Evaluation of long-dated investments under uncertain growth trend, volatility and catastrophes

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Abstract
In this paper, we examine the term structures of interest rates and risk premia when the random walk of economic growth is affected by some parametric uncertainty. Using a time-consistent expected utility framework, we show that parametric uncertainty does not affect assets prices of short maturities. We also show that the same arguments proposed in the literature to justify a decreasing term structure for the safe discount rate also apply to justify an increasing term structure for the risk premium. Another important consequence of parametric uncertainty is that the risk premium is not proportional to the beta of the investment. We apply these general results to the case of an uncertain probability of macroeconomic catastrophes à la Barro (2006), and to the case of an uncertain trend or volatility of growth à la Weitzman (2007). Finally, we apply our findings to the evaluation of climate change policy. We argue in particular that the beta of actions to mitigate climate change is relatively large, so that the term structure of the risk-adjusted discount rates should be increasing.

Keywords: asset prices, term structure, risk premium, decreasing discount rates, uncertain growth, climate beta, rare events, long-term risk.

JEL Codes: G11, G12, E43, Q54.

1. Introduction

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Do we do enough for the distant future? This question is implicit in many policy debates, from the fight against climate change to the speed of reduction of public deficits, investments in research and education, or the protection of the environment and of natural resources for example. The discount rate used to evaluate investments is the key determinant of our individual and collective efforts in favor of the future. Since Weitzman (1998), an intense debate has emerged among economists about whether one should use different discount rates for different time horizons *t*. It is however well-known that the term structure of efficient discount rates is flat and constant through time if we assume that the representative agent has a constant relative risk aversion and that the growth rate of consumption follows a random walk. In this benchmark specification, if a rate of 3% is efficient to discount cash flows occurring in 12 months, it is also efficient to use that rate of 3% to discount cash flows occurring in 200 years. This yields a present value of a given benefit that is exponentially decreasing with its maturity.

Compared to this benchmark, a decreasing term structure of discount rates would bias the economic evaluation of investments towards those with more distant benefits. Using Jensen inequality, Weitzman (1998, 2001) and Newell and Pizer (2003) justified such a decreasing structure of the "certainty equivalent discount rates" by relying on the observation that the future return of capital is unknown, and that the NPV is decreasing and convex in it. Gollier (2002, 2008, 2012) and Weitzman (2007) used standard consumption-based asset pricing theory to explore the same question. The basic idea underlying these papers is that the large uncertainty associated to aggregate consumption in the distant future should induce the prudent representative agent to use lower rates to discount more distant cash flows.[2] Under constant relative risk aversion (CRRA), the various dynamic processes that support this result include for example mean-reversion, Markov regime-switches, and parametric uncertainty on the trend of a Brownian motion.[3] Gollier (2008) demonstrates that the positive serial correlation (or persistence) of the growth of consumption that is inherent to these stochastic processes is the driving force of the result, together with prudence. For growth processes with persistent shocks, aggregate uncertainty accumulates faster with respect to longer time horizons than in a pure random walk with the same instantaneous volatility. Prudent people want to bias their

[2] Prudence is a concept defined by Kimball (1990) to characterize the willingness to save more when the future becomes more uncertain.
[3] Persistent movements in expected growth rates of aggregate consumption are documented for the U.S. by Bansal and Yaron (2004) for example.

investments towards those which yield more sure benefits for these horizons. This bias is implemented by using a decreasing term structure.

With the notable exception of Weitzman (2013), this recent literature focuses on rates r_{ft} at which *safe* cash flows should be discounted. In reality, most investment projects yield uncertain future costs and benefits. For marginal projects, we know that idiosyncratic risks should not be priced, because they will be washed out in diversified portfolios. In public economics, this result is usually referred to as the Arrow-Lind Theorem (Arrow and Lind (1970)), but this is a well-known feature of the consumption-based capital asset pricing model (Rubinstein (1976), Lucas (1978), Breeden (1979)). More generally, the discount rate ρ_t to be used to evaluate risky projects depends upon their beta which measures the elasticity of net cash flows to changes in aggregate consumption. A positive beta justifies a positive risk premium $\pi_t(\beta) = \rho_t(\beta) - r_{ft}$. In the benchmark specification with CRRA and a random walk for the growth rate of consumption, the term structure of the risk premium is flat. However, the arguments listed above in favor of a downward sloping term structure of safe discount rates are also in favor of an upward sloping term structure of the aggregate risk premia. If we assume that the stochastic process of the growth rate of consumption exhibits some positive serial correlation, the annualized measure of aggregate risk will have an increasing term structure. Under risk aversion, the term structure of the risk premia will inherit this property, as shown for example by Hansen (2012).

With positively serially correlated growth rates, the project-specific discount rate $\rho_t(\beta) = r_{ft} + \pi_t(\beta)$ for positive betas is thus the sum of a prudence-driven decreasing function r_{ft} and of a risk-aversion-driven increasing function $\pi_t(\beta)$ of the time horizon t. In standard models, this risk premium is proportional to the beta. Thus, this term structure will be downward sloping if the project-specific beta is small enough. This implies that the recent literature on the discount rate that has been advocating decreasing discount rates is potentially misleading. If the beta of some green projects is positive and large, $\rho_t(\beta)$ may well be increasing with maturity. From that point of view, this paper provides a more balanced discussion about the shape of the term structure of discount rates.

In financial economics, this risk-return trade-off has been examined mostly for small maturities. More recent researches have been aimed at characterizing long-run risk prices (Hansen, Heaton and Li (2008), Hansen and Scheinkman (2009), Hansen (2012), Martin (2012)). One of the contributions of this paper is to build a bridge between these recent developments of the theory of asset pricing to this debate on the socially efficient long-term discount rates. Contrary to Hansen, Heaton and Li (2008), Hansen and Scheinkman (2009) and Hansen (2012) who derived general decomposition properties of the long-term risk prices under various auto-regressive frameworks, we examine in this paper a specific economy in which the growth of log consumption is known to be stationary and serially independent, but its stochastic process is subject to some parameter uncertainty. For maturities measured in decades and centuries, we believe that it is crucial to adapt the CCAPM by recognizing that the stochastic process governing the growth of aggregate consumption is affected by parametric uncertainties. A simple specification is examined in Section 4, in which we assume that log consumption follows an arithmetic Brownian motion, but its trend or volatility is uncertain. Observe that the uncertainty on the trend of growth implies that the unconditional growth rates are positively correlated, thereby magnifying the uncertainty affecting consumption in the long run. This explains again why the term structures of the risk free discount rate and of the risk premium are respectively downward and upward sloping in this specification. This analysis is generalized to the case of mean-reversion in Section 5.

In Section 6, we assume that the economy may face macroeconomic catastrophes at low frequency. In normal time, the growth of log consumption is Gaussian, but a large drop in aggregate consumption strikes the economy at infrequent dates. Our modeling duplicates the one proposed by Barro (2006, 2009), at the notable exception that we take seriously a critique formulated by Martin (2013). Martin convincingly demonstrates that it is very complex to estimate the true probability of infrequent catastrophes, and that a small modification in the choice of parameters values has a huge effect on asset prices. We take this into account by explicitly introducing ambiguity about this probability into the model. This implies the same kind of positive serial correlation in the unconditional growth rates of the economy. An important consequence of introducing uncertainty on the probability of catastrophes is that, contrary to Barro (2006, 2009), the term structures of discount rates are generally not flat. In a calibration exercise based on Barro (2009), we show that the risk free rate and the risk premium for long maturities are strongly impacted by the uncertainty affecting the probability of catastrophes.

The mathematical methodology used in this paper was suggested by Martin (2013). We followed here this author's suggestion to use the properties of Cumulant-Generating Functions (CGF) to characterize asset prices when the growth of log consumption is not Gaussian. This paper provides various illustrations of the power of this method, whose basic elements are presented in sections 2 and 3. It can also be inferred from Martin (2013) that abandoning the Gaussian assumption for growth introduces a new source of complexity in asset pricing theory. Namely, without the Gaussian assumption, the risk premium is generally not proportional to the beta of the asset, or to the variance of log consumption. We show in this paper that the nonlinearity is quite impressive in realistic calibrations of our specifications.

Our goal in this paper is normative. We show that the same ingredient, i.e., parametric uncertainty affecting the growth process, can explain why one should use a downward sloping risk free discount rate and an upward sloping risk premium to value investment projects. One should however recognize that markets do not follow these recommendations. The yield curve is indeed increasing in normal time. And it has recently been discovered that the term structure of the equity premium is downward sloping (e.g. Binsbergen, Brandt and Koijen (2012)). However, these puzzles may be explained by hyperbolic discounting and a time consistency problem for the yield curve (Laibson (1997)), and by some limited information problem for the term structure of the risk premia (Croce, Lettau, and Ludvigson (2012)).

This normative goal justifies using the standard exponential Discounted Expected Utility model, which has an appealing normative ground. In particular, this implies that the representative agent is neutral to the ambiguity affecting the parameters of the growth process. This is a clear difference with respect to the important recent literature of asset pricing under ambiguity aversion (see for example Hansen and Sargent (2010) and Gollier (2011a)). As is well-known, using the DEU model necessarily implies the time-consistency of risk management and asset prices: The price of an asset today is the risk-adjusted discounted value of its information-dependent price tomorrow.[4] However, in this paper, we don't examine the dynamic of prices, which entails learning and the sensitiveness of prices to future information. These effects exist and are anticipated by the representative agent, and our results are perfectly compatible with these phenomena not described in the paper.

[4] In the terminology of Hansen and Scheinkman (2009), this means that our pricing operator satisfies the semigroup property.

Our paper provides important new insights about how public policies should be evaluated around the world. It is worrying to observe that it is common practice in public administrations to use a single discount rate to evaluate public investments independent of their riskiness and time horizons. In the U.S. for example, the Office of Management and Budget (OMB) recommend to use a flat discount rate of 7% since 1992. It was argued that the "*7% is an estimate of the average before-tax rate of return to private capital in the U.S. economy*" (OMB (2003)). In 2003, the OMB also recommended the use of a discount rate of *3%*, in addition to the *7%* mentioned above as a sensitivity. The *3%* corresponds to the average real rate of return of the relatively safe 10-year Treasury notes between 1973 and 2003. Interestingly enough, the recommended use of 3% and 7% is not differentiated by the nature of the underlying risk, and is independent of the time horizon of the project. In another field, guidelines established by the Government Accounting Standards Board (GASB) recommend that state and local governments discount their pension liabilities at expected returns on their plan assets, which is usually estimated around 8%, independent of their maturities.[5] The absence of risk-and-maturity-based price signals has potentially catastrophic consequences for the allocation of capital in the economy.[6] This paper provides recommendations about the changes in evaluation tools that should be implemented.

