

Slowing Learning Down^{*}

(very preliminary version)

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February 27, 2014

Abstract

We investigate the dynamic signaling incentives of an entrepreneur who is willing to sell her firm. The underlying value of the firm is known only to the seller, and potential buyers learn through noisy signals like sales or dividends. Before the firm is sold the entrepreneur decides between efficient management or inefficient but revenue-generating activities, such as secret price cuts, covered loans, etc. We provide a full characterization of the equilibrium set of the model, which is set in discrete time and with infinite horizon. We find that, in all equilibria, when the underlying value is high, the entrepreneur efficiently manages her firm. When, instead, the underlying value is low, the entrepreneur keeps switching between efficient managing and inefficient signaling in order to slow down the learning about the value, and therefore increasing the probability of receiving a high offer. By mapping our model into a reputations model, we show that reputation may be a permanent phenomenon even under imperfect monitoring, and it can be sustained without building-milking reputation phases.

Keywords: Dynamic Signaling, Dynamic Moral Hazard, Endogenous Effort.

JEL Classification: D82, D83, C73, J24.

^{*}Work in progress. I thank George Mailath, Andrew Postlewaite and Qingmin Liu for their comments and suggestions, as well as the participants in the UPenn Micro Lunch, Columbia's Theory Colloquium and BGSE Microworkshop.

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1 Introduction

Entrepreneurs have private information about their firms. Not only do they have inside information about their firms' underlying value, but also most of their management decisions cannot be observed by outsiders. When they try to sell their firms, potential buyers make inferences about such private information using noisy signals (such as sales, dividends, etc.). Hence, entrepreneurs face a trade-off between efficient management to ensure a high flow payoff and signal-enhancing inefficient activities (such as secret price cuts, suboptimal advertising, etc.) to generate high offers. This paper studies how these incentives interact over time and how they affect trade dynamics.

In our model, an entrepreneur (seller) wants to sell her firm (asset), which can have a low or a high underlying value (quality). Only the seller observes the quality of her asset, and she can exert costly management effort that generates observable noisy returns. We assume that the optimal management effort is different for different qualities of the asset. Short-lived buyers arrive over time, observe the history of returns and make an offer to the seller. If she accepts the offer, the asset is sold, and the game ends. Otherwise, the entrepreneur continues managing her firm until the arrival of the next buyer.

We show that Diamond's paradox holds in a dynamic (noisy) signaling model. Namely, buyers extract all surplus from the high-type seller in all equilibria. As a consequence, in equilibrium, buyers never offer more than the reservation (autarchy) value of the high-quality seller, and there is pooling on the decision to accept offers across the types of seller. Also, the high-quality seller has strict incentives to manage her firm optimally, even in the presence of signaling motives. Hence, most of the action in our model is driven by the incentives of the low-quality seller to slow the learning about her asset type down.

At every period, the low-quality seller balances choosing the optimal (low) managerial effort versus masquerading her type by exerting a suboptimal (high) effort. These two incentives exactly balance at most of the histories. The reason is that when the low-quality entrepreneur is supposed to efficiently manage her firm, the signal becomes very informative. Indeed, if the seller does not undertake cost-inefficient (but revenue-generating) activities, low returns strongly indicate that the asset is of low quality, while high returns convince future buyers that the quality of the asset is high. Therefore, the low-quality seller can greatly increase the posterior of future buyers about the quality being high at a relatively low cost, so she has incentives to masquerade herself. The reverse is also true when she is supposed to put effort into signaling. We find that the effort put into masquerading is small when buyers' beliefs about the quality are close to being degenerated, since then learning is slow.

One of the main contributions of this paper is a full characterization of the set of Nash equilibria in a fully dynamic noisy signaling model.¹ To do this, we develop new techniques

¹Different from most of the previous literature (see the literature review below), we do not impose any restriction on players' behavior (for example, measurability conditions on strategies, etc.) and we do not restrict our focus to

that allow us to construct the “equilibrium payoffs correspondence,” which provides the equilibrium continuation payoffs for each prior about the quality of the firm being high. To do this, we separate the continuation payoffs dynamics from the beliefs dynamics. This is crucial for the solvability of our model since, despite the discreteness of types, effort, signals and time, endogenous learning implies that the space of posteriors cannot be discretized, since mixing probabilities cannot be restricted. We also show that the equilibrium public outcome (distribution) is generically unique.

We provide insights on why dynamic learning models are, in general, difficult to solve in discrete time. We show that when the action undertaken for one of the agents (in our case the buyers) is discontinuous as a function of some relevant variable (in our case the posterior about the quality of the asset), this generates a discontinuity in the possible continuation payoffs that replicates itself an infinite number of times. This generates a step structure that may have the form of a devil’s staircase, that is, a non-constant continuous function that is flat almost everywhere. As a consequence, we find that if we endogenize the composition of the market by introducing an entry fee, the equilibrium market composition is highly discontinuous with respect to the entry fee.

We show that our model is equivalent to a reputations model where a seller repeatedly sells goods at a given fixed price to short lived-customers, who only observe noisy signals (internet reviews, awards, etc.) about the quality of the product sold in the previous periods. In this case, efficient and inefficient management correspond, respectively, to “reputation milking” and “reputation building.” We show that in equilibrium there are no reputation building-milking cycles. Also, under some parametric assumptions, we find that reputation can be a permanent phenomenon, even when monitoring is imperfect.

In the next section we review the literature related to our paper. In Section 2, we set our base model and the main results of the paper. Section 3 concludes. The Appendix provides the proofs omitted in the previous sections and some omitted developments.

1.1 Literature Review

Recently there has been an interest in dynamic noisy signaling models. These models analyze how the revelation of information prevents the Swinkels (1999) inexistence result from holding.² Among others, Kremer and Skrzypacz (2007), Kim (2012), Kaya and Kim (2013) and Daley and Green (2012) analyze the dynamic effects of introducing exogenous information in a lemons market. Our goal is to endogenize the information released and make the speed of learning consistent with incentives. Also, our model features repeated signaling instead of a repeated screening, since the rejection of offers is (endogenously) not informative, while types

a particular equilibrium subset (Markov strategies, refinements.)

²Swinkels (1999) shows that if there is no signal other than the time spent in school and offers are private and frequent enough, the only equilibrium of the dynamic version of the standard Spence (1973) model is pooling.

separate on the effort choice.

As in Dilmé (2013), we introduce a hidden action that is necessary for the signal to be informative. Nevertheless, signaling is productive in our model, and buyers randomly arrive over time. This implies that, in our model, the action is driven by the incentives of the low-quality sellers to mimic the high-quality sellers, rather than the incentives of the high-quality sellers to separate. This allows us to construct a model that is more tractable, without the need to restrict the focus to a subset of equilibria.

In the previous literature, results have been proven to sharply depend on the existence of public or private offers. For example, Nöldeke and van Damme (1990) show that if offers are public, the dynamic version of the Spence (1973) model has separating equilibria. Hörner and Vieille (2009) show that the assumption of public or private offers has a high effect on the trade dynamics in lemon markets. We abstract from this distinction by assuming that there are no gains from trade from the sale of low-quality assets. In this environment, Diamond’s (1971) paradox makes the different types of seller pool on the decision to accept offers that do not generate losses to the buyer, independently of whether if they are public or private.

Our model is also related to the reputations literature, inaugurated by the seminal works of Kreps and Wilson (1982) and Milgrom and Roberts (1982), since it can be reinterpreted as a reputations model. In particular, we show that the vanishing reputations result in Cripps, Mailath and Samuelson (2004) may not hold when short-lived players’ payoff is a function of the type, not the action. So, we show that reputation may be sustained without building-milking cycles (in the previous literature, among other mechanisms, building-milking cycles were generated through bounded recall (Liu, 2011), through replenishing types (Mailath and Samuelson, 2001, Board and Meyer-ter-Vehn, 2013) and through moral hazard (Dilmé, 2012)).

