Dynamic Equilibrium with Rare Disasters and Heterogeneous Epstein-Zin Investors*

Georgy Chabakauri
London School of Economics
Houghton Street
London WC2A 2AE
United Kingdom
Tel: +44 (0)20 7107 5374
G.Chabakauri@lse.ac.uk

Preliminary and Incomplete

This Version: November 2013

*I am grateful to Christian Julliard, Igor Makarov, Ian Martin, Marzena Rostek, David Webb, and seminar participants at the London School of Economics for helpful comments. All errors are mine.

Dynamic Equilibrium with Rare Disasters and Heterogeneous Epstein-Zin Investors

Abstract

We consider a general equilibrium Lucas (1978) economy with one consumption good and two heterogeneous Epstein-Zin investors. The output is subject to rare disasters or, more generally, can have non-lognormal distribution with higher cumulants. We demonstrate that the heterogeneity in preferences generates excess stock return volatilities, procyclical price-dividend ratios and interest rates, and countercyclical market prices of risk when the elasticity of intertemporal substitution (EIS) is greater than one. We show that the latter results cannot be jointly replicated in a model where investors have EIS ≤ 1. Our model produces endogenous time-variation in equilibrium processes without assuming time-varying probabilities of disasters, as in the recent literature with homogeneous investors. We propose new approach for finding general equilibrium, and characterize optimal portfolios and consumptions in terms of tractable backward equations. Finally, we extend analysis to the case of heterogeneous beliefs about disaster probabilities.

Journal of Economic Literature Classification Numbers: D53, G11, G12.
Keywords: heterogeneous investors, Epstein-Zin preferences, rare disasters, equilibrium, portfolio choice.

Rare economic disasters such as unexpected large decreases of aggregate output lead to significant welfare losses, and mere anticipation of disasters can have signifiant effect on asset prices in normal times. In particular, growing economic literature argues that accounting for the effects of disasters in general equilibrium models helps reconcile the magnitudes and properties of equity premia, risk free rates and other equilibrium processes with empirical findings [e.g., Rietz (1988); Barro (2006, 2009); Gabaix (2012), among others]. The extant literature primarily studies economies with homogeneous investors. Despite the fact that the heterogeneity in preferences is a salient feature of financial markets, its implications for equilibrium and risk sharing in the presence of rare disasters are relatively unexplored. In this paper, we demonstrate that heterogeneity in Epstein-Zin preferences can generate excess stock return volatility and asset prices dynamics consistent with the empirical literature.

We consider Lucas (1978) economy with one consumption good generated by a Lucas tree and two investors with Epstein-Zin preferences, which differ in their risk aversions and elasticities of intertemporal substitutions. The tree is subject to rare disasters, which lead to large decreases in aggregate consumption. More generally, our model allows for non-lognormal distributions with higher cumulants for the aggregate consumption process. The financial market is complete, and the investors share risks by trading in a riskless bond, one stock, which is a claim to the aggregate consumption, and insurances against rare disasters. In our baseline analysis, we consider the case when investors differ in risk aversions but have the same EIS, and explore the heterogeneity in EIS separately. In particular, we compare the cases when EIS is greater than unity, less than unity, and when both investors have constant relative risk aversion (CRRA) preferences, in which case EIS is a reciprocal of the risk aversion.

The equilibrium processes are derived as functions of the consumption share of the more risk averse investor, which is the only state variable in the model. This variable turns out to be countercyclical in the sense that it is negatively correlated with the changes in aggregate consumption in normal times when there are no disasters. Intuitively, positive shocks to aggregate consumption benefit the less risk averse investors because these investors hold larger fraction of wealth in stocks, and hence the consumption share of the more risk averse investor decreases, and the opposite happens when the dividend shock is negative. The heterogeneity in preferences gives rise to asset prices dynamics similar to those in models with homogeneous investors and time-varying probabilities of disasters [e.g., Gabaix (2012); Wachter (2013)]. However, the time-variation in our model is endogenously generated by risk sharing effects, and hence provides a new channel for explaining the dynamics of asset prices. Below, we discuss our main results.

First, we study the effect of rare events and heterogeneity in preferences for riskless rates and Sharpe ratios in the economy. We demonstrate that when investors have homogeneous EIS parameter the interest rates are procyclical and Sharpe ratios are countercyclical in normal times, consistent with the empirical literature. That is, the latter decreases and the former increases when aggregate consumption shocks are positive, and vice versa when the shocks are

negative. Moreover, the anticipation of disasters makes these processes more sensitive to shocks. For the case of CRRA preferences we observe similar dynamic patterns only in the presence of disasters, whereas without disasters, counterfactually, riskless rates become countercyclical. Furthermore, disaster risk decreases the interest rates and increases Sharpe ratios and bring their magnitudes in line with the empirically observed ones, as in the previous literature [e.g., Rietz (1988); Barro (2006); Gabaix (2012), among others]. Intuitively, as the consumption share of the more risk averse investor increases, Sharpe ratio increases to provide adequate compensation for risk whereas the interest rate decreases due to precautionary savings as this investor "flies to quality". We also find that the dependence of Sharpe ratios on EIS is very weak, and they look almost the same for different levels of EIS.

Second, we demonstrate that the stock price-dividend ratios are procyclical when EIS $>$ 1, consistent with the literature [e.g., Campbell and Cochrane (1999)], countercyclical when EIS $<$ 1, and have mixed pattern when investors have CRRA preferences. Furthermore, the anticipation of disasters decreases price-dividend ratios when EIS $>$ 1, and increases them when EIS $<$ 1 and when investors have CRRA preferences. Dynamic properties of price-dividend ratios determine those of stock return volatilities. We focus on the volatilities in normal times, that is, between rare disasters. We show that stocks are more volatile than dividends when EIS $>$ 1, consistent with the data [e.g., Shiller (1981)], less volatile when EIS $<$ 1, and the pattern is mixed when investors have CRRA preferences. Intuitively, when EIS $>$ 1 stock return volatility increases because the procyclical price-dividend ratio and the dividend change in the same direction amplifying the effects of each other. The opposite happens when EIS $<$ 1 when price-dividend ratio is countercyclical. The anticipation of disasters strengthens the dynamic patterns and quantitative effects for all processes by increasing their sensitivity to shocks.

Third, we explore the portfolio choice and risk sharing of investors. Less risk averse investor invests more wealth in stocks than the more risk averse investor. The fraction of wealth that the less risk averse investor allocates to stocks depends on the amount of liquidity available for borrowing, similarly to the related literature on heterogeneous investors [e.g., Longstaff and Wang (2012); Chabakauri (2013), among others]. The leverage is highest when the economy is dominated by the more risk averse investor because this investor holds large fraction of wealth in riskless assets, and hence provides liquidity at low cost. We also derive investors' holdings of insurance contracts and demonstrate that in equilibrium it is the more risk averse investor that sells the insurance to the less risk averse investor. This happens because the less risk averse investor holds a very large fraction of wealth in stocks, and hence needs the insurance the most. Dieckmann and Gallmeyer (2005) find a similar result in a simpler model where one investor is logarithmic while the other has CRRA preferences with risk aversion of either 2 or 0.5.

Fourth, we derive closed-form solutions in the economy where investors have identical risk aversions but different EIS. We demonstrate that differences in EIS affect the interest rates but have no impact on Sharpe ratios even in the presence of disasters. Moreover, all the equilib-

rium processes are deterministic functions of time. Based on our analysis we conclude that the difference in risk aversions is the main source of risk sharing between the investors. This result explains the finding that Sharpe ratios only weakly respond to changes in EIS in the general model, as discussed above. We note that Isaenko (2008) and Gârleanu and Panageas (2012) have similar result on the irrelevance of heterogeneity in EIS, but only in settings without rare events.

Finally, we consider an extension of the model in which the investors can additionally disagree about the probability of disasters in the economy. Such disagreement may naturally arise due to the fact that the probability of disasters is difficult to estimate because these events are rare. Introducing the heterogeneity of beliefs improves the performance of the model. In particular, making the more risk averse investor pessimistic further decreases the interest rates, increases Sharpe ratios and stock return volatilities. When EIS > 1 it also makes the equilibrium processes more sensitive to economic shocks.

We also contribute to the literature by proposing a new method for solving models with heterogeneous Epstein-Zin investors and rare events. We consider a discrete-time model with finite horizon and aggregate consumption following a multinomial process, which converges to a continuous-time Lévy process as the interval between two dates shrinks to zero. The discreteness of time allows for distributions with non-lognormal returns and makes the analysis more tractable. In the calibration we take the horizon equal to 200 years, and the interval between dates is one trading day. The model also permits taking continuous-time limits, which we derive in some special cases. First, we solve for the partial equilibrium, in which investors take asset prices as given. We characterize optimal consumption and portfolio policies in terms of investor's wealth-consumption ratio, which satisfies a discrete backward equation. Time-t solution of this equation is an explicit function of the solution at the previous step, and hence is found simply by backward iteration without solving any nonlinear equations. In continuous time, the wealth-consumption ratios satisfy integro-differential equations, which are more difficult to solve.

Then, we solve for general equilibrium in the model. From the condition that the marginal rates of substitution of investors should be equal when the financial market is complete we derive a system of equations for consumption shares of the more risk averse investor in all one-period ahead discrete states of the economy. This system of equation is easily solved by Newton's algorithm [e.g., Judd (1998)]. Then, we characterize the state price density and all the equilibrium processes in terms of these consumption shares. The tractability of our analysis comes from the fact the multinomial process for aggregate consumption treats normal and disaster shocks equally. The difference between these shocks emerges only in continuous-time limit. We note that discrete-time models are commonly employed economics [e.g., Mehra and Prescott (1985); Rietz (1988); He (1991), among others].

There is growing economic literature that studies the effects of rare disasters in general equilibrium. In particular, the literature explores the implications of the idea of Rietz (1988) that the anticipation of rare disasters can explain the equity premium puzzle of Mehra and

Prescott (1985). Barro (2006) further develops this idea by showing that the sizes and the frequency of disasters in the twentieth century are sufficient to explain high equity premia and low riskless rates in a Lucas (1978) with homogeneous CRRA investors. Barro (2009) extends the analysis in Barro (2006) to the case of Epstein-Zin preferences. Gabaix (2012), Gourio (2012), and Wachter (2013) consider models with time-varying disaster risks, and explain numerous asset pricing puzzles and the magnitudes of equity return volatilities, risk premia, and interest rates. Martin (2013a, 2013b) studies asset pricing with rare disasters in economies Epstein-Zin and CRRA investors, respectively. Backus, Chernov and Zin (2013) demonstrate that jumps have powerful effects on entropy. In contrast to the above literature, we allow for heterogeneous investors, which generates endogenous time-variation of equilibrium processes.

Ma (1993) provides a theoretical analysis of models with heterogeneous Epstein-Zin investors and derives conditions for the existence and uniqueness of equilibrium. Dieckmann and Gallmeyer (2005) consider a model similar to ours, but with CRRA investors, where one investor has logarithmic preferences while the other has risk aversion of 2 or 0.5. Dieckmann (2011) considers a similar model with incomplete markets. Chen, Joslin, and Tran (2012) consider an economy where investors have heterogeneous beliefs regarding the intensity of disasters in the economy. They also provide caution for extracting probabilities from asset prices, and demonstrate that ignoring the risk sharing of heterogeneous investors substantially underestimates the disaster risk. Backus, Chernov, and Martin (2011) and Julliard and Ghosh (2012) provide the evidence that the probability of disasters might be smaller than the one estimated in previous literature. The probability of disasters then might not be sufficient to explain equity premia.

Our paper is also related to the growing literature on asset pricing implications of investor heterogeneighty, although this literature does not study the effects of rare events. Most closely related to our paper is the work by Gârleanu and Panageas (2012), which considers an overlapping generations model with heterogeneous Epstein-Zin investors. Other related works on with heterogeneous investors include (but are not limited to) Basak (2000, 2005), Bhamra and Uppal (2013), Borovička (2012), Buss, Uppal, and Vilkov (2011), Chabakauri (2013a, 2013b), Chan and Kogan (2002), Cvitanić, Jouini, Malamud, and Napp (2012), Cvitanić and Malamud (2011), Detemple and Murthy (1994), Dumas and Lyasoff (2012), Gallmeyer and Hollifield (2008), Gollier and Zeckhauser (2005), Isaenko (2008), Longstaff and Wang (2012), Xiouros and Zapatero (2010), Yan (2008). Related works by Ait-Sahalia, Cacho-Diaz and Hurd (2009), Liu, Longstaff, and Pan (2003), among others, study optimal portfolio allocations with event risk.

The paper is organized as follows. Section 1 discusses the economic setup and defines the equilibrium. In Section 2, discusses optimal consumption and portfolio choice in partial equilibrium, and then provides the characterization of equilibrium processes. In Section 3, we provide the results of calibrations, the analysis of equilibrium, and discuss the economic intuition. Section 4 extends the model to the case of heterogeneous beliefs. Section 5 concludes, and Appendix A provides the proofs.

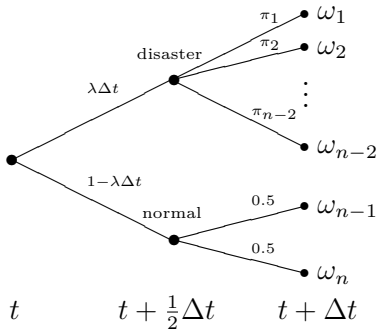

Figure 1
States of the Economy
After time t the economy moves to disaster state with small probability $\lambda\Delta t$ and to normal state with probability $1-\lambda\Delta t$. Conditional on being in disaster state the economy can further move to states ω_1, ..., ω_{n-2} at time $t+\Delta t$ with conditional probabilities π_1, ..., π_{n-2}. Conditional on being in normal state the economy moves to either ω_{n-1} or ω_n with equal probabilities.

