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Complete quenching for a quasilinear parabolic equation

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ABSTRACT

We study the homogeneous Dirichlet problem for the quasilinear parabolic equation with the singular absorption term

$$\partial_t u - \Delta_p u + \mathbb{1}_{\{u>0\}} u^{-\beta} = f(x,u) \quad \text{in } Q_T = (0,T)\times\Omega.$$

Here $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, is a bounded domain, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator and $\beta \in (0,1)$ is a given parameter. It is assumed that the initial datum satisfies the conditions

$$u_0 \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \quad u_0 \geqslant 0 \text{ a.e. in } \Omega.$$

The right-hand side $f : \Omega\times\mathbb{R} \to [0,\infty)$ is a Carathéodory function satisfying the power growth conditions: $0 \leqslant f(x,s) \leqslant \alpha|s|^{q-1} + C_\alpha$ with positive constants α, C_α and $q \geqslant 1$. We establish conditions of local and global in time existence of nonnegative solutions and show that if $q \leqslant p$ and α and C_α are sufficiently small, then every global solution vanishes in a finite time a.e. in Ω.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, be an open set with the Lipschitz-continuous boundary $\partial\Omega$ and $Q_T = (0,T)\times\Omega$ be the cylinder of height $T < \infty$ with the lateral boundary $\Gamma_T = (0,T)\times\partial\Omega$. We study the homogeneous Dirichlet problem for the quasilinear parabolic equation with the singular absorption term

$$(P)\quad \begin{cases} \partial_t u - \Delta_p u + \mathbb{1}_{\{u>0\}} u^{-\beta} = f(x,u) & \text{in } Q_T,\\ u = 0 & \text{on } \Gamma_T,\\ u(0,\cdot) = u_0 & \text{in } \Omega. \end{cases}$$

Here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator and $\beta \in (0,1)$ is a given constant. By $\mathbb{1}_{\{u>0\}}$ we denote the characteristic function of the set $\{u > 0\}$,

$$\mathbb{1}_{\{u>0\}} = \begin{cases} 1 & \text{if } u > 0,\\ 0 & \text{if } u \leqslant 0. \end{cases}$$

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By convention, throughout the paper we assume that $\mathbb{1}_{\{u>0\}}u^{-\beta}=0$ whenever $u=0$. It is assumed that the initial function u_0 satisfies the conditions

$$u_0 \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega) \quad \text{and} \quad u_0 \geqslant 0 \quad \text{a.e. in } \Omega. \tag{1.1}$$

The right-hand side $f(x,s)$ is assumed to satisfy the conditions:

$$(F1) \quad \begin{cases} f : \Omega\times\mathbb{R}\to[0,\infty) \text{ is a Carathéodory function,} \\ f(x,s) \text{ is uniformly in } x\in\Omega \text{ locally Lipschitz-continuous in } s\in\mathbb{R}, \\ \forall \text{ a.e. } x\in\Omega, \quad f(x,0)=0 \end{cases}$$

and the following growth condition:

$$(F2) \quad \begin{cases} \text{there exist constants } q\geqslant 1,\ \alpha\geqslant 0,\ C_\alpha\geqslant 0 \text{ such that,} \\ \forall x\in\Omega,\ \forall s\in\mathbb{R}, \quad 0\leqslant f(x,s)\leqslant \alpha|s|^{q-1}+C_\alpha. \end{cases}$$

Problem (P) appears as the limit case of a mathematical model describing enzymatic kinetics (Banks [4]), or in the Langmuir–Hinshelwood model of the heterogeneous chemical catalyst (Cho, Aris and Carr [6] and also Díaz [11]). It has already been studied for the heat equation, *i.e.* in the case $p=2$, by Deng and Levine [10], Fila, Hulshof and Quittner [14], Fila and Kawohl [15], Fila, Levine and Vázquez [16] and Levine [19] under the Dirichlet boundary condition $u=1$ on Γ. The Cauchy problem for equation (P) was studied by Phillips [25]. Parabolic equations not in divergence form and with singular absorption terms were studied by Winkler [29,30].

The singular absorption term may cause a striking phenomenon: even if the solution of the semilinear problem (P) is generated by a strictly positive initial function, it may vanish in a finite time on a set of nonzero measure. Such a behavior, usually referred to as ***quenching***, was first observed in the pioneering paper by Kawarada [18]. We refer to the above cited works for a detailed discussion of the possibility of quenching in solutions of various parabolic equations and for the study of certain properties of these solutions, such as asymptotic behavior, uniqueness, stability and evolution of the solution profile near the quenching point.

In [7] Davila and Montenegro have studied the semilinear problem (P) with $p=2$ under the assumptions $u_0\in L^\infty(\Omega)\cap C(\Omega)$ and $u_0\geqslant 0$ a.e. in Ω. A weak solution was obtained as the pointwise limit of a sequence of solutions to the problems with a regularized singular term. It is proved in [7] that in the case of sublinear growth of the source term $f(u)$ the solution may exhibit the quenching behavior: the measure of the vanishing set $\{(t,x)\in[0,+\infty)\times\Omega:\ u(t,x)=0\}$ is positive. It is also shown that the possibility of quenching in solutions of the semilinear problem is tightly related to the nonexistence of positive solutions of the stationary counterpart of problem (P). The properties of stationary solutions to problem (P) with $p=2$ are studied in Davila and Montenegro [8,9] and Díaz, Hernández and Mancebo [12]. In particular, it is proved in [8] that under additional restrictions on f problem (P) admits stationary solutions with compact support. In the recent paper [23] M. Montenegro proved that under stronger conditions on the initial data the solutions of problem (P) for the semilinear equation with $p=2$ may exhibit the property of ***complete quenching***, that is, $u(t,\cdot)=0$ a.e. in Ω for all t beginning with some T_*.

In the present paper, we study problem (P) for the quasilinear equation with $1<p<\infty$ and the nonnegative source f subject to conditions $(F1)$–$(F2)$. We prove first the existence of a local in time weak solution. Under the additional assumption of the subcritical growth of $f(x,s)$ as $s\to\infty$, $q<p$, we show that the local solution can be continued to the arbitrary time interval. The same is true if the growth rate of f is critical, $q=p$, and α is sufficiently small. We show then that in both cases every weak solution possesses the property of complete quenching in a finite time, provided that the constants α and C_α are sufficiently small. To be precise, we prove that $u(t,x)=0$ a.e. in Ω for all $t\geqslant T_*$ and estimate the value of T_* through $\|u_0\|_{2,\Omega}$, $\|u_0\|_{\infty,\Omega}$, d, p, α, C_α and the first eigenvalue of the Dirichlet problem for the p-Laplace operator in Ω.

We finally show that the condition on the growth rate of f is in a certain sense necessary for the global in time existence: in case of the supercritical growth of f, $q>p$, problem (P) still has a local in time weak solution, but there is a subset of initial data satisfying (1.1) such that the corresponding weak solutions blow up in a finite time.

2. Definitions and main results

Let us introduce the function space

$$\mathcal{U} := \left\{ v\in L^\infty\left(0,T;W_0^{1,p}(\Omega)\right)\cap L^\infty(Q_T) \,\middle|\, \partial_t v\in L^2(Q_T)\right\}.$$

By convention we use the notation $z=(t,x)$ for the points of the cylinder $Q_T=(0,T)\times\Omega$.

Definition 2.1. A function $u(t,x)$ is ***a weak solution of problem (P)*** if:

1. $u\in\mathcal{U}\cap C([0,T];L^2(\Omega))$, $u\geqslant 0$ a.e. in Q_T,

2. for every test-function $\varphi \in \mathcal{U}$ the inclusion $\mathbb{1}_{\{u>0\}} u^{-\beta}\varphi \in L^1(Q_T)$ holds and

$$\int_{Q_T} \partial_t u\,\varphi\,dz + \int_{Q_T} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dz + \int_{Q_T} \mathbb{1}_{\{u>0\}} u^{-\beta}\varphi\,dz = \int_{Q_T} f(x,u)\varphi\,dz, \tag{2.1}$$

3. $u(0,\cdot) = u_0$ a.e. in Ω.

Theorem 2.1 *(Local in time existence of a weak solution). Let us assume that u_0 satisfies condition (1.1) and f satisfies conditions $(F1)$–$(F2)$. Then there exists $T^* > 0$ such that for every $T < T^*$ problem (P) has at least one weak solution in the sense of Definition 2.1, which satisfies the energy relations: for every $t_1, t_2 \in [0,T]$*

$$\frac{1}{2}\|u(t_2)\|^2_{2,\Omega} - \frac{1}{2}\|u(t_1)\|^2_{2,\Omega} + \int_{t_1}^{t_2}\int_\Omega |\nabla u|^p\,dz + \int_{t_1}^{t_2}\int_\Omega u^{1-\beta}\,dz = \int_{t_1}^{t_2}\int_\Omega f(x,u)u\,dz \tag{2.2}$$

and for almost every $t \in (0,T)$

$$\begin{aligned} &\|\partial_t u\|^2_{2,Q_t} + \frac{1}{p}\|\nabla u(t)\|^p_{p,\Omega} + \frac{1}{1-\beta}\int_\Omega u^{1-\beta}(t)\,dx - \int_0^{u(t)} f(x,s)\,ds\,dx \\ &\quad \leqslant \frac{1}{p}\|\nabla u_0\|^p_{p,\Omega} + \frac{1}{1-\beta}\int_\Omega u_0^{1-\beta}\,dx - \int_0^{u_0} f(x,s)\,ds\,dx. \end{aligned} \tag{2.3}$$

Here and throughout the rest of the paper we use the notations $u(t) := u(t,\cdot)$ a.e. in Ω and $Q_t := (0,t)\times\Omega$.

A local in time weak solution of problem (P) is obtained by means of a suitable regularization of the singular term in equation (P) with the consequent passage to the limit with respect to the regularization parameters. The key point of the proof is a special choice of approximations for the discontinuous and nonmonotone term $\mathbb{1}_{\{s>0\}}s^{-\beta}$ (see formula (3.2) below) and a careful analysis of their convergence properties.

The next issues are the possibility of continuation of the constructed local in time solution to an arbitrary cylinder $(0,T)\times\Omega$ and the study of possible quenching. Let us denote by λ_1 the first eigenvalue of the Dirichlet problem for the p-Laplace operator:

$$\lambda_1 := \inf\left\{\int_\Omega |\nabla v|^p\,dx:\ v \in W_0^{1,p}(\Omega),\ \int_\Omega |v|^p\,dx = 1\right\}. \tag{2.4}$$

Theorem 2.2 *(Global in time existence and complete quenching). Let the conditions of Theorem 2.1 be fulfilled. Assume that $f(x,s)$ satisfies the growth condition*

$$\forall s \in \mathbb{R}, \quad 0 \leqslant f(x,s) \leqslant \alpha|s|^{q-1} + C_\alpha, \tag{2.5}$$

with constants $1 \leqslant q \leqslant p$, $\alpha \geqslant 0$ and $C_\alpha \geqslant 0$. If

$$\alpha < \lambda_1,$$

then problem (P) has a global in time bounded weak solution $u \in \mathcal{U}$. Moreover, if

$$\alpha + C_\alpha < \min\{1, \lambda_1\},$$

then every weak solution $u \in \mathcal{U}$ vanishes in a finite time: there exists $T_* > 0$, depending on p, d, $|\Omega| :=$ meas Ω, α, λ_1, $\|u_0\|_{2,\Omega}$ and $\|u\|_{\infty,\Omega}$, such that:

$$\forall t \geqslant T_*, \quad u(t) = 0 \quad \text{a.e. in } \Omega.$$

The possibility of continuation of a local solution to an arbitrary time interval relies on the uniform L^∞-estimates for the solutions of the regularized problems, which are obtained by means of comparison with suitable barrier functions independent of t. In the case of critical growth of f, $q = p$, smallness of the parameter α turns out to be crucial for such a comparison. Conversely, for any given α we can fulfill the same restriction by claiming that λ_1 is sufficiently big (or, equivalently, that diam Ω is suitably small). It is worth noting that the evolution p-Laplace equation with a continuous low-order term of critical growth admits global solutions only if the domain is sufficiently small – see [31] for a discussion

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of this issue in the case of power growth and [27] for the results on the global existence of solutions for equations with nonpower source terms, as well as for further references.