2 Basic Model

2.1 Setting

Time is discrete, $t = 0, 1, \dots$. There is an entrepreneur (seller) who wants to sell a firm (or asset.) The asset is either of low quality ($\theta = L$) or high quality ($\theta = H$.) The seller discounts future payoffs at a discount factor $\delta \in (0, 1)$.

There is a pool of homogeneous short-lived buyers. At every period, there is a probability $1 - \lambda$ that no buyer arrives, and a probability λ that (exactly) one buyer arrives.³ Buyers value an asset of quality $\theta \in \{L, H\}$ at U_θ , with $U_L < U_H$. If a buyer arrives at t , he makes a take-it-or-leave-it offer to the seller $P_t \in \mathbb{R}$. If the seller accepts the offer, the asset is sold and the game ends. Otherwise, the game continues.

³The random arrival of buyers has recently been introduced in bargaining models (for example Fuchs and Skrzypacz (2007)) and dynamic lemons markets (for example, Kim (2013)), where it is interpreted as a search friction.

At every period t , the entrepreneur decides on the effort $e_t \in \{0, 1\}$ put into managing the firm. If the effort exerted at t is e_t , the asset generates returns at t equal to $\pi_G \equiv \pi > 0$ with probability νe_t and $\pi_B \equiv 0$ with probability $1 - \nu e_t$, for some $\nu \in (0, 1)$. We use $\xi_t \in \{B, G\}$ to denote the realization of the returns at time t , where B denotes that returns were low at t (“bad” signal), while G denotes that returns were high at t (“good” signal). The timing of the game is schematically displayed in Figure 1.

The cost of providing effort e is type-dependent and normalized to $c_\theta \nu e$ for $\theta \in \{L, H\}$. Note that c_θ can be interpreted as “the cost per unit of the probability of generating high returns.” We assume that $c_H < \pi < c_L$. This implies that, in autarchy, high effort is optimal for the H -seller, but not for the L -seller. We define $\bar{V}_L \equiv 0$ and $\bar{V}_H \equiv \nu \frac{\pi - c_H}{1 - \delta}$ as the autarchy values of the L -seller and the H -seller, respectively. Also, we assume $U_H > \bar{V}_H$ (gains from trade for the H -asset) and $U_L < \bar{V}_L$ (no gains from trade for the L -asset).⁴

A (unterminated) *public history* is an element of $\mathcal{H} \equiv \cup_{t=0}^{\infty} \{B, G\}^t$ and it encodes the returns realized in the past. A (unterminated) *private history* is an element of $\tilde{\mathcal{H}} \equiv \cup_{t=0}^{\infty} (\{B, G\} \times \{0, 1\} \times (\{-\infty\} \cup \mathbb{R}))^t$, that is, a public history plus the effort choices by the seller and the offers made by the buyers, where an offer equal to $-\infty$ at time t corresponds to no buyer arriving in this period. A *terminated private history* $(\tilde{h}^t, P) \in \tilde{\mathcal{H}} \times \mathbb{R}$ is composed of a private history and the offer accepted after it (at time $t + 1$).

A strategy of a buyer who arrives at time t with public history h^t is given by a distribution over the price offers $\tilde{P}(h^t) \in \Delta(\mathbb{R})$. A strategy by the θ -seller, for $\theta \in \{L, H\}$, is an acceptance decision rule $\beta_\theta : \tilde{\mathcal{H}} \times \mathbb{R} \rightarrow [0, 1]$, where $\beta_\theta(\tilde{h}^t, P)$ is the probability of accepting an offer P at history $\tilde{h}^t$, and an effort choice $\alpha_\theta : \tilde{\mathcal{H}} \times (\mathbb{R} \cup \{-\infty\}) \rightarrow [0, \nu]$, where α/ν is the probability of choosing effort equal to 1.⁵ Given a terminated private history $((\xi^t, e^t, P^t), P_t)$, the payoff of the corresponding payoff for the θ -seller at time 0 is given by

$$V_{\theta,0}(\tilde{h}^t, P_t) = \sum_{s=0}^{t-1} \delta^s (\pi_{\xi_s} - \nu c_\theta e_s) + \delta^t P_t. \quad (2.1)$$

The payoff of a never-terminated path of play (i.e., no offer is accepted by the seller) is defined as usual. Given a strategy profile, the expected payoff at 0 is defined as the expected payoff over the terminated and the never-terminated paths of play.

We use $\psi_t \equiv \psi_t(h^t) \equiv \Pr(\theta = H|h^t)$ to denote the posterior of the buyers about the type of the seller being H at t . Also, we use $V_{\theta,t} \equiv V_\theta(h^t)$ to denote the highest continuation value

⁴Since $\bar{V}_L = 0$ we have $U_L < 0$, which may seem counter-intuitive, especially given the usual assumption of free disposal of the asset. Nevertheless, note that $\pi_B = 0$ is just a normalization, so in general $\bar{V}_L = \frac{\pi_B}{1-\delta}$. Furthermore, transaction (legal/taxes) costs may reduce the buyers’ valuation of the asset.

⁵It is notationally convenient to use the probability of generating high returns as the choice of the seller, instead of the effort. It is clear that if the probability of issuing a dividend is restricted to belong to $[0, \nu]$, the two modeling choices are equivalent. Also, since the action at period t is taken after the rejection of an offer (or if there has been no offer at all; see Figure 1), α is a function of $\tilde{\mathcal{H}} \times (\mathbb{R} \cup \{-\infty\})$.

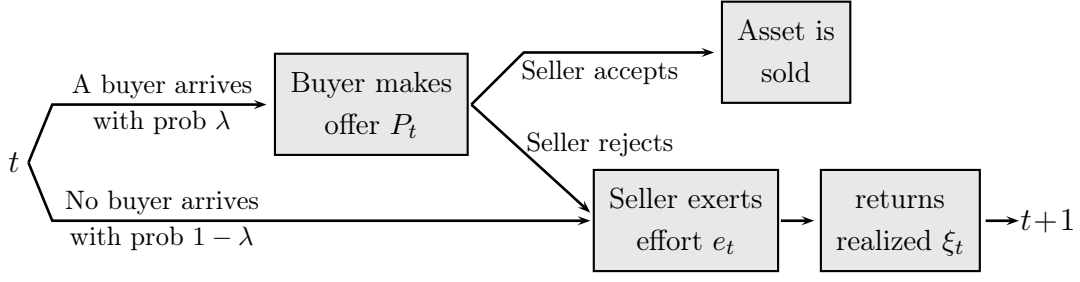


Figure 1: Timing of the model.

of the seller at time t . Note that it only depends on the public history and the type, even if the seller follows a non-public strategy. The reason is that previous previous effort choices or offers received are not observed by future buyers, so the distribution of future offers depends only on the current (and future) public history. Also, past effort choices do not change the payoffs of a continuation play.

Definition 2.1. A *Nash equilibrium (NE)* (or just an *equilibrium*) is a strategy profile for the seller $(\beta_\theta^*, \alpha_\theta^*)_{\theta \in \{L, H\}}$ and a strategy for the buyers $\tilde{P}^*$ that satisfy:

1. $\tilde{P}^*(h^t)$ solves the following problem:

$$\tilde{P}^* \in \arg \max_{\tilde{P}'} \mathbb{E}_t [\psi_t \beta_{H,t}(\tilde{P}') U_H + (1 - \psi_t) \beta_{L,t}(\tilde{P}') U_L - \tilde{P}' | \tilde{P}'] . \quad (2.2)$$

2. $(\beta_\theta^*, \alpha_\theta^*)$ are optimal policy functions of the θ -seller's problem, for $\theta \in \{L, H\}$:

$$V_\theta(h^t) = \lambda \mathbb{E}_t \left[\max_{\beta_\theta \in [0,1]} (\beta_\theta \tilde{P}_t + (1 - \beta_\theta) W_\theta(h^t)) \right] + (1 - \lambda) W_\theta(h^t) , \quad (2.3)$$

$$W_\theta(h^t) = \max_{\alpha \in [0, \nu]} (\alpha (\pi - c_\theta) + \delta (\alpha V_\theta(h^t, G) + (1 - \alpha) V_\theta(h^t, B))) . \quad (2.4)$$

Remark 2.1. Note that in our model signaling is productive, in the sense that high managerial effort generates high returns that are valuable for the seller. Nevertheless, this is very different from the usual productive signaling models of education, where the signal increases the productivity of the student, and therefore the value that she has for the uninformed part of the market.