1. Economic Setup

We consider a discrete-time Lucas (1978) economy with dates $t = 0, \Delta t, 2\Delta t, \ldots, T$, one consumption good produced by an exogenous Lucas tree, and two heterogeneous investors, A and B, with Epstein-Zin preferences over consumption. At date t the tree generates $D_t\Delta t$ units of consumption good, where D_t follows a discrete-time multinomial process with n states $\omega_1, \ldots, \omega_n$ [see Figure 1]:

$$\Delta D_t = D_t[m_D\Delta t + \sigma_D\Delta w_t + J_D(\omega)\Delta j_t], \tag{1}$$

where m_D and σ_D are constants, $J_D(\omega)$ is a random variable which gives the size of a drop in aggregate output following a rare disaster, and $\Delta D_t = D_{t+\Delta t} - D_t$. Parameters m_D, σ_D, $J_D(\omega)$ and Δt are such that $D_t > 0$ at all times. Processes w_t and j_t follow dynamics $w_{t+\Delta t} = w_t + \Delta w_t$ and $j_{t+\Delta t} = j_t + \Delta j_t$, respectively, where increments Δw_t and Δj_t are given by:

$$\Delta w_t = \begin{cases} +\sqrt{\Delta t}, & \text{in state } \omega_{n-1}, \\ -\sqrt{\Delta t}, & \text{in state } \omega_n, \end{cases} \qquad \Delta j_t = \begin{cases} 1, & \text{in states } \omega_1, \ldots \omega_{n-2}, \\ 0, & \text{in states } \omega_{n-1}, \omega_n. \end{cases} \tag{2}$$

The structure of uncertainty is illustrated on Figure 1. From the current time-t state, the economy moves to time-$(t+\Delta t)$ disaster states with small probability $\lambda\Delta t$ or to normal states with probability $1-\lambda\Delta t$. Disaster states ω_1, ..., ω_{n-2} have conditional probabilities $\text{Prob}(\omega = \omega_k|\text{disaster}) = \pi_k$ whereas normal states ω_{n-1} and ω_n have conditional probabilities $\text{Prob}(\omega = \omega_k|\text{normal}) = 0.5$. Processes w_t and j_t are analogues of continuous-time Brownian motion and Poisson processes, respectively. Lemma A.1 in the Appendix demonstrates that process (1) converges to a Lévy process when $\Delta t \to 0$. Conveniently, $\mathbb{E}_t[\Delta w_t|\text{normal}] = 0$ and $\text{var}_t[\Delta w_t|\text{normal}] = \Delta t$, similarly to a Brownian motion, where $\mathbb{E}_t[\cdot]$ and $\text{var}_t[\cdot]$, denote expectation and variance operators conditional on time-t information, respectively.

The discreteness of time has several advantages. First, it is more realistic to assume that

investors make consumption and portfolio choice decisions discretely. Moreover, consumption data are not available at high frequencies. Second, it allows for modeling distributions with non-lognormal consumption processes. Such processes can be easily approximated by multinomial dynamics. Finally, we demonstrate in subsequent sections that the discrete-time model is more tractable, and allows for passing to continuous time. We also note, that discrete-time models with binomial or multinomial processes for consumption and asset prices are widely employed in the literature [e.g., Mehra and Prescott (1985); Rietz (1985); He (1991); Pliska (1997); Buss, Uppal, and Vilkov (2011); Dumas and Lyasoff (2012), among others]. However, the literature typically sets $\Delta t = 1$, whereas in our calibration we take much smaller $\Delta t = 1/250$ and long horizon $T = 200$ and also take continuous-time limits in various special cases.

1.1. Securities Markets

The financial market is complete, and the investors can trade n securities: a riskless bond in zero net supply, one stock in net supply of one unit, which is the claim to the stream of dividends generated by the Lucas tree, and $n-2$ zero net supply catastrophe insurances each paying one unit of consumption in rare disaster states $\omega_1, \ldots, \omega_{n-2}$, respectively. All trades happen at discrete dates $t = 0, \Delta t, 2\Delta t, \ldots, T$. We consider Markovian equilibria in which bond prices, B, ex-dividend stock prices, S, and catastrophe insurance prices, P_k, follow dynamics:

$$\Delta B_t = B_t r_t \Delta t, \tag{3}$$

$$\Delta S_t + D_{t+\Delta t}\Delta t = S_t[m_{S,t}\Delta t + \sigma_{S,t}\Delta w_t + J_{S,t}(\omega)\Delta j_t], \tag{4}$$

$$\Delta P_{k,t} + 1_{\{\omega=\omega_k\}} = P_{k,t}[m_{P_k,t}\Delta t + \sigma_{P_k,t}\Delta w_t + J_{P_k,t}(\omega)\Delta j_t], \tag{5}$$

where $k = 1, \ldots, n-2$.[1] The drift and volatility processes $m_{S,t}$, $\sigma_{S,t}$, $m_{P_k,t}$, and $\sigma_{P_k,t}$, and jump sizes $J_{S,t}$ and $J_{P_k,t}$ are determined in equilibrium.

We denote the vector of drifts by $m_t = (m_{S,t}, \ldots, m_{P_{n-2},t})^\top$, the vector of risky asset expected returns by $\mu_t = (\mu_{S,t}, \mu_{P_1,t}, \ldots, \mu_{P_{n-2},t})^\top$, and the volatility matrix by $\Sigma = (\Sigma_S, \Sigma_{P_k}, \ldots, \Sigma_{P_{n-2}})^\top \in \mathbb{R}^{(n-1)\times(n-1)}$, where $\Sigma_S = (\sigma_{S,t}, J_{S,t}(\omega_1), \ldots, J_{S,t}(\omega_{n-2}))^\top$, $\Sigma_{P_k} = (\sigma_{P_k,t}, J_{P_k,t}(\omega_1), \ldots, J_{P_k,t}(\omega_{n-2}))^\top$, for $k = 2, \ldots, n-2$. We note that expected risky asset returns are given by $\mu_t = m_t + \lambda\Sigma_t(0, \pi_1, \ldots, \pi_{n-2})^\top$. Finally, we define the *state price density* ξ_t as a strictly positive process such that asset prices have representations

$$B_t = \mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t} B_{t+\Delta t}\Big], \tag{6}$$

$$S_t = \mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}(S_{t+\Delta t} + D_{t+\Delta t}\Delta t)\Big], \tag{7}$$

$$P_{k,t} = \mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}(P_{k,t+\Delta t} + 1_{\{\omega=\omega_k\}})\Big], \tag{8}$$

[1] We note that the multinomial dynamics for stock prices and catastrophe insurances can always be written as processes (4)–(5) with Δt, Δw, and Δj terms because the vector of time-$(t+\Delta t)$ asset returns in states $\omega_1, \ldots, \omega_n$ can be uniquely decomposed as a linear combination of vectors $(\Delta t, \ldots, \Delta t)^\top$, $(\sqrt{\Delta t}, -\sqrt{\Delta t}, 0, \ldots, 0)^\top$, and $n-2$ linearly independent vectors of the form $(0, 0, \ldots, 1, \ldots, 0)^\top$.

where $S_T = 0$ and $P_{k,T} = 0$ because $D_\tau = 0$ and $D_{k,\tau} = 0$ for $\tau > T$.

1.2. Investor Optimization

The investors have recursive utility U_t over consumption $c_{i,t}$ [e.g., Epstein and Zin (1989)], which satisfies the following backward equation[2]

$$U_{i,t} = \Big[(1 - e^{-\rho\Delta t})c_{i,t}^{1-1/\psi_i} + e^{-\rho\Delta t}\Big(\mathbb{E}_t[U_{i,t+\Delta t}^{1-\gamma_i}]\Big)^{\frac{1-1/\psi_i}{1-\gamma_i}}\Big]^{\frac{1}{1-1/\psi_i}}, \tag{9}$$

where $i = A, B$, γ_i and ψ_i denote investor i's risk aversion and elasticity of intertemporal substitution (EIS), respectively, and $\rho > 0$ is a time-discount parameter. In general, the investors have different risk aversions and EIS. Each period, investor i allocates fractions $\alpha_{i,t}$ and $\theta_{i,t} = (\theta_{i,S,t}, \theta_{i,P_1,t}, \ldots, \theta_{i,P_{n-2},t})^\top$ of wealth $W_{i,t}$ to riskless bonds and risky securities, respectively, so that $W_{i,t} = \alpha_{i,t}W_{i,t} + \theta_{i,t}^\top \underline{1} W_{i,t} + c_{i,t}\Delta t$, where $\underline{1} = (1, \ldots, 1)^\top \in \mathbb{R}^{n-1}$. At time 0 investor A is endowed with b units of bond, s units of stock, and p_k units of catastrophe insurance k, whereas investor B is endowed with $-b$ units of bond, $1 - s$ units of stock, and $-p_k$ units of insurance k. Investors solve the following dynamic programming problem [e.g., Epstein and Zin (1989)]:

$$V_{i,t} = \max_{c_{i,t},\theta_{i,t}} \Big[(1 - e^{-\rho\Delta t})c_{i,t}^{1-1/\psi_i} + e^{-\rho\Delta t}\Big(\mathbb{E}_t[V_{i,t+\Delta t}^{1-\gamma_i}]\Big)^{\frac{1-1/\psi_i}{1-\gamma_i}}\Big]^{\frac{1}{1-1/\psi_i}}, \tag{10}$$

where $V_{i,t}$ is investor i's value function, subject to a self-financing dynamic budget constraint[3]

$$\Delta W_{i,t} = W_{i,t}\Big(r_t + \theta_{i,t}^\top(m_t - r_t)\Big)\Delta t + W_{i,t}\theta_{i,t}^\top \Sigma_t \Delta\tilde{w}_t - c_{i,t}\Delta t(1 + r_t\Delta t), \quad W_T = c_{i,T}\Delta t, \tag{11}$$

where $i = A, B$, $\Delta\tilde{w} = (\Delta w, 1_{\{\omega=\omega_1\}}, \ldots, 1_{\{\omega=\omega_{n-2}\}})^\top$, and $1_{\{\omega=\omega_k\}}$ is an indicator function.

1.3. Equilibrium

Definition. *An equilibrium is a set of processes* $\{r_t, \mu_t, \Sigma_t\}$*, and of consumption and investment policies* $\{c_{i,t}^*, \alpha_{i,t}^*, \theta_{i,t}^*\}_{i\in\{A,B\}}$ *which solve dynamic optimization problem* (10) *for each investor, given processes* $\{r_t, \mu_t, \Sigma_t\}$*, and consumption and securities markets clear, that is,*

$$c_{A,t}^* + c_{B,t}^* = D_t, \tag{12}$$

$$\alpha_{A,t}^* W_{A,t}^* + \alpha_{B,t}^* W_{B,t}^* = 0, \tag{13}$$

$$\theta_{A,S,t}^* W_{A,t}^* + \theta_{B,S,t}^* W_{B,t}^* = S_t, \tag{14}$$

$$\theta_{A,P_k,t}^* W_{A,t}^* + \theta_{B,P_k,t}^* W_{B,t}^* = 0, \tag{15}$$

[2] Gârleanu and Panageas (online appendix, 2012) and Skiadas (2012) consider similar formulations with arbitrarily small interval Δt between dates and derive continuous-time formulations by passing to limits $\Delta t \to 0$. Furthermore, the model in Skiadas (2012) allows for rare events, similarly to the present paper.

[3] Investors' time t and $t+\Delta t$ wealths are given by $W_t = \alpha_t W_t + \theta_t^\top \underline{1} W_t + c_t\Delta t$ and $W_{t+\Delta t} = W_t + \alpha_t W_t \Delta B_t/B_t + \theta_{S,t}(\Delta S_t + D_{t+\Delta t}\Delta t)/S_t + \sum \theta_{P_k,t}(\Delta P_{k,t} + D_{k,t+\Delta t}\Delta t)/P_{k,t}$, respectively. Substituting $\alpha_t W_t = W_t - \theta_t^\top \underline{1} W_t - c_t\Delta t$ into the latter equation and using asset price dynamics (4)–(5) we obtain budget constraint (11).

where $k = 1, \ldots, n-2$, and $W^*_{A,t}$ and $W^*_{B,t}$ denote wealths under optimal strategies.

In addition to asset returns μ, we also study their *risk premia* $\mu - r$. We also derive stock *price-dividend* and *wealth-consumption* ratios $\Psi = S/D$ and $\Phi_i = W^*_i/c^*_i$, respectively. We derive a Markovian equilibrium in which the *consumption share* $y = c^*_B/D$ of investor B in aggregate consumption D is an endogenous state variable, as in the related literature [e.g., Chen, Joslin, and Tran (2012), Gârleanu and Pedersen (2012); among others]. We demonstrate later that in a Markovian equilibrium consumption share y_t follows a process

$$\Delta y_t = y_t[m_{y,t}\Delta t + \sigma_{y,t}\Delta w_t + J_{y,t}(\omega)\Delta j_t], \tag{16}$$

where the drift m_y, volatility σ_y, and jump sizes $J_{y,t}(\omega)$ are determined in equilibrium.

Throughout the paper we restrict preferences and Lucas tree parameters γ_i, ψ_i, ρ, m_D, σ_D, $J_D(\omega)$, λ, π_k, and Δt to be such that the following two technical conditions are satisfied:

$$g_{i,1} \equiv e^{-\rho\Delta t}\left(\mathbb{E}_t\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma_i}\Big]\right)^{\frac{1-1/\psi_i}{1-\gamma_i}} < 1, \tag{17}$$

$$g_{i,2} \equiv e^{-\rho\Delta t}\left(\mathbb{E}_t\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma_i}\Big]\right)^{\frac{\gamma_i-1/\psi_i}{1-\gamma_i}} \mathbb{E}_t\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{-\gamma_i}\Big] < 1, \tag{18}$$

where $i = A, B$, and $g_{i,1}$ and $g_{i,2}$ are constants. We demonstrate in Section 2.2 that under these conditions the equilibrium processes are bounded as $T \to \infty$ in homogeneous agent economies, and hence are analogous to dividend growth restrictions in continuous-time models.

2. Characterization of Equilibrium

2.1. Consumption and Portfolio Choice

In this section, we derive optimal investment and consumption policies of investors in a partial equilibrium, that is, taking the asset prices dynamics (3)–(5) as given. We propose new method for solving problems with non-lognormal asset returns, which retains the tractability and the structure of asset allocation rules in continuous-time settings. For the time being, we do not take a stand on state variables in the economy, and assume that processes r_t, m_t, and Σ_t are functions of an unspecified Markovian variable z_t. We start by deriving the discrete-time process for state price density ξ_t in terms of processes r_t, μ_t, and Σ_t. Lemma 1 reports the result.

