ESTIMATING CHANGES IN MARGINAL UTILITY FROM DISAGGREGATED EXPENDITURES

ETHAN LIGON

1. Introduction

The models economists use to describe dynamic consumer behavior almost invariably boil down to a description of how consumers' marginal utilities evolve over time. A central example involves the canonical Euler equation, which describes how consumers smooth consumption over time when they have access to credit markets (Bewley, 1977; Hall, 1978) or some more general set of asset markets (Hansen and Singleton, 1982); this same Euler equation tells us how to price assets using the mechanics of the Consumption Capital-Asset Pricing Model (Lucas, Jr., 1978; Breeden, 1979).

So, taking dynamic models of consumer behavior to the data means measuring marginal utilities. But marginal utilities are not directly observable. The usual approach to measuring these indirectly involves constructing measures of consumers' total consumption expenditures, and then plugging these total expenditures into a parametric utility function, where marginal utilities may (Hansen and Singleton, 1982; Ogaki and Atkeson, 1997) or may not Hall (1978) also depend on unknown parameters which have to be estimated. A possible justification for this approach comes the Marshallian treatment of consumer demand: Provided that consumer's intertemporal preferences are additively time-separable, then Marshallian intratemporal demand systems are functions of (all) prices within a period and (all) expenditures within that same period; further, the consumer's indirect utility can be written as a function of the same two arguments. Thus modeling demand and welfare using the Marshallian apparatus then seems to call for *measuring* all prices and total expenditures.

There are problems with this approach, both in principle and in practice. In principle, the usual practice involves plugging total expenditures into a *direct*, rather than an indirect utility function. This is

Date: April 25, 2014.

This is an incomplete draft of a paper describing research in progress. Please do not distribute without first consulting the author.

defensible only if Engel curves are all linear, and we know that they are not (Engel, 1857; Houthakker, 1957). In practice the exercise of collecting the necessary data on quantities and prices of all consumption goods and services is extremely difficult, and even well-financed efforts by well-trained, ingenious economists and statisticians have yielded less than satisfactory results.

Understanding the behavior of households in low income countries through the lens of economic theory involves thinking about consumer demand. Provided that consumer's intertemporal preferences are additively time-separable, then Marshallian intratemporal demand systems are functions of (all) prices within a period and (all) expenditures within that same period. But in practice the exercise of collecting the necessary data on quantities and prices of all consumption goods and services is extremely difficult, and even well-financed efforts by well-trained, ingenious economists and statisticians have yielded less than satisfactory results.

In this paper we describe an alternative approach to measuring changes in households' marginal utilities which completely avoids the task of trying to measure total expenditures on all goods and services. Instead we measure disaggregate expenditures on selected goods. Avoiding aggregation allows us to also avoid the most serious of the problems described above. The key to our approach is to take advantage of the variation in the composition of different consumers' consumption bundles; this is variation ignored by the usual approach. Our approach is also practical: we simply don't need to use data on goods or services for which data or prices are suspect; and we entirely avoid the difficulties of constructing comprehensive aggregate; and it's simply unnecessary to construct price indices to recover "real" expenditures; nominal expenditures are all that we need.

This paper proceeds by first sketching a simple model of household demand behavior, and using this model to derive a set of "Frischian" demands, using a strategy that is quite close to that taken by Attfield and Browning (1985). But where their final demand system resembles a Frischian version of the Rotterdam demand system (which they take to time series data), ours more closely resembles a Frischian version of the AID system of Deaton and Muellbauer (1980). We take this demand system to household-level data, so as to exploit differences in the composition of households' expenditure bundles across the wealth distribution.

We next use the differential Frischian demand system we derive to develop a demand system which can be estimated using household panel data, in a specification involving first differences in logarithms of expenditures. This estimator delivers estimates of changes in each household's marginal utility over time, along with a estimates of the value of a function of shadow prices which summarizes the influence of aggregate changes on demand, and estimates of the effects of various observable household characteristics on demand.

We subsequently discuss an attractive practical feature of our demand system: it's very simple and natural to estimate a reduced incomplete demand system. In particular, if there are some goods which seem poorly measured or which are uninformative regarding household marginal utility, we can simply ignore these. We give two more formal criteria for making decisions about what goods ought to be included in the system being estimated.

Finally, in an appendix we illustrate our methods using data from two rounds of surveys in Uganda. We're able to obtain workable estimates of both the parameters of the demand system and of changes in household marginal utility. We're also able to show something important: namely that our estimates of household marginal utility are quite robust to changes in the number and type of goods being included in the estimated demand system. The chief difficulty we encounter with the Ugandan data is that many observations feature zero expenditures for many goods. We conclude with some discussion of several specific food goods that seem well suited to on-going monitoring of marginal utility in Uganda.

2. A Wrong Turn

In the standard case in which utility takes a von Neumann-Morgenstern form and is thought to be separable across periods, the Euler equation for a consumer j might be written

$$u'(c_{jt}) = \beta_j \mathrm{E}_t R_{t+1} u'(c_{jt+1}), \tag{1}$$

where u is a momentary utility function, β_j is the discount factor for the jth consumer, R_t are returns to some asset realized at time t, and where c_{jt} is a measure of total expenditures or consumption by consumer j at time t, so that $u'(c_{jt})$ is the marginal utility of consumption for the jth household at time t.

These same marginal utilities are often used to characterize not only intertemporal behavior of a representative consumer often featured in the macroeconomic literature, but also tests of risk sharing across households in the US (Mace, 1991; Cochrane, 1991), other high income countries (Deaton and Paxson, 1996), and low income countries (Townsend, 1994; Ligon, 1998; Ligon et al., 2002; Angelucci and Giorgi, 2009).

Estimating or testing models using the kinds of restrictions of which (1) is an example requires one to take a stand on just what $u'(c)$ is. The very notation seems to imply that $u' : \mathbb{R} \rightarrow \mathbb{R}$; that is that marginal utility depends only on a scalar quantity. To construct measures of marginal utilities, empirical papers of the sorts mentioned above typically begin by constructing a *consumption aggregate*, which is typically designed to capture total expenditures on non-durable goods and services over some period of time; and then plug that consumption aggregate into some parametric direct momentary utility function. For example, Hall (1978) substitutes annual per capita US consumption into a quadratic utility function; Runkle (1991) substitutes household-level non-durable expenditures into the Constant Relative Risk Aversion (CRRA) power utility function; Townsend (1994) substitutes household-level "adult-equivalent" consumption into an exponential utility function; and Ogaki and Zhang (2001) use household-level measures of consumption expenditures into a power utility function, but with a translation to allow for the possibility that relative risk aversion might vary with wealth.

2.1. **Expenditure Data.** What data is collected to support the construction of a consumption aggregate? Expenditures (or consumptions) are better than income, because they're a better measure of *permanent* income or wealth than is realized income in a particular year.

Careful surveys of consumer expenditures are conducted occasionally in many countries, often with the aim of collecting the data necessary to calculate consumer price indices of some sort (which typically rely on estimates of the composition of consumption bundles). Such surveys are, however, in particularly widespread use in low income countries. Of particular note are the Living Standards Measurement Surveys (LSMS) first designed and introduced by researchers at the World Bank in 1979 (Deaton, 1997). These surveys typically feature quite comprehensive modules designed to collect data on expenditures of nondurable consumption and services. The World Bank has had great success in using expenditures over time using its LSMS family of surveys. The main complaints about these are simply that there aren't enough of them, and that too seldom do they form a panel. Both of these complaints presumably have a great deal to do with the associated costs; Lanjouw and Lanjouw (2001) report that the cost of fielding a single round of an LSMS survey ranges from \$300,000 to \$1,500,000, or about \$300 per household.

The LSMS expenditure modules typically collect data on expenditures on dozens or even hundreds of different goods and services. However, it's unusual for these disaggregate data to feature directly in any intertemporal analysis. Instead, the usual practice is to use these disaggregated data to construct a comprehensive expenditure aggregate (Deaton and Zaidi, 2002). This is intended to be a measure of all current consumption expenditures (including all goods and services).

There is nothing wrong with these consumption aggregates in principle: to the contrary, theory suggests that period-by-period total expenditures on non-durables and services are exactly the object that we ought to think intertemporally-maximizing households are making decisions about. However, in practice constructing such aggregates may be rather like making sausage. It's not that the issues, both practical and theoretical, haven't been carefully considered ({**?**, provide what amounts to an instruction manual)Deaton-Zaidi02}. The problem instead is simply that the demands of this exercise on the data are extreme. To indicate just a couple of the challenges: Even when the list of goods and services is comprehensive, it may be extremely difficult to back out the value of services from assets. The value of housing services is a particular problem, particularly since in many low income countries houses may be sold or exchanged very infrequently, but in general finding the right prices to go with different consumption items may be very challenging. This problem of measuring prices may be particularly acute when a good or service isn't acquired in a market; for example, inferring the value of home-produced goods may be a serious problem.

2.2. **From Direct to Indirect Utility.** For the moment, let us set aside the problem of constructing a consumption aggregate. In what world does it even make sense to model consumer preferences in this way? The assumption that momentary utility depends only on the quantity of total expenditures (perhaps adjusted for household size or composition) is, on its face, an odd one. Nobody really thinks consumers are just consuming a single numeraire good, denominated in some currency units. Instead, we should think of u as an indirect utility function.

Provided preferences time separable, we can think of $u : \mathbb{R}^n \to \mathbb{R}$, and of indirect utility:

$$v(x, p) = \max_{c \in \mathbb{R}^n} u(c) \qquad \text{such that } p'c \leq x.$$

But now a problem with using the indirect utility function emerges: one needs not only data on total expenditures, but also information about (all) prices. However, for many datasets (including most surveys in the LSMS family) data on prices is collected, sometimes both a the community and at the household level. Further, the same analysts who constructed the consumption aggregate are also likely to have constructed a price index, say $\pi(p)$, which we assume to be a continuously differentiable function of prices, and which is further assumed *not* to depend on an individual's expenditures x. Then to complete our justification for using a consumption aggregate, it's required that

$$v(x,p) \equiv v(x/\pi(p),1),$$

substituting a measure of *real* expenditures for nominal expenditures.

Now, when will using a simple price index like this be valid, so that $v(x,p) \equiv v(x/\pi(p),1)$ hold? And in particular, what restrictions does this place on the underlying direct utility function? Roy's identity tells that we can write the Marshallian demand for good i as

$$c_i(x,p) = \frac{\partial v/\partial p_i}{\partial v/\partial x} = \frac{v'(x/\pi)\frac{x\pi_i}{\pi}}{v'(x/\pi)\frac{1}{\pi}} = x\frac{\pi_i}{\pi},$$

where π_i is the partial derivative of the price index with respect to the price of the ith good. This tells us that demands are all linear in total expenditures x, and pass through the origin. And this is the case if and only if the utility function is homothetic.

2.3. **The Dead End.** The scenario we've described (homothetic, time-separable preferences) is the *only* scenario in which it is correct to use deflated expenditure aggregates in dynamic consumer analysis. If utility is in fact homothetic, then Engel curves must be linear, and the linear expenditure system is the correct way of describing consumer demand. But the first fact implies unitary expenditure (income) elasticities for all goods, and is thus at odds with Engel's Law, while the second flies in the face of decades of empirical rejections of the linear expenditure system.

Of course, though the assumptions an empirical researcher must make to use the usual deflated consumption aggregates in dynamic analysis seem implausible, unrealistic assumptions on their own needn't deter a dedicated economist (Friedman, 1953). And even if those assumptions seem to lead to predictions that are sharply at odds with one set of stylized empirical facts (e.g., Engel's Law), they may nevertheless allow the researcher to explain *other* empirical facts (Kydland and Prescott, 1996).

However, it's far from clear that homothetic utility and aggregating consumption is important for explaining *any* of the important facts. And for all the convenience they may offer the econometrician (only a single random variable needs to measured), the *construction* of realizations of that random variable is extremely difficult, expensive, and involves some intractable measurement problems. The household surveys and analysis necessary to collect comprehensive data on expenditures are very complicated and are hard to systematize across different environments. Heroic assumptions are typically required to value flows of services (Deaton and Zaidi, 2002), or deal with variation in quality (Deaton and Kozel, 2005). Additional heroics are required to construct price indices (Boskin et al., 1998).

Is there a better way? In this paper I'll argue that by starting with aggregated consumption we've taken a serious *wrong turn*, and that by simply backing up and making *disaggregated* data the center of our analytical focus we can make important progress without complicating our dynamic analysis.

3. A Frisch Approach

In the rest of this paper we'll describe an alternative approach to measuring marginal utilities which which is theoretically consistent; which uses Engel-style facts about the composition of differently-situated consumers' consumption bundles; which has comparatively modest data requirements; which allows us to simply *ignore* expenditures on goods and services which are too difficult or expensive to measure well; and which completely avoids the price index problem by simply avoiding the need to construct price indices. The approach imposes fewer restrictions on the demand system than is usual; avoids the usual sausage factory from which consumption aggregates are extruded; should allow for much less expensive data collection; and directly yields measures of both household marginal utility and functions of shadow prices which can be used in subsequent analysis and model testing.

What we're calling "marginal utility" has a very precise theoretical interpretation: it's the rate at which household utility would increase if the household received a small increase in its resources in a given period. Provided that the household has a concave momentary utility function then (the usual assumption) then marginal utility will decrease as resources increase. This same quantity goes by other names, but all of them awkward: the "marginal utility of income" (inaccurate, since a change in income will generally affect utility in several different periods); the "Lagrange multiplier on the budget constraint" (mathematically accurate, but devoid of intuition regarding the consequences for the household)"; the "marginal utility of expenditures" (perhaps the best of a bad lot, and a term that Browning (1986) abbreviates to *mue*—but this is confusing for us since we use the Greek letter λ for this quantity).[1] We settle for imprecision, and simply use the term "marginal utility"; where we mean something like the "marginal utility of rice" we'll be explicit about the good.

