Quenching phenomenon of singular parabolic problems with L^1 initial data.

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Abstract. We extend some previous existence results for quenching type parabolic problems involving a negative power of the unknown in the equation to the case of merely integrable initial data. We show that $L^1(\Omega)$ is the suitable framework in order to get the continuous dependence with respect to some norm of the initial datum, giving answer, in this way, to this question raised by several authors in the previous literature.We aslo show the global and local quenching phenomena for such type of initial datum.
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1 Introduction.

The main purpose of this paper is to study the existence of nonnegative mild solution and the "quenching phenomenon" of the singular parabolic equation:

$$\begin{cases} \partial_t u - \Delta u + \chi_{\{u>0\}} u^{-\beta} = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0(\cdot) & \text{on } \Omega, \end{cases} \tag{1}$$

where $\beta \in (0,1)$, Ω is a smooth bounded domain in $\mathbb{R}^N$, $u_0 \geq 0$ and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x,t) where $u(x,t) > 0$. Parabolic equations involving as zero order term a negative exponent of the unknown are quite common in the literature since 1960. The pioneering paper by Fulks and Maybee [17] was motivated by the study of the heat conduction in an electric medium but in the modelling the singular term was of a sourcing nature and so in the right hand side of the equation: the differences between the behavior of solutions of such model with respect to our problem (1) are today well-known. Perhaps, one of the first papers dealing with the first equation of (1) was [22] in the study of Electric Current Transient in Polarized Ionic Conductors (in fact for $\beta = 1$). The literature on this type of problems increased then very quickly and models arising in other contexts were mentioned by different authors, specially when regarding the first equation of (1) as the limit case of models in chemical catalyst kinetics (Langmuir-Hinshelwood model) or of models in enzyme kinetics (see [10, 14] for the elliptic case and [3, 28] for the parabolic equation). See also many references in the survey [21] and the monograph [19]). Many other variants of the equation were formulated in terms of possible doubly nonlinear diffusion operators of the form

$$\partial_t w - \Delta_p(|w|^{m-1} w) + \frac{\chi_{\{w\neq 0\}}}{w^k} = \lambda(w^q \chi_{\{w\neq 0\}} + g(x,t)), \tag{2}$$

for some $p > 1, m, q \in \mathbb{R}$ and $k, \lambda > 0$. Here the above unknown u was formally replaced by $|w|^{m-1} w$ and $\Delta_p h$ denotes the usual p-Laplacian operator $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$: see[24] for $p = 2$ and $m > 1$, and [8, 20] for the case of $p\neq 2$ and $m = 1$. We also mention the formulation considered in [31, 32] in terms of a non-divergential equation. For instance, the case $p = 2$ and $m \neq 1$, and $k \in (0, 1)$ was associated to some models arising in plasma physics, and nonlinearities on the gradient appear in some geometrical problems (see references and examples in [23]). We also mention that this equation also arises in the context of the study of the so-called Euler-Poisson system in Maxwell-Vlasov problems (see [1]) and in hydrodynamic quantum fluids (see [18]). As a last mention in the modelling, we want to point out that similar problems, but with $\beta \geq 1$ (and mostly with initial conditions and initial data implying the strict positiveness of the solution u) arise in the so-called MEMS materials (Micro Electromechanical Systems): see, e.g., the monograph [27]. Obviously, what makes specially interesting equations like (1) and (2) is the fact that the solutions may raise to a free boundary defined as the boundary of the set $\{(x, t)\colon\ u(x, t) > 0\}$. In many contexts the boundary conditions are not zero but, for instance, $u = 1$ and thus, the terminology of "quenching problem" was used in the literature to denote the appearance of blow-up result on $\partial_t u$ for the first time in which $u = 0$ (see, e.g., [22, 26, 28]).

In spite of such a long list of references, most of the theory in the literature deals with bounded (quite often even assumed continuous) initial data. We must add that even so, it is today well-known that the uniqueness of solution fails (see [33]) except for the case in which there is not a free boundary (see [9]). In particular, it is know that the solution is not necessarily continuously dependent on the norm $\|u_0\|_{L^\infty(\Omega)}$. The main purpose of this work is to deal with initial data satisfying merely

$$0 \leq u_0 \in L^1(\Omega)$$

and to show that $L^1(\Omega)$ is the suitable framework in order to get the continuous dependence with respect some norm of the initial datum, giving answer, in this way, to this question raised by several authors in the previous literature. To be more precise, we introduce the notion of solution we shall use in this paper:

Definition 1 *A function $u \in \mathcal{C}([0, T); L^1(\Omega))$ is called a mild solution of (1) if $\chi_{\{u>0\}}u^{-\beta} \in L^1(\Omega \times (0, T))$ and u fulfils the identity*

$$u(\cdot, t) = S(t)u_0(\cdot) - \int_0^t S(t - s)\chi_{\{u>0\}}u^{-\beta}(\cdot, s)ds \quad \text{in } L^1(\Omega), \tag{3}$$

where $S(t)$ is the $L^1(\Omega)$-semigroup corresponding to the Laplace operator with homogeneous Dirichlet boundary conditions.

We recall that the $L^1(\Omega)$-semigroup $S(t)$ corresponding to the Laplace operator with homogeneous Dirichlet boundary conditions was considered by many authors since the seventies (or even earlier) of the past century and that the associated weak solutions $S(t)u_0$ can be characterized by multiplying by suitable test functions (see, e.g., [4, 6, 7] and the exposition made in Chapter 4 of [10]). In particular, we know that any mild solution u belongs to the space $L^s(0,T;W_0^{1,s}(\Omega))$, for any $s \in (1, \frac{N+2}{N+1})$, and satisfies that

$$\int_\Omega u(x,t)\psi(x,t)dx + \int_0^t \int_\Omega \nabla u(x,s) \cdot \nabla \psi(x,s)dxds$$
$$+ \int_0^t \int_\Omega \chi_{\{u>0\}} u^{-\beta}(x,s)\psi(x,s)dxds = \int_0^t \int_\Omega u(x,s)\partial_t \psi(x,s)dxds + \int_\Omega u_0(x)\psi(x,0)dx$$

for any test function $\psi \in W^{1,\infty}(0,T;L^1(\Omega)) \cap L^\infty(0,T;W_0^{1,\infty}(\Omega))$ and almost every $t \in (0,T)$.

The main results of this paper are the following:

Theorem 2 *Let $0 \leq u_0 \in L^1(\Omega)$. Then, there exists the (global) maximal nonnegative mild solution u of* (1)*, i.e. such that for any other mild solution v of* (1) *we have $0 \leq v \leq u$ in $\Omega \times [0,T]$. Moreover, for any $0 < \tau < T$, $u \in L^2(\tau,T;W_0^{1,2}(\Omega)) \cap L^\infty(\Omega \times (\tau,T))$.*

Concerning the quenching phenomenon, we recall that since there is lack of uniqueness of solutions, it looks difficult to apply, directly, super- and sub-solutions methods to study it and the free boundary defined as the boundary of the set $\{(x,t)\colon u(x,t) > 0\}$. Our alternative is the application of local energy methods available for many types of evolution equations and systems for the last thirty years of the last century (see, e.g., the monograph [2] and its many references) but with the new fact that our initial datum does not need to be in the natural energy space defined over $L^2(\Omega)$.