Lemma 1 (State Price Density). *The state price density ξ_t follows a multinomial process*

$$\Delta\xi_t = -\frac{\xi_t}{1 + r_t\Delta t}\Big[r_t\Delta t + \Big(\Sigma_t^{-1}(\mu_t - r_t\underline{1})\Big)^\top\Big(\tfrac{1}{\Delta t}\operatorname{var}_t[\Delta\tilde{w}_t]\Big)^{-1}\Big(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t]\Big)\Big], \tag{19}$$

where $\mu_t = m_t + \Sigma_t\mathbb{E}_t[\Delta\tilde{w}_t]/\Delta t$ is the vector of expected risky asset returns, $\underline{1} = (1, \ldots, 1)^\top \in \mathbb{R}^{n-1}$, and $\Delta\tilde{w}_t$, $\mathbb{E}_t[\Delta\tilde{w}_t]$, and $\operatorname{var}_t[\Delta\tilde{w}_t]$ are given by

$$\Delta\tilde{w}_t = (\Delta w_t, 1_{\{\omega=\omega_1\}}, \ldots, 1_{\{\omega=\omega_{n-2}\}})^\top, \tag{20}$$

$$\mathbb{E}_t[\Delta\tilde{w}_t] = (0, \lambda\pi_1, \ldots, \lambda\pi_{n-2})^\top \Delta t, \tag{21}$$

$$\mathrm{var}_t[\Delta\tilde{w}_t] = \mathrm{diag}\{1-\lambda\Delta t, \lambda\pi_1, \ldots, \lambda\pi_{n-2}\}\Delta t - \mathbb{E}_t[\Delta\tilde{w}_t]\mathbb{E}_t[\Delta\tilde{w}_t]^\top, \tag{22}$$

where $1_{\{w=w_k\}}$ *is an indicator function and* $\mathrm{diag}\{1-\lambda\Delta t, \lambda\pi_1, \ldots, \lambda\pi_{n-2}\}$ *is a diagonal matrix.*

The state price density process (19) preserves the structure of the familiar continuous-time process for ξ when there is no disaster risk. In particular, as in continuous-time, the drift and volatility terms of process (19) are driven by the interest rate r_t and the *market prices of risk* $\Sigma^{-1}(\mu_t - r_t)$, respectively. In a model without disasters $\Delta\tilde{w}_t = \Delta w_t$, $\mathbb{E}_t[\Delta\tilde{w}_t] = 0$, $\mathrm{var}[\Delta\tilde{w}_t] = \Delta t$, and hence from equation (19) we obtain dynamics $\Delta\xi_t = -\xi_t[r_t\Delta t + (\mu_t - r_t)/\sigma_t \Delta w_t]/(1 + r_t\Delta t)$. Consequently, as $\Delta t \to 0$, the dynamics for ξ_t converges (under some technical assumptions) to the well-known process $d\xi_t = -\xi_t[r_t dt + (\mu_t - r_t)/\sigma_t dw_t]$ [e.g., Duffie (2001)], where μ_t and σ_t are stock mean-return and volatility, respectively.

Next, we derive optimal consumption and investment policies by solving dynamic programming problem (10). In particular, from the first order conditions we obtain the expressions for consumption growths $c^*_{i,t+\Delta t}/c^*_{i,t}$ and optimal portfolio weights $\theta^*_{i,t}$ in terms of state price density ξ_t and wealth-consumption ratios $\Phi_{i,t}$. Proposition 1 reports the results.

Proposition 1 (Optimal Consumption and Investment Policies). *The investors' wealth-consumption ratios* $\Phi_{i,t} = \Phi_i(z_t, t)$ *satisfy backward equations*

$$\Phi_{i,t} = e^{-\rho\psi_i\Delta t}\left(\mathbb{E}_t\Big[\Big(\frac{\xi_{t+\Delta t}}{\xi_t}\Big)^{\frac{\gamma_i-1}{\gamma_i}}\Phi_{i,t+\Delta t}^{\frac{\gamma_i-1}{\gamma_i(1-\psi_i)}}\Big]\right)^{\frac{\gamma_i(1-\psi_i)}{\gamma_i-1}} + \Delta t, \quad \Phi_{i,T} = \Delta t. \tag{23}$$

The value functions $V_{i,t}$*, consumption growths* $c^*_{i,t+\Delta t}/c^*_{i,t}$ *and portfolio weights* $\theta^*_{i,t}$ *are given by:*

$$V_{i,t} = \Big(\Phi_{i,t}\frac{1-e^{-\rho\Delta t}}{\Delta t}\Big)^{\frac{1}{1-1/\psi_i}}\frac{W_{i,t}}{\Phi_{i,t}}, \tag{24}$$

$$\frac{c^*_{i,t+\Delta t}}{c^*_{i,t}} = e^{-\psi_i\rho\Delta t}\Big(\frac{\xi_{t+\Delta t}}{\xi_t}\Big)^{-\frac{1}{\gamma_i}}\Big(\Phi_{i,t+\Delta t}\Big)^{\frac{\gamma_i\psi_i-1}{\gamma_i(1-\psi_i)}}\left(\mathbb{E}_t\Big[\Big(\frac{\xi_{t+\Delta t}}{\xi_t}\Big)^{\frac{\gamma_i-1}{\gamma_i}}\Big(\Phi_{i,t+\Delta t}\Big)^{\frac{\gamma_i-1}{\gamma_i(1-\psi_i)}}\Big]\right)^{\frac{\gamma_i\psi_i-1}{1-\gamma_i}}, \tag{25}$$

$$\theta^*_i(z_t,t) = (\Sigma_t^{-1})^\top\mathbb{E}_t\Big[\frac{\Phi_{i,t+\Delta t}}{\Phi_{i,t}}\frac{c^*_{i,t+\Delta t}}{c^*_{i,t}}\,\mathrm{var}_t[\Delta\tilde{w}_t]^{-1}\Big(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t]\Big)\Big], \tag{26}$$

where $i = A, B$*,* $\Delta\tilde{w}_t$*,* $\mathbb{E}_t[\Delta\tilde{w}_t]$*, and* $\mathrm{var}_t[\Delta\tilde{w}_t]$ *are given by equations* (20)–(22) *Furthermore, the state price density* ξ_t *is related to consumption growths* $c^*_{i,t+\Delta t}/c^*_{i,t}$ *as follows:*

$$\frac{\xi_{t+\Delta t}}{\xi_t} = e^{-\rho\Delta t}\Big(\frac{c^*_{i,t+\Delta t}}{c^*_{i,t}}\Big)^{-\gamma_i}\Phi_{i,t+\Delta t}^{\frac{\gamma_i\psi_i-1}{1-\psi_i}}\left(\mathbb{E}_t\Big[\Big(\frac{c^*_{i,t+\Delta t}}{c^*_{i,t}}\Big)^{1-\gamma_i}\Phi_{i,t+\Delta t}^{\frac{(\gamma_i-1)\psi_i}{1-\psi_i}}\Big]\right)^{\frac{\gamma_i\psi_i-1}{(1-\gamma_i)\psi_i}}, \quad i = A, B, \tag{27}$$

and marginal rates of substitution, defined as $MRS_{i,t+\Delta t}(\omega_k) = \Big(\partial U_{i,t}/\partial c_{i,t+\Delta t}(\omega_k)\Big)/(\partial U_{i,t}/\partial c_{i,t})$*, are given by* $MRS_{i,t+\Delta t}(\omega_k) = (1-\lambda)\pi_k\xi_{t+\Delta t}(\omega_k)/\xi_t$*.*

Equations (23)–(26) demonstrate that consumption and portfolio choice problem can be solved by backward induction. In particular, wealth-consumption ratio $\Phi_{i,t}$ can be obtained by backward induction starting from the terminal date $t = T$. In particular, $\Phi_{i,t}$ appears to be an explicit function of $\Phi_{i,t+\Delta t}$ from the previous step, and hence its calculation does not require solving any equations. The wealth-consumption ratios can then be used to calculate consumption growths $c^*_{i,t+\Delta t}/c^*_{i,t}$ in all states $\omega_1, \ldots, \omega_n$ from equation (25). The growth rates $c^*_{i,t+\Delta t}/c^*_{i,t}$ are then used to calculate portfolio weights $\theta^*_{i,t}$ from equations (26).

We note that all the equations in Proposition 1 significantly simplify when the investors have CRRA preferences. In particular, substituting $c^*_{i,t+\Delta t}/c^*_{i,t}$ from (25) into portfolio weight (26), after simple algebra, we obtain the following expression

$$\begin{aligned}\theta^*_t = &\underbrace{e^{-\rho/\gamma_i \Delta t}(\Sigma_t^{-1})^\top \operatorname{cov}_t\Big[\Big(\frac{\xi_{t+\Delta t}}{\xi_t}\Big)^{-\frac{1}{\gamma_i}}, \operatorname{var}_t[\Delta\tilde{w}_t]^{-1}\Big(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t]\Big)\Big]}_{\text{myopic demand}} \\ &+ \underbrace{e^{-\rho/\gamma_i \Delta t}(\Sigma_t^{-1})^\top \operatorname{cov}_t\Big[\frac{\Phi_{i,t+\Delta t} - \Phi_{i,t}}{\Phi_{i,t}}\Big(\frac{\xi_{t+\Delta t}}{\xi_t}\Big)^{-\frac{1}{\gamma_i}}, \operatorname{var}_t[\Delta\tilde{w}_t]^{-1}\Big(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t]\Big)\Big]}_{\text{hedging demand}}.\end{aligned} \tag{28}$$

Optimal weight (28) preserves the structure of weights in continuous-time analysis. In particular, the first term in equation (28) can be interpreted as myopic demand and the second term as hedging demand, as in continuous-time portfolio choice [e.g., Merton (1973); Liu (2007)].

2.2. General Equilibrium

In this Section, we use the results of Proposition 1 to characterize the equilibrium. Equation (27) provides two expressions for the state price density ξ_t in terms of investor A's and investor B's consumptions, respectively. Equating these expressions and substituting consumptions in terms of consumption share y_t, that is, $c^*_{A,t} = (1-y_t)D_t$ and $c^*_{B,t} = y_t D_t$, we obtain the following system of equations from which $y_{t+\Delta t}$ can be obtained as a function of y_t and state ω:

$$\begin{aligned}\frac{\xi_{t+\Delta t}}{\xi_t} &= e^{-\rho\Delta t}\Big(\frac{1-y_{t+\Delta t}}{1-y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{-\gamma_A}\Phi_{A,t+\Delta t}^{\frac{\gamma_A\psi_A-1}{1-\psi_A}}\left(\mathbb{E}_t\Big[\Big(\frac{1-y_{t+\Delta t}}{1-y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma_A}\Phi_{A,t+\Delta t}^{-\frac{(1-\gamma_A)\psi_A}{1-\psi_A}}\Big]\right)^{\frac{\gamma_A\psi_A-1}{(1-\gamma_A)\psi_A}} \\ &= e^{-\rho\Delta t}\Big(\frac{y_{t+\Delta t}}{y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{-\gamma_B}\Phi_{B,t+\Delta t}^{\frac{\gamma_B\psi_B-1}{1-\psi_B}}\left(\mathbb{E}_t\Big[\Big(\frac{y_{t+\Delta t}}{y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma_B}\Phi_{B,t+\Delta t}^{-\frac{(1-\gamma_B)\psi_B}{1-\psi_B}}\Big]\right)^{\frac{\gamma_B\psi_B-1}{(1-\gamma_B)\psi_B}}.\end{aligned} \tag{29}$$

Intuitively, equation (29) holds because investors' marginal rates of substitution, derived in Proposition 1, are equal due to market completeness.

We solve equation (29) using Newton's method [e.g., Judd (1998)], and derive time-$(t + \Delta t)$ consumption shares $y_{t+\Delta t}(y_t; \omega_k)$ in states ω_1, ..., ω_n. Substituting shares $y_{t+\Delta t}$ back into equation (29) we obtain state price density process $\xi_{t+\Delta t}/\xi_t$ as function of share y_t and time-$(t + \Delta t)$ state ω. Then, we use $\xi_{t+\Delta t}/\xi_t$ to obtain asset prices by solving equations (7)–(8)

backwards from the terminal date T. The equilibrium processes r_t, $\mu_t - r_t\underline{1}$, and Σ_t are determined as drifts and volatilities of processes ξ_t, S_t, and $P_{k,t}$, and are reported in Proposition 2 below.

Proposition 2 (Equilibrium Processes). *Investors' optimal time-t portfolio weights and wealth-consumption ratios are functions of state variable $y_t = c^*_{B,t}/D_t$, and are given by equations (26)–(23) in which $c^*_{A,t} = (1-y_t)D_t$ and $c^*_{B,t} = y_t D_t$. The equilibrium interest rate r_t, risk premium $\mu_t - r_t\underline{1}$, and volatility matrix Σ_t are functions of consumption share y_t, given by*

$$r_t = \Big(\frac{1}{\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t]} - 1\Big)\frac{1}{\Delta t}, \tag{30}$$

$$\mu_t - r_t\underline{1} = -\frac{\Sigma_t \operatorname{cov}_t(\xi_{t+\Delta t}/\xi_t, \Delta\tilde{w}_t)}{\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t]\Delta t}, \tag{31}$$

$$\Sigma_t = \mathbb{E}_t\Big[R_{t+\Delta t}\Big(\operatorname{var}_t[\Delta\tilde{w}_t]^{-1}(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])\Big)^\top\Big], \tag{32}$$

where $\Delta\tilde{w}$, $\mathbb{E}_t[\Delta\tilde{w}]$, and $\operatorname{var}_t[\Delta\tilde{w}]$ are given by equations (20)–(22), ratio $\xi_{t+\Delta t}/\xi_t$ is given by equation (29), $\underline{1} = (1,\ldots,1)^\top \in \mathbb{R}^{n-1}$, time-$(t+\Delta t)$ risky assets returns $R_{t+\Delta t}$ are given by

$$R_{t+\Delta t} = \Big(\frac{\Psi_{t+\Delta t} + \Delta t}{\Psi_t}\frac{D_{t+\Delta t}}{D_t}, \frac{P_{1,t+\Delta t} + D_{1,t+\Delta t}}{P_{1,t}}, \ldots, \frac{P_{n-2,t+\Delta t} + D_{n-2,t+\Delta t}}{P_{n-2,t}}\Big)^\top - \underline{1}. \tag{33}$$

Asset returns $R_{t+\Delta t}$ are functions of share y_t, time t, and state ω. Catastrophe insurances prices $P_{k,t}$ and price-dividend ratio Ψ_t are functions of share y_t and time t. Furthermore, prices $P_{k,t}$ solve backward equations (8), whereas price-dividend ratio Ψ_t solves backward equation

$$\Psi_t = \mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\frac{D_{t+\Delta t}}{D_t}\Big(\Psi_{t+\Delta t} + \Delta t\Big)\Big], \quad \Psi_T = 0. \tag{34}$$

Similarly, consumption share y_t follows process (16) where $m_{y,t} = \mathbb{E}_t[\Delta y_t/y_t|\text{normal}]/\Delta t$ and $(\sigma_{y,t}, J_y(\omega_1), \ldots, J_{y,t}(\omega_n))^\top = \mathbb{E}_t[y_{t+\Delta t}/y_t \operatorname{var}_t[\Delta\tilde{w}_t]^{-1}(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])]$.