The question of how marginal utility is related to consumer demand and welfare was extensively considered by Ragnar Frisch (see esp. Frisch, 1959, 1964, 1978), and demand systems which depend on prices and marginal utility were apparently given the moniker "Frisch demands" by Martin Browning (Browning et al., 1985). However, previous approaches to estimating Frisch demand systems have generally imposed much more structure on the underlying consumer preferences than is necessary for estimating marginal utility.

Our plan is to follow the "sequential approach" advocated by (Blundell, 1998) to estimating and testing dynamic models. We take disaggregate data from one or more rounds of a household expenditure survey to estimate a Frischian demand system (demands which depend on prices and marginal utilities). Estimating such a system allows us to more or less directly recover estimates of some demand elasticities and households' marginal utilities in each round, which can then be used as an input to a subsequent (possibly dynamic) analysis.[2]

There is, of course, a vast literature on different approaches to estimating demand systems, so it's been surprising to discover that none of these approaches seems well-suited to our problem. The first issue is simply that almost all existing approaches are aimed at estimating Marshallian demands, rather than Frischian.[3] Related, demand systems which are nicely behaved (e.g., linear in parameters,) in a Marshallian setting are typically ill-behaved in a Frischian. This includes essentially all of the standard demand systems based on a dual approach (e.g., the AID system).

---

[1]The problem of naming this quantity has a long history; Irving Fisher was already complaining about it in 1917. Fisher himself offers the coinage "wantab" (Fisher, 1927).

[2]It would, of course, also be possible to estimate the demand system and the dynamic model jointly (as in, e.g., {?)Browning-etal85}. But since we so often are able to reject the dynamic models we estimate, joint estimation seems likely to result in a mis-specified system; here, we prefer to not impose any dynamic restrictions on the expenditure data so as to allow ourselves to remain comfortably agnostic about what the 'right' dynamic model ought to be.

[3]Notable exceptions include Browning et al. (1985); Kim (1993) and Blundell (1998)

Other existing demand systems can be straight-forwardly adapted to estimating Frischian demand systems, such as the Linear Expenditure System (LES), which can be derived from the primal consumer's problem when that consumer has e.g., Cobb-Douglas utility. But such systems are too restrictive, imposing a linearity in demand which is sharply at odds with observed demand behavior.

4. Model of Household Behavior

In this section we give a simple description of a Frischian function λ, which at the same time maps prices and resources into a welfare function (higher values mean that the household is in greater need), and which also serves as the central object for making predictions regarding *future* welfare.

4.1. **The household's one-period consumer problem.**

To fix concepts, suppose that in a particular period t a household faces a vector of prices for goods p_t and has budgeted a quantity of the numeraire good x_t to spend on contemporaneous consumption, from which it derives utility via an increasing, concave, continuously differentiable utility function U. Within that period, the household uses this budget to purchase non-durable consumption goods and services $c \in X \subseteq \mathbb{R}^n$, solving the classic consumer's problem

$$V(p_t, x_t) = \max_{\{c_i\}_{i=1}^n} U(c_1, \dots, c_n) \tag{2}$$

subject to a budget constraint

$$\sum_{i=1}^{n} p_{it} c_i \leq x_t. \tag{3}$$

The solution to this problem is characterized by a set of n first order conditions which take the form

$$U_i(c_1, \dots, c_n) = \lambda_t p_{it} \tag{4}$$

(where U_i denotes the ith partial derivative of the momentary utility function U), along with the budget constraint (3), with which the Karush-Kuhn-Tucker multiplier λ_t is associated.

So long as U is strictly increasing the solution to this problem delivers a set of demand functions, the Marshallian indirect utility function V, and a Frischian measure of the marginal value of additional resources to the household $\lambda_t = \lambda(p_t, x_t)$.

It is this last object which is of central interest for our purposes. By the envelope theorem, the quantity $\lambda_t = \partial V / \partial x_t$; it's thus positive but decreasing in x_t, so that marginal utility decreases as the total value of per-period expenditures increase.

4.2. **The household's intertemporal problem.** Of course, we're interested in the welfare of households in a stochastic, dynamic environment. But it turns out to be simple to relate the solution to the static one-period consumer's problem to a multi-period stochastic problem; at the same time we introduce a simple form of (linear) production.

We assume that households have time-separable von Neumann-Morgenstern preferences, and that households discount future utility using a common discount factor β. As above, within a period t, a household is assumed to assumed to allocate funds for total expenditures in that period obtaining a total momentary utility described by the Marshallian indirect utility function $V(p_t, x_t)$, where p_t are time t prices, and x_t are time t expenditures.

The household brings a portfolio of assets with total value $R_t b_t$ into the period, and realizes a stochastic income y_t. Given these, the household decides on investments b_{t+1} for the next period, leaving x_t for consumption expenditures during period t. More precisely, the household solves

$$\max_{\{b_{t+1+j}\}_{j=1}^{T-t}} \mathrm{E}_t \sum_{j=0}^{T-t} \beta^j V(p_{t+j}, x_{t+j})$$

subject to the intertemporal budget constraints

$$x_{t+j} = R_{t+j} b_{t+j} + y_{t+j} - b_{t+1+j}$$

and taking the initial b_t as given.

The solution to the household's problem of allocating expenditures across time will satisfy the Euler equation

$$\frac{\partial V}{\partial x}(p_t, x_t) = \beta^j \mathrm{E}_t R_{t+j} \frac{\partial V}{\partial x}(p_{t+j}, x_{t+j}).$$

But by definition, these partial derivatives of the indirect utility function are equal to the functions λ evaluated at the appropriate prices and expenditures, so that we have

$$\lambda(p_t, x_t) = \beta^j \mathrm{E}_t R_{t+j} \lambda(p_{t+j}, x_{t+j}). \tag{5}$$

This expression tells us, in effect, that the household's marginal utility or marginal utility of expenditures λ_t satisfies a sort of martingale restriction, so that the current value of λ_t play a central role in predicting *future* values λ_{t+j}.

When we estimate Frisch demands, we will typically also directly obtain estimates of the consumer's λ_t. And notice that once we have these estimated $\{\lambda_t\}$ in hand restrictions such as (5) are *linear* in these variables. This can simplify estimation, and perhaps also make dealing with measurement error a comparatively straight-forward procedure.

4.3. **Differentiable Demand Systems.** We now turn our attention to the practical problem of specifying a Frischian demand relation that can be estimated using the kinds of data we have available on disaggregated expenditures. Attfield and Browning (1985) take a so-called "differentiable demand" approach to a related problem; their method yields Frischian (aggregate) demands without requiring separability. These demands will, in general, depend on all prices, yet one need only estimate demand equations for a select set of goods.

Our analysis here follows that of Attfield and Browning (1985) in outline, but where they arrive at a Rotterdam-like demand system in quantities, we obtain something importantly different in expenditures.

It's easiest here to work with the consumer's profit function,

$$\pi(p, r) = \max_c rU(c) - pc,$$

where r has the interpretation of being the "price" of utility. Let subscripts to the π function denote partial derivatives. Some immediate properties of importance: the price r is equal to the quantity $1/\lambda$ from our earlier analysis; the profit function is linearly homogeneous in p and r; by the envelope theorem $\pi_i(p, r) = -c_i$ for all $i = 1, \ldots, n$; and (since we want to work with expenditures) $-p_i\pi_i = x_i$.

Using this last fact and taking the total derivative yields

$$dx_i = -\pi_i dp_i - p_i \sum_{j=1}^{n} \pi_{ij} dp_j - p_i \pi_{ir} dr.$$

Now, since $d \log x = dx/x$ for $x > 0$, this can be written as

$$x_i d \log x_i = -\pi_i p_i d \log p_i - p_i \sum_{j=1}^{n} \pi_{ij} p_j d \log p_j - p_i \pi_{ir} r d \log r.$$

Recalling that $-\pi_i p_i = x_i$

$$d \log x_i = d \log p_i + \sum_{j=1}^{n} \frac{\pi_{ij}}{\pi_i} p_j d \log p_j + \frac{\pi_{ir}}{\pi_i} r d \log r. \tag{6}$$

Now, let $\theta_{ij} = -\frac{\pi_{ij}}{\pi_i} p_j$ denote the (cross-) price elasticities of demand holding r constant (Frisch, 1959, called these "want elasticities"), and let $\beta_i = \frac{\pi_{ir}}{\pi_i} r$ denote the elasticity of demand with respect to r. Note in passing that this is exactly equal to minus the elasticity of demand with respect to λ, so we can rewrite this as

$$d\log x_i = d\log p_i - \sum_{j=1}^n \theta_{ij} d\log p_j - \beta_i d\log\lambda. \tag{7}$$

Using the linear homogeneity of the profit function, it follows that $\beta_i = \sum_{j=1}^n \theta_{ij}$. Also, provided only that the utility function is twice continuously differentiable, then by Young's theorem we know that $\theta_{ij} = \theta_{ji}$.

Equation (7) gives us an exact description of how expenditures will change in response to infinitesimal changes in prices. Now we make two further assumptions: first, that the elasticities $\{\theta_{ij}\}$ (and so $\{\beta_i\}$) are *constant*. Because the β_i are simply equal to the row sums of the matrixo of elasticities $\Theta = (\theta_{ij})$, in this case the Θ matrix summarizes all the pertinent information for understanding changes in demand; we call Θ the matrix of "Frisch elasticities."

Second, we assume that (given this constancy) (7) will also give us a good approximation of how demand changes with respect to larger changes in prices (see (Mountain, 1988) for a critical discussion of related issues in the Rotterdam demand system). Allowing also for household characteristics z_t to serve as demand shifters, we can then write a discrete-time version of (7)

$$\Delta\log x_{it} = \Delta\log p_{it} - \sum_{j=1}^n \theta_{ij}\Delta\log p_{jt} + \beta_i\delta_i^\intercal\Delta z_t - \beta_i\Delta\log\lambda_t + \Delta\xi_{it}, \tag{8}$$

where $\Delta\xi_{it}$ is an approximation error (For the case of the Rotterdam system Mountain (1988) argues that this approximation error must be of second or higher order; a similar argument seems likely to pertain here).

4.4. **From "Changes in" to "Levels of" Demand.** Setting aside the possibility of error when the matrix of parameters Θ is constant (8), we can integrate to obtain an exact expression for the *level* of expenditures and demand. In particular, let $\log\tilde\alpha_i(z)$ arise as a constant of integration, where we make explicit a possible dependence of this constant on household characteristics z. Then the Frischian demand for good i is given by

$$c_i = \tilde\alpha_i(z)\left[\lambda^{\beta_i}\prod_{j=1}^n p_j^{\theta_{ij}}\right]^{-1}. \tag{9}$$

One way of thinking about the richness of this demand system is to consider its *rank* (Lewbel, 1991). The marginal utility λ can be regarded as a function of total expenditures x and prices p. Then the budget constraint can be written in the form

$$\sum_{i=1}^{n} a_i(p)\lambda^{-\beta_i} = x,$$

with the function $\lambda(p,x)$ the solution to this equation. Using the same notation, expenditures for good i are $x_i(p,x) = a_i(p)\lambda(p,x)^{-\beta_i}$. Expressed in matrix form, the right hand side of this equation takes the form $\mathbf{a}(p)\mathbf{g}(p,x)$, with $\mathbf{g}(p,x)$ a diagonal matrix with rank equal to the number of distinct values of β_i. Thus, the rank of a demand system with n goods may be as great as n.

5. The Constant Frisch Elasticity (CFE) Utility Function

From the demand relation (9), we can easily obtain an expression for the marginal utility function, provided only that the matrix Θ has an inverse.

Lemma 1. *If the matrix $-\Theta$ has an inverse Γ and consumer demands are given by (9), then consumers' marginal utility of consumption of good i is*

$$U_i(c_1,\ldots,c_n) = \prod_{j=1}^{n} \left(\tilde{\alpha}_j(z)/c_j\right)^{\gamma_{ij}}, \tag{10}$$

where γ_{ij} is the (i,j) element of the inverse matrix Γ.

Proof. After integrating (7) and re-writing in terms of quantities rather than expenditures, we obtain in matrix form

$$\log c = \Theta \log p + \log\tilde{\alpha}(z) - \log\lambda\Theta\iota,$$

where ι is a column vector of n ones. Θ is invertible by assumption, so

$$\log p + \log\lambda = \Theta^{-1}\left[\log c - \log\alpha(z)\right].$$

From the first order conditions to the consumer's problem we know that $\log U_i(c_1,\ldots,c_n)$ is equal to the left-hand side of this equation; taking anti-logs then yields the result. □

Thus, *marginal* utility is homogeneous and has a Cobb-Douglas structure.

The same is not true of the utility function. Though (with some modest restrictions on Θ) the results of Hurwicz and Uzawa (1971) imply that this demand system can be integrated to obtain a well-behaved utility function, I have been able to obtain an explicit analytical expression for this utility function only for two special cases. The first of these is the "want independent" case of Frisch (1959).

Proposition 1. *If the matrix Θ is diagonal and its diagonal elements all negative, then the demand system (9) will be the demands for a consumer with a utility function*

$$U(c_1, \dots, c_n; z) = \sum_{i=1}^{n} \alpha_i(z) \frac{c_i^{1-\gamma_i} - 1}{1 - \gamma_i}, \tag{11}$$

where the parameters $\gamma_i \equiv \gamma_{ii} = -1/\theta_{ii}$, and where $\alpha_i(z) \equiv \tilde{\alpha}_i(z)^{\gamma_i}$.

Proof. When Θ is diagonal, we know from Lemma 1 that the marginal utilities of the different goods i will be equal to $\alpha_i(z)c_i^{-\gamma_i}$, which coincide with the partial derivatives of (11). $\square$

Restating this result, if the matrix of Frisch elasticities Θ is constant, negative definite, and diagonal ("want independence") then the consumer utility function takes the form (11). The form of this is similar to the constant elasticity of substitution utility function (e.g., Brown and Heien, 1972); the critical difference is that the curvature parameters γ_i are permitted to vary across different goods.