2.2 Equilibrium Characterization

Existence

We begin with a result stating the existence of equilibria.

Proposition 2.1. *For all ψ_0 , an equilibrium exists.*

Proof. Proof in the Appendix on page 20. □

Buyers' Behavior

Let ψ^* be the lowest posterior such that if a buyer offers $\bar{V}_H$ and it is accepted by the seller for sure (independently of her type), the buyer makes non-negative profits. Formally:

$$\psi^* U_H + (1 - \psi^*) U_L - \bar{V}_H = 0 \quad \Rightarrow \quad \psi^* \equiv \frac{\bar{V}_H - U_L}{U_H - U_L} \in (0, 1) .$$

The next proposition establishes the equilibrium behavior of the buyers. In this proposition, as in those in the rest of the paper, the phrase “In any equilibrium” will be omitted.

Proposition 2.2. *Fix a public history $h^t \in \mathcal{H}$.*

1. *If $\psi(h^t) < \psi^*$ then no equilibrium offer at h^t is accepted.*
2. *If $\psi(h^t) = \psi^*$ then an equilibrium offer $P(h^t)$ is accepted only if $P(h^t) = \bar{V}_H$.*
3. *If $\psi(h^t) > \psi^*$ then $\Pr(P(h^t) = \bar{V}_H) = 1$.*

Proof. Proof in the Appendix on page 20. □

Part 1 derives from the assumption $U_L < \bar{V}_L$. Indeed, an offer that is accepted only by the L -seller is clearly suboptimal, since there are no gains from trade for the L -quality asset. Also, since $\psi(h^t) < \psi^*$, attracting the H -seller (by offering at least $\bar{V}_H$) generates losses. The rationale behind part 3 is very similar to “Diamond’s Paradox” and can be explained as follows. Assume that the maximum continuation value for the H -seller is $V_H^* > \bar{V}_H$. This is only possible if there is an equilibrium offer higher than V_H^* . Nevertheless, a buyer can always offer slightly less than V_H^* and, given that the seller discounts the future, she accepts the offer for sure, which generates a profitable deviation.

High-Quality Seller

Let’s now determine the behavior of the high-quality buyer.

Proposition 2.3. *The H -seller always exerts high effort and accepts all offers equal to $\bar{V}_H$.*

Proof. Proof in the Appendix on page 21. □

The first part of Proposition 2.3 is a consequence of Proposition 2.2. Indeed, given that equilibrium offers do not exceed $\bar{V}_H$, exerting low effort lessens the continuation payoff below $\bar{V}_H$. Since the H -seller can guarantee herself a payoff of $\bar{V}_H$, exerting low effort is strictly suboptimal. The second part comes from the fact that if the H -seller accepts an equilibrium offer with probability less than one, the buyer can increase it an arbitrarily small amount so that the seller accepts it for sure. This would be a profitable deviation.

Pooling on Accepting Offers, Separation on Effort Choices

Given that the behavior of the H -seller is fully determined by Proposition 2.3, we now focus on the behavior of the L -seller. In order to save notation we will drop the subindexes in the continuation values and strategies of the L -seller, using V_t to denote $V_{L,t}$ and α_t to denote $\alpha_{L,t}$.

A trivial corollary of Propositions 2.2 and 2.3 is that the L -seller accepts with probability one equilibrium offers equal to $\bar{V}_H$ and rejects all other equilibrium offers. Indeed, given that $c_L > c_H$, it is clear that $V(h^t) < \bar{V}_H$ for all histories $h^t \in \mathcal{H}$. Therefore, at every history, the posterior is updated using only the returns ξ_{t-1} and the expected equilibrium effort of the L -seller $\alpha(h^{t-1})$.⁶ In particular, $\psi(h^t) = \psi_{\xi_{t-1}}(\psi(h^{t-1}), \alpha(h^{t-1}))$ for all $h^t \in \mathcal{H}$, where⁷

$$\psi_G(\psi, \alpha) \equiv \frac{\nu \psi}{\nu \psi + \alpha(1-\psi)} \quad \text{and} \quad \psi_B(\psi, \alpha) \equiv \frac{(1-\nu) \psi}{(1-\nu) \psi + (1-\alpha)(1-\psi)}, \quad (2.5)$$

If $\alpha = \nu$ (i.e. the L -seller pools with the H -seller) the signal is totally uninformative, so $\psi_\xi(\psi, \nu) = \psi$ for all $\xi \in \{B, G\}$. This is likely to happen when the cost of mimicking is low. The following proposition establishes a sufficient condition in order to have separation in the effort decision:

Proposition 2.4. *If $c_L - \pi > \delta \lambda \bar{V}_H$, the L -seller never fully mimics the H -seller, i.e., $\alpha(h^t) < \nu$ for all $h^t \in \mathcal{H}$.*

Proof. Proof in the Appendix on page 21. □

Proposition 2.4 is very helpful in simplifying the arguments and intuitions. The reason is that, at each history, one could interpret the strategy of the L -seller “as if” she could choose the realization of the returns (generating high returns at cost c_L), that is, as if she had a perfect control over the realized public history. In order to provide a neat characterization of the equilibrium set in our model and clear intuitions about our results, in the remainder of the paper we will assume that $c_L - \pi > \delta \lambda \bar{V}_H$, except for Section 2.4, where we will relax it.

Continuation Payoffs Dynamics

Let $\underline{v} \equiv \lambda \bar{V}_H$ be the expected revenue from selling the asset at a given period, provided that if a buyer arrives he offers $\bar{V}_H$ to the seller. Note that, by Proposition 2.2, if $\psi(h^t) > \psi^*$ then $V(h^t) \geq \underline{v}$. Also, let $\bar{v} \equiv \frac{\lambda \bar{V}_H}{1-(1-\lambda)\delta}$ be the upper bound on the payoff that can be achieved by

⁶We use $\alpha(h^t) \equiv \mathbb{E}[\alpha(\tilde{h}^{t-1}, \tilde{P}(h^{t-1})) | h^{t-1}]$ to denote the “expected equilibrium effort” of the L -seller. Note that even though the seller could possibly be using a non-public strategy (and therefore conditioning the mixing probability on the private history), only the expected equilibrium effort is relevant for updating the posterior.

⁷Note that, since $\nu \in (0, 1)$, Bayes’ rule is well defined except for the case $\psi = 0$ and $\alpha \in \{0, 1\}$. Nevertheless, note that buyers are never perfectly convinced that the type of the seller is L . Indeed, there is no public history that has 0 probability when the type of the seller is H .

the L -seller. This consists of the continuation payoff for the L -seller if she does not exert any effort in the continuation play, and whenever a buyer arrives, he offers $\bar{V}_H$ to the seller.

For any $v \in \mathbb{R}$, let's define

$$V_B(v) \equiv \frac{v - \underline{v}}{\delta} \quad \text{and} \quad V_G(v) \equiv \frac{v + (1 - \lambda)(c_H - \pi) - \underline{v}}{\delta}. \quad (2.6)$$

It is easy to show that if $v \in [0, \bar{v}]$ then $V_L(v) < v < V_H(v)$. These functions characterize the dynamics of the continuation values of the L -seller, as the following result states.