Proposition 2 demonstrates how the equilibrium interest rates r, risk premia $\mu - r$, volatilities Σ, and price dividend ratios Ψ can be derived given the state price density ξ_t, which is obtained from equation (29). To provide further intuition for the role of rare events, we obtain closed-form expressions for the equilibrium processes when investors have identical risk aversions, and when the economy is populated by homogeneous agents. When $\gamma_A = \gamma_B$ the analysis is simplified by the fact that aggregate consumption growths $D_{t+\Delta t}/D_t$ cancel out from both sides of equation (29) for consumption share $y_{t+\Delta t}$. Consequently, share y_t turns out to be a deterministic function of time. As a result, all the other equilibrium processes are also deterministic processes. To provide tractable expressions, we pass to continuous time limit. Corollary 1 reports the results.

Corollary 1 (Closed-Form Solutions). 1) *If the investors have the same risk aversion, that is $\gamma_i = \gamma$ for $i = A, B$, the equilibrium risk premium $\mu_{S,t}, -r_t$ and the market prices of risk*

$\Sigma_t^{-1}(\mu_t - r_t\underline{1})$ *are constant and do not depend on intertemporal elasticities of substitution* ψ_i. *Furthermore, as* $\Delta t \to 0$ *processes* r_t, $\Sigma_t^{-1}(\mu_t - r_t\underline{1})$, $\mu_{S,t} - r_t$ *and* $\Sigma_{S,t}$ *converge to*

$$r_t = \rho + \gamma m_D - \frac{\gamma(1+\gamma)}{2}\sigma_D^2 - \lambda\Big(\mathbb{E}\Big[(1+J_D(\omega))^{-\gamma}|\text{disaster}\Big] - 1\Big) + \Big(\frac{1}{\psi_B y_t + \psi_A(1-y_t)} - \gamma\Big)\Big(m_D - \frac{\gamma}{2}\sigma_D^2 + \lambda\mathbb{E}\Big[\frac{(1+J_D(\omega))^{1-\gamma}-1}{1-\gamma}\Big|\text{disaster}\Big]\Big), \tag{35}$$

$$\Sigma_t^{-1}(\mu_t - r_t\underline{1}) = \Big(\gamma\sigma_D, -\lambda\pi_1((1+J_1)^{-\gamma}-1), \ldots, -\lambda\pi_{n-2}((1+J_{n-2})^{-\gamma}-1)\Big)^\top, \tag{36}$$

$$\mu_{S,t} - r_t = \gamma\sigma_D^2 - \lambda\mathbb{E}\Big[(1+J_D(\omega))^{-\gamma}J_D(\omega)|\text{disaster}\Big] + \lambda\mathbb{E}\Big[J_D(\omega)|\text{disaster}\Big], \tag{37}$$

$$\Sigma_{S,t} = (\sigma_D, J_D(\omega_1), \ldots, J_D(\omega_{n-2}))^\top. \tag{38}$$

Consumption share y_t *is a deterministic function of* t, *which solves the equation*

$$\Big(\frac{y_t}{y_0}\Big)^{1/\psi_B} = \Big(\frac{1-y_t}{1-y_0}\Big)^{1/\psi_A}\Big(\mathbb{E}_t\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma}\Big]\Big)^{\frac{1/\psi_A - 1/\psi_B}{1-\gamma}\frac{t}{\Delta t}}. \tag{39}$$

2) *If the economy is populated by investor* i *only, price-dividend ratio* Ψ *and insurance prices* P_k *are given by* $\Psi_t = \Big(1 - g_{i,1}^{(T-t+\Delta t)/\Delta t}\Big)/(1-g_{i,1})g_{i,1}\Delta t$, $P_{k,t} = \Big(1 - g_{i,2}^{(T-t)/\Delta t}\Big)/(1-g_{i,2})b_k\lambda\Delta t$ *where* $g_{i,1}$ *and* $g_{i,2}$ *are given by equations* (17)–(18), *and* b_k *is given in the Appendix. Ratio* Ψ *and prices* P_k *are bounded functions of* T *if* $g_{i,1} < 1$ *and* $g_{i,2} < 1$, *and as* $\Delta t \to 0$ *converge to*

$$\Psi_t = \frac{1 - e^{-(r+(\mu_S - r) - \mu_D)(T-t)}}{r + (\mu_S - r) - \mu_D}, \quad P_{k,t} = \lambda\pi_k(1+J_k)^{-\gamma_i}\frac{1-e^{-r(T-t)}}{r}, \tag{40}$$

where r *is given by equation* (35) *with* $\psi_A = \psi_B = \psi_i$, $\mu_D = m_D + \lambda\mathbb{E}_t[J_D(\omega)|\text{disaster}]$ *is dividend expected growth rate, and* $\mu_S - r$ *is given by equation* (37).

Corollary 1 demonstrates that the heterogeneity in intertemporal elasticities of substitution affects only interest rates in the economy, whereas the market prices of risk, Sharpe ratio, and stock return volatility are unaffected by EIS, similarly to settings without rare events [e.g., Gârleanu and Panageas (2012)]. We observe that risk premium (37) is an increasing function of intensity λ, because $J_D < 0$. In the case of homogeneous-investor economy equilibrium processes (35)–(40) can be interpreted as boundary conditions for the case with heterogeneous investors, and provide good approximation for the equilibrium processes when the economy is dominated by either investor A (i.e, $y \approx 0$) or investor B (i.e., $y \approx 1$). Consequently, the intuition for the effect of EIS on rate r and price-dividend ratio Ψ in the homogeneous investor case provides insights about the general case. We provide this intuition below.

Setting $\psi_A = \psi_B = \psi$ in equation (35) we obtain interest rate in a homogeneous-investor economy, which after some algebra can be rewritten as follows:

$$r = \rho + \frac{1}{\psi}m_D - \frac{\gamma(1+\psi)}{2\psi}\sigma_D^2 + \Big(\frac{1}{\psi} - 1\Big)\lambda\mathbb{E}_t\Big[\frac{(1+J_D)^{1-\gamma}-1}{1-\gamma}|\text{disaster}\Big] + \lambda\mathbb{E}_t\Big[(1+J_D)^{-\gamma}J_D|\text{disaster}\Big]. \tag{41}$$

Equation (41) highlights the effect of EIS ψ on interest rate r. The second term in (41) captures the consumption smoothing effect. This term decreases with higher ψ because investors with higher EIS tend to save more for consumption smoothing purposes, which pushes down the interest rates. The third term in (41) captures the effect of precautionary savings, and coefficient $\gamma(1+\psi)$ is investor's prudence parameter for small risks Δw_t [e.g., Kimball and Weil (2009)]. The impact of the third term diminishes with higher ψ because the investor with higher EIS saves more for consumption smoothing, and hence has lower demand for precautionary savings. The last two terms in (41) quantify the effect of precautionary savings in the case of large disaster risks. Similarly, higher ψ diminishes the impact of the last two terms. In particular, because $((1+J_D)^{1-\gamma}-1)/(1-\gamma) < 0$ and $(1+J_D)^{-\gamma}J_D < 0$, when $\psi > 1$ the fourth term is positive and the last term is negative, and hence these terms partially offset each other. Finally, from equation (41), after some algebra, we find that rare disasters unambiguously decrease interest rates.[4]

Next, substituting interest rate r and equity premium $\mu_S - r$ from equations (41) and (37) into equation (40) for price-dividend ratio Ψ we obtain:

$$\Psi = \frac{1}{\rho - \Big(1 - \frac{1}{\psi}\Big)\Big(m_D - \frac{\gamma}{2}\sigma_D^2 + \lambda \mathbb{E}_t\Big[\frac{(1+J_D)^{1-\gamma}-1}{1-\gamma}|\text{disaster}\Big]\Big)}. \tag{42}$$

Equation (42) analytically demonstrates that the economic uncertainty, captured both by volatility σ_D and the probability of disaster λ, increases (decreases) price-dividend ratio for $\psi < 1$ ($\psi > 1$), as also pointed out in the related literature [e.g., Bansal and Yaron (2004)].

3. Analysis of Equilibrium

In this Section, we derive equilibrium processes numerically and study the interaction between investor heterogeneity and rare disasters. Table 1 provides the exogenous parameters for the aggregate dividend process (1). We take dividend growth rates m_D and volatilities σ_D in normal times from Cambell (2003), and the probability of disaster λ from Barro (2005). We also fix $\Delta t = 1/250$ and $T = 200$ so that the results are not affected by the discreteness and the horizon. For our calibrations we set investor risk aversions to $\gamma_A = 3$ and $\gamma_B = 6$, and compare the equilibrium processes for different values of EIS ψ.

Figure 2 reports equilibrium interest rates r, Sharpe ratios κ, excess volatilities $(\sigma_t - \sigma_D)/\sigma_D$, and price-dividend ratios Ψ as functions of consumption share y_t in the economy without rare disasters (solid lines) and with rare disasters (dashed line). Left, middle and right panels of Figure 2 show the equilibrium processes for the cases $\psi_A = \psi_B = 1.5$, $\psi_A = \psi_B = 0.5$ and CRRA preferences, respectively. For brevity, we do not report drifts and volatilities of process (16) for

[4]We note that the sum of the last two terms in equation (41) is negative. This is because $(1/\psi - 1)\mathbb{E}_t[((1 + J_D)^{1-\gamma} - 1)/(1-\gamma)] + \lambda\mathbb{E}_t[(1+J_D)^{-\gamma}J_D] < -\mathbb{E}_t[((1+J_D)^{1-\gamma}-1)/(1-\gamma)] + \lambda\mathbb{E}_t[(1+J_D)^{-\gamma}J_D] < 0$. The latter inequality holds because function $g(J_D) = ((1+J_D)^{1-\gamma}-1)/(1-\gamma)$ is concave, and hence $g(J_D) \le g(0) + g'(J_D)J_D$. Therefore, the interest rate r in equation (41) is lower than that in the economy without disasters, i.e., when $\lambda = 0$.

Description	Notation	Value
dividend mean growth	m_D	1.8%
dividend volatility	σ_D	3.6%
jump intensity	λ	1.7%
dividend jump in state ω_1	$J_D(\omega_1)$	-10%
dividend jump in state ω_2	$J_D(\omega_2)$	-25%
dividend jump in state ω_3	$J_D(\omega_3)$	-40%
conditional probability of state ω_1	π_1	0.5
conditional probability of state ω_2	π_2	0.4
conditional probability of state ω_3	π_3	0.1
time interval	Δt	1/250
horizon	T	200

Table 1
Parameters of Aggregate Consumption Process
Table 1 presents the exogenous parameters of the dividend process (1). dividend growth rates m_D and volatilities σ_D in normal times from Cambell (2003), and the probability of disaster λ from Barro (2005).

consumption share y_t. We note, however, that in our model volatility of share y_t in normal times σ_y is negative, and hence changes Δy_t are negatively correlated with dividend changes ΔD_t, conditional on being in normal times. Therefore, following the related literature [e.g., Gârleanu and Panageas (2012), Longstaff and Wang (2012)], we call process y_t countercyclical. Intuitively, negative shocks to dividend process D_t shift consumption and wealth from the less risk averse investor A to more risk averse investor B because the former holds larger fraction of wealth in stocks and hence is more sensitive to dividend shocks. As a result, B's consumption share y_t increases after negative shocks and decreases after positive shocks to dividends.

Panels (A.L), (A.M), and (A.R) of Figure (2) show interest rates for the cases $\psi = 1.5$, $\psi = 0.5$, and CRRA preferences, respectively. Comparing the results on these panels, we observe that higher EIS decreases interest rates because investors with higher EIS save more for consumption smoothing purposes. Furthermore, the fear of rare disasters increases investors' precautionary savings and decreases interest rates, consistent with the literature [e.g., Barro (2006)]. The latter effect is stronger for more risk averse investor, and hence the interest rates on panels (A.L) and (A.M) are lower when consumption share y_t of risk averse investor B is high in the economy, which makes rates r procyclical. However, when both investors have CRRA preferences, interest rates become countercyclical because investor B has lower EIS given by $\psi_B = 1/\gamma_B$ and hence saves less for consumption smoothing, which counterbalances the precautionary savings motive and pushes the interest rates up when investor B's share y_t is large.

Panels (B.L), (B.M), and (B.R) show stock Sharpe ratios $\kappa = (\mu_{S,t} - r_t)\Delta t / (\Sigma_{S,t}^\top \operatorname{var}_t[\Delta \tilde{w}_t] \Sigma_{S,t})$, defined as ratios of risk premia $(\mu_{S,t} - r_t)\Delta t$ and stock return volatilities, given by $\Sigma_{S,t}^\top \operatorname{var}_t[\Delta \tilde{w}_t] \Sigma_{S,t}$.