Thus, this is a richer parameterization of utility functions than is usual found in applied work, and is neither necessarily PIGLOG nor Gorman-aggregable. The parameters $\{\gamma_i\}$ govern the curvature of the n sub-utility functions associated with consumption of the various n goods. We assume that $\gamma_i \geq 0$, and in the usual way use the fact that $\lim_{\gamma_i \searrow 1} \frac{x_i^{1-\gamma_i}-1}{1-\gamma_i} = \log x_i$ to interpret values of $\gamma_i = 1$ as though the corresponding sub-utility function is logarithmic. The functions $\{\alpha_i(z)\}$ govern the weight of the n sub-utilities in total momentary utility.

5.1. **Want-Independent Constant Frisch Elasticity Demands.**

An important feature of these CFE preferences is that if different goods are associated with curvature parameters (i.e., there exists an (i, j) such that $\gamma_i \neq \gamma_j$) then the preferences are not Gorman-aggregable; indeed, while Marshallian demand functions exist for these preferences, except for some special cases these Marshallian demands won't have closed form solutions.

However, a more general approach to characterizing the demand system is available to us. Instead of deriving the Marshallian demands, we'll instead work with the Frisch demand system (Browning et al., 1985). Instead of expressing demands as a function of expenditure and prices, as in the Marshallian demand system, the Frisch system expresses demands as a function of prices and the marginal utility of expenditures. Though the result is standard, it's not as well known as it might be, and so we give it here in the form of the following proposition.

Proposition 2. *Let $V(p, x)$ denote the indirect utility function associated with the problem of choosing consumption goods so as to maximize (11) subject to a budget constraint $\sum_{i=1}^n p_i c_i \leq x$, and let $\lambda(p, x) = \frac{\partial V}{\partial x}(p, x)$. Then the Frisch demand for good i can be written as some function $c_i(p, \lambda)$, which is related to the Marshallian demand via the identity $c_i^m(p, x) \equiv c_i(p, \lambda(p, x))$.*

For the particular case at hand, Frisch demands are

$$c_i(p, \lambda) = \left(\frac{\alpha_i}{\lambda p_i}\right)^{1/\gamma_i} - \phi_i \tag{12}$$

for $i = 1, \ldots, n$, and the Frischian counterpart to the indirect utility function (which we'll unimaginatively call the 'Frischian indirect utility function') is given by

$$\mathcal{V}(p, \lambda) = \sum_{i=1}^n \frac{1}{1 - \gamma_i} \alpha_i^{1/\gamma_i} \left(\frac{1}{\lambda p_i}\right)^{1/\gamma_i - 1} - \sum_{i=1}^n \frac{\alpha_i}{1 - \gamma_i}, \tag{13}$$

or $\mathcal{V}(p, \lambda) = \lambda \sum_{i=1}^n \frac{p_i}{1-\gamma_i} \left(\frac{\alpha_i}{p_i \lambda}\right)^{1/\gamma_i} - \sum_{i=1}^n \frac{\alpha_i}{1-\gamma_i}$.

5.1.1. *Elasticities.*

(1) Demand Elasticities, Relative Risk Aversion, and Pigou's Law
Even when preferences are not separable (want-independent), the first order conditions from the standard consumer's problem imply that $u_i'(c_i) = p_i \lambda$ for $i = 1, \ldots, n$, and we have the identity

$$u_i'(c_i(p, \lambda)) \equiv p_i \lambda.$$

It follows that $u_i''(c_i) \frac{\partial c_i}{\partial p_i} = \lambda$, and that the elasticity of Frisch demand for good i with respect to p_i is equal to

$$\frac{u_i'(c_i)}{u_i''(c_i) c_i},$$

which can be interpreted as the reciprocal of either the elasticity of the marginal utility of the ith good, or as (minus) the Arrow-Pratt relative risk aversion of the consumer to variation in c_i.

A similar argument establishes that the elasticity of Frischian demands to changes in λ is *also* equal to (minus) Arrow-Pratt relative risk aversion, and thus equal to the price elasticity of Frisch demands.

In the context of Marshallian demands, Deaton (1974) argues that when the demand system is reasonably large then the price elasticity of demand will be approximately equal to its expenditure elasticity, and calls this approximation "Pigou's Law". Here we see that in the context of separable utility and Frisch demands that an *exact* version of Pigou's law holds, regardless of the number of commodities. Browning (2005) calls this an "exact version" of Pigou's law.

(2) Frischian Indirect Utility The response of utility to changes in λ provides some important information about the curvature of preferences—if the marginal utility of the consumer increases, how does utility change?

The derivative of the Frischian indirect utility in the separable case is

$$\frac{\partial \mathcal{V}}{\partial \lambda}(p,\lambda) = \sum_{i=1}^{n} u_i'(x_i)\frac{\partial x_i}{\partial \lambda},$$

or $\frac{\partial \mathcal{V}}{\partial \lambda}(p,\lambda) = \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial \lambda}$.

This expression has an interesting interpretation. Consider just the summation—this the decrease in expenditures associated with a small increase in marginal utility. Thus the effect of a small increase in marginal utility on total utility is approximately equal to marginal utility times the reduction in expenditures.

The elasticity, in turn, takes the simple form

$$\lambda^2 \frac{\sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial \lambda}}{\sum_{i=1}^{n} u_i(x_i)}.$$

This elasticity looks a bit peculiar: setting aside the λ^2 factor, the numerator is in the same units as expenditures, while the denominator is measured in utils. Thus the elasticity is proportional to the reduction in expenditures associated with increased marginal utility divided by total utility.

In the want-independent CFE case, the derivative of the Frischian indirect utility takes the particularly nice form

$$\frac{\partial \mathcal{V}}{\partial \lambda}(p,\lambda) = -\sum_{i=1}^{n} \frac{p_i}{\gamma_i}\left(\frac{\alpha_i}{p_i \lambda}\right)^{1/\gamma_i}.$$

The corresponding elasticity takes the less elegant form

$$-\lambda \frac{\sum_{i=1}^n \frac{p_i}{\gamma_i}\left(\frac{\alpha_i}{p_i\lambda}\right)^{1/\gamma_i}}{\sum_{i=1}^n \frac{p_i}{1-\gamma_i}\left(\frac{\alpha_i}{p_i\lambda}\right)^{1/\gamma_i} - \frac{1}{\lambda\sum_{i=1}^n \frac{\alpha_i}{1-\gamma_i}}}.$$

5.2. **Marshallian Demands.** To move from the Frischian system of demands to the Marshallian, note that total expenditures are equal to

$$\sum_{i=1}^n p_i\left[\left(\frac{\alpha_i}{\lambda p_i}\right)^{1/\gamma_i} - \phi_i\right] = x. \tag{14}$$

We can use this expression for total expenditures to construct an identity relating the usual (Marshallian) indirect utility function to its Frischian counterpart:

$$\mathcal{V}(p,\lambda) \equiv V\left(p, \sum_{i=1}^n p_i\left[\left(\frac{\alpha_i}{\lambda p_i}\right)^{1/\gamma_i} - \phi_i\right]\right). \tag{15}$$

We can also use (14) to solve for λ as a function of prices p, total expenditures x, and the preference parameters $(\alpha_i, \gamma_i, \phi_i)_{i=1}^n$. The form in which the ϕ parameters affect utility make it convenient to write the implicit function λ which solves (14) as $\lambda(p, x + p^\intercal\phi; \alpha, \gamma, \phi) = \lambda(p, x; \alpha, \gamma, 0)$.

5.2.1. *Properties of* $\lambda(p,x)$. The variable λ has an immediate interpretation as marginal utility, or the marginal (indirect) utility of income, of course. But what more can we say about the relationship between λ and total expenditures x?

The connection between these two quantities is determined by the budget constraint. Substituting Frischian demands into that constraint gives us

$$\sum_{i=1}^n p_i\left(\frac{\alpha_i}{p_i}\right)^{1/\gamma_i}\lambda^{-1/\gamma_i} = x + \sum_{i=1}^n p_i\phi_i.$$

5.2.2. *Budget Shares.* Let β_i denote the expenditure share of good i. For these preferences, these shares take the form

$$\beta_i = \frac{p_i^{1-1/\gamma_i}(\alpha_i/\lambda)^{1/\gamma_i} - p_i\phi_i}{\sum_{j=1}^n\left(p_j^{1-1/\gamma_j}(\alpha_j/\lambda)^{1/\gamma_j} - p_j\phi_j\right)}. \tag{16}$$

Note that (unlike the usual CES case) expenditure shares depend not only on the parameters $\{\alpha_i\}$, but also on the curvature parameters $\{\gamma_i\}$

(though if these are all equal these curvature parameters all cancel out) and on the "marginal utility" parameter λ.

Proposition 3. *The expenditure share of good i is an increasing function of total expenditures x if and only if*

$$p_i\phi_i \sum_{j=1}^{n} \left(\frac{\gamma_i}{\gamma_j}\right)\left(\frac{p_j(\alpha_j/p_j)^{1/\gamma_j}}{p_i(\alpha_i/p_i)^{1/\gamma_i}}\right)\mu^{1/\gamma_j-1/\gamma_i} - \sum_{j=1}^{n} p_j\phi_j > \sum_{j=1}^{n}\left(\frac{\gamma_i}{\gamma_j}-1\right)p_j\left(\frac{\alpha_j}{p_j}\right)^{1/\gamma_j}\mu^{1/\gamma_j}.$$

Proof. To conserve ink, let $b_i = p_i\left(\frac{\alpha_i}{p_i}\right)^{1/\gamma_i}$, and let $\psi = \sum_{j=1}^{n} p_j\phi_j$ denote the expenditures necessary to satisfy all 'subsistence' requirements. Then we can rewrite (16) as

$$\beta_i = \frac{b_i\mu^{1/\gamma_i} - p_i\phi_i}{\sum_{j=1}^{n} b_j\mu^{1/\gamma_j} - \psi}$$

Now, μ is a strictly increasing, differentiable function of x, say $\mu(x)$; thus the sign of partial derivative of β_i with respect to x will be equal to the sign of the partial derivative with respect to $\mu(x)$. This latter derivative is

$$\frac{\partial\beta_i}{\partial\mu} = \frac{(1/\gamma_i)b_i\mu^{1/\gamma_i-1}}{\sum_{j=1}^{n} b_j\mu^{1/\gamma_j} - \psi} - \frac{\left(b_i\mu^{1/\gamma_i} - p_i\phi_i\right)\sum_{j=1}^{n} b_j\mu^{1/\gamma_j-1}}{\left[\sum_{j=1}^{n} b_j\mu^{1/\gamma_j} - \psi\right]^2}.$$

The sign of this expression will be positive if and only if

$$b_i\mu^{1/\gamma_i}\left[\sum_{j=1}^{n} b_j\mu^{1/\gamma_j} - \psi\right] > \left(b_i\mu^{1/\gamma_i} - p_i\phi_i\right)\sum_{j=1}^{n}\frac{\gamma_i}{\gamma_j}b_j\mu^{1/\gamma_j}.$$

Rearranging this inequality yields the result. □

If $\gamma_i = \gamma_j$ for all i, j, then whether or not the budget share is increasing or decreasing turns out to depend *only* on the total subsistence cost (which may be positive or negative); in this case, *all* budget shares must be either increasing or decreasing together with total expenditures.

The most interesting case occurs when γ_i is large relative to other curvature parameters—it's in this case that the budget share of good i will eventually fall with total expenditures. However, when the total subsistence cost is large and positive, then even if γ_i is large then budget shares may be increasing at low levels of expenditure.

5.2.3. *Expenditure elasticities.* As can be seen from above, objects such as budget shares may be fairly complicated objects in the VES system, and so may be income elasticities. However, the elasticities of demand

and of Frischian indirect utility with respect to λ are relatively simple to express, as is the elasticity of λ with respect to income.

(1) Expenditure elasticity of λ

Proposition 4. *The income elasticity of $\lambda(p, x)$ is equal to*
$$-\frac{x}{\sum_{i=1}^n \frac{x_i}{\gamma_i}}.$$

Thus, by the implicit function theorem $\lambda(p, x)$ is the solution to an equation of the form
$$\sum_{i=1}^n a_i(p)\lambda^{b_i} = x.$$
This resembles an ordinary polynomial, except that the exponents b_i are all negative real numbers. Because the coefficients $a_i(p)$ are all positive, it follows $\lambda(p, x)$ is montonically decreasing in x, and because $\partial a_i/\partial p_i > 0$, that $\lambda(p, x)$ is monotonically increasing in every price p_i.

Proof. From (14) we have
$$x = \sum_{i=1}^n x_i(p, \lambda) = \sum_{i=1}^n a_i(p)\lambda^{1/\gamma_i} - \sum_{i=1}^n p_i\phi_i,$$
so that
$$\frac{\partial x}{\partial \lambda} = -\sum_{i=1}^n p_i \frac{a_i(p)}{\gamma_i}\lambda^{-(1+1/\gamma_i)}.$$
Then by the inverse function theorem
$$\frac{\partial \lambda}{\partial x} = -\left[\sum_{i=1}^n p_i \frac{a_i(p)}{\gamma_i}\lambda^{-(1+1/\gamma_i)}\right]^{-1}.$$
Substituting this into the usual formula for an elasticity then gives the result. □

(2) Expenditure elasticity of demand

Let η_i denote the expenditure elasticity of demand for good i.

Proposition 5. *When consumer preferences are given by (11) and quantities demanded by that consumer are given by $\{c_i\}$, then the elasticity of demand for good i can be expressed as*
$$\eta_i = \frac{c_i + \phi_i}{\gamma_i c_i} \frac{\sum_{j=1}^n p_j(c_j - \phi_j)}{\sum_{i=1}^n \frac{p_j}{\gamma_j} c_j}.$$

Proof. Let $a_i(p) = (\alpha_i/p_i)^{1/\gamma_i}$. Observe that

$$\frac{\partial c_i}{\partial x} = \frac{a_i(p)}{\gamma_i}\lambda^{-(1+1/\gamma_i)}\frac{\partial \lambda}{\partial x}.$$

Then using the result of Proposition 4 along with the usual formula for an elasticity yields the result. □

A note in passing: as we'll see in the next section, sometimes we may only be able to estimate the parameters γ_i up to a common factor of proportion, obtaining $\gamma_i\xi$ with ξ unknown. Fortunately, for our calculation of income elasticities this doesn't matter, as the factor ξ will cancel out of the fraction which defines the income elasticity of demand for good i.

