(R_n)$, which is weakly dominated by taking C_n. Moreover, given that H randomizes between C_n and R_{n+1}, his expected payoff is given by (16), and by perfection of the auxiliary equilibrium, the strategy $\{\mu_n\}$ is a best response to the sequence of contracts.

Consider now the agent's strategy after a deviation by P. In Case a), if L chooses $\bar{C}_n$, his utility is bounded above by $\max\{u_L(\bar{C}_n), u_L(\hat{C}_n)\}$, which is less than $u_L(\bar{R}_{n+1})$, by definition of Case a). Similarly, if L picks any other contract R, then either P puts probability 1 on L, in which case L gets utility $u_L(R)$, which is less than $u_L(\bar{R}_{n+1})$, by definition of $\bar{R}_{n+1}$, or P puts probability 1 on H, but in this case L cannot benefit from this erroneous belief. The same reasoning applies to H: it is optimal for that type to choose $\bar{C}_n$.

In Case b), L prefers $\bar{R}_{n+1}$ over any other contract in $M_n \cup \{R_n\}$, by an argument similar to Case a). Now consider H's response to P's deviation. First, H cannot benefit from choosing a contract R other $\bar{C}_n$ and $\bar{R}_{n+1}$, for the reason explained in Case a). Moreover, given the continuation play, which is defined by the auxiliary equilibrium, randomizing according the probability μ_n coming from the auxiliary equilibrium in which $\bar{R}_{n+1}$ and $\hat{C}_n$ are proposed is optimal.[40]

Case c) is similar to Case b).

There remains to show that P's strategy is optimal. By construction of the auxiliary equilibrium, P's strategy is optimal among all strategies that propose contracts (R_{n+1}, C_n) satisfying (12), (13), and (14). As shown by Lemma 3 (whose proof is independent of Theorem 1), P can never benefit from any deviation in which L accepts a contract that is not in the H-Rent configuration. Moreover, any contract R accepted by H with positive probability and that is *not* in the H-Rent configuration immediately results, at the next round, in an H-efficient contract that gives H the same utility as R and is less costly to P than R. We can therefore, without loss of generality, consider deviations in which P proposes one H-efficient contract, $\bar{C}_n$, and a number of contracts in the H-Rent configuration, among which $\bar{R}_{n+1}$ maximizes L's utility, and such that $u_L(\bar{C}_n) \leq u_L(R_{n+1})$. Given the agent's strategy, the menu is equivalent to just proposing $\bar{C}_n$ and $\bar{R}_{n+1}$, which is a feasible strategy in the auxiliary equilibrium and thus has to be weakly dominated by the equilibrium menu, by subgame perfection of that menu in the auxiliary game.

B Results holding for all friction levels

B.1 Statements

PROPOSITION 1 *The following holds for any PBE and η:*

i) If the prior β puts probability 1 on some type θ, P immediately proposes the θ-efficient contract that leaves θ's utility unchanged and θ accepts it.

ii) If R_0 is θ-efficient, P immediately proposes $E_{\theta'}(R_0)$ ($\theta' \neq \theta$), and θ' accepts it.

[40] Because $u_H(\hat{C}_n) = u_H(\bar{C}_n)$, H gets exactly the same utility as in the auxiliary equilibrium, even though the contract $\bar{C}_n$ is not in the H-Rent configuration.

iii) If R_0 is in the No-Rent configuration, P immediately proposes $E_L(R_0)$ and $E_H(R_0)$, and each type θ accepts $E_\theta(R_0)$.

iv) If R_0 is in the H-Rent (L-Rent) configuration, H's (L's) expected utility is bounded above by $u_H(E_H(R_0))$ ($u_L(E_L(R_0))$).

The next result is crucial for the analysis: for any PBE and round n, P can always propose the contracts $E_H(R_n)$ and $E_L(R_n)$ and have them accepted by types H and L, respectively. This deviation puts an upper bound on P's continuation cost as a function of the current contract R_n. The deviation will henceforth simply be referred to as the "jump."

Let β_n denote the probability, at the beginning of round n, that P assigns to type H.

Lemma 2 (Jump) *If R_n is in the H-Rent configuration and P proposes the contracts $E_H(R_n)$ and $E_L(R_n)$, with $E_H(R_n)$ augmented by an arbitrarily small amount $\varepsilon > 0$, then H accepts $E_H(R_n)$ with probability 1 and L accepts $E_L(R_n)$ with probability 1. Therefore, P's continuation cost is bounded above by $\bar{Q}_n = \beta_n Q(E_H(R_n)) + (1-\beta_n)Q(E_L(R_n))$*

Proof. The result follows from Part iv) of Proposition 1: $E_H(R_n)$ plus any small amount gives a strictly higher utility to H than what he can get under any continuation utility, and also gives him strictly more utility than $E_L(R_n)$. Therefore, H accepts the contract with probability 1. Because L strictly prefers $E_L(R_n)$ to $E_H(R_n)$ in the H-Rent configuration, and because the agent's type is revealed in round n unless L takes the strictly suboptimal contract $E_H(R_n)$, it is optimal for L to accept $E_L(R_n)$. ■

Lemma 3 *If R_0 is in the H-Rent configuration, then in any PBE, L accepts only contracts that are in the H-Rent configuration.*

Given any PBE, any contract sequence $\{R_n\}$ that is accepted by L with positive probability (until the exogenous negotiation breakdown) will be called a **choice sequence**. When R_0 is in the H-Rent configuration, choice sequences will play a particular role: we will see that, without loss of generality, any accepted contract sequence is a choice sequence, until H accepts an H-efficient contract. Moreover, choice sequences have several important properties. First, as indicated by Lemma 3, any choice sequence consists of contracts that are in the H-Rent configuration. Other properties are described by the following proposition.

Proposition 2 *Suppose that R_0 is in the H-Rent configuration. Along any choice sequence $\{R_n\}$, i) β_n converges to zero and ii) R_n converges to an L-efficient contract, denoted $\bar{C}_L$.*

B.2 Proofs

Proof of Proposition 1

Part i) Let $\bar{u}$ denote the agent's supremum over his expected utility, given his type θ, over all possible continuation PBEs starting from R_0 at which P puts probability 1 on type θ, and let $u = u_\theta(R_0)$. Suppose by contradiction that $\bar{u} > u$. By time homogeneity, $\bar{u}$ will be the same in the next round if the agent rejects new offers from P in round 0 and P continues to assign probability 1 on facing type θ. In such case, the agent's continuation payoff is bounded above by $\tilde{u} = \eta u + (1-\eta)\bar{u} < \bar{u}$. Consider any PBE that gives θ an expected utility $u_0 \in (\tilde{u}, \bar{u})$ (such PBE must exist, by definition of $\bar{u}$). Suppose that the principal deviates by proposing the θ-efficient contract C that give θ a utility level u' in $(\tilde{u}, u_0)$. By definition of a PBE,[41] P continues to assign probability 1 to type θ after his *own* deviation. If the agent accepts C with probability 1, the deviation is strictly profitable to P since C is the cheapest way of providing utility $u' < u_0$ to the agent. If the agent rejects the offer with positive probability, then by Bayes rule, P must continue to assign probability 1 to type θ, which implies that his continuation utility is bounded above by $\bar{u}$. Therefore, the agent's rejection is strictly suboptimal, implying that the agent must accept C with probability 1 and the deviation is profitable.[42] Let $\underline{Q}$ denote the cost of the θ-efficient contract, $\underline{C}$, that provides utility u to θ. Clearly, any PBE must cost exactly $\underline{Q}$, otherwise P has a profitable deviation which is to propose the θ-efficient contract that gives θ slightly more than u and costs less than following the PBE. Moreover, the only way of achieving $\underline{Q}$ is to propose $\underline{C}$ in the first round and have it accepted with probability one.

Part ii) Suppose without loss that $\theta = L$ (the opposite case is treated identically). Let $u_L = u_L(R_0)$ and $u_H = u_H(R_0)$. Also let $\bar{u}_H(\beta)$ denote the supremum utility that H can achieve over any continuation PBE starting from R_0 when P assigns probability β to H, and let $\bar{u}_H = \sup_{\beta\in[0,1]} \bar{u}_H(\beta)$. Suppose by contradiction that $\bar{u}_H > u_H$. Then, for any small $\varepsilon > 0$, there exists $\bar{\beta}$ and an associated PBE for which H's continuation utility is above $\bar{u}_H - \varepsilon > u_H$. For that PBE, because L gets at least u_L and C is L-efficient, $\bar{Q}_L \geq Q$, where $Q = Q(R_0)$, and $\bar{Q}_L$ is P's expected cost in that PBE conditional on facing θ_L. Since not proposing any new contract is always feasible for P, and costs Q, the continuation cost $\bar{Q}_H$ conditional on facing H must satisfy $\bar{Q}_H \leq Q$ to offset the weakly higher cost conditional on facing L. Suppose that P deviates from that PBE by proposing the H-efficient contract that gives θ_H utility $\bar{u}_H - \varepsilon - \epsilon$, for arbitrarily small ϵ. Because, for small enough ε and ϵ, $\bar{u}_H - \varepsilon - \epsilon > \eta u_H + (1-\eta)\bar{u}_H$, H accepts this proposal with probability 1. For any strategy that θ_L chooses and continuation equilibrium, this proposal strictly reduces P's expected cost (since $\bar{Q}_H \leq Q$), yielding a contradiction. This shows that $\bar{u}_H(\beta) = u_H$ for all β.[43] To conclude, suppose that P proposes the H-efficient contract that gives H utility $u_H + \tilde{\epsilon}$, for $\tilde{\epsilon}$ arbitrarily small. From the previous observation, H must accept that contract regardless of L's strategy. This shows that P can and, hence, does achieve the full-commitment optimal cost under any PBE. This proves Part ii).

Part iii) Suppose without loss that $Q_L \geq Q_H$, where $Q_\theta = Q(E_\theta(R_0))$ (the opposite case is proved symmetrically). Let $\bar{Q}$ denote the maximal expected cost incurred by P over all PBEs and beliefs $\beta \in [0,1]$, starting from R_0.

We start by showing that $\bar{Q} \leq Q_L$. Suppose by contradiction that $\bar{Q} > Q_L$ and consider any PBE that achieves $\bar{Q}$.[44] Now suppose that P deviates by proposing the pair $\tilde{C}_L, \tilde{C}_H$ of contracts such that $\tilde{C}_\theta$ is efficient for θ and costs $\bar{Q} - \varepsilon$ for some ε arbitrarily small compared to η. Those contracts maximize each type's utility subject to costing P at most $\bar{Q} - \varepsilon$. Because these contracts are efficient and incentive compatible,

[41] See Fudenberg and Tirole (1991), part iii) of the definition.

[42] The continuation play after P's deviation must be a PBE of the corresponding continuation game. Therefore, if θ's continuation strategy, after P's deviation, is to reject the proposed deviation with positive probability, Bayes rule applies. I am grateful to Marcin Peski for proposing the current version of this argument.

[43] If $\beta = 0$, P does not propose anything new, from i) and L-efficiency of R_0, and the result trivially holds in that case too.

[44] If the supremum $\bar{Q}$ is not achieved, the argument below can easily be adapted by considering a PBE whose expected cost is arbitrarily close to $\bar{Q}$.

Part ii) guarantees that no type ever chooses the contract meant for the other type. Moreover, no matter what belief and continuation PBE follows rejection of these contracts, P's continuation cost must be less than $\bar{Q}$, by definition of $\bar{Q}$. But this latter bound implies that there must be at least one type θ of the agent who is getting a lower payoff if he rejects $\tilde{C}_\theta$ than if he accepts it: conditional on rejection P has to be spending weakly less on at least one type of the agent than under $\tilde{C}_\theta$ (up to ε, which is negligible compared to η). Moreover, the contract $\tilde{C}_\theta$ maximizes that type's utility subject to P spending less than $\tilde{C}_\theta$. Since rejection leads to a renegotiation breakdown with a probability η, which gives that type a strictly lower utility than $\tilde{C}_\theta$, accepting $\tilde{C}_\theta$ is strictly more profitable than rejection for that type, and thus he accepts $\tilde{C}_\theta$ with probability 1. As a result, a rejection fully reveals that the agent is of the other type. From Part i), that agent gets $u_\theta(C)$ after rejection, which is strictly less than the utility he gets from $\tilde{C}_\theta$ (since that contract maximizes the agent's utility subject to a higher cost than what P incurs with C). Therefore, both types accept their contract, and this reduces the cost of the principal strictly below $\bar{Q}$, showing that this is a profitable deviation. Thus, necessarily, $\bar{Q} \leq Q_L$.

Since L cannot get utility less than $u_L(R_0)$, under any PBE, and Q_L is the cheapest cost of providing that utility, this means that in all PBEs starting with $\beta \in (0,1)$, P must spend weakly less than $\bar{Q}$ on the high type, in order to guarantee that $\bar{Q} \leq Q_L$. Let $\bar{u}_H$ denote the supremum expected utility that H gets over all PBEs and beliefs $\beta > 0$. Since P spends less than Q_L on H, $\bar{u}_H$ is bounded by the utility $\hat{u}_H$ obtained from the H-efficient contract $\hat{C}_H$ that costs Q_L. We will show that $\bar{u}_H = u_H(E_H(R_0))$. Suppose by contradiction that $\bar{u}_H > u_H(E_H(R_0))$, and consider a PBE that achieves $\bar{u}_H$.[45] The expected cost Q from that PBE must be above $\beta Q(\bar{C}_H) + (1-\beta)Q_L$, where $\bar{C}_H$ is the H-efficient contract that gives utility $\bar{u}_H$ to H. Suppose that P deviates by proposing the contracts $\tilde{C}_L, \tilde{C}_H$ such that $\tilde{C}_L$ is L-efficient and gives utility $u_L(C) + \varepsilon^2$ to L and $\tilde{C}_H$ is H-efficient and gives utility $\bar{u}_H - \varepsilon$ to H, for ε small compared to η. H accepts $\tilde{C}_H$, since rejection leads to a continuation utility bounded above by $\bar{u}_H$ and to a strictly lower payoff in case of a breakdown. Given that, L also accepts, since rejection will reveal his type, and, by Part i), result in a utility of $u_L(E_L(R_0))$. The cost reduction on the high type is of order ε compared to $Q(\bar{C}_H)$, while the cost increase on the low type is of order ε^2, compared to Q_L. Therefore, this deviation is strictly profitable for ε small enough. This shows that $\bar{u}_H = u_H(E_H(R_0))$. Proceeding as in the end of the proof of Part i), this shows that L's maximal utility across all PBEs for $\beta \in (0,1)$ is $u_L(E_L(R_0))$.

Part iv) The argument is similar to the proof of Part iii). Let $\bar{Q}$ denote P's maximal expected cost over all PBEs and beliefs, starting from R_0. We will start by showing that $\bar{Q} \leq Q(E_L)$, where $E_L = E_L(R_0)$. Suppose by contradiction that $\bar{Q}$ is strictly greater than $Q(E_L)$ and achieved for some PBE and belief,[46] and consider the following deviation: P proposes the contracts $\tilde{C}_\theta$ that are efficient for each type and cost $\bar{Q} - \varepsilon$ for ε arbitrarily small. It is easily shown that these contracts are IC, and by a similar argument as in Part iii), rejecting those contracts is always a strictly dominated strategy for one of the two types, and hence for both types. This is a strictly profitable deviation for P, yielding a contradiction. Hence, $\bar{Q} \leq Q(E_L)$. Since L gets an expected utility of at least $u_L(R_0)$ in all PBEs, and providing that utility costs at least $Q_L = Q(E_L)$ to P, this means that P spends at most Q_L on H, in all PBEs, and for all initial beliefs $\beta > 0$. This implies that H's expected utility is bounded above by the utility it achieves with the H-efficient contract that costs Q_L. We now show that H's expected utility is bounded above by $u_H(E_L)$. Suppose not, and consider a PBE

[45] Again, the proof is easily adapted if the supremum is not achieved, by considering a PBE that gets very close to providing $\bar{u}_H$.