Another aim in this paper is to make recommendations about which discount rates should be used to evaluate environmental policies, in particular those associated to climate change. This raises the question of the beta of climate change, which we believe to be crucial for the determination of the so-called "social cost of carbon" (SCC). Sandsmark and Vennemo (2007) claim that the beta of mitigation investments is negative, so that the term structure of discount rates should be low and decreasing, thereby yielding a large SCC. They consider a simplified version of the standard integrated assessment model by Nordhaus' DICE model (Nordhaus and Boyer (2000)). They assumed that the only source of aggregate fluctuations originates from climate change, with an uncertain climate sensitivity affecting socioeconomic damages to the economy.[7] Under this assumption, a large climate sensitivity yields at the same time a low consumption (due to the

[5] The European Union is currently debating about the new solvency regulation of insurance companies (Solvency 2). In the most recent consultation paper (European Insurance and Occupational Pensions Authority (2012)), it is proposed to discount safe liabilities using the yield curve up to 20-year maturities, and a real discount rate tending to 2% ("Ultimate Forward Rate") for longer maturities.

[6] In 2005, France has adopted a decreasing real discount rate from 4% to 2% for safe projects. This rule has been complemented in 2011 by an aggregate risk premium of 3% (Gollier (2011)).

[7] The climate sensitivity is a physical parameter that measures the relationship between the concentration of greenhouse gases in the atmosphere and the average temperature of the earth.

climate damages) and a large social benefit from early mitigation. This explains the negative beta of their model. But suppose alternatively that the climate sensitivity is known, but the growth rate of aggregate consumption is unknown. Because emissions are increasing in consumption, a larger growth rate of consumption goes together with a larger concentration of CO2. Because the damage function is assumed to be convex with the concentration of greenhouse gases, it also goes with larger damages, and with a larger societal benefit from early mitigation. This justifies a positive beta. We show in Section 8 that any credible calibration of a model combining the two sources of aggregate fluctuations yields a positive and large beta of mitigation. From our discussion above, this is compatible with using increasing discount rates to measure SCC. This provides a radical reversal in the trend of the literature on discounting. This suggests that rather than focusing on climate change, one should rather invest in negative-beta projects whose largest benefits materialize in the most catastrophic scenarii of the destiny of humankind on this planet.

This paper is organized as follows. In Section 2, we restate the classical pricing model with constant relative risk aversion and an arithmetic Brownian motion for the logarithm of aggregate consumption. We also introduce the CGF method in that section. The core of the paper is in Section 3, where we examine the general properties of discount rates when the random walk is affected by parametric uncertainty. We apply these results in two different specifications: uncertain trend or volatility of growth (Section 4), and uncertain frequency of catastrophes (Section 6). In Section 5, we extend the Gaussian specification of Section 4 to mean-reversion. We show how these results allow us to evaluate a broad class of risky projects with non-constant betas in Section 7. An application to climate change is presented in Section 8.

2. The benchmark model without parametric uncertainty

We evaluate a marginal investment project whose cash flow of net benefits is represented by a continuous-time stochastic process $\{F_t : t \geq 0\}$. In order to evaluate the social desirability of such a project, we measure its impact on the intertemporal social welfare[8]

[8] Contrary to Hansen and Sargent (2010) for example, we assume in this paper that the representative agent is neutral to ambiguity. This is because our aim is mostly normative, and because Subjective Expected Utility and its associated Sure Thing Principle have a strong normative appeal. Observe that, although our representative agent is

$$W_0 = E\left[\int_0 e^{-\delta t} u(c_t)dt \middle| I_0\right] \tag{0.1}$$

where u is the increasing and concave utility function of the representative agent, δ is her rate of pure preference of the present, $\{c_t | t \geq 0\}$ is the continuous-time process of consumption of the representative agent, and I_0 is the information set available at $t = 0$. We assume that this integral exists. For the sake of a simple notation, we hereafter make the contingency of the expectation operator to I_0 implicit. Because the investment project is marginal, its implementation increases intertemporal social welfare if and only if

$$\int_0 E\left[\frac{e^{-\delta t} F_t u'(c_t)}{u'(c_0)}\right] dt \geq 0. \tag{2}$$

This can be rewritten as a standard NPV formula:

$$\int_0 e^{-\rho_t(F_t)t} EF_t dt \geq 0, \tag{3}$$

where $\rho_t(F_t)$ is the rate at which the expected cash flow occurring in t years should be discounted. It is important to observe at this stage that the asset is decomposed into a bundle of horizon-specific payoffs or dividends, and that a specific discount rate $\rho_t(F_t)$ is used for each of them. These discount rates are characterized by the following equation:

$$\rho_t(F_t) = \delta - \frac{1}{t} \ln \frac{EF_t u'(c_t)}{u'(c_0)EF_t} = r_{ft} + \pi_t(F_t). \tag{4}$$

It is traditional in the CCAPM to decompose the project-specific discount rate $\rho_t(F_t)$ into a risk free discount rate r_{ft} and a project-specific risk premium $\pi_t(F_t)$. From (4), we define these two components of the discount rate as follows:

$$r_{ft} = \delta - \frac{1}{t} \ln \frac{Eu'(c_t)}{u'(c_0)}, \tag{5}$$

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neutral to ambiguity, we show that the ambiguity affecting the parameters of the stochastic process has a strong impact on long-dated asset prices.

$$\pi_t(F_t) = -\frac{1}{t}\ln\frac{EF_t u'(c_t)}{EF_t Eu'(c_t)}. \tag{6}$$

Observe that the risk premium $\pi_t(F)$ is zero when the project is safe and more generally when its future cash flow is independent of future aggregate consumption. This implies that r_{ft} is indeed the rate at which safe projects should be discounted. The CCAPM also characterizes the project-specific risk premium $\pi_t(F_t)$. Throughout the paper, we assume that $u'(c) = c^{-\gamma}$ and that

$$F_t = \xi_t c_t^{\beta} \tag{7}$$

where ξ_t is a random variable with $E\xi_t = 1$ that is independent of c_t, and β is the CCAPM beta of the project.[9] Because the idiosyncratic risk ξ_t is not priced, we hereafter identify a project $\{F_t\}$ by its single parameter β. When β is positive, implementing the project raises the risk on aggregate consumption. When β is negative, the project has an insurance component since it pays more on average in the worse macroeconomic scenarii.

We characterize the properties of asset prices by using a standard tool in statistics. Let $\chi(a,x) = \ln E\exp(ax)$ denote the Cumulant-Generating Function (CGF) associated to random variable x evaluated at $a \in \mathbb{R}$. CGF has recently been used by Martin (2013) to explore asset prices under non Gaussian economic growth processes. The CGF, if it exists, is the log of the better known moment-generating function. In this paper, we use the following properties of CGF (see Billingsley (1995)).

Lemma 1 : *If it exists, the CGF function $\chi(a,x) = \ln E\exp(ax)$ has the following properties:*

i. $\chi(a,x) = \sum_{n=1}^{\infty} \kappa_n^x a^n / n!$ *where κ_n^x is the nth cumulant of random variable x. If m_n^x denotes the centered moment of x, we have that $\kappa_1^x = Ex$, $\kappa_2^x = m_2^x$, $\kappa_3^x = m_3^x$, $\kappa_4^x = m_4^x - 3(m_2^x)^2$,…*

ii. *The most well-known special case is when x is $N(\mu,\sigma^2)$, so that $\chi(a,x) = a\mu + 0.5a^2\sigma^2$.*

[9] See for example Campbell (1986) and Martin (2013). In Gollier (2013), I show that this specification for the net benefit can be derived in a two-good economy with a CES utility function. Weitzman (2012) considers an alternative risk profile $F_t = a + b_t c_t$ which can be interpreted as a portfolio containing *a* units of the risk free asset and an exponentially-decreasing equity share b_t.

iii. $\chi(a,x+y)=\chi(a,x)+\chi(a,y)$ *when x and y are independent random variables.*

iv. $\chi(0,x)=0$ *and* $\chi(a,x)$ *is infinitely differentiable and convex in* a.

v. $a^{-1}\chi(a,x)$ *is increasing in a, from Ex to the supremum of the support of x when a goes from zero to infinity.*

vi. The cumulant of the nth order is homogeneous of degree n: $\kappa_n^{\lambda x}=\lambda^n\kappa_n^x$ *for all* $\lambda\in\mathbb{R}$.

Property *i* explains why χ is called the cumulant-generating function, and it links the sequence of cumulants to those of the centered moments. The first cumulant is the mean. The second, the third and the fourth cumulants are respectively the variance the skewness and the excess kurtosis of the random variable. Because the cumulants of the normal distribution are all zero for orders *n* larger than 2, the CGF of a normally distributed *x* is a quadratic function of *a*, as expressed by property *ii.*[10] This property also implies that the CGF of a Dirac distribution degenerated at $x=x_0\in\mathbb{R}$ is equal to ax_0. Property *v* will play a crucial role in this paper because of the assumption of an i.i.d. process for the growth of log consumption. It is a consequence of property *iv*, which is itself an illustration of the Cauchy-Schwarz inequality.

Under our specification, asset pricing formulas (5) and (6) can now be rewritten using CGF functions:

$$r_{ft}=\delta-t^{-1}\chi\left(-\gamma,G_t\right), \tag{8}$$

$$\pi_t(\beta)=t^{-1}\left(\chi\left(\beta,G_t\right)-\chi\left(\beta-\gamma,G_t\right)+\chi\left(-\gamma,G_t\right)\right), \tag{9}$$

where $G_t=\ln c_t/c_0$ is log consumption growth. Equation (9) expresses risk premia for different maturities. Notice that, ignoring δ, $\chi\left(\beta,G_t\right)$, $\chi\left(\beta-\gamma,G_t\right)$ and $-\chi\left(-\gamma,G_t\right)$ can be interpreted respectively as the log of expected payoff, the log of price, and the log of the risk free return. This means that the right side of equation (9) is indeed the different between the expected annualized return $t^{-1}\left(\chi\left(\beta,G_t\right)-\chi\left(\beta-\gamma,G_t\right)\right)$ of F_t over the risk free rate. The modern theory of finance used to focus the analysis on the short-term market risk premium $\pi_0(1)=\lim_{t\to 0}\pi_t(1)$. We

[10] It should be noticed that the normal distribution is the only distribution that has a finite sequence of non-zero cumulants. This implies that the Gaussian case is the only one in which the equation in property *i* in Lemma 1 can be used as an exact solution to the CGF.

are interested in characterizing the full term structure of risk premia $\pi_t(\beta)$ in parallel to the term structure of risk free rates r_{ft}. As noticed by Hansen (2012) for example, the simplest configuration is when G_t is normally distributed, so that we can use property *ii* of Lemma 1 to rewrite equation (9) as follows:

$$\pi_t(\beta) = \beta\gamma \frac{Var(G_t)}{t}. \tag{10}$$

In this specific Gaussian specification, the term structure of risk premia is proportional to the term structure of the annualized variance of log consumption contingent to current information. Hansen, Heaton, and Li (2008) pursued the same goal of describing the term structure of risk prices in a Gaussian world with persistent shocks to growth. Hansen and Scheinkman (2009) and Hansen (2012) generalized this approach to non-Gaussian specifications.