The proof of the complete quenching of every global solution is based on the analysis of an ordinary differential inequality satisfied by the function $\|u(t)\|_{2,\Omega}$. To derive such an inequality we rely on the energy identity (2.2) and an interpolation inequality of Gagliardo–Nirenberg type. The main steps of the proof are similar to those in [1, Ch. 2] where this method was proposed for the study of the finite-time extinction of solutions of parabolic diffusion-absorption equations. Nonetheless, the specific kind of nonlinearity in equation (P) makes impossible a direct application of the known results.

Organization of the paper. In Section 3 we introduce the regularized problem $(P_{\varepsilon,\eta})$ with the singular term approximated by a sequence of continuous functions depending on two positive parameters. If the growth rate of the source term f is either subcritical, or critical and the parameter α is sufficiently small, then problem $(P_{\varepsilon,\eta})$ has a global in time weak solution. This assertion holds under the minimal assumptions on the regularity of the initial function: $u_0 \in L^2(\Omega)$. For $u_0 \in L^\infty(\Omega)$ we show that the solution is locally in time bounded and, moreover, if the growth of f is either subcritical, or is critical and $\alpha < \lambda_1$, then the maximum of the solution does not depend on time and the parameters of regularization, which allows one to continue the constructed solution to the maximal interval of existence. Finally, for $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ we prove that the sequence of solutions to the regularized problems is uniformly bounded in the norm of $\mathcal{U}$.

Section 4 is devoted to justification of the limit passage with respect to the regularization parameters. This is done in two steps, the first one uses the uniform boundedness of the nonmonotone approximations of the singular term, while the second step relies on the monotonicity of the approximating sequence.

In Section 5 we derive the ordinary differential inequality satisfied by the function $\|u(t)\|_{2,\Omega}$ and show that under suitable conditions on the data every nonnegative function satisfying this inequality vanishes in a finite time, which means the complete quenching. Finally, we prove that the growth conditions on f, sufficient for the finite time quenching, are in a certain sense necessary for the global existence: if $f = \alpha|u|^{q-2}u$ with any $q > \max\{2, p\}$ and $\alpha > 0$, and if

$$\int_\Omega \left(\frac{1}{p}|\nabla u_0|^p + \frac{1}{1-\beta}u_0^{1-\beta} - \frac{\alpha}{q}u_0^q\right) dx < 0,$$

then every weak solution of problem (P) blows up in a finite time: there exists a finite $\widetilde{T} > 0$ such that $\|u(t)\|_{2,\Omega} \to +\infty$ as $t \to \widetilde{T}_-$.

3. Regularized problems

To prove Theorem 2.1 we consider the family of regularized problems. For every $\varepsilon > 0$ we introduce the function

$$g_\varepsilon(s) = \begin{cases} 0 & \text{if } s \leqslant 0, \\ \varepsilon^{-\beta} & \text{if } s \in (0, \varepsilon), \\ s^{-\beta} & \text{if } s \geqslant \varepsilon \end{cases} \tag{3.1}$$

and consider the problems

$$(P_\varepsilon) \quad \begin{cases} \partial_t u_\varepsilon - \Delta_p u_\varepsilon = h_\varepsilon(x, u_\varepsilon) & \text{in } Q_T, \\ u_\varepsilon = 0 & \text{on } \Gamma_T, \\ u_\varepsilon(0,\cdot) = u_0 & \text{in } \Omega \end{cases}$$

with $h_\varepsilon(x,s) := f(x,s) - g_\varepsilon(s) : \Omega \times \mathbb{R} \to \mathbb{R}$. The function $g_\varepsilon(s)$ is bounded but discontinuous. Let us approximate $g_\varepsilon(s)$ by the sequence of continuous functions

$$g_{\varepsilon,\eta}(s) = \begin{cases} \varepsilon^{-\beta}\eta^{-1}s & \text{if } s < \eta, \\ \varepsilon^{-\beta} & \text{if } s \in [\eta, \varepsilon), \qquad \eta \in (0, \varepsilon), \\ s^{-\beta} & \text{if } s \geqslant \varepsilon, \end{cases} \tag{3.2}$$

and consider the sequence of solutions to the problems with two regularization parameters:

$$(P_{\varepsilon,\eta}) \quad \begin{cases} \partial_t u_{\varepsilon,\eta} - \Delta_p u_{\varepsilon,\eta} = h_{\varepsilon,\eta}(x, u_{\varepsilon,\eta}) & \text{in } Q_T, \\ u_{\varepsilon,\eta} = 0 & \text{on } \Gamma_T, \\ u_{\varepsilon,\eta}(0,\cdot) = u_0 & \text{in } \Omega. \end{cases}$$

The nonlinear term $h_{\varepsilon,\eta}(x,s) := f(x,s) - g_{\varepsilon,\eta}(s) : \Omega \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz-continuous with respect to s, which allows us to make use of the known results on the solvability of problem $(P_{\varepsilon,\eta})$.

The double regularization of the discontinuous nonlinear term $\mathbb{1}_{\{u>0\}}u^{-\beta}$ requires an explanation. The sequence of the regularized functions $g_\varepsilon(u_\varepsilon)$ is monotone increasing as $\varepsilon \downarrow 0$, but is unbounded and discontinuous. At the same time, the sequence $g_{\varepsilon,\eta}(u_{\varepsilon,\eta})$ that approximates $g_\varepsilon(u_\varepsilon)$ is bounded and continuous but not monotone. This difference allows us to use different tools for the proofs of convergence of the sequences of solutions of problems $(P_{\varepsilon,\eta})$ and (P_ε) to the solutions of the corresponding limit problems.

3.1. Solvability of problem $(P_{\varepsilon,\eta})$

A solution of the regularized problem is constructed under weaker assumptions on the data. Let us define the function space

$$\mathcal{V} := \{v \in L^p(0,T; W_0^{1,p}(\Omega)) \mid \partial_t v \in L^{p'}(0,T; W^{-1,p'}(\Omega))\}.$$

Definition 3.1. A function $u_{\varepsilon,\eta} \in \mathcal{V}$ is called ***weak solution of problem*** $(\boldsymbol{P_{\varepsilon,\eta}})$ if:

1. $u_{\varepsilon,\eta} \geqslant 0$ a.e. in Q_T,
2. for every test-function $\varphi \in L^p(0,T; W_0^{1,p}(\Omega))$

$$\int_{Q_T} (\varphi \partial_t u_{\varepsilon,\eta} + |\nabla u_{\varepsilon,\eta}|^{p-2} \nabla u_{\varepsilon,\eta} \cdot \nabla \varphi)\, dz = \int_{Q_T} h_{\varepsilon,\eta}(x, u_{\varepsilon,\eta}) \varphi\, dz, \tag{3.3}$$

3. $u_{\varepsilon,\eta}(0,\cdot) = u_0$ a.e. in Ω.

Theorem 3.1 *(Global in time weak solution). Let $u_0 \in L^2(\Omega)$ and $f(x,s)$ satisfies conditions $(F1)$–$(F2)$. Then problem $(P_{\varepsilon,\eta})$ has a global in time weak solution if either $q = \max\{2, p-\delta\}$ with some $\delta > 0$, or $q = p$ and $\alpha < \lambda_1$.*

Problem $(P_{\varepsilon,\eta})$ with the continuous low-order term $h_{\varepsilon,\eta}(x,s)$ has been studied by many authors and various assertions similar to Theorem 3.1 are available in the literature – see, *e.g.*, [28,31] for the case $p \geqslant 2$, or [27] for the anisotropic p-Laplace equation with nonpower low-order terms. We will follow here the proof given in [2,3], which is an adaptation of the classical Faedo–Galerkin method for nonlinear parabolic equations – [20, Ch. 2]. Let $\{\psi_k\}$ be the orthonormal basis of $L^2(\Omega)$ composed of the eigenfunctions of the operator

$$(\psi_k, w)_{H_0^s(\Omega)} = \lambda_k (\psi_k, w)_{2,\Omega} \quad \forall w \in H_0^s(\Omega)$$

with $s \geqslant 1 + d(\frac{1}{2} - \frac{1}{p})$. The approximate solutions to problem $(P_{\varepsilon,\eta})$ are sought in the form

$$u^{(m)}(z) = \sum_{k=1}^{m} c_k^{(m)}(t) \psi_k(x), \tag{3.4}$$

where the coefficients $c_k^{(m)}(t)$ are defined from the relations

$$(\partial_t u^{(m)}, \psi_k)_{2,\Omega} = -(|\nabla u^{(m)}|^{p-2} \nabla u^{(m)}, \nabla \psi_k)_{2,\Omega} + (h_{\varepsilon,\eta}(x, u^{(m)}), \psi_k)_{2,\Omega}, \tag{3.5}$$

$k = 1, \dots, m$. Equalities (3.5) generate the system of m ordinary differential equations for the coefficients $c_k^{(m)}(t)$,

$$\begin{cases} (c_k^{(m)})' = F_k(t, c_1^{(m)}(t), \dots, c_m^{(m)}(t)), \\ c_k^{(m)}(0) = (u_0, \psi_k)_{2,\Omega}, \quad k = 1, \dots, m, \end{cases}$$

which is solvable for any natural m. Uniform *a priori* estimates on the functions $\{u^{(m)}\}$ and the compactness results of [26] allow one to extract a subsequence which converges to a weak solution $u_{\varepsilon,\eta}$ of problem $(P_{\varepsilon,\eta})$:

$$\begin{aligned} &u^{(m)} \underset{m\to\infty}{\rightharpoonup} u_{\varepsilon,\eta} \quad \text{in } L^p(0,T; W_0^{1,p}(\Omega)) \quad \text{and} \quad u^{(m)} \underset{m\to\infty}{\longrightarrow} u_{\varepsilon,\eta} \quad \text{a.e. in } Q_T, \\ &\partial_t u^{(m)} \underset{m\to\infty}{\rightharpoonup} \partial_t u_{\varepsilon,\eta} \quad \text{in } L^{p'}(0,T; W^{-1,p}(\Omega)), \\ &|\nabla u^{(m)}|^{p-2} \nabla u^{(m)} \underset{m\to\infty}{\rightharpoonup} |\nabla u_{\varepsilon,\eta}|^{p-2} \nabla u_{\varepsilon,\eta} \quad \text{in } L^{p'}(Q_T). \end{aligned}$$