Theorem 3 *Let $0 \leq u_0 \in L^1(\Omega)$. Then, if v is any nonnegative mild solution of* (1)*, there exists a finite time, $T^* > 0$ such that $v(\cdot,t)$ vanishes in* a.e. *in Ω for $t > T^*$. Moreover, T^* only depends on $\|u_0\|_{L^1(\Omega)}$, N and $|\Omega|$.*

Finally, concerning the spatial behavior of the free boundary we have

Theorem 4 *(Instantaneous shrinking of the support) Let $0 \leq u_0 \in L^1(\Omega)$ and let v be any nonnegative mild solution of* (1)*. Then, there exist a point $x_0 \in \Omega$, a parameter $\mu \in (0,1)$ and a finite time $t^* \leq T^*$, only depending on $\|u_0\|_{L^1(\Omega)}$ and N, such that $u(x,t) = 0$ in the paraboloid $\{(x,t) : |x - x_0| < (t - t^*)^\mu, t \in (t^*, T^*)\}$.*

Some remarks about global statements of the above result will be given later. The paper is organized as follows: section 2 is devoted to the proof of Theorem 2 and the proofs of Theorem 3 and Theorem 4 will be given in section 3.

The consideration of the more sophisticated equation (2) in the $L^1(\Omega)$-framework requires sharper gradient estimates and is the main object of the paper [8].

2 Proof of Theorem 2

We shall follow a scheme of approximation similar to the one used in [33]. We start by considering the problem

$$\begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon + g_\varepsilon(u_\varepsilon) = 0 & \text{in } \Omega \times (0,+\infty), \\ u_\varepsilon = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u_\varepsilon(\cdot,0) = u_0(\cdot) & \text{on } \Omega, \end{cases} \tag{4}$$

with

$$g_\varepsilon(s) := \begin{cases} 0 & \text{if } s \le 0, \\ \psi_\varepsilon(s) s^{-\beta} & \text{if } s > 0. \end{cases}$$

where $\psi_\varepsilon(s) = \psi(\frac{s}{\varepsilon})$ and $\psi \in \mathcal{C}^\infty(\mathbb{R})$ is a non-decreasing function on $\mathbb{R}$ such that $\psi(s) = 0$ for $s \le 1$, $\psi(s) = 1$ for $s \ge 2$. The main idea of the proof is to construct the maximal solution of (4) by passing to the limit in the solutions u_ε of (4) as $\varepsilon \to 0$. Our proof differs, and offers an alternative, to the approach to the existence of solution presented in [20].

First of all, we observe that for any fixed $\varepsilon > 0$, g_ε is a global Lipschitz-continuous function. Then, we get easily to the following result:

Theorem 5 *There exists a unique nonnegative mild solution $u_\varepsilon \in \mathcal{C}([0,+\infty); L^1(\Omega))$ to problem (4),i.e. satisfying, for any $t > 0$,*

$$u_\varepsilon(t) = S(t)u_0 - \int_0^t S(t-\sigma) g_\varepsilon(u_\varepsilon(s)) ds. \tag{5}$$

Moreover, for any $0 < \tau < T < +\infty$, and for some $\alpha \in (0,1)$, we have $u_\varepsilon \in \mathcal{C}_{x,t}^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times (\tau, T))$.

Proof: Concerning the existence, we shall follow some rather classical arguments and so, we give its proof in the appendix section at the end of the paper. Concerning the uniqueness, the proof is an immediate consequence from the lemma below. □

Lemma 6 *For any $0 < \tau < T$, let $v_1 \in L^\infty(\Omega \times (\tau, T)) \cap L^2(\tau, T; W_0^{1,2}(\Omega))$ (resp. v_2) be a mild sub-solution (resp super-solution) of (4). Then, we have $v_1 \le v_2$, in $\Omega \times (0,T)$.*

Proof: *(of Lemma 6)* We shall use a L^2-technique. We introduce the truncation function

$$T_k(s) := \begin{cases} s & \text{if } |s| \le k, \\ sign(s)k & \text{if } |s| > k, \end{cases}$$

and its primitive integral

$$S_k(u) := \int_0^u T_k(s) ds = \frac{1}{2}|u|^2 \chi_{\{|u|<k\}} + k\left(|u| - \frac{1}{2}k\right)\chi_{\{|u|\ge k\}}.$$

Let us consider the equation satisfied by the difference between v_1 and v_2

$$\partial_t(v_1 - v_2) - \Delta(v_1 - v_2) + g_\varepsilon(v_1) - g_\varepsilon(v_2) \le 0.$$

Then, using the test function $T_1(v_+)$, with $v := v_1 - v_2$, we get for any $0 < \tau < t$,

$$\int_\Omega S_1(v_+(t))dx + \int_\tau^t \int_\Omega |\nabla v_+|^2 dxds + \int_\tau^t \int_\Omega (g_\varepsilon(v_1) - g_\varepsilon(v_2)) T_1(v_+) dxds \le \int_\Omega S_1(v_+(\tau))dx.$$

Since g_ε is a global Lipschitz-continuous function, it follows from the last inequality that

$$\int_\Omega S_1(v_+(t))dx \le C(\varepsilon) \int_\tau^t \int_\Omega |v| T_1(v_+) dxds + \int_\Omega S_1(v_+(\tau))dx. \tag{6}$$

By passing to the limit as $\tau \to 0$ in (6), noting that $\int_\Omega S_1(v_+(\tau))dx \xrightarrow[\tau\to 0]{} 0$, we obtain

$$\int_\Omega S_1(v_+(t))dx \le C(\varepsilon)\int_0^t\int_\Omega |v|T_1(v_+)dxds. \tag{7}$$

On the other hand, we observe that

$$|v|T_1(v_+) \le 2S_1(v_+). \tag{8}$$

Combining (7) and (8) we deduce

$$\int_\Omega S_1(v_+(t))dx \le 2C(\varepsilon)\int_0^t\int_\Omega S_1(v_+)dxds.$$

Then, if we define $y(t) := \int_\Omega S_1(v_+(t))dx$, we obtain the ordinary differential inequality

$$\begin{cases} \dfrac{d}{dt}y(t) \le 2C(\varepsilon)y(t), \\ y(0) = 0. \end{cases}$$

Thus, by Gronwall's inequality, for any $t > 0$, $y(t) = 0$, and so $v_+(t) = 0$, which completes the proof of the lemma. $\square$

Now, we shall show the existence of a solution of (1) by passing to the limit as $\varepsilon \to 0$.

Theorem 7 *For any fixed $\varepsilon > 0$, let u_ε be the unique solution of (4). Then, there is a subsequence of $\{u_\varepsilon\}_\varepsilon$ (still denoted as $\{u_\varepsilon\}_\varepsilon$) such that, for any $T > 0$, u_ε converges to a function u in $L^r(0,T;W_0^{1,r}(\Omega))$, for any $r \in (1, \frac{N+2}{N+1})$, and $g_\varepsilon(u_\varepsilon)$ converges to $u^{-\beta}\chi_{\{u>0\}}$ in $L^1(\Omega\times(0,T))$ as $\varepsilon \to 0$. Furthermore, u is a mild solution of (1).*

Proof: It follows from (5), the fact that $g_\varepsilon(u_\varepsilon) \ge 0$ and the regularizing effect of the $L^1(\Omega)-$semigroup (see, e.g., [30] and [5, Proposition 2.1]) that, for any $t > 0$,

$$0 \le u_\varepsilon(x,t) \le S(t)u_0(x) \le Ct^{-\frac{N}{2}}\|u_0\|_{L^1(\Omega)}. \tag{9}$$