(a) Form of Demands and Reconciliation with Earlier Literature

Browning (2005) was first distributed as a working paper in 1985, and seems to have been an important influence on many subsequent papers working with Frisch demand structures. Among other things, he proposes a particular form for the Frisch expenditure function, given by

$$\log x = (1+\beta_0)\log b(p) - \beta_0 \log \lambda.$$

A centerpiece of the paper is Browning's Proposition 2, in which he shows that if the consumer's intertemporal elasticity of substitution is a constant, then Frisch expenditure functions must take a particular form; the intratemporal consumer's utility function must take the PIGL form; and the converse (so that all three conditions are equivalent). We've seen by construction (Proposition 1) that even in the simple want-independent case the preferences we're working with are not generally PIGL, so we can infer in our case both that the intertemporal elasticity of substitution isn't generally constant and that the expenditure function doesn't take the form assumed by Browning.

It may be useful to identify the points at which Browning's argument fails for our case. The Frisch demands (9) we've worked with above can be written in the form

$$\log c_i = \log \psi_i(p) - \theta_{ii}\log p_i - \beta_i \log \lambda,$$

where $\psi_i(p) = \log\tilde{\alpha}_i - \sum_{i\neq j}\theta^{ij}\log p_j$. Note that in the want independent case the diagonal element θ_{ii} will be equal to $-\beta_i$, but otherwise will not be.

Browning (p. 313) shows that when preferences are *not* want independent, then for a very similar form

$$\log c_i = \log \tilde{\psi}_i(p) + \beta_i \log p_i - \beta_i \log \lambda \tag{17}$$

the corresponding preferences must be homothetic and expenditure elasticities must be equal. There are two things to note about this claim. First is that the argument he presents does not apply to the separable want-independent case, since the symmetry condition (6.3) he exploits is trivially satisfied in this case. Second is that in the more general non-separable case the restriction that our $\theta_{ii} = -\beta_i$ will hold if and only if utility from good i *is* additively separable from other goods, since by symmetry we have $\theta_{ij} = \theta_{ji}$ and by the homogeneity of the demand function we have $\sum_{j=1}^{n} \theta_{ij} = -\beta_i$. Browning's form (17) can only arise for all goods i if we have want-independence, and in this case his form does not imply homothetic preferences.
For want-independent preferences to give rise to the expenditure function proposed by Browning, it's necessary and sufficient for the matrix $\Theta = \beta_0 I$; that is, for Θ to be diagonal, and for each diagonal element to be equal to β_0; this case obviously does imply homothetic preferences and unitary expenditure elasticities.
Two later papers, Browning (1986) and Browning et al. (1985), work with a different set of expenditure functions, in which either levels of expenditures or levels quantities are additive in some function of λ which is independent of prices. This case also implies strong restrictions on consumer preferences; namely that they're quasi-homothetic. We simply note here that this is *not* our case. For us, *logarithms* of quantities are additive in a function of λ, not levels.

(3) Indirect Utility

A key question related to the evaluation of welfare has to do with the increase in utility when an wealth increases. To this end we calculate the income (or wealth) elasticity of indirect utility, using (15). In particular, we have

Proposition 6. *When consumer preferences are given by (11) and quantities demanded by that consumer are given by* $\{c_i\}$*,* *then the elasticity of indirect utility can be expressed as*

$$\frac{\partial \log V(p,x)}{\partial \log x} = \frac{x}{\sum_{i=1}^n \frac{x_i}{1-\gamma_i} - \frac{1}{\lambda(p,x)}\sum_{i=1}^n \frac{1}{1-\gamma_i}}.$$

Proof. By the envelope condition associated with the consumer's primal problem, $\partial V/\partial x = \lambda(p,x)$, so the elasticity we're interested is equal to $x\lambda/V(p,x)$. Substituting from (13) then gives the result. □

6. Estimation Using a Panel

Suppose we have data on expenditures at two or more different periods (but lack data on prices). We want to use these data to estimate the parameters of (8). However, that equation describes only the demand system for a single household. Adapting it, let $j = 1, \ldots, N$ index different households, and assume that household characteristics for the jth household at time t include both observable characteristics z_t^j and time-varying unobservable characteristics ϵ_{it}^j. Then we can write our structural estimating equation as

(18)
$$\Delta \log x_{it}^j = \left(\Delta \log p_{it} - \sum_{k=1}^n \theta_{ij}\Delta \log p_{kt}\right) + \beta_i\delta_i^\intercal \Delta z_t^j - \beta_i \Delta \log \lambda_t^j + \Delta\xi_{it}^j + \beta_i\Delta\epsilon_{it}^j.$$

We assume that prices are unknown to the econometrician, but that all households face the same prices. Expressed as a reduced form, we write

$$y_{it}^j = a_{it} + b_i^\intercal(\Delta z_t^j - \overline{\Delta z}_t) + c_i w_t^j + e_{it}^j, \tag{19}$$

where

$$\begin{aligned}
y_{it}^j &= \Delta \log x_{it}^j \\
a_{it} &= \left[\Delta \log p_{it} - \sum_{k=1}^n \theta_{ij}\Delta \log p_{kt}\right] - \beta_i\overline{\Delta \log \lambda}_t + \beta_i\overline{\Delta\epsilon}_{it} + \overline{\Delta\xi}_{it} \\
b_i &= \beta_i\delta_i \\
e_{it}^j &= \beta_i(\Delta\epsilon_{it}^j - \overline{\Delta\epsilon}_{it}) + (\Delta\xi_{it}^j - \overline{\Delta\xi}_{it}) \\
c_i w_t^j &= -\beta_i(\Delta \log \lambda_t^j - \overline{\Delta \log \lambda}_t).
\end{aligned}$$

We obtain the reduced form parameters (a_{it}, b_i) simply by using least squares to estimate (19), treating the a_{it} as a set of good-time effects.

6.1. **Identification of the Parameters of Interest.** What parameters and unobserved quantities can we identify? There are three different groups of objects which are likely to be of interest: the $n(n+1)/2$ parameters of the demand system Θ; the marginal utilities $\{\lambda_t^j\}$; and the nT prices $\{p_{it}\}$. We consider each group in turn.

6.1.1. *Marginal Utilities.* The residuals are then equal to $c_i w_t^j + e_{it}^j$. The first term of this sum is what we're interested in. Arrange the residuals as an $n \times NT$ matrix $\mathbf{Y}$. The first term in the equation captures the role that variation in marginal utility λ plays in explaining variation in expenditures. Because it's computed as the outer product of two vectors, this first term is at most of rank one. The second term captures the further role that changes in household observables play in changes in demand; if there are ℓ such observables, then this second term is of at most rank $\bar{\ell} = \min(\ell, n-1)$.

We proceed by considering the singular value decomposition of $\mathbf{Y} = \mathbf{U}\Sigma\mathbf{V}^\intercal$, where $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices, and where Σ is a diagonal matrix of the singular values of $\mathbf{Y}$, ordered from the largest to the smallest. Then the rank one matrix that depends on λ is $\mathbf{g}\mathbf{w}^\intercal = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^\intercal$, while the second matrix (of at most rank $\bar{\ell}$) is $\mathbf{d}\mathbf{Z}^\intercal = \sum_{k=2}^{\bar{\ell}} \sigma_k \mathbf{u}_k \mathbf{v}_k^\intercal$, where σ_k denotes the kth singular value of $\mathbf{Y}$, and where the subscripts on $\mathbf{u}$ and $\mathbf{v}$ indicate the column of the corresponding matrices $\mathbf{U}$ and $\mathbf{V}$. The sum of these matrices is the optimal $1+\bar{\ell}$ rank approximation to $\mathbf{Y}$, in the sense that by the Eckart-Young theorem this is the solution to the problem of minimizing the Frobenius distance between $\mathbf{Y}$ and the approximation; accordingly, this is also the least-squares solution (Golub and Reinsch, 1970).

The singular value decomposition thus identifies the structural parameters β_i and changes in log marginal utility up to a factor ϕ, so that we obtain estimates of $\phi\beta_i$ and of $(\Delta \log \lambda_t^j - \overline{\Delta \log \lambda_t})/\phi$. We adopt a normalization which chooses ϕ so that it's equal to the the reciprocal of the mean of the estimated β_i across goods in the estimated demand system.

6.1.2. *Additional Parameters of the Demand System.* To completely identify the demand system, we'd like to estimate the $n(n+1)/2$ Frisch elasticities Θ_{ij} (which in turn pin down the n elasticities β_i); and the $n\ell$ effects of household observables on demands δ_i.

In the estimation procedure described above, we first obtain a set of estimated good-time effects $\{a_{it}\}$; this is the only place in which the full set of coefficients in Θ appear, so the question of identifying the Frisch matrix of elasticities is the question of being able to compute Θ

from the matrix equation

$$\mathbf{A} = (\boldsymbol{\Delta} \log \mathbf{p})(\mathbf{I} - \Theta) + \mathbf{E}$$

where $\mathbf{A}$ and $\Delta \log p$ are $n \times T$ matrices, and where $\mathbf{E} = \beta_i \overline{\Delta \log \lambda}_t + \beta_i \overline{\Delta \epsilon}_{it} + \overline{\Delta \xi}_{it}$ is a matrix of residuals averaged over households.

We've already established that the elasticities β_i can be separately identified up to an unknown scalar. This imposes n adding-up restrictions on the matrix Θ, leaving $n(n-1)/2$ degrees of freedom. If we do not observe prices, little more can be said about Θ. If we observe n prices in each period, then we must have $T \geq (n-1)/2$ if we are to estimate Θ (Larson, 1966, using, e.g., the methods of).

In the application of this paper this requirement is *not* satisfied, and so we cannot estimate without additional restrictions. This means that we will not be in a position to discuss the intra-temporal substitution elasticities between different goods.

6.1.3. *Prices.* If we have independent information on the matrix Θ, it may be possible to draw inferences regarding changes in prices, using the same relationship between the latent good-time effects A and prices described above. If $\mathbf{I} - \Theta$ is invertible, then we have

$$\boldsymbol{\Delta} \log \mathbf{p} = (\mathbf{A} - \mathbf{E})(\mathbf{I} - \Theta)^{-1}.$$

If we know less about Θ but are willing to assume some additional structure we may also be able to make progress. For example, if Θ is known (or assumed) to be diagonal, then we can use estimates of the β_i to construct an estimated $\hat{\Theta}$, and then proceed as above.

7. Selecting Particular Goods

One of the attractive features of Frisch demand systems is that it's very simple to estimate *incomplete* demand systems, featuring just a few goods, something which isn't easy or straightforward in a Marshallian demand context unless one invokes some very strong assumptions regarding separability.

The upshot of this is that if we're interested in measuring marginal utility, we don't have to know about expenditures for all goods; instead, we can simply choose a *few* goods. The cost of using a smaller set of goods is simply that one ignores a possible additional useful source of information on marginal utility. However, this cost may be quite small for some goods. First, if the elasticity parameter β_i is small, then variation in expenditures simply isn't closely related to variation in marginal utility. Second, the demand equations we estimate may be a poor fit to the expenditure data, whether because of measurement error, a high importance of unobservable household characteristics in determining demand for a particular good, large approximation error, or simply because the specification of demand is particularly poor for some goods. In any of these cases the fit of the estimated demand system will tend to be poor, and variation in expenditures may depend more on the error term in the regression than on variation in marginal utility.

The preceding considerations suggest two simple criteria for determining what goods to use. The first merely involves examining the magnitude of the estimated elasticities β_i, and preferring goods with larger (absolute) elasticities. The second involves simply considering a measure of equation fit, such as the simple R^2 statistic.

8. Conclusion

In this paper we've outlined some of the key methodological ingredients needed in a recipe to measure what we've termed a household's marginal utility of contemporaneous expenditures in a manner which is theoretically coherent, which lends itself to straightforward statistical inference and hypothesis testing, and which is very parsimonious in its data requirements.

Our goal is to devise procedures to easily *measure* and *monitor* households' marginal utility over both different environments and across time. To this end, we've described an approach which involves estimating an incomplete demand system of a new sort which features a highly flexible relationship between total expenditures and demand. The method worked out in this paper involves using a panel of data on household expenditures on different goods and/or services, and requires at least two periods of data. It seems likely that related methods could be developed to permit the analysis of repeated cross-sections of data, but this is left for future research.

In an application developed in an appendix to this paper we illustrate the use of our methods using two rounds of data from Uganda. We focus on food expenditures in this dataset; initially estimating a system of 29 demands, we find that a much smaller system of goods suffices to estimate household marginal utility. In particular, a scheme of monitoring marginal utility by carefully tracking expenditures on just 21 food items would do just as well as the current collection of 61 food items, and could do much better if care was taken to reduce the proportion of households reporting zero expenditures for these items. Of particular importance are just a handful of food goods with high elasticities of marginal utility: fresh cassava, fresh sweet potatoes, bread, rice, fresh milk, tea, beef, sugar, cooking oil, onions, and tomatoes.

The presence of zeros in households' reported expenditures is the chief difficulty in our empirical exercise. These zeros cause problems for two reasons. First, the construction of the differential demand system (7) assumes that expenditures are positive, and simply isn't valid for settings in which expenditures are sometimes zero. Addressing this problem in the theory may be possible, by imposing non-negativity constraints on the demands for consumers. How this will change the estimating equation is unknown. Second, some of the recorded zeros may be the result of a particular kind of measurement error, perhaps related to the fatigue or inattention of enumerators or respondents. If it's possible to change data collection practices (perhaps by focusing on fewer goods) so as reduce this possible source of error, then perhaps it would be possible to avoid the theoretical difficulties posed by zero expenditures simply by focusing on goods for which few households report zeros.