[46] As before, one can use a PBE that yields a cost arbitrarily close to $\bar{Q}$, in case it is not exactly achieved.

that gives H its highest utility, across PBEs and beliefs, denoted $\bar{u}_H > u_H(E_L)$. The expected cost Q from that PBE must be above $\beta Q(\bar{C}_H) + (1-\beta)Q_L$, where $\bar{C}_H$ is the H-efficient contract that gives utility $\bar{u}_H$ to H. Suppose that P deviates by proposing the contracts $\tilde{C}_L, \tilde{C}_H$ such that $\tilde{C}_L$ is L-efficient and gives utility $u_L(C) + \varepsilon^2$ to L, and $\tilde{C}_H$ is H-efficient and gives utility $\bar{u}_H - \varepsilon$ to H, for ε arbitrarily small. Because $\tilde{C}_H$ gives strictly more to H than $\bar{u}_H$, H will accept $\tilde{C}_H$ and, hence L will accept $\tilde{C}_L$. Repeating the proof of Part iii), one can show that this deviation is strictly profitable, establishing the desired contradiction. The only difference with that earlier proof lies in showing that the proposed contracts are incentive compatible. This is indeed true, for ε small enough, because $\bar{u}_H > u_H(E_L)$ so H does not want to mimic L.[47]

Proof of Lemma 3

Consider any PBE starting with R_0 in the H-Rent configuration. Consider, by contradiction, the first round n such that i) R_n is the H-Rent configuration and ii) L accepts with positive probability a contract R_{n+1} that is in a different configuration. Suppose that R_{n+1} is in the No-Rent configuration. Then $u_L(n) = u_L(R_{n+1})$, by Part iii) of Proposition 1. This immediately implies that $u_L(R_n) \leq u_L(R_{n+1})$: R_{n+1} is on a weakly higher iso-utility curve of u_L than R_n. Moreover, because H can always accept R_{n+1}, $u_H(n) \geq u_H(R_{n+1}) > u_H(E_H(R_n))$, where the strict inequality comes from the fact that u_H is increasing along the iso-utility curve of u_L in the direction of $\mathcal{E}_H$.[48] This implies that the continuation cost for P is strictly above $\beta_n Q(E_H(R_n)) + (1-\beta_n)Q(E_L(R_n))$, which contradicts Lemma 2. Now suppose that R_{n+1} is in the L-Rent configuration. Part iv) of Proposition 1 applied to the L-Rent configuration implies that, by choosing R_{n+1}, L gets a continuation utility of at most $u_L(E_L(R_{n+1})$. Therefore, $u_L(E_L(R_{n+1})$ must be weakly greater than $u_L(R_n)$. This, along with the single-crossing property, implies that $u_H(R_{n+1})$ is strictly greater than $u_H(E_H(R_n))$ and contradicts Part iv) of Proposition 1 applied to H. ■

Proof of Proposition 2

i) Observe, first, that negotiation cannot end endogenously at a finite round N in the sense that $\beta_n = \beta_N > 0$ and $R_n = R_N \in \mathcal{H}$ for all $n \geq N$. If that were the case, P could strictly reduce his cost at round N by proposing the H-efficient contract $E_H(R_N)$ and have it accepted by H with probability 1, by Part iv) of Proposition 1. Hence, consider the case in which P keeps proposing new contracts until renegotiation is exogenously interrupted, and suppose by contradiction that there is a choice sequence with an associated belief subsequence $\{\beta_{n(k)}\}_{k\in\mathbb{N}}$ that converges to $\beta^* > 0$. Let $u_H^* = \sup\{u_H(R_n)\}$ where the supremum is taken among all contracts in the choice sequence. For H to accept R_n with positive probability infinitely often, $u_H(R_n)$ must converge to u_H^* for any subsequence, including along the subsequence $\{n(k)\}$.[49] However, that implies that proposing the H-efficient contract C_H that gives u_H^* to H is a strictly profitable deviation as $\beta_{n(k)}$ gets arbitrarily close to β^*: it does not change P's cost conditional on facing L but it strictly reduces P's expected cost by an amount arbitrarily close to $\beta^*[Q(C_L) - Q(C_H)]$, where C_θ is the θ-efficient contract

[47] It is straightforward to show that L does not want to mimic H, since P spends less on H than on L, and L is already getting his maximal utility given the cost that P incurs conditional on facing L.

[48] More explicitly, we have $u_H(R_{n+1}) > u_H(E_L(R_{n+1})) \geq u_H(E_L(R_n)) = u_H(E_H(R_n))$.

[49] Otherwise, there must exist a subsequence of rounds for which $u_H(R_{m+1})$ is bounded above away from u_H^* by some constant $\delta > 0$. However, H's continuation utility, $u_H(m)$, is nondecreasing and becomes arbitrarily close to u_H^*. (Monotonicity comes from the fact that H can always hold on to the last accepted contract and is proved formally in Lemma 5.) When H's continuation gets within $\varepsilon\eta$ of u_H^* for some ε arbitrarily small, this implies that accepting R_{m+1} causes a loss of order $\eta\delta$, due to the probability of an immediate breakdown, and contradicts the fact that $u_H(m)$ is within $\varepsilon\eta$ of u_H^*.

that provides H with utility u_H^*.[50]

ii) Suppose that there exists $\varepsilon > 0$ and a subsequence of rounds, indexed by m, for which $Q(R_m) - Q(E_L(m)) \geq \varepsilon$. For m large enough, β_m converges to zero, from part i), and is thus bounded above by $\frac{\eta\varepsilon}{2\Delta_Q}$, where $\Delta_Q = \max_{C\in\mathcal{C}} Q(C) - \min_{C\in\mathcal{C}} Q(C)$. Therefore, P can deviate by proposing $E_L(m), E_H(R_m)$, which are respectively accepted by L and H. This deviation yields an immediate gain of $\eta\varepsilon$ on L and a loss of at most $\frac{\eta\varepsilon}{2}$ on H, given the upper bound on β_m, and is thus strictly profitable. This shows that the limit points of $\{R_n\}$ are all L-efficient. Let $u_L^* = \sup\{u_L(R_n)\}$. There is a subsequence $\tilde{m}$ for which $u_L(R_{\tilde{m}})$ converges to u_L^*. Moreover, since L can always hold on to any contract R_n along the choice sequence, and thus in particular for contracts occurring along the subsequence $\{\tilde{m}\}$, $u_L(R_n)$ must converge to u_L^* for all subsequences. Combining these observations, $\{R_n\}$ must converge to the L-efficient contract $\bar{C}_L$ such that $u_L(\bar{C}_L) = u_L^*$. ■

C Proof of Theorem 2

Without loss of generality, it suffices to prove the theorem when R_0 is in the H-Rent configuration: Proposition 1 already addresses the case in which R_0 is in the No-Rent configuration, and the L-Rent configuration can be proved by symmetry. Let us thus assume that $R_0 \in \mathcal{H}$. From Lemma 3, L accepts only contracts in $\mathcal{H}$. Moreover, any contract C_n that is accepted only by H in equilibrium can be replaced by an H-efficient contract $\tilde{C}_n$ that gives H the same utility, without affecting anyone's incentive (assuming that H accepts $\tilde{C}_n$ with the same probability as the one with which he was accepting C_n in the initial equilibrium) and reduces P's cost. Therefore, we can without loss of generality focus on PBEs in which P only proposes, at each round, a number of contracts in $\mathcal{H}$, and the H-efficient contract that gives H his continuation utility. This assumption is maintained throughout the analysis.

Organization of the proof

The proof of Theorem 2 (Statement A) proceeds by contradiction. We will suppose that there exists $\varepsilon > 0$, a decreasing sequence $\{\eta_m\}_{m\in\mathbb{N}}$ of breakdown probabilities converging to zero, and a PBE associated to each η_m for which H's expected utility $u_H(0)$ at round 0 is below $u_H(E_H(R_0)) - \varepsilon$. Throughout, $u_\theta(n)$ will denote θ's continuation utility at the beginning of round n.

In what follows, we focus entirely on that sequence of η's and corresponding PBEs. The expression "as η goes to zero" will refer to the elements of that sequence and corresponding PBEs.[51]

The difference $w_0 = u_H(E_H(R_0)) - u_H(0)$ can be thought of as a *rent extraction index* for type H. It defines how much rent P is extracting from H, relative to the immediate jump: $u_H(0)$ is H's continuation utility while $u_H(E_H(R_0))$ is the maximal utility that P can give H in any equilibrium, as shown by Proposition 1, part iv).

The proof consists of the following steps.

[50] Since $u_H(R_n)$ gets arbitrarily close to u_H^* and R_n lies in $\mathcal{H}$, $Q(R_n)$ becomes arbitrarily close to (or above) $Q(C_L)$ as n gets large.

[51] Without loss of generality, we focus on ε small enough so that the constant $D(2\varepsilon)$ defined by (57) is strictly positive.

Step 1: For each PBE of the sequence, show that one can construct a choice sequence ending at some finite round $\tilde{N}$ for which the *augmented* rent extraction index,

$$\bar{w}_{\tilde{N}} = \max_{m \leq \tilde{N}} \{u_H(E_H(\tilde{m}))\} - u_H(\tilde{N}),$$

is of order η, and there exists $d > 0$ such that either

a) $\beta_{\tilde{N}} \geq \eta^d$ and $w_0 \leq \hat{w}\sqrt{\eta}$, where $\hat{w} > 0$ is exogenous, or

b) $\beta_{\tilde{N}} < \eta^d$.

Proving that step is the object of Part I below. Of course, Case a) above implies that w_0 could not have been greater than ε, for η small enough. Therefore, it suffices to rule out Case b).

Step 2: Show that, in Case b), there must exist a round $N \geq \tilde{N}$ for which $\bar{w}_N \leq \frac{\eta D}{2a}$ but $\bar{w}_N \geq \underline{w}\eta$ and $\beta_N \leq \eta^d$, for some exogenous constants $a, D, \underline{w}$. This is done in Part II.

Step 3: Show that at round N, one must have $\bar{w}_N \leq \bar{w}\eta^{1+d}$ for some $\bar{w} > 0$. This contradicts, for η small enough, the inequality of Step 2 involving $\underline{w}$, which rules out Case b). (Part III).

To avoid cluttering the exposition, the longest proofs for each of these parts are given in separate appendices (Appendix E for Parts I and II and Appendix F for Part III). Several of these proofs are based on geometric inequalities and other inequalities based on incentive constraints which are collected in Appendix D. Once Statement A of Theorem 2 has been proven, showing Statement B is relatively straightforward; the proof is provided in Appendix G.

Part I: Block Construction

The strategy of the proof is to build a sequence of *blocks* (each consisting of finitely many rounds), and a choice sequence going through these blocks, with the following properties: i) within each block, for the PBE to be profitable to P compared to an immediate jump, H must accept H-efficient contracts with a high enough probability, which drives the posterior β closer to zero, by a controlled amount, ii) P's potential gain, conditional on facing type H, shrinks geometrically across blocks. This construction ends at some terminal block, K, such that P's maximal potential gain on H is of order η, and the posterior $\hat{\beta}_K$ is bounded above by $g^K\beta_0$ for some factor $g < 1$. When $\hat{\beta}_K > \eta^d$ for some power $d > 0$ that is judiciously chosen, this yields an upper bound on the number K of blocks which, by using the geometric series backwards, implies that the initial gain on H must have been small as well, for η small enough, contradicting the existence of a sequence $\{\eta_m\}$ and corresponding PBEs for which the initial rent index w_0 always exceeds the constant ε. The ulterior parts (Parts II and III) of the proof establish that $\hat{\beta}_K > \eta^d$ is the only possible case, provided that η is small enough.

For any round n and choice sequence up to round n, let $\bar{e}_n = \max\{u_H(E_H(R_m)) : m \leq n\}$ and $\bar{w}_n = \bar{e}_n - u_H(n)$.

Notation Throughout the analysis, variables with upper bars, such as $\bar{w}_n$, refer to specific *rounds*, while variables with hats, such as $\hat{w}_k$, refer to specific *blocks*.

We begin the proof by the following observation.

Lemma 4 *If $u_H(0) < u_H(E_H(R_0)) - \varepsilon$, there exists a choice sequence and a round n_0 such that i) $\beta_{n_0} \le \beta_0$ and ii) $\bar{w}_{n_0} \in [\varepsilon/2, \varepsilon]$.*

Block 1 starts after the choice sequence and round n_0 coming from Lemma 4. At round n_0, one has by construction $\bar{w}_{n_0} = \bar{e}_{n_0} - u_H(n_0) \in (\varepsilon/2, \varepsilon)$. Let $\hat{u}_0 = u_H(n_0)$, $\hat{e}_0 = \bar{e}_{n_0}$, and $\hat{\beta}_0 = \beta_{n_0} \le \beta_0$. The last round of Block 1 is determined as follows: let $\hat{u}_1$ be defined by the equation

$$\frac{\hat{e}_0 - \hat{u}_0}{\hat{u}_1 - \hat{u}_0} = t > 1$$

where $t > 1$ is a fixed threshold to be determined shortly and let $n_1 = \inf\{n : u_H(n) \ge \hat{u}_1\}$ denote the first round at which H's continuation utility exceeds the threshold $\hat{u}_1$. We set the last round of Block 1 equal to n_1. Because $\hat{u}_1 < \hat{e}_0$, Lemma 14 (Appendix E) guarantees that n_1 is finite with probability 1.[52]

To get a control on how much β must have dropped within Block 1, let μ_0 denote the probability, evaluated at round n_0, that H accepts only contracts in $\mathcal{H}$ until round n_1 (i.e., the probability that H does not fully reveal himself during Block 1). Lemma 15 (Appendix E) shows that there must exist a *pushdown* choice sequence such that, upon observing that sequence up until H's utility passes $\hat{u}_1$, the posterior probability $\hat{\beta}_1$ of facing H satisfies

$$\hat{\beta}_1 \le \frac{\hat{\beta}_0 \mu_0}{\hat{\beta}_0 \mu_0 + (1 - \hat{\beta}_0)}. \tag{18}$$

At round n_0, P can always implement the jump to the contracts $(E_H(R_{n_0}), E_L(R_{n_0}))$, by Lemma 2. For that deviation to be suboptimal, the ***net gain*** from extracting some rent from H, compared to the jump, must outweigh the ***net loss*** resulting from a negotiation breakdown at an inefficient contract. In the argument below, it suffices to exploit the inefficiency loss on H. (The loss on L is exploited later in the proof of Theorem 2.)

We now compute an upper bound on this gain and a lower bound on the loss. Comparing these bounds will yield an upper bound on P's posterior belief of facing H after the first block, following the pushdown choice sequence. The following lemma will be helpful to derive these bounds, as well as in later parts of the proof.

Lemma 5 *Along any choice sequence, H's continuation utility at the beginning of round n, denoted $u_H(n)$, is nondecreasing in n and satisfies*

$$u_H(n+1) - u_H(n) \le \eta \Delta_H,$$

where $\Delta_H = \max_{C \in \mathcal{C}} u_H(C) - \min_{C \in \mathcal{C}} u_H(C)$.

Proof. Given the current contract R_n at round n, let R_{n+1} denote any contract chosen by H with positive probability among $R_n \cup \{M_n\}$. H's utility satisfies the dynamic equation[53]

$$u_H(n) = \eta u_H(R_{n+1}) + (1 - \eta) u_H(n+1). \tag{19}$$

[52] Indeed, $u_H(R_n)$ must eventually exceed any utility level below $\max\{u_H(E_H(R_m)) : m \le n_0\}$, along any choice sequence, as n gets large enough.

[53] More generally, H's utility satisfies the Bellman equation $u_H(n) = \max_{R \in \{R_n\} \cup M_n}\{\eta u_H(R) + (1-\eta) u_H(n+1)\}$. Equation (19) then follows for all contracts that are optimal for H in round n.

Therefore, $u_H(n)$ is a convex combination of $u_H(R_{n+1})$ and $u_H(n+1)$. Because H can hold on to R_{n+1} in all rounds $m \geq n+1$, $u_H(n+1)$ is bounded below by $u_H(R_{n+1})$. Combining these observations yields $u_H(n) \leq u_H(n+1)$. Since $u_H(n+1) - u_H(n) = \eta(u_H(n+1) - u_H(R_{n+1}))$, the second claim of the lemma follows. The intuition is simple: if the utility jump was higher between two rounds, H would prefer to wait until the next round rather than accept any contract today. ■

The net gain, between rounds n_0 and n_1, is bounded above by $\hat{\beta}_0(1-\mu_0)a(\hat{e}_0 - \hat{u}_0)$ for some Lipschitz constant $a > 0$. Indeed, $\hat{\beta}_0(1-\mu_0)$ is the probability that the agent is of type H *and* that he accepts some H-efficient contract at some round of the first block. Because H accepts only H-efficient contracts that give him at least his continuation utility,[54] and because that continuation utility is nondecreasing, by Lemma 5, the smallest utility that H when choosing H-efficient contract within that first block is $\hat{u}_0$. By contrast, $\hat{e}_0 \geq e_{n_0}$ is an upper bound on the utility that P provides to H if he chooses the immediate jump. Therefore, the maximum rent that P can extract from H is $\hat{e}_0 - \hat{u}_0$. The constant a is a Lipschitz constant that "translates" utility differences for H along $\mathcal{E}_H$ into cost differences for P, and is derived in the "Inequalities" section of the appendix (Lemma 10).