As a reminder, let us first describe the classical CCAPM specification. The simplest stochastic process is such that log consumption follows an arithmetic Brownian motion with trend μ and volatility σ. This implies that G_t is normally distributed with mean μt and variance $\sigma^2 t$. Using property *ii* in Lemma 1 implies that equations (8) and (9) can be rewritten as follows

$$r_{ft} = \delta + \gamma\mu - 0.5\gamma^2\sigma^2, \tag{11}$$

and

$$\pi_t(\beta) = \beta\gamma\sigma^2 = \beta\pi, \tag{12}$$

Equation (11), which is often referred to as the extended Ramsey rule, holds independent of the maturity of the cash flow. In other words, the term structure of the safe discount rate is flat under this benchmark specification. Its level is determined by three elements: impatience, a wealth effect and a precautionary effect. The wealth effect comes from the observation that investing for the future in a growing economy does increase intertemporal inequality. Because of inequality aversion (which is equivalent to risk aversion under the veil of ignorance), this is desirable only if the return of the project is large enough to compensate for this adverse effect on welfare. From (11), this wealth effect is equal to the product of the expected growth of log consumption by the degree γ of concavity of the utility function which measures inequality aversion. The precautionary effect comes from the observation that consumers want to invest more for the future when this future is more uncertain (Drèze and Modigliani (1972), Kimball (1990)). This tends to reduce the discount rate. The precautionary effect is proportional to the volatility of the growth of log consumption.[11]

Equation (12) tells us that the project-specific risk premium $\pi_t(\beta)$ is just equal to the product of the project-specific beta by the CCAPM aggregate risk premium $\pi = \gamma\sigma^2$. Under this standard specification, the risk premium associated to benefit $F_t \sim c_t^{\beta}$ is independent of its maturity *t*. The standard calibration of these two equations yields a too large risk free rate (risk free rate puzzle (Weil (1989))) and a too small risk premium (equity premium puzzle (Grossman and Shiller (1981), Hansen and Singleton (1983), Mehra and Prescott (1985))) compared to historical market data.

Because both the risk free rate and the risk premium of the project are independent of the maturity in this benchmark specification, their sum $\rho_t(\beta) = r_{ft} + \pi_t(\beta)$ is also independent of *t*. The term structure of risk-adjusted discount rates is flat in this case:

$$\rho_t(\beta) = \delta + \gamma\mu + \gamma(\beta - 0.5\gamma)\sigma^2. \tag{13}$$

Notice that the risk-adjusted discount rate can be either increasing or decreasing in the aggregate uncertainty measured by σ^2 depending upon whether the β of the project is larger or smaller than $\gamma/2$. Two competing effects are at play here. First, a large aggregate risk induces the representative agent to save more for the future (precautionary saving motive). That reduces the risk free discount rate. Second, ceteris paribus, a larger aggregate risk increases the project-

[11] In an intriguing paper, Martin (2012) compared the price $P_t = \exp(-r_{ft}t)$ of payoff $F_t = 1$ to its realized value $M_t(c_t) = \exp(-\delta t)u'(c_t)/u'(c_0)$. He demonstrated that this ratio tends to zero in probability when *t* tends to infinity, although its expectation remains equal to one at any finite maturity. It is easy to check that the expected rate of growth of $M_t(c_t)/P_t$ is equal to the precautionary term $-0.5\gamma^2\sigma^2$. In this consumption-based model, this intriguing result is indeed due to the precautionary effect: Because of the uncertain future, prudent individuals accept a risk-free rate that is smaller than the mean risk-free rate prevailing if c_t would be known at *t=0*. Martin (2012) showed that, although one will almost surely regret marginal sure investments ex post $(M_t(c_t) < P_t)$, this is compensated by rare catastrophic evolutions of consumption in which these sure investments are particularly valuable. Martin developed a similar analysis for risky assets.

specific risk and the associated risk premium. This risk aversion effect is proportional to the beta of the project. The two effects counterbalance each other perfectly when $\beta = \gamma / 2$.[12]

3. *Asset prices with an uncertain random walk for the growth of log consumption*

Following Weitzman (2007) and Gollier (2008), we now characterize the term structure of the risk free rate and of the risk premium when there is some uncertainty about the true value of some of the parameters of the growth process. This uncertainty takes the form of the stochastic process of G_t being a function of an unknown parameter $\theta \in \Theta$. The current beliefs about θ are represented by some distribution function with support in Θ.[13] Parameter uncertainty usually implies that the unconditional distribution of log consumption is not normal. This is an important source of complexity because, contrary to the Gaussian case examined in the previous section, equation (9) will entail cumulants of order larger than 2.

In this paper, we crucially assume that, conditional to any $\theta \in \Theta$, $G_t = \ln c_t / c_0$ is a Lévy process. This means that, conditional to θ, temporal increments in G_t are serially independent and stationary, and that the process is continuous in probability. It is well-known that Brownian motions, Poisson processes and any mixture of them belong to the family of Lévy processes. In this section, we examine the properties of asset prices under this general assumption. This is a clear difference with respect to recent papers in mathematical finance (Hansen, Heaton, and Li (2008), Hansen and Scheinkman (2009), Hansen (2012)) in which the focus is on the effect of unambiguous auto-regressive components of the growth process on the long-term risk prices.

In this continuous-time model, let us consider intervals of arbitrarily small duration Δ, and let $g_{t,\Delta} = G_{t+\Delta} - G_t = \ln c_{t+\Delta} / c_t$ denote the growth of log consumption during the interval of time $[t, t+\Delta]$. We can then rewrite equation (8) as follows:

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[12] We are concerned here by an increase in uncertainty measured by an increase in the variance of changes in log consumption. If the increased uncertainty would be measured by the variance of changes in consumption, the two effects would counterbalance each other when $\beta = 0.5(\gamma + 1)$.

[13] The observation of future changes in consumption will allow for a Bayesian updates of beliefs which will impact prices. In this framework with exponential Discount Expected Utility, agents' decisions (and so asset prices) will be time consistent. In this paper, we are interested in characterizing asset prices *today*. We do not examine the dynamic of prices, so we do not determine how beliefs are updated. We just do not need to do this for our purpose.

$$
\begin{aligned}
r_{ft} &= \delta - t^{-1}\ln E\left[E\left[e^{-\gamma \sum_{i=1}^{t/\Delta} g_{(i-1)\Delta,\Delta}} \middle| \theta \right]\right] \\
&= \delta - t^{-1}\ln E\left[E\left[e^{-\gamma g_\Delta} \middle| \theta \right]^{t/\Delta} \right] \\
&= \delta - t^{-1}\ln E e^{t\chi(-\gamma, g_\Delta|\theta)/\Delta} \\
&= \delta - t^{-1}\chi\left(t, \chi\left(-\gamma, g_\Delta \middle| \theta\right)/\Delta\right),
\end{aligned}
$$

with $g_\Delta = \ln c_\Delta / c_0$. The second equality above is a consequence of the assumption that $G_t|\theta$ is a Lévy process, so that the $g_{j\Delta,\Delta}$ are i.i.d. variables. The last two equalities are direct consequences of the definition of the CGF. Observe now that this safe discount rate r_{ft} is independent of the duration Δ of time intervals. Indeed, because $G_t|\theta$ is a Lévy process, we know that, conditional to θ, g_Δ is the sum of *n* i.i.d. random variables $g_{\Delta'}$, with $\Delta' = \Delta / n$. Using property *iii* of Lemma 1, this implies that

$$
\chi\left(-\gamma, g_\Delta \middle| \theta\right)/\Delta = n\chi\left(-\gamma, g_{\Delta'} \middle| \theta\right)/\Delta = \chi\left(-\gamma, g_{\Delta'} \middle| \theta\right)/\Delta'
$$

for all $\theta \in \Theta$. Thus, we can arbitrarily take $\Delta = 1$. Denoting $g = G_1 = \ln(c_1 / c_0)$, we obtain that

$$
r_{ft} = \delta - t^{-1}\chi\left(t, \chi\left(-\gamma, g \middle| \theta\right)\right). \tag{14}
$$

This means that the term structure of the risk free rate is determined by a sequence of two CGF operations. One must first compute $x|\theta = \chi(-\gamma, g|\theta)$, which is the CGF of consumption growth g conditional to θ. One must then compute $\chi(t,x)$ by using the distribution of θ that characterizes current beliefs about this unknown parameter. This pricing process has three immediate consequences. First, when there is no parametric uncertainty, $x|\theta$ is a constant, which implies that $r_{ft} = \delta - t^{-1}\chi(t,x) = \delta - x$ has a flat term structure, as shown in the previous section. Second, using property *v* of Lemma 1 implies that

$$
\begin{aligned}
r_{f0} &= \delta - E\chi(-\gamma, g|\theta) \\
&= \delta - E\left[\sum_{n=1}^{\infty} \frac{(-\gamma)^n}{n!} \kappa_n^{g|\theta} \right] = \delta - \sum_{n=1}^{\infty} \frac{(-\gamma)^n}{n!} E\left[\kappa_n^{g|\theta} \right].
\end{aligned} \tag{15}
$$

This tells us that the short-term discount rate is not affected by parametric uncertainty in the sense that only the *expected* cumulants of the growth of log consumption matter to compute it. This is reminiscent of a result by Hansen and Sargent (2010) who showed that parameter uncertainty does not contribute to local uncertainty prices in a Bayesian analysis. But parameter uncertainty does affect the discount rate associated to longer maturities. Indeed, by application of property *v* of Lemma 1, the risk free discount rate has a decreasing term structure when there is some parametric uncertainty. This observation generalizes a result obtained by Gollier (2008) in the special case of a geometric Brownian motion for aggregate consumption with an uncertain trend. Combining the last two observations implies that parameter uncertainty affects long discount rates negatively. We summarize our findings about the safe discount rate in the following proposition.