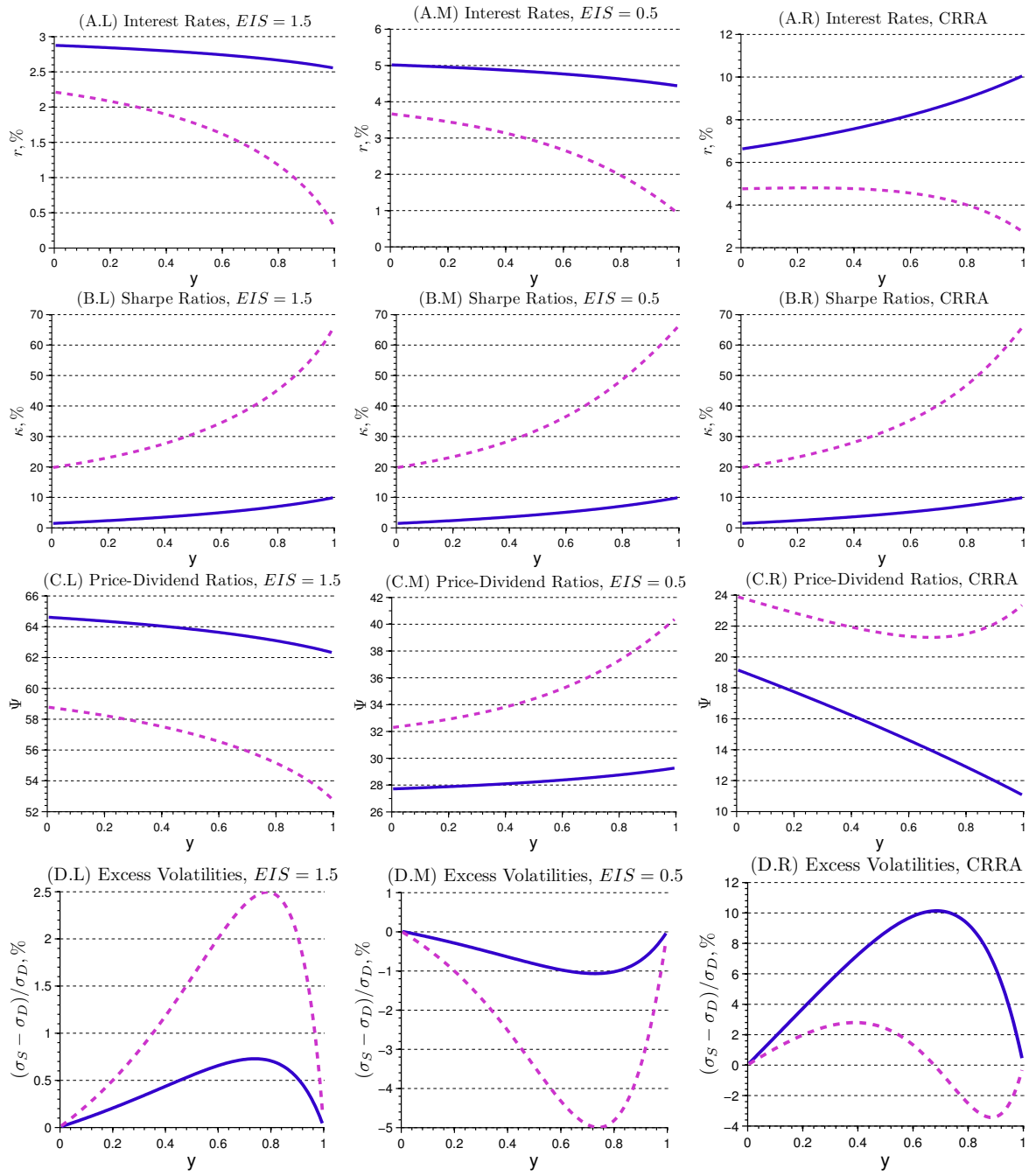

Figure 2
Equilibrium Processes

Solid lines show the processes for the case of no disaster risk, whereas dashed lines show the same processes in the economy with rare disasters. Investors have risk aversions $\gamma_A = 3$ and $\gamma_B = 6$, and time discount parameter $\rho = 2\%$. Left, middle, and right panels show the equilibrium processes when investors have $\psi_A = \psi_B = 1.5$, $\psi_A = \psi_B = 0.5$, and $\psi_A = 1/\gamma_A$ and $\psi_B = 1/\gamma_B$, respectively.

We observe that EIS has only small impact on κ, consistent with the results of Section 2. The fear of rare disasters leads to significant increases in Sharpe ratios, bringing them in line with the empirical estimates of 36% [e.g., Campbell (2003)]. Furthermore, the Sharpe ratios are countercyclical, consistent with the empirical findings [e.g., Ferson and Harvey (1991)]. Similarly, stock risk premia $\mu_{S,t} - r_t$ are also countercyclical, although they are not shown for brevity. We note that despite large Sharpe ratios stock risk premia remain low in the economy because of low stock return volatilities, as discussed below.

The dynamic behavior of interest rates and Sharpe ratios and the EIS determine the dynamic properties of price-dividend ratios Ψ, shown on Panels (C.L), (C.M), and (C.R). The fear of rare disasters decreases wealth-consumption ratios when $\psi > 1$ and increases them when $\psi < 1$, consistent with equation (42) in Section 2, derived for the homogeneous-investor economy. Furthermore, we observe that Ψ is procyclical when $\psi > 1$, consistent with empirical evidence [e.g., Campbell and Cochrane (1999)]. However, for $\psi < 1$ and CRRA preferences Ψ is countercyclical. We also observe that rare disasters make price-dividend ratios more procyclical for $\psi > 1$ and more countercyclical for $\psi < 1$.

Panels (D.L), (D.M), and (D.R) show the excess volatilities of stock returns over the volatilities of dividends conditional on being in normal times, $(\sigma_t - \sigma_{_D})/\sigma_{_D}$. The dynamic properties of volatilities are determined by those of price-dividend ratios because stock price is given by $S_t = \Psi_t D_t$. Consequently, when $\psi > 1$, and hence Ψ_t is procyclical [see Panel (C.L)], both Ψ_t and D_t move in the same direction, which gives rise to positive excess volatility, consistent with the data [e.g., Shiller (1981); Schwert (1989); Campbell and Cochrane (1999)]. Furthermore, volatilities turn out to be countercyclical over a large interval of consumption shares y_t. When $\psi < 1$ ratio Ψ_t is countercyclical, and hence its variation cancels the variation in dividends, leading to lower volatility. As noted above, the fear of disasters enhances countercyclicality or procyclicality of price-dividend ratios, depending on the EIS. Therefore, the presence of disaster risk leads to larger excess volatilities in absolute terms. We note, that the results for total volatilities $\Sigma_{S,t}^{\top} \operatorname{var}_t[\Delta\tilde{w}_t]\Sigma_{S,t}$ are qualitatively the same, and hence are not reported for brevity.

As demonstrated on Panels (D.L), (D.M), and (D.R) disasters and the heterogeneity in risk aversions generates only small increases of volatilities relative to the volatilities of dividends. We note that the difficulty of matching the magnitudes of stock return volatility is common for general equilibrium models [e.g., Heaton and Lucas (1996), Barro (2006)]. In a model with homogeneous investors and rare disasters Barro (2006) proposes to use exogenous levered claims on consumption to increase stock return volatilities. However, lest to complicate the model and to focus on the dynamic properties of asset prices, we adhere to the classic Lucas (1978) model.

Finally, we look at portfolio strategies of investors for the case when investors have risk aversions $\gamma_{_A} = 3$ and $\gamma_{_B} = 6$ and EIS $\psi_{_A} = \psi_{_B} = 1.5$. Panels (A.L) and (A.R) of Figure 3 show the fractions of wealth that investors A and B invest in stocks, respectively, and Panels (B.L) and (B.R) show the fractions of wealth that investors A and B invest in rare disaster insurance.

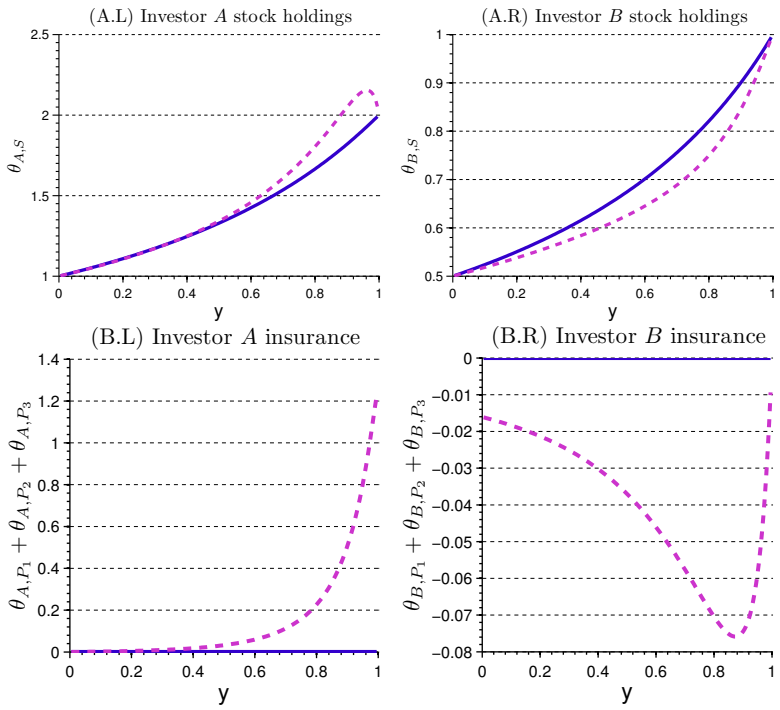

Figure 3
Portfolio Strategies
Solid lines show the processes for the case of no disaster risk, whereas dashed lines show the same processes in the economy with rare disasters. Investors have risk aversions $\gamma_A = 3$ and $\gamma_B = 6$, EIS $\psi_A = \psi_B = 1.5$, and time discount parameter $\rho = 2\%$.

Dashed and solid lines show the strategies in the economies with and without rare disasters, respectively. The results on Figure 3 demonstrate that investor A increases the investment in stocks to take advantage of high risk premia in the economy whereas more risk averse investor B decreases the investment in stocks. The increase in the stockholding of investor A is financed by leverage, and hence the share of wealth invested in stocks exceeds unity, $\theta_{A,S} > 1$, as in models with heterogeneous investors and no disaster risks [e.g., Longstaff and Wang (2012); Chabakauri (2013), among others]. Moreover, as investor A's consumption share decreases, that is, $1 - y_t$ goes down, investor A decreases investment in stocks and increases the investment in insurance contracts. This is because when $1 - y_t$ is low, investor A's consumption and wealth are low, and hence the investor becomes more sensitive to disaster risk.

Panels (B.L) and (B.R) demonstrate that, surprisingly, in the economy with rare disasters more risk averse investor B sells insurance to less risk averse investor A because the latter has very high exposure to disaster risk. Dieckmann and Gallmeyer (2005) find a similar result in economies with two CRRA investors with risk aversions $\gamma_A = 1$ and $\gamma_B = 0.5$ and $\gamma_A = 1$ and $\gamma_B = 2$. We note that investor B allocates only a small fraction of wealth to short positions in insurance securities. The fraction invested by investor A in long positions in insurance securities

is larger, but only in states where their consumption share $1 - y_t$ small. Consequently, the insurance trading has small impact on risk sharing in the economy.

4. Extension with Heterogeneous Beliefs

In this Section, we study an extension of the model in Section 2 in which investors agree on observed prices and dividends but disagree on the intensity λ of disasters, because the latter are difficult to estimate due to insufficient number of observations [e.g., Chen, Joslin, Tran (2012)]. We assume that investor A has correct estimate of intensity λ, whereas investor B beliefs that the intensity is λ_B. For the sake of tractability, we assume that investor B does not update intensity λ_B. Due to the heterogeneity in beliefs each investor prices the assets using investor-specific state price density ξ_i, where $i = A, B$, and hence equation (29), which equates the marginal rates of substitution, is not satisfied. Proposition 3 below generalizes equation (29) to the case of heterogeneous beliefs and derives the equilibrium processes.

Proposition 3 (Equilibrium Processes Under Heterogeneous Beliefs). *Investors' state price densities satisfy equation* $\xi_{A,t+\Delta t}/\xi_{A,t} = \eta(\omega_k)\xi_{B,t+\Delta t}/\xi_{B,t}$ *in states* ω_k*, where* $\eta(\omega_k)$ *is Radon-Nikodym derivative of investor B's subjective probability measure* $\mathbb{Q}$ *with respect to the correct measure* $\mathbb{P}$*. Investor B's consumption shares* $y_{t+\Delta t}(y_t; w_k)$ *at time* $t + \Delta t$ *in states* w_k *as functions of time-t share* y_t *satisfy the following system of equations:*

$$\Big(\frac{1 - y_{t+\Delta t}(y_t;\omega_k)}{1 - y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{-\gamma_A}\Phi_{A,t+\Delta t}^{\frac{\gamma_A\psi_A - 1}{1-\psi_A}}\left(\mathbb{E}_t\Big[\Big(\frac{1 - y_{t+\Delta t}}{1 - y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma_A}\Phi_{A,t+\Delta t}^{-\frac{(1-\gamma_A)\psi_A}{1-\psi_A}}\Big]\right)^{\frac{\gamma_A\psi_A - 1}{(1-\gamma_A)\psi_A}} =$$

$$\eta(\omega_k)\Big(\frac{y_{t+\Delta t}(y_t;\omega_k)}{y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{-\gamma_B}\Phi_{B,t+\Delta t}^{\frac{\gamma_B\psi_B - 1}{1-\psi_B}}\left(\mathbb{E}_t^B\Big[\Big(\frac{y_{t+\Delta t}}{y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma_B}\Phi_{B,t+\Delta t}^{-\frac{(1-\gamma_B)\psi_B}{1-\psi_B}}\Big]\right)^{\frac{\gamma_B\psi_B - 1}{(1-\gamma_B)\psi_B}}, \quad (43)$$

where $\eta(\omega_1) = \ldots = \eta(\omega_{n-2}) = \lambda_B/\lambda$*,* $\eta(\omega_{n-1}) = \eta(\omega_n) = (1 - \lambda_B)/(1 - \lambda)$*, and* $\mathbb{E}_t^B[\cdot]$ *is expectation under investor B's probability measure. The equilibrium interest rate* r_t*, risk premia under correct beliefs* $\mu_t - r_t\underline{1}$*, and the volatility matrix* Σ *are given by equations* (30)–(32)*, in which all expectations are under the correct beliefs of investor A, and the state price density is that of investor A, and* ξ_A *is given by the first equality in equation* (29)*.*

We solve the system of equations (43) numerically, using Newton's method [e.g., Judd (1998)] and obtain the dynamics of consumption shares $y_{t+\Delta t}$. We obtain these consumption shares in all time-$(t+\Delta t)$ states $\omega_1, \ldots, \omega_n$, then, we obtain state price density ξ_A from equation (29), and finally, derive the equilibrium processes from the expressions in Proposition 2. The processes for interest rate r_t, Sharpe ratios κ, stock return excess volatilities $(\sigma - \sigma_D)/\sigma_D$, and price-dividend ratios Ψ, are given on Figure 4.