All this said, while the existence of a large numbers of reported zeros clearly leads to problems of bias in our estimated elasticities, in practice our estimates of changes in households' marginal utility seems not to be very sensitive to zeros. After fairly extensive experimentation with including or excluding different goods from the estimation, we find that estimated changes in household marginal utility are quite highly correlated across these different experiments. Indeed, it seems very likely that estimates of changes in marginal utility will be much less sensitive to this sort of reduction in the set of expenditures being considered than changes in poverty (based on the construction of corresponding expenditure 'sub-aggregates') would be.

9. References

Angelucci, M. and G. D. Giorgi (2009, March). Indirect effects of an aid program: How do cash transfers affect non-eligibles' consumption. *The American Economic Review 99*(1), 486–508.

Attfield, C. L. and M. J. Browning (1985). A differential demand system, rational expectations and the life cycle hypothesis. *Econometrica 53*(1), 31–48.

Bewley, T. (1977). The permanent income hypothesis: A theoretical formulation. *Journal of Economic Theory 16*, 252–92.

Blundell, R. (1998). Consumer demand and intertemporal allocations: Engel, Slutsky, and Frisch. *Econometric Society Monographs 31*, 147–168.

Boskin, M. J., E. R. Dulberger, R. J. Gordon, Z. Griliches, and D. W. Jorgenson (1998). Consumer prices, the consumer price index, and the cost of living. *Journal of Economic Perspectives 12*(1), 3–26.

Breeden, D. T. (1979). An intertemporal asset pricing model with stochastic consumption and investment opportunities. *Journal of Financial Economics 7*(3), 265–296.

Brown, M. and D. Heien (1972). The *s*-branch utility tree: A generalization of the linear expenditure system. *Econometrica 40*(4), 737–747.

Browning, M. (1986). The costs of using Frisch demand functions that are additive in the marginal utility of expenditure. *Economics Letters 21*(3), 205–207.

Browning, M. (2005). A working paper from April 1985: Which demand elasticities doe we know and which do we need to know for policy analysis? *Research in Economics 59*, 293–320.

Browning, M., A. Deaton, and M. Irish (1985). A profitable approach to labor supply and commodity demands over the life cycle. *Econometrica 53*, 503–543.

Cochrane, J. H. (1991). A simple test of consumption insurance. *Journal of Political Economy 99*, 957–976.

Deaton, A. (1974). A reconsideration of the empirical implications of additive preferences. *The Economic Journal 84*(334), 338–348.

Deaton, A. (1997). *The Analysis of Household Surveys*. Baltimore: The John Hopkins University Press.

Deaton, A. and V. Kozel (2005). Data and dogma: The great Indian poverty debate. *Oxford University Press 20*, 177–199.

Deaton, A. and J. Muellbauer (1980, June). An almost ideal demand system. *The American Economic Review 70*(3), 312–326.

Deaton, A. and C. Paxson (1996). Economies of scale, household size, and the demand for food. manuscript.

Deaton, A. and S. Zaidi (2002). *Guidelines for Constructing Consumption Aggregates for Welfare Analysis*. World Bank Publications.

Engel, E. (1857). Die productions-und consumtionsverhaltnisse des königreichs sachsen. *Zeitschrift des Statistischen Biireaus des Koniglich Sachsischen Ministeriums des Innern* (8 & 9), 1–54.

Fisher, I. (1918). Is "utility" the most suitable term for the concept it is used to denote? *American Economic Review 8*(2), 335–337.

Fisher, I. (1927). A statistical method for measuring "marginal utility" and testing the justice of a progressive income tax. In *Economic*

Essays in Honor of John Bates Clark, pp. 157–193. Macmillan Co.

Friedman, M. (1953). The methodology of positive economics. In *Essays on Positive Economics*, pp. 3–43. Chicago: University of Chicago Press.

Frisch, R. (1959, April). A complete scheme for computing all direct and cross demand elasticities in a model with many sectors. *Econometrica 27*(2), 177–196.

Frisch, R. (1964). Dynamic utility. *Econometrica*, 418–424.

Frisch, R. (1978). *New methods of measuring marginal utility*. Porcupine Press.

Golub, G. H. and C. Reinsch (1970). Singular value decomposition and least squares solutions. *Numerische Mathematik 14*(5), 403–420.

Hall, R. E. (1978). Stochastic implications of the life cycle–permanent income hypothesis: Theory and evidence. *Journal of Political Economy 86*, 971–987.

Hansen, L. P. and K. J. Singleton (1982). Generalized instrumental variables estimation of nonlinear rational expectations models. *Econometrica 50*, 1269–1286.

Houthakker, H. S. (1957). An international comparison of household expenditure patterns, commemorating the centenary of Engel's law. *Econometrica 25*(4), 532–551.

Hurwicz, L. and H. Uzawa (1971). On the integrability of demand functions. In J. Chipman and H. Sonnenschein (Eds.), *Preferences, utility and demand*, pp. 114–148. NY: Harcourt, Brace Jovanovich.

Kim, H. Y. (1993). Frisch demand functions and intertemporal substitution in consumption. *Journal of Money, Credit and Banking 25*(3), pp. 445–454.

Kydland, F. E. and E. C. Prescott (1996). The computational experiment: An econometric tool. *Journal of Economic Perspectives 10*(1), 69–85.

Lanjouw, J. O. and P. Lanjouw (2001). How to compare apples and oranges: poverty measurement based on different definitions of consumption. *Review of Income and Wealth 47*(1), 25–42.

Larson, H. J. (1966). Least squares estimation of the components of a symmetric matrix. *Technometrics 8*(2), 360–362.

Lewbel, A. (1991). The rank of demand systems: theory and nonparametric estimation. *Econometrica: Journal of the Econometric Society 59*(3), 711–730.

Ligon, E. (1998). Risk-sharing and information in village economies. *Review of Economic Studies 65*, 847–864.

Ligon, E., J. P. Thomas, and T. Worrall (2002). Informal insurance arrangements with limited commitment: Theory and evidence from village economies. *The Review of Economic Studies 69*(1), 209–244.

Lucas, Jr., R. E. (1978). Asset prices in an exchange economy. *Econometrica 46*, 1429–1446.

Mace, B. J. (1991). Full insurance in the presence of aggregate uncertainty. *Journal of Political Economy 99*, 928–956.

Mountain, D. (1988). The Rotterdam model: an approximation in variable space. *Econometrica 56*(2), 477–484.

Ogaki, M. and A. Atkeson (1997). Rate of time preference, intertemporal elasticity of substitution, and level of wealth. *Review of Economics and Statistics 79*(4), 564–572.

Ogaki, M. and Q. Zhang (2001). Decreasing relative risk aversion and tests of risk sharing. *Econometrica 69*(2), 515–526.

Runkle, D. E. (1991). Liquidity constraints and the permanent-income hypothesis: Evidence from panel data. *Journal of Monetary Economics 27*, 73–98.

Townsend, R. M. (1994, May). Risk and insurance in village India. *Econometrica 62*(3), 539–591.

Appendix A. Application in Uganda

To illustrate some of the methods and issues discussed above, we use data from two rounds of surveys conducted in Uganda (in 2005–06 and 2009–10).[4] We first give a descriptive account of some of the data on expenditures from these surveys, and then supply data on estimates of household marginal utility and the marginal utility elasticities (β_i). We finally supply some discussion of the effects of using only selected goods in our estimation procedure.

A.1. **Data on Expenditures & Results from Estimation.** Excluding durables, taxes, fees & transfers, there are 99 categories of expenditure in the data, of which 61 are different food items or categories, and 38 are nondurables or services. We consider some different collections of goods, first describing all the goods, then turning our attention to a group of what we call "slightly aggregated" foods.

A.1.1. *All Expenditures.* Figure 1 paints a picture of aggregate expenditure shares across these categories, listing mean and aggregate expenditure shares for all goods which had an aggregate expenditure share greater than one percent in 2005. A glance reveals that shares of aggregates is fairly stable across the two rounds, with an increase in the

[4]Data on expenditures was provided by Thomas Pavesohnesen, and on income and household characteristics by Jonathan Kaminski. My thanks to them both for sharing their work.

share of housing from 8% to 10% the most notable change. It should be noted, however, that stability of shares over time is not a prediction of theory—changes in incomes or relative prices could easily be responsible for changes in shares.

Table 1. Shares of Aggregate Expenditures in Uganda (2005 and 2009), for all goods with aggregate shares greater than 1% in 2005.

Expenditure Item	Agg. 2005	Shares 2009	Mean 2005	Shares 2009
Imputed rent of owned house	0.078	0.100	0.060	0.067
Matoke (Bunch)	0.050	0.054	0.044	0.050
Sweet potatoes (Fresh)	0.042	0.045	0.051	0.056
Maize (flour)	0.040	0.038	0.052	0.049
Medicines etc	0.038	0.041	0.034	0.038
Water	0.033	0.030	0.044	0.033
Food (restaurant)	0.033	0.033	0.030	0.031
Beef	0.033	0.033	0.027	0.028
Sugar	0.031	0.029	0.030	0.029
Beans (dry)	0.031	0.033	0.040	0.044
Taxi fares	0.027	0.023	0.020	0.016
Firewood	0.027	0.030	0.040	0.042
Hospital/ clinic charges	0.026	0.020	0.021	0.018
Cassava (Fresh)	0.024	0.026	0.029	0.034
Fresh Milk	0.022	0.024	0.018	0.021
Air time & services fee for owned fixed/mobile phones	0.022	0.035	0.011	0.023
Cassava (Dry/ Flour)	0.020	0.024	0.028	0.032
Rent of rented house	0.019	0.017	0.017	0.015
Rice	0.015	0.015	0.012	0.013
Fresh Fish	0.013	0.012	0.012	0.012
Paraffin (Kerosene)	0.013	0.011	0.016	0.015
Cooking oil	0.012	0.011	0.015	0.013
Washing soap	0.012	0.012	0.016	0.015
Charcoal	0.012	0.014	0.010	0.011
Barber and Beauty Shops	0.011	0.011	0.008	0.008
Tomatoes	0.011	0.010	0.012	0.011
Petrol, diesel etc	0.011	0.015	0.003	0.006

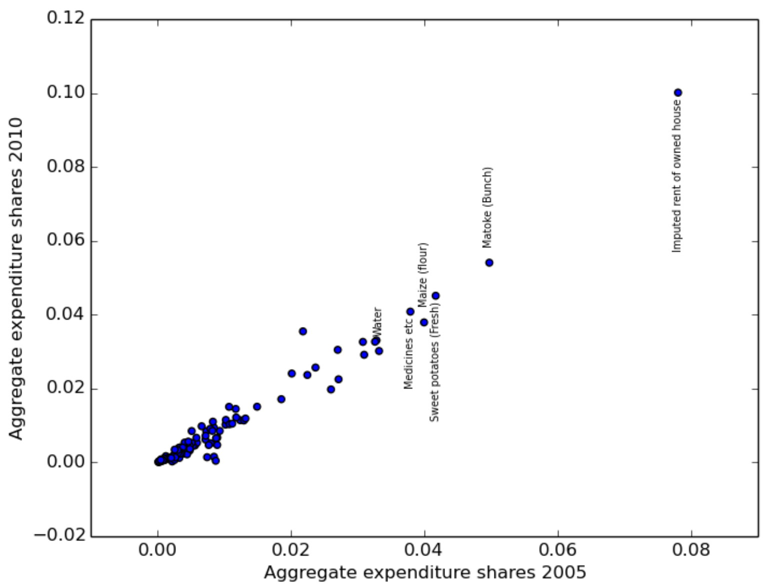

FIGURE 1. Aggregate expenditure shares in 2005 and 2009, using all expenditure items.

Table 1 describes the share of aggregate expenditures on different consumption items in each of the two rounds of the survey, for goods with an expenditures share greater than 1% in 2005. The second pair columns instead provides the average expenditure shares; that is, averaging shares across households, instead of summing across households and then computing shares. These two different measures of shares give an interesting indication of what shares are more important for wealthy or poor households, since wealthy households are over-represented in the calculation of aggregate shares relative to poor households. This general point is made perhaps more effectively by Figure 2. In this figure goods are ordered by the log of the *ratio* of the mean share to the aggregate share in 2005. Accordingly, this statistic for goods with an outsized share of wealthy households' total expenditures take values less than zero, while goods that take a larger share in poor households total expenditures take values greater than one. Some of the goods at extremes are labeled: it appears that wealthier households tend to spend proportionally more on servants and motor fuels, for example, while poorer households spend proportionally more on foods (consistent with Engel's law) such as sorghum and maize.

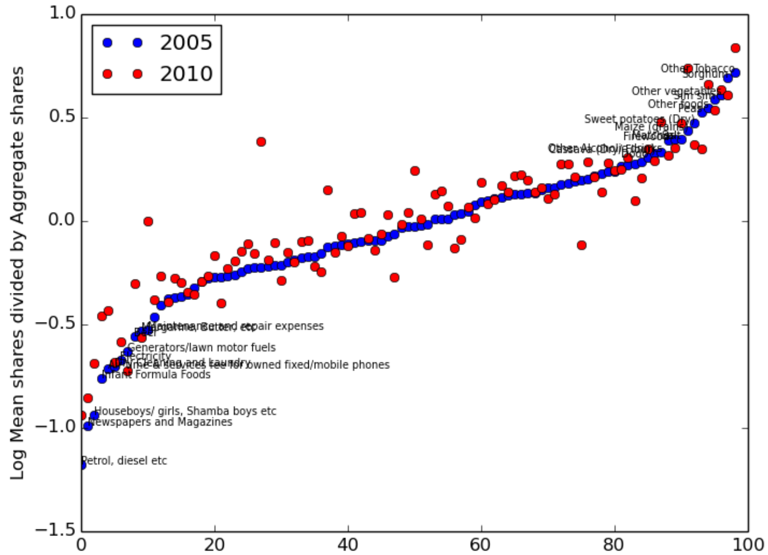

FIGURE 2. Ratio of mean shares to aggregate shares. Items on the left form a disproportionately large share of the budget of the rich; items on the right a disproportionately large budget share of the needy.