Similarly, the expected net gain made after round n_1, but seen from round n_0, is bounded above by $\hat{\beta}_0\mu_0 a(\hat{e}_0 - \hat{u}_1)$, because $\hat{\beta}_0\mu_0$ is the probability of facing H *and* reaching round n_1, and $\hat{u}_1$ is the smallest utility that P must provide to H at any round following n_1.

To get a lower bound on the net loss, the intuition is that, as long as H accepts contracts in $\mathcal{H}$, he is getting contracts that are inefficient, and hence costly to P relative to the immediate jump to $E_H(R_{n_0})$. Lemma 16 shows that this inefficiency cost must be greater than some constant $D > 0$ whenever the rent extraction index at the beginning of a block is less than 2ε, which holds without loss of generality (see Remark 3 in Appendix E). This cost is only incurred in case of a breakdown. To compute the probability of a breakdown between rounds n_0 and n_1, the key is to observe that H's utility jumps, at each round, by at most $\eta\Delta_H$, by Lemma 5. Therefore, there must be at least $\underline{n}(1) = \lfloor(\hat{u}_1 - \hat{u}_0)/\eta\Delta_H)\rfloor$ steps to get to $\hat{u}_1$, for any choice sequence.

This implies that the breakdown probability is bounded below by[55]

$$1 - (1-\eta)^{\underline{n}(1)} = 1 - \exp\left(\underline{n}(1)\ln(1-\eta)\right) \geq -\underline{n}(1)\ln(1-\eta) - \frac{1}{2}\underline{n}(1)^2(\ln(1-\eta))^2. \tag{20}$$

Because the gain is of order ε, which is small, while the loss conditional on a breakdown is of order D, the probability of a breakdown must be (at most) of order ε, which means that $\underline{n}(1)\ln(1-\eta)$ must also be small. The quadratic term of (20) is therefore negligible. Moreover, because we are focusing on the case where η is

[54]Indeed, by accepting such contract, H reveals his type, and his continuation utility is exactly the one provided by the last accepted contract, by Proposition 1, Part i).

[55]The inequality comes from the standard inequality $1 - \exp(x) \geq -x - x^2/2$, valid for all $x \leq 0$, which may be shown as follows. The function $x \mapsto \exp(x) - 1 - x - \frac{x^2}{2}$ vanishes at 0, as do its first and second derivatives. Since its third derivative is positive (equal to $\exp(x)$), its first derivative is convex and, from the previous observations, must have a minimum at zero. This implies that the function itself is increasing and, since it vanishes at 0, that it is negative for $x \leq 0$.

small, $\ln(1-\eta)$ can be approximated by $-\eta$. Combining these bounds on gains and losses yields[56]

$$\beta a[(\hat{e}_0-\hat{u}_0)(1-\mu_0)+(\hat{e}_0-\hat{u}_1)\mu_0] \geq \beta\mu_0 D\frac{\hat{u}_1-\hat{u}_0}{\Delta_H}. \tag{21}$$

Recall that $\hat{u}_1$ was defined in terms of an arbitrary threshold t. We now define t precisely by the following equation[57]

$$t^2 = \frac{a + D/\Delta_H}{a} > 1.$$

With this value of t, we have

$$\mu_0 \leq \frac{a}{a+D/\Delta_H}\frac{\hat{e}_0-\hat{u}_0}{\hat{u}_1-\hat{u}_0} = t^{-1}.$$

Combining this inequality with (18) implies that, upon observing the constructed pushdown choice sequence until round n_1, the posterior $\hat{\beta}_1$ satisfies

$$\hat{\beta}_1 \leq \frac{\mu_0\hat{\beta}_0}{\mu_0\hat{\beta}_0+(1-\hat{\beta}_0)} \leq \hat{\beta}_0\frac{t^{-1}}{\hat{\beta}_0 t^{-1}+(1-\hat{\beta}_0)} = g\hat{\beta}_0.$$

where $g = \frac{t^{-1}}{\beta_0 t^{-1}+(1-\beta_0)}$ and we used the inequality $\hat{\beta}_0 \leq \beta_0$. Because $t^{-1} < 1$, g is strictly less than 1. This achieves the goal of guaranteeing that the posterior $\hat{\beta}_1$ drops by some fixed factor along the first block, for some choice sequence.

To initiate the second block, we use the value $\hat{u}_1$ that was defined as part of Block 1.[58] Note that the actual value of $u_H(n_1)$ may be slightly above $\hat{u}_1$, but by no more than $\Delta_H\eta$, by Lemma 5. This observation is useful to bound below the number of rounds in each block. The level $\hat{e}_1 = \max_{m\leq n_1}\{u_H(E_H(R_m))\}$ is the maximum value that H gets if P jumps at any round $m \leq n_1$ along the particular choice sequence constructed so far. Having defined $\hat{u}_1$ and $\hat{e}_1$, we define $\hat{u}_2 \in (\hat{u}_1, \hat{e}_1)$ by

$$\frac{\hat{e}_1-\hat{u}_1}{\hat{u}_2-\hat{u}_1} = t.$$

Let μ_1 denote the probability, seen from round n_1 and following the pushdown choice sequence used for Block 1, that H takes a contract in $\mathcal{H}$ at all rounds $n \geq n_1$ until $\hat{u}_2$ is reached. Repeating the previous analysis, there exists a pushdown choice sequence for Block 2 such that, upon observing that sequence up to $\hat{u}_2$, the probability $\hat{\beta}_2$ of facing H satisfies $\hat{\beta}_2 \leq \frac{\hat{\beta}_1\mu_1}{\hat{\beta}_1\mu_1+(1-\hat{\beta}_1)}$. Let n_2 denote the round at which $\hat{u}_2$ is first exceeded. Repeating the analysis used for the first block, we have

$$\hat{\beta}_2 \leq \frac{\mu_1\hat{\beta}_1}{\mu_1\hat{\beta}_1+(1-\hat{\beta}_1)} \leq \hat{\beta}_1\frac{t^{-1}}{\hat{\beta}_1 t^{-1}+(1-\hat{\beta}_1)} \leq g^2\hat{\beta}_0.$$

The value of $\hat{e}_2$ is determined by the pushdown sequence of the second block, by $\hat{e}_2 = \max_{m\leq n_2}\{u_H(E_H(R_m))\}$.

By induction, this defines a sequence of blocks indexed by k. To each block k corresponds a terminal round, n_k, as well as values $\hat{u}_k, \hat{e}_k$ and $\hat{\beta}_k = \beta_{n_k}$, which is P's belief at the end of the k^{th} block following

[56] For expositional simplicity, the "floor" operator is dropped. This change is negligible because $\underline{n}(1)$ is large, since $\hat{u}_1-\hat{u}_0 = \frac{1}{t}(\hat{e}_0-\hat{u}_0) \gg \eta\Delta_H$, for η small. That observation applies to each block k; see Footnote 59.

[57] D is defined independently of t (and of this entire block construction), so there is no circularity in the definition.

[58] The next block is defined only following the pushdown choice sequence that we constructed in Block 1: what matters to us is to understand what happens along a particular choice sequence constructed by piecing together pushdown sequences constructed for each block.

the successive pushdown sequences. The potential overshoot of $u_H(n_k)$ above $\hat{u}_k$, $\Delta_H\eta$ is negligible when computing the lower bound on number of blocks, because we stop the block construction when $\hat{u}_{k+1} - \hat{u}_k$ is still large relative to $\Delta_H\eta$, as explained in the next paragraph. Upon observing the pushdown choice sequence across blocks 1 to k, we get

$$\hat{\beta}_k \leq g^k \hat{\beta}_0 \leq g^k \beta_0.$$

The construction stops at the first block, K, such that $\hat{e}_k - \hat{u}_k < \bar{W}\eta$ for some constant $\bar{W}$ such that $\bar{W} > \max\{\frac{t-1}{t}(1+\Delta_H), \frac{\hat{W}+\Delta_H}{t\Delta_H}\}$, where $\hat{W}$ is an arbitrarily large constant.[59] Such a block must exist, because $\hat{w}_k = \hat{e}_k - \hat{u}_k$ converges to zero, as shown in Part ii) of Lemma 14. Let ρ be defined by $g^{-\rho} = \frac{t}{t-1}$. Since the ratio is greater than 1 and $g < 1$, ρ is strictly positive. Also let $d = \frac{1}{2}\min\{\frac{1}{\rho}, 1\} \in (0, 1/2]$.

As explained earlier, the key to showing Theorem 2 is the following proposition, whose proof is the object of Parts II and III.

Proposition 3 *There exists $\tilde{\eta} > 0$ such that $\hat{\beta}_K > \eta^d$ for all $\eta < \tilde{\eta}$.*

It must be emphasized that the proposition holds for any $d > 0$. However, we only need it for the value of d defined above. Taking Proposition 3 as given for now, we compute an upper bound on the initial rent, by backward induction. For each block $k \leq K$, we have

$$\hat{e}_K - \hat{u}_k = (\hat{e}_K - \hat{u}_{k+1}) + (\hat{u}_{k+1} - \hat{u}_k) \leq (\hat{e}_K - \hat{u}_{k+1}) + \frac{1}{t-1}(\hat{e}_k - \hat{u}_{k+1}) \leq \frac{t}{t-1}(\hat{e}_K - \hat{u}_{k+1}).$$

By construction, moreover, $\hat{e}_K - \hat{u}_K \leq \bar{W}\eta$, which implies that

$$\hat{e}_K - \hat{u}_0 \leq \left(\frac{t}{t-1}\right)^K \bar{W}\eta. \tag{22}$$

Since $\hat{\beta}_K \geq \eta^d$ and $\hat{\beta}_K \leq g^K \hat{\beta}_0 < 1$, we must also have

$$\frac{1}{g^K}\eta^d \leq 1.$$

Combining these inequalities yields

$$\hat{e}_K - \hat{u}_0 \leq \left(\frac{t}{t-1}\right)^K \bar{W}\eta = \bar{W}\eta\left(\frac{1}{g}\right)^{\rho K} \leq \eta\bar{W}\eta^{-\rho d} \leq \bar{W}\eta^{1/2}.$$

Since $\hat{e}_K \geq \hat{e}_0$, this shows that $\hat{e}_0 - \hat{u}_0 = O(\eta^{1/2})$, which contradicts the existence of the sequence of $\{\eta_m\}$, converging to zero, and corresponding PBEs for which $\hat{e}_0 - \hat{u}_0 \in (\varepsilon/2, \varepsilon)$.

[59] The number of rounds in each block $k \leq K$ is bounded below by $\frac{\hat{u}_k - \hat{u}_{k-1} - \Delta_H\eta}{\Delta_H\eta} \geq \frac{1}{t\Delta_H\eta}(\hat{e}_{k-1} - \hat{u}_{k-1} - \Delta_H\eta) \geq \frac{\bar{W}\eta - \Delta_H\eta}{t\eta\Delta_H} > \hat{W}$, which can be made arbitrarily large by choosing $\hat{W}$ appropriately. The reason for also requiring that $\bar{W} > \frac{t-1}{t}(1+\Delta_H)$ is explained at the beginning of Part II.

Part II: Bridging Argument

The objective of Parts II and III is to prove Proposition 3. Suppose, by contradiction, that $\hat{\beta}_K < \eta^d$. By definition of K, the previous block $K-1$ must satisfy $\hat{e}_{K-1} - \hat{u}_{K-1} > \bar{W}\eta$. This implies that[60]

$$\hat{w}_K = \hat{e}_K - \hat{u}_K \geq \underline{W}\eta \tag{23}$$

where $\underline{W} = \frac{t-1}{t}\bar{W}$. Since we chose $\bar{W} > \frac{t}{t-1}(1+\Delta_H)$, we have $\underline{W} > 1+\Delta_H$. Combining this with Lemma 5, we obtain, for the augmented index evaluated at round n_K,[61]

$$\hat{w}_K = \hat{e}_K - u_H(n_K) \geq (\underline{W} - \Delta_H)\eta \geq \eta. \tag{24}$$

Part III (Proposition 5) establishes that *if* $\hat{w}_K \leq \frac{D\eta}{2a}$, then $\hat{w}_K \leq \hat{w}\eta^{1+d}$, which contradicts (24) for η small enough.

Unfortunately, nothing guarantees that $\hat{w}_K$ lies below $\frac{D\eta}{2a}$. The purpose of Part II is to bridge Parts I and III when $\hat{w}_K \in \left(\frac{D\eta}{2a}; \bar{W}\eta\right)$. We will analyze the dynamics of β_n and $\bar{w}_n$ along some appropriate choice sequence between the levels $\hat{w}_K$ and $\frac{\eta D}{2a}$, and establish the following result.

Proposition 4 (Bridge) *Suppose that $\hat{\beta}_K < \eta^d$. Then, letting $N \geq n_K$ denote the first round for which $\bar{w}_N \leq \frac{\eta D}{2a}$, there exists a choice sequence such that*
1. $\bar{w}_N \geq \frac{\eta D}{2a} - o(\eta)$
2. $\beta_N = O(\eta^d)$.

Thus, if $\hat{\beta}_K < \eta^d$, there must exist a choice sequence and a round N to which the contradiction argument of Part III can be applied.

To construct a choice sequence that yields the two conclusions of Proposition 4, we start by exploiting P's IC constraint, similarly to what was done in Part I. This time, however, there are no blocks: the equation is used at every single round n, and exploits the losses on *both* types. For each $R_{n+1} \in M_n \cup \{R_n\}$, let $\mu_n^\theta(R_{n+1})$ denote the probability that θ accepts R_{n+1}. Because P can always jump to $E_L(R_n), E_H(R_n)$ (cf. Lemma 2), P's IC constraint implies as explained below that

$$\begin{aligned} w_n a \beta_n &\geq \sum_{R_{n+1} \in (M_n \cup \{R_n\}) \cap \mathcal{H}} \beta_n \mu_n^H(R_{n+1})\eta D + (1-\beta_n)\mu_n^L(R_{n+1})\eta(Q(R_{n+1}) - Q(E_L(R_n))) & (25) \\ &= \sum_{R_{n+1} \in (M_n \cup \{R_n\}) \cap \mathcal{H}} \mu_n^L(R_{n+1}) \left[\beta_n \mu_n(R_{n+1})\eta D + (1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(R_n)))\right], & (26) \end{aligned}$$

where $\mu_n(R_{n+1}) = \mu_n^H(R_{n+1})/\mu_n^L(R_{n+1})$ and D is the lower bound on the loss on H given in Lemma 11.[62]

[60] This inequality comes the fact that $\hat{e}_{K-1} - \hat{u}_{K-1} = t(\hat{u}_K - \hat{u}_{K-1})$, by construction of the blocks in Part I, and the fact that $\hat{e}_K \geq \hat{e}_{K-1}$.

[61] The reason for using $u_H(n_K)$ instead of $\hat{u}_K$ is that H's continuation utility at round n_K need not be exactly equal to $\hat{u}_K$: it lies between $\hat{u}_K$ and $\hat{u}_K + \Delta_H\eta$.

[62] We can assume without loss of generality that $\mu_n^L(R_{n+1})$ is strictly positive for all $R_{n+1} \in (M_n \cup \{R_n\}) \cap \mathcal{H}$: first, if any contract in that set is not chosen with any probability, we can construct an equilibrium in which those

The left-hand side of (25) is an upper bound on the gain, relative to the immediate jump, made on the high type: given his continuation utility $u_H(n)$, the lowest achievable cost that provides this utility is the cost of the H-efficient contract that gives $u_H(n)$. From Lemma 11 of Appendix D, this gain is bounded above by $a(u_H(E_H(R_n)) - u_H(n)) = aw_n$ (that bound is computed using a 'best-case scenario' for P, in which H accepts with probability 1 the H-efficient contract C_n providing $u_H(n)$).[63] The first term of the right-hand side is the net loss on H if he accepts a contract in the H-Rent configuration and, hence, far from efficient, in case a breakdown occurs at the end of round n. This loss is bounded by D as long as $w_n \leq 2\varepsilon$, which will be true along the choice sequence that we consider. The last term is the net loss on L in case of such a breakdown.