Moreover, under the imposed growth conditions on f this solution exists globally in time. The proof of the continuous embedding $\mathcal{V} \hookrightarrow C([0,T]; L^2(\Omega))$ can be found in Barbu [5, Lemma 4.1, Th. 4.2, pp. 167–168], or [20, pp. 158–161]. Moreover, for every $v, w \in \mathcal{V}$ and every $t_1, t_2 \in [0,T]$

$$\int_\Omega v(t_1) w(t_1)\, dx - \int_\Omega v(t_2) w(t_2)\, dx = \int_{t_1}^{t_2} \int_\Omega w \partial_t v\, dz + \int_{t_1}^{t_2} \int_\Omega v \partial_t w\, dz.$$

In particular, in the special case $v = w$

$$\frac{1}{2}\|v(t_2)\|_{2,\Omega}^2 - \frac{1}{2}\|v(t_1)\|_{2,\Omega}^2 = \int_{t_1}^{t_2}\int_\Omega v\,\partial_t v\,dz. \tag{3.6}$$

Theorem 3.2 *(Local in time bounded weak solution). Let $u_0 \in L^\infty(\Omega)$, $u_0 \geqslant 0$ a.e. in Ω, and let $f(x,s)$ satisfy conditions $(F1)$–$(F2)$ with an arbitrary $q \geqslant 1$. Then there exists $T^* > 0$ such that for every $T < T^*$ the solution of problem $(P_{\varepsilon,\eta})$ satisfies the estimate $0 \leqslant v \leqslant M$ a.e. in Q_T with an independent of ε and η constant M.*

Proof. We begin with checking nonnegativity of solutions to problem $(P_{\varepsilon,\eta})$. Let $u_0 \geqslant 0$ a.e. in Ω. Given a solution v of problem $(P_{\varepsilon,\eta})$, we take $\varphi_- = \min\{0, v\} \leqslant 0$ for the test-function in (3.3). Since $g_{\varepsilon,\eta} \geqslant 0$, $f \geqslant 0$, $\varphi_-(t,\cdot) \leqslant 0$ and $\varphi_-(0,\cdot) = 0$, it follows that in every cylinder $Q_t = (0,t)\times\Omega$ with $t < T^*$

$$\frac{1}{2}\|\varphi_-(t)\|_{2,\Omega}^2 \leqslant -\int_{Q_t}\left(|\nabla\varphi_-|^p + g_{\varepsilon,\eta}(v)\varphi_-\right)dz + \int_{Q_t} f(x,v)\varphi_-\,dz \leqslant 0,$$

whence $\min\{0,v\} = 0$ a.e. in Q_t for every $t < T^*$.

Without loss of generality we may assume that condition $(F2)$ is fulfilled with $q \geqslant p$, otherwise we make use of Young's inequality to get

$$0 \leqslant f(x,s) \leqslant \alpha|s|^{q-1} + C_\alpha \leqslant \alpha\frac{q-1}{p-1}|s|^{p-1} + \frac{p-1}{p-q} + C_\alpha.$$

Let us fix an arbitrary constant $L > 1$ and consider the auxiliary problem

$$\begin{cases} \partial_t u - \Delta_p u + g_{\varepsilon,\eta}(u) = f_L(x,u) & \text{in } Q_T, \\ u = 0 & \text{on } \Gamma_T, \\ u(0,\cdot) = u_0 & \text{in } \Omega \end{cases} \tag{3.7}$$

with the function f_L defined by

$$f_L(x,u) = \begin{cases} f(x,u) & \text{if } |u| < L, \\ f(x, L\operatorname{sign} u) & \text{if } |u| \geqslant L. \end{cases}$$

This function satisfies the growth condition

$$0 \leqslant f_L(x,u) \leqslant \alpha\min\left\{|u|^{q-1}, L^{q-1}\right\} + C_\alpha \leqslant \alpha L^{q-2}|u| + C_\alpha. \tag{3.8}$$

By Theorem 3.1 for every $L > 1$ problem (3.7) has a local in time solution v. Set

$$\Psi(t) = K\,e^{\delta t}, \qquad K = \|u_0\|_{\infty,\Omega}, \qquad \delta = \alpha L^{q-2} + \frac{C_\alpha}{\|u_0\|_{\infty,\Omega}}.$$

It is easy to see that

$$\partial_t v - \Delta_p v \leqslant \alpha L^{q-2}|v| + C_\alpha \quad \text{in } L^{p'}\left(0,T; W^{-1,p'}(\Omega)\right),$$

and

$$\begin{cases} \partial_t\Psi - \Delta_p\Psi = \delta K e^{\delta t} \geqslant \alpha L^{q-2}\Psi + C_\alpha & \text{in } (0,T]\times\Omega, \\ \Psi - \|u_0\|_{\infty,\Omega} \geqslant 0 \quad \text{in } \Omega, \quad \Psi > 0 \quad \text{on } \Gamma. \end{cases}$$

It follows that for every nonnegative test-function $\varphi \in L^p(0,T; W_0^{1,p}(\Omega))$

$$\int_{Q_T}\left(\partial_t(v-\Psi)\varphi + \left(|\nabla v|^{p-2}\nabla v - |\nabla\Psi|^{p-2}\nabla\Psi\right)\cdot\nabla\varphi\right)dz \leqslant \alpha L^{q-2}\int_{Q_T}\left(|v| - \Psi\right)\varphi\,dz.$$

Choosing $\varphi_+ := \max\{0, v-\Psi\} \in L^p(0,T; W_0^{1,p}(\Omega))$ for the test-function and applying the well-known inequality

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)\cdot(\xi - \eta) \geqslant 0 \tag{3.9}$$

we conclude that

$$\frac{1}{2}\|\varphi_+(t)\|_{2,\Omega}^2 \leqslant \alpha L^{q-2}\int_0^t \|\varphi_+(s)\|_{2,\Omega}^2\,ds.$$

Since $\|\varphi_+(0)\|_{2,\Omega}=0$, it follows from the Gronwall inequality that $\varphi_+=0$ a.e. in Q_T. Let us choose $L=1+\|u_0\|_{\infty,\Omega}$ and then fix T by the condition

$$L=1+\|u_0\|_{\infty,\Omega}\geqslant\Psi(T)=\|u_0\|_{\infty,\Omega}e^{\delta T}\quad\Leftrightarrow\quad T=\frac{1}{\delta}\ln\left(1+\frac{1}{\|u_0\|_{\infty,\Omega}}\right). \tag{3.10}$$

For every $t\in[0,T]$ we have $0\leqslant v(t,x)\leqslant L$ a.e. in Ω, which means that v is in fact a solution of problem (3.7) with the right-hand side f independent of L, that is, a nonnegative solution of problem $(P_{\varepsilon,\eta})$. Taking $v(T,x)$ for the initial datum and repeating the comparison procedure with the new function

$$\Psi(t)=\big\|v(T)\big\|_{\infty,\Omega}e^{\delta(t-T)},\qquad\delta=\alpha L^{q-2}+\frac{C_\alpha}{\|u(T)\|_{\infty,\Omega}},\qquad L'=1+\big\|v(T)\big\|_{\infty,\Omega},$$

we extend the solution $v(t,x)$ to the cylinder $[T,T']\times\Omega$ with T' calculated from $\|v(T)\|_{\infty,\Omega}$ and L' chosen according to (3.10), and conclude that for every $t\in[T,T']$ and a.e. $x\in\Omega$ we have $0\leqslant v(t,x)\leqslant L'$. This process is continued until the interval $(0,T^*)$ is exhausted. □

Theorem 3.3 *(Global in time bounded weak solution). Let in the conditions of Theorem 3.2 $p=q$ and*

$$0\leqslant\alpha<\lambda_1. \tag{3.11}$$

Then the constructed solution v of problem $(P_{\varepsilon,\eta})$ is global in time and there exists a constant M such that $0\leqslant v\leqslant M$ a.e. in Q_T for every $T>0$. The constant M depends on p, $\|u_0\|_{\infty,\Omega}$, α, λ_1, but is independent of ε, η and T.

Proof. By Theorem 3.2 the solutions of problem $(P_{\varepsilon,\eta})$ are nonnegative. By virtue of (3.3) for every nonnegative function $\varphi\in L^p(0,T;W_0^{1,p}(\Omega))$ the solution of problem $(P_{\varepsilon,\eta})$ satisfies the inequality

$$\int_{Q_T}\big[\partial_t v\varphi+|\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\big]\,dz=\int_{Q_T}\big(f(x,v)-g_{\varepsilon,\eta}(v)\big)\varphi\,dz\leqslant\int_{Q_T}\big(\alpha|v|^{p-1}+C_\alpha\big)\varphi\,dz. \tag{3.12}$$

Let $\widetilde{\Omega}\subset\mathbb{R}^d$ be a regular domain that contains Ω: $\Omega\Subset\widetilde{\Omega}$. Denote by ψ and $\lambda_1(\widetilde{\Omega})$ the first nonnegative normalized eigenfunction and the first eigenvalue of the problem

$$\Delta_p\psi+\lambda_1(\widetilde{\Omega})|\psi|^{p-2}\psi=0\quad\text{in }\widetilde{\Omega},\qquad\psi=0\quad\text{on }\partial\widetilde{\Omega},\qquad\int_\Omega|\psi|^p\,dx=1.$$

It is known that ψ is strictly positive in Ω and $\lambda_1(\widetilde{\Omega})\leqslant\lambda_1(\Omega)$. Moreover, $\lambda_1(\widetilde{\Omega})$ continuously depends on $\widetilde{\Omega}$ and $\lambda_1(\widetilde{\Omega})\to\lambda_1(\Omega)$ as $\widetilde{\Omega}\to\Omega$ in the Hausdorff complementary topology [21, Th. 3.2]. The last property together with the assumption (3.11) allow us to choose the domain $\widetilde{\Omega}$ in such a way that

$$\alpha<\alpha+\delta\leqslant\lambda_1(\widetilde{\Omega})\leqslant\lambda_1(\Omega),\quad\text{with some }\delta>0.$$

Let us denote $\mu=\inf_\Omega\psi>0$ and consider the function $\Psi=K\psi$ with $K=const>0$ such that

$$\begin{cases}\Psi(x)=K\psi(x)\geqslant K\mu\geqslant\|u_0\|_{L^\infty(\Omega)}\quad\text{in }\Omega,\\ \delta(K\mu)^{p-1}\geqslant C_\alpha.\end{cases} \tag{3.13}$$

For every nonnegative $\varphi\in L^p(0,T;W_0^{1,p}(\Omega))$,

$$\begin{aligned}\int_{Q_T}\big[\partial_t\Psi\varphi+|\nabla\Psi|^{p-2}\nabla\Psi\cdot\nabla\varphi\big]\,dz&=\frac{\lambda_1(\widetilde{\Omega})}{\alpha}\left(\alpha\int_{Q_T}\Psi^{p-1}\varphi\,dz\right)\\&\geqslant\alpha\int_{Q_T}\Psi^{p-1}\varphi\,dz+\delta(K\mu)^{p-1}\int_{Q_T}\varphi\,dz\\&\geqslant\alpha\int_{Q_T}\Psi^{p-1}\varphi\,dz+C_\alpha\int_{Q_T}\varphi\,dz.\end{aligned} \tag{3.14}$$