Proposition 1: *Suppose that the log consumption conditional to* θ *follows a Lévy process, for all* $\theta \in \Theta$. *The parametric uncertainty affecting the stochastic process of consumption growth has no effect on the risk free rate for maturities close to zero. Moreover, it makes the term structure of the risk free rate decreasing.*

A similar exercise can be performed on equation (9), which implies the following pair of equations:

$$\pi_t(\beta) = t^{-1}\Big(\chi\big(t, \chi(\beta, g|\theta)\big) - \chi\big(t, \chi(\beta-\gamma, g|\theta)\big) + \chi\big(t, \chi(-\gamma, g|\theta)\big)\Big) \tag{16}$$

$$\rho_t(\beta) = \delta + t^{-1}\Big(\chi\big(t, \chi(\beta, g|\theta)\big) - \chi\big(t, \chi(\beta-\gamma, g|\theta)\big)\Big). \tag{17}$$

This implies that, as for the risk free rate, the risk premium and the risk-adjusted discount rate have a flat term structure when there is no parametric uncertainty. However, these equations also show that the shape of their term structures is more difficult to characterize under parametric uncertainty.

For small maturities, the application of Lemma 1 *v* applied to the above three equations implies that

$$\pi_0(\beta) = E\left[\chi(\beta, g|\theta) - \chi(\beta-\gamma, g|\theta) + \chi(-\gamma, g|\theta)\right] \tag{18}$$

$$\rho_0(\beta) = \delta + E\left[\chi\left(\beta, g\,|\theta\right) - \chi\left(\beta-\gamma, g\,|\theta\right)\right]. \tag{19}$$

If we keep in mind again that χ is a weighted sum of the different cumulants of $g\,|\theta$, these equations tell us that, as for the short risk free rate, only the expectation of the cumulants of $g\,|\theta$ matters to determine short-lived risk prices.

Proposition 2: *Suppose that the log consumption conditional to θ follows a Lévy process, for all $\theta \in \Theta$. The parametric uncertainty affecting the stochastic process of consumption growth has no effect on risk premia for maturities close to zero.*

This result may explain why parametric uncertainty has not been much studied in asset pricing theory. In particular, this proposition shows that parametric uncertainty cannot solve the risk free rate puzzle and the equity premium puzzle, at least for short-dated assets. The remainder of this section demonstrates that parametric uncertainty has more radical effects on prices of long-dated assets. Observe that property *i* of Lemma 1 implies that

$$\left.\frac{\partial \pi_t(\beta)}{\partial t}\right|_{t=0} = 0.5\Big(Var\left(\chi\left(\beta, g\,|\theta\right)\right) + Var\left(\chi\left(-\gamma, g\,|\theta\right)\right) - Var\left(\chi\left(\beta-\gamma, g\,|\theta\right)\right)\Big). \tag{20}$$

Similar equations can be obtained for the slope of the discount rates r_{ft} and $\rho_t(\beta)$ at $t=0$. Using property *i* of Lemma 1 again yields the following proposition in which we examine the effect of the uncertainty relative to a specific cumulant of *g*. By property *iii* of Lemma 1, if more than one cumulant is uncertain, there effects are additive as long as they are statistically independent. We will come back on this point later on in this paper.

Proposition 3: *Suppose that the log consumption conditional to θ follows a Lévy process, for all $\theta \in \Theta$. Suppose also that the uncertainty about the distribution of g is concentrated in its nth cumulant κ_n^g, $n \geq 1$. This implies that*

$$\left.\frac{\partial r_{ft}}{\partial t}\right|_{t=0} = -\frac{\gamma^{2n}}{2\left(n!\right)^2} Var\left(\kappa_n^{g|\theta}\right) \tag{21}$$

$$\left.\frac{\partial \pi_t(\beta)}{\partial t}\right|_{t=0} = \frac{\beta^{2n} + \gamma^{2n} - (\beta-\gamma)^{2n}}{2\left(n!\right)^2} Var\left(\kappa_n^{g|\theta}\right) \tag{22}$$

$$\left.\frac{\partial \rho_t(\beta)}{\partial t}\right|_{t=0} = \frac{\beta^{2n} - (\beta-\gamma)^{2n}}{2(n!)^2} Var\left(\kappa_n^{g|\theta}\right). \tag{23}$$

Equation (22) immediately implies that the sign of $\partial \pi_t(\beta)/\partial t$ evaluated at $t=0$ coincides with the sign of β. This implies in particular that the aggregate risk premium has an increasing term structure, at least for small maturities. This means that the uncertainty argument presented in the recent literature to justify a *decreasing* term structure for the risk free discount rate (as seen from equation (21)) also implies an *increasing* term structure for the risk premium. It is therefore unclear whether this argument actually raises the global willingness to invest in the future. If the betas of investment projects are large enough, the parametric uncertainty should reduce the intensity of investments, because it will raise the risk-adjusted discount rate.

It is interesting to determine the critical beta at which the risk-adjustd discount rate $\rho_t(\beta) = r_{ft} + \pi_t(\beta)$ has a flat term structure in the neighborhood of *t*=1. From equation (23), this beta is equal to half the degree of relative risk aversion. It yields the following corollary.

Corollary 1: *Under the assumptions of Proposition 3, the term structure of the aggregate risk premium* $\pi_t(1)$ *is increasing for small maturities. Moreover, the term structure of the risk-adjusted discount rate* $\rho_t(\beta)$ *is decreasing (increasing) for small maturities if* $\beta < (>) \gamma/2$.

There is a simple intuition for this result. It combines the observation that the parametric uncertainty magnifies the uncertainty affecting the future level of development of the economy with the observation made at the end of the previous section that risk decreases or increases the discount rate depending upon whether β is smaller or larger than $\gamma/2$.

Observe that the result in Corollary 1 is independent of the rank *n* of the uncertain cumulant. By application of property *iii* of Lemma 1, the results of Corollary 1 are thus robust to the multiplicity of the cumulants of g being uncertain, as long as they are statistically independent. They are not robust to the introduction of correlation among two or more cumulants. To show this, suppose that cumulants of degrees *m* and *n* are the only two uncertain cumulants of *g*. The above analysis can easily be extended to yield the following results:

$$
\begin{aligned}
\left.\frac{\partial \pi_t(\beta)}{\partial t}\right|_{t=0} &= \frac{\beta^{2m}+\gamma^{2m}-(\beta-\gamma)^{2m}}{2\left(m!\right)^2} Var\left(\kappa_m^{g|\theta}\right) + \frac{\beta^{2n}+\gamma^{2n}-(\beta-\gamma)^{2n}}{2\left(n!\right)^2} Var\left(\kappa_n^{g|\theta}\right) \\
&+ \frac{\beta^{m+n}+(-\gamma)^{m+n}-(\beta-\gamma)^{m+n}}{m!n!} Cov(\kappa_m^{g|\theta},\kappa_n^{g|\theta})
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\left.\frac{\partial \rho_t(\beta)}{\partial t}\right|_{t=0} &= \frac{\beta^{2m}-(\beta-\gamma)^{2m}}{2\left(m!\right)^2} Var\left(\kappa_m^{g|\theta}\right) + \frac{\beta^{2n}-(\beta-\gamma)^{2n}}{2\left(n!\right)^2} Var\left(\kappa_n^{g|\theta}\right) \\
&+ \frac{\beta^{m+n}-(\beta-\gamma)^{m+n}}{m!n!} Cov(\kappa_m^{g|\theta},\kappa_n^{g|\theta}).
\end{aligned} \tag{25}
$$

Let us first discuss equation (25) in the special case of $\beta=\gamma/2$. In that case, the first two terms in the RHS of this equation are zero. But the coefficient of the covariance in the third term is clearly positive if $m+n$ is an odd integer. This implies that the term structure of the risk-adjusted discount rate with $\beta=\gamma/2$ will not be flat when the uncertain cumulants are correlated. In particular, a positive correlation between two subsequent cumulants tends to make this term structure of $\rho_t(\beta=\gamma/2)$ increasing, at least at small maturities.

Let us now discuss equation (24) in the case of the aggregate risk ($\beta=1$). In the absence of a covariance between the two uncertain cumulants, the term structure of the risk premia is increasing. Bansal and Yaron (2004) obtained a similar result in a different framework with persistent shocks to the growth of log consumption. But a growing literature documents evidence that the term structure of the equity premium is downward sloping for time horizons standard for financial markets.[14] Observe however that equation (24) is not incompatible with a downward sloping term structure for $\pi_t(1)$ if the covariance term is sufficiently negative.

In the standard CCAPM model with a normal distribution for g, the risk premium $\pi_t(\beta)$ is proportional to β. Equation (22) reveals that this property does not hold when there is some parametric uncertainty about the cumulants of g. We will show later on in this section that the nonlinearity of the risk premium with respect to the asset's beta may be sizeable.

The methodology based on the CGF proposed by Martin (2013) is also useful to explore the curvature of the term structures. Equation (17) implies that

[14] See for example Binsbergen, Brandt and Koijen (2012) and the references mentioned in that paper.

$$\left.\frac{\partial^2 \rho_t(\beta)}{\partial t^2}\right|_{t=0} = \frac{1}{3} Skew\left(\chi(\beta, g\,|\theta) - \chi(\beta-\gamma, g\,|\theta)\right), \tag{26}$$

where $Skew(x)$ is the skewness of *x*. If the uncertainty on the distribution of log consumption is concentrated on the *n*th cumulant, then this equation simplifies to

$$\left.\frac{\partial^2 \rho_t(\beta)}{\partial t^2}\right|_{t=0} = \frac{\beta^{3n} - (\beta-\gamma)^{3n}}{3\left(n!\right)^3} Skew\left(\kappa_n^{g|\theta}\right). \tag{27}$$

This means for example that the term structure of the discount rate for the aggregate risk ($\beta = 1$) is convex at *t*=0 if the trend (*n=1*) of growth is uncertain and positively skewed. Pursuing in the same vein for larger derivatives of ρ_t would allow us to fully describe the shape of the term structure of discount rates from the sign of the successive cumulants of g.

In the spirit of Hansen and Scheinkman (2009) and Hansen (2012), we now determine the asymptotic values of the discount rates in the context of parameter uncertainty. Property *v* of Lemma 1 immediately yields the following result.