Table 2: Estimated parameters of the demand system for all food items. The first two columns report the values of $\hat{\delta}_i$ associated with household characteristics. The third column reports estimates of the elasticity of marginal utility, up to a common factor ϕ. The fourth column reports the proportion of variance in the residual term accounted for by variation in $\Delta \log \lambda_t^j$; the final column indicates the proportion of observations with zero recorded for the expenditure category.

Expenditure category	HHSize	Rural	$\phi\beta_i$	R^2	% Zeros
Other foods	0.05	-1.30	-0.22	0.00	0.92
Imputed rent of free house	2.14	-23.52	-0.06	0.00	0.94
Food (restaurant)	1.77	17.26	-0.04	0.00	0.83
Beer (restaurant)	0.68	0.01	-0.04	0.00	0.99
Dry Cleaning and Laundry	0.81	0.05	-0.02	0.00	0.99
Sweet potatoes (Dry)	-0.38	-33.56	-0.02	0.00	0.98

Continued on next page

Table 2: Estimated parameters of the demand system for all food items. The first two columns report the values of $\hat{\delta}_i$ associated with household characteristics. The third column reports estimates of the elasticity of marginal utility, up to a common factor ϕ. The fourth column reports the proportion of variance in the residual term accounted for by variation in $\Delta \log \lambda_t^j$; the final column indicates the proportion of observations with zero recorded for the expenditure category.

Expenditure category	HHSize	Rural	$\phi\beta_i$	R^2	% Zeros
Others (Rent of rented house/fuel/power)	7.40	234.88	-0.01	0.00	0.98
Infant Formula Foods	-7.83	180.65	-0.00	0.00	1.00
Other Tobacco	6.85	-47.64	0.01	0.00	0.91
Other Meat	1.51	-0.42	0.02	0.00	0.98
Other juice	-2.61	-13.27	0.05	0.00	0.96
Matoke (Others)	1.32	1.48	0.05	0.00	0.98
Stamps, envelopes, etc.	-0.01	13.14	0.05	0.00	0.99
Generators/lawn motor fuels	0.54	-6.54	0.06	0.00	0.99
Others (Transport and communication)	0.60	10.24	0.06	0.00	0.98
Traditional Doctors fees/ medicines	0.51	0.37	0.08	0.00	0.98
Consultation Fees	0.47	0.52	0.09	0.00	0.96
Ground nuts (in shell)	0.78	-5.26	0.13	0.00	0.97
Soda (restaurant)	-1.12	0.47	0.13	0.00	0.96
Rent of rented house	-0.80	-47.92	0.13	0.00	0.86
Other drinks	-0.10	-4.10	0.14	0.00	0.96
Bus fares	0.48	-8.40	0.17	0.00	0.95
Houseboys/ girls, Shamba boys etc	-0.32	3.34	0.19	0.01	0.96
Matoke (Heap)	-0.05	-22.52	0.20	0.00	0.92
Others (Non-durable and personal goods)	0.11	6.54	0.20	0.00	0.95
Maize (grains)	0.22	3.01	0.22	0.00	0.96
Margarine, Butter, etc	-0.14	2.93	0.24	0.01	0.96
Petrol, diesel etc	0.55	-3.91	0.25	0.01	0.95
Maintenance and repair expenses	0.24	-7.30	0.26	0.00	0.92
Handbags, travel bags etc	0.15	-2.24	0.26	0.01	0.96
Ghee	0.16	-2.57	0.26	0.01	0.95
Cigarettes	0.26	-4.30	0.30	0.01	0.91
Beer	-0.01	6.51	0.33	0.01	0.95
Electricity	0.42	-2.16	0.33	0.01	0.88
Others (Health and Medical Care)	0.34	3.02	0.36	0.01	0.91
Newspapers and Magazines	-0.07	-3.58	0.37	0.02	0.94

Continued on next page

Table 2: Estimated parameters of the demand system for all food items. The first two columns report the values of $\hat{\delta}_i$ associated with household characteristics. The third column reports estimates of the elasticity of marginal utility, up to a common factor ϕ. The fourth column reports the proportion of variance in the residual term accounted for by variation in $\Delta \log \lambda_t^j$; the final column indicates the proportion of observations with zero recorded for the expenditure category.

Expenditure category	HHSize	Rural	$\phi\beta_i$	R^2	% Zeros
Sports, theaters, etc	0.42	3.47	0.37	0.01	0.96
Sorghum	0.16	5.55	0.38	0.01	0.86
Coffee	0.37	2.71	0.43	0.01	0.93
Matoke (Cluster)	0.19	3.24	0.46	0.01	0.95
Peas	0.29	-2.89	0.47	0.01	0.91
Beans (fresh)	-0.09	1.17	0.51	0.01	0.84
Goat Meat	-0.11	1.44	0.51	0.02	0.94
Pork	-0.12	-0.04	0.52	0.02	0.94
Other Alcoholic drinks	0.50	2.71	0.52	0.01	0.83
Other vegetables	0.18	8.23	0.55	0.01	0.66
Mangos	0.50	3.61	0.56	0.01	0.88
Maize (cobs)	0.14	2.24	0.57	0.01	0.86
Hospital/ clinic charges	0.29	0.13	0.60	0.01	0.79
Ground nuts (shelled)	-0.06	0.26	0.60	0.01	0.89
Sim sim	0.25	3.13	0.65	0.02	0.89
Oranges	-0.07	1.72	0.66	0.02	0.92
Tires, tubes, spares, etc	0.34	1.96	0.71	0.02	0.84
Dodo	0.23	1.54	0.87	0.01	0.67
Cassava (Dry/ Flour)	0.12	0.70	0.90	0.03	0.73
Chicken	0.13	-0.71	0.94	0.04	0.92
Expenditure on phones not owned	0.02	4.82	0.96	0.03	0.84
Washing soap	0.15	0.81	1.00	0.09	0.04
Water	0.03	-0.01	1.02	0.02	0.22
Imputed rent of owned house	0.38	5.30	1.04	0.04	0.23
Irish Potatoes	0.11	-1.31	1.05	0.05	0.88
Millet	0.15	2.64	1.07	0.04	0.84
Matches	0.16	0.52	1.07	0.11	0.05
Medicines etc	0.80	-1.62	1.12	0.02	0.49
Paraffin (Kerosene)	0.18	1.52	1.17	0.06	0.14
Firewood	0.34	3.55	1.20	0.05	0.28

Continued on next page

Table 2: Estimated parameters of the demand system for all food items. The first two columns report the values of $\hat{\delta}_i$ associated with household characteristics. The third column reports estimates of the elasticity of marginal utility, up to a common factor ϕ. The fourth column reports the proportion of variance in the residual term accounted for by variation in $\Delta \log \lambda_t^j$; the final column indicates the proportion of observations with zero recorded for the expenditure category.

Expenditure category	HHSize	Rural	$\phi\beta_i$	R^2	% Zeros
Soda	-0.02	-3.35	1.24	0.06	0.90
Eggs	0.02	-0.63	1.29	0.06	0.87
Airtime & fees for owned phones	0.18	-1.72	1.33	0.04	0.65
Fresh Fish	0.09	1.21	1.36	0.05	0.80
Charcoal	0.02	-2.62	1.45	0.10	0.75
Passion Fruits	0.07	0.93	1.46	0.09	0.88
Dry/ Smoked fish	0.23	-2.51	1.47	0.04	0.75
Matoke (Bunch)	0.21	-0.36	1.50	0.04	0.66
Other Fruits	0.09	0.43	1.56	0.06	0.82
Sweet Bananas	0.04	0.29	1.58	0.08	0.85
Batteries (Dry cells)	0.10	-0.34	1.61	0.05	0.47
Salt	0.22	0.20	1.70	0.19	0.08
Ground nuts (pounded)	0.15	1.16	1.70	0.06	0.70
Boda boda fares	0.11	0.27	1.72	0.06	0.73
Cabbages	0.12	0.32	1.85	0.09	0.81
Maize (flour)	0.25	-3.35	1.98	0.06	0.46
Tooth paste	0.01	-0.95	2.04	0.11	0.52
Cassava (Fresh)	0.17	3.60	2.05	0.07	0.57
Bathing soap	0.04	-1.38	2.13	0.11	0.68
Bread	-0.01	-1.51	2.20	0.14	0.78
Beans (dry)	0.31	1.01	2.23	0.08	0.37
Taxi fares	0.00	0.38	2.24	0.10	0.69
Rice	0.13	0.33	2.25	0.12	0.74
Sweet potatoes (Fresh)	0.23	2.08	2.42	0.10	0.46
Cosmetics	0.15	0.44	2.44	0.11	0.32
Fresh Milk	0.07	0.25	2.60	0.13	0.65
Barber and Beauty Shops	0.11	1.05	2.79	0.14	0.55
Tea	0.09	0.61	3.03	0.22	0.39
Beef	0.07	-0.62	3.28	0.19	0.67
Cooking oil	0.17	0.08	3.84	0.30	0.42

Continued on next page

Table 2: Estimated parameters of the demand system for all food items. The first two columns report the values of $\hat{\delta}_i$ associated with household characteristics. The third column reports estimates of the elasticity of marginal utility, up to a common factor ϕ. The fourth column reports the proportion of variance in the residual term accounted for by variation in $\Delta \log \lambda_t^j$; the final column indicates the proportion of observations with zero recorded for the expenditure category.

Expenditure category	HHSize	Rural	$\phi\beta_i$	R^2	% Zeros
Tomatoes	0.10	-0.75	3.85	0.32	0.33
Onions	0.15	-0.19	3.88	0.33	0.31
Sugar	0.11	0.09	3.90	0.31	0.36

Table 2 reports estimates and some diagnostics from the estimation of the complete system of 99 food and nondurable goods and services, ordered by the size of the critical estimated marginal utility elasticities (up to a unknown scale parameter) $\phi\beta_i$. The table also reports estimated coefficients associated with the household characteristics of household size (number of people in the household) and a dummy variable indicating whether the household is rural. Table 2 also reports a statistic labeled R^2 which reports the proportion of variation in the residual term accounted for by variation in changes in log marginal utility.

As discussed above, the goods and services most useful goods for measuring marginal utility are those with larger estimated elasticities and with smaller measurement error. The R^2 statistics here provide a measure of the variance of measurement error (smaller R^2 statistics indicate more measurement error). The goods in Table 2 with the largest estimated elasticities also tend to have larger R^2 statistics; they also happen to be food items. We next explore this association in some detail by focusing *just* on food items.

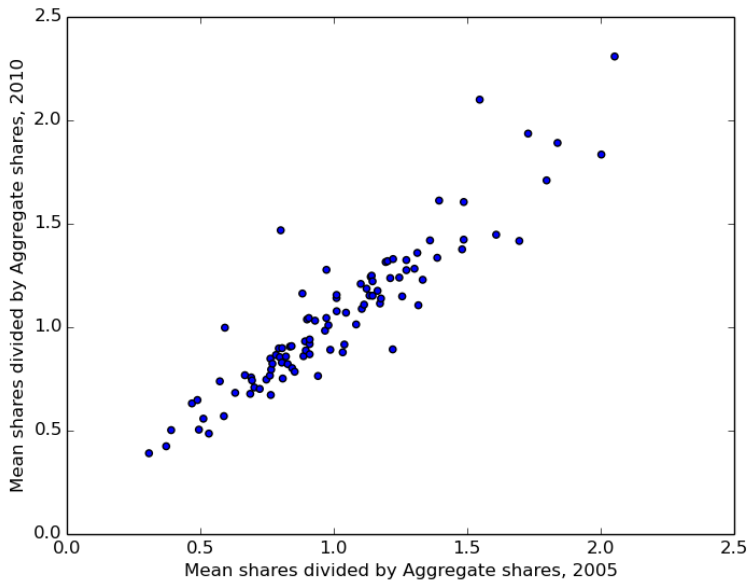

FIGURE 3. Changes in ratio of mean shares to aggregate shares across survey rounds, using all expenditure categories.

A.1.2. *All Food Categories.*

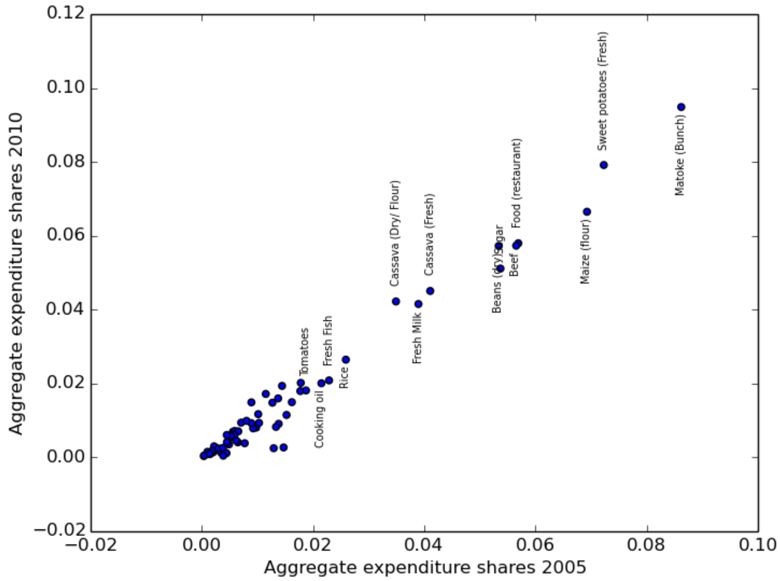

FIGURE 4. Aggregate expenditure shares in 2005 and 2009, using all food items.

Table 3 reports statistics obtained from the estimation of a demand system consisting of just the 61 different food items. As in Table 2 expenditure categories are ordered according to the estimated marginal utility elasticity β_i. The ordering of foods in Table 3 is extremely close to the ordering observed in Table 2.