Taking in (3.12), (3.14) $\varphi_+ = \max\{0, v - \Psi\} \geqslant 0$ and subtracting the results we arrive at the inequality

$$\frac{1}{2}\|\varphi_+(t)\|^2_{2,\Omega} \leqslant \alpha \int_0^t \int_\Omega \left(|v|^{p-1} - \Psi^{p-1}\right)\varphi_+ \, dz - \int_0^t \int_\Omega \left(|\nabla v|^{p-2}\nabla v - |\nabla \Psi|^{p-2}\nabla \Psi\right) \cdot (\nabla v - \nabla \Psi)\, dz. \tag{3.15}$$

Let us consider separately the cases $p \in (1, 2]$ and $p > 2$. Assume first that $p \in (1, 2]$. The second term on the right-hand side of (3.15) is nonpositive because of (3.9). The first term is estimated as follows: at every point where $\varphi_+ > 0$,

$$\begin{aligned} v^{p-1} - \Psi^{p-1} &= \int_0^1 \frac{d}{d\theta}\left(\theta v + (1-\theta)\Psi\right)^{p-1} d\theta \\ &= (p-1)\left(\int_0^1 \left(\theta v + (1-\theta)\Psi\right)^{p-2} d\theta\right)\varphi_+ \\ &\leqslant \varphi_+ \begin{cases} (p-1)v^{p-2} & \text{if } p > 2, \\ (p-1)\Psi^{p-2} & \text{if } 1 < p \leqslant 2. \end{cases} \end{aligned} \tag{3.16}$$

Plugging (3.16) into (3.15) we have that in the case $p \in (1, 2]$ for all $t \in (0, T)$

$$\begin{cases} \frac{1}{2}\|\varphi_+(t)\|^2_{2,\Omega} \leqslant \alpha(p-1)\inf_\Omega \Psi^{p-2}\|\varphi_+\|^2_{2,Q_t} \leqslant \alpha(p-1)(K\mu)^{p-2}\|\varphi_+\|^2_{2,Q_t}, \\ \|\varphi_+(0)\|^2_{2,\Omega} = 0. \end{cases}$$

By Gronwall's inequality $\|\varphi_+(t)\|_{2,\Omega} = 0$, whence $\varphi_+ = 0$ a.e. in Q_T and $v \leqslant \Psi$ a.e. in Q_T. Let us assume now that $p > 2$. By Theorem 3.2 the solution is locally in time bounded and there exist $T > 0$ and M such that $0 \leqslant v(t, x) \leqslant M$ for every $t \in [0, T]$ and a.e. $x \in \Omega$. Gathering (3.15), (3.16) we arrive at Gronwall's inequality for $\|\varphi_+\|^2_{2,Q_t}$: for every $t \in (0, T)$

$$\frac{1}{2}\|\varphi_+(t)\|^2_{2,\Omega} \leqslant \alpha(p-1)\|v(t)\|^{p-2}_{\infty,\Omega}\|\varphi_+\|^2_{2,Q_t} \leqslant \alpha(p-1)M^{p-2}\|\varphi_+\|^2_{2,Q_t}.$$

It follows that $\|\varphi_+(t)\|_{2,\Omega} = 0$ for all $t \in [0, T]$. Let us consider problem $(P_{\varepsilon,\eta})$ with $v(T, x)$ taken for the initial datum. Since the barrier function $\Psi(x)$ is independent of t, both conditions in (3.13) are already fulfilled and the same arguments show that the inequality $0 \leqslant v \leqslant \Psi(x)$ a.e. in Ω holds on the interval $[T, 2T]$. Iterating, we extend the same estimate to the cylinder of arbitrary height. □

Corollary 3.1. *Let in the conditions of Theorem 3.1 $u_0 \in L^\infty(\Omega)$, $u_0 \geqslant 0$ a.e. in Ω. If $q < p$, then for every $T > 0$ and arbitrary positive α, C_α the solution of problem $(P_{\varepsilon,\eta})$ satisfies the estimate $0 \leqslant v \leqslant M$ in Q_T with an independent of ε, η and T constant M.*

Proof. By Young's inequality, for every $\gamma > 0$

$$\left|f(x, s)\right| \leqslant \alpha|s|^{q-1} + C_\alpha \leqslant \gamma|s|^{p-1} + C(\gamma).$$

The assertion follows from Theorem 3.3 if we choose γ according to condition (3.11). □

3.2. Higher regularity of solutions to problem $(P_{\varepsilon,\eta})$

Theorem 3.4. *Let the conditions of Theorem 3.2 be fulfilled. If we additionally assume that $u_0 \in W^{1,p}_0(\Omega)$, then $u_{\varepsilon,\eta} \in \mathcal{U}$ and for a.e. $t \in (0, T^*)$*

$$\|\partial_t u_{\varepsilon,\eta}\|^2_{2,Q_t} + \frac{1}{p}\|\nabla u_{\varepsilon,\eta}(t)\|^p_{p,\Omega} + \int_\Omega \int_0^{u_{\varepsilon,\eta}(t)} g_{\varepsilon,\eta}(s)\, ds\, dx - \int_\Omega \int_0^{u_{\varepsilon,\eta}(t)} f(x, s)\, ds\, dx \leqslant \frac{1}{p}\|\nabla u_0\|^p_{p,\Omega} + \int_\Omega \int_0^{u_0} g_{\varepsilon,\eta}(s)\, ds\, dx - \int_\Omega \int_0^{u_0} f(x, s)\, ds\, dx. \tag{3.17}$$

Proof. The solution of problem $(P_{\varepsilon,\eta})$ is obtained as the limit of the sequence $\{u^{(m)}\}$ defined by (3.4). Multiplying each of Eqs. (3.5) by $(c^{(k)})'$, summing in $k = 1, 2, \dots, m$ and integrating the result over the interval $(0, t)$ with $t < T^*$, we obtain the equality

$$\|\partial_t u^{(m)}\|^2_{2,Q_t} + \frac{1}{p}\|\nabla u^{(m)}(t)\|^p_{p,\Omega} + \int_\Omega \int_0^{u^{(m)}(t)} g_{\varepsilon,\eta}(s)\,ds\,dx - \int_\Omega \int_0^{u^{(m)}(t)} f(x,s)\,ds\,dx$$
$$= \frac{1}{p}\|\nabla u_0^{(m)}\|^p_{p,\Omega} + \int_\Omega \int_0^{u_0^{(m)}} g_{\varepsilon,\eta}(s)\,ds\,dx - \int_\Omega \int_0^{u_0^{(m)}} f(x,s)\,ds\,dx. \tag{3.18}$$

By the definition $0 \leqslant g_{\varepsilon,\eta}(s) \leqslant s^{-\beta}$ for $s > 0$ and $g_{\varepsilon,\eta}(s) = 0$ if $s \leqslant 0$. Using Young's inequality we estimate

$$\int_0^{u_0^{(m)}} g_{\varepsilon,\eta}(s)\,ds \leqslant \frac{1}{1-\beta}\big(\max\{0, u_0^{(m)}\}\big)^{1-\beta} \leqslant C\big(1 + (u_0^{(m)})^2\big) \leqslant C\big(1 + \|u_0\|^2_{2,\Omega}\big).$$

The solutions of problem $(P_{\varepsilon,\eta})$ are uniformly bounded on the interval $[0, t]$ and can be obtained as the solution of problem (3.7) with the auxiliary dummy parameter L. This means that the last term on the right-hand side of (3.18) has to be estimated only for a function f_L satisfying (3.8). Since

$$\int_\Omega \int_0^v f(x,s)\,ds\,dx \leqslant \big(\alpha L^{q-2} + C_\alpha\big) \int_\Omega \big(v^2 + |v|\big)\,dx \leqslant C\big(1 + \|v\|^2_{2,\Omega}\big),$$

the last term on the right-hand side of (3.18) is estimated by $\|u_0\|^2_{2,\Omega}$ and $\|u^{(m)}\|^2_{2,\Omega}$. To estimate the latter we multiply each of Eqs. (3.5) by $c_k(t)$ and sum up the results. This leads to the inequality

$$\frac{1}{2}\|u^{(m)}(t)\|_{2,\Omega} + \|\nabla u^{(m)}\|^p_{p,Q_t} \leqslant \frac{1}{2}\|u_0\|^2_{2,\Omega} + \int_{Q_t} u^{(m)} f_L\big(x, u^{(m)}\big)\,dz.$$

The uniform in m and $t \in [0, T]$ estimate on $\|u^{(m)}(t)\|_{2,\Omega}$ follows from the Gronwall lemma. By virtue of (3.18) $\partial_t u^{(m)}$ are uniformly bounded in $L^2(Q_T)$ and $|\nabla u^{(m)}|$ are uniformly bounded in $L^\infty(0, T; L^p(\Omega))$. By [26, Th. 5] the sequence $\{u^{(m)}\}$ contains a subsequence which converges in $L^q(Q_T)$ with some $q > 1$ and a.e. in Q_T to a function u. By construction, this function coincides with $u_{\varepsilon,\eta}$, the solution of problem $(P_{\varepsilon,\eta})$. Letting $m \to \infty$ in (3.18), using the pointwise convergence $u^{(m)} \to u_{\varepsilon,\eta}$ and applying the Fatou lemma and the dominated convergence theorem, we obtain (3.17). □

3.3. Comparison and monotonicity

Let T^* be the value from the conditions of Theorem 3.1 and $T < T^*$ any fixed number.

Lemma 3.1. *Under the conditions of Theorem 3.2 the sequence $\{u_{\varepsilon,\eta}\}$ is monotone decreasing as $\eta \to 0$: if $\eta > \eta' > 0$, then $u_{\varepsilon,\eta} \geqslant u_{\varepsilon,\eta'}$ a.e. in Q_T.*

Proof. Let $0 < \eta' < \eta$. Denote by $u_{\varepsilon,\eta}, u_{\varepsilon,\eta'} \in \mathcal{U}$ the corresponding solutions of problem $(P_{\varepsilon,\eta})$. The function $(u_{\varepsilon,\eta'} - u_{\varepsilon,\eta})_+ = \max\{0, u_{\varepsilon,\eta'} - u_{\varepsilon,\eta}\} \in \mathcal{U}$ is an admissible test-function in the integral identity (3.3). By the definition, $g_{\varepsilon,\eta}(s)$ is monotone decreasing as a function of η: for every $\eta' < \eta$ we have $g_{\varepsilon,\eta'}(s) \geqslant g_{\varepsilon,\eta}(s)$ for all $s \in \mathbb{R}$. This yields the inequality

$$\big(g_{\varepsilon,\eta'}(u_{\varepsilon,\eta'}) - g_{\varepsilon,\eta}(u_{\varepsilon,\eta})\big)(u_{\varepsilon,\eta'} - u_{\varepsilon,\eta})_+ \geqslant \big(g_{\varepsilon,\eta}(u_{\varepsilon,\eta'}) - g_{\varepsilon,\eta}(u_{\varepsilon,\eta})\big)(u_{\varepsilon,\eta'} - u_{\varepsilon,\eta})_+$$
$$\geqslant -\frac{\beta}{\varepsilon^{\beta+1}}(u_{\varepsilon,\eta'} - u_{\varepsilon,\eta})^2_+.$$