The constant C in (9) merely depends on $N, |\Omega|$. Then, u_ε is bounded locally in time. For any $0 < \tau < T$, integrating equation (4) on $\Omega \times (\tau, T)$ yields

$$\int_\Omega u_\varepsilon(x,T)dx - \int_\tau^T\int_{\partial\Omega}\nabla u_\varepsilon.\mathbf{n}d\sigma ds + \int_\tau^T\int_\Omega g_\varepsilon(u_\varepsilon)dxds = \int_\Omega u(x,\tau)dx,$$

where $\mathbf{n}$ is the unit outward normal vector of $\partial\Omega$. Since $\nabla u_\varepsilon.\mathbf{n} \le 0$ on $\partial\Omega\times(\tau,T)$, we get

$$\int_\Omega u_\varepsilon(x,T)dx + \int_\tau^T\int_\Omega g_\varepsilon(u_\varepsilon)dxds \le \int_\Omega u(x,\tau)dx,$$

Passing to the limit as $\tau \to 0$ in the last inequality asserts

$$\int_\Omega u_\varepsilon(x,T)dx + \int_0^T\int_\Omega g_\varepsilon(u_\varepsilon)dxds \le \|u_0\|_{L^1(\Omega)}. \tag{10}$$

By using [4, Lemma 3.3], we obtain

$$\|u_\varepsilon\|_{L^s(0,T;W_0^{1,r}(\Omega))} \le C(s,r,T,\Omega)\left(\|g_\varepsilon(u_\varepsilon)\|_{L^1(\Omega\times(0,T))} + \|u_0\|_{L^1(\Omega)}\right), \tag{11}$$

with $s,r \ge 1$ such that $\frac{2}{s} + \frac{N}{r} > N+1$. Combining (10) and (11) we get

$$\|u_\varepsilon\|_{L^r(0,T;W_0^{1,r}(\Omega))} \le C(r,T,\Omega)\|u_0\|_{L^1(\Omega)}, \tag{12}$$

with $r = s \in [1, \frac{N+2}{N+1})$. Thus, for any $r \in (1, \frac{N+2}{N+1})$, $\{\partial_t u_\varepsilon\}_\varepsilon$ is bounded in $L^1(0,T;W^{-1,r'}(\Omega)) + L^1(\Omega \times (0,T))$ by a constant independent of ε. Then, the sequence $\{u_\varepsilon\}_\varepsilon$ is relatively compact in $L^1(\Omega\times(0,T))$ (see [29]) and there is a subsequence of $\{u_\varepsilon\}_\varepsilon$ (still denoted as $\{u_\varepsilon\}_\varepsilon$) such that

$$u_\varepsilon \xrightarrow[\varepsilon\to 0]{} u \quad \text{in } L^1(\Omega\times(0,T)). \tag{13}$$

(We denote that any passage to the limit is up to a subsequence in the sequel). Next, we claim that

$$u_\varepsilon(x,t) \downarrow u(x,t), \quad \text{for a.e. } (x,t) \in \Omega\times(0,T). \tag{14}$$

Indeed, it is enough to show that $\{u_\varepsilon\}_\varepsilon$ is a non-decreasing sequence. We have for any $\varepsilon > \varepsilon' > 0$, $g_\varepsilon \le g_{\varepsilon'}$ on $\mathbb{R}$. Then,

$$\partial_t u_\varepsilon - \Delta u_\varepsilon + g_{\varepsilon'}(u_\varepsilon) \ge \partial_t u_\varepsilon - \Delta u_\varepsilon + g_\varepsilon(u_\varepsilon) = 0.$$

This implies that u_ε is a super-solution of the equation satisfied by $u_{\varepsilon'}$. Thanks to Lemma 6, we get $u_\varepsilon(x,t) \ge u_{\varepsilon'}(x,t)$, for a.e. $(x,t) \in \Omega\times(0,T)$, likewise we get the claim (14).
Next, we shall show the convergence of the gradients. Let us first demonstrate that

$$\nabla u_\varepsilon \xrightarrow[\varepsilon\to 0]{} \nabla u \quad \text{in } L^1(\Omega\times(0,T)). \tag{15}$$

For any $\varepsilon, \varepsilon' > 0$, we consider function $v_{\varepsilon,\varepsilon'} := u_\varepsilon - u_{\varepsilon'}$ and the difference between the equations satisfied by u_ε and $u_{\varepsilon'}$

$$\partial_t v_{\varepsilon,\varepsilon'} - \Delta v_{\varepsilon,\varepsilon'} + g_\varepsilon(u_\varepsilon) - g_{\varepsilon'}(u_{\varepsilon'}) = 0. \tag{16}$$

For any $\delta > 0$ and any $0 < T_0 < +\infty$, we take $T_\delta(v_{\varepsilon,\varepsilon'})$ as a test function for (16). Then, we get

$$\int_\Omega S_\delta(v_{\varepsilon,\varepsilon'}(T_0))dx + \int_0^{T_0}\int_\Omega |\nabla T_\delta(v_{\varepsilon,\varepsilon'})|^2 dxds + \int_0^{T_0}\int_\Omega \left(g_\varepsilon(u_\varepsilon) - g_{\varepsilon'}(u_{\varepsilon'})\right) T_\delta(v_{\varepsilon,\varepsilon'})dxds = \int_\Omega S_\delta(v_{\varepsilon,\varepsilon'}(0))dx. \tag{17}$$

It follows from (17) that

$$\int_0^{T_0}\int_\Omega |\nabla T_\delta(v_{\varepsilon,\varepsilon'})|^2 dxds \le \delta \int_0^{T_0}\int_\Omega g_\varepsilon(u_\varepsilon) + g_{\varepsilon'}(u_{\varepsilon'})dxds. \tag{18}$$

Combining (10) and (18) yields

$$\int_{\{|v_{\varepsilon,\varepsilon'}(x,t)|<\delta\}\cap\Omega\times(0,T_0)} |\nabla v_{\varepsilon,\varepsilon'}|^2 dxds \le 2\delta\|u_0\|_{L^1(\Omega)}. \tag{19}$$

Next, we have from Holder's inequality

$$\int_{\{|v_{\varepsilon,\varepsilon'}(x,t)|<\delta\}\cap\Omega\times(0,T_0)} |\nabla v_{\varepsilon,\varepsilon'}|dxds$$
$$\leq mes\{\Omega\times(0,T_0)\}^{\frac{1}{2}}\left(\int_{\{|v_{\varepsilon,\varepsilon'}(x,t)|<\delta\}\cap\Omega\times(0,T_0)} |\nabla v_{\varepsilon,\varepsilon'}|^2 dxds\right)^{\frac{1}{2}}. \quad (20)$$

From (19) and (20), we get

$$\int_{\{|v_{\varepsilon,\varepsilon'}(x,t)|<\delta\}\cap\Omega\times(0,T_0)} |\nabla v_{\varepsilon,\varepsilon'}|dxds \leq C\sqrt{\delta}, \quad (21)$$

where $C = C(|\Omega|, T_0, \|u_0\|_{L^1(\Omega)})$. On the other hand, using Hölder's inequality again yields

$$\int_{\{|v_{\varepsilon,\varepsilon'}(x,t)|\geq\delta\}\cap\Omega\times(0,T_0)} |\nabla v_{\varepsilon,\varepsilon'}|dxds \leq \left(\int_{\{|v_{\varepsilon,\varepsilon'}(x,t)|\geq\delta\}\cap\Omega\times(0,T_0)} |\nabla v_{\varepsilon,\varepsilon'}|^r dxds\right)^{\frac{1}{r}}$$
$$\times\, mes\left(\{|v_{\varepsilon,\varepsilon'}(x,t)|\geq\delta\}\cap\Omega\times(0,T_0)\right)^{1-\frac{1}{r}},$$

with some value $r \in (1, \frac{N+2}{N+1})$. By inserting (12) into the last inequality, we obtain

$$\int_{\{|v_{\varepsilon,\varepsilon'}(x,t)|\geq\delta\}\cap\Omega\times(0,T_0)} |\nabla v_{\varepsilon,\varepsilon'}|dxds \leq$$
$$C(r,|\Omega|,T_0,\|u_0\|_{L^1(\Omega)})\times mes\left(\{|v_{\varepsilon,\varepsilon'}(x,t)|\geq\delta\}\cap\Omega\times(0,T_0)\right)^{1-\frac{1}{r}}. \quad (22)$$