Table 3: Estimated parameters of the demand system for all food items. The first two columns report the values of $\hat{\delta}_i$ associated with household characteristics. The third column reports estimates of the elasticity of marginal utility, up to a common factor ϕ. The fourth column reports the proportion of variance in the residual term accounted for by variation in $\Delta \log \lambda_t^j$; the final column indicates the proportion of observations with zero recorded for the expenditure category.

Food category	HHSize	Rural	$\phi\beta_i$	R^2	% Zeros
Other foods	0.07	-0.74	-0.30	0.01	0.92
Food (restaurant)	0.31	-0.01	-0.24	0.00	0.83
Beer (restaurant)	0.63	0.10	-0.03	0.00	0.99

Continued on next page

Table 3: Estimated parameters of the demand system for all food items. The first two columns report the values of $\hat{\delta}_i$ associated with household characteristics. The third column reports estimates of the elasticity of marginal utility, up to a common factor ϕ. The fourth column reports the proportion of variance in the residual term accounted for by variation in $\Delta \log \lambda_t^j$; the final column indicates the proportion of observations with zero recorded for the expenditure category.

Food category	HHSize	Rural	$\phi\beta_i$	R^2	% Zeros
Infant Formula Foods	-4.14	98.29	-0.01	0.00	1.00
Sweet potatoes (Dry)	0.44	-135.09	-0.00	0.00	0.98
Other juice	-53.77	-328.61	0.00	0.00	0.96
Matoke (Others)	3.46	3.34	0.02	0.00	0.98
Other Tobacco	2.22	-18.50	0.03	0.00	0.91
Other Meat	0.77	-0.15	0.04	0.00	0.98
Soda (restaurant)	-2.28	11.03	0.06	0.00	0.96
Other drinks	-0.30	-4.73	0.11	0.00	0.96
Ground nuts (in shell)	0.86	-4.89	0.13	0.00	0.97
Matoke (Heap)	0.06	-24.08	0.18	0.00	0.92
Cigarettes	0.54	-6.44	0.18	0.00	0.91
Maize (grains)	0.10	3.09	0.20	0.01	0.96
Ghee	0.19	-2.97	0.21	0.01	0.95
Margarine, Butter, etc	-0.20	2.29	0.28	0.01	0.96
Beer	0.01	6.73	0.30	0.01	0.95
Coffee	0.43	3.18	0.35	0.01	0.93
Other Alcoholic drinks	0.73	3.74	0.36	0.01	0.83
Sorghum	0.22	5.39	0.37	0.01	0.86
Peas	0.36	-2.94	0.43	0.01	0.91
Goat Meat	-0.09	1.51	0.45	0.02	0.94
Matoke (Cluster)	0.08	3.06	0.45	0.02	0.95
Mangos	0.53	2.67	0.47	0.01	0.88
Pork	-0.10	-1.47	0.48	0.02	0.94
Beans (fresh)	-0.06	1.07	0.52	0.01	0.84
Maize (cobs)	0.12	2.28	0.52	0.01	0.86
Sim sim	0.25	3.56	0.54	0.02	0.89
Other vegetables	0.26	6.61	0.55	0.01	0.66
Ground nuts (shelled)	-0.07	0.25	0.57	0.02	0.89
Oranges	-0.05	0.63	0.64	0.03	0.92
Cassava (Dry/ Flour)	0.19	0.81	0.75	0.03	0.73

Continued on next page

Table 3: Estimated parameters of the demand system for all food items. The first two columns report the values of $\hat{\delta}_i$ associated with household characteristics. The third column reports estimates of the elasticity of marginal utility, up to a common factor ϕ. The fourth column reports the proportion of variance in the residual term accounted for by variation in $\Delta \log \lambda_t^j$; the final column indicates the proportion of observations with zero recorded for the expenditure category.

Food category	HHSize	Rural	$\phi\beta_i$	R^2	% Zeros
Dodo	0.23	1.56	0.82	0.02	0.67
Chicken	0.16	-1.50	0.88	0.05	0.92
Millet	0.18	2.88	0.92	0.04	0.84
Irish Potatoes	0.18	-1.32	0.96	0.05	0.88
Soda	0.03	-4.74	1.08	0.07	0.89
Eggs	0.04	-0.63	1.19	0.07	0.87
Other Fruits	0.09	0.51	1.26	0.05	0.82
Fresh Fish	0.11	1.23	1.27	0.06	0.80
Passion Fruits	0.07	0.99	1.30	0.09	0.88
Dry/ Smoked fish	0.20	-2.63	1.31	0.05	0.75
Sweet Bananas	0.06	0.33	1.36	0.08	0.85
Matoke (Bunch)	0.28	-0.81	1.39	0.05	0.66
Ground nuts (pounded)	0.12	1.71	1.43	0.06	0.70
Salt	0.24	0.20	1.52	0.21	0.08
Cabbages	0.11	0.35	1.64	0.09	0.81
Maize (flour)	0.27	-3.26	1.89	0.08	0.45
Beans (dry)	0.35	0.81	1.93	0.08	0.37
Cassava (Fresh)	0.18	3.11	2.01	0.09	0.57
Sweet potatoes (Fresh)	0.28	2.34	2.03	0.10	0.46
Bread	0.01	-1.24	2.07	0.16	0.78
Rice	0.17	0.04	2.10	0.15	0.74
Fresh Milk	0.07	0.52	2.33	0.15	0.65
Tea	0.10	0.66	2.71	0.24	0.39
Beef	0.06	-0.82	3.14	0.23	0.67
Sugar	0.13	0.11	3.34	0.32	0.36
Cooking oil	0.18	0.09	3.43	0.34	0.42
Onions	0.17	-0.20	3.49	0.37	0.31
Tomatoes	0.09	-0.76	3.56	0.37	0.33

The parameter estimates reported in Table 3 mostly seem reasonable enough, though one should bear in mind that the critical β_i parameters

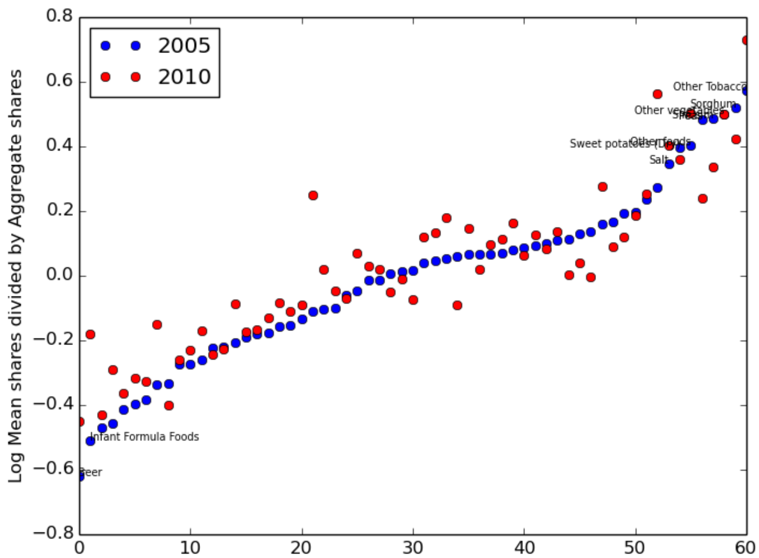

FIGURE 5. Ratio of mean shares to aggregate shares, using all food items. Items on the left form a disproportionately large share of the budget of the rich; items on the right a disproportionately large budget share of the needy.

are only identified up to a scale parameter (so that one can't really say that some goods are "luxuries" while others are "necessities", for example). However, one can rank the β_i, and most of the goods that have the smallest elasticities also have a very high proportion of "zero" recorded for expenditures. Moving from the lowest elasticity up (the rows are orded by the estimated $\phi\beta_i$), with a single exception ("Other vegetables") one has to move more than halfway down the table before reaching a good (Cassava dry/flour) for which more than 20% of the observations report positive expenditures. What seems to be happening for at least some of these goods is that our estimator is trying to fit a large proportion of zeros, and these zeros occur across a large part of the marginal utility distribution, leading to very low estimates of β_i for these goods.

Once one gets down the table to goods that have a proportion of zeros less than one half the relative elasticities seem to be fairly plausible: it's easy to believe that demands for tomatoes, onions, cooking oil, sugar

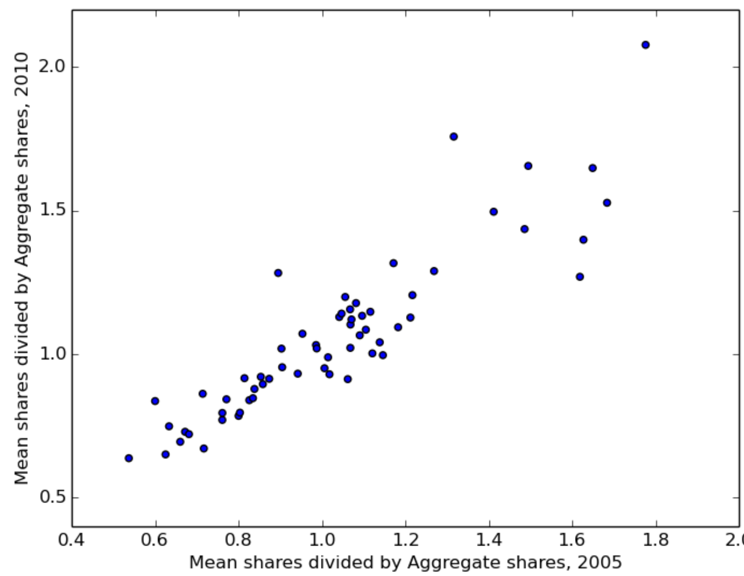

FIGURE 6. Changes in ratio of mean shares to aggregate shares across survey rounds, using all food items.

and beef are more elastic than demands for maize flour, dry beans, and fresh cassava, for example.

Recall that the goods which are most useful for drawing inferences regarding marginal utility are those which either have large (relative) values of β_i, or small errors (large R^2 statistics). The fact that we earlier offered two distinct criteria for choosing goods created the possibility that these criteria might conflict. However, Table 3 indicates a very high degree of correlation between these two criteria, suggesting that the conflict may not arise often in practice.

Figure 7 provides a histogram of estimated changes in log marginal utility. These estimates have a mean of zero by construction using this estimator—time varying differences common to all households are swept out with the good-time effects of the estimating equation (19). Further, the scale is only determined by an arbitrary normalization of the β_i, so the information in this histogram all pertains to the shape of the distribution. In this case the distribution appears to be reasonably symmetric, though slightly skew right (the third central moment is positive) and with slightly fatter tails than the normal distribution (the fourth central moment is approximately 55, while if the distribution were normal we'd expect this moment to be approximately 40).

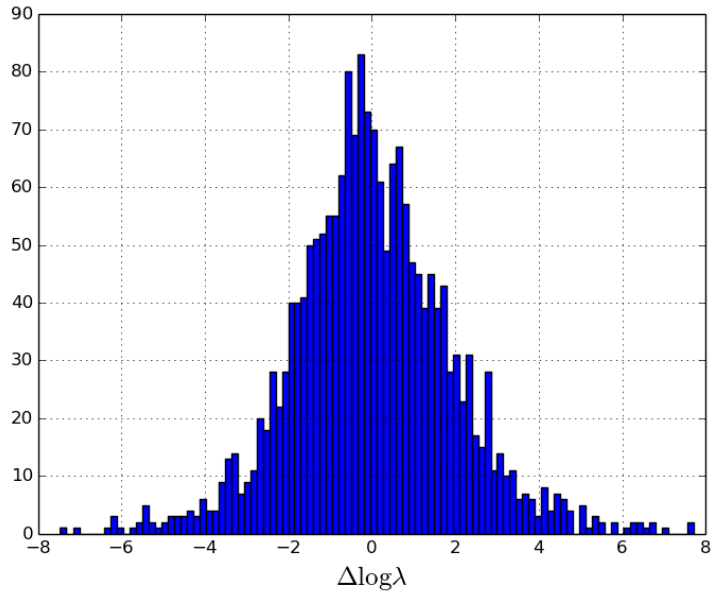

FIGURE 7. Histogram of $\Delta \log \lambda_t^j$ using all food items.

These deviations seem fairly minor, however; taking the distribution of changes in marginal utility to be log-normal would not seem to do too much violence to the data.

One might wonder what would happen if we simply dropped the goods with a high proportion of zeros. In this case dropping any good with a proportion of zeros greater than eighty per cent reduces the number of goods in the demand system from 61 to 21. The (Pearson) correlation between the estimated $\Delta \log \lambda_t^j$ in the 61 good and the 21 good system is 0.98, and the ordering of the goods by elasticity is almost unchanged (bread and milk are slightly less relatively elastic in the 21 good system). Dropping an additional 10 goods, leaving the 11 most elastic, yields a correlation of 0.94 between changes in log marginal utility estimated using the 61 and 11 good systems.

A.1.3. *Food in Slightly Aggregated Groups.* Is there a better way to deal with the problem of the most detailed expenditure categories having a high proportion of zeros? Though there are 61 different food expenditure categories (including two different categories for tobacco) many of these are categories that seem that they must be very close substitutes. For instance, the four different forms in which Matoke is acquired (bunches, clusters, heaps, and others) are all elicited as separate expenditure items. Other staple items are also elicited in detail, with two different expenditure items each for sweet potatoes and cassava (fresh and dry), three for maize and ground nut, and so on. Expenditures on five different kinds of fresh fruit and five different kinds of fresh vegetables are also collected.

This level of detail in expenditure isn't a problem in principle, but in practice many of the detailed categories feature zero expenditures for many households. Supposing these to be "true" zeros (rather than measurement error), we'd interpret these as corner solutions for the households where they occur, but the requirement that expenditures be non-negative isn't something that's taken into account in the derivation of our demand system. Adding this to our modeling exercise is undoubtedly the correct way to proceed, but is sufficiently difficult that we try to reduce the impact of this by aggregating goods that seem likely to be close substitutes.