Proposition 2.5. *If $\psi(h^t) > \psi^*$ then $V(h^t, \xi) = \min\{\bar{v}, V_\xi(V(h^t))\}$ for all $\xi \in \{B, G\}$ and $h^t \in \mathcal{H}$.*

Proof. Proof in the Appendix on page 22. □

Intuitively, when the continuation value of the L -seller is not too high (not too low), then she has to be indifferent about generating high (low) returns. The reason is that, otherwise, high (low) returns would convince future buyers that the type of the seller is H , effectively providing her with a continuation value of $\bar{v}$. The indifference conditions of the L -seller impose the restriction that the next period's continuation payoff is obtained using the functions $V_\xi(\cdot)$ for $\xi \in \{B, G\}$. When the continuation value of the seller is high (i.e. $V(h^t) \geq V_G^{-1}(\bar{v})$), the L -seller can convince future buyers that her type is H by generating high returns, but she does not do this because the increment on her continuation value does not compensate for the cost of generating high returns.

Proposition 2.5 is very useful because it isolates the dynamics of the posterior from the dynamics of the continuation payoff. Indeed, the continuation payoff in the next period is only a function of the current continuation payoff and the dividend issuance on the current period, independently of the previous history or equilibrium played.

The Set of Equilibrium Payoffs

Let's now characterize the equilibrium continuation payoffs. As we will see, this will be useful in order to characterize the equilibrium strategies. The *equilibrium payoffs correspondence* $\hat{V} : [0, 1] \rightarrow [0, \bar{v}]$ is given by

$$v \in \hat{V}(\psi) \Leftrightarrow \text{when } \psi_0 = \psi \text{ there exists a NE with } V(\emptyset) = v. \quad (2.7)$$

The following result establishes some basic properties of $\hat{V}(\cdot)$:

Proposition 2.6. *$\hat{V}(\psi) = \{0\}$ if $\psi < \psi^*$, $\hat{V}(\psi^*) = [0, \underline{v}]$ and $\hat{V}(\psi) \subset [\underline{v}, \bar{v}]$ if $\psi > \psi^*$.*

Proof. Proof in the Appendix on page 23. □

Using Proposition 2.6 we can schematically depict the equilibrium payoffs correspondence as in Figure 2. The first statement of Proposition 2.6 is a consequence of the fact that, by Proposition 2.4, the L -seller never has strict incentives to mimic the H -seller. In particular, the L -seller is indifferent about exerting 0 effort forever, which implies that returns are low thereafter. In this outcome, if $\psi(h^t) < \psi^*$, the posterior remains below ψ^* forever. Therefore, no future buyer makes a positive offer to the seller, so the L -seller gets its autarchy value $\bar{V}_L = 0$.

The intuition behind the second statement is that if $\psi(h^t) = \psi^*$ and a buyer arrives, he is indifferent about offering $\bar{V}_H$ or an unacceptable offer, so he may potentially randomize. Therefore, the flow expected revenue from selling the asset in this period can be any value in $[0, \underline{v}]$. Also, by Proposition 2.4 the L -seller does not fully mimic the H -seller; therefore, low returns lower the posterior below ψ^* , which provides a continuation value equal to 0, which implies that the continuation value is no higher than $\underline{v}$.⁸

The third statement is a direct consequence of Proposition 2.2. If $\psi(h^t) > \psi^*$ and a buyer arrives, he is going to offer $\bar{V}_H$ for sure. Since a buyer arrives with probability λ , the expected payoff is at least $\underline{v} = \lambda \bar{V}_H$.

Beliefs Dynamics

For a given v , $\underline{\psi}(v)$ (resp. $\bar{\psi}(v)$) provides us with the lowest (resp. highest) initial prior where an equilibrium with an initial continuation payoff equal to v exists. Formally:

$$\underline{\psi}(v) \equiv \inf \{ \psi \in [0, 1] \mid v \in \hat{V}(\psi) \} \quad \text{and} \quad (2.8)$$

$$\bar{\psi}(v) \equiv \sup \{ \psi \in [0, 1] \mid v \in \hat{V}(\psi) \} . \quad (2.9)$$

In Figure 2 we have a graphical depiction of both concepts. For a given continuation value v , $\underline{\psi}(v)$ corresponds to the horizontal infimum at height v of the graph of $\hat{V}$, while $\bar{\psi}(v)$ corresponds to the horizontal supremum. The following result establishes that if (ψ_t, V_t) is part of the boundary of the graph of $\hat{V}$, with $V_t > \underline{v}$, then (ψ_{t+1}, V_{t+1}) also is.

Lemma 2.1. *If $\psi(h^t) = \underline{\psi}(V(h^t))$ for some h^t and $V(h^t) > \underline{v}$, then $\psi(h^t, \xi) = \underline{\psi}(V(h^t, \xi))$ for both $\xi \in \{B, G\}$. The same holds for $\bar{\psi}(\cdot)$.*

Proof. Proof in the Appendix on page 24. □

Let's provide some intuition as to why Lemma 2.1 is true. Assume, by contradiction, that there is some equilibrium and h^t (normalize it to $h^t \equiv \emptyset$) such that $\psi(\emptyset) = \underline{\psi}(V(\emptyset))$ and $\psi(G) > \bar{\psi}_+$ for some $\bar{\psi}_+ \in \hat{V}^{-1}(V(\emptyset))$ (the case with B is analogous). It is then easy to find $\tilde{\psi} < \psi(\emptyset)$ and $\tilde{\alpha} \in (0, 1)$ such that $\psi_B(\tilde{\psi}, \tilde{\alpha}) = \psi(B)$ and $\psi_H(\tilde{\psi}, \tilde{\alpha}) = \tilde{\psi}_+$. This implies that

⁸While the first and the third parts of Proposition 2.6 still hold when $c_L - \pi < \tilde{\delta} \lambda \bar{V}_H$, the second part only holds under this assumption. Section 2.4 discusses the $c_L - \pi < \tilde{\delta} \lambda \bar{V}_H$ case.

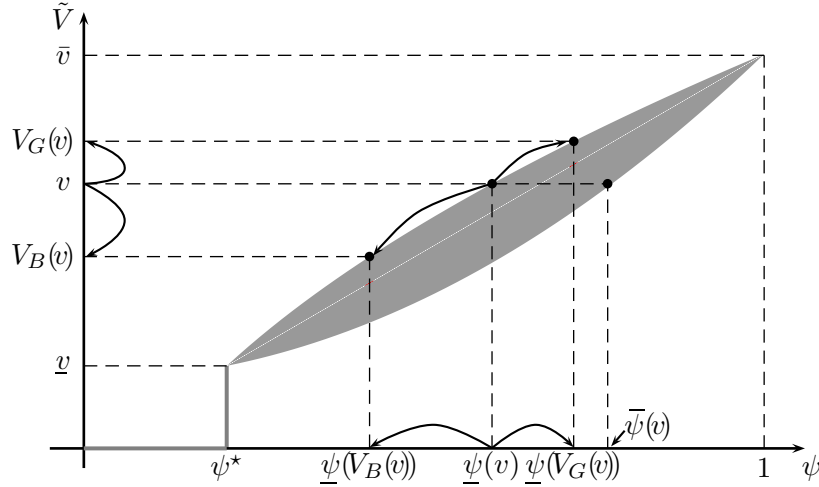


Figure 2: Schematic depiction of the equilibrium payoffs correspondence. By Proposition 2.6 we have that $\hat{V}(\psi) = \{0\}$ when $\psi < \psi^*$, and $\hat{V}(\psi^*) = [0, \underline{v}]$. The arrows indicate the equilibrium change in the posterior and continuation payoff at $(\underline{\psi}(v), v)$. By Lemma 2.1, in all equilibria, if $\psi(h^t) = \underline{\psi}(v)$ and $V(h^t) = v$, then $\psi(h^t, \xi_t) = \underline{\psi}(V_{\xi_t}(v))$ and $V(h^t, \xi_t) = V_{\xi_t}(v)$.

when $\psi_0 = \tilde{\psi}$, there exists an equilibrium providing a continuation value equal to $V(\emptyset)$ to the L -seller, so $\tilde{\psi} \in \hat{V}^{-1}(V(\emptyset))$. This contradicts our initial assumption that $\psi(\emptyset) = \underline{\psi}(V(\emptyset))$.