Combining (19) and (22) induces

$$\int_0^{T_0}\int_\Omega |\nabla v_{\varepsilon,\varepsilon'}|dxds \leq C\left(\sqrt{\delta} + mes\left(\{|v_{\varepsilon,\varepsilon'}(x,t)|\geq\delta\}\cap\Omega\times(0,T_0)\right)^{1-\frac{1}{r}}\right). \quad (23)$$

We observe that $v_{\varepsilon,\varepsilon'}$ converges to 0 in measure by (13) or (14). Then, letting $\varepsilon,\varepsilon'\to 0$ in (23) leads to

$$\limsup_{\varepsilon,\varepsilon'\to 0}\int_0^{T_0}\int_\Omega |\nabla v_{\varepsilon,\varepsilon'}|dxds \leq C\sqrt{\delta}.$$

The last inequality holds for any $\delta > 0$, so we obtain (15).
Let us show now a sharper convergence: for any $r \in (1, \frac{N+2}{N+1})$,

$$u_\varepsilon \xrightarrow[\varepsilon\to 0]{} u \quad \text{in } L^r(0,T_0;W_0^{1,r}(\Omega)). \quad (24)$$

Indeed, the conclusion (24) just follows from (12), (13), (15) and Vitali's theorem.
Next, we show that for any $T_0 > 0$, there is a subsequence of $\{g_\varepsilon(u_\varepsilon)\}_\varepsilon$ such that

$$g_\varepsilon(u_\varepsilon) \xrightarrow[\varepsilon\to 0]{} u^{-\beta}\chi_{\{u>0\}} \quad \text{in } L^1(\Omega\times(0,T_0)). \quad (25)$$

More precisely, we claim that the above subsequence satisfies, from Fatou's lemma, that

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon) = u^{-\beta}\chi_{\{u>0\}} \quad \text{in } L^1(\Omega\times(0,T)) \quad (26)$$

and that

$$u \in \mathcal{C}([0, T_0]; L^1(\Omega)). \tag{27}$$

Let us skip the proof of (25) (or (26)) for the moment and let us first show (27) if (25) holds. For any $0 < t < T_0$, we use the argument of (17) with $\delta = 1$ to get

$$\int_\Omega S_1(v_{\varepsilon,\varepsilon'})(t)dx + \int_0^t \int_\Omega |\nabla T_1(v_{\varepsilon,\varepsilon'})|^2 dxds + \int_0^t \int_\Omega (g_\varepsilon(u_\varepsilon) - g_{\varepsilon'}(u_{\varepsilon'}))\, T_1(v_{\varepsilon,\varepsilon'})dxds = 0;$$

and so

$$\int_\Omega S_1(v_{\varepsilon,\varepsilon'})(t)dx \leq \int_0^T \int_\Omega |g_\varepsilon(u_\varepsilon) - g_{\varepsilon'}(u_{\varepsilon'})|dxds. \tag{28}$$

On the other hand, we observe from the expression of S_1 that

$$\int_\Omega |v_{\varepsilon,\varepsilon'}(t)|\chi_{\{|v_{\varepsilon,\varepsilon'}(t)|\geq 1\}}dx \leq 2\int_\Omega S_1(v_{\varepsilon,\varepsilon'})(t)dx,$$

and using Hölder's inequality it yields

$$\int_\Omega |v_{\varepsilon,\varepsilon'}(t)|\chi_{\{|v_{\varepsilon,\varepsilon'}(t)|<1\}}dx \leq |\Omega|^{\frac{1}{2}} \left(\int_\Omega |v_{\varepsilon,\varepsilon'}(t)|^2\chi_{\{|v_{\varepsilon,\varepsilon'}(t)|<1\}}dx\right)^{\frac{1}{2}} \leq \left(2|\Omega|\int_\Omega S_1(v_{\varepsilon,\varepsilon'})(t)dx\right)^{\frac{1}{2}}.$$

Thus, we obtain

$$\int_\Omega |v_{\varepsilon,\varepsilon'}(t)|dx \leq 2\int_\Omega S_1(v_{\varepsilon,\varepsilon'}(t)dx + \left(2|\Omega|\int_\Omega S_1(v_{\varepsilon,\varepsilon'}(t)dx\right)^{\frac{1}{2}}. \tag{29}$$

It follows from (25), (28) and (29) that

$$\lim_{\varepsilon,\varepsilon'\to 0} \|v_{\varepsilon,\varepsilon'}(t)\|_{L^1(\Omega)} = 0, \quad \text{uniformly in } [0, T_0].$$

In other words, we have

$$\lim_{\varepsilon\to 0} \|u_\varepsilon(t) - u(t)\|_{L^1(\Omega)} = 0, \quad \text{uniformly in } [0, T_0]. \tag{30}$$

Thanks to (30), for any $\delta > 0$ (small) there exists a positive number $\varepsilon_\delta > 0$ such that

$$\sup_{t\in[0,T_0]} \|u_{\varepsilon_\delta}(t) - u(t)\|_{L^1(\Omega)} < \delta. \tag{31}$$

Now, we consider for any $t, t_0 \in [0, T_0]$

$$\|u(t) - u(t_0)\|_{L^1(\Omega)} \leq \|u(t) - u_{\varepsilon_\delta}(t)\|_{L^1(\Omega)} + \|u_{\varepsilon_\delta}(t) - u_{\varepsilon_\delta}(t_0)\|_{L^1(\Omega)} + \|u_{\varepsilon_\delta}(t_0) - u(t_0)\|_{L^1(\Omega)}.$$

By (31), we get from the last inequality

$$\|u(t) - u(t_0)\|_{L^1(\Omega)} \leq 2\delta + \|u_{\varepsilon_\delta}(t) - u_{\varepsilon_\delta}(t_0)\|_{L^1(\Omega)}$$

Letting $t \to t_0$ and noting that $u_{\varepsilon_\delta} \in \mathcal{C}([0, T]; L^1(\Omega))$, we get for any $\delta > 0$,

$$\limsup_{t\to t_0} \|u(t) - u(t_0)\|_{L^1(\Omega)} \leq 2\delta + \limsup_{t\to t_0} \|u_{\varepsilon_\delta}(t) - u_{\varepsilon_\delta}(t_0)\|_{L^1(\Omega)} = 2\delta.$$

This implies the conclusion of (27).

Now, to prove (26), we shall use a suitable gradient estimate which will be obtained by the so-called Bernstein technique in a similar way to [33, Lemma 3.1] (see also [9, Lemma 2.4]).