Figure 4 shows the equilibrium processes for calibrated model parameters as in Section 3. We assume that investor A has correct beliefs so that $\lambda_A = \lambda = 1.7\%$. We study the equilibrium in

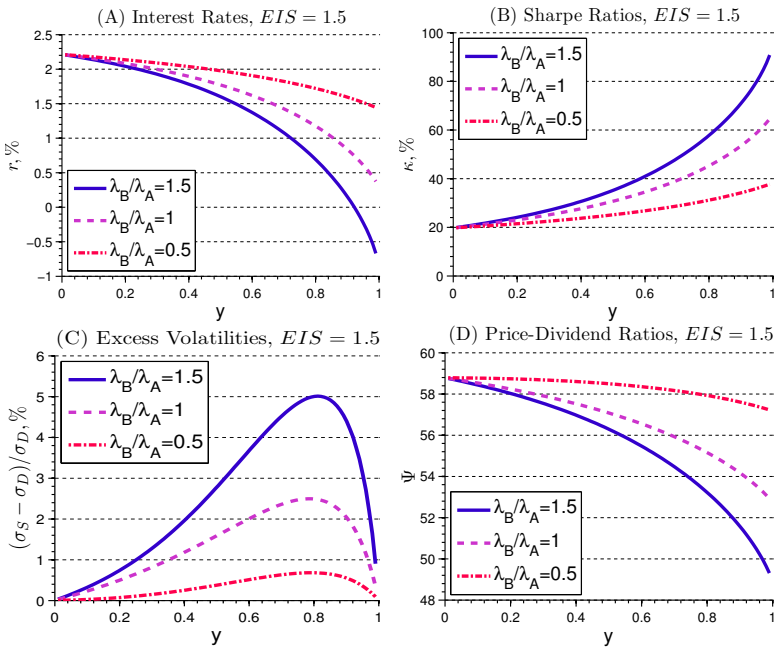

Figure 4
Equilibrium Processes under Heterogeneous Beliefs
This Figure shows equilibrium processes for different ratios of investors' estimates of the intensity of disasters λ_B/λ_A. Investors have risk aversions $\gamma_A = 3$ and $\gamma_B = 6$, time discount parameter $\rho = 2\%$, and EIS $\psi_A = \psi_B = 1.5$, the parameters for process D_t are given in Table 1, and $\lambda_A = 1.7\%$.

three cases, when investor B is pessimistic (i.e., $\lambda_B = 1.5\lambda$), has correct beliefs (i.e., $\lambda_B = \lambda$), and is optimistic (i.e., $\lambda_B = 0.5\lambda$). We provide the analysis only for the case where both investors have the same EIS $\psi_A = \psi_B = 1.5$, because only $\psi_i > 1$ generates realistic dynamic properties of asset prices, as demonstrated in Section 3.

We find that making investor B 50% more pessimistic or optimistic than investor A has significant effect on equilibrium. In particular, making investor B pessimistic, that is, increasing λ_B, decreases interest rates, increases Sharpe ratios and volatilities, and decreases wealth-consumption-ratios. Making investor B pessimistic has the opposite effects. However, the dynamic properties of equilibrium are not affected by the ratio of intensities λ_B/λ_A. In particular, interest rates and price-dividend ratios are procyclical, whereas Sharpe ratios are countercyclical, consistent with the empirical evidence, which is discussed in Section 3. The intuition for these results can be analyzed similarly to Section 3.

5. Conclusion

In this paper we study the asset pricing implications of rare disasters and investor heterogeneity in pure exchange Lucas (1978) economy. We demonstrate that EIS has significant impact on asset prices. Based on our analysis we conclude that the model with EIS > 1 provides the best match with the data. This model generates low procyclical interest rates, large countercyclical Sharpe ratios, procyclical price-dividend ratios, and excess volatility. We show that the anticipation of rare events decreases price dividend ratios and increases stock price volatilities when EIS > 1 and has the opposite effects when EIS < 1. We also find that in equilibrium the more risky investor provides the insurance to the less risky risky investor, because the latter holds a very large fraction of wealth in stocks.

We develop new methodology which provides tractable approach for finding optimal consumptions, portfolio strategies, and other equilibrium processes. The tractability of the solution method allows us to obtain closed-form expressions for the equilibrium processes in the setting where both investors have identical risk aversions but different EIS. In particular, we demonstrate analytically that the heterogeneity in EIS does not affect asset risk premia when investors have identical risk aversions. We note that our methodology can be extended to solve models with heterogeneous beliefs, two Lucas trees, and with portfolio constraints.

Appendix A

Lemma A.1 (Convergence of Multinomial Processes). *In the continuous time limit $\Delta t \to 0$ the distribution of consumption variable D_t which follows process* (1) *converges to the distribution of a variable following a continuous time Lévy process, given by*

$$dD_t = D_t[m_D dt + \sigma_D dw_t + J_D(\omega)\Delta j_t], \tag{A1}$$

where w_t is a Brownian motion and j_t is a Poisson process with intensity λ.

Proof of Lemma A.1. Consider a characteristic function $\varphi_{\Delta t}(p) = \mathbb{E}[e^{ip\ln(D_t/D_0)}]$ of random variable $\ln(D_t)$, where D_t follows process (1). From the fact that ΔD_t are i.i.d. we obtain

$$\begin{aligned}
\varphi_{\Delta t}(p) &= \Big(\mathbb{E}\Big[(1 + m_D\Delta t + \sigma_D\Delta w_t + J_D(\omega)\Delta j_t)^{ip}\Big]\Big)^{\frac{t}{\Delta t}} \\
&= \Big((1-\lambda\Delta t)\mathbb{E}\Big[(1 + m_D\Delta t + \sigma_D\Delta w_t)^{ip}|\text{normal}\Big] \\
&\qquad + \lambda\Delta t\,\mathbb{E}\Big[(1 + J_D(\omega))^{ip}|\text{disaster}\Big]\Big)^{\frac{t}{\Delta t}} \\
&= \Big(1 + ip\,m_D\Delta t + \frac{ip(ip-1)}{2}\sigma_D^2\Delta t \\
&\qquad + \lambda\Delta t\,\mathbb{E}_t\Big[(1 + J_D(\omega))^{ip} - 1|\text{disaster}\Big] + o(\Delta t)\Big)^{\frac{t}{\Delta t}}.
\end{aligned} \tag{A2}$$

Taking the limit $\Delta t \to 0$ we obtain that $\varphi_{\Delta t}(p)$ point-wise converges to function $\varphi(p)$, given by

$$\varphi(p) = \exp\Big(ip\,tm_D + \frac{ip(ip-1)}{2}t\sigma_D^2 + \lambda t\,\mathbb{E}_t[(1 + J_D(\omega))^{ip} - 1]\Big). \tag{A3}$$

It can be easily verified that function (A3) is a characteristic function for Lévy process (A1) [e.g., Shreve (2004)]. Therefore, the distribution function for the discrete-time process D_t converges to the distribution of Lévy process (A1) by Lévy's continuity theorem [e.g., Shiryaev (1996)]. ■

Proof of Lemma 1. Suppose, state price density follows process $\Delta\xi_t = \xi_t[a_t\Delta t + b_t^\top(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])]$, where $\Delta\tilde{w}_t$ is given by equation (20). Next, we find coefficients a_t and b_t from the condition that equations (6)–(8) for asset prices are satisfied. The vector of time-$(t+\Delta t)$ risky asset returns can be written as $R_{t+\Delta t} = 1 + \mu_t\Delta t + \Sigma_t(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])$, where $\mu_t = m_t + \Sigma_t\mathbb{E}_t[\Delta\tilde{w}_t]/\Delta t$ is the vector of risky assets expected returns. The equations (6)–(8) for asset prices imply that $\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t B_{t+\Delta t}/B_t] = 1$ and $\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t R_{t+\Delta t}] = 1$. Substituting $B_{t+\Delta t}/B_t$ and $R_{t+\Delta t}$ into the latter equations, we obtain two equations for a_t and b_t:

$$(1 + a_t\Delta t)(1 + r_t\Delta t) = 1, \tag{A4}$$

$$(1 + a_t\Delta t)(1 + \mu_t\Delta t) + \Sigma_t\,\mathrm{var}_t[\Delta\tilde{w}_t]b_t = 1. \tag{A5}$$

Solving the system of equations (A4)–(A5) we obtain process (19) for the state price density. ■

Proof of Proposition 1. Suppose, all processes are functions of a Markovian state variable z_t. The investor solves the following dynamic programming problem:

$$V_i(W_t, z_t, t) = \max_{c_{i,t}, \theta_{i,t}} \Big[(1 - e^{-\rho\Delta t}) c_{i,t}^{1-1/\psi_i} + e^{-\rho\Delta t}\Big(\mathbb{E}_t[V_i(W_{t+\Delta t}, z_{t+\Delta t}, t+\Delta t)^{1-\gamma_i}]\Big)^{\frac{1-1/\psi_i}{1-\gamma_i}}\Big]^{\frac{1}{1-1/\psi_i}}. \tag{A6}$$

For simplicity, we omit subscript i for the rest of the proof. Next, we substitute $W_{t+\Delta t}$ from budget constraint (11) into optimization (A6), and taking derivatives with respect to $c_{i,t}$ and $\theta_{i,t}$ we obtain the following first order conditions:

$$e^{-\rho\Delta t}\left(\mathbb{E}_t\Big[\frac{V_{t+\Delta t}^{1-\gamma}}{V_t^{1-\gamma}}\Big]\right)^{\frac{\gamma-1/\psi}{1-\gamma}} \mathbb{E}_t\Big[\frac{\partial V_{t+\Delta t}}{\partial W_{t+\Delta t}} \frac{V_{t+\Delta t}^{-\gamma}}{V_t^{-\gamma}}(1 + r_t\Delta t)\Delta t\Big] = (1 - e^{-\rho\Delta t})\Big(\frac{V_t}{c_t}\Big)^{1/\psi}, \tag{A7}$$

$$\mathbb{E}_t\Big[\frac{\partial V_{t+\Delta t}}{\partial W_{t+\Delta t}} \frac{V_{t+\Delta t}^{-\gamma}}{V_t^{-\gamma}}\Big((m_t - r_t)\Delta t + \Sigma_t\Delta\tilde{w}_t\Big)\Big] = 0. \tag{A8}$$

To proceed further, we conjecture that $c_t^* = W_t/\Phi(z_t, t)$ and that θ_t^* does not depend on W_t, which can be verified by backward induction starting at terminal date T, where $W_T = c_T^*\Delta t$, and hence $\Phi_i(z_T, T) = \Delta t$. To find $\partial J/\partial W$, we substitute c_t^* and θ_t^* into equation (A6), and differentiating V_t in (A6) with respect to W_t we obtain:

$$\begin{aligned}\frac{\partial V_t}{\partial W_t} = V_t^{1/\psi}\Bigg((1 - e^{-\rho\Delta t})c_t^{-1/\psi}\frac{1}{\Phi_t} + e^{-\rho\Delta t}\Big(\mathbb{E}_t\Big[V_{t+\Delta t}^{1-\gamma}\Big]\Big)^{\frac{\gamma-1/\psi}{1-\gamma}} \mathbb{E}_t\Bigg[\frac{\partial V_{t+\Delta t}}{\partial W_{t+\Delta t}} V_{t+\Delta t}^{-\gamma} \\ \times\Big(1 + r_t\Delta t + (\theta_t^*)^\top(m_t - r_t)\Delta t + (\theta_t^*)^\top\Sigma_t\Delta\tilde{w}_t - \frac{(1 + r_t\Delta t)\Delta t}{\Phi_t}\Big)\Bigg]\Bigg).\end{aligned} \tag{A9}$$

Using the first order conditions (A7)–(A8) to simplify equation (A9) we find that

$$\frac{\partial V_t}{\partial W_t} = \frac{1 - e^{-\rho\Delta t}}{\Delta t}\left(\frac{V_t}{c_t}\right)^{1/\psi}. \tag{A10}$$

Substituting equation (A10) back into equations (A7)–(A8), after some algebra, we obtain:

$$e^{-\rho\Delta t}\Big(\mathbb{E}_t\Big[\Big(\frac{V_{t+\Delta t}}{V_t}\Big)^{1-\gamma}\Big]\Big)^{\frac{\gamma-1/\psi}{1-\gamma}} \mathbb{E}_t\Big[\Big(\frac{V_{t+\Delta t}}{V_t}\Big)^{1/\psi-\gamma}\Big(\frac{c_{t+\Delta t}^*}{c_t^*}\Big)^{-1/\psi}\Big(1 + r_t\Delta t\Big)\Big] = 1, \tag{A11}$$

$$e^{-\rho\Delta t}\Big(\mathbb{E}_t\Big[\Big(\frac{V_{t+\Delta t}}{V_t}\Big)^{1-\gamma}\Big]\Big)^{\frac{\gamma-1/\psi}{1-\gamma}} \mathbb{E}_t\Big[\Big(\frac{V_{t+\Delta t}}{V_t}\Big)^{1/\psi-\gamma}\Big(\frac{c_{t+\Delta t}^*}{c_t^*}\Big)^{-1/\psi}\Big(1 + \mu_t\Delta t + \Sigma_t\Delta\tilde{w}_t\Big)\Big] = 1. \tag{A12}$$

Substituting $1 + R_{t+\Delta t} = 1 + \mu_t\Delta t + \Sigma_t\Delta\tilde{w}_t$ and $B_{t+\Delta t}/B_t = 1 + r_t\Delta t$ into equations (A11)–(A12), where $R_{t+\Delta t}$ is the vector of risky asset returns, and comparing the resulting equations (6)–(8) with the representations for asset prices in terms of state price density ξ_t, we obtain that

$$\frac{\xi_{t+\Delta t}}{\xi_t} = e^{-\rho\Delta t}\Big(\mathbb{E}_t\Big[V_{t+\Delta t}^{1-\gamma}\Big]\Big)^{\frac{\gamma-1/\psi}{1-\gamma}} V_{t+\Delta t}^{1/\psi-\gamma}\Big(\frac{c_{t+\Delta t}^*}{c_t^*}\Big)^{-1/\psi}. \tag{A13}$$

Next, we prove the equation (24) for the value function V_t. Multiplying both sides of equation (A13) by $(V_{t+\Delta t})^{1-1/\psi}(c_{t+\Delta t}^*/c_t^*)^{1/\psi}$ and taking expectation $\mathbb{E}_t[\cdot]$ on both sides we obtain that

$$e^{-\rho\Delta t}\Big(\mathbb{E}_t\Big[V_{t+\Delta t}^{1-\gamma}\Big]\Big)^{\frac{1-1/\psi}{1-\gamma}} = \mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\Big(\frac{c_{t+\Delta t}^*}{c_t^*}\Big)^{1/\psi} V_{t+\Delta t}^{1-1/\psi}\Big]. \tag{A14}$$

Rewriting equation (A6) for V_t in terms of $(V_t/c_t)^{1-1/\psi}$ and using equation (A14) we find that $(V_t/c_t)^{1-1/\psi}$ solves the equation

$$\begin{aligned}\Big(\frac{V_t}{c_t^*}\Big)^{1-1/\psi} &= 1 - e^{-\rho\Delta t} + \frac{1}{(c_t^*)^{1-1/\psi}}\Big(\mathbb{E}_t\Big[V_{t+\Delta t}^{1-\gamma}\Big]\Big)^{\frac{1-1/\psi}{1-\gamma}} \\ &= 1 - e^{-\rho\Delta t} + \mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\frac{c_{t+\Delta t}^*}{c_t^*}\Big(\frac{V_{t+\Delta t}}{c_{t+\Delta t}^*}\Big)^{1-1/\psi}\Big].\end{aligned} \tag{A15}$$

Furthermore, because the market is complete, wealth W_t is given by the martingale representation $W_t = c_t\Delta t + \mathbb{E}_t[(\xi_{t+\Delta t}/\xi_t)W_{t+\Delta t}]$. Rewriting the latter equation in terms of wealth-consumption ratio $\Phi_t = W_t/c_t$ we obtain a recursive equation for Φ_t:

$$\Phi_t = \Delta t + \mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\frac{c_{t+\Delta t}^*}{c_t^*}\Phi_{t+\Delta t}\Big]. \tag{A16}$$

Comparing the latter equation with equation (A15) we conclude that $(V_t/c_t^*)^{1-1/\psi} = (1 - e^{\rho\Delta t})\Phi_t/\Delta t$. Substituting consumption $c_t^* = W_t/\Phi_t$, after simple algebra, we obtain expression (24) for the value function. Next, substituting equation (24) for V_t into equation (A13) for state price density ξ_t, after simple algebra, we prove expression (27) for ξ_t in Proposition 1. Optimal consumption growths (25) can be obtained by solving equation (A13), which provides ξ_t in terms of $c_{t+\Delta t}^*/c_t^*$. We omit the details, but note that it can be directly verified by substitution that $c_{t+\Delta t}^*/c_t^*$ in equation (25) satisfies equation (A13). Backward equation (23) for Φ_t can be obtained by substituting $c_{t+\Delta t}^*/c_t^*$ given by equation (25) into equation (A16) for Φ_t.