Proposition 4: *Suppose that the log consumption conditional to* θ *follows a Lévy process, for all* $\theta \in \Theta$. *This implies that*

$$\lim_{t\to\infty} r_{ft} = \delta - \sup \chi(-\gamma, g\,|\theta). \tag{28}$$

$$\lim_{t\to\infty} \pi_t(\beta) = \sup \chi\left(\beta, g\,|\theta\right) + \sup \chi\left(-\gamma, g\,|\theta\right) - \sup \chi\left(\beta-\gamma, g\,|\theta\right) \tag{29}$$

$$\lim_{t\to\infty} \rho_t(\beta) = \delta + \sup \chi\left(\beta, g\,|\theta\right) - \sup \chi\left(\beta-\gamma, g\,|\theta\right). \tag{30}$$

The suprema are taken with respect to $\theta \in \Theta$. In the next section, we illustrate these findings in different special cases.

4. Application : Brownian growth process with an unknown trend or volatility

In this section, we examine the special case of a geometric Brownian motion for consumption. More specifically, we assume here that log consumption follows an arithmetic Brownian motion with unknown constant drift μ and volatility σ.

Let us first consider the case in which the trend is known, but the volatility is ambiguous. Weitzman (2007) examined this question by assuming that σ^2 has an inverted Gamma distribution. This implies that the unconditional g is a Student's t-distribution rather than a normal, yielding fat tails, a safe discount rate of $-\infty$ and a market risk premium of $+\infty$. In our terminology, because the Inverse Gamma distribution for σ^2 has no real CGF, equations (14) and (16) offer another proof of Weitzman (2007)'s inexistence result. One can use Lemma 1 for distributions of σ^2 that have a real CGF. It yields

$$r_{ft} = \delta + \gamma\mu - \frac{1}{2}\gamma^2\kappa_1^{\sigma^2} - \frac{1}{8}\gamma^4\kappa_2^{\sigma^2}t - \frac{1}{48}\gamma^6\kappa_3^{\sigma^2}t^2 - ... \tag{31}$$

$$\pi_t(\beta) = \beta\gamma\kappa_1^{\sigma^2} + \frac{1}{2}\beta\gamma\kappa_2^{\sigma^2}\left[\beta^2 - \frac{3}{2}\beta\gamma + \gamma^2\right]t + ... \tag{32}$$

Equation (32) provides an illustration of the non-linearity of the risk premium $\pi_t(\beta)$ with respect to the asset's beta.

Suppose alternatively that only the drift of growth is uncertain. Suppose moreover that the current beliefs about the true value of the drift can be represented by a normal distribution: $\mu \sim N(m_1^\mu, m_2^\mu)$. This implies the following characterization:

$$r_{ft} = r_{f0} - 0.5\gamma^2 m_2^\mu t \tag{33}$$

$$\pi_t(\beta) = \beta\gamma\left(\sigma^2 + m_2^\mu t\right) \tag{34}$$

Equation (34) can be seen as an application of equation (10) since G_t is Gaussian in this case with $Var(G_t) = \sigma^2 t + m_2^\mu t^2$. The following observations should be made in relation with these equations. First, the term structure of the safe discount rate is linearly decreasing (Gollier (2008)), and one more year in maturity implies a reduction of the discount rate by $0.5\gamma^2$ times the variance of the trend μ. Second, the risk premium is linearly increasing with *t*. One more year of maturity has an effect on the risk premium that is equivalent to an increase in the volatility of the log consumption growth σ^2 by the variance of the trend. Third, because the support of normal distributions are unbounded, these results are compatible with Proposition 4. Observe finally from (34) that the risk premium is proportional to β in this specification of parametric uncertainty.

The linearity of the term structures of discount rates exhibited in the above equations strongly relies on the tails of the distribution of μ. In the remainder of this section, we explore the case of non-Gaussian distributions for the trend of growth. Equation (17) can be rewritten as follows:

$$\rho_t(\beta) = \rho_0(\beta) + t^{-1}\chi(t, \beta(\mu - m_1^{\mu})) - t^{-1}\chi(t, (\beta - \gamma)(\mu - m_1^{\mu})). \tag{35}$$

Because $\chi(t, kx) = \chi(tk, x)$, the above equation implies that

$$\frac{\partial \rho_t(\beta)}{\partial t \partial \gamma} = (\beta - \gamma)\frac{\partial^2 \chi}{\partial t^2}(t, (\beta - \gamma)(\mu - m_1^{\mu})). \tag{36}$$

Because χ is convex in its first argument, this implies that $\partial \rho_t(\beta)/\partial t$ is decreasing in γ for all $\beta \leq \gamma$. If μ has a symmetric distribution, equation (35) also implies that $\rho_t(\beta = 0.5\gamma)$ is constant in t. Combining these results, we obtain the following result.

Proposition 5: *Suppose that log consumption follows an arithmetic Brownian motion with a known volatility* $\sigma \in \mathbb{R}^+$ *and with an unknown trend* μ *of bounded support. When* μ *is symmetrically distributed, the term structure of the discount rate* $\rho_t(\beta = 0.5\gamma)$ *is flat at* $\delta + \gamma m_1^{\mu}$*. It is decreasing when* β *is smaller than* 0.5γ*, and it is increasing when* β *is in interval* $[0.5\gamma, \gamma]$.

We now characterize the asymptotic properties of the term structure of discount rates when the distribution of the trend μ has a bounded support $[\mu_{\min}, \mu_{\max}]$. We first rewrite condition (35) as

$$\rho_t(\beta) = \delta + \gamma\sigma^2(\beta - 0.5\gamma) + t^{-1}\chi(t, \beta\mu) - t^{-1}\chi(t, (\beta - \gamma)\mu). \tag{37}$$

Property v of Lemma 1 tells us that $a^{-1}\chi(a, x)$ tends to the supremum of the support of x when a tends to infinity. If β is negative, the supremum of the support of $\beta\mu$ is $\beta\mu_{\min}$, and the supremum of the support of $(\beta - \gamma)\mu$ is $(\beta - \gamma)\mu_{\min}$. This implies that the sum of the last two terms of the

RHS of the above equality tends to $\gamma\mu_{\min}$. The other two cases are characterized in in the same way to finally obtain the following result:

$$\rho_\infty(\beta)=\begin{cases}\delta+\gamma\sigma^2(\beta-0.5\gamma)+\gamma\mu_{\min} & if \quad \beta\le 0\\ \delta+\gamma\sigma^2(\beta-0.5\gamma)+(\gamma-\beta)\mu_{\min}+\beta\mu_{\max} & if \quad 0<\beta\le\gamma\\ \delta+\gamma\sigma^2(\beta-0.5\gamma)+\gamma\mu_{\max} & if \quad \beta>\gamma\end{cases} \tag{38}$$

For distant futures, the ambiguity affecting the trend is crucial for the determination of the discount rate. The long term wealth effect is equal to the product of γ by a growth rate of consumption belonging to its support $\left[\mu_{\min},\mu_{\max}\right]$. Its selection depends here upon the beta of the project. When β is negative, the wealth effect should be computed on the basis of the smallest possible growth rate $\mu_{\min}$ of the economy. On the contrary, when β is larger than γ, the wealth effect should be computed on the basis of the largest possible rate $\mu_{\max}$.

Equation (38) also tells us that the condition of a symmetric distribution for μ in Proposition 4 cannot be relaxed. Indeed, equation (38) implies that $\rho_0(\beta=\gamma/2)$ and $\rho_\infty(\beta=\gamma/2)$ are equal only if $E\mu=m_1^\mu$ and $(\mu_{\min}+\mu_{\max})/2$ coincide. Most asymmetric distributions will not satisfy this condition, which implies that the constancy of $\rho_t(\beta=\gamma/2)$ with respect to *t* will be violated.

In Figure 1, we illustrate some of the above findings through the following numerical example. We assume that $\delta=0$, $\gamma=2$, $\sigma=4\%$ and μ is uniformly distributed on interval $\left[0\%,3\%\right]$. The term structure is flat for $\beta=\gamma/2=1$. The sensitiveness of the discount rate to changes in the beta is increasing in the maturity of the cash flows. The aggregate risk premium is increased tenfold from $\pi_0(1)=0.3\%$ to $\pi_\infty(1)=3.3\%$ when maturity goes from 0 to infinity. This numerical example also illustrates the property that project-specific risk premia are in general not proportional to the project-specific beta. For example, consider a time horizon of 400 years. For this maturity, the risk premia associated to $\beta=1$ and $\beta=4$ are respectively equal to $\pi_{400}(1)=2.5\%$ and $\pi_{400}(4)=6.3\%<4\pi_{400}(1)$.

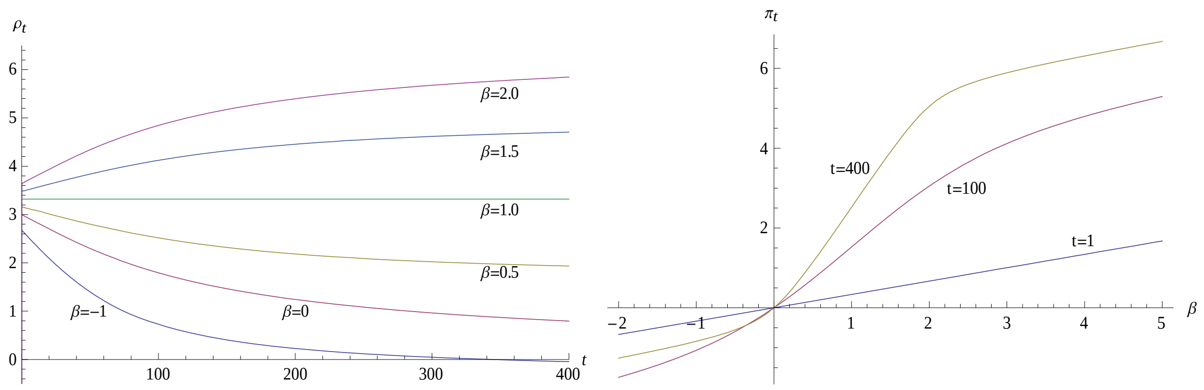

Figure 1: The discount rate as a function of maturity (left) and of the beta of the project (right). We assume that $\delta = 0$, $\gamma = 2$, $\sigma = 4\%$ and μ is $U[0\%, 3\%]$. Rates are in %.

Figure 1 clearly shows that the elasticity of the risk-adjusted discount rate to changes in the beta of the project is increasing in the maturity. This observation has important consequences. First, it indicates that it is particularly crucial to estimate the beta of projects having long-term impacts on the economy. Projects whose main benefits are to reduce emissions of greenhouse gases illustrate this point. It is thus particularly disappointing that we know almost nothing about the "climate beta". Section 8 is devoted to exploring this question. Second, this observation should reinforce the need to promote the integration of risk measures in the evaluation of public policies with long-term impacts. Up to our knowledge (Gollier (2011b)), France is the only OECD country in which public institutions are requested to estimate the beta of the public investment projects.[15]

In the next two sections, we examine two generalizations of this model. First, we allow for mean-reversion, a clear departure from the stationarity assumption made up to now. Second, we allow for catastrophes.