Lemma 8 *There is a positive constant $C>0$ such that for any fixed $\tau>0$, we have*

$$\left|\nabla u_\varepsilon^{\frac{\beta+1}{2}}(x,t)\right| \le C\left(1+\left\|u_\varepsilon(\tau)^{\frac{\beta+1}{2}}\right\|_{L^\infty(\Omega)}\right)\left(1+(t-\tau)^{-\frac{1}{2}}+d(x)^{-1}\right), \quad \text{in } \Omega\times(\tau,+\infty), \tag{32}$$

with $d(x)=\inf\limits_{y\in\partial\Omega}\|x-y\|_{\mathbb{R}^N}$, the distance from x to the boundary of the domain Ω. Moreover, if $u:=\lim\limits_{\varepsilon\to 0}u_\varepsilon$, then $\nabla u_\varepsilon \xrightarrow[\varepsilon\to 0]{} \nabla u$, in $L^2_{\text{loc}}(\Omega\times(0,+\infty))$ and we have the estimate

$$|\nabla u(x,t)| \le C(N,|\Omega|,\beta)u^{\frac{1-\beta}{2}}\left(1+\left(\tau^{-\frac{N}{2}}\|u_0\|_{L^1(\Omega)}\right)^{\frac{\beta+1}{2}}\right)\left(1+(t-\tau)^{-\frac{1}{2}}+d(x)^{-1}\right). \tag{33}$$

Proof: *(of Lemma 8)* In fact, we observe that, for any $\tau>0$, $u_\varepsilon(\tau)\in\mathcal{C}_0(\Omega)$. Then, we can mimic the proof of [33, Lemma 3.3] but by considering $u_\varepsilon(\tau)$ as the initial condition instead of u_0 in order to get (31). In $\Omega\times(\tau,+\infty)$, we rewrite the estimate (32) as follows:

$$|\nabla u_\varepsilon(x,t)| \le C(\beta)u_\varepsilon^{\frac{1-\beta}{2}}\left(1+\left\|u_\varepsilon(\tau)^{\frac{\beta+1}{2}}\right\|_{L^\infty(\Omega)}\right)\left(1+(t-\tau)^{-\frac{1}{2}}+d(x)^{-1}\right). \tag{34}$$

Combining (9) and (34) we deduce that

$$|\nabla u_\varepsilon(x,t)| \le C(N,|\Omega|,\beta)u_\varepsilon^{\frac{1-\beta}{2}}\left(1+\left(\tau^{-\frac{N}{2}}\|u_0\|_{L^1(\Omega)}\right)^{\frac{\beta+1}{2}}\right)\left(1+(t-\tau)^{-\frac{1}{2}}+d(x)^{-1}\right). \tag{35}$$

Passing to the limit as $\varepsilon\to 0$ in (35), we get (33). By (24) and (35), we conclude that $\nabla u_\varepsilon \xrightarrow[\varepsilon\to 0]{} \nabla u$, in $L^2_{\text{loc}}(\Omega\times(0,+\infty))$. □

Remark 9 *The uniqueness result in Lemma 6 plays an important role in the proof of Lemma 8. Indeed, the proof of Lemma 8 (following [33, Lemma 3.3]) uses a regularization of u_ε, say $u_{\varepsilon,\eta}$ to get (31) in the terms of $u_{\varepsilon,\eta}$. After that, passing to the limit as $\eta\to 0$, we get the conclusion of Lemma 8. The uniqueness result ensures that $u_{\varepsilon,\eta}$ converges to the unique solution u_ε mentioned above.*

Now, it is sufficient to show the claim (26). Indeed, using (10) and Fatou's lemma asserts that there is a non-negative function $\Phi\in L^1(\Omega\times(0,T))$ such that

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon)=\Phi \quad \text{in } L^1(\Omega\times(0,T)). \tag{36}$$

Furthermore, we observe that

$$g_\varepsilon(u_\varepsilon)(x,t)\ge g_\varepsilon(u_\varepsilon)\chi_{\{u>0\}}(x,t), \quad \text{for a.e. } (x,t)\in\Omega\times(0,T),$$

which implies that

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon)(x,t)\ge u^{-\beta}\chi_{\{u>0\}}(x,t), \quad \text{for a.e. } (x,t)\in\Omega\times(0,T). \tag{37}$$

It follows from the Lebesgue's dominated convergence theorem that

$$u^{-\beta}\chi_{\{u>0\}}\le\Phi \quad \text{and} \quad u^{-\beta}\chi_{\{u>0\}}\in L^1(\Omega\times(0,T)). \tag{38}$$

Now we shall use a L^1-technique. For any fixed $\eta > 0$, we use the test function $\psi_\eta(u_\varepsilon)\phi$, $\phi \in \mathcal{C}_c^\infty(\Omega \times (0,T))$ to the equation satisfied by u_ε. Then, an integration by part gives us

$$\int_{Supp(\phi)} \left(-\Psi_\eta(u_\varepsilon)\partial_t\phi + \frac{1}{\eta}|\nabla u_\varepsilon|^2 \psi'\left(\frac{u_\varepsilon}{\eta}\right)\phi + \psi_\eta(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\phi + g_\varepsilon(u_\varepsilon)\psi_\eta(u_\varepsilon)\phi \right) dxds = 0,$$

where

$$\Psi_\eta(u) = \int_0^u \psi_\eta(s)ds.$$

By (14), (33), we can pass to the limit as $\varepsilon \to 0$ in the last inequality in order to get

$$\int_{Supp(\phi)} \left(-\Psi_\eta(u)\partial_t\phi + \frac{1}{\eta}|\nabla u|^2 \psi'\left(\frac{u}{\eta}\right)\phi + \psi_\eta(u)\nabla u \cdot \nabla\phi + u^{-\beta}\psi_\eta(u)\phi \right) dxds = 0, \tag{39}$$

From (33), (38) and the Lebesgue's dominated convergence theorem, it is not difficult to verify that

$$\lim_{\eta\to 0} \int_{Supp(\phi)} \left(-\Psi_\eta(u)\partial_t\phi + \psi_\eta(u)\nabla u \cdot \nabla\phi + u^{-\beta}\psi_\eta(u)\phi \right) dxds$$
$$= \int_{Supp(\phi)} \left(-u\partial_t\phi + \nabla u \cdot \nabla\phi + u^{-\beta}\chi_{\{u>0\}}\phi \right) dxds, \tag{40}$$

with any term of the left hand side converges to any term of the right hand side in order.
On the other hand, it follows from (33) that

$$\begin{aligned} \frac{1}{\eta}\int_{Supp(\phi)} |\nabla u|^2 \left|\psi'\left(\frac{u}{\eta}\right)\phi\right| dxds &\le C(\phi)\frac{1}{\eta}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{1-\beta}dxds \\ &\le 2C(\phi)\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{-\beta}dxds. \end{aligned}$$

Then, by the Lebesgue's dominated convergence theorem and (38), we obtain

$$\lim_{\eta\to 0}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{-\beta}dxds = 0.$$

This leads to

$$\lim_{\eta\to 0}\int_{Supp(\phi)} \frac{1}{\eta}|\nabla u|^2\psi'\left(\frac{u}{\eta}\right)\phi dxds = 0. \tag{41}$$

Combining (39), (40) and (41) yields

$$\int_{Supp(\phi)} \left(-u\partial_t\phi + \nabla u \cdot \nabla\phi + u^{-\beta}\chi_{\{u>0\}}\phi \right) dxds = 0. \tag{42}$$

Note that (42) says that u is a weak solution of (1) in $\Omega \times (0, +\infty)$. However, this is not enough to conclude that u is a mild solution of (1).
Since u_ε is a weak solution of (4), we have

$$\int_{Supp(\phi)} \left(-u_\varepsilon\partial_t\phi + \nabla u_\varepsilon \cdot \nabla\phi + g_\varepsilon(u_\varepsilon)\phi \right) dxds = 0.$$