It remains to prove expression (26) for θ^*. First, we rewrite budget constraint (11) under optimal strategies θ^* and c^* as $\Delta W_t = (\ldots)\Delta t + (\theta_t^*)^\top\Sigma(\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])$. Multiplying both sides by $(\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])^\top$ and then taking expectations, we obtain that $\mathbb{E}_t[(W_{t+\Delta t}/W_t)(\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])^\top] = (\theta_t^*)^\top\Sigma_t\,\mathrm{var}_t[\Delta\tilde{w}_t]$. Next, replacing $W_{t+\Delta t}$ and W_t by $\Phi_{t+\Delta t}c_{t+\Delta t}^*$ and $\Phi_t c_t^*$, respectively, and solving for θ_t^* we obtain expression (26) in Proposition 1.

Finally, we find the marginal rate of substitution $MRS_{t+\Delta t}(\omega_k) = \Big(\partial U_t/\partial c_{t+\Delta t}(\omega_k)\Big)/(\partial U_t/\partial c_t)$:

$$\begin{aligned}MRS_{t+\Delta t}(\omega_k) &= \frac{\partial U_t}{\partial U_{t+\Delta t}}\frac{\partial U_{t+\Delta t}/\partial c_{t+\Delta t}}{U_t/\partial c_t} \\ &= e^{-\rho\Delta t}\Big(\mathbb{E}_t\Big[U_{t+\Delta t}^{1-\gamma}\Big]\Big)^{\frac{\gamma-1/\psi}{1-\gamma}}U_{t+\Delta t}^{1/\psi-\gamma}\Big(\frac{c_{t+\Delta t}}{c_t}\Big)^{-1/\psi}\mathrm{Prob}_t(\omega_k).\end{aligned} \tag{A17}$$

Under optimal strategies θ_t^* and c_t^*, we obtain that $U_t = V_t$, and hence, from equation (A13) we obtain that $MRS_{t+\Delta t}(\omega_k) = (1-\lambda)\pi_k\xi_{t+\Delta t}(\omega_k)/\xi_t$. ■

Proof of Proposition 2. Taking expectation $\mathbb{E}_t[\cdot]$ on both sides of equation (19) for ξ_t we find that $\mathbb{E}_t[\xi_{t+\Delta t}] = 1/(1 + r_t\Delta t)$. Solving the latter equation we obtain r_t in equation (30). Next, multiplying both sides of equation (19) by $(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])^\top$ and taking expectations, we obtain that $\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])^\top] = -(\Sigma_t^{-1}(\mu_t - r_t\underline{1}))^\top/(1 + r_t\Delta t)/\Delta t$. Solving for $(\mu_t - r_t\underline{1})$, we obtain equation (31) for the risk premia.

To obtain Σ_t, from the dynamics of asset prices (4)–(5), we observe that asset returns $R_{t+\Delta t}$, defined by equation (33), are given by $R_{t+\Delta t} = \mu_t \Delta t + \Sigma_t(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])$. Multiplying both sides by $(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])^\top$ and taking expectations, we obtain $\mathbb{E}_t[R_{t+\Delta t}(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])^\top] = \Sigma_t \operatorname{var}_t[\Delta\tilde{w}_t]$. Solving the latter equation, we obtain equation (32) for Σ. Next, we derive backward equation (34) for the price-dividend ratio by substituting $S_t = \Psi_t D_t$ into equation (7). Finally, we note that if equation (29) for consumption share $y_{t+\Delta t}$ has solution $y_{t+\Delta t}(y_t;\omega)$, then $\xi_{t+\Delta t}/\xi_t$ is also is a function of y_t and ω. Consequently, from equations (30)–(34) we obtain that all the equilibrium processes are functions y_t, and returns $R_{t+\Delta t}$ are also functions of state ω. The drift and volatility of consumption share y_t are found analogously. ■

Proof of Corollary 1. 1) From equation (29) for consumption share $y_{t+\Delta t}$ we note that when the risk aversions are the same, $\gamma_i = \gamma$, terms involving $D_{t+\Delta t}/D_t$ cancel out from both sides of the equation. As a result, $y_{t+\Delta t}$, and hence all the equilibrium processes, are deterministic functions of time. Factoring out terms with $y_{t+\Delta t}$, $\Phi_{i,t}$ from the expectation operators in equation (29) and canceling terms, we obtain that $y_{t+\Delta t}$ satisfies equation

$$\Big(\frac{y_{t+\Delta t}}{y_t}\Big)^{1/\psi_B} = \Big(\frac{1-y_{t+\Delta t}}{1-y_t}\Big)^{1/\psi_A}\Big(\mathbb{E}_t\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma}\Big]\Big)^{\frac{1/\psi_A-1/\psi_B}{1-\gamma}}. \tag{A18}$$

Next, we write equation (A18) for all $t = 0, \Delta t, \ldots, T$, and multiplying right and left sides of these equations, respectively, we obtain equation (39) for y_t. Using similar algebra, from equation (29) we find that ξ_t is given by:

$$\frac{\xi_{t+\Delta t}}{\xi_t} = e^{-\rho\Delta t}\Big(\frac{y_{t+\Delta t}}{y_t}\Big)^{-1/\psi_B}\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{-\gamma}\Big(\mathbb{E}_t\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma}\Big]\Big)^{\frac{\gamma\psi_B-1}{(1-\gamma)\psi_B}}. \tag{A19}$$

Now, we pass to the limit when $\Delta t \to 0$. First, we substitute $D_{t+\Delta t}/D_t$ from the aggregate consumption process (1) into $\mathbb{E}_t[(D_{t+\Delta t}/D_t)^\alpha]$, and obtain the following expansion:

$$\begin{aligned}
\mathbb{E}_t\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^\alpha\Big] &= \mathbb{E}_t\Big[\Big(1 + m_D\Delta t + \sigma_D\Delta w_t + J_D(\omega)\Delta j_t\Big)^\alpha\Big] \\
&= \frac{1-\lambda\Delta t}{2}\Big[(1 + m_D\Delta t + \sigma_D\sqrt{\Delta t})^\alpha + (1 + m_D\Delta t - \sigma_D\sqrt{\Delta t})^\alpha\Big] \\
&\quad + \lambda\Delta t\,\mathbb{E}\Big[\Big(1 + m_D\Delta t + J_D(\omega)\Big)^\alpha|\text{disaster}\Big] \\
&= 1 + \Big(\alpha m_D + \frac{\alpha(\alpha-1)}{2}\sigma_D^2 + \lambda\Big\{\mathbb{E}_t\Big[(1+J_D(\omega))^\alpha\Big] - 1\Big\}\Big)\Delta t + o(\Delta t).
\end{aligned} \tag{A20}$$

Next, substituting expansions $(y_{t+\Delta t}/y_t)^{1/\psi_B} = 1 + (1/\psi_B)(\Delta y_t/y_t)\Delta t + o(\Delta t)$ and $((1-y_{t+\Delta t})/(1-y_t))^{1/\psi_A} = 1 - (1/\psi_A)(\Delta y_t/(1-y_t))\Delta t + o(\Delta t)$ into equation (A18), we obtain a linear equation for Δy_t. Using expansion (A20), after some algebra, we obtain expansion:

$$\Delta y_t = \frac{(\psi_B - \psi_A)y_t(1-y_t)}{\psi_B y_t + \psi_A(1-y_t)}\Big(m_D - \frac{\gamma}{2}\sigma_D^2 + \frac{\lambda}{1-\gamma}\Big\{\mathbb{E}_t\Big[\Big(1+J_D(\omega)\Big)^{1-\gamma}|\text{disaster}\Big] - 1\Big\}\Big)\Delta t + o(\Delta t). \tag{A21}$$

Using expansions (A20) and (A21), we obtain expansion for $\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t]$, where $\xi_{t+\Delta t}/\xi_t$ is given by equation (A19). Then, we derive an expansion for interest r_t, given by (30), and passing to the limit $\Delta t \to 0$, after some algebra, we obtain closed-form solution (35). The expression for the market price of risk (36) is obtained similarly, using the same expansions, and equation (31) for the risk premia in Proposition 1.

Finally, we derive the stock risk premium. Writing down the dynamics for stock prices (7) in states ω_{n-1} and ω_n, and using the fact that price-dividend ratio $\Psi_{t+\Delta t}$ is deterministic, after some algebra we obtain expressions for the drift of the stock price:

$$1 + m_{S,t}\Delta t = \frac{\Psi_{t+\Delta t} + \Delta t}{2\Psi_t}\Big(\frac{D_{t+\Delta t}(\omega_{n-1})}{D_t} + \frac{D_{t+\Delta t}(\omega_n)}{D_t}\Big), \tag{A22}$$

where $D_{t+\Delta t}(\omega_{n-1})$ and $D_{t+\Delta t}(\omega_n)$ denote time-t+Δt dividend in states ω_{n-1} and ω_n, respectively. Moreover, from equation (30) $1+r_t\Delta t = 1/\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t]$ and from equation (34) $(\Psi_{t+\Delta t}+\Delta t)/\Psi_t = 1/\mathbb{E}_t[(\xi_{t+\Delta t}/\xi_t)(D_{t+\Delta t}/D_t)]$. Using all the above equations, we obtain

$$m_{S,t} - r = \frac{1}{2}\Big(\frac{D_{t+\Delta t}(\omega_{n-1})}{D_t} + \frac{D_{t+\Delta t}(\omega_n)}{D_t}\Big)/\mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\frac{D_{t+\Delta t}}{D_t}\Big] - 1/\mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\Big]. \tag{A23}$$

Risk premium is then found as $\mu_{S,t} - r_t = m_{S,t} - r_t + \Sigma_{S,t}\mathbb{E}_t[\tilde{w}_t]/\Delta t$. We also note that because Ψ_t is deterministic, the volatility $\sigma_{S,t}$ and jump sizes $J_{S,t}(\omega)$ of stock prices are the same as those of dividend process (1). Therefore, $\Sigma_{S,t} = (\sigma_D, J_D(\omega_1), \ldots, J_D(\omega_{n-2}))^\top$. Substituting $\xi_{t+\Delta t}/\xi_t$ from equation (A19) into equation (A23) and noting from the dividend dynamics (1) that $D_{t+\Delta t}(\omega_{n-1})/D_t + D_{t+\Delta t}(\omega_n)/D_t = 2 + 2m_D\Delta t$, using expansions (A20) and (A21), after some algebra we obtain risk premium (37).

2) Now, we consider the case of homogeneous investors, that is, $\psi_A = \psi_B = \psi$, $\gamma_A = \gamma_B = \gamma$. From the equation (34) for price-dividend ratio Ψ_t and the fact that it is deterministic, we obtain

$$\begin{aligned}\Psi_t &= (\Psi_{t+\Delta t} + \Delta t)\mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\frac{D_{t+\Delta t}}{D_t}\Big] \\ &= (\Psi_{t+\Delta t} + \Delta t)e^{-\rho\Delta t}\Big(\mathbb{E}_t\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma}\Big]\Big)^{\frac{1-1/\psi}{1-\gamma}},\end{aligned} \tag{A24}$$

where the second equality is obtained by substituting ξ_t from equation (A19) into equation (A24) and noting that $y_{t+\Delta t} = y_t$ in homogeneous investor economy. Solving backward equation (A24) we obtain that $\Psi_t = \Big(1 - g_{i,1}^{(T-t+\Delta t)/\Delta t}\Big)/(1 - g_{i,1})g_{i,1}\Delta t$ where $g_{i,1}$ is given by equation (17). As $T \to \infty$, the solution converges to a stationary one iff $g_{i,1} < 1$.

Next, we obtain another representation for Ψ_t in terms of rate r and risk premium $\mu_S - r$. Using the expression for $\xi_{t+\Delta t}$ from equation (19) we obtain:

$$\begin{aligned}\mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\frac{D_{t+\Delta t}}{D_t}\Big] &= \frac{1}{1+r\Delta t}\mathbb{E}_t\Big[\Big(1 - (\Sigma^{-1}(\mu - r))^\top(\mathrm{var}_t[\Delta\tilde{w}_t]/\Delta t)^{-1}(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])\Big) \\ &\qquad \times\Big(1 + m_D\Delta t + \Sigma_D^\top\mathbb{E}_t[\Delta\tilde{w}_t] + \Sigma_D^\top(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])\Big)\Big] \\ &= \frac{1 + \Big(m_D + \Sigma_D^\top\mathbb{E}_t[\Delta\tilde{w}_t]/\Delta t - (\Sigma^{-1}(\mu - r))^\top)\Sigma_D\Big)\Delta t}{1 + r\Delta t},\end{aligned} \tag{A25}$$

where $\Sigma_D = (\sigma_D, J_D(\omega_1), \ldots, J_D(\omega_{n-2}))^\top$. Using formula (36) for $\Sigma^{-1}(\mu - r)$, after some algebra, as $\Delta t \to 0$, we obtain $(\Sigma^{-1}(\mu - r))^\top \Sigma_D = \gamma\sigma_D^2 - \lambda\mathbb{E}_t[(1 + J_D(\omega))^{-\gamma} J_D(\omega)|\text{disaster}] + \lambda\mathbb{E}_t[J_D(\omega)|\text{disaster}]$. Furthermore, it can be shown by straightforward algebra that $\Sigma_D^\top \mathbb{E}_t[\Delta\tilde{w}_t] = \lambda\Delta t\mathbb{E}_t[J_D(\omega)|\text{disaster}]$. Substituting the latter expressions back into equation (A25) we obtain

$$\mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\frac{D_{t+\Delta t}}{D_t}\Big] = 1 - (r + (\mu_S - r) - m_D)\Delta t + o(\Delta t). \tag{A26}$$

Substituting (A26) back into equation (A24) we obtain that in continuous time $\Psi'(t) - (r + (\mu_S - r) - m_D)\Psi(t) + 1 = 0$, subject to $\Psi(T) = 0$. Solving the ODE we obtain Ψ_t in equation (40).