By Theorem 3.2 $u_{\varepsilon,\eta}$, $u_{\varepsilon,\eta'}$ are uniformly bounded in the cylinder Q_T by a constant M. Since $f(x, s)$ is locally Lipschitz-continuous, it is globally Lipschitz-continuous on the interval $s \in [0, M]$:

$$\big(f(x, u_{\varepsilon,\eta'}) - f(x, u_{\varepsilon,\eta})\big)(u_{\varepsilon,\eta'} - u_{\varepsilon,\eta})_+ \leqslant L\,(u_{\varepsilon,\eta'} - u_{\varepsilon,\eta})^2_+$$

with an independent of η and ε constant L. Plugging these inequalities into identities (3.3) for $u_{\varepsilon,\eta}$ and $u_{\varepsilon,\eta'}$ and dropping the terms of constant sign, we find that for every $t \in (0, T]$

$$\frac{1}{2}\int_{Q_t} \partial_t(u_{\varepsilon,\eta'} - u_{\varepsilon,\eta})^2_+\,dz \leqslant \left(L + \frac{\beta}{\varepsilon^{\beta+1}}\right)\int_{Q_t} (u_{\varepsilon,\eta'} - u_{\varepsilon,\eta})^2_+\,dz.$$

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Let us introduce the function $\Phi(t)=\|(u_{\varepsilon,\eta'}-u_{\varepsilon,\eta})_+(t)\|_{2,\Omega}^2$. Since $u_{\varepsilon,\eta'}(0,x)=u_{\varepsilon,\eta}(0,x)$, then $\Phi(0)=0$ and

$$0\leqslant\Phi(t)\leqslant 2\left(L+\frac{\beta}{\varepsilon^{\beta+1}}\right)\int_0^t\Phi(s)\,ds,\quad\text{for a.e. } t\in(0,T).\tag{3.19}$$

By the Gronwall lemma $\int_0^t\Phi(s)\,ds=0$, thence $\Phi(t)=0$ for a.e. $t\in(0,T)$. □

Corollary 3.2. *Under the conditions of Theorem 3.4 the solution of problem $(P_{\varepsilon,\eta})$ is unique.*

Proof. The assertion follows from the proof of Lemma 3.1: if problem $(P_{\varepsilon,\eta})$ admits two different solutions $u^{(1)}_{\varepsilon,\eta}$, $u^{(2)}_{\varepsilon,\eta}$, the function $\Phi(t)=\|(u^{(1)}_{\varepsilon,\eta}-u^{(2)}_{\varepsilon,\eta})_+(t)\|^2_{2,\Omega}$ satisfies (3.19). □

4. Existence of weak solutions of problems (P_ε) and (P)

4.1. Weak solution of problem (P_ε)

We are now in position to prove the following assertion.

Theorem 4.1. *Let the functions u_0 and f satisfy conditions (1.1) and $(F1)$–$(F2)$. Then there exists $T^*>0$ such that for every $T<T^*$, problem (P_ε) has a unique weak solution $u_\varepsilon\in\mathcal{U}$. Moreover, u_ε satisfies the following energy relations: $\forall t_1,t_2\in[0,T]$,*

$$\frac12\big\|u_\varepsilon(t_2)\big\|^2_{2,\Omega}-\frac12\big\|u_\varepsilon(t_1)\big\|^2_{2,\Omega}+\int_{t_1}^{t_2}\int_\Omega|\nabla u_\varepsilon|^p\,dz+\int_{t_1}^{t_2}\int_\Omega g_\varepsilon(u_\varepsilon)u_\varepsilon\,dz=\int_{t_1}^{t_2}\int_\Omega f(x,u_\varepsilon)u_\varepsilon\,dz\tag{4.1}$$

and for a.e. $t\in(0,T)$,

$$\begin{aligned}&\|\partial_t u_\varepsilon\|^2_{2,Q_t}+\frac1p\int_\Omega|\nabla u_\varepsilon|^p(t)\,dx+\int_\Omega\int_0^{u_\varepsilon(t)}g_\varepsilon(s)\,ds\,dx-\int_\Omega\int_0^{u_\varepsilon(t)}f(x,s)\,ds\,dx\\&\quad\leqslant\frac1p\int_\Omega|\nabla u_0|^p\,dx+\int_\Omega\int_0^{u_0}g_\varepsilon(s)\,ds\,dx-\int_\Omega\int_0^{u_0}f(x,s)\,ds\,dx.\end{aligned}\tag{4.2}$$

4.2. The limit as $\eta\to0$

The uniform in η estimates on the solutions of problem $(P_{\varepsilon,\eta})$ allow us to extract a subsequence such that

$$u_{\varepsilon,\eta}\ \underset{\eta\to0}{\overset{*}{\rightharpoonup}}\ u_\varepsilon\quad\text{in } L^\infty\big(0,T;W_0^{1,p}(\Omega)\big)\cap L^\infty(Q_T),\tag{4.3}$$

$$\partial_t u_{\varepsilon,\eta}\ \underset{\eta\to0}{\rightharpoonup}\ \partial_t u_\varepsilon\quad\text{in } L^2(Q_T),\tag{4.4}$$

$$u_{\varepsilon,\eta}\underset{\eta\to0}{\longrightarrow}u_\varepsilon\quad\text{in } C\big([0,T];L^2(\Omega)\big),\tag{4.5}$$

$$u_{\varepsilon,\eta}\underset{\eta\to0}{\longrightarrow}u_\varepsilon\quad\text{a.e. in } Q_T,\tag{4.6}$$

$$|\nabla u_{\varepsilon,\eta}|^{p-2}\nabla u_{\varepsilon,\eta}\ \underset{\eta\to0}{\rightharpoonup}\ V_\varepsilon\quad\text{in } L^{p'}(Q_T)^d.\tag{4.7}$$

By convention we use the same notation for the sequence and the extracted subsequences. The functions $g_{\varepsilon,\eta}(u_{\varepsilon,\eta})$ are bounded uniformly with respect to η, which is why there exists $\phi_\varepsilon\in L^\infty(Q_T)$ such that

$$g_{\varepsilon,\eta}(u_{\varepsilon,\eta})\ \underset{\eta\to0}{\overset{*}{\rightharpoonup}}\ \phi_\varepsilon\quad\text{in } L^2(Q_T).\tag{4.8}$$

Since f is a Carathéodory function, it follows from (4.6) that

$$f(x,u_{\varepsilon,\eta})\underset{\eta\to0}{\longrightarrow}f(x,u_\varepsilon)\quad\text{in } L^{p'}(Q_T).\tag{4.9}$$

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Gathering (4.4), (4.7), (4.8), (4.9) and passing to the limit as $\eta \to 0$ in (3.3), we conclude that for every test-function $\varphi \in L^p(0,T;W_0^{1,p}(\Omega))$

$$\int_{Q_T} \partial_t u_\varepsilon \varphi \, dz + \int_{Q_T} V_\varepsilon \cdot \nabla\varphi \, dz + \int_{Q_T} \phi_\varepsilon \varphi \, dz = \int_{Q_T} f(x,u_\varepsilon)\varphi \, dz. \tag{4.10}$$

The limit vector-valued function V_ε is identified as follows:

Proposition 4.1. $V_\varepsilon = |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon$ a.e. *in* Q_T.

The proof uses the classical "Minty argument" (Minty [22]) based on monotonicity of the p-Laplace operator. It is well-known (see, *e.g.*, [20, pp. 160–161], [2,27]) and can be omitted.

4.3. Proof of Theorem 4.1

By (4.10) and Proposition 4.1 $u_\varepsilon \in \mathcal{U}$ and satisfies

$$\partial_t u_\varepsilon - \Delta_p u_\varepsilon + \phi_\varepsilon = f(x,u_\varepsilon) \quad \text{in } L^{p'}\big(0,T,W^{-1,p'}(\Omega)\big). \tag{4.11}$$

Let us check that $\phi_\varepsilon = g_\varepsilon(u_\varepsilon)$ a.e. in Q_T. By (4.6)

$$g_{\varepsilon,\eta}(u_{\varepsilon,\eta}) \xrightarrow[\eta\to 0]{} g_\varepsilon(u_\varepsilon) \quad \text{a.e. in}\, Q_T \cap \{u_\varepsilon > 0\}, \tag{4.12}$$

which allows us to represent the limit function ϕ_ε in the form

$$\phi_\varepsilon = g_\varepsilon(u_\varepsilon) + \mathbb{1}_{\{u_\varepsilon=0\}}\chi_\varepsilon \quad \text{a.e. in } Q_T \tag{4.13}$$

with a function χ_ε to be defined. Since $g_{\varepsilon,\eta}$ is nonnegative, it follows from (4.8) that ϕ_ε is also nonnegative a.e. in Q_T. Thus, $\mathbb{1}_{\{u_\varepsilon=0\}}\chi_\varepsilon \geqslant 0$ a.e. in Q_T. Let us take $g_{\varepsilon,\eta}(u_\varepsilon) \in L^p(0,T,W_0^{1,p}(\Omega))$ for the test-function in identity (4.11) for u_ε,

$$\int_{Q_T} \partial_t u_\varepsilon g_{\varepsilon,\eta}(u_\varepsilon)\, dz + \int_{Q_T} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla\big(g_{\varepsilon,\eta}(u_\varepsilon)\big)\, dz + \int_{Q_T} \phi_\varepsilon g_{\varepsilon,\eta}(u_\varepsilon)\, dz = \int_{Q_T} f(x,u_\varepsilon) g_{\varepsilon,\eta}(u_\varepsilon)\, dz,$$

introduce the functions

$$G_{\varepsilon,\eta}(w) := \int_0^w g_{\varepsilon,\eta}(v)\, dv, \qquad G_\varepsilon(w) := \int_0^w g_\varepsilon(v) dv$$

and rewrite the previous equality in the form

$$\int_\Omega G_{\varepsilon,\eta}\big(u_\varepsilon(T)\big)\, dx - \int_\Omega G_{\varepsilon,\eta}(u_0)\, dx + \int_{Q_T} |\nabla u_\varepsilon|^p g'_{\varepsilon,\eta}(u_\varepsilon)\, dz + \int_{Q_T} \phi_\varepsilon g_{\varepsilon,\eta}(u_\varepsilon)\, dz = \int_{Q_T} f(x,u_\varepsilon) g_{\varepsilon,\eta}(u_\varepsilon)\, dz. \tag{4.14}$$

By the Lebesgue dominated convergence theorem

$$\int_\Omega G_{\varepsilon,\eta}\big(u_\varepsilon(T)\big)\, dx \xrightarrow[\eta\to 0]{} \int_\Omega G_\varepsilon\big(u_\varepsilon(T)\big)\, dx \quad \text{and} \quad \int_\Omega G_{\varepsilon,\eta}(u_0)\, dx \xrightarrow[\eta\to 0]{} \int_\Omega G_\varepsilon(u_0)\, dx, \tag{4.15}$$

$$\int_{Q_T} \phi_\varepsilon g_{\varepsilon,\eta}(u_\varepsilon)\, dz \xrightarrow[\eta\to 0]{} \int_{Q_T} \big(g_\varepsilon(u_\varepsilon)\big)^2\, dz, \tag{4.16}$$

$$\int_{Q_T} f(x,u_\varepsilon) g_{\varepsilon,\eta}(u_\varepsilon)\, dz \xrightarrow[\eta\to 0]{} \int_{Q_T} f(x,u_\varepsilon) g_\varepsilon(u_\varepsilon)\, dz. \tag{4.17}$$