The passage to the limit as $\varepsilon \to 0$ provides us

$$\int_{Supp(\phi)} (-u\partial_t \phi + \nabla u \cdot \nabla \phi)\, dxds + \lim_{\varepsilon \to 0} \int_{Supp(\phi)} g_\varepsilon(u_\varepsilon)\phi\, dxds = 0. \tag{43}$$

By (42) and (43), we get

$$\lim_{\varepsilon \to 0} \int_0^{+\infty} \int_\Omega g_\varepsilon(u_\varepsilon)\phi dxds = \int_0^{+\infty} \int_\Omega u^{-\beta} \chi_{\{u>0\}} \phi dxds. \tag{44}$$

Thanks to Fatou's lemma, (36) and (44), we obtain for any non-negative $\phi \in \mathcal{C}_c^\infty(\Omega \times (0, +\infty))$,

$$\int_0^{+\infty} \int_\Omega u^{-\beta} \chi_{\{u>0\}} \phi dxds \geq \int_0^{+\infty} \int_\Omega \Phi \phi dxds.$$

We deduce, from this last inequality and (38), that

$$u^{-\beta} \chi_{\{u>0\}} = \Phi, \quad \text{a.e. in } \Omega \times (0, +\infty);$$

in other words, that claim (26) holds.
Now, it is clear that u is a mild solution of (1) since we have

$$u_\varepsilon(t) = S(t)u_0 - \int_0^t S(t-s) g_\varepsilon(u_\varepsilon(s))ds. \tag{45}$$

and the conclusion follows by passing to the limit as $\varepsilon \to 0$ in (45) with the help of (26). $\square$

Finally, we shall show that the solution u constructed above is the maximal solution of (1).

Theorem 10 *Let v be any mild solution of* (1). *Then, we have*

$$v \leq u \quad \text{in } \Omega \times (0, +\infty).$$

Proof: First of all, we claim that any mild solution v of (1) satisfies

$$v \in L^2(\tau, T; W_0^{1,2}(\Omega)) \cap L^\infty(\Omega \times (\tau, +\infty)), \quad \text{for } 0 < \tau < T < +\infty. \tag{46}$$

Since the arguments are rather classical we skip the proof of (46) for the moment but we shall prove it in the Appendix section. Then, assumed (46), we have for any $\varepsilon > 0$,

$$0 = \partial_t v - \Delta v + v^{-\beta} \chi_{\{v>0\}} \geq \partial_t v - \Delta v + + g_\varepsilon(v).$$

This implies that v is a sub-solution of (4). Applying Lemma 6 to v and u_ε we get

$$v \leq u_\varepsilon \quad \text{in } \Omega \times (0, +\infty).$$

Letting $\varepsilon \to 0$ we arrive to the desired conclusion. $\square$

Remark 11 *It can be shown (see, e.g., [9, Lemma* 2.4*], [33, Lemma* 3.1*]) that if $\beta \geq 1$, no mild solution can exist. Nevertheless, by extending the notion of solution (by requiring merely that $\chi_{\{u>0\}} u^{-\beta} \in L^1(0, T; L^1(\Omega, d))$), it seems possible to show, as in the elliptic case (see [12, 13]), that such extended notion of solution exists even if $\beta \in (0, 2)$ and also for suitable initial data merely in $u_0 \in L^1(\Omega, d)$. Here again, $d(x)$ is the distance from x to the boundary of the domain Ω (this could follow the same lines of proof as in the elliptic case [13]).*

Remark 12 *It is easy to see that the above proof allows to get some extra information on the regularity of the mild solution. In the following, we list some of such properties but since the arguments for their proof are today rather classical we leave the details to the interested reader (see also the proof of Theorem 13 bellow):*

1. $u^{1-\beta} \in L^{\frac{1}{1-\beta}}(\Omega \times (0,T))$, $t^{\frac{N}{2}}u_0 \in L^\infty(0,T;L^1(\Omega))$, $u \in C(\Omega \times (0,T))$ *and for any* $\tau > 0$, $u \in L^\infty(\tau, T; H_0^1(\Omega))$ *and* $u_t \in L^2(\tau, T; L^2(\Omega))$.
2. *If* $u_0 \in L^2(\Omega)$, *then* $u \in L^2(0,T; H_0^1(\Omega))$ *and if we define*

$$D(u,0,T) := \operatorname*{ess\,sup}_{s\in(0,T)} \int_\Omega |u(x,s)|^2\, dx + \int_{\Omega\times(0,T)} \left(|\nabla u|^2 + |u|^{1-\beta}\right) dxdt, \tag{47}$$

   then we have $D(u,0,T) < +\infty$.
3. *Finally, if* $u_0 \in H_0^1(\Omega)$, *then* $\partial_t u \in L^2(0,T;L^2(\Omega))$ *and* $u \in L^\infty(0,+\infty; H_0^1(\Omega)) \cap L^2(0,+\infty; H_0^1(\Omega))$.

3 Quenching phenomenon in a finite time

It is well known (see, e.g., [16]) that since, for any $\tau > 0$, the maximal solution u belongs to $L^\infty(\tau,+\infty; H_0^1(\Omega)) \cap L^2(\tau,+\infty; H_0^1(\Omega))$, $u(x,t) \to 0$ as $t \to +\infty$. In this section, we shall show a stronger property: in fact any mild solution of (1) vanishes after a finite time, likewise Theorem 4 will state. As a previous comment, we recall that thanks to Theorem 10, it is enough to show this property only for the maximal solution u.

Theorem 13 *Let u be as in Theorem 7. Then, $u(t)$ vanishes on the whole domain Ω after a finite time $T^*> 0$. Moreover, T^* only depends on $\|u_0\|_{L^1(\Omega)}, N$ and $|\Omega|$.*

Proof: First of all, we establish the energy equation for u (local in time). By multiplying the equation of u_ε by u_ε and integrating by parts, we get for any $0 < \tau < t < +\infty$,

$$\frac{1}{2}\int_\Omega (|u_\varepsilon(t)|^2 - |u_\varepsilon(\tau)|^2)dx + \int_\tau^t \int_\Omega |\nabla u_\varepsilon|^2 dxds + \int_\tau^t \int_\Omega g_\varepsilon(u_\varepsilon)u_\varepsilon dxds = 0.$$

By passing to the limit in the last equation as $\varepsilon \to 0$, we deduce that

$$\frac{1}{2}\int_\Omega (|u(t)|^2 - |u(\tau)|^2)dx + \int_\tau^t \int_\Omega |\nabla u|^2 dxds + \int_\tau^t \int_\Omega u^{1-\beta} dxds = 0. \tag{48}$$

Then, usual variational arguments leads to the fact that for a.e. $t \in (0,+\infty)$,,

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |u(t)|^2 dx\right) + \int_\Omega |\nabla u(t)|^2 dx + \int_\Omega u^{1-\beta}(t)dx = 0. \tag{49}$$

On the other hand, from the Gagliardo-Nirenberg inequality, we have

$$\|u(t)\|_{L^2(\Omega)} \le C(N,\theta)\|\nabla u(t)\|^\theta_{L^2(\Omega)}\|u(t)\|^{1-\theta}_{L^1(\Omega)}, \tag{50}$$

with $\theta = \frac{N}{N+2}$, and $C(N,\theta) = C(N)$. Moreover, for any fixed $\tau > 0$, (9) yields

$$\sup_{t\ge\tau} \|u(t)\|_{L^\infty(\Omega)} \le C(N,|\Omega|)\tau^{-\frac{N}{2}}\|u_0\|_{L^1(\Omega)} := M_\tau.$$