To establish Proposition 4, we introduce the variable $y_n = u_H(E_H(R_n)) - u_H(R_{n+1})$, i.e., H's utility gap, for any choice R_{n+1}, between the immediate jump and his utility in case of a negotiation breakdown at round n (the breakdown occurs *after* the agent has chosen the new contract, R_{n+1}, which explains the index). This quantity y_n is important for the analysis, because it provides a control on the decrements of w_n and makes sure that we do not overshoot the threshold $\frac{\eta D}{2a}$ by too much. Indeed, subtracting $u_H(E_H(R_n))$ from (19) and rearranging (and recalling that $w_n = u_H(E_H(R_n)) - u_H(n)$) leads, along any choice sequence, to

$$w_{n+1} = w_n - \eta y_n + \eta w_{n+1} + (1-\eta)(u_H(E_H(R_{n+1})) - u_H(E_H(R_n))).$$

These concepts are represented on Figure 4

Proposition 4 is based on the following lemma, which is proved in Appendix E. Fix a positive integer $\bar{N}$, positive constants $\bar{\beta}$ and $\bar{w}$, and a small positive constant $\bar{\varepsilon}$.

Lemma 6 *Suppose that negotiations reach a round $\bar{n}$ such that $\beta_{\bar{n}} \leq \bar{\beta}\eta^d$ and $w_{\bar{n}} \leq \bar{w}\eta$. There exist functions $\bar{W}(\bar{N})$ and $k(\bar{N})$ of $\bar{N}$ (only) such that the following holds. Let $\mathcal{S}$ denote the event that the agent chooses at all rounds $n \in \{\bar{n}+1, \ldots, \bar{n}+\bar{N}\}$ contracts such that $y_n = O(\eta^{d/4})$, $\beta_n \leq \beta_{\bar{n}}\bar{\varepsilon}^{\,-(n-\bar{n})}$, and $w_n \leq \bar{W}(\bar{N})\eta$. For η small enough, the probability of $\mathcal{S}$ is greater than $1 - k(\bar{N})\bar{\varepsilon}$.*

We now modify the analysis of Part I to study blocks consisting of $\bar{N}$ rounds, indexed by $\bar{n}+1$ to $\bar{n}+\bar{N}$, where $\bar{N}$ will be determined shortly. The first such block starts with $\bar{n} = n_K$, the second of these blocks starts with $\bar{n} = n_K + \bar{N}$, etc. These blocks are different from those of Part I, because the number $\bar{N}$ of rounds in each block is fixed and, unlike the blocks of Part I, H's utility at the end of each block is not precisely controlled.

The analysis of Part I is modified as follows. First, notice that P's IC constraint at round $\bar{n}$, looking ahead over the next $\bar{N}$ rounds, implies that

$$\beta_{\bar{n}} a \left\{(1-\mu_{\bar{n}})(e_{\bar{n}} - u_H(\bar{n})) + \mu_{\bar{n}}(e_{\bar{n}} - E[u_H(\bar{n}+\bar{N})])\right\} \geq \beta_{\bar{n}}\mu_{\bar{n}} D\eta\bar{N} - \beta_{\bar{n}}\delta_Q k(\bar{N})\bar{\varepsilon},$$

where $\mu_{\bar{n}}$ is the probability, seen from round $\bar{n}$, that H rejects all H-efficient contracts between rounds $\bar{n}$ and $\bar{n}+\bar{N}$. The argument for this equation is the same as in Part I, the only difference being that we are now

contracts are removed. And if any contract R'_{n+1} in that set is chosen only by H with positive probability, then Proposition 1 implies that H gets the H-efficient contract C that gives him the same utility as R'_{n+1}, so that the equilibrium can be modified by having P propose C instead of R'_{n+1}. That change reduces P's cost and does not affect incentives.

[63] This is an upper bound on the gain, since C_n is the cheapest way of providing H with his continuation utility.

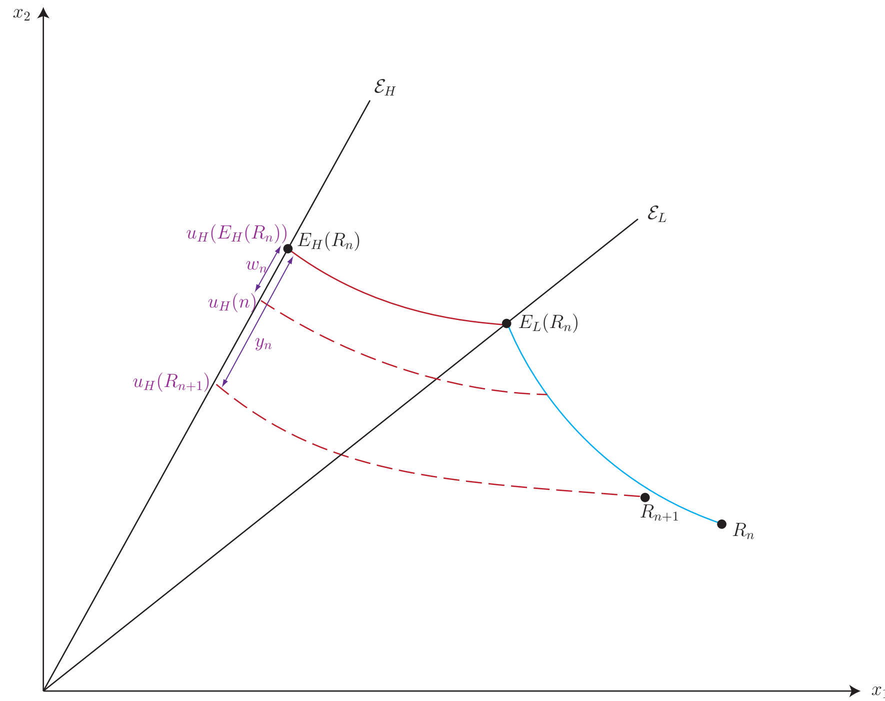

Figure 4: Rent extraction concepts (contracts are in black, utilities are in purple)

taking the expectation of $u_H(\bar{n}+\bar{N})$ because we do not know its value (before, we had precisely defined the end of the block as the first time that u_H crosses some level, but now $\bar{N}$ is exogenous). The lower bound D on the loss is valid with probability $1-k(\bar{N})\varepsilon$ because conditional on $\mathcal{S}$ occurring, w_n is small throughout the block by Lemmas 6 and the lower bound on the loss follows from Lemma 16. On the complement of $\mathcal{S}$, whatever cost is incurred by P conditional on facing H is bounded above by some constant, by compactness of the contract space. The difference between D and that constant is captured by $\delta_Q > 0$ (last term of the above equation), which is independent of η, $\bar{N}$, and $\bar{\varepsilon}$.

This implies that

$$\mu_{\bar{n}} \leq \frac{a(e_{\bar{n}} - u_H(\bar{n})) + \delta_Q k(\bar{N})\bar{\varepsilon}}{a(E[u_H(\bar{n}+\bar{N})] - u_H(\bar{n})) + D\eta\bar{N}} \leq \frac{a\bar{W} + \delta_Q k(\bar{N})\bar{\varepsilon}}{D\bar{N}},$$

where the second inequality comes from i) the fact that $u_H(n)$ is nondecreasing across all paths which implies, taking expectations, that $Eu_H(\bar{n}+\bar{N}) \geq u_H(\bar{n})$ and ii) $\bar{w}_{\bar{n}} = e_{\bar{n}} - u_H(\bar{n}) \leq \bar{W}\eta$ (this inequality holds for all blocks of Part II, without loss of generality, see Remark 2 below). Now let $\mu_{\bar{n}}^{\mathcal{S}}$ (resp. $\mu_{\bar{n}}^{\mathcal{B}}$) denote the probability that H rejects all H-efficient contracts, *conditional* on event $\mathcal{S}$ (resp. conditional on its complement, $\mathcal{B}$), and let $p_{\mathcal{S}}$ (resp. $p_{\mathcal{B}}$) the probability of $\mathcal{S}$ ($\mathcal{B}$). We have $\mu_{\bar{n}} = p_{\mathcal{S}}\mu_{\bar{n}}^{\mathcal{S}} + p_{\mathcal{B}}\mu_{\bar{n}}^{\mathcal{B}}$. Since $p_{\mathcal{S}} \geq 1-k(\bar{N})\bar{\varepsilon}$, we conclude that

$$\mu_{\bar{n}}^{\mathcal{S}} \leq \frac{(a\bar{W} + \delta_Q k(\bar{N})\bar{\varepsilon})(1+k(\bar{N})\bar{\varepsilon})}{D\bar{N}}. \tag{27}$$

We now choose $\bar{\varepsilon}$ and $\bar{N}$ so that this ratio is less than $\frac{1}{2}$: first choose $\bar{N}$ so that $\frac{a\bar{W}}{D\bar{N}} < \frac{1}{8}$, then choose $\bar{\varepsilon}$ so that $k(\bar{N})\bar{\varepsilon} < 1$ and $\delta_Q k(\bar{N})\bar{\varepsilon} < a\bar{W}$ so that the numerator of (27) is less than $4a\bar{W}$.

Proceeding as in Part I, there must exist a pushdown choice sequence contained in event $\mathcal{S}$ such that the ex post probability that H has not chosen an H-efficient contract is weakly less than $\mu_{\bar{n}}^{\mathcal{S}}$. Along that sequence, i) y_n is of order $O(\eta^{d/4})$, and ii) $\beta_{\bar{n}+\bar{N}} \leq \frac{\beta_{\bar{n}}}{2}$. We have thus built a choice sequence over $\bar{N}$ rounds, starting from $\bar{n}$, such that y_n and β_n stay small, and β_n ends up *smaller* than at the beginning of the block.[64]

Starting from round n_K, we build a sequence of $\bar{N}$-sized blocks as described above. Because $\bar{w}_n$ converges to zero (by part ii) of Lemma 14), it will eventually cross $\frac{D\eta}{2a}$. Let N denote the first round at which $\bar{w}_N$ drops below that threshold. From (52) of Lemma 11 (Appendix D), we have $w_{n+1}(1 - b\beta_{n+1}) \geq w_n - \eta y_n$, or

$$w_{n+1} - w_n \geq b\beta_{n+1}w_{n+1} - \eta y_n. \tag{28}$$

The blocks were constructed in such a way that $y_n = O(\eta^{d/4})$ and $\beta_n = O(\eta^d)$ at each round of each block. Applying these observations to (28) at round $N-1$, we obtain

$$w_N - w_{N-1} \geq -o(\eta).$$

Finally, we have

$$\begin{aligned} \bar{w}_N - \bar{w}_{N-1} &= (w_N - w_{N-1}) + (\max\{e_k : k \leq N\} - e_N) - (\max\{e_k : k \leq N-1\} - e_{N-1}) \\ &\geq -o(\eta) - (e_N - e_{N-1}), \end{aligned}$$

since the different of maxima is nonnegative. From (48), the difference $e_N - e_{N-1}$ is bounded above by $\frac{\alpha\beta_{N-1}}{1-\beta_{N-1}}w_{N-1}$ which is $o(\eta)$. Since $\bar{w}_{N-1} > \frac{\eta D}{2a}$ by definition of N, we conclude that

$$\bar{w}_N \geq \bar{w}_{N-1} - o(\eta) \geq \frac{\eta D}{2a} - o(\eta) \geq \frac{\eta D}{3a}.$$

This concludes the proof of Proposition 4 and implies that we have reached a round N such that i) $\bar{w}_N$ is above $\hat{w}\eta$, for some $\hat{w} = \frac{D}{3a} > 0$ independent of η, and ii) $\beta_N = O(\eta^d)$. Part III will show that this is impossible.

Remark 2 *It is a priori possible that $\bar{w}_n$ goes above $\bar{W}\eta$ at the end of some block. If that happens, the bound $D = D(2\varepsilon)$ need not be valid for the next block. At the end of such block, should it occur, β_n is of order $\eta^d \leq \beta_0$. We can restart the blocks of Part I as if n were the initial round. Since β decreases along the blocks of Part I, we have to reach again a round at which $\bar{w}_n$ drops below $\bar{W}\eta$. At that point, we necessarily have $\beta_n \leq \eta^d$. Because $\bar{w}_n$ converges to zero along any sequence (by Lemma 14), and thus also along the sequences constructed through Parts I and II, the back and forth between blocks of Part I and Part II has to stop in finite time at some round N of the type above, i.e., with $\bar{w}_N \in (\hat{w}\eta, \frac{\eta D}{2a})$ and $\beta_N \leq \eta^d$. The logic of the argument is explained in more detail in Remark 3.*

[64]Notice that β_n can increase up to $\beta_{\bar{n}}\bar{\varepsilon}^{-\bar{N}}$ along such a block. However, because $\bar{N}$ is fixed, it still remains of order $O(\eta^d)$ along the sequence, and drops in any case below $\beta_{\bar{n}}/2$ when round $\bar{n}+\bar{N}$ is reached, for the pushdown sequence.

Part III: Asymptotic Level

The purpose of this section is to establish the following proposition:

Proposition 5 *There exist a constant $\check{w} > 0$ and a threshold $\check{\eta} > 0$ such that the following holds for all $\eta < \check{\eta}$: if one reaches a round N for which $\beta_N \leq \eta^d$ and $\bar{w}_N \leq \frac{\eta D}{2a}$, then $\bar{w}_N \leq \check{w}\eta^{1+d}$.*

The proof proceeds in three steps:

1. Show, starting from round N, that one can build a choice sequence along which β_n is decreasing and, at each round, a simplified version of P's ex ante IC constraint (before the agent chooses R_{n+1}) is also satisfied ex post (after R_{n+1} is chosen). This step is achieved by Lemma 7;
2. Show that along such a sequence, one must necessarily have $w_n \leq \hat{c}\eta\beta_n$ for all $n \geq N$, where $\hat{c} > 0$ is independent of η (Proposition 6);
3. Show that $\bar{w}_N - w_N = O(\eta^{1+2d})$ (Proposition 7).

Combining these steps (with 2. applied to $n = N$) along with the fact that $\beta_N \leq \eta^d$ then proves Proposition 5.[65]

To express P's IC constraint, recall from Part II that

$$w_n a\beta_n \geq \sum_{R_{n+1}\in(M_n\cup\{R_n\})\cap\mathcal{H}} \mu_n^L(R_{n+1})\left[\beta_n\mu_n(R_{n+1})\eta D + (1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(R_n)))\right] \tag{29}$$

where $\mu_n(R_{n+1}) = \mu_n^H(R_{n+1})/\mu_n^L(R_{n+1})$ and D is the lower bound on the loss on H obtained in Lemma 11.

In particular, the RHS of (29) is a convex combination of terms indexed by R_{n+1}, and there must exist $R_{n+1} \in (M_n \cup \{R_n\}) \cap \mathcal{H}$ such that

$$w_n a\beta_n \geq \beta_n\mu_n(R_{n+1})\eta D + (1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(R_n))). \tag{30}$$

Therefore, there exists a choice sequence that satisfies (30) for all $n \geq N$. In what follows we entirely focus on that sequence, which will be called a **regular** choice sequence.

We have $Q(R_{n+1}) - Q(E_L(R_n)) \geq Q(E_L(R_{n+1})) - Q(E_L(R_n)) \geq -k\beta_{n+1}w_{n+1}$, where the second inequality comes (50) of Lemma 11 (Appendix D). Letting $\mu_n = \mu_n(R_{n+1})$, (30) implies that

$$\beta_n w_n a \geq \beta_n\mu_n\eta D - \eta k\beta_{n+1}w_{n+1},$$

which may be re-expressed as

$$\mu_n \leq \frac{w_n a}{\eta D} + k\frac{\beta_{n+1}}{\beta_n D}w_{n+1}. \tag{31}$$

The first step, in order to exploit this equation, is to show that β_n remains small for all $n \geq N$. This is achieved by the following lemma, which guarantees that β_n is actually decreasing along the regular sequence (see Appendix F for the proof).

[65] Indeed, we have $\bar{w}_N \leq w_n + O(\eta^{1+2d}) \leq \hat{c}\eta\beta_N + O(\eta^{1+2d}) \leq \hat{c}\eta^{1+d} + O(\eta^{1+2d})$, where the first inequality comes from Step 3., the second comes from Step 2. applied to $n = N$, and the third one comes from the assumption that $\beta_N \leq \eta^d$. Taking $\check{w}$ slightly above $\hat{c}$ then yields the proposition.