Lemma 4.1. *For every fixed* $\varepsilon > 0$

$$\int_{Q_T} |\nabla u_\varepsilon|^p g'_{\varepsilon,\eta}(u_\varepsilon)\, dz \xrightarrow[\eta\to 0]{} \int_{Q_T} \mathbb{1}_{\{u_\varepsilon>0\}} |\nabla u_\varepsilon|^p g'_\varepsilon(u_\varepsilon)\, dz. \tag{4.18}$$

Proof. By the definition $0 \leqslant g_{\varepsilon,\eta}(u_\varepsilon) \leqslant \varepsilon^{-\beta}$ and

$$g'_{\varepsilon,\eta}(u_\varepsilon) = \begin{cases} \varepsilon^{-\beta}\eta^{-1} & \text{if } u_\varepsilon < \eta, \\ 0 & \text{if } u_\varepsilon \in [\eta, \varepsilon), \\ -\beta u_\varepsilon^{-(\beta+1)} & \text{if } u_\varepsilon \geqslant \varepsilon, \end{cases} \qquad g''_{\varepsilon,\eta}(u_\varepsilon) = \begin{cases} 0 & \text{if } u_\varepsilon < \varepsilon, \\ \beta(\beta+1)u_\varepsilon^{-(\beta+2)} & \text{if } u_\varepsilon \geqslant \varepsilon, \end{cases} \tag{4.19}$$

which means that to prove (4.18) it suffices to show that

$$I_{\varepsilon,\eta} := \varepsilon^{-\beta}\eta^{-1} \int\limits_{Q_T \cap \{0 \leqslant u_\varepsilon \leqslant \eta\}} |\nabla u_\varepsilon|^p \, dz \xrightarrow[\eta \to 0]{} 0.$$

Let us fix some $\eta \in (0, \varepsilon)$ and take $g'_{\varepsilon,\eta}(u_\varepsilon) g_{\varepsilon,\eta}(u_\varepsilon) \in L^p(0, T, W_0^{1,p}(\Omega))$ for the test-function in (4.11):

$$\frac{1}{2}\int\limits_\Omega g^2_{\varepsilon,\eta}\big(u_\varepsilon(T)\big)\, dx + \int\limits_{Q_T} |\nabla u_\varepsilon|^p \big(g_{\varepsilon,\eta}(u_\varepsilon) g''_{\varepsilon,\eta}(u_\varepsilon) + \big(g'_{\varepsilon,\eta}(u_\varepsilon)\big)^2\big)\, dz + \int\limits_{Q_T} \phi_\varepsilon g'_{\varepsilon,\eta}(u_\varepsilon) g_{\varepsilon,\eta}(u_\varepsilon)\, dx$$
$$= \frac{1}{2}\int\limits_\Omega g^2_{\varepsilon,\eta}(u_0)\, dx + \int\limits_{Q_T} f(x, u_\varepsilon) g'_{\varepsilon,\eta}(u_\varepsilon) g_{\varepsilon,\eta}(u_\varepsilon)\, dz.$$

By assumption $f(x, s)$ is nonnegative, locally Lipschitz-continuous with respect to s and $f(x, 0) = 0$. Since the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty(Q_T)$, there exists a finite constant $L > 0$, independent of ε, such that $|f(x, u_\varepsilon)| = |f(x, u_\varepsilon) - f(x, 0)| \leqslant L|u_\varepsilon|$. Using (4.19), (4.12) and Lipschitz-continuity of f, and dropping the sign-definite terms in the previous equality, we arrive at the inequality

$$\frac{\varepsilon^{-2\beta}}{\eta^2} \int\limits_{Q_T \cap \{0 < u_\varepsilon \leqslant \eta\}} |\nabla u_\varepsilon|^p \, dz \leqslant \frac{1}{2}\int\limits_\Omega g^2_{\varepsilon,\eta}(u_0)\, dx + \frac{\beta}{\varepsilon^{2\beta+1}} \int\limits_{Q_T \cap \{u_\varepsilon \geqslant \varepsilon\}} \phi_\varepsilon \, dz + \frac{1}{\varepsilon^{2\beta}} \int\limits_{Q_T \cap \{0 < u_\varepsilon \leqslant \eta\}} \frac{f(x, u_\varepsilon)}{\eta}\, dz$$
$$\leqslant C|\Omega|\big(1 + (1 + L)T\big)\varepsilon^{-2\beta}\big(1 + \varepsilon^{-(1+\beta)}\big)$$

with an independent of η and ε constant C. It follows that $I_{\varepsilon,\eta} \to 0$ as $\eta \to 0$, as required. □

Letting $\eta \to 0$ in (4.14) and taking into account (4.15)–(4.18), we obtain

$$\int\limits_\Omega G_\varepsilon\big(u_\varepsilon(T)\big)\, dx - \int\limits_\Omega G_\varepsilon(u_0)\, dx + \int\limits_{Q_T} \mathbb{1}_{\{u_\varepsilon > 0\}} |\nabla u_\varepsilon|^p g'_\varepsilon(u_\varepsilon)\, dz + \int\limits_{Q_T} \big(g_\varepsilon(u_\varepsilon)\big)^2\, dz = \int\limits_{Q_T} f(x, u_\varepsilon) g_\varepsilon(u_\varepsilon)\, dz.$$

At the same time, (3.3) with the test-function $g_{\varepsilon,\eta}(u_{\varepsilon,\eta}) \in L^p(0, T; W_0^{1,p}(\Omega))$ gives

$$\int\limits_\Omega G_{\varepsilon,\eta}\big(u_{\varepsilon,\eta}(T)\big)\, dx - \int\limits_\Omega G_{\varepsilon,\eta}(u_0)\, dx + \int\limits_{Q_T} |\nabla u_{\varepsilon,\eta}|^p g'_{\varepsilon,\eta}(u_{\varepsilon,\eta})\, dz + \int\limits_{Q_T} \big(g_{\varepsilon,\eta}(u_{\varepsilon,\eta})\big)^2\, dz$$
$$= \int\limits_{Q_T} f(x, u_{\varepsilon,\eta}) g_{\varepsilon,\eta}(u_{\varepsilon,\eta})\, dz.$$

Using (4.6) and the assumption $f(x, 0) = 0$ and then applying the Lebesgue dominated convergence theorem we find that

$$\int\limits_\Omega G_{\varepsilon,\eta}\big(u_{\varepsilon,\eta}(T)\big)\, dx \xrightarrow[\eta \to 0]{} \int\limits_\Omega G_\varepsilon\big(u_\varepsilon(T)\big)\, dx,$$
$$\int\limits_\Omega G_{\varepsilon,\eta}(u_0)\, dx \xrightarrow[\eta \to 0]{} \int\limits_\Omega G_\varepsilon(u_0)\, dx,$$
$$\int\limits_{Q_T} f(x, u_{\varepsilon,\eta}) g_{\varepsilon,\eta}(u_{\varepsilon,\eta})\, dz \xrightarrow[\eta \to 0]{} \int\limits_{Q_T} f(x, u_\varepsilon) g_\varepsilon(u_\varepsilon)\, dz.$$

From (4.8),

$$\int\limits_{Q_T} \phi_\varepsilon^2\, dz = \int\limits_{Q_T} \big(\mathbb{1}_{\{u_\varepsilon = 0\}} \chi_\varepsilon^2 + g_\varepsilon(u_\varepsilon)^2\big)\, dz \leqslant \liminf_{\eta \to 0} \int\limits_{Q_T} \big(g_{\varepsilon,\eta}(u_{\varepsilon,\eta})\big)^2\, dz. \tag{4.20}$$

Furthermore, repeating the arguments used to prove (4.18), we conclude that

$$\liminf_{\eta\to 0}\int_{Q_T} |\nabla u_{\varepsilon,\eta}|^p g'_{\varepsilon,\eta}(u_{\varepsilon,\eta})\,dz \geqslant \lim_{\eta\to 0}\int_{Q_T} \mathbb{1}_{\{u_\varepsilon>0\}}|\nabla u_{\varepsilon,\eta}|^p g'_{\varepsilon,\eta}(u_{\varepsilon,\eta})\,dz$$
$$= \int_{Q_T} \mathbb{1}_{\{u_\varepsilon>0\}}|\nabla u_\varepsilon|^p g'_\varepsilon(u_\varepsilon)\,dz.$$

It follows now that

$$\limsup_{\eta\to 0}\int_{Q_T} \big(g_{\varepsilon,\eta}(u_{\varepsilon,\eta})\big)^2\,dz \leqslant \int_{Q_T} \big(g_\varepsilon(u_\varepsilon)\big)^2\,dz. \tag{4.21}$$

Finally, combining inequalities (4.20) and (4.21), we conclude that $\mathbb{1}_{\{u_\varepsilon>0\}}\chi_\varepsilon = 0$ a.e. in Q_T, which means that $\phi_\varepsilon = g_\varepsilon(u_\varepsilon)$ a.e. in Q_T. Uniqueness of the weak solution to problem (P_ε) follows as in Corollary 3.2.

Proposition 4.2. *The function $u_\varepsilon = \lim_{\eta\to 0} u_{\varepsilon,\eta}$ satisfies the energy relations (4.1) and (4.2).*

Proof. Let us take $u_\varepsilon \in \mathcal{U} \subset L^\infty(0,T;W_0^{1,p}(\Omega))$ for the test-function in the integral identity (3.3). Using the convergence properties (4.3)–(4.7) we pass to the limit as $\eta\to 0$ and obtain (4.1) applying (3.6). To get (4.2) we let $\eta\to 0$ in (3.17) and make use of the Fatou lemma, the uniform bound $0\leqslant u_{\varepsilon,\eta}\leqslant M$ a.e. in Q_T and the pointwise convergence $u_{\varepsilon,\eta}\to u_\varepsilon$. □

The proof of Theorem 4.1 is completed.

4.4. Existence of solution of problem (P): Proof of Theorem 2.1

Using the uniform in ε and η estimates on the solutions of problems $(P_{\varepsilon,\eta})$ we may find functions $u\in\mathcal{U}$ and $V\in L^{p'}(Q_T)^d$ such that the sequences $\{u_\varepsilon\}$, $\{|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\}$ converge to u and V in the sense of (4.3)–(4.7). Literally repeating the proof of Theorem 4.1 we can justify the limit passage as $\varepsilon\to 0$ in every term of the integral identity

$$\int_{Q_T}\big(\partial_t u_\varepsilon\varphi + |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\cdot\nabla\varphi + g_\varepsilon(u_\varepsilon)\varphi\big)\,dz = \int_{Q_T} f(x,u_\varepsilon)\varphi\,dz, \quad \varphi\in L^p\big(0,T;W_0^{1,p}(\Omega)\big),$$

except for the term $g_\varepsilon(u_\varepsilon)\varphi$, which becomes singular as $\varepsilon\to 0$.

The proof of convergence of the corresponding term in the problems $(P_{\varepsilon,\eta})$ relied on boundedness of $g_{\varepsilon,\eta}(s)$ with respect to η, which is no longer valid if we allow ε to approach zero. The proof of convergence of the sequence $g_\varepsilon(u_\varepsilon)$ is based on monotonicity.