Step Structure

Let $\mathcal{V} \subset [\underline{v}, \bar{v}]$ be the smallest set that contains $\underline{v}$ and

$$v \in \mathcal{V} \cup [0, \underline{v}) \iff V_{\xi}(v) \in \mathcal{V} \text{ for some } \xi \in \{B, G\}.$$

Note that $\mathcal{V}$ is countable, and therefore $[0, \bar{v}] \setminus \mathcal{V}$ is dense in $[0, \bar{v}]$. Intuitively, $\mathcal{V}$ is composed of the continuation values v such that there exists some ψ_0 and history h^t such that $V(\emptyset) = v$ and $V(h^t) = \underline{v}$, satisfying $V(h^s) \geq \underline{v}$ for all $s < t$.

Proposition 2.7. *For all $v \in (\underline{v}, \bar{v})$ we have $\bar{\psi}(v) = \inf\{\underline{\psi}(v') | v' > v \text{ \& } v' \in \mathcal{V}\}$ and $\underline{\psi}(v) = \sup\{\bar{\psi}(v') | v' < v \text{ \& } v' \in \mathcal{V}\}$. Furthermore, $\underline{\psi}(v) < \bar{\psi}(v)$ if and only if $v \in \mathcal{V} \cup \{0\}$.*

Proof. Proof in the Appendix on page 24. □

A corollary of Proposition 2.7 is that $\underline{\psi}(\cdot)$ and $\bar{\psi}(\cdot)$ are increasing. Indeed, if there existed $\underline{v} \leq v < v' \leq \bar{v}$ such that $\underline{\psi}(v) > \underline{\psi}(v')$ then for any $v'' \in (v, v')$ we would have $\bar{\psi}(v'') \leq \underline{\psi}(v') < \underline{\psi}(v) \leq \bar{\psi}(v) \leq \underline{\psi}(v'')$, which is a contradiction. Intuitively, the existence of an equilibrium where the initial continuation payoff is high requires the prior about the type

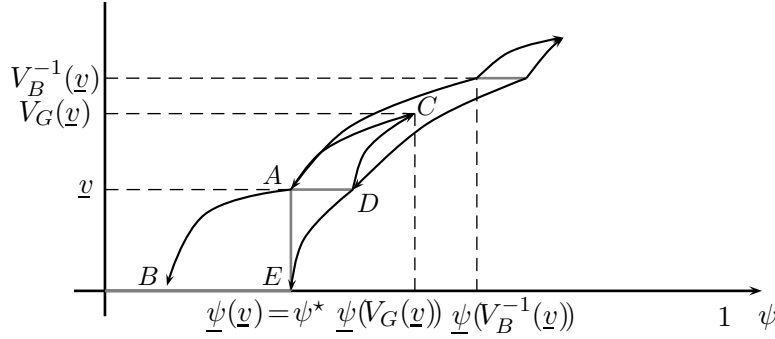


Figure 3: Intuition of the self-replication of the steps in the equilibrium payoffs correspondence. Since $\underline{v} \in \hat{V}(\psi^*)$ (point A in the picture), we can find two continuation equilibria with posteriors and continuation values at points B and C . Then, there exists an equilibrium with posterior and continuation value D that uses $E \equiv (\psi^*, 0)$ and C as continuation equilibria. This generates the first step (all points between A and D). The next step is constructed similarly using the first step.

being H to be also high. Also, since by Proposition 2.1 $\hat{V}(\psi_0)$ is non-empty for all $\psi_0 \in [0, 1]$, we have that $\hat{V}^{-1}(v) = [\underline{\psi}(v), \bar{\psi}(v)]$, which implies that the graph of $\hat{V}(\cdot)$ has a step structure.

The fact that, by Proposition 2.7, if $v \in \mathcal{V} \cup \{0\}$ then $\underline{\psi}(v) < \bar{\psi}(v)$, highlights one of the main difficulties of considering discrete (time/types/signals/effort choices) models: the jumps in the posterior belief. Since at every period, after the realization of the signals, there is a jump in the posterior, the continuation value jumps accordingly. In our model, due to the discontinuity in the behavior of the buyers (established in Proposition 2.2), the set of continuation payoffs suddenly increases at ψ^* (see Proposition 2.6.) The effect of this discontinuity is replicated for higher posteriors and payoffs and makes $\hat{V}$ have non-standard properties, as described below.

The intuition behind Proposition 2.7 is depicted in Figure 3. Assume $\psi_0 = \psi^*$ and $V(\emptyset) = \underline{v}$. By Proposition 2.4 we have that in such an equilibrium $\psi(B) < \psi^*$, and therefore, by Proposition 2.6, $V(B) = 0$. It is then easy to show that if we increase ψ_0 above but close to ψ^* , there exists an equilibrium (denoted using tildes) where $\tilde{\psi}(B) < \psi^*$ and $\tilde{\psi}(G) = \psi(G)$ (constructed using the same continuation plays as in our original equilibrium). The same argument can be applied to $V_B^{-1}(\underline{v})$, and by induction to all elements in $\mathcal{V}$.

Given that, by Proposition 2.7, the set $\mathcal{V}$ determines the heights of the “steps” in the graph of $\hat{V}$, its density plays a crucial role in determining its properties. The following proposition establishes the properties of $\mathcal{V}$.

- Proposition 2.8.** *1. If $V_G^{-1}(\bar{v}) \leq \underline{v}$ then $\bar{v}$ is the only accumulation point of $\mathcal{V}$.*
2. If $V_G^{-1}(\bar{v}) \in (\underline{v}, V_B^{-1}(\underline{v})]$ then $\mathcal{V}$ is not dense in $(\underline{v}, \bar{v}]$ but has many accumulation points.
3. If $V_G^{-1}(\bar{v}) > V_B^{-1}(\underline{v})$ then $\mathcal{V}$ is dense in $(\underline{v}, \bar{v}]$.

Proof. Proof in the Appendix on page 27. $\square$

Note that when $\mathcal{V}$ is not dense (i.e., in the first two cases of Proposition 2.8) there exist continuation values in the interior of $[\underline{v}, \bar{v}] \setminus \mathcal{V}$. This implies that $\hat{V}$ is not a function on $(\psi^*, 1]$, i.e., $\hat{V}(\psi)$ is multivalued for some $\psi > \psi^*$. In order to see this, assume that $(v, v') \cap \mathcal{V} = \emptyset$ for some $v, v' \in (\underline{v}, \bar{v})$ with $v < v'$. Then, Proposition 2.7 establishes that $\underline{\psi}(\cdot)$ and $\bar{\psi}(\cdot)$ are constant in (v, v') , i.e., $(v, v') \subset \hat{V}(\underline{\psi}(v))$.

In Figure 4 we depict the graph of $\hat{V}(\cdot)$ when $\mathcal{V}$ is not dense. The reason for the existence of vertical segments in the graph (other than the one situated at ψ^*) is, as it was for the existence of horizontal steps, the discrete nature of our model. For the sake of clarity, focus on the case of $\underline{v} \geq V_G^{-1}(\bar{v})$. By Proposition 2.6, continuation values in $(0, \underline{v})$ can only be achieved if the posterior is equal to ψ^* . Since when $\underline{v} \geq V_G^{-1}(\bar{v})$ the only effort exerted in equilibrium by the L -seller is 0,⁹ $\psi(B^t) = \psi^*$ for some t only if ψ_0 belongs to $\underline{\psi}(\mathcal{V})$, which is a discrete, non-dense set.