Lemma 7 *There exist $\hat{\eta} > 0$ and $\hat{w} > 0$ such that, for $\eta < \hat{\eta}$ i) β_n is decreasing in n, ii) $\mu_n \leq 3/4$ for all $n \geq N$, and iii) $w_n \leq \hat{w}\eta$ for all $n \geq N$.*

Part ii) of Lemma 7 implies that the second term in the right-hand side of (31) is of order w_{n+1} (since $\beta_{n+1} \sim \beta_n \mu_n$ from Bayes rule, see (38)) and is thus negligible compared to the first term, of order $\frac{w_n}{\eta}$, because w_{n+1} is bounded above by $w_n\left(1 + \frac{\alpha\beta_n}{1-\beta_n}\right)$ (see (49) in Appendix D) and $\eta \ll 1$. Therefore, by slightly increasing a, whose specific value does not matter in any case for the proof, we get

$$\mu_n \leq \frac{w_n a}{\eta D}, \qquad (IC_n^{LL}) \tag{32}$$

Moreover, (30) also implies that

$$\beta_n w_n a \geq (1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(R_n))) \qquad (IC_n^{LH}) \tag{33}$$

Relaxation

The previous arguments have shown that any PBE must entail, given the assumptions of Proposition 5, a regular choice sequence satisfying (32) and (33) for all $n \geq N$. Moreover, Appendix D contains a list of inequalities which must also be satisfied at all these rounds. Finally, w_n converges to zero along that regular sequence, by Proposition 2.

From now on, we entire focus on all these inequalities, which arise along the regular sequence. Those inequalities involve the contracts chosen along that sequence as well as the beliefs β_n's and likelihood ratios μ_n's for $n \geq N$. We work with these inequalities in complete isolation from the rest of the PBE. The objective here is to prove some properties of the *contracts* involved in this sequence by modifying the *beliefs* β_n and likelihood ratios μ_n.

The transformation works as follows. Starting with round N, one increases the likelihood ratio μ_N so as to satisfy (32) as an equality. The interpretation of this change is that H becomes relatively more likely than before the change, conditional on observing the contract R_{N+1} arising in the regular sequence. We maintain standard Bayesian updating, but applied to this new likelihood ratio, so that β_{N+1} is given by

$$\beta_{N+1} = \frac{\beta_N \mu_N}{\beta_N \mu_N + (1-\beta_N)}.$$

With only that change, the posteriors β_n's at all rounds $n \geq N+1$ of the regular sequence are weakly increased as a result of Bayesian updating. Therefore, the inequality (33) at all rounds $n \geq N+1$ is preserved (in fact, looser) along the regular sequence. After this is done, one can increase the likelihood ratio μ_{N+1} pertaining to the contract R_{N+2} chosen in the regular sequence at round $N+1$, so as to make IC_{N+1}^{LL} tight. All beliefs β_n for $n \geq N+2$ are then weakly increased as a result of Bayesian updating. This preserves the inequalities (33) for $n \geq N+2$ and does not perturb (33) and (32) at round N. Increasing μ_n inductively for all $n \geq N$ along the regular choice sequence, the new $\{\mu_n\}_{n \geq N}$ and $\{\beta_n\}_{n \geq N+1}$ satisfy (33) as well as

$$\mu_n = \frac{w_n a}{\eta D}. \tag{34}$$

Moreover, because all the posteriors β_n's have been weakly increased compared to the initial belief sequence, while the contracts involved in the regular sequence are left completely intact by the transformation, all the inequalities appearing in Appendix D still hold, because they only get looser for higher values of β_n.[66]

The above transformation has thus led to a new sequence of beliefs and likelihood ratios which, along with the contracts of the regular sequence satisfy all the previous inequalities and now satisfy (32) as equality (34). Because the contracts have not been changed, moreover, w_n must still converge to zero. From (32), this implies that the sequence of μ_n's must also goes to zero and, hence, that β_n goes to zero even after the relaxation.

We are now ready to perform the Steps 2 and 3 needed to prove Proposition 5. In Appendix D (Lemma 11), it is shown that

$$u_H(E_H(R_{n+1}) - u_H(E_H(R_n))) \geq -\hat{b}\beta_{n+1}w_{n+1} \tag{35}$$

for some constant $\hat{b} > 0$. As in Part II, we also use the the following relation between w_n and w_{n+1} (see (54) for the proof):

$$w_{n+1} = w_n - \eta y_n + \eta w_{n+1} + (1-\eta)(u_H(E_H(R_{n+1})) - u_H(E_H(R_n))).$$

Combining this with (35) yields

$$(1-\eta)w_{n+1} \geq w_n - \eta y_n - \hat{b}\beta_{n+1}w_{n+1}. \tag{36}$$

Multiplying both sides of (36) by $\frac{a}{\eta D}$ and using (34), we obtain for $n \geq N$

$$\mu_{n+1} \geq (1-\eta)\mu_{n+1} \geq \mu_n - \frac{a}{D}y_n - \tilde{b}\beta_{n+1} \tag{37}$$

for some constant $\tilde{b} > 0$. (To get $\tilde{b}$ in the last term, the derivation used the inequality $w_n \leq \hat{w}\eta$ for $n \geq N$, from Part iii) of Lemma 7.)

The Bayesian updating equation

$$\beta_{n+1} = \frac{\beta_n\mu_n}{\beta_n\mu_n + (1-\beta_n)}$$

implies that[67]

$$\frac{\beta_{n+1}}{\beta_n} \geq \mu_n \geq \frac{\beta_{n+1}}{\beta_n} - \mu_n\beta_n + \mu_n O(\beta_n^2) \geq \frac{\beta_{n+1}}{\beta_n} - \beta_{n+1} + o(\beta_{n+1}). \tag{38}$$

Lemma 17 of Appendix F shows that $y_n^2 \leq \frac{\bar{A}\beta_{n+1}}{1-\beta_0}$. Intuitively, this equation means that the loss on L in round n, which is of order ηy_n^2, must be smaller than the gain on H, which is of order $\beta_n w_n$ (i.e., the probability of facing H times the maximum gain).[68]

[66] The only inequality which does not get looser is (52). However, this inequality was only used in Part II, is not involved in the derivation of any other inequality of Appendix D and does not appear anywhere in the analysis of regular sequences.

[67] We have $\frac{\beta_{n+1}}{\beta_n} = \mu_n \frac{1}{1-\beta_n(1-\mu_n)} = \mu_n(1+\beta_n(1-\mu_n)) + \mu_n O(\beta_n^2)$. Rearranging yields the second inequality.

[68] Dividing by η, we get $y_n^2 \leq C\beta_n w_n/\eta$ for some constant C. Since w_n/η is proportional to μ_n and $\mu_n\beta_n$ is roughly equal to β_{n+1}, which provides some intuition for how the equation was derived.

Combining this upper bound for y_n^2 with (37) and (38), we obtain the following dynamic equation for β_n, for all $n \geq N$:

$$\frac{\beta_{n+2}}{\beta_{n+1}} \geq \frac{\beta_{n+1}}{\beta_n} - c\sqrt{\beta_{n+1}} - (1+\tilde{b})\beta_{n+1} \tag{39}$$

where $c = \frac{a}{D}\sqrt{\frac{\bar{A}}{1-\beta_0}}$. For β_{n+1} small enough, the last term is negligible compared to the penultimate term. Therefore, by slightly increasing the value of c, whose precise value does not affect the proof, we obtain

$$\frac{\beta_{n+2}}{\beta_{n+1}} \geq \frac{\beta_{n+1}}{\beta_n} - c\sqrt{\beta_{n+1}}. \tag{40}$$

Let, for all n, $q_n = \frac{\beta_{n+1}}{\beta_n}$. We have $\prod_0^n q_k = \frac{\beta_{n+1}}{\beta_0}$ and (40) may be rewritten as

$$q_{n+1} \geq q_n - c'\sqrt{\Pi_0^n q_k} \tag{41}$$

where $c' = \sqrt{\beta_0}c$. Note that because q_n is proportional to μ_n and hence w_n, it must converge to zero as n goes to infinity.

Proposition 6 *Along the regular choice sequence, we have $w_n \leq \hat{c}\eta\beta_n$ for all $n \geq N$.*

Proof. The proposition is based on the following two lemmas, proved in Appendix F.

Lemma 8 *Suppose that there exists a round $\hat{N} > N$ such that*

$$\beta_{\hat{N}+1} \geq 4c^2\beta_{\hat{N}}^2, \tag{42}$$

$$\beta_{\hat{N}}^{1/4} \leq \frac{1}{2\sqrt{c}}, \tag{43}$$

Then,

$$\liminf_{n\to+\infty} \frac{q_{n+1}}{q_n} \geq 1.$$

Lemma 9 *Suppose that $\{q_n\}$ is a strictly positive sequence such that*

$$q_n - q_{n+1} \leq c'\sqrt{\Pi_0^n q_k}$$

and $\liminf_n \frac{q_{n+1}}{q_n} \geq 1$. Then, $\{q_n\}$ does not converge to zero.

To conclude the proof of Proposition 6, suppose by contradiction that there exists $\hat{N} \geq N$ such that $w_{\hat{N}} > \hat{c}\eta\beta_{\hat{N}}$. From (34), this implies that $\mu_{\hat{N}} > \frac{a}{D}\hat{c}\beta_{\hat{N}}$, and from the first inequality of (38) this implies, using the definition of $\hat{c}$, that (42) holds for $\hat{N}$. Moreover, from Lemma 7, $\beta_{\hat{N}}$ clearly satisfies (43), for η small enough. Therefore the hypotheses of Lemma 8 are satisfied and, hence, $\liminf_{n\to+\infty} \frac{q_{n+1}}{q_n} \geq 1$. Combining this with Lemma 9 then implies that w_n cannot converge to zero, which contradicts Proposition 2, since w_n converges to zero along any choice sequence. ■

Proposition 7 *There exist $\hat{w} > 0$ and $\bar{\eta} > 0$ such that $\bar{w}_N - w_N \leq \hat{w}\eta^{1+2d}$ for all for $\eta \leq \bar{\eta}$.*

Proof. Recalling the definition of $\bar{C}_H$ as the H-efficient contract that provides H with its asymptotic utility $\lim_n u_H(n)$ (see Proposition 2), we have

$$u_H(\bar{C}_H) - u_H(E_H(R_N)) = \sum_{n\geq N} u_H(E_H(R_{n+1})) - u_H(E_H(R_n)) \leq 2\alpha \sum_{n\geq N} \beta_n w_n,$$

where the last inequality comes from (48). From Proposition 6, we have $w_n \leq \hat{c}\eta\beta_n$ for all $n \geq N$. Therefore,

$$u_H(\bar{C}_H) - u_H(E_H(R_N)) \leq \tilde{K}\eta \sum_{n\geq N} \beta_n^2,$$

where $\tilde{K} = 2\alpha\hat{c}$.

We have $\beta_{n+1} \leq 2\mu_n\beta_n = \frac{2aw_n}{D\eta}$, by (34). Using again the inequality $\frac{w_n}{\eta} \leq \hat{c}\beta_n$, which holds for all $n \geq N$, we have, letting $\hat{K} = \frac{2a\hat{c}}{D}$,

$$\beta_n \leq \beta_N \prod_{k=N+1}^{n-1} (\hat{K}\beta_k) \leq \beta_N(\hat{K}\beta_N)^{n-N}.$$

For $\beta_N < \frac{1}{\sqrt{2}\hat{K}}$, this implies that

$$\sum_{n\geq N} \beta_n^2 \leq \sum_{n\geq N} 2^{-(n-N)}\beta_N^2 = 2\beta_N^2.$$

We then obtain

$$u_H(\bar{C}_H) - u_H(E_H(R_N)) \leq 2\tilde{K}\eta\beta_N^2 \leq 2\tilde{K}\eta^{1+2d}, \tag{44}$$

where the last inequality comes from the fact that $\beta_N \leq \eta^d$.

To conclude, note that $\bar{w}_N - w_N = \max\{e_k : k \leq N\} - e_N = \max\{u_H(E_H(k)) : k \leq N\} - u_H(E_H(R_N))$. Since $\max\{u_H(E_H(k)) : k \leq N\} \leq u_H(\bar{C}_H)$, by an argument that is similar to the proof of Lemma 14), (44) yields the result. ■

D Inequalities

Lemma 10 (Regularity Bounds) *There exist positive constants $\underline{a}, a, \underline{b}, b$ such that for any $C, \hat{C} \in \mathcal{E}_H$ such that $u_H(C) < u_H(\hat{C})$, we have*

$$\underline{a}(u_H(\hat{C}) - u_H(C)) \leq Q(\hat{C}) - Q(C) \leq a(u_H(\hat{C}) - u_H(C)) \tag{45}$$

$$\underline{b}(Q(\hat{D}) - Q(D)) \leq Q(\hat{C}) - Q(C) \leq b(Q(\hat{D}) - Q(D)), \tag{46}$$

where D (resp. $\hat{D}$) is the L-efficient contract that gives H the same utility as C (resp. $\hat{C}$).

Proof. Consider two contracts C and $\hat{C}$ on $\mathcal{E}_H$ ordered as in the statement of the lemma. The efficiency curve $\mathcal{E}_H$ can be parameterized by a univariate parameter λ such that, letting $C(\lambda) = (x_1(\lambda), x_2(\lambda))$ denote the H-efficient contract corresponding to parameter λ, the map $\lambda \mapsto C(\lambda)$ is continuous, one-to-one, and onto from the parameter set Λ (a compact interval of $\mathbb{R}$) to $\mathcal{E}_H$. We can assume without loss that Λ contains $[0, 1]$

and that $C(0) = C$ and $C(1) = \hat{C}$. We choose the parametrization to be regular, i.e., such that $\lambda \mapsto C(\lambda)$, seen as a function from Λ to $\mathbb{R}^2$, is smooth and does not go "too slow" or "too fast" along $\mathcal{E}_H$.[69] We have

$$\begin{aligned} Q(\hat{C}) - Q(C) &= \int_0^1 \frac{dQ(x_1(\lambda), x_2(\lambda))}{d\lambda} \cdot dC(\lambda) \\ &= \int_0^1 \left(\frac{\partial Q(C(\lambda))}{\partial x_1} \frac{dx_1}{d\lambda} + \frac{\partial Q(C(\lambda))}{\partial x_2} \frac{dx_2}{d\lambda} \right) d\lambda. \end{aligned}$$

Similarly, we have

$$\begin{aligned} u_H(\hat{C}) - u_H(C) &= \int_0^1 \frac{du_H(x_1(\lambda), x_2(\lambda))}{d\lambda} \cdot dC(\lambda) \\ &= \int_0^1 \left(\frac{\partial u_H(C(\lambda))}{\partial x_1} \frac{dx_1}{d\lambda} + \frac{\partial u_H(C(\lambda))}{\partial x_2} \frac{dx_2}{d\lambda} \right) d\lambda. \end{aligned}$$

By assumption, the partial derivatives of Q and u_H are strictly positive and continuous on the compact domain $\mathcal{C}$, and hence bounded below away from zero as well as bounded above. Therefore, there exist positive constants $\underline{a} < a$ such that $\underline{a}\frac{\partial u_H}{\partial x_i} \leq \frac{\partial Q}{\partial x_i} \leq a\frac{\partial u_H}{\partial x_i}$ for $i = 1, 2$. Using these inequalities into the previous integral representations of $Q(\hat{C}) - Q(C)$ and $u_H(\hat{C}) - u_H(C)$ then shows (45).