Simply adding up expenditures which, according to their descriptions, seem as though they might be closely related yields a system of 29 goods; we'll refer to this as our "slightly aggregated food" system. Expenditure shares from this system are reported in Table **??**, paralleling the complete set of foods described in Table **??**. Similar figures and tables also parallel those for the complete set of foods.

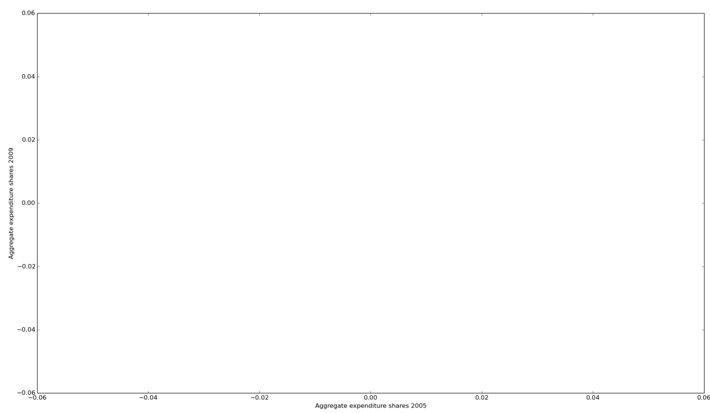

FIGURE 8. Aggregate expenditure shares in 2005 and 2009, using slightly aggregated food groups.

Figure 8 reveals aggregate shares that are reasonably stable over time, though Figure 9 indicates that the ratio of mean to aggregate shares shows some greater variability. The most prominent examples are the goods with the largest such ratio in 2005: salt, "other foods", peas, sim sim (a sweetener), and sorghum, all goods that feature more prominently in the budgets of poorer rather than than wealthier households.

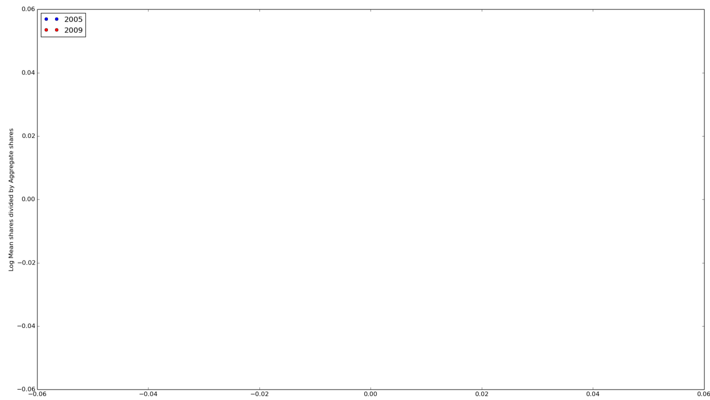

FIGURE 9. Ratio of mean shares to aggregate shares, using slightly aggregated food groups. Items on the left form a disproportionately large share of the budget of the rich; items on the right a disproportionately large budget share of the needy.

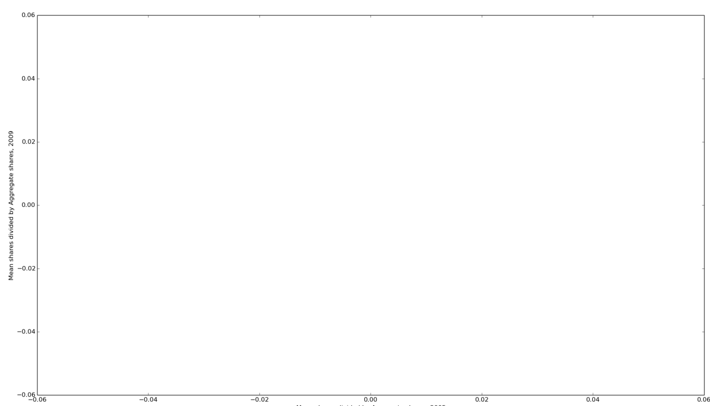

FIGURE 10. Changes in the ratio of mean shares to aggregate shares across survey rounds, using slightly aggregated food groups.

Continued on next page

Table 4: Estimated parameters of the demand system for "slightly aggregated" food items. The first two columns report the values of $\hat{\delta}_i$ associated with household characteristics. The third column reports estimates of the elasticity of marginal utility, up to a common factor ϕ. The fourth column reports the proportion of variance in the residual term accounted for by variation in $\Delta \log \lambda_t^j$; the final column indicates the proportion of observations with zero recorded for the expenditure category.

Food category HHSize Rural $\phi\beta_i$ R^2 % Zeros

Table 4: Estimated parameters of the demand system for "slightly aggregated" food items. The first two columns report the values of $\hat{\delta}_i$ associated with household characteristics. The third column reports estimates of the elasticity of marginal utility, up to a common factor ϕ. The fourth column reports the proportion of variance in the residual term accounted for by variation in $\Delta \log \lambda_t^j$; the final column indicates the proportion of observations with zero recorded for the expenditure category.

Food category	HHSize	Rural	$\phi\beta_i$	R^2	% Zeros
Other foods	0.11	-1.53	-0.15	0.00	0.92
Restaurant meals	1.11	-0.10	-0.11	0.00	0.83
Infant formula	-17.86	543.66	-0.00	0.00	1.00
Tobacco	1.26	-10.27	0.12	0.00	0.83
Coffee	0.63	6.27	0.17	0.01	0.93
Peas	0.63	-5.24	0.24	0.01	0.91
Sorghum	0.40	7.28	0.28	0.01	0.86
Sim sim	0.46	6.14	0.31	0.02	0.89
Millet	0.44	5.03	0.53	0.03	0.84
Irish Potatoes	0.21	-2.05	0.62	0.06	0.88
Eggs	0.05	-0.97	0.77	0.07	0.87
Salt	0.43	0.36	0.86	0.18	0.08
Beans	0.49	0.72	1.12	0.11	0.25
Drinks	0.15	-0.96	1.13	0.08	0.67
Cassava	0.27	3.79	1.20	0.10	0.42
Sweet potatoes	0.42	4.45	1.21	0.09	0.45
Vegetables	0.38	-0.14	1.23	0.24	0.10
Bread	0.06	-2.01	1.27	0.16	0.78
Matoke	0.34	-3.14	1.27	0.13	0.53

Continued on next page

Table 4: Estimated parameters of the demand system for "slightly aggregated" food items. The first two columns report the values of $\hat{\delta}_i$ associated with household characteristics. The third column reports estimates of the elasticity of marginal utility, up to a common factor ϕ. The fourth column reports the proportion of variance in the residual term accounted for by variation in $\Delta \log \lambda_t^j$; the final column indicates the proportion of observations with zero recorded for the expenditure category.

Food category	HHSize	Rural	$\phi\beta_i$	R^2	% Zeros
Rice	0.31	0.07	1.30	0.15	0.74
Fish	0.22	-0.05	1.36	0.13	0.60
Maize	0.28	-3.51	1.40	0.12	0.38
Ground nut	0.22	1.40	1.41	0.12	0.58
Fresh milk	0.19	0.84	1.47	0.16	0.65
Tea	0.17	1.12	1.57	0.23	0.39
Sugar	0.25	0.19	1.97	0.31	0.36
Fruits	0.22	1.53	2.00	0.23	0.58
Oils	0.27	-0.13	2.03	0.31	0.37
Meat	0.09	-1.09	2.41	0.33	0.57

This aggregation helps with the problem of "zeros" to a considerable extent; now one only has to go to the eleventh good reported in Table 4 to find one with proportions of zeros less than 80% (salt). And the ordering of goods by elasticity seems quite consistent with the ordering using the 61-good system of all food expenditures; as before, meat, oils, fruit, sugar, and tea are among the most elastic goods. Less optimistically, it's still the case that even with this aggregation there are a great number of zeros—more than half of all households report no expenditures on any kind of meat, for example. Knowing whether this is accurate or evidence of measurement error requires evidence outside of the dataset, but it seems possible that some of these zeros may be due to respondent (or enumerator) fatigue. Proposing to collect data on future expenditures only on the eighteen slightly aggregated catetories (salt through meat) would involve inquiring about 47 distinct expenditure categories but with accompanying instructions which reduced the proportion of recorded zeros might be wise (and further reductions in the number of categories are surely possible if the proportion of zeros can be reduced).

However, perhaps the most the critical question is whether this aggregation harms our estimates of household marginal utility. The answer illustrates our last point. If we eliminate those expenditure categories for which more than eighty percent of observations are zero, we're left with 18 of 29 categories; the correlation between estimates of changes in marginal utility for this 18 good category with the 29 good category is 0.9964, so there's essentially no loss here. Again, asking about this relatively limited set of expenditure categories while eliciting fewer zeros might be a worthwhile trade-off.

A.2. **Distribution of Changes in Marginal Utility.** Figure 11 illustrates the relationship between our estimated changes in (minus) log marginal utility and changes in aggregate expenditures. Though there's obviously a strong positive relationship, it's also apparent that the relationship is considerably less than perfect.

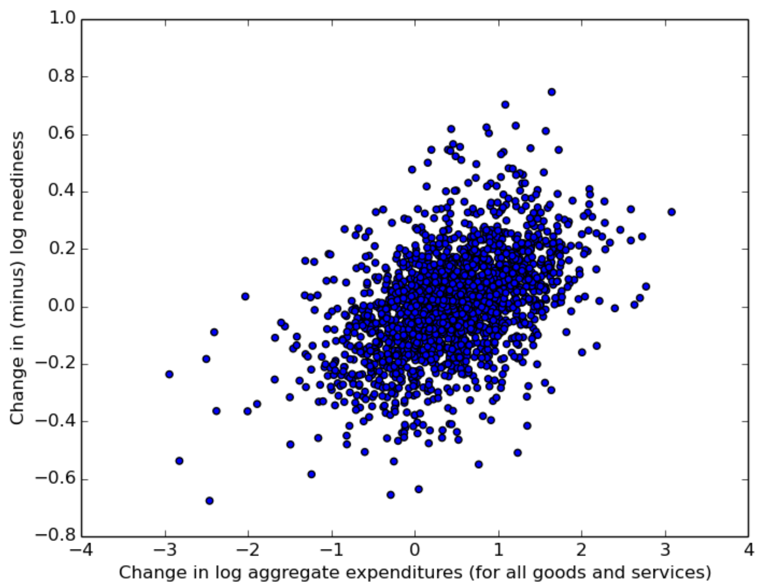

FIGURE 11. Relationship between changes in (the logarithm of) a consumption aggregate and changes in estimated marginal utility, using estimates from slightly aggregated foods.

In this appendix we have discussed a variety of different approaches to estimating (changes in log) marginal utility. Table 5 reports on the relationship between these different approaches, by reporting correlations between them. The first column (and row) use as a measure

TABLE 5. Correlations between estimates of changes in marginal utility using different demand systems. Pearson correlation coefficients are below the diagonal; Spearman correlation coefficients above. Diagonal elements are estimates of the proportion of total residual variation accounted for by variation in marginal utility. "Agg. Exp." is aggregate expenditures (across all food and nondurable goods and services); remaining columns are estimated $\Delta \log \lambda_t^j$. "All" uses the 99 good demand system of all food and nondurables; "All Food" uses all 61 food expenditure categories; "S.A. Food" is the 'slightly aggregated' demand system of 29 categories. Numbers in parentheses indicate that categories are only included if the proportion of zeros is less than that number.

	Agg. Exp.	All	All Food	All Food (0.8)	All Food (0.5)	S.A. Food	S.A. Food (0.8)
Agg. Exp.	—	-0.490	-0.481	-0.441	-0.334	-0.508	-0.504
All	-0.489	0.074	0.937	0.906	0.804	0.868	0.862
All Food	-0.471	0.945	0.095	0.971	0.852	0.921	0.915
All Food (0.8)	-0.432	0.921	0.978	0.150	0.901	0.870	0.876
All Food (0.5)	-0.343	0.837	0.883	0.925	0.280	0.718	0.725
S.A. Food	-0.500	0.897	0.942	0.904	0.774	0.142	0.995
S.A. Food (0.8)	-0.497	0.893	0.938	0.909	0.782	0.996	0.174

changes in log total expenditures; this is the "consumption aggregate" approach typically used for measuring poverty at the World Bank. The remaining columns (and rows) use measures of $\Delta \log \lambda_t^j$ estimated using the different demand systems described above. "All" is the 99 good system of foods, non-durables and services; "All Food" is the 61 good system of food (and tobacco) expenditures; "S.A. Food" is the "slightly aggregated" system of 29 different food expenditure categories. Where a number appears in parentheses the demand system has been reduced by eliminating any goods for which the proportion of zeros exceeds the parenthetical number; for example, "All Food (0.8)" excludes the food expenditure categories for which fewer than 20% of observations have a positive value.

The table has three parts. Statistics below the diagonal are Pearson correlation coefficients, while statistics above the diagonal are Spearman rank correlation coefficients. These two different measures of correlation are generally in fairly close agreement, reflecting the roughly log-normal distribution of estimated $\Delta \log \lambda_t^j$ noted in our discussion of Figure 7. Along the diagonal is a measure of the overall ability of the demand system to fit the data: the statistics reported here use the singular values obtained when we computed the β_i and $\Delta \log \lambda_t^j$, and are the ratio of the square of the first singular value to the sum of the squares of all the singular values. This ratio can be interpreted as the proportion of total residual variance accounted for by variation in household marginal utility.

Increases in log aggregate expenditures are negatively correlated with changes in log marginal utility, as we'd expect, though the correlation is far from perfect (echoing the lesson of Figure 11), taking values of at most (minus) 0.50. It's not at all clear that it *should* be perfect—even in the complete absence of measurement error marginal utility is generally a highly nonlinear (though monotonic) function of total expenditures. Neither is it clear whether aggregate expenditures or estimates of marginal utility are more affected by measurement error. Measures of aggregate expenditures will be more sensitive to the problem of many expenditure categories having zero expenditures for a large proportion of observations than will our marginal utility estimates, since our estimation approach tends to assign low weights (in the form of the estimated β_i) to goods with large proportions of zeros.