Thus, we have for any $t \geq \tau$,

$$\int_\Omega u^{1-\beta}(t)dx \geq M_\tau^{-\beta} \int_\Omega u(t)dx. \tag{51}$$

Combining (50) and (51), we deduce that

$$\begin{aligned} M_\tau^{-\beta(1-\theta)} \|u(t)\|_{L^2(\Omega)} &\leq C(N) \left(\int_\Omega |\nabla u(t)|^2 dx \right)^{\frac{\theta}{2}} \left(M_\tau^{-\beta} \int_\Omega u(t)dx \right)^{1-\theta} \\ &\leq C(N) \left(\int_\Omega |\nabla u(t)|^2 dx \right)^{\frac{\theta}{2}} \left(\int_\Omega u^{1-\beta}(t)dx \right)^{1-\theta} \\ &\leq C(N) \left(\int_\Omega |\nabla u(t)|^2 dx + \int_\Omega u^{1-\beta}(t)dx \right)^{\frac{\theta}{2}+1-\theta}. \end{aligned}$$

Then,

$$M_\tau^{-\frac{2\beta(1-\theta)}{2-\theta}} \left(\int_\Omega |u(t)|^2 dx \right)^\gamma \leq C(N) \left(\int_\Omega |\nabla u(t)|^2 dx + \int_\Omega u^{1-\beta}(t)dx \right), \tag{52}$$

with $\gamma := \frac{1}{2-\theta} = \frac{N+2}{N+4}$. Hence, from (49) and (52), we get for any $t \geq \tau$,

$$\frac{d}{dt} w(t) + K(\tau) w^\gamma(t) \leq 0. \tag{53}$$

where

$$w(t) := \int_\Omega |u(t)|^2 dx \quad \text{ and } \quad K(\tau) := 2C(N)^{-1} M_\tau^{-\frac{2\beta(1-\theta)}{2-\theta}}.$$

Thus, since $u(\tau) \in L^2(\Omega)$, for any $\tau > 0$, and since $\gamma \in (0,1)$, $w(t)$ vanishes after a finite time (see, e.g., [2]).

Finally, we shall show that the vanishing time (*i.e.* the quenching time) of $u(t)$ can be estimated by a constant only depending on $\|u_0\|_{L^1(\Omega)}$ and N, $|\Omega|$. In fact, by the smoothing effect estimate (see [5, 30]), we know that

$$w(\tau)^{\frac{1}{2}} = \|u(\tau)\|_{L^2(\Omega)} \leq C\tau^{-\frac{N}{4}} \|u_0\|_{L^1(\Omega)}.$$

Thus, by integrating in the ODE (53), we get for any $t \geq \tau$,

$$w^{1-\gamma}(t) + (1-\gamma)K(\tau)(t-\tau) \leq \left(C\tau^{-\frac{N}{4}} \|u_0\|_{L^1(\Omega)} \right)^{2(1-\gamma)}. \tag{54}$$

Let T_{min} be a minimum vanishing time of $u(t)$. According to (54), we have for any $\tau > 0$,

$$T_{min} \leq T(\tau) = \tau + C_1(N, \gamma, |\Omega|) \tau^{-\frac{N}{2}(1-\gamma)} K(\tau)^{-1} \|u_0\|_{L^1(\Omega)}^{2(1-\gamma)}.$$

By a computation based on the definition of $K(\tau)$ and $M(\tau)$, we obtain

$$T(\tau) = \tau + C_2(N, \gamma, |\Omega|) \tau^{-\left(\frac{N(1-\gamma)}{2} + N\beta(1-\theta)\gamma \right)} \|u_0\|_{L^1(\Omega)}^{2\beta(1-\theta)\gamma + 2(1-\gamma)} := \tau + C_2 \tau^{-\alpha_1} \|u_0\|_{L^1(\Omega)}^{\alpha_2}.$$

But

$$\min_{\tau>0}\{\tau + C_2\tau^{-\alpha_1}\|u_0\|^{\alpha_2}_{L^1(\Omega)}\} = \tau_0 + C_2\tau_0^{-\alpha_1}\|u_0\|^{\alpha_2}_{L^1(\Omega)},$$

with $\tau_0^{\alpha_1+1} := \alpha_1 C_2\|u_0\|^{\alpha_2}_{L^1(\Omega)}$. Then, the previous equality gives us

$$\min_{\tau>0}\{\tau + C_2\tau^{-\alpha_1}\|u_0\|^{\alpha_2}_{L^1(\Omega)}\} = C_3\|u_0\|^{\frac{\alpha_2}{\alpha_1+1}}_{L^1(\Omega)} := T^*,$$

with $C_3 = C_3(N, \gamma, |\Omega|)$. Then, $T_{min} \le T^*$, which completes the proof. $\square$

Remark 14 *The quenching property was established in the previous literature (see, e.g., [20, 33]) only for the special case of bounded initial data or $u_0 \in L^2(\Omega)$; and so, the obtained quenching time T^* was always depending on $\|u_0\|_{L^\infty(\Omega)}$ or $\|u_0\|_{L^2(\Omega)}$. Thus, our result is sharper in the sense that we merely require that $u_0 \in L^1(\Omega)$.*

Remark 15 *The phenomenon of extinction in finite time was earlier well known for the case of absorption terms with a positive power less than one (which corresponds to the case $\beta \in (-1, 0]$ in our formulation). See the exposition and references quoted in [2].*

We shall end this paper by showing that the quenching phenomenon has a local nature in the sense that according to the initial data, the solution may start to vanish in a small subset of Ω.

Proof: *(of Theorem 4)* Thanks to the additional regularity mentioned in a previous Remark the proof is a direct application of [11, Theorem 4.2]. Indeed, even if $u_0 \in L^1(\Omega)$ the maximal solution satisfies that

$$D(u, \tau, T) := \operatorname*{ess\,sup}_{s\in(\tau,T)} \int_\Omega |u(x,s)|^2\, dx + \int_{\Omega\times(\tau,T)} \left(|\nabla u|^2 + |u|^{1-\beta}\right) dxds < +\infty,$$

for any $\tau > 0$. Moreover, we can choose $t^* \in (0, T)$ such that energy $D(u, t^*, T)$ may be as small as we wish and thus, the conclusion comes from the application of [11, Theorem 4.2]. $\square$

Remark 16 *Some estimates on the behavior of $\chi_{\{u>0\}}u^{-\beta}(t)$ near the quenching time T^* can be found in [15] (see also its references). We also point out that in our framework (i.e., for $\beta \in (0, 1)$) the gradient estimate (33) implies that on the free boundary (defined as the boundary of the set $\{(x, t)\colon u(x, t) > 0\}$), we have not only that $u = 0$ but that also $\nabla u = 0$. This property fails if $\beta \ge 1$ (see [15, 22]).*

Remark 17 *Other qualitative properties of the free boundary (as the "finite speed of propagation" or the "finite waiting time" properties) can also be obtained as application of the results of [11, Theorem 1] under additional information on the initial datum u_0.*

4 Appendix

4.1 Proof of Theorem 5

Let us regularize the initial condition u_0 by considering a nonnegative sequence $\{u_{0,k}\}_k \subset \mathcal{C}_c^\infty(\Omega)$ such that $u_{0,k} \xrightarrow[k\to+\infty]{} u_0$ in $L^1(\Omega)$, and consider the problem

$$\begin{cases} \partial_t v_k - \Delta v_k + g_\varepsilon(v_k) = 0, & \text{in } \Omega \times (0,T), \\ v_k = 0, & \text{on } \partial\Omega \times (0,T), \\ v_k(\cdot, 0) = u_{0,k}(\cdot) & \text{on } \Omega. \end{cases} \tag{55}$$