In the third case of Proposition 2.8, instead, $\hat{V}(\psi)$ is univaluated when $\psi \in (\psi^*, 1)$. In this case, the region of the continuation payoffs space where the seller is indifferent about the effort choice (given by $(\underline{v}, V_G^{-1}(\bar{v}))$) is wide enough so that all its elements can be approached by iteratively applying $V_B^{-1}(\cdot)$ and $V_G^{-1}(\cdot)$ to $\underline{v}$. Since learning is endogenous and mixing probabilities cannot be discretized, the size of the jumps in the posterior is endogenous and non-discretizable. Therefore, mixing probabilities are adjusted in such a way that even though $\underline{\psi}(v) = \psi^*$ for all $v \in (0, \underline{v}]$, $\underline{\psi}(V_B^{-1}(v))$ is increasing in v when $v \in (0, \underline{v}]$.

Effort Choice

Let's now establish the (generic) uniqueness of the equilibrium:

Proposition 2.9. *The equilibrium distribution over public outcomes is generically¹⁰ unique.*

Proof. Proof in the Appendix on page 27. $\square$

When $\underline{v} \geq V_G^{-1}(\bar{v})$ (which happens if, for example, $c_L - \pi$ is large, δ is small or λ is large) the equilibrium choice is always 0, that is, the L -seller does not put any effort into masquerading. The reason is that either the seller has a high cost of signaling, current costs of masquerading are highly valued compared with the future reward or the seller is very confident that a buyer will arrive soon, so there is no need to keep the posterior high.

⁹Indeed, $V(v) > \bar{v}$ for all $v > \underline{v}$, so high effort is never exerted in equilibrium.

¹⁰By generically we mean that it is unique except for a set of measure zero in the parameter space. For example, for any choice of parameters $(\psi_0, \lambda, \delta, c_H, \pi, U_L, U_H)$, there is a unique equilibrium for all choices of c_L satisfying our assumptions except maybe for a countable (i.e., measure-zero) set.

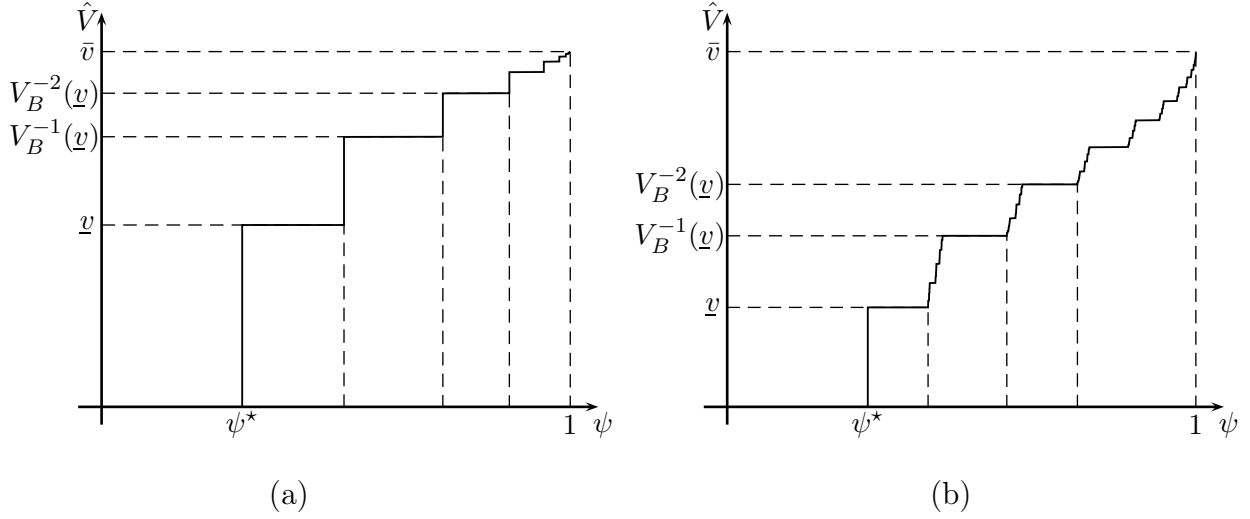


Figure 4: Graph of $\hat{V}(\cdot)$ when $\mathcal{V}$ is not dense. (a) corresponds to the first case in Proposition 2.8, while (b) corresponds to the second case. While in (a) we have a “regular” staircase, in (b) we see that each step has a substructure, provided by the existence of multiple accumulation points in $\mathcal{V}$.

Devil’s Staircase and Ubiquitously Discontinuous Market Composition

Let’s assume $V_B^{-1}(v) < V_G^{-1}(\bar{v})$. In this case, by Proposition 2.8, the set of heights of the “steps” in $\mathcal{V}$ defined previously is dense. Therefore, the graph of $\hat{V}$ looks like an upward going “staircase” with an infinite number of steps continuously put one after the other. The next lemma states this formally:

Lemma 2.2. *If $\mathcal{V}$ is dense, the $\hat{V}(\psi)$ is a singleton for all $\psi \in (\psi^*, 1]$. In this case, $\hat{V} : (\psi^*, 1] \rightarrow \mathbb{R}$ is a devil’s staircase, i.e., it is continuous, differentiable with derivative equal to 0 almost everywhere and globally increasing.*

Proof. Proof in the Appendix on page 27. □

Figure 5 (a) shows the graph of $\hat{V}$ when $\mathcal{V}$ is dense (i.e., $V_B^{-1}(v) < V_G^{-1}(\bar{v})$.) As we see, the steps in the staircase make the continuation payoffs correspondence flat almost everywhere. This implies, as the following Lemma 2.3 below states, that its inverse is not well behaved.

Figure 5 (b) depicts the mixing probability in the effort choice in the generically unique equilibrium when $\mathcal{V}$ is dense. We see that it is low in the extremes (i.e. when learning is slow). The effort choice peaks around ψ^* , since if a B signal is observed, the posterior falls below ψ^* ; so in the following period no high offer will be made. Intuitively, the L -seller gets “scared,” so the equilibrium expected effort increases.

Lemma 2.3. *If $V_B^{-1}(v) < V_G^{-1}(\bar{v})$ then any function $f : (\psi^*, 1] \rightarrow [v, \bar{v}]$ that satisfies $f(\psi_0) \in \hat{V}^{-1}(\psi_0)$ for all $\psi_0 \in (\psi^*, 1]$ is discontinuous in a dense set.*

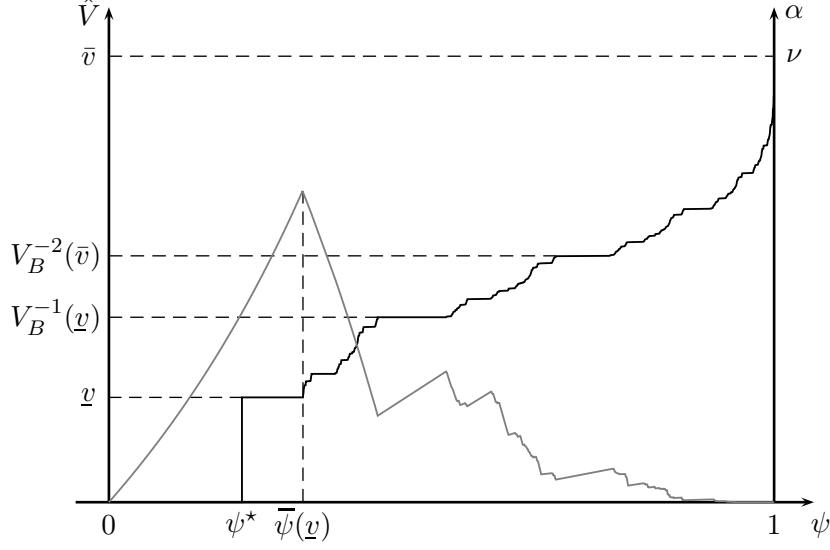


Figure 5: In black, graph of $\hat{V}$ when $V_B^{-1}(\underline{v}) < V_G^{-1}(\bar{v})$. For $\psi > \psi^*$ this is the graph of a devil's staircase, that is, a continuous, non-constant and increasing function that is flat almost everywhere. In gray (right vertical axis), equilibrium effort choice in the (generically) unique equilibrium. We plot them together in order to see the relationship between the steps and the peaks of both curves.