Proposition 4.3. *If $\varepsilon > \varepsilon' > 0$, then for any $t\in(0,T)$, $u_\varepsilon(t)\geqslant u_{\varepsilon'}(t)$ a.e. in Ω.*

Proof. Since $g_\varepsilon(u_\varepsilon)\in L^\infty(Q_T)\subset L^{p'}(0,T;W^{-1,p'}(\Omega))$, by virtue of the equation we also have $\partial_t u_\varepsilon\in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and may take $u_{\varepsilon'}-u_\varepsilon$ for the test-function. We adapt the proof of Lemma 3.1. Let us notice first that

$$\forall\varepsilon>\varepsilon'>0,\ \forall s\in\mathbb{R},\quad g_{\varepsilon'}(s)\geqslant g_\varepsilon(s).$$

Subtracting the identity for u_ε from the one for $u_{\varepsilon'}$ and taking into account the inequality

$$\big(g_{\varepsilon'}(u_{\varepsilon'})-g_\varepsilon(u_\varepsilon)\big)(u_{\varepsilon'}-u_\varepsilon)_+ \geqslant \big(g_\varepsilon(u_{\varepsilon'})-g_\varepsilon(u_\varepsilon)\big)(u_{\varepsilon'}-u_\varepsilon)_+ \geqslant -\frac{\beta}{\varepsilon^{\beta+1}}(u_{\varepsilon'}-u_\varepsilon)_+^2,$$

we arrive at the integral inequality (3.19) for the function $\|(u_{\varepsilon'}-u_\varepsilon)_+(t)\|_{2,\Omega}^2$. □

Proposition 4.4. *For every $\varphi\in\mathcal{U}$ we have $\mathbb{1}_{\{u>0\}}u^{-\beta}\varphi\in L^1(Q_T)$ and*

$$\int_{Q_T} g_\varepsilon(u_\varepsilon)\varphi\,dz \xrightarrow[\varepsilon\to 0]{} \int_{Q_T}\mathbb{1}_{\{u>0\}}u^{-\beta}\varphi\,dz. \tag{4.22}$$

Proof. Let us take an arbitrary monotone decreasing sequence $\{\varepsilon_k\}$, $\varepsilon_k\to 0$ as $k\to\infty$. Each of the functions $u_{\varepsilon_k}=\lim u_{\varepsilon_k,\eta}$ as $\eta\to 0$ is defined almost everywhere in Q_T, which allows us to remove from Q_T a zero-measure set ω_T in such a

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way that $u = \lim u_{\varepsilon_k}$ as $k \to \infty$ is defined at every point $(t, x) \in \widetilde{Q}_T = Q_T \setminus \omega_T$. Let $\varphi \in \mathcal{U}$ and $(t_0, x_0) \in \widetilde{Q}_T$. We consider separately the following two possibilities.

1. There exists $K \in \mathbb{N}$ such that $u_{\varepsilon_K}(t_0, x_0) = 0$. By virtue of Proposition 4.3 the sequence $\{u_{\varepsilon_k}(t_0, x_0)\}$ is monotone decreasing as $k \to \infty$, which entails $u_{\varepsilon_k}(t_0, x_0) = 0$ for all $k \geqslant K$. It follows that $u(t_0, x_0) = 0$ and

$$g_{\varepsilon_k}\big(u_{\varepsilon_k}(t_0, x_0)\big) = 0 \xrightarrow[\varepsilon_k \to 0]{} 0 = \mathbb{1}_{\{u>0\}} u(t_0, x_0)^{-\beta}.$$

2. Let us assume that $u_{\varepsilon_k}(t_0, x_0) > 0$ for all $k \in \mathbb{N}$. Since $g_{\varepsilon_k}(s)$ is nonincreasing as a function of s for $s > 0$, it follows from Proposition 4.3 that the sequence $\{g_{\varepsilon_k}(u_{\varepsilon_k}(t_0, x_0))\}$ is nondecreasing. We may now define the measurable function $g : Q_T \to [0, +\infty]$ as follows:

$$\forall (t, x) \in \widetilde{Q}_T, \quad g(t, x) := \lim_{k \to \infty} g_{\varepsilon_k}\big(u_{\varepsilon_k}(t, x)\big) \in [0, +\infty].$$

For every nonnegative test-function $\varphi \in \mathcal{U}$, we have from (3.3) that

$$\int_{Q_T} g_{\varepsilon_k}(u_{\varepsilon_k})\varphi \, dz = \int_{Q_T} f(x, u_{\varepsilon_k})\varphi \, dz - \int_{Q_T} |\nabla u_{\varepsilon_k}|^{p-2} \nabla u_{\varepsilon_k} \cdot \nabla \varphi \, dz - \int_{Q_T} \partial_t u_{\varepsilon_k} \varphi \, dz. \tag{4.23}$$

Recall that $\{u_{\varepsilon_k}\}$ is uniformly bounded in $L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$, while $\{\partial_t u_{\varepsilon_k}\}$ is uniformly bounded in $L^2(Q_T)$. It follows that the right-hand side of the above equation is bounded independently of ε_k, whence by the monotone convergence theorem,

$$\int_{Q_T} g\varphi \, dz = \lim_{k \to \infty} \int_{Q_T} g_{\varepsilon_k}(u_{\varepsilon_k})\varphi \, dz < +\infty. \tag{4.24}$$

Thus, $g\varphi \in L^1(Q_T)$ for every a.e. nonnegative function $\varphi \in L^p(0, T; W_0^{1,p}(\Omega))$. This means that

$$\operatorname{meas}\big\{(t, x) \in Q_T \bigm| g(t, x) = +\infty\big\} = 0.$$

In the case $g(t_0, x_0) > 0$ the equality $g(t_0, x_0) = u^{-\beta}(t_0, x_0)$ holds by virtue of monotonicity. Thus, $g_{\varepsilon_k}(u_{\varepsilon_k})\varphi \xrightarrow[\varepsilon \to 0]{} \mathbb{1}_{\{u>0\}} u^{-\beta}\varphi$ a.e. in Q_T and $\mathbb{1}_{\{u>0\}} u^{-\beta}\varphi \in L^1(Q_T)$. By Proposition 4.3 and (4.24) we have that a.e. in Q_T

$$\big|g_{\varepsilon_k}(u_{\varepsilon_k})\varphi\big| \leqslant \mathbb{1}_{\{u>0\}} u^{-\beta} \max\{0, \varphi\} \in L^1(\Omega).$$

Applying the Lebesgue's dominated convergence theorem we obtain (4.22). □

Remark 4.1. For every $\varepsilon > 0$ the weak solution $u_\varepsilon \in \mathcal{U}$ of problem (P_ε) is locally Hölder-continuous (see [13, Ch. 3, Th. 1.1]), which means that the argument used in the proof of Proposition 4.4 is in fact valid for every point $(t_0, x_0) \in Q_T$.

The energy relations (2.2) and (2.3) follow as in the proof of Proposition 4.2 with the help of Proposition 4.4. The proof of Theorem 2.1 is completed.

5. Quenching in a finite time

5.1. *The energy inequality*

Identity (2.2) and condition (2.5) yield the inequality: for every $t_1, t_2 \in [0, T]$

$$\frac{1}{2}\big\|u(t_2)\big\|_{2,\Omega}^2 - \frac{1}{2}\big\|u(t_1)\big\|_{2,\Omega}^2 + \int_{t_1}^{t_2}\int_\Omega \big(|\nabla u|^p + u^{1-\beta}\big) \, dz \leqslant \int_{t_1}^{t_2}\int_\Omega \big(\alpha u^q + C_\alpha u\big) \, dz. \tag{5.1}$$

Let us take $t_1 = t$, $t_2 = t + h$ with $t, t + h \in [0, T]$ and write (5.1) in the form

$$\frac{1}{2h}\big\|u(t+h)\big\|_{2,\Omega}^2 - \frac{1}{2h}\big\|u(t)\big\|_{2,\Omega}^2 + \frac{1}{h}\int_t^{t+h}\int_\Omega \big(|\nabla u|^p + u^{1-\beta}\big) \, dz \leqslant \frac{1}{h}\int_t^{t+h}\int_\Omega \big(\alpha u^q + C_\alpha u\big) \, dz.$$

Since $u \in \mathcal{U}$ and satisfies (2.2), the inclusions hold

$$\int_\Omega \big(|\nabla u|^p + u^{1-\beta}\big) \, dx \in L^1(0, T) \quad \text{and} \quad \int_\Omega \big(\alpha u^q + C_\alpha u\big) \, dx \in L^1(0, T).$$

By the Lebesgue differentiation theorem for a.e. $t \in (0, T)$ there exist

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\int_\Omega \left(|\nabla u|^p + u^{1-\beta}\right) dx\, ds = \int_\Omega \left(\left|\nabla u(t)\right|^p + u(t)^{1-\beta}\right) dx$$

and

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\int_\Omega \left(\alpha u^q + C_\alpha u\right) dx\, ds = \int_\Omega \left(\alpha u(t)^q + C_\alpha u(t)\right) dx.$$

By virtue of (5.1) the following inequality is fulfilled: $\forall$ a.e. $t \in (0, T)$

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{2,\Omega}^2\right) + \int_\Omega \left|\nabla u(t)\right|^p dx + \int_\Omega u(t)^{1-\beta} dx \leqslant \alpha \int_\Omega u(t)^q dx + C_\alpha \int_\Omega u(t)\, dx.$$

5.2. Ordinary differential inequality for the energy function

Let us introduce the function $z(t) = \|u(t)\|_{2,\Omega}^2$ and write the previous inequality in the form: $\forall$ a.e. $t \in (0, T)$,

$$\frac{1}{2}z'(t) + \int_\Omega \left|\nabla u(t)\right|^p dx + \int_\Omega u(t)^{1-\beta} dx \leqslant \alpha \int_\Omega u(t)^q dx + C_\alpha \int_\Omega u(t)\, dx. \tag{5.2}$$

Notice that for every $t \in (0, T)$ and every $s \in [1, p]$

$$\begin{aligned}\int_\Omega u^s\, dx &= \int_{\Omega\cap\{u>1\}} u^s\, dx + \int_{\Omega\cap\{0\leqslant u\leqslant 1\}} u^s\, dx \\ &\leqslant \int_{\Omega\cap\{u>1\}} u^p\, dx + \int_{\Omega\cap\{0\leqslant u\leqslant 1\}} u^{1-\beta}\, dx \\ &\leqslant \int_\Omega \left(u^p + u^{1-\beta}\right) dx.\end{aligned} \tag{5.3}$$

Using (5.3) to estimate each of the terms on the right-hand side of (5.2) and applying the inequality $\lambda_1\|u\|_{p,\Omega}^p \leqslant \|\nabla u\|_{p,\Omega}^p$ we find that

$$\frac{1}{2}z'(t) + D\int_\Omega \left(\left|\nabla u(t)\right|^p + u^{1-\beta}(t)\right) dx \leqslant 0 \tag{5.4}$$

with the constant

$$D = 1 - (\alpha + C_\alpha)\min\left\{1, \frac{1}{\lambda_1}\right\}. \tag{5.5}$$

Now we make use of the well-known interpolation inequality of Gagliardo–Nirenberg type.