Since g_ε is a global Lipschitz-continuous function, the classical result ensures the existence and the uniqueness of a classical solution v_k. Moreover, v_k fulfils that for any $t > 0$,

$$v_k(t) = S(t)u_{0,k} - \int_0^t S(t-s)g_\varepsilon(v_k(s))ds. \tag{56}$$

Next, we claim that, for any $T > 0$, $v_k \geq 0$ in $\Omega \times (0,T)$. Indeed, it is sufficient to show that

$$\min_{(x,t)\in\Omega\times(0,T)} v_k(x,t) \geq 0.$$

We can assume by contradiction that there is a point $(x_0, t_0) \in \Omega \times (0,T)$ such that

$$\min_{\Omega\times(0,T)} v_k(x,t) = v_k(x_0,t_0) < 0.$$

Let $\overline{v}_k(x,t) := v_k(x,t) + \delta t$, with $\delta > 0$ small enough such that $\overline{v}_k(x_0,t_0) = v_k(x_0,t_0) + \delta t_0 < 0$. This implies that $\overline{v}_k$ attains its minimum at a point inside of $\Omega \times (0,T)$, say $(x_1,t_1) \in \Omega \times (0,T)$, and $\overline{v}_k(x_1,t_1) \leq \overline{v}_k(x_0,t_0) < 0$. Then, we have $\partial_t \overline{v}_k(x_1,t_1) = 0$ and $\Delta \overline{v}_k(x_1,t_1) \geq 0$, so

$$0 = \partial_t v_k(x_1,t_1) - \Delta v_k(x_1,t_1) + g_\varepsilon(v_k(x_1,t_1)) = (\partial_t \overline{v}_k(x_1,t_1) - \delta) - \Delta \overline{v}_k(x_1,t_1) + 0.$$

This leads to a contradiction. Thus, we get the claim.

Next, we proceed as in the proof of Theorem 7 to get $v_k \to u_\varepsilon$, in $L^r(0,T; W_0^{1,r}(\Omega))$, as $k \to +\infty$ (up to a subsequence if necessary), and that $u_\varepsilon \in \mathcal{C}([0,T]; L^1(\Omega))$. Then, it suffices to pass to the limit in (56) as $k \to +\infty$ in order to get (5).
It remains to show now that $u_\varepsilon \in \mathcal{C}_{x,t}^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times (\tau,T))$ for any $0 < \tau < T < +\infty$, with some $\alpha \in (0,1)$. Indeed, applying the result of [25] to v_k we obtain that $\partial_t v_k,\ \nabla v_k,\ D^2_{x_i x_j} v_k \in L^p(\Omega \times (\tau,T))$, for $p > 1$. When p is large enough (such as $p{>}N+2$), we have that $v_k \in \mathcal{C}_{x,t}^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times (\tau,T))$, for some $\alpha \in (0,1)$. Note that v_k is bounded in $\mathcal{C}_{x,t}^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times (\tau,T))$ by a constant independent of k. Therefore, Ascoli's theorem implies that there is a subsequence (still denoted $\{v_k\}_k$) such that

$$v_k \xrightarrow[k\to+\infty]{} u_\varepsilon \quad \text{in } \mathcal{C}_{x,t}^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times (\tau,T)).$$

On the other hand, u_ε satisfies equation

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = -g_\varepsilon(u_\varepsilon).$$

But, since g_ε is Lipschitz-continuous, we have that $g_\varepsilon(u_\varepsilon) \in \mathcal{C}_{x,t}^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times (\tau,T))$. Then, the conclusion $u_\varepsilon \in \mathcal{C}_{x,t}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega} \times (\tau,T))$ follows from the α-Holder regularity of parabolic equations.

4.2 Proof of claim (46)

Let v be a mild solution of (1) and let us consider the following problem:

$$\begin{cases} \partial_t \overline{v} - \Delta \overline{v} + f = 0, & \text{in } \Omega \times (0,T), \\ \overline{v} = 0, & \text{on } \partial\Omega \times (0,T), \\ \overline{v}(\cdot,0) = u_0(\cdot) & \text{on } \Omega. \end{cases} \tag{57}$$

where $f{:=}v^{-\beta}\chi_{\{v>0\}} \in L^1(\Omega\times(0,T))$ and $0 < T < +\infty$. Then, a classical result (see for example [4, Lemma 3.3]) ensures that there is a unique mild (or weak) solution $\overline{v}$ of (57). Moreover, [4, Lemma 3.4] asserts $v = \overline{v}$ in $\Omega \times (0,T)$. To prove (46), it is enough to show that, for any $0 < \tau < T < +\infty$, $\overline{v} \in L^2(\tau, T; W_0^{1,2}(\Omega))$. Indeed, let $\{f_n\}_n \subset \mathcal{C}_c^\infty(\Omega \times (0,+\infty))$ be a sequence converging to f in $L^1(\Omega \times (0,+\infty))$ as $n \to +\infty$. Then, there exists a unique classical solution of the following equation:

$$\begin{cases} \partial_t \overline{v}_n - \Delta \overline{v}_n + f_n = 0, & \text{in } \Omega \times (0,T), \\ \overline{v}_n = 0, & \text{on } \partial\Omega \times (0,T), \\ \overline{v}_n(\cdot,0) = u_0(\cdot) & \text{on } \Omega. \end{cases}$$

Consider the difference between two equations satisfied by $\overline{v}_n$ and $\overline{v}_m$:

$$\partial_t(\overline{v}_n - \overline{v}_m) - \Delta(\overline{v}_n - \overline{v}_m) + f_n - f_m = 0,$$

Multiplying the above equation with $\overline{v}_{n,m}{:=}\overline{v}_n - \overline{v}_m$ and integrating by parts we get

$$\frac{1}{2}\int_\Omega (\overline{v}_{n,m})^2(T)dx + \int_\tau^T \int_\Omega |\nabla \overline{v}_{n,m}|^2 dxds = \int_\tau^T \int_\Omega (f_m - f_n)\overline{v}_{n,m} dxds + \frac{1}{2}\int_\Omega (\overline{v}_{n,m})^2(\tau)dx.$$

This implies

$$\int_\tau^T \int_\Omega |\nabla \overline{v}_{n,m}|^2 dxds \le \int_\tau^T \int_\Omega |f_m - f_n|.|\overline{v}_{n,m}| dxds + \frac{1}{2}\int_\Omega (\overline{v}_{n,m})^2(\tau)dx.$$

The fact that $(f_n - f_m)$ converges to 0 in $L^1(\Omega \times (0,T))$ as $n, m \to +\infty$, and that $\{v_n\}_n$ is bounded by (9) assert that

$$\lim_{n,m\to+\infty} \int_\tau^T \int_\Omega |f_m - f_n|.|\overline{v}_{n,m}| dxds = 0.$$

Moreover, using the same compactness argument as in the proof of Theorem 7, we get

$$\lim_{n,m\to+\infty} \int_\Omega (\overline{v}_{n,m})^2(\tau)dx = 0.$$

Finally, combining the last three inequalities, we deduce that

$$\lim_{n,m\to+\infty} \int_\tau^T \int_\Omega |\nabla \overline{v}_{n,m}|^2 dxds = 0.$$

Then, the uniqueness result implies that $\{\nabla \overline{v}_n\}_n$ converges to ∇v in $L^2(\Omega\times(\tau,T))$ and we reach the conclusion.

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