For the second part of the lemma, consider the parameterizations of $\mathcal{E}_H$ and $\mathcal{E}_L$ for which the parameter corresponds to the utility that each contract gives to H (thus, $u_H(C(\lambda)) = \lambda$), with elements $C(\lambda) = (x_1^H(\lambda), x_2^H(\lambda))$ for $\mathcal{E}_H$ and $D(\lambda) = (x_1^L(\lambda), x_2^L(\lambda))$ for $\mathcal{E}_L$. Because the partial derivatives of u_H are strictly positive on the compact domain $\mathcal{C}$ and because the curves $\mathcal{E}_\theta$ are both nondecreasing in $\mathcal{C}$, that parameterization is well defined and regular (in the sense of the previous paragraph) for both curves. Consider two contracts C and $\hat{C}$ of $\mathcal{E}_H$ with provide H with utilities $u_H < \hat{u}_H$ and let D and $\hat{D}$ denote the contracts of $\mathcal{E}_L$ corresponding to utilities u_H and $\hat{u}_H$. Repeating the argument of the previous paragraph, we have

$$Q(\hat{C}) - Q(C) = \int_{u_H}^{\hat{u}_H} \left(\frac{\partial Q(C(\lambda))}{\partial x_1} \frac{dx_1^H}{d\lambda} + \frac{\partial Q(C(\lambda))}{\partial x_2} \frac{dx_2^H}{d\lambda} \right) d\lambda$$

and

$$Q(\hat{D}) - Q(D) = \int_{u_H}^{\hat{u}_H} \left(\frac{\partial Q(D(\lambda))}{\partial x_1} \frac{dx_1^L}{d\lambda} + \frac{\partial Q(D(\lambda))}{\partial x_2} \frac{dx_2^L}{d\lambda} \right) d\lambda.$$

Because the paramaterizations are regular and the curves are nondecreasing, there must exist positive constants $\underline{x} < \bar{x}$ such that $0 < \underline{x} \max\{dx_1^H/d\lambda, dx_2^H/d\lambda\} \leq \max\{dx_1^L/d\lambda, dx_2^L/d\lambda\} \leq \bar{x} \max\{dx_1^H/d\lambda, dx_2^H/d\lambda\}$. Moreover, since Q has strictly positive derivatives, bounded below away from zero and bounded above, there also exist positive constants $\underline{q} < \bar{q}$ such that $\underline{q}\partial Q(C(\lambda))/\partial x_i \leq \partial Q(D(\lambda))/\partial x_i \leq \bar{q}\partial Q(C(\lambda))/\partial x_i$ for all $\lambda \in [u_H, \hat{u}_H]$ and $i = 1, 2$. Combining these inequalities with the previous integral representations implies, as is easily checked, that there exist positive constants $\underline{b} < b$ such that

$$\underline{b}(Q(\hat{D}) - Q(D)) \leq Q(\hat{C}) - Q(C) \leq b(Q(\hat{D}) - Q(D)),$$

which concludes the proof. ■

For the next result, let Q_θ denote P's expected continuation cost at the beginning of round n, conditional on facing type θ. (We omit dependence on n for simplicity).

[69] Formally, this means that the norm of the gradient of the function $\lambda \mapsto C(\lambda)$ is uniformly bounded below and above by strictly positive constants.

LEMMA 11 (INCENTIVE BOUNDS) *Given any PBE and choice sequence $\{R_n\}$, there exist positive constants α, γ, $\hat{b}$, and b, such that*

$$Q_L \leq Q(E_L(R_n)) + \frac{\beta_n}{(1-\beta_n)} a w_n, \tag{47}$$

$$u_H(E_H(R_{n+1})) - u_H(E_H(R_n)) \leq \frac{\alpha\beta_n}{1-\beta_n} w_n, \tag{48}$$

$$w_{n+1} \leq w_n \left(1 + \frac{\alpha\beta_n}{1-\beta_n}\right), \tag{49}$$

$$u_L(R_n) - u_L(R_{n+1}) \leq \gamma\beta_{n+1} w_{n+1}, \tag{50}$$

$$u_H(E_H(R_{n+1}) - u_H(E_H(R_n))) \geq -\hat{b}\beta_{n+1} w_{n+1}, \tag{51}$$

$$w_{n+1}(1 - b\beta_{n+1}) \geq w_n - \eta y_n. \tag{52}$$

Proof. Lemma 2 implies that

$$\beta_n Q_H + (1-\beta_n) Q_L \leq \beta_n Q(E_H(R_n)) + (1-\beta_n) Q(E_L(R_n)).$$

Moreover, Q_H is bounded below by the cost of the H-efficient contract $C_H(n)$ that provides utility $u_H(n)$ to H, since that is the cheapest way of providing H with his continuation utility (by convexity of the cost function Q). This implies that $Q_L \leq Q(E_L(R_n)) + \frac{\beta_n}{1-\beta_n}(Q(E_H(R_n)) - Q(C_H(n))$. The contracts $E_H(R_n)$ and $C_H(n)$ both lie on $\mathcal{E}_H$. Equation (45) implies that $Q(E_H(R_n)) - Q(C_H(n)) \leq a(u_H(E_H(R_n)) - u_H(n)) = a w_n$. This shows (47).

From (47), R_{n+1} cannot give L a utility greater than the L-efficient contract that costs $Q(E_L(R_n)) + \frac{a\beta_n}{1-\beta_n} w_n$. This implies that $Q(E_L(R_{n+1})) - Q(E_L(R_n))$ is bounded above by $\frac{a\beta_n}{1-\beta_n} w_n$. Combining this with (46) yields[70]

$$Q(E_H(R_{n+1})) - Q(E_H(R_n)) \leq \frac{ab\beta_n}{1-\beta_n} w_n.$$

This, along with the first part of (45) yields (48). We have

$$\begin{aligned} w_{n+1} &= u_H(E_H(R_{n+1})) - u_H(n+1) = [u_H(E_H(R_{n+1})) - u_H(E_H(R_n))] + u_H(E_H(R_n)) - u_H(n+1) \\ &\leq [u_H(E_H(R_{n+1})) - u_H(E_H(R_n))] + u_H(E_H(R_n)) - u_H(n) \\ &\leq w_n \left(\frac{\alpha\beta_n}{1-\beta_n} + 1\right) \end{aligned}$$

where the first inequality comes from the monotonicity of $u_H(n)$ in n, and the second inequality comes from (48). This shows (49).

Because L can hold on forever to R_n, his continuation utility $u_L(n)$ is bounded below by $u_L(R_n)$. At round $n+1$, P's expected cost conditional on facing L is bounded above by $Q(E_L(R_{n+1})) + \frac{\beta_{n+1}}{1-\beta_{n+1}} a w_{n+1}$, from (47) applied to round $n+1$. By the same argument that yielded (45), there exists $\alpha_L > 0$ such that $u_L(E) - u_L(E') \leq \alpha_L(Q(E) - Q(E'))$ for all $E, E' \in \mathcal{E}_L$. Therefore, the highest utility which may be achieved

[70] Equation 46 applies if $Q(E_H(R_{n+1})) - Q(E_H(R_n)) \geq 0$. In the opposite case, the inequality holds trivially since the left-hand side is negative and the right-hand side is positive.

at that cost is bounded above by $u_L(R_{n+1}) + \hat{a}\beta_{n+1}/(1-\beta_{n+1})w_{n+1}$, for some proportionality constant $\hat{a}$, and

$$u_L(R_n) \le u_L(n) \le u_L(n+1) \le u_L(R_{n+1}) + \hat{a}\beta_{n+1}/(1-\beta_{n+1})w_{n+1},$$

which yields (50).

In general, $u_H(E_H(R_{n+1}) - u_H(E_H(R_n)))$ may be negative. To provide a lower bound for that case, we use the relations

$$u_L(E_L(R_n)) - u_L(E_L(R_{n+1})) = u_L(R_n) - u_L(R_{n+1}) \le \hat{a}\beta_{n+1}/(1-\beta_{n+1})w_{n+1}, \tag{53}$$

where the equality simply comes from the definition of $E_L(R_n)$ and $E_L(R_{n+1})$ and the inequality comes from (50). Replicating the argument that yielded (45), but using $\mathcal{E}_L$ instead of $\mathcal{E}_H$ and the pair of functions (u_H, u_L) instead of (u_H, Q), we get

$$u_H(E_L(R_n)) - u_H(E_L(R_{n+1})) \le \tilde{\alpha}\left[u_L(E_L(R_n)) - u_L(E_L(R_{n+1}))\right]$$

for some positive constant $\tilde{\alpha}$. Since $u_H(E_L(R_n)) = u_H(E_H(R_n))$ and $u_H(E_L(R_{n+1})) = u_H(E_H(R_{n+1}))$, the previous equation combined with (53) proves (51).[71]

For the last equation, subtracting $u_H(E_H(R_n))$ from (19) and rearranging (recalling that $w_n = u_H(E_H(R_n)) - u_H(n)$) leads, along any choice sequence, to

$$w_{n+1} = w_n - \eta y_n + \eta w_{n+1} + (1-\eta)(u_H(E_H(R_{n+1})) - u_H(E_H(R_n))). \tag{54}$$

Combining this with (51) yields

$$w_{n+1} - w_n \ge \eta w_{n+1} - \eta y_n - b\beta_{n+1}w_{n+1}$$

and hence (52). ■

Lemma 12 (Geometric Bound) *There exists $\underline{q} > 0$ such that for any C on $\mathcal{E}_L$ and R in $\mathcal{H}$ such that $u_L(R) = u_L(C)$,*

$$Q(R) - Q(C) \ge \underline{q}(u_H(C) - u_H(R))^2.$$

Proof. Fix some $C \in \mathcal{E}_L$ and consider the referential centered at C whose x-axis is the common tangent of u_L and Q at C, oriented towards $\mathcal{H}$, and whose y-axis is the normal vector pointing northeast in $\mathcal{C}$. The components of a contract, in this referential, are denoted x_t and x_n, respectively, with C being the origin.

[71]Intuitively, that equation comes from two observations. First, L's utility from the current contract, R_n, cannot decrease by too much between consecutive rounds. Indeed, recall that β_{n+1} is the probability of facing H in round $n+1$, while w_{n+1} is a measure of the maximum rent that P can extract from H at round $n+1$. If the product $\beta_{n+1}w_{n+1}$ is small, it means that, comes round $n+1$, P has very little incentive to extract rents from H, which implies, intuitively, that his continuation strategy must be similar to what he would do if he only faced L, namely to jump to the L efficient contract $E_L(R_{n+1})$, which gives L utility $u_L(R_{n+1})$. Anticipating this, however, L is willing to forgo the current contract R_n only if R_{n+1} gives him a utility that is not much lower than R_n. The second observation is that the H-efficient contracts $E_H(R_n)$ and $E_H(R_{n+1})$ are constructed based on the utility that L gets from R_n and R_{n+1}. Using a suitable Lipschitz relation between utility differences then yields (35).

We parameterize the contract set $\mathcal{U}_L(C) = \{R \in \mathcal{C} : u_L(\tilde{C}) = u_L(C)\}$ in terms of x_t: $\{C(x_t) = (x_t, x_n(x_t))\}$. With this parameterization, we have $C(0) = C$ and $C(x_t) \in \mathcal{H}$ if and only if $x_t \geq 0$.[72]

Let $Q(x_t) = Q(C(x_t))$ and $u_H(x_t) = u_H(C(x_t))$ denote the cost and utility of H along $\mathcal{U}_L(C)$, as a function of the parameter x_t. By L-efficiency of C, we have $Q'(0) = 0$.[73] Because u_L is concave and Q is convex, the iso-utility curve of u_L going through C is convex and corresponds to positive values of x_n, while the isocost curve going through C is concave and corresponds to negative values of x_n. Moreover, by assumption at least one of these curves has a nonzero curvature at C. In the (x_t, x_n) space, this means that either $d^2u_L/dx_t^2 > 0$ or $d^2Q/dx_t^2 < 0$. We wish to show the existence of a constant $\hat{q} > 0$ such that $Q(x_t) - Q(0) \geq \hat{q}x_t^2$ for x_t in a right neighborhood of 0. Suppose first that $d^2u_L/dx_t^2 > 0$. This implies that $x_n(x_t) \geq q_x x_t^2$ for some $q_x > 0$ and x_t in a neighborhood of zero. Therefore, $Q(x_t) \geq Q(0) + q_x\|\nabla Q(C)\|x_t^2$ for that neighborhood. Now suppose that $d^2Q/dx_t^2 < 0$. In that case, let $D(x_t)$ denote the contract of the isocost curve with x-value x_t in the new referential (hence, just below $C(x_t)$ in the new referential), so that $Q(D(x_t)) = Q(C)$ for all x_t. By tangency of the curves, we have $\|C(x_t) - D(x_t)\| = o(x_t)$. Moreover, $Q(x_t) = Q(C(x_t)) = Q(D(x_t) + \nabla Q(D(x_t)) \cdot (C(x_t) - D(x_t)) + O(\|C(x_t) - D(x_t)\|^2)$ by a standard Taylor expansion. Finally, for x_t in a neighborhood of 0, $\nabla Q(D(x_t)) = \nabla Q(C) + O(\|D(x_t) - C\|) = \nabla Q(C) + o(x_t)$. Combining this, we get $Q(x_t) = Q(D(x_t)) + \nabla Q(C) \cdot (C(x_t) - D(x_t)) + o(x_t)(\|C(x_t) - D(x_t)\| + \|D(x_t) - C\|)$. Since $d^2Q/dx_t^2 < 0$, the y-value of $D(x_t)$ in the new referential satisfies $x_n^D(x_t) \leq -\hat{q}_x x_t^2$ for some $\hat{q}_x > 0$. Hence, $\nabla Q(C) \cdot (C(x_t) - D(x_t)) \geq \tilde{q}_x(x_n(x_t) - x_n^D(x_t)) \geq \check{q}_x x_t^2$ for some positive constants $\tilde{q}_x, \check{q}_x$. Combining all this implies that

$$Q(x_t) \geq Q(C) + \hat{q}x_t^2 + o(x_t^2), \tag{55}$$

proving the result for that case too.

By compactness and convexity of $\mathcal{U}_L(C)$, moreover, $\hat{q}$ may be chosen small enough so that the inequality

$$Q(x_t) - Q(0) \geq \hat{q}x_t^2$$

holds for all nonnegative x_t. Since u_H has bounded derivatives, there must exists $\bar{u} > 0$ such that $|u_H(x_t) - u_H(0)| \leq \bar{u}x_t$ (the single-crossing property between u_H and u_L imply that $u_H(x_t) \leq u_H(0)$ for all $x_t \geq 0$ and that $\nabla u_H(C) \cdot (C(x_t) - C) \neq 0$ for x_t in a neighborhood of 0). Combining these inequalities, there exists $\underline{q}(C) > 0$ such that

$$Q(C_\lambda) - Q(C) \geq \underline{q}(C)\left(u_H(C) - u_H(C_\lambda)\right)^2.$$

Moreover, $\underline{q}(C)$ can clearly be chosen to vary continuously in $C \in \mathcal{E}_L$.[74] By compactness of $\mathcal{E}_L$, $\underline{q} = \min_{C \in \mathcal{E}_L} \underline{q}(C)$ is strictly positive and yields the desired inequality. ■

Lemma 13 *There exist positive constants k_2 and k_3 such that*

$$y_n^2 \leq k_2[Q(R_{n+1}) - Q(E_L(R_n))] + k_3(\max\{(\beta_n w_n/(1-\beta_n))^2, (\beta_{n+1}w_{n+1})^2\} + \beta_{n+1}w_{n+1}) \tag{56}$$

[72]The parameterization is well defined, because $\mathcal{U}_L(C)$ can only have one point for each x_t, by strict monotonicity of u_L in the original coordinates (x_1, x_2) and the fact that increasing x_n corresponds to increasing both x_1 and x_2 and at least one of these increases is strict, since the normal vector defining x_n points northeastwards.

[73]Formally, we have $Q'(x_t) = \frac{\partial Q}{\partial x_1}\frac{dx_1}{dx_t} + \frac{\partial Q}{\partial x_2}\frac{dx_2}{dx_t}$. Since C is L-efficient, Q and u_L are tangent at C. This implies that the tangent vector $(dx_1/dx_t, dx_2/dx_t)$ is orthogonal to the normal vector $(\partial Q/\partial x_1, \partial Q/\partial x_2)$ at C.

[74]Indeed, all the constants involved in the previous steps are based on the curvature of the isoutility and isocost curves at C, which only involve the second derivative of the utility and cost functions at C. These functions were assumed to be C^2 over $\mathcal{C}$.