Proof. Proof in the Appendix on page 28. $\square$

The following remark explains why the previous result may generate highly discontinuous functions if we endogenize the initial prior about the quality of the asset.

Remark 2.2. Consider an extension of our model where we endogenize the initial composition of the qualities of the asset (i.e., ψ_0) in the following way. We assume that there is a big pool of entrepreneurs with low-value ideas and a much smaller pool of entrepreneurs with high-value ideas.¹¹ Setting up a firm requires paying a fixed cost $\bar{V}_0 \in (\underline{v}, \bar{v})$. Since $\bar{V}_H > \bar{v}$, all entrepreneurs with highly valuable ideas set up the firm for sure. Heuristically, free entry imposes the restriction that the mass of entrepreneurs with low-value ideas that set up the firm is such that $V(\emptyset) = \bar{V}_0$. In this case, from Lemma 2.2, it is easy to show that any $\psi_0 : (\underline{v}, \bar{v}) \rightarrow (\psi^*, 1]$ is strictly increasing and discontinuous in a dense set. This makes the composition of the market extremely sensitive to changes in the entry costs by the firms.

2.3 Reputations Interpretation

In order to interpret our model as a reputations model, let's first establish the following result:

¹¹We interpret the quality of the firm as the value of the patents it owns. So, entrepreneurs first get (good or bad) ideas, then set up the firm, and finally sell (or not) the firm.

Proposition 2.10. *Fix any equilibrium $((\alpha_\theta, \beta_\theta)_{\theta \in \{L, H\}}, \tilde{P})$ of our model, and let $\mu(h^t) \equiv \Pr(\tilde{P}(h^t) = \bar{V}_H)$ for all $h^t \in \mathcal{H}$. Then the continuation value of the L -seller solves*

$$V(h^t) = \max_{\alpha \in [0, \nu]} (\mu(h^t) \underline{v} - \alpha c + \tilde{\delta} (\alpha V(h^t, G) + (1 - \alpha) V(h^t, B))) \quad (2.10)$$

where $\tilde{\delta} \equiv \delta(1 - \lambda)$ and $c \equiv (1 - \lambda)(c_L - \pi)$. Conversely, if (α, μ, ψ) solve (2.10), with ψ updated following Bayes' rule (2.5) and $\mu(h^t) = \mathbb{I}_{\psi(h^t) > \psi^*}$ whenever $\psi(h^t) \neq \psi^*$, then they are equilibrium strategies in our model.

Proof. Proof in the Appendix on page 28. □

In the previous proposition, $\underline{v}$ is interpreted as the flow payoff provided by the fact that when $\psi(h^t) > \psi^*$, there is a positive probability (equal to λ) that an offer $\bar{V}_H$ will be received. c is the net cost of choosing high effort per unit of probability, adjusted for the fact that it is incurred only when no offer is accepted. Finally, $\tilde{\delta}$ is the effective discount factor, which incorporates the fact that the L -seller discounts the future, but also that buyers arrive randomly, so it takes some time for offers to arrive.

This renormalization has an interpretation of the model as a reputations model (see Remark 2.3). It is particularly useful because, even though the probability that the game ends by accepting an offer changes depending on the posterior about the type of the seller, the effective discount rate ($\tilde{\delta}$) is independent of the probability of receiving a high offer.

Remark 2.3. Our model can be reinterpreted as a reputations model in which a seller (for example, a restaurant) repeatedly sells a good to short-lived customers at price $\underline{v}$. Customers buy the good only if they are reasonably convinced that the quality of the good is high (i.e., $\psi_t \geq \psi^*$). In this case, the returns in our original model can be reinterpreted as a noisy signal (online reviews, etc.) about the quality of the signal. While a high-quality product “looks good” (i.e., good signals appear frequently), making a low-quality product “look good” (i.e. send good signals as if it were high-quality) is costly. In this case, the continuation value of the L -seller would follow equation (2.10).

2.4 Mimicking Case

Let's now relax the condition made in Proposition 2.4 (assumed in the subsequent results in Section 2.2). Assume, therefore, that $c_L - \pi \leq \delta \lambda \bar{V}_H$.

In this case, we are not going to repeat the previous analysis, although most results still apply.¹² Instead, we will provide a result that exposes the main qualitative difference when we assume $c_L - \pi \leq \delta \lambda \bar{V}_H$ instead of $c_L - \pi > \delta \lambda \bar{V}_H$

¹²For example, it is not difficult to show that the equilibria that we found are still equilibria under the assumption $c_L - \pi \leq \delta \lambda \bar{V}_H$.

Proposition 2.11. *Assume $c_L - \pi \leq \delta \lambda \bar{V}_H$. Then there exist equilibria where $\psi(h^t) \geq \psi^*$ for all $h^t \in \mathcal{H}$. In those equilibria, $\lim_{t \rightarrow \infty} \Pr(\psi_t = \psi^*) = 1$.*

Proof. Proof in the Appendix on page 29. □

Proposition 2.11 states that there are equilibria where the amount of information released is limited. Even though it is not efficient for the L -seller to exert effort (we still assume $\pi - c_L < 0$), the prospect of obtaining a high offer makes her willing to incur payoff losses for arbitrarily long periods of time. As the second part of the proposition hints, incentives to exert high effort are provided when the posterior is ψ^* by buyers mixing over the offer. When good signals are observed, (indifferent) buyers make a high offer with high probability, while if bad signals are observed, the probability of offering a high price decreases.

This result is in sharp contrast to the result in Cripps, Mailath and Samuelson (2004), especially given the reputations interpretation of our model. Indeed, they show that under imperfect monitoring, reputation asymptotically disappears. Hence, a continuation strategy of the game with perfect information is played with probability 1 in the long run. Note that in our model with perfect information the L -asset is not sold in any equilibrium, since there are no gains from trade for this asset.

The reason why our model features equilibria with permanent reputation effects is the following. In classical reputation models the uninformed (sequence of short-lived) players' payoff depends on the actions taken by the informed players and not directly on the types. In our model, instead, their payoff is a function of the type, not the action. In the restaurant story, in the classical reputations model, a customer who believes that the seller exerts high effort would buy the good independently of the type of the seller. In our model, instead, if a customer is convinced enough that the type of the seller is L , he does not buy, independently of the effort exerted by the seller.

3 Conclusions

The analysis of the dynamic trade of heterogeneous assets involves studying the seller's incentives to disguise the quality of her asset in order to increase the transaction price. When we endogenize the information that a signal conveys, the incentive to mimic forces learning not to be too fast or too slow. Fast learning induces low types to masquerade, which slows learning down. Slow learning, instead, reduces their incentives to mimic, which speeds learning up.

We characterize all Nash equilibria in our model, without any restriction on players' strategies or beliefs. We find that even though our problem is not stationary (information accumulates over time), public outcomes are generically Markov with respect to the posterior about the quality of the asset. By showing that our dynamic signaling model can be mapped into a reputation model, we find that this implies that no reputation cycles or building-milking

phases appear in any equilibrium. Also, under some parametric assumptions, reputation may be a permanent effect.

Our model highlights the technical challenges that dynamic learning models pose. Even though it is arguably one of the simplest non-trivial fully dynamic models of repeated noisy signaling, the characterization of the equilibrium set is technically demanding. We develop new techniques to characterize the equilibrium continuation payoffs correspondence. We find that when the behavior of some player(s) is discontinuous in the posterior, a self-replicating step structure arises that may take the form of a devil’s staircase. This structure is likely to be present in discrete time versions of other models in the literature and may imply that the market composition is highly discontinuous with respect to the entry fee.