5. Extension to mean-reversion

In the benchmark specification with a CRRA utility function and a Brownian motion for log consumption, the term structures of risk-adjusted discount rates are flat and constant through time. In the specification with some parametric uncertainty on this Brownian motion that we examined in the previous section, they are monotone with respect to maturity and they move

[15] Over the last 15 years, Norway used risk-sensitive social discount rates, but, sadly enough, a recent report (Official Norwegian Report (2012)) recommends giving up this source of complexity in the evaluation procedure.

smoothly through time due to the revision of beliefs about the true values of the uncertain parameters. But this specification ignores the cyclicality of the economic activity. The introduction of predictable changes in the trend of growth introduces a new ingredient to the evaluation of investments. When expectations are temporarily diminishing, the discount rate associated to short horizons should be reduced to bias investment decisions toward projects that dampen the forthcoming temporary recession. Long termism is a luxury that should be favored only in periods of economic prosperity with pessimistic expectations for the distant future. More generally, when expectations are cyclical, it is important to frequently adapt the price signals contained in the term structure of discount rates to the moving macroeconomic expectations. To illustrate, it is clearly inefficient to maintain the U.S. official discount rate unchanged since 1992.

In this section, we propose a simple model in which the economic growth is cyclical, with some uncertainty about the parameter governing this process. Following Bansal and Yaron (2004) and Hansen, Heaton and Li (2008) for example, we assume in this section that the change in log consumption follows an auto-regressive process: [16]

$$\begin{aligned} &\ln c_{t+1}/c_t = x_t \\ &x_t = \mu_\theta + y_t + \varepsilon_{xt}, \\ &y_t = \phi y_{t-1} + \varepsilon_{yt,} \end{aligned} \tag{39}$$

for some initial (potentially ambiguous) state characterized by y_{-1}, where ε_{xt} and ε_{yt} are independent and serially independent with mean zero and variance σ_x^2 and σ_y^2, respectively. Parameter ϕ, which is between 0 and 1, represents the degree of persistence in the expected growth rate process. When ϕ is zero, then the model returns to a pure random walk as in Section 4. We hereafter allow the trend of growth μ_θ to be uncertain. By forward induction of (39), it follows that:

$$G_t = \ln c_t / c_0 = \mu_\theta t + y_{-1}\phi \frac{1-\phi^t}{1-\phi} + \sum_{\tau=0}^{t-1} \frac{1-\phi^{t-\tau}}{1-\phi} \varepsilon_{y\tau} + \sum_{\tau=0}^{t-1} \varepsilon_{x\tau}. \tag{40}$$

It implies that, conditional to θ, G_t is normally distributed with annualized variance

[16] A more general model entails a time-varying volatility of growth as in Bansal and Yaron (2004) and Hansen (2012). Mean-reversion in volatility is useful to explain the cyclicality of the market risk premium.

$$t^{-1}Var(\ln c_t / c_0) = \frac{\sigma_y^2}{(1-\phi)^2}\left[1 - 2\phi\frac{\phi^t - 1}{t(\phi-1)} + \phi^2\frac{\phi^{2t}-1}{t(\phi^2-1)}\right] + \sigma_x^2. \tag{41}$$

Using property *ii* of Lemma 1, equations (8) and (9) imply that

$$\begin{aligned}\rho_t(\beta) &= t^{-1}\left(\chi\left(\beta, G_t\right) - \chi\left(\beta-\gamma, G_t\right)\right) \\ &= \delta + \gamma y_{-1}\phi\frac{1-\phi^t}{t\left(1-\phi\right)} + \gamma(\beta - 0.5\gamma)\left(\frac{\sigma_y^2}{(1-\phi)^2}\left[1-2\phi\frac{\phi^t-1}{t(\phi-1)} + \phi^2\frac{\phi^{2t}-1}{t(\phi^2-1)}\right] + \sigma_x^2\right) \\ &\quad + t^{-1}\left(\chi(t, \beta\mu) - \chi(t, (\beta-\gamma)\mu)\right)\end{aligned} \tag{42}$$

One can then treat the last term in the RHS of this equation as in the previous section. Bansal and Yaron (2004) consider the following calibration of the model, using annual growth data for the United States over the period 1929-1998. Taking the month as the time unit, they obtained, $\mu = 0.0015$, $\sigma_x = 0.0078$, $\sigma_y = 0.00034$, and $\phi = 0.979$. Using this ϕ yields a half-life for macroeconomic shocks of 32 months. Let us assume that $\delta = 0$, and let us introduce some uncertainty about the historical trend of growth from the sure $\mu = 0.0015$ to the uncertain context with two equally likely trends $\mu_1 = 0.0005$ and $\mu_2 = 0.0025$. In Figure 2, we draw the term structures of discount rates for three different positions y_{-1} in the business cycle. In the set of curves starting from below, the current growth rate of the economy is 0.6% per year, well below its unconditional expectation of 1.8%. In this recession phase, the short term discount rate is a low 1%, but the expectation of a recovery makes the term structure steeply increasing for low maturities. For low betas, the term structure is non-monotone because of the fact that for very distant maturities, the effect of parametric uncertainty eventually dominates. In the set of curves starting from above, the current growth rate is 1.2% per year above its unconditional expectation. In this expansionary phase of the cycle, the short term discount rate is large at around 6%, but is steeply decreasing for short maturities because of the diminishing expectations. At the middle of the business cycle, when expectations are in line with the historical average ($y_{-1} = 0$), the term structure corresponds to the set of curves starting from the center of the vertical axe. Although short-term discount rates are highly sensitive to the position in the business cycle, the long-term discount rates are not.

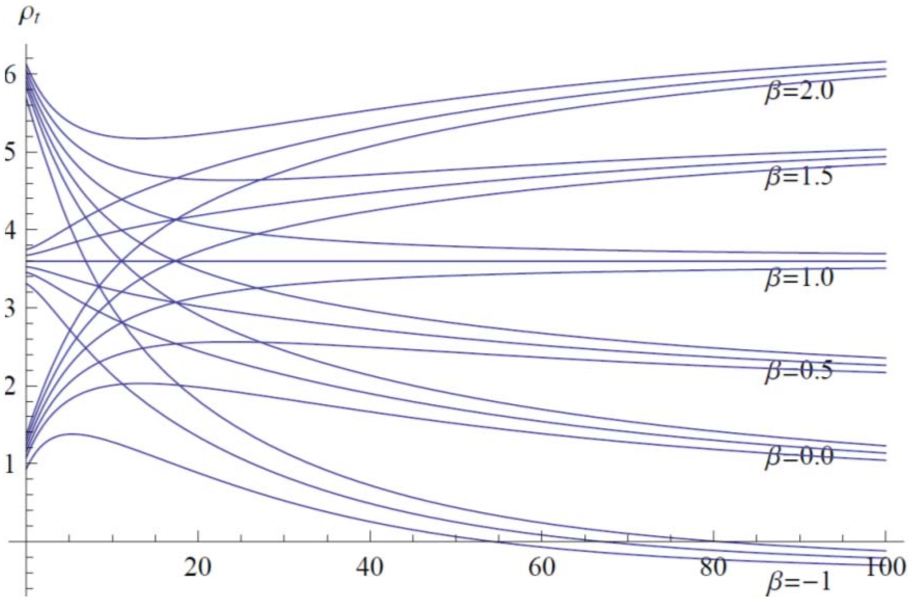

Figure 2: The discount rate (in % per year) as a function of the maturity (in years) in recession for different betas. Equation (42) is calibrated with $\delta = 0$, $\gamma = 2$, $\sigma_x = 0.0078$, $\sigma_y = 0.00034$, $\phi = 0.979$, two equally likely trends $\mu_1 = 0.0005$ and $\mu_2 = 0.0025$, and $y_{-1} = -0.001$ (bottom), $y_{-1} = 0$ (center), and for $y_{-1} = 0.001$(top).

One can also examine a model in which the current state variable y_{-1} is uncertain. It is easy to generalize equation (42) to examine this ambiguous context. We obtain the following pricing formula:

$$\begin{aligned}\rho_t(\beta) &= \delta + \gamma(\beta - 0.5\gamma)\left(\frac{\sigma_y^2}{(1-\phi)^2}\left[1 - 2\phi\frac{\phi^t - 1}{t(\phi - 1)} + \phi^2\frac{\phi^{2t} - 1}{t(\phi^2 - 1)}\right] + \sigma_x^2\right) \\ &+ t^{-1}\left(\chi\left(t, \beta\mu + \beta y_{-1}\phi\left(1 - \phi^t\right)/t(1-\phi)\right) - \chi\left(t, (\beta - \gamma)\mu + (\beta - \gamma)y_{-1}\phi\left(1 - \phi^t\right)/t(1-\phi)\right)\right).\end{aligned} \tag{43}$$

Observe again that when $\beta = 0.5\gamma$, the term structure of discount rates is flat if both μ and y_{-1} have a symmetric distribution function. We also observe that the ambiguity on y_{-1} plays a role similar to the ambiguity on μ to shape the term structure. Our numerical simulations (available upon request) show that the hidden nature of the state variable does not modify the general characteristics of the term structures described above.

6. *Unknown probability of a macroeconomic catastrophe*

In this section, in the spirit of Barro (2006, 2009), Backus, Chernov and Martin (2011) and Martin (2013), we examine a discrete version of a mixture of Brownian and Poisson processes. Rare events have recently been recognized for being a crucial determinant of assets prices. The underlying discrete-time model is such that the growth of log consumption follows an i.i.d. process, so that equations (14), (16) and (17) are relevant. We assume in this section that the per-period change in log consumption compounds two normal distributions:

$$g \sim (h_1, 1-p; h_2, p) \quad \textit{with} \quad h_i \sim N(\mu_i, \sigma_i^2), \tag{44}$$

The log consumption growth compounds a "business-as-usual" random variable $h_1 \sim N(\mu_1, \sigma_1^2)$ with probability $1-p$, with a catastrophe event $h_2 \sim N(\mu_2, \sigma_2^2)$ with probability p and $\mu_2 << 0 < \mu_1$ and $\sigma_2 \geq \sigma_1$. Barro (2006, 2009) convincingly explains that the risk free puzzle and the equity premium puzzle can be explained by using credible values of the intensity μ_2 of the macro catastrophe and of its frequency *p*. However, Martin (2013) shows that the levels of the r_f and $\pi(1)$ are highly sensitive to the frequency p of rare events, and that this parameter p is extremely difficult to estimate. In this section, we contribute to this emerging literature by integrating this source of parametric uncertainty into the asset pricing model.