Lemma 5.1. *(See [17,24].) Let $1 < p < +\infty$ and $r \in [1, +\infty)$ if $p \geqslant d$, and $r \in [1, \frac{dp}{d-p}]$ if $d > p$. Then there exists a constant $C > 0$, depending only on p, r, d and $|\Omega|$, such that for every $v \in W_0^{1,p}(\Omega)$*

$$\|v\|_{r,\Omega} \leqslant C\|\nabla v\|_{p,\Omega}^\theta \|v\|_{1,\Omega}^{1-\theta} \quad \text{with } \theta = \frac{1-\frac{1}{r}}{\frac{1}{d}-\frac{1}{p}+1} \in (0,1). \tag{5.6}$$

Lemma 5.2. *Let $u \in \mathcal{U}$ be a weak solution of problem (P) satisfying (2.2). The function $z(t) = \|u(t)\|_{2,\Omega}^2$ satisfies the differential inequality*

$$\begin{cases} z'(t) + Kz^\gamma(t) \leqslant 0, & \text{for a.e. } t \in (0, T) \\ z(0) = \|u_0\|_{2,\Omega}^2 \end{cases} \tag{5.7}$$

with the constants

$$\gamma = \frac{1}{r(\frac{\theta}{p} + 1 - \theta)} \in (0,1), \qquad \theta = \frac{1 - \frac{1}{r}}{\frac{1}{d} - \frac{1}{p} + 1} \in (0,1),$$

$$K = M^{\gamma(r-2)}\left(C^{-1} D^{\frac{\theta}{p}} \left(DM^{-\beta}\right)^{1-\theta}\right)^{\gamma r}, \qquad r = \begin{cases} 2 & \text{if } p \geqslant \frac{2d}{d+2}, \\ \frac{dp}{d-p} & \text{otherwise}, \end{cases}$$

$M = \|u\|_{\infty, Q_T}$, D given in (5.5) and the constant C from (5.6).

Proof. Let us assume first that $p \geqslant \frac{2d}{d+2}$. In this case $\frac{dp}{d-p} \geqslant 2$ and inequality (5.6) holds with $r = 2$. Let us denote $M = \|u\|_{\infty, Q_T}$. Applying (5.6) we may estimate: for a.e. $t \in (0,T)$

$$\begin{aligned} D^{\frac{\theta}{p}} \left(DM^{-\beta}\right)^{1-\theta} \|u(t)\|_{2,\Omega} &\leqslant D^{\frac{\theta}{p}} \left(DM^{-\beta}\right)^{1-\theta} \left[C \left\|\nabla u(t)\right\|_{p,\Omega}^{\theta} \left\|u(t)\right\|_{1,\Omega}^{1-\theta}\right] \\ &= C\left(D \int_\Omega \left|\nabla u(t)\right|^p \mathrm{d}x\right)^{\frac{\theta}{p}} \left(D \int_\Omega M^{-\beta} u(t)\, \mathrm{d}x\right)^{1-\theta} \\ &\leqslant C\left(D \int_\Omega \left|\nabla u(t)\right|^p \mathrm{d}x + D \int_\Omega M^{-\beta} u(t)\, \mathrm{d}x\right)^{\frac{\theta}{p}+1-\theta}. \end{aligned} \tag{5.8}$$

Since

$$\int_\Omega u(t)^{1-\beta}\, \mathrm{d}x \geqslant M^{-\beta} \int_\Omega u(t)\, \mathrm{d}x,$$

we now have

$$\left(C^{-1} D^{\frac{\theta}{p}} \left(DM^{-\beta}\right)^{1-\theta}\right)^2 z(t) \leqslant \left(D \int_\Omega \left|\nabla u(t)\right|^p \mathrm{d}x + D \int_\Omega u(t)^{1-\beta}\, \mathrm{d}x\right)^{2(\frac{\theta}{p}+1-\theta)}.$$

Plugging this inequality into (5.4) we obtain (5.7). Let $1 < p < \frac{2d}{d+2}$. Since $u \leqslant M$ and $r < 2$ we may estimate

$$z(t) = \left\|u(t)\right\|_{2,\Omega}^2 = \int_\Omega u^{2-r} u^r\, \mathrm{d}x \leqslant M^{2-r} \int_\Omega u^r\, \mathrm{d}x = M^{2-r} \left\|u(t)\right\|_{r,\Omega}^r.$$

It follows now from (5.6) and (5.8) that

$$\begin{aligned} z(t) = \left\|u(t)\right\|_{2,\Omega}^2 &\leqslant M^{2-r}\left(\left\|u(t)\right\|_{r,\Omega}\right)^r \\ &\leqslant M^{2-r}\left(C\|\nabla v\|_{p,\Omega}^{\theta} \|v\|_{1,\Omega}^{1-\theta}\right)^r \\ &\leqslant M^{2-r}\left(D^{\frac{\theta}{p}} \left(DM^{-\beta}\right)^{1-\theta}\right)^{-r} C^r \left(D \int_\Omega \left|\nabla u(t)\right|^p \mathrm{d}x + D \int_\Omega u(t)^{1-\beta}\, \mathrm{d}x\right)^{\frac{1}{\gamma}} \end{aligned}$$

with $\theta = (1 - \frac{1}{r})/(\frac{1}{d} - \frac{1}{p} + 1)$ and $\frac{1}{\gamma} = r(\frac{\theta}{p} + 1 - \theta)$. It is easy to see that $\gamma < 1$ because for every $p > 1$ and $r > 1$

$$\frac{1}{\gamma} = r\left(\frac{\theta}{p} + 1 - \theta\right) > 1 \quad \Leftrightarrow \quad 1 - \frac{1}{r} > \frac{(1 - \frac{1}{r})(1 - \frac{1}{p})}{1 - \frac{1}{p} + \frac{1}{d}} \quad \Leftrightarrow \quad d > 0.$$

Thus,

$$M^{\gamma(r-2)}\left(C^{-1} D^{\frac{\theta}{p}} \left(DM^{-\beta}\right)^{1-\theta}\right)^{\gamma r} z^{\gamma}(t) \leqslant D \int_\Omega \left|\nabla u(t)\right|^p \mathrm{d}x + D \int_\Omega u(t)^{1-\beta}\, \mathrm{d}x. \qquad \square$$

5.3. Proof of Theorem 2.2

We are now in position to complete the proof of Theorem 2.2. The assertion of Theorem 2.2 is an immediate byproduct of the following lemma.

Lemma 5.3. *Let $z(t)$ be a nonnegative function satisfying inequality (5.7) with $\gamma \in (0,1)$. Then*

$$z(t) = 0 \quad \forall t \geqslant T_*, \tag{5.9}$$

where $T_* = z_0^{1-\gamma}[K(1-\gamma)]^{-1}$ and K is defined in Lemma 5.2.

Proof. First of all, let us notice that (5.9) is surely true if $z_0 = 0$. If $z_0 > 0$ there exists an interval $(0,\tau)$ such that $z(t) > 0$ for all $t \in [0,\tau)$. Let us assume, for contradiction, that

$$\xi = \sup\{\tau > 0\colon\ z(t) > 0,\ \forall t \in [0,\tau)\} > T_*.$$

Dividing the both terms of inequality (5.7) by $z^\gamma(t)$, we get the inequality

$$\frac{1}{1-\gamma}\big(z^{1-\gamma}(t)\big)' \leqslant -K.$$

Integrating it over the interval $(0,t)$ with $t \in (T_*,\xi)$ we have:

$$z^{1-\gamma}(t) \leqslant z_0^{1-\gamma} - K(1-\gamma)t.$$

By virtue of (5.7) $z'(t) \leqslant 0$ for a.e. t and $z(t)$ is a nonincreasing function. On the other hand, since $z(t)$ is nonnegative and $t \mapsto z_0^{1-\gamma} - K(1-\gamma)t$ is monotone decreasing, we have

$$\forall t \geqslant T_*, \quad 0 \leqslant z(t) \leqslant z_0^{1-\gamma} - K(1-\gamma)t < 0,$$

which is impossible unless $T_* \geqslant \xi$. Thus, $z(T_*) = 0$ and the assertion follows. □

5.4. Finite time blow-up

Let us finally show that the critical or subcritical growth of f is in fact necessary for the global in time existence.

Proposition 5.1 *(Finite time blow-up). Let in the conditions of Theorem 2.1*

$$f(x,u) = \alpha|u|^{q-2}u \quad \text{with } q > \max\{p,2\},\ \alpha > 0, \tag{5.10}$$

and

$$E(0) := \int_\Omega \left(\frac{1}{p}|\nabla u_0|^p + \frac{1}{1-\beta}u_0^{1-\beta} - \frac{\alpha}{q}u_0^q\right)\mathrm{d}x < 0. \tag{5.11}$$

Then every solution of problem (P) blows up in a finite time:

$$y(t) = \|u(t)\|_{2,\Omega}^2 \to \infty \quad \text{as } t \to \frac{|\Omega|^{\frac{q}{2}-1}}{(\frac{q}{2}-1)p\alpha(\frac{1}{p}-\frac{1}{q})\|u_0\|_{2,\Omega}^{q-2}} := \widetilde{T}.$$

Proof. Let u be a weak solution of problem (P). By virtue of (2.2) for every $t, t+h \in [0,T]$

$$\frac{1}{2h}\|u(t+h)\|_{2,\Omega}^2 - \frac{1}{2h}\|u(t)\|_{2,\Omega}^2 + \int_t^{t+h}\int_\Omega \big(|\nabla u|^p + u^{1-\beta} - \alpha u^q\big)\,\mathrm{d}z = 0.$$

Arguing as in the derivation of the differential inequality (5.2) we let $h \to 0$ and arrive at the relation: $\forall$ a.e. $t \in (0,T)$

$$\frac{1}{2}y'(t) = -\int_\Omega \big(|\nabla u(t)|^p + u^{1-\beta}(t) - \alpha u^q(t)\big)\,\mathrm{d}x.$$

Let us introduce the function

$$E(t) := \int_\Omega \left(\frac{1}{p}|\nabla u(t)|^p + \frac{1}{1-\beta}u^{1-\beta}(t) - \frac{\alpha}{q}u^q(t)\right)\mathrm{d}x$$

and notice that $E(t) \leqslant 0$ due to (2.3) and assumption (5.11). Since $q > \max\{p,2\}$

$$\begin{aligned}\frac{1}{2p}y'(t) &\geqslant \frac{1}{2p}y'(t)+E(t)\\ &= -\left(\frac{1}{p}-\frac{1}{1-\beta}\right)\int_\Omega u^{1-\beta}(t)\,\mathrm{d}x+\alpha\left(\frac{1}{p}-\frac{1}{q}\right)\|u(t)\|_{q,\Omega}^q\\ &> \alpha\left(\frac{1}{p}-\frac{1}{q}\right)\|u(t)\|_{q,\Omega}^q.\end{aligned}$$

Applying the Hölder inequality we arrive at the following differential inequality for the function $y(t)$:

$$y'(t)\geqslant \frac{2p\alpha}{|\Omega|^{\frac{q}{2}-1}}y^{\frac{q}{2}}(t) \quad \text{for } t\in(0,T), \qquad y(0)=\|u_0\|_{2,\Omega}^2.$$

The straightforward integration shows that $y^{\frac{q}{2}-1}(t)\to\infty$ as $t\to\widetilde{T}_-$. $\square$

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