Proof. We have

$$\begin{aligned}
y_n^2 &= [(u_H(E_H(R_n)) - u_H(E_H(R_{n+1}))) + (u_H(E_H(R_{n+1})) - u_H(R_{n+1}))]^2 \\
&\leq 2[u_H(E_H(R_n)) - u_H(E_H(R_{n+1}))]^2 + 2[u_H(E_H(R_{n+1})) - u_H(R_{n+1})]^2 \\
&\leq k_1 \left(\max\{\beta_n w_n/(1-\beta_n), \beta_{n+1}w_{n+1}\}\right)^2 + 2[u_H(E_H(R_{n+1})) - u_H(R_{n+1})]^2 \\
&\leq k_1 \max\{(\beta_n w_n/(1-\beta_n))^2, (\beta_{n+1}w_{n+1})^2\} + k_2[Q(R_{n+1}) - Q(E_L(R_{n+1}))] \\
&= k_1 \max\{(\beta_n w_n/(1-\beta_n))^2, (\beta_{n+1}w_{n+1})^2\} + k_2[Q(E_L(R_n)) - Q(E_L(R_{n+1}))] + k_2[Q(R_{n+1}) - Q(E_L(R_n))].
\end{aligned}$$

The first inequality is standard ($(a+b)^2 \leq 2a^2+2b^2$). The second inequality comes from (48) and (51), which taken together imply an upper bound on $|u_H(E_H(R_n) - u_H(E_H(R_{n+1}))|$. The third inequality comes from the equality $u_H(E_H(R_{n+1})) = u_H(E_L(R_{n+1}))$ and Lemma 12 applied to the contracts $C = E_L(R_{n+1})$ and $R = R_{n+1}$. The difference $Q(E_L(R_n)) - Q(E_L(R_{n+1}))$ is bounded above in proportion to $u_L(R_n) - u_L(R_{n+1})$ (by a simple transposition to $\mathcal{E}_L$ of the proof of (45)), and that latter difference is bounded above by $\gamma\beta_{n+1}w_{n+1}$, from (50). This shows that

$$y_n^2 \leq k_2[Q(R_{n+1}) - Q(E_L(R_n))] + k_3(\max\{(\beta_n w_n/(1-\beta_n))^2, (\beta_{n+1}w_{n+1})^2\} + \beta_{n+1}w_{n+1})$$

which yields the result. ■

E Proofs for Parts I and II

Proof of Lemma 4 Fix any choice sequence and let n_0 denote the first round along that sequence such that $\bar{w}_{n_0} \leq \varepsilon$. By construction, $\bar{w}_{n_0-1} > \varepsilon$. From Lemma 5 (whose proof, in the main text, is independent of this lemma), we have $u_H(n_0) \leq u_H(n_0-1) + \eta\Delta_H$. Therefore,

$$\bar{w}_{n_0} \geq \bar{w}_{n_0-1} + u_H(n_0-1) - u_H(n_0) \geq \varepsilon - \eta\Delta_H.$$

Since we can always select a choice sequence along which β_n is weakly decreasing, we also get $\beta_{n_0} \leq \beta_0$. ■

Lemma 14 *i) For any round n_0, $\breve{\varepsilon} > 0$, and choice sequence, there exists a round $n > n_0$ such that $u_H(R_n) \geq \max\{u_H(E_H(R_m)) : m \leq n_0\} - \breve{\varepsilon}$. ii) The augmented rent index $\bar{w}_n = \max\{u_H(E_H(R_m)) : m \leq n\} - u_H(n)$ converges to zero as n goes to infinity, along any choice sequence.*

Proof. i) Fix $\breve{\varepsilon} > 0$. Proposition 2 guarantees that, along any choice sequence, R_n converges to an L-efficient $\bar{C}_L$. Continuity of $u_H(\cdot)$ implies that there exists a round $\breve{n}$ such that $u_H(R_n) \geq u_H(\bar{C}_L) - \breve{\varepsilon}$ for all $n \geq \breve{n}$. Therefore, it suffices to show that $u_H(\bar{C}_L) \geq \max\{u_H(E_H(R_m)) : m \leq n_0\}$ for all n_0. Equivalently, we must show that $u_H(\bar{C}_L) \geq \max\{u_H(E_L(R_m)) : m \leq n_0\}$ for all n_0 since, by construction, $E_H(R)$ and $E_L(R)$ give the same utility to H for any $R \in \mathcal{H}$. For contracts C, C' on the L-efficiency curve $\mathcal{E}_L$, $u_H(C) \leq u_H(C')$ if and only if $u_L(C) \leq u_L(C')$. Therefore, it suffices to show that $u_L(\bar{C}_L) \geq \max_{m\in\mathbb{N}}\{u_L(E_L(R_m))\}$. By construction, $u_L(E_L(R)) = u_L(R)$ for all $R \in \mathcal{H}$, since $E_L(R)$ is the L-efficient contract that gives L the same utility as R. Therefore, we have reduced the problem to showing that

$$u_L(\bar{C}_L) \geq \max_{m\in\mathbb{N}}\{u_L(R_m)\}.$$

We recall that for all n, $u_L(n) \geq u_L(R_n)$ since holding on to R_n is always a feasible strategy for L, and that $u_L(n)$ is nondecreasing in n for all choice sequences (see Lemma 5; the argument also there applies to L). Since R_n converges to $\bar{C}_L$, $u_L(n)$ must converge to $u_L(\bar{C}_L)$. Finally, because $u_L(n)$ is nondecreasing, we get

$$u_L(R_n) \leq u_L(n) \leq u_L(\bar{C}_L),$$

which concludes the proof of i). To prove ii), it suffices to notice that $\max\{u_H(E_H(R_m)) : m \leq n\}$ and $u_H(R_n)$ both converge to $u_H(\bar{C}_L)$, from the previous reasoning. ■

Lemma 15 *There exists a pushdown sequence at Block 1.*

Proof. Let $\mu^\theta(\{\tilde{R}_n\})$ denote the probability, conditional on facing type θ, of observing choice sequence $\{\tilde{R}_n\}$ until $\hat{u}_1$ is reached. By definition, summing over all choice sequence with elements in $\mathcal{H}$ and truncated at the first round at which H's continuation utility reaches $\hat{u}_1$, we have $\sum_{\{\tilde{R}_n\}} \mu^H(\{\tilde{R}_n\}) = \mu_0$. Because L always chooses contracts in $\mathcal{H}$, we also have $\sum_{\{\tilde{R}_n\}} \mu^L(\{\tilde{R}_n\}) = 1$. These two equations immediately imply that there exists a choice sequence $\{R_n^0\}$ such that $\mu^H(\{R_n^0\})/\mu^L(\{R_n^0\}) \leq \mu_0$. Conditional on observing that choice sequence, the posterior is given by Bayesian updating

$$\hat{\beta}_1 = \frac{\mu^H(\{R_n^0\})\hat{\beta}_0}{\mu^H(\{R_n^0\})\hat{\beta}_0 + \mu^L(\{R_n^0\})(1-\hat{\beta}_0)}.$$

Dividing by $\mu^L(\{R_n^0\})$ and using that $\mu^H(\{R_n^0\})/\mu^L(\{R_n^0\}) \leq \mu_0$ yields the result. ■

Let $\underline{u}_H = u_H(R_0)$ and, for any $\tilde{\epsilon} \geq 0$,

$$D(\tilde{\epsilon}) = \inf\{Q(C) - Q(E) : C \in \mathcal{H}, E \in \mathcal{E}_H : \underline{u}_H \leq u_H(E) \leq u_H(E_H(C)) + \tilde{\epsilon}\}. \tag{57}$$

$D(\tilde{\epsilon})$ is nonincreasing in $\tilde{\epsilon}$, as a higher $\tilde{\epsilon}$ merely increases the set of (C, E) pairs over which the objective is minimized. Because R_0 is regular, the contracts C arising in (57) are bounded away from $\mathcal{E}_H$ for $\tilde{\epsilon}$ small enough, and this implies that $D(\tilde{\epsilon})$ is strictly positive for $\tilde{\epsilon}$ small enough. Intuitively, C and E must provide almost the same utility to H, and E is a strictly cheaper way than C of doing so. For such values of $\tilde{\epsilon}$, $D(\tilde{\epsilon})$ defines a lower bound on the inefficiency of contracts in $\mathcal{H}$ conditional on facing H.

Lemma 16 *If at the beginning of any block k, $w_{n_{k-1}} \leq \varepsilon$, then for all rounds n of block k,*

$$Q(R_n) \geq Q(E_H(R_{n(k-1)})) + D(\varepsilon)$$

Proof. Let C denote the L-efficient contract that gives H utility $u_H(n_{k-1})$. Since $u_H(n)$ is nondecreasing, we have for any round n of block k, $u_H(n_{k-1}) \leq u_H(n)$. From part iv) of Proposition 1, this implies that R_n must cost weakly more than C: otherwise, we would have $u_L(E_L(R_n)) < u_L(C)$ and hence $u_H(E_H(R_n)) < u_H(E_H(C))$, which would imply that $u_H(n) \leq u_H(E_H(R_n)) < u_H(E_H(C)) = u_H(n_{k-1})$, a contradiction. By assumption, we have $u_H(E_H(R_{n(k-1)})) - u_H(C) = w_{n_{k-1}} \leq \varepsilon$. By definition of $D(\varepsilon)$, this implies that $Q(C) \geq Q(E_H(R_{n_{k-1}})) + D(\varepsilon)$. Since $Q(R_n) \geq Q(C)$, this proves the lemma. ■

We will consider ε such that $D(2\varepsilon) > 0$, we let $D = D(2\varepsilon)$ denote the lower bound on the loss that is used throughout the proof.

Remark 3 *In principle, one could reach a block k for which w_{k-1}, and hence $\hat{w}_{k-1}$, is greater than 2ε, which would imply that the lower bound D on the loss is not guaranteed to hold for that block. If that is the case, however, Lemma 4 guarantees that one can find a later round n for which $\bar{w}_n$ lies in $(\varepsilon/2, \varepsilon)$, and one can restart the analysis from that round (i.e., this is our new "n_0"). Moreover, $\hat{\beta}_{k-1} \leq \beta_0$, so the two conclusions of Lemma 4 hold. Re-starting Part I from the new round n_0, one may encounter a block for which this problem arises again, in which case one re-initialize the analysis again, starting from a yet later round. Since $\bar{w}_n$ converges to zero along any choice sequence as n goes to infinity, by Lemma 14, there can only be finitely many such initializations: there must exist a round n_0 such that i) $\bar{w}_{n_0} \in (\varepsilon/2, \varepsilon)$, ii) $\beta_{n_0} \leq \beta_0$, and iii) $\hat{w}_k$ remains below 2ε for all blocks constructed from n_0.*[75]

Proof of Lemma 6

Consider any round n and contract R_{n+1} in $M_n \cup \{R_n\}$. If $\mu_n^L(R_{n+1}) \geq \bar{\varepsilon}$, then $\mu_n(R_{n+1}) \leq \frac{1}{\bar{\varepsilon}}$ and, hence, $\beta_{n+1} \leq \frac{\beta_n}{\bar{\varepsilon}}$, since $\beta_{n+1} \sim \mu_n(R_{n+1})\beta_n$ by Bayesian updating (see (38); the term $\beta_{n+1}\beta_n$ can be neglected). The set of contracts R_{n+1} for which $\mu_n^L(R_{n+1}) < \bar{\varepsilon}$ has probability at most $G\bar{\varepsilon}$, where G is the upper bound on the size of the menu. Therefore, with probability at least $1 - G\bar{\varepsilon}$,

$$\beta_{n+1} \leq \frac{\beta_n}{\bar{\varepsilon}}$$

At round $\bar{n}$, we have $\beta_{\bar{n}} \leq \eta^d$ and $w_{\bar{n}} \leq \bar{w}\eta$. Therefore, $\beta_{\bar{n}+1} \leq \eta^d/\bar{\varepsilon}$ with probability at least $1 - G\bar{\varepsilon}$. From (49), $w_{n+1} \leq w_n \left(1 + \frac{\alpha\beta_n}{1-\beta_n}\right)$. Therefore, we also have $w_{\bar{n}+1} \leq k_1\eta$ for some constant k_1. This implies that with probability at least $1 - G\bar{\varepsilon}$, the lower bound $D = D(2\varepsilon)$ on the loss is valid for round $\bar{n}+1$, because $w_{n+1} \leq 2\varepsilon$. The previous reasoning can be applied by induction to rounds $n = \bar{n}, \ldots, \bar{n} + \bar{N} - 1$. It implies that with probability $1 - k(\bar{N})\bar{\varepsilon}$, we have

$$\beta_n \leq (\bar{\varepsilon})^{-\bar{N}}\bar{\beta}\eta^d \tag{58}$$

$$w_n \leq \bar{W}(\bar{N})\eta \tag{59}$$

for all $n \in \{\bar{n}, \ldots, \bar{n} + \bar{N}\}$, for some constants $k(\bar{N})$ and $\bar{W}(\bar{N})$ independent of $\bar{\varepsilon}$ and η.

Consider any choice sequence such that β_n and w_n satisfy the above inequalities throughout the block, which occur with probability $1 - k(\bar{N})\bar{\varepsilon}$. There remains to show the claim that $y_n = O(\eta^{d/4})$ throughout the block for such sequences. We begin by showing the result for round $n = \bar{n}$. The first step is to show that $Q(R_{n+1}) - Q(E_L(R_n))$ must be of order $O\left(\frac{\beta_n}{\mu_n^L(R_{n+1})}\right)$ for that round. If each term in the sum entering P's IC constraint (26) is nonnegative, this result comes from the inequality[76]

$$w_n a\beta_n \geq \mu_n^L(R_{n+1})[\beta_n\mu_n(R_{n+1})\eta D + (1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(R_n)))]$$

which implies that

$$Q(R_{n+1}) - Q(E_L(R_n)) \leq \frac{aw_n}{\eta(1-\beta_n)}\frac{\beta_n}{\mu_n^L(R_{n+1})}. \tag{60}$$

In general, while some terms

$$\mu_n^L(R_{n+1})(1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(R_n))) \tag{61}$$

[75]Indeed, the contrapositive is that there exists a choice sequence such that for all n, there is a round $n' > n$ for which $w_{n'} \geq 2\varepsilon$, which clearly contradicts the convergence of w_n to zero along all choice sequences.

[76]The inequality holds because each term in the sum is nonnegative, and $w_n a\beta_n$ is bigger than the sum.

involved in the sum of (26) may be negative, they can only be very slightly so: indeed, we have

$$Q(R_{n+1}) - Q(E_L(R_n)) \geq Q(E_L(R_{n+1})) - Q(E_L(R_n)) \geq -k\beta_{n+1}w_{n+1}, \tag{62}$$

where the second inequality comes (50) of Lemma 11. Moreover, $w_{n+1} \leq w_n\left(1 + \frac{\alpha\beta_n}{1-\beta_n}\right)$, from (49), and $\mu_n^L(R_{n+1})\beta_{n+1}$ is of order β_n. Therefore, the lower bound of (60) is of order $w_n\beta_n$, and hence each negative term which may arise in (26) are of order $\eta w_n\beta_n$. Since there are at most G of them, we conclude that (60) holds up to a term of order $\eta\beta_n$, which is negligible compared to the first term.

From (60), we see that for all R_{n+1} such that $\mu_n^L(R_{n+1}) \geq \sqrt{\beta_n}$, the difference $Q(R_{n+1}) - Q(E_L(R_n))$ is at most of order $\sqrt{\beta_n}$. By Lemma 13 of the Appendix, this implies that y_n is $O(\eta^{d/4})$: Indeed, y_n^2 is bounded above by terms proportional to $Q(R_{n+1}) - Q(E_L(R_n))$ and a term proportional to $\max\{(\beta_n w_n/(1-\beta_n))^2, (\beta_{n+1}w_{n+1})^2\} + \beta_{n+1}w_{n+1}$. The first term is of order $\sqrt{\beta_n} = O(\eta^{d/2})$, while the latter is of order $\beta_n\eta$, and is thus negligible compared to the first. Moreover, the set of contracts R_{n+1} for which $\mu_n^L(R_{n+1}) < \sqrt{\beta_n}$, is negligible: it arises with probability at most $G\sqrt{\beta_n}$. Since $\sqrt{\beta_n} = O(\eta^{d/2})$ is small compared to $\bar{\varepsilon}$, for η small enough, we conclude that with probability $1 - O(\bar{\varepsilon})$, $\beta_n = O(\eta^d)$, $w_n = O(\eta)$ and $y_n = O(\eta^{d/4})$ for round $\bar{n}$ and, by induction, for all rounds of the block. ■

F Proofs for Part III

Proof of Lemma 7

By assumption, $\beta_N \leq \eta^d$ so β_N becomes arbitrarily small as η gets small. We recall equation (49) from Lemma 11:

$$w_{n+1} \leq w_n\left(1 + \frac{\alpha\beta_n}{1-\beta_n}\right), \tag{63}$$

where $\alpha > 0$. From Bayesian updating, we have $\beta_{n+1} = \frac{\mu_n\beta_n}{\mu_n\beta_n + (1-\beta_n)}$. Since β_N is arbitrarily small, the denominator is arbitrarily close to 1 for $n = N$. More generally we have

$$\beta_{n+1} \leq \mu_n\beta_n(1+\epsilon) \tag{64}$$

where ϵ is a small positive constant, as long as β_n remains small. At N, we have $\beta_N \leq \eta^d$ and $w_N = e_N - u_N \leq \bar{e}_N - u_N \leq \frac{\eta D}{2a}$, which implies from (63) that $w_{N+1} \leq \frac{\eta D}{2a}(1 + \alpha\beta_n/(1-\beta_n))$. From (31), this implies that $\mu_N \leq \frac{1}{2} + O(\eta) \leq \frac{3}{5}$.