In the absence of parametric uncertainty, equations (14), (16) and (17) imply that the term structures of discount rates are flat. We hereafter assume alternatively that p is uncertain. We calibrate this model as in the EU benchmark version of Barro (2006) and Martin (2013). We assume that $\delta = 3\%$ and $\gamma = 4$. In the business-as-usual scenario, the trend of growth is $\mu_1 = 2.5\%$ and its volatility is equal to $\sigma_1 = 2\%$. In case of a catastrophe, the trend of growth is $\mu_2 = -39\%$ and the volatility is $\sigma_2 = 25\%$. Finally, we assume that the probability of catastrophe is either 1.2% or 2.2% with equal probabilities.[17]

[17] This corresponds to the two sensitivity analyses performed by Martin (2013) around Barro's estimation of *p=1.7%.*

	$t = 1$	$t = 10$	$t = 100$	$t \to \infty$
$\beta = -1.0$	-11.0	-12.7	-16.5	-16.9
$\beta = 0.0$	0.5	-0.1	-2.2	-2.9
$\beta = 0.5$	3.9	3.6	2.2	1.6
$\beta = 1.0$	6.4	6.2	5.4	4.8
$\beta = 2.92$	11.0	11.0	11.0	11.0
$\beta = 5.0$	12.8	12.8	12.9	13.0

Table 1: The discount rate $\rho_t(\beta) = r_{ft} + \pi_t(\beta)$ (in %) when the probability of macroeconomic catastrophe is uncertain. Rates are in % per year.

In the calibration of this section, we obtain that the slope of the term structure of the discount rate at t=0 is equal to -0.06% per year for $\beta = 0$, which is quite large. It increases to -0.02% for $\beta = 1$, thereby showing that Corollary 1 does not applied here because more than one cumulant of g is uncertain. In fact, the slope of the term structure of $\rho_t(\beta)$ for short maturities goes to zero at $\beta = 2.92$, and is positive for larger betas. Observe that the term structure of $\rho_t(\beta)$ is decreasing at the critical level $\beta = \gamma / 2$ expressed in Corollary 1 because of the negative correlation between the first two cumulants of the distribution of g. This is a consequence of equation (25).

We also obtain that $r_{f\infty} = -2.86\%$, a rate that should be compared to $r_{f\infty} = 0.46\%$ that holds in the absence of parametric uncertainty. We also have that $\pi_0(1) = 5.87\%$ and $\pi_\infty(1) = 7.71\%$. In Table 1, we report the discount rates for different maturities and asset's beta. It is interesting to observe that the risk premium $\pi_t(\beta)$ is not proportional to β. There are two reasons that explain this feature of asset prices in this context. Both are linked to the non-Gaussian nature of G_t. The first one comes from the fat tail induced by rare events. The second reason comes from the uncertainty affecting the probability of rare events, which also fatten the tails of the distribution.

7. *Pricing projects with a non-constant beta*

Although it is standard in the literature on asset pricing, specification (7) is critical for our results, in particular because it allows us to use the properties of CGF functions that appears in all pricing formulas used in this paper. Although specification (7) is restrictive, exploring the pricing of a project satisfying it opens the path to examining the pricing of a much larger class of projects of the form

$$F_t = \sum_{i=1}^{n} \alpha_{it} F_{it}, \quad \text{with } F_{it} = \xi_{it} c_t^{\beta_i}, \tag{45}$$

where random variables ξ_{it} are independent of c_t. This project can be interpreted as a portfolio of n different projects, each project i having a constant beta. Of course, the resulting projects portfolio has a non-constant beta. By a standard arbitrage argument, the value of this portfolio is the sum of the values of each of its beta-specific components. We can thus rely on our results in this paper for the evaluation of projects with a non-constant beta. If $\rho_t(\beta_i)$ is the efficient discount rate to evaluate the $\beta_i -$component of the project, the global value of the project will be equal to

$$\sum_{i=1}^{n} e^{-\rho_t(\beta_i)t} \alpha_{it} EF_{it}, \tag{46}$$

where we assumed without loss of generality that $E\xi_{it} = 1$. In other words, the discount factors – rather than the discount rates – must be averaged to determine the discount factor to be used to evaluate the cash flows of the global portfolio. In the absence of parametric uncertainty, Weitzman (2013) examines the case in which the project under scrutiny is a time-varying portfolio of a risk free asset ($\beta_1 = 0$) and of a risky project with a unit beta ($\beta_2 = 1$).

8. The beta of CO_2 projects

The purpose of this section is to heuristically derive a crude numerical estimate for term structure of discount rates to be used for the evaluation of climate policies. To do this, we need to answer the following often overlooked question: What is the beta of investments whose main objective is to abate emissions of greenhouse gases? To explore this question, let us consider a simple two-date version of the DICE model of Nordhaus (2008) and Nordhaus and Boyer (2000):

$$T = \omega_1 E \quad (47)$$

$$E = \omega_2 Y - I_0 \quad (48)$$

$$D = \theta_1 T^{\theta_2} \quad (49)$$

$$Q = e^{-D} Y \quad (50)$$

$$C = \alpha Q \quad (51)$$

All parameters of the model are assumed to be nonnegative. T is the increase in temperature and E is the emission of greenhouse gases from date 0 to date 1. It is assumed in equation (47) that the increase in temperature is proportional to the emission of these gases. By equation (48), emissions are proportional to the pre-damage production level Y, but they can be reduced by investing I_0 in a green technology at date 0. In equation (49), we assume that the damage D is an increasing power function of the increase in temperature. Equation (50) defines damage D as the logarithm of the ratio Y/Q of pre-damage and post-damage production levels.[18] We hereafter refer to D as the relative damage. Finally, consumption C is proportional to the post-damage production Q. This model yields the following reduced form:

$$C = \alpha Y \exp\left[-\theta_1 \left(\omega_1 (\omega_2 Y - I_0) \right)^{\theta_2} \right]. \quad (52)$$

We consider the beta of a green investment I_0. Such an investment has the benefit to raise consumption in the future by

$$\frac{\partial C}{\partial I_0} = \alpha Y \theta_1 \theta_2 \xi^{\theta_2} (\omega_2 Y - I_0)^{\theta_2 - 1} \exp\left[-\theta_1 \left(\omega_1 (\omega_2 Y - I_0) \right)^{\theta_2} \right]. \quad (53)$$

This is the future cash flow F of the investment. We assume that this investment is marginal, so that our model can be rewritten as:

[18] Equation (50) is traditionally expressed as $Q_1 = (1 - D_1)Y_1$. However, for high temperatures, this specification could lead to a negative after-damage production.

$$\begin{cases} C = \alpha Y \exp\left[-\theta_1^* Y^{\theta_2}\right] \\ F = \dfrac{\theta_1^* \theta_2 \alpha}{\omega_2} Y^{\theta_2} \exp\left[-\theta_1^* Y^{\theta_2}\right], \end{cases} \tag{54}$$

where $\theta_1^* = \theta_1 \omega_1^{\theta_2} \omega_2^{\theta_2}$ can be interpreted as a synthetic climate-sensitivity parameter.

A critical parameter for this model is θ_2. When θ_2 is equal to unity, the relative damage is just proportional to the change in temperature and in the concentration of greenhouse gases. The absolute damage $Y-Q$ is thus convex in Y in that case. When θ_2 is larger than unity, the relative damage is itself convex, thereby bringing even more convexity to the absolute damage as a function of Y. Let us first assume that $\theta_2 = 1$. In that case, we can derive from system (54) that

$$F = \theta_1 \omega_1 C. \tag{55}$$

Let us further assume that the only source of uncertainty is about the growth of pre-damage production. This implies that $\theta_1 \omega_1$ is a constant. We can thus conclude in this case that the green investment project under scrutiny in this section satisfies condition (7) with $\beta = 1$. This proves the following proposition.

Proposition 6: *Consider the simplified integrated assessment model* (47)-(51) *with* $\theta_2 = 1$ *and without uncertainty about the climate parameter* $v = \theta_1 \omega_1$*.Under this specification, any project whose benefits are to reduce emissions of greenhouse gases has a constant beta equaling unity.*

However, this simple result raises two difficulties. First, although all experts in the field recognize the scarcity of evidence to infer θ_2, most of them agree that the relation $D = f(T)$ should be convex, yielding $\theta_2 > 1$. Nordhaus and Boyer (2000) used $\theta_2 = 2$,[19] whereas Cline (1992) used $\theta_2 = 1.3$. The Monte-Carlo simulations of the PAGE model used in the Stern (2007) Review draw θ_2 from an asymmetric triangular probability density function with support in $[1,3]$, giving a mean of about 1.8 (See Dietz, Hope and Patmore, 2007). Although there is no consensus on the value of this parameter, this suggests a more consensual θ_2 somewhere between

[19] Nordhaus (2007, 2011) used a quadratic function, yielding a similar degree of convexity of the damage function in the relevant domain of increases in concentration.

1 and 2. Compared to the result in Proposition 6, a larger θ_2 tends to increase the benefits of reducing emissions in good states (large Y), and to reduce them in bad states (low Y). Intuitively, this should raise the beta of green projects above unity. To see this, observe that a local estimation of the beta from (7) can be obtained by fully differentiating system (54) with respect to Y. We obtained

$$\beta \approx \frac{d \ln F / dY}{d \ln C / dY} = \theta_2 \frac{1 - \theta_1^* Y^{\theta_2}}{1 - \theta_1^* Y} \tag{56}$$

which is close to θ_2 when θ_2 is close to unity. Observe that the beta of the project is not constant when θ_2 is not equal to unity. In other words, the expected benefit function conditional to C is not a power function of C.