Similarly, given that prices $P_{k,t}$ are deterministic, from equation (8) we obtain:

$$\begin{aligned}
P_{k,t} &= P_{k,t+\Delta t}\mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\Big] + \lambda\pi_k\Delta t\frac{\xi_{t+\Delta t}(\omega_k)}{\xi_t} \\
&= P_{k,t+\Delta t}\frac{1}{1 + r\Delta t} + \lambda\pi_k\Delta t e^{-\rho\Delta t}\Big(\frac{D_{t+\Delta t}(\omega_k)}{D_t}\Big)^{-\gamma}\mathbb{E}_t\Big(\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma}\Big]\Big)^{\frac{\gamma\psi-1}{(1-\gamma)\psi}} \\
&= e^{-\rho\Delta t}P_{k,t+\Delta t}\mathbb{E}_t\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{-\gamma}\Big]\mathbb{E}_t\Big(\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma}\Big]\Big)^{\frac{\gamma\psi-1}{(1-\gamma)\psi}} \\
&\quad + \lambda\pi_k\Delta t e^{-\rho\Delta t}\Big(\frac{D_{t+\Delta t}(\omega_k)}{D_t}\Big)^{-\gamma}\mathbb{E}_t\Big(\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma}\Big]\Big)^{\frac{\gamma\psi-1}{(1-\gamma)\psi}}.
\end{aligned} \tag{A27}$$

Iterating backward it can be demonstrated that $P_{k,t} = \Big(1 - g_{i,2}^{(T-t)/\Delta t}\Big)/(1 - g_{i,2})b_k\lambda\Delta t$, where $g_{i,2}$ is given by equation (18), and b_k is given by:

$$b_k = \lambda\pi_k e^{-\rho\Delta t}\Big(\frac{D_{t+\Delta t}(\omega_k)}{D_t}\Big)^{-\gamma}\mathbb{E}_t\Big(\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma}\Big]\Big)^{\frac{\gamma\psi-1}{(1-\gamma)\psi}}. \tag{A28}$$

Passing to continuous time limit in the second equality in equation (A27), similarly to price-dividend ratios Ψ_t, we obtain the insurance prices in equation (40). ■

Proof of Proposition 3. Because the investors agree on observed asset prices, using equations (6)–(8) for asset prices in terms of the state price density, we obtain:

$$\begin{aligned}
B_t &= \mathbb{E}_t\Big[\frac{\xi_{A,t+\Delta t}}{\xi_{A,t}}B_{t+\Delta t}\Big] \qquad\qquad (A29)\\
&= \mathbb{E}_t^B\Big[\frac{\xi_{B,t+\Delta t}}{\xi_B, t}B_{t+\Delta t}\Big] \\
S_t &= \mathbb{E}_t\Big[\frac{\xi_{A,t+\Delta t}}{\xi_{A,t}}(S_{t+\Delta t} + D_{t+\Delta t}\Delta t)\Big] \qquad\qquad (A30)\\
&= \mathbb{E}_t^B\Big[\frac{\xi_{B,t+\Delta t}}{\xi_B, t}(S_{t+\Delta t} + D_{t+\Delta t}\Delta t)\Big], \\
P_{k,t} &= \mathbb{E}_t\Big[\frac{\xi_{A,t+\Delta t}}{\xi_{A,t}}(P_{k,t+\Delta t} + D_{k,t+\Delta t})\Big] \qquad\qquad (A31)\\
&= \mathbb{E}_t^B\Big[\frac{\xi_{B,t+\Delta t}}{\xi_{B,t}}(P_{k,t+\Delta t} + D_{k,t+\Delta t})\Big].
\end{aligned}$$

The expectations under investor B's subjective probability measure in equations (A32)–(A32) can be rewritten in terms of the expectations under the correct measure of investor A and Radon-Nikodym derivative $\eta_{t+\Delta t}(\omega)$ to obtain:

$$\mathbb{E}_t\Big[\frac{\xi_{A,t+\Delta t}}{\xi_{A,t}}B_{t+\Delta t}\Big] = \mathbb{E}_t\Big[\eta_{t+\Delta t}\frac{\xi_{B,t+\Delta t}}{\xi_B, t}B_{t+\Delta t}\Big],$$

$$\mathbb{E}_t\Big[\frac{\xi_{A,t+\Delta t}}{\xi_{A,t}}(S_{t+\Delta t} + D_{t+\Delta t}\Delta t)\Big] = \mathbb{E}_t\Big[\eta_{t+\Delta t}\frac{\xi_{B,t+\Delta t}}{\xi_B, t}(S_{t+\Delta t} + D_{t+\Delta t}\Delta t)\Big],$$

$$\mathbb{E}_t\Big[\frac{\xi_{A,t+\Delta t}}{\xi_{A,t}}(P_{k,t+\Delta t} + D_{k,t+\Delta t})\Big] = \mathbb{E}_t\Big[\eta_{t+\Delta t}\frac{\xi_{B,t+\Delta t}}{\xi_{B,t}}(P_{k,t+\Delta t} + D_{k,t+\Delta t})\Big].$$

From the latter equations and from the uniqueness of the state price density under the correct expectations, demonstrated in Lemma 1, we obtain that $\xi_{A,t+\Delta t}/\xi_{A,t} = \eta_{t+\Delta t}\xi_{B,t+\Delta t}/\xi_{B,t}$. Next, using the latter equality and equation (27) for the state price density in terms of investors consumptions, similarly to equation (29) we obtain a system of equations (43) for consumption shares $y_{t+\Delta t}(y_t; w_k)$. Because the time is discrete, the Radon-Nikodym derivative is simply given by the ratio of subjective investor B's and real probabilities of states $\omega_1, \ldots \omega_n$. Therefore, the Radon-Nikodym derivative does not depend on time, and hence can be written as $\eta(\omega)$. ■

References

Ait-Sahalia, Y., J. Cacho-Diaz, and T. R. Hurd, 2009, "Portfolio Choice with Jumps: A Closed-Form Solution," *Annals of Applied Probability*, 19, 556-84.

Backus, D., M. Chernov, and I. Martin, 2011, "Disasters Implied by Equity Index Options," *Journal of Finance*, 66, 1969-2012.

Backus, D., M. Chernov, and S. Zin, 2013, "Sources of Entropy in Representative Agent Models," *Journal of Finance*, forthcoming.

Bansal, R., and A. Yaron, 2004, "Risk for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 59, 1481-509.

Barro, R., 2006, "Rare Disasters and Asset Markets in the Twentieth Century," *Quarterly Journal of Economics*, 121, 243-64.

Barro, R., 2009, "Rare Disasters, Asset Prices, and Welfare Costs," *American Economic Review*, 99, 823-66.

Basak, S., 2000, "A Model of Dynamic Equilibrium Asset Pricing with Heterogeneous Beliefs and Extraneous Risk," *Journal of Economic Dynamics and Control*, 24, 63-95.

Basak, S., 2005, "Asset Pricing with Heterogeneous Beliefs," *Journal of Banking and Finance*, 29, 2849-81.

Bhamra, H. S., and R. Uppal, 2013, "Asset Prices with Heterogeneity in Preferences and Beliefs," *Review of Financial Studies*, forthcoming.

Borovička, J., 2012, "Survival and Long-run Dynamics with Heterogeneous Beliefs under Recursive Preferences," Working Paper, New York University.

Buss, A., R. Uppal, G. Vilkov, 2011, "Asset Prices in General Equilibrium with Transactions Costs and Recursive Utility," Working Paper, INSEAD.

Campbell, J. Y., 2003, "Consumption-Based Asset Pricing," *Handbook of Economics and Finance*, G. Constantinides, M. Harris, and R. Stulz (eds.), Amsterdam: North Holland.

Campbell, J. Y., and J. H. Cochrane, 1999, "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy*, 107, 205-51.

Campbell, J. Y., and R. J. Shiller, 1988, "The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors," *Review of Financial Studies*, 3, 195-227.

Chabakauri, G., 2013a, "Asset Pricing with Heterogeneous Preferences, Beliefs, and Portfolio Constraints," Working Paper, London School of Economics.

Chabakauri, G., 2013b, "Dynamic Equilibrium with Two Stocks, Heterogeneous Investors, and Portfolio Constraints," *Review of Financial Studies*, 12, 3104-41.

Chan, Y. L., and L. Kogan, 2002, "Catching Up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices," *Journal of Political Economy*, 110, 1255-85.

Chen, H., S. Joslin, and N.-K. Tran, 2012, "Rare Disasters and Risk Sharing with Heterogeneous Beliefs," *Review of Financial Studies*, 25, 2189-224.

Cvitanić, J., E. Jouini, S. Malamud, and C. Napp, 2012, "Financial Markets Equilibrium with Heterogeneous Agents," *Review of Finance*, 16, 285-31.

Cvitanić, J., and S. Malamud, 2011, "Price Impact and Portfolio Impact," *Journal of Financial Economics*, 100, 201-25.

Detemple, J. B., and S. Murthy, 1994, "Intertemporal Asset Pricing with Heterogeneous Beliefs," *Journal of Economic Theory* 62, 294-320.

Dieckmann, S., 2011, "Rare Even Risk and Heterogeneous Beliefs: The Case of Incomplete Markets," *Journal of Economic Dynamics and Control*, 46, 459-88.

Dieckmann, S., and M. Gallmeyer, 2005, "The Equilibrium Allocation of Diffusive and Jump Risks with Heterogeneous Agents," *Journal of Economic Dynamics and Control*, 29, 1547-76.

Duffie, D., 2001, *Dynamic Asset Pricing Theory*, Princeton, NJ: Princeton University Press.

Dumas, B., and A. Lyasoff, 2012, "Incomplete-market equilibria solved recursively on an event tree," *Journal of Finance*, 67, 1897-941.

Epstein, L., and S. Zin, 1989, "Substitution, Risk Aversion, and the Temporal Behavior of COnsumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937-69.

Ferson, W. E, and C. R. Harvey, 1991, "The Variation of Economic Risk Premiums," *Journal of Political Economy*, 99, 385-415.

Gabaix, X., 2012, "Variable Rare Disasters: An Exactly Solved Framework For Ten Puzzles in Macro-Finance," *Quarterly Journal of Economics*, 127, 645-700.

Gallmeyer, M., and B. Hollifield, 2008, "An Examination of Heterogeneous Beliefs with a Short-Sale Constraint in a Dynamic Equilibrium," *Review of Finance*, 12, 323-64.

Gârleanu, N., and S. Panageas, 2010, "Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing," Working Paper, University of California, Berkeley.

Gourio, F., 2012, "Disaster Risk and Business Cycle," *American Economic Review*, 102, 2734-66.

Gollier, C., and R. Zheckhauser, 2005, "Aggregation of Heterogeneous Time Preferences," *Journal of Political Economy*, 113, 878-96.

He, H., 1991, "Optimal Consumption-Portfolio Policies: A Convergence from Discrete to Continuous Time Models," *Journal of Economic Theory*, 55, 340-63.

Isaenko, S., 2008, "The Term Structure of Interest Rates in a Pure Exchange Economy Where Investors Have Heterogeneous Recursive Preferences," *Quarterly Review of Economic and Finance*, 48, 457-81.

Judd, K. L., 1998, *Numerical Methods in Economics*, Cambridge: MIT Press.

Julliard, C., and A. Ghosh, 2012, "Can Rare Events Explain the equity Premium Puzzle?" *Review of Financial Studies*, 25, 3037-76.

Kimball, M., and P. Weil, 2009, "Precautionary Saving and Consumption Smoothing across Time and Possibilities," *Journal of Money, Credit and Banking*, 41, 245-84.

Liu, J., 2007, "Portfolio Selection in Stochastic Environments," *Review of Financial Studies*, 20, 1-39.

Liu, J., Longstaff, F., and J. Pan, 2003, "Dynamic Asset Allocation with Event Risk," *Journal of Finance*, 58, 231-59.

Longstaff, F. A., and J. Wang, 2012, "Asset Pricing and Credit Markets," *Review of Financial Studies*, 25, 3169-214.

Lucas, R. E., 1978, "Asset Prices in an Exchange Economy", *Econometrica*, 46, 1429-45.

Ma, C., 1993, "Market Equilibrium with Heterogeneous Recursive-Utility-Maximizing Agents," *Economic Theory*, 3, 243-66.

Martin, I., 2013a, "Consumption-Based Asset Pricing with Higher Cumulants," *Review of Economic Studies*, 80, 745-73.

Martin, I., 2013b, "The Lucas Orchard," *Econometrica*, 81, 55-111.

Mehra, R., and E. C. Prescott, 1985, "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15, 145-61.

Merton, R. C., 1973, "An Intertemporal Capital Asset Pricing Model," *Econometrica*, 41, 867-87.

Pliska, S., 1997, *Introduction to Mathematical Finance: Discrete Time Models*, Blackwell Publishing.

Rietz, T. A., 1988, "The Equity Premium: A Solution," *Journal of Monetary Economics*, 22, 117-31.

Shiller, R. J., 1981, "Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends?" *American Economic Review*, 71, 421-36.

Shiryaev, A. N., 1996, *Probability*, New York: Springer.

Skiadas, C., "Smooth Ambiguity Aversion Toward Small Risk and Continuous-Time Recursive Utility," *Journal of Political Economy*, forthcoming.

Wachter, J., 2013, "Can Time-Varying Risk of Rare Disasters Explain Aggregate Stock Market Volatility?" *Journal of Finance*, 68, 987-1035.

Xiouros, C., and F. Zapatero, 2010, "The Representative Agent of an Economy with External Habit Formation and Heterogeneous Risk Aversion," *Review of Financial Studies*, 23, 3017-47.

Yan, H., 2008, "Natural Selection in Financial Markets: Does it Work?" *Management Science*, 54, 1935-50.