Consider the first round $M > N$ for which $\mu_M \geq 3/4$. The probability β_n is decreasing[77] until at least round M. Proceeding by induction, from round N to round M, the previous inequalities imply that

$$w_{N+m} \leq w_N \prod_{i=1}^{m}(1 + \alpha(1+\varepsilon)\beta_{N+i}) \tag{65}$$

and

$$\beta_{N+i} \leq \beta_N \prod_{j=0}^{i-1}\left(\mu_{N+j}(1+\epsilon)\right), \tag{66}$$

[77]This comes from the Bayesian updating equation $\beta_{n+1} = \frac{\mu_n\beta_n}{\mu_n\beta_n+(1-\beta_n)}$, which is nondecreasing in μ_n. Taking $\mu_n = 1$ shows that $\beta_{n+1} \leq \beta_n$ as long as $\mu_n \leq 1$.

and hence also that (64) is valid for all rounds $n \in \{N, \dots, M\}$. From (66), we have

$$\beta_{N+i} \le \left(\frac{3(1+\epsilon)}{4}\right)^i \beta_N$$

Therefore, (65) implies that

$$w_M \le w_N \prod_{i=1}^{M-N} \left(1 + \alpha(1+\varepsilon)\eta^d \left(\frac{3(1+\epsilon)}{4}\right)^i\right)$$

The product

$$\prod_{i=1}^{\infty} \left(1 + \alpha(1+\varepsilon)\eta^d \left(\frac{3(1+\epsilon)}{4}\right)^i\right) \tag{67}$$

is finite for η small enough, and converges to 1 as η goes to zero.[78] Therefore, for η small, w_M is bounded above by $\frac{5}{4}w_N \le \frac{5\eta D}{8a}$. From (31), this implies that μ_M is bounded above by $5/8 + O(\eta) < 3/4$, so M cannot be finite. This shows that for η below some threshold $\hat{\eta}$, μ_n is bounded above by 3/4 for all $n \ge N$ and, from (64), that β_n is decreasing. Since w_n is bounded above by $\frac{3}{2}w_N$ and $w_N \le \frac{\eta D}{2a}$, the last claim follows easily. ∎

Lemma 17 *There exists a positive constant $\bar{A}$ such that*

$$y_n^2 \le \frac{\bar{A}\beta_{n+1}}{1-\beta_0} \tag{68}$$

Proof. Equation (33) implies that $Q(R_{n+1}) - Q(E_L(R_n)) \le \frac{\beta_n w_n a}{\eta(1-\beta_0)}$, since $\beta_n \le \beta_0$. This, along with (34), yields[79]

$$Q(R_{n+1}) - Q(E_L(R_n)) \le \frac{D\beta_{n+1}}{1-\beta_0}.$$

Combining this inequality with Lemma 13, we get

$$y_n^2 \le k_2 \frac{D\beta_{n+1}}{1-\beta_0} + k_3(\max\{(\beta_n w_n/(1-\beta_n))^2, (\beta_{n+1}w_{n+1})^2\} + \beta_{n+1}w_{n+1}).$$

Since $w_{n+1} \le \frac{\eta D}{2a} \ll 1$, the last term is negligible compared to β_{n+1}. Taking $\bar{A}$ slightly greater than $k_2 D$ proves the lemma. ∎

Proof of Lemma 8

Taking the square root of (42) and multiplying the result by $\frac{\sqrt{\beta_{\hat{N}+1}}}{\beta_{\hat{N}}}$, we get

$$\frac{\beta_{\hat{N}+1}}{\beta_{\hat{N}}} \ge 2c\sqrt{\beta_{\hat{N}+1}}.$$

[78] Indeed, taking the logarithm of that product, we obtain a sequence that is approximately geometric with geometric factor 3/4 and, hence converges, uniformly in η. Moreover, each term of the sequence is of order η^d, which converges to 0 as η goes to zero. This implies that all partial sums converge to zero and, by uniform convergence, that the sequence converges to zero as well. By continuity of the exponential function, the product itself thus converges to 1 as η goes to zero.

[79] We are using $\beta_{n+1} \ge \mu_n \beta_n$, which comes from the first inequality of (38).

Combining this with (40) yields

$$\frac{\beta_{\hat{N}+2}}{\beta_{\hat{N}+1}} \geq c\sqrt{\beta_{\hat{N}+1}}.$$

Taking the square root of this expression and dividing both sides by $\sqrt{\beta_{\hat{N}+1}}$, we get

$$\frac{\sqrt{\beta_{\hat{N}+2}}}{\beta_{\hat{N}+1}} \geq \frac{\sqrt{c}}{\beta_{\hat{N}+1}^{1/4}} \tag{69}$$

Combining this with (43) (and using that $\beta_{\hat{N}+1} \leq \beta_{\hat{N}}$) shows that (42) holds at round $\hat{N}+1$. Since β_n is non-increasing in n for $n \geq \hat{N}$ and hence satisfies (43) for all $n \geq \hat{N}$, we can apply the previous argument by induction to conclude that (42) and (69) hold for all $n \geq \hat{N}$. Multiplying (40) by $\frac{\beta_n}{\beta_{n+1}}$, we obtain

$$\frac{q_{n+1}}{q_n} \geq 1 - \frac{c\beta_n}{\sqrt{\beta_{n+1}}}.$$

From (69) applied to round n (instead of $\hat{N}+1$), the last term is bounded above by $\frac{c\beta_n^{1/4}}{\sqrt{c}}$, which converges to zero as n goes to infinity. ■

Proof of Lemma 9

Suppose by contradiction that $\{q_n\}$ converges to zero. This along with the second assumption of the lemma implies the existence, for any fixed $\varepsilon > 0$, of an integer $\bar{N}$ such that i) $\frac{q_{n+1}}{q_n} \geq 1-\varepsilon$ and ii) $q_n \leq q_{\bar{N}} \leq \varepsilon$ for all $n \geq \bar{N}$.[80] Convergence of $\{q_n\}$ to zero also implies that $\max_N \Pi_0^N q_k$ is bounded above by some constant $\bar{\Pi}$. Letting $\tilde{\varepsilon} = \sqrt{q_{\bar{N}}}$, we have $\Pi_{\bar{N}+1}^{\bar{N}+k} q_k \leq \tilde{\varepsilon}^{2k}$ for all integers $k \geq 1$. Therefore, for any integer $K \geq 1$, we have

$$q_{\bar{N}+K} = q_{\bar{N}+K} - q_\infty = \sum_{n \geq \bar{N}+K} (q_n - q_{n+1}) \leq \tilde{c}\tilde{\varepsilon}^K \sum_{k \geq 0} \tilde{\varepsilon}^k,$$

where $\tilde{c} = c'\sqrt{\bar{\Pi}}$ and the last inequality comes from the first hypothesis of the lemma. Taking $K=3$ and using that $\sum_{k\geq 0} \tilde{\varepsilon}^k = 1/(1-\tilde{\varepsilon})$, this yields

$$q_{\bar{N}+3} \leq \frac{c'}{1-\tilde{\varepsilon}} q_{\bar{N}}^{3/2} \leq 2c' q_{\bar{N}}^{3/2}. \tag{70}$$

Applying inequality i) above to $n = \bar{N}$, $\bar{N}+1$, and $\bar{N}+2$, yields

$$q_{\bar{N}+3} \geq q_{\bar{N}}(1-\varepsilon)^3. \tag{71}$$

Combining (70) and (71), we get$(1-\varepsilon)^3 \leq 2c' q_{\bar{N}}^{1/2} \leq 2c'\varepsilon^{1/2}$, which is impossible if we choose ε small enough. This yields the desired contradiction. ■

G Proof of Theorem 2, Statement B

Fix an initial belief $\beta_0 \in (0,1)$ and suppose without loss that $R_0 \in \mathcal{H}$. We start by showing that the probability $p_{\mathcal{H}}$ that H ends up with a contract in $\mathcal{H}$ converges to zero as η goes to zero. Let $\hat{Q}(u,p)$

[80] Indeed, there exist N_1 such that i) holds for all $n \geq N_1$ and N_2 such that $q_n \leq \varepsilon$ for all $n \geq N_2$. Letting $N = \max\{N_1, N_2\}$, any $\bar{N} \in \arg\max_{n \geq N}\{q_n\}$ satisfies conditions i) and ii).

denote the minimal expected cost of providing an expected utility u to H with a contract distribution that puts probability at least p on contracts lying in $\mathcal{H}$. We have $\hat{Q}(u_H(E_H(R_0)), 0) = Q(E_H(R_0))$, and $\hat{Q}(u_H(E_H(R_0)), p)$ is strictly increasing for p in a neighborhood of zero because contracts in $\mathcal{H}$ are inefficient for H.[81] Statement A of Theorem 2 guarantees that H must get a utility arbitrarily close to $u_H(E_H(R_0))$ and that the cost to P conditional on facing H must be arbitrarily close to $Q(E_H(R_0))$ as η goes to zero.[82] For any $\varepsilon > 0$, this implies that there exists a threshold $\tilde{\eta}(\varepsilon)$ such that $p_{\mathcal{H}} < \varepsilon$ for all PBEs corresponding to any $\eta < \tilde{\eta}(\varepsilon)$.

For the remainder of the proof, we fix some small[83] $\varepsilon > 0$ and focus on η's below the threshold $\tilde{\eta} = \tilde{\eta}(\varepsilon^4)$, so that $p_{\mathcal{H}} \leq \varepsilon^4$.

From Statement A, there exists a threshold $\hat{\eta}$ such that θ's expected utility at the beginning of the game is bounded below by $v_\theta(\varepsilon) = u_\theta(E_\theta(R_0)) - \varepsilon^4$ for all η's below that threshold.[84] Moreover, the cheapest contract $E_\theta(v_\theta(\varepsilon))$ that provides this utility costs $Q(E_\theta(R_0)) - O(\varepsilon^4)$ and lies within ε^4 of $E_\theta(R_0)$. We also recall from Lemma 2 that P's expected cost is bounded above by $\beta_0 Q(E_H(R_0)) + (1-\beta_0)Q(E_L(R_0))$ for any η and PBE. Fix a PBE associated with some $\eta \leq \min\{\hat{\eta}, \tilde{\eta}\}$ and let Q_θ denote P's expected cost conditional on facing θ and u_θ denote θ's expected utility in that PBE. The previous observations imply that $|Q_\theta - Q(E_\theta(R_0)| = O(\varepsilon^4)$ (see Footnote 82).

Let E_ε denote the L-efficient contract that gives H a utility of $u_H(v_H(\varepsilon))$. That contract lies within $O(\varepsilon^4)$ of $E_L(R_0)$. Part iv) of Proposition 1 implies that H never accepts a contract R' such that $u_H(R') < u_H(E_\varepsilon)$. Moreover, L as well would reject such a contract since it would reveal his type and lead to a lower utility, from Part i) of Proposition 1. Thus, such a contract does not arise in equilibrium. Let $\mathcal{C}(\varepsilon) = \{R' \in \mathcal{C} : u_H(R') \geq u_H(E_\varepsilon)\}$ denote the set of contracts which may arise in equilibrium. Also let $B(\theta)$ denote the ε-ball of $\mathbb{R}^2$ centered at $E_\theta(R_0)$.

The set of contracts which θ may end up with can be split between $B(\theta)$ and its complement $C(\theta) = \mathcal{C}(\varepsilon) \setminus B(\theta)$. Let p_θ denote the probability that θ ends up with a contract in $C(\theta)$, and let u_θ^B and u_θ^C denote the expected probabilities of θ conditional on ending up with a contract in $B(\theta)$ and $C(\theta)$, respectively. Our objective is to establish that $p_\theta \leq \varepsilon$ for η small enough.

From above, we already know that $p_{\mathcal{H}} = O(\varepsilon^4)$. Consider the probability p that H ends up with a contract in $D(H) = C(H) \setminus \mathcal{H}$. We show that p is of order ε^2. Because these contracts lie in $\mathcal{C}(\varepsilon) \setminus \mathcal{H}$, they must provide H with utility at least $u_H(R_0) - \varepsilon^4$. However, because they are outside of $B(H)$, they must cost at least $Q(E_H(R_0)) + q_H\varepsilon^2 + O(\varepsilon^4)$ from Lemma 12 (again, applying it to $\mathcal{E}_H$ instead of $\mathcal{E}_L$) where $q_H > 0$. Therefore, the expected cost $\tilde{Q}_H$ conditional on contracts being in $D(H)$ is bounded below by $Q_H + q_H\varepsilon^2 + O(\varepsilon^4)$. Moreover, the expected cost $\hat{Q}_H$ conditional on the contracts being in $B(H)$ is bounded below by $Q_H - O(\varepsilon^4)$. Let $Q_{\mathcal{H}}$ denote P's expected cost conditional on the joint event that $\theta = H$ and that H ends up with a

[81] In fact, Lemma 12 (which can be reproduced for $\mathcal{E}_H$ instead of $\mathcal{E}_L$) already implies this for contracts that must lie outside of any fixed ball centered at $E_H(R_0)$.

[82] Indeed, each type gets a utility arbitrarily close to $u_\theta(E_\theta(R_0))$ but P's expected cost is bounded above by $\beta_0 Q(E_H(R_0)) + (1-\beta_0)Q(E_L(R_0))$, from Lemma 2. Since $E_\theta(R_0)$'s are efficient, the claim follows.

[83] It suffices to show the claim for all ε small enough, as it immediately implies that claim for higher values of ε.

[84] We have proved the result for $\theta = H$, and the result is also trivially true for L, without the ε^4, since $u_L(E_L(R_0)) = u_L(R_0)$ is a lower bound on L's utility.

contract in $\mathcal{H}$. From $Q_H = p\tilde{Q}_H + p_{\mathcal{H}}Q_{\mathcal{H}} + (1-p-p_{\mathcal{H}})\hat{Q}_H$ we get

$$p = \frac{Q_H - \hat{Q}_H + p_{\mathcal{H}}(Q_{\mathcal{H}} - \hat{Q}_H)}{\tilde{Q}_H - \hat{Q}_H}$$

Since the numerator is bounded above by $O(\varepsilon^4)$ while the denominator is bounded below by a factor ε^2, we conclude that $p = O(\varepsilon^2)$ and, hence, that $p_H = p + p_{\mathcal{H}} \leq k_H\varepsilon^2$ for some $k_H > 0$.

There remains to show the result for p_L. We repeat the argument of the previous paragraph. Any contract in $\mathcal{C}(\varepsilon)$ provides L with utility at least $u_L(E_L(R_0)) - O(\varepsilon^4)$. However, because contracts in $C(L)$ lie outside of $B(L)$, they must cost at least $Q(E_L(R_0)) + q_L\varepsilon^2 + O(\varepsilon^4)$. Therefore, the expected cost $\tilde{Q}_L$ conditional on contracts being in $C(L)$ is bounded below by $Q_L + q_L\varepsilon^2 + O(\varepsilon^4)$ where $q_L > 0$. Moreover, the expected cost $\hat{Q}_L$ conditional on the contracts being in $B(L)$ is bounded below by $Q_L - O(\varepsilon^4)$. From $Q_L = p_L\tilde{Q}_L + (1-p_L)\hat{Q}_L$ we get

$$p_L = \frac{Q_L - \hat{Q}_L}{\tilde{Q}_L - \hat{Q}_L}$$

Since the numerator is bounded above by $O(\varepsilon^4)$ while the denominator is bounded below by a factor ε^2, we conclude that $p_L \leq k_L\varepsilon^2$ for some $k_L > 0$.

For ε small enough, $\max\{k_L\varepsilon^2, k_H\varepsilon^2\} \leq \varepsilon$. The threshold $\min\{\tilde{\eta}, \hat{\eta}\}$ delivers the conclusions of Statement B.

H Notation

- $u_\theta(n)$: type θ's continuation utility at the beginning of round n.
- $E_\theta(R)$: If R is the H-Rent configuration, $E_L(R)$ is the L-efficient contract that gives L the same utility as R and $E_H(R)$ is the H-efficient contract that gives H the same utility as $E_L(R)$ (see the definition preceding Theorem 2).
- $w_n = u_H(E_H(R_n)) - u_H(n)$.
- $\bar{w}_n = \max\{u_H(E_H(m)) : m \leq n\} - u_H(n)$.
- $y_n = u_H(E_H(R_n)) - u_H(R_{n+1})$.
- $\bar{C}_\theta = \lim_{n\to+\infty} u_\theta(n)$.

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