Second, when $\nu = \theta_1 \omega_1$ is random, it will in general be correlated with C, as shown by the first equation in system (54). In that case, equation (55) cannot anymore be interpreted as describing a project with a unit beta. To illustrate this point, consider the extreme case where economic growth Y is certain, together with ω_1, ω_2 and θ_2, but the climate sensitivity parameter θ_1 is uncertain. In that case, eliminating θ_1 from system (54) yields

$$F = \frac{\theta_2}{\omega_2} \frac{C}{Y} \ln \frac{\alpha Y}{C}. \tag{57}$$

In this case, cash flow F is a deterministic function of C. Because it is not a power function, the beta of green projects is not constant in this specification. We can approximate it through the following formula:

$$\beta \approx \frac{d \ln F / d\theta_1}{d \ln C / d\theta_1} = 1 - \frac{1}{D}, \tag{58}$$

with $D = \theta_1^* Y^{\theta_2}$. This typically yields a negative beta, which is large in absolute value. Indeed, if we assume a range of damages between 5% and 20% of the aggregate production, we obtain a beta in the range between -4 and -19. In this story based on the uncertain climate sensitivity, a large sensitivity yields at the same time large damages, low consumption, and large benefits of mitigation. This explains the negative beta obtained under this specification. This story is similar to the one proposed by Sandsmark and Vennemo (2007) who claim that the beta of mitigation investments should be negative.[20] Their argument is based on the climate variability as being the only source of fluctuation in the economy. Murphy and Topel (2013) developed a similar argument.

To sum up, the result presented in Proposition 6 suffers from two major deficiencies: The assumed $\theta_2 = 1$ is unrealistically small, and it does not recognize that there is still much uncertainty about the sensitivity of the climate to an increase in concentration of greenhouse gases in the atmosphere. The difficulty is that improving the model to allow for $\theta_2 \neq 1$ or for an uncertain climate sensitivity implies that the benefit F cannot be written anymore as a power function of C as in equation (7), or as a sum of power functions of C as in equation (45).

We can conclude from this discussion that the beta of investments whose main benefits are a reduction of emissions of greenhouse gases is non-constant. Its average level is determined by the relative intensity of two sources of uncertainty, the one coming from the future economic prosperity, and the one due to the unknown intensity of the climatic problem. We believe that the economic source of variability has an order of magnitude larger than the climatic source of variability. When the annual growth rate of the economy varies between 0% and 3%, aggregate consumption in 100 years is between 0% and 1800% larger than today. This should be compared to climate damages for this time horizon which are usually estimated between 0% and 5% of GDP (see for example Stern (2007) and IPCC (2007)). To make this argument more concrete, let us consider the calibration of the simple above model as described in Table 2.[21]

Variable	*Value*	*Remark*
t	50 years	Time horizon between dates 0 and 1.
$Y = \mathrm{e}^{\sum_{i=1}^{t} x_i}$	$x_i\ iid \sim N(\mu,\sigma^2)$ $\mu = 1.5\%, \sigma = 4\%$	Y_0 is normalized to unity. The growth rate of production follows a normal random walk.
ω_2	1	Normalization

[20] They obtain β=–0.004, which is much closer to zero that what we obtain here. However, notice that these authors consider another definition of the beta, which is equal to the ratio of the covariance of (C_1,F_1) to the variance of C_1. In our model, the beta is equal to the ratio of the covariance of $(\ln C_1, \ln F_1)$ to the variance of $\ln C_1$.
[21] Pindyck (2013) has recently suggested that the main ingredient for the determination of the social cost of carbon is the risk of climate catastrophe, which we do not consider in this calibration. Doing so would require to determine how the probability of catastrophe is affected by economic growth.

ω_1	0.45	This implies that the expected increase in temperature in the next 50 years equals $\xi EY = 1°C$.
θ_2	1.5	Center of the "consensus interval" [1,2].
θ_1	$\sim U[0\%, 5\%]$	This means that the damage at the average temperature increase of 1°C is uniformly distributed on [0%, 5%] of pre-damage production.
α	0.75	Consumption equals 75% of post-damage production.

Table 2: Calibration of the two-date IAM model

Using the Monte-Carlo method, we generated 50 000 independent random selections of the pair (Y, θ_1) and the corresponding outcomes in terms of aggregate consumption C and benefits of mitigation F, as defined by system (54). Using these data, we regressed $\ln F$ on $\ln C$. The OLS estimation of the beta equals $\hat{\beta} = 1.32$ with a standard deviation of 0.016.[22] Because of the predominance of the uncertainty about economic growth in the long run, and because of the convexity of the cost function, there is a positive correlation between economic growth and the benefits of mitigation, yielding a large positive beta. This is in line with the recent results by Nordhaus (2011), in which the author summarizes the outcome of Monte-Carlo simulations of the much more sophisticated RICE-2011 model with 16 sources of uncertainty: " *Those states in which the global temperature increase is particularly high are also ones in which we are on average richer in the future.*"

In Figure 3, we draw the term structures of discount rates prevailing for $\beta = 1.32$ in three different phases of the macroeconomic cycle under the calibration used in Section 5. The discount rate to be used for super-long maturities is around 4.6%, whereas the short-term discount rate fluctuates along the business cycle from around 1.3% when the expected instantaneous trend is 1.2% per annum below its historical mean, to around 6% when the expected instantaneous trend is 1.2% above its historical mean.

[22] Increasing the degree of uncertainty affecting θ_1 has a sizeable impact on this estimation. For example, if we replace the assumption that the interval [0%, 5%] on which it is uniformly distributed by [0%, 10%], the OLS estimation of the beta goes down to 1.14%. However, it is hard to imagine damages amounting to 10% of the world production due to a 1°C increase in temperature.

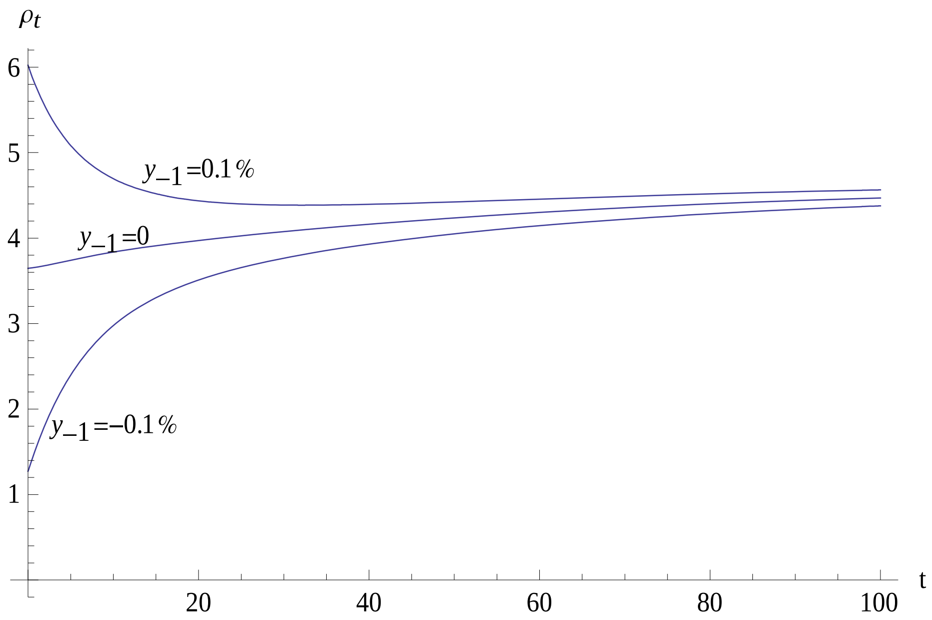

Figure 3: The discount rate (in % per year) as a function of the maturity (in years) for $\beta = 1.32$ in different phases of the cycle. Equation (42) is calibrated on a monthly basis with $\delta = 0$, $\gamma = 2$, $\sigma_x = 0.0078$, $\sigma_y = 0.00034$, $\phi = 0.979$, and two equally likely trends $\mu_1 = 0.0005$ and $\mu_2 = 0.0025$.

The work presented in this section, together with Sandsmark and Vennemo (2007), should be considered with caution. Our model is particularly simple, and its calibration should require more investigation and counter-expertise. We must recognize that, in spite of its crucial importance in the debate on climate change, the climate beta is still extremely ambiguous and uncertain. In Gollier (2013), we take seriously this source of uncertainty by examining how the uncertainty affecting a project's beta does affect its economic value.

9. Concluding remarks

By focusing on the riskiness of future benefits and costs, this paper contributes to the debate on the discount rate for climate change in several directions. Following Weitzman (2007) and Gollier (2008), we assumed that consumption growth follows a random walk, but its distribution entails parametric uncertainties. Our main messages in this framework are as follows. First, we showed that the shape of the term structure of risk-adjusted discount rates for risky projects is determined by the relative intensity of a precautionary effect that pushes towards a decreasing term structure of the risk free discount rate, and of a risk aversion effect that pushes towards an increasing term structure of the risk premium. For small maturities, the term structure is flat, decreasing or increasing depending upon whether the beta of the project is respectively equal, smaller or larger than half the relative risk aversion of the representative agent. This implies that the sensitiveness of the risk-adjusted discount rate to changes in the beta is increasing in the maturity of the cash flow. This implies that the standard recommendation existing in the United States or Norway (Official Norwegian Report (2012, Chapter 5)) for example to use a single discount rate to evaluate public policies independent of their risk profile is particularly problematic for policies having long-lasting socioeconomic impacts.

Second, we showed that the risk premium associated with a project is generally not proportional to its beta, which implies that knowing the aggregate risk premium and the project's beta is not enough to compute the project-specific risk premium. We derived simple formulas to compute the project-specific risk premium as a function of the project's beta.

We also calibrated our general model with various specifications of the parametric uncertainty affecting growth. We have examined in particular a model à la Barro (2006) in which the growth rate of consumption has fat tails, because of a small probability of a macroeconomic catastrophe that is added to the otherwise Gaussian business-as-usual growth process. The recognition of the intrinsic ambiguity that affects the frequency of catastrophes provides another justification for the risk free discount rate and the aggregate risk premium to be respectively decreasing and increasing with maturity.

Finally, we have shown that there are reasons to believe that the beta of projects whose main benefits are to reduce emissions of greenhouse gases is relatively large, around 1.3. This allows us to conclude that the discount rates to be used to evaluate public policies to fight climate change should be increasing with respect to maturities. Given the current global economic crisis in the western world, we are in favor of using a real discount rate for climate change around 1.3% for short horizons, and up to 4.6% for maturities exceeding 100 years.

A word of caution should be added to this conclusion. The use of price signals like discount rates and risk premia is possible only for investment projects that are marginal, i.e., for actions that do not affect expectations about the growth of the economy. The reader should be aware that this assumption does not hold when considering the *global* strategy to fight climate change. When thinking globally, one needs to take into account the general equilibrium effects that the chosen strategy will have on the stochastic growth process, hence on the discount rates that are used to evaluate this strategy. The right evaluation approach for global projects relies on the direct measure of the impact of the global action on the intergenerational social welfare function, as done for example in Stern (2007) and Nordhaus (2008).

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