Overall, repeated signaling provides us with valuable insights into the dynamic incentives of traders of heterogeneous assets. Diamond’s paradox generates stickiness in the accepted price offers and disciplines buyers’ beliefs updating. As a consequence, high-quality sellers behave optimally, while low-quality sellers partially masquerade themselves at every history, making information revelation slow but persistent. At a more technical level, we show that even though discrete dynamic models are attractive in terms of the understandability of the assumptions and non-restriction of the strategies, they may generate economically relevant equilibrium objects with undesirable mathematical properties. The generalization of our insights and techniques to more general models is left to future research.

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A Omitted Proofs

Proof of Proposition 2.1 (page 6)

Note that using Lemma A.1 we can prove that an equilibrium exists for all ψ_0 . Indeed, it is easy to show that the structure established in Lemma A.1 for $\underline{\psi}(\cdot)$ and $\bar{\psi}(\cdot)$ guarantees that if $\psi_0 \in [\underline{\psi}(v), \bar{\psi}(v)]$ then there exists some α such that $\psi_\xi(\psi_0, \alpha) \in [\underline{\psi}(V_\xi(v)), \bar{\psi}(V_\xi(v))]$. So, given the properties of $\underline{\psi}(\cdot)$ and $\bar{\psi}(\cdot)$ established in Proposition 2.7, it follows that we can construct a continuation play consistent with equilibrium for any $\psi_0 \in [0, 1]$.

Proof of Proposition 2.2 (page 7)

Proof. Let's first show that no offer exceeds $\bar{V}_H$. Let's define $v_H^{\max} \equiv \sup_{h^t \in \mathcal{H}} V_H(h^t)$. Note that $v_H^{\max} \leq U_H$, since equilibrium offers never exceed U_H . Assume that $v_H^{\max} > \bar{V}_H = \frac{\nu(\pi - c_H)}{1 - \delta}$ and let's define $\varepsilon \equiv (1 - \delta) v_H^{\max} - \nu(\pi - c_H) > 0$. Let h^t be such that $V_H(h^t) \in (v_H^{\max} - \varepsilon, v_H^{\max}]$. Then, there is some $\alpha_H \in [0, \nu]$ such that

$$\begin{aligned} W_H(h^t) &= \overbrace{\alpha_H(\pi - c_H)}^{\leq \nu(\pi - c_H)} + \delta \left(\overbrace{\alpha_H V_H(h^t, G) + (1 - \alpha_H(h^t)) V_H(h^t, B)}^{\leq v_H^{\max}} \right) \\ &< v_H^{\max} - \varepsilon. \end{aligned}$$

Therefore, it must be the case that, in equilibrium, if a buyer arrives at h^t the distribution of offers $\tilde{P}(h^t)$ is such that

$$v_H^{\max} - \varepsilon < V_H(h^t) \leq \lambda \mathbb{E}[\max\{\tilde{P}(h^t), v_H^{\max} - \varepsilon\}] + (1 - \lambda)(v_H^{\max} - \varepsilon).$$

In particular, this requires that $\Pr[\tilde{P}_t > v_H^{\max} - \varepsilon] > 0$. Nevertheless, an offer $P_t > v_H^{\max} - \varepsilon$ is dominated by offering $v_H^{\max} - \varepsilon$,¹³ which is accepted by the H -seller for sure. So, $v_H^{\max} \leq \bar{V}_t$.

Assume first $\psi_t > \psi^*$. Assume that a buyer arrives at t . If the H -seller accepts $\bar{V}_H$ for sure then it is optimal to offer $\bar{V}_H$. Otherwise, assume that the H -seller accepts an offer $\bar{V}_H$ with probability $\gamma \in [0, 1]$. Then, if the buyer offers $\bar{V}_H + \varepsilon$, with $\varepsilon > 0$, the offer is accepted by the H -seller for sure (since her continuation payoff is $\bar{V}_H$), so this guarantees a payoff to the buyer equal to

$$\psi_t \bar{V}_H + (1 - \psi_t) \bar{V}_L - (\bar{V}_H + \varepsilon).$$

Note that since $\psi_t > \psi^*$, the sum of the first two terms is strictly higher than $\bar{V}_H$. Therefore, if $\varepsilon > 0$ is small enough, the whole expression is positive, and higher than $\psi_t \gamma \bar{V}_H + (1 - \psi_t) \bar{V}_L - \bar{V}_H$. Since the payoff is decreasing in ε , all such offers are strictly dominated. Therefore, in equilibrium, $\bar{V}_H$ is offered and accepted with probability one.

¹³It is strictly dominated because since $V_H(h^t) \geq V_L(h^t)$, in both cases, the offer is going to be accepted by all types of seller.

Assume now $\psi_t \leq \psi^*$. Note that the first part of the proof $W_L(h^t) < W_H(h^t) = \bar{V}_H$ for all histories. Therefore, an offer strictly lower than $\bar{V}_H$ would attract (at most) only the L -seller. So it would never be profitable (given that $U_L < \bar{V}_L$). Also, if a buyer offers $\bar{V}_H$ the L -seller accepts for sure. Therefore, the profit for the buyer from offering $\bar{V}_H$ is no larger than $\psi_t U_H + (1 - \psi_t) U_L - \bar{V}_H$. If $\psi_t < \psi^*$, this is negative, so no such offer is made. Only if $\psi_t = \psi^*$ and the H -seller accepts an offer equal to $\bar{V}_H$, the buyer is indifferent about offering it or not, so he could potentially randomize. $\square$

Proof of Proposition 2.3 (page 7)

Proof. Note that Proposition 2.2 implies that $V_H(h^t) = \bar{V}_H$ for all histories. Assume that there is a history where the H -seller exerts effort 1 with probability $\alpha_H(h^t)/\nu \in [0, 1)$. Then,

$$V_H(h^t) = \overbrace{\alpha_H(h^t)(\pi - c_H)}^{< \nu(\pi - c_H)} + \delta \left(\overbrace{\alpha_H(h^t)V_H(h^t, G) + (1 - \alpha_H(h^t))V_H(h^t, B)}^{=\bar{V}_H} \right).$$

Trivially, $V_H(h^t) < \bar{V}_H$, which is a contradiction. Finally, in the proof of Proposition 2.2 we see that an offer equal to $\bar{V}_H$ is made only if it is accepted with probability one by the H -seller. $\square$

Proof of Proposition 2.4 (page 8)

We prove this result by first proving a simpler version of our model, referred as to the perfect-monitoring model, where we allow the L -seller to choose G for sure (so high returns are generated for sure) at cost c_L (we assume this in all proofs of Propositions 2.5-2.10 [note that none of the proofs of these propositions uses Proposition 2.4]). We then verify (in this proof) that for all equilibria and histories of the simpler model $\alpha(h^t) \leq \nu$, so all equilibria found under the model problem are also equilibria of the original model. We finally show (in this proof) that the set of equilibrium payoffs in both models is the same, so our result holds. We recommend the reader first go over the proofs of Propositions 2.5-2.10 before reading this proof.

We first prove that $\alpha(h^t) < \nu$ for all h^t . Note that if $V(h^t) = v \in \mathcal{V}$ then $\bar{\psi}(V_B(v)) \leq \underline{\psi}(v) < \bar{\psi}(v) \leq \underline{\psi}(V_G(v))$ (using Proposition 2.7,) so $\alpha(h^t) < \nu$. If, instead, $v \notin \mathcal{V}$ then note that there exists $v' \in \mathcal{V}$ such that $V_B(v) < v' < v$. The reason is that $V_L^{-n}(\underline{v}) \in \mathcal{V}$ for all $n \in \mathbb{N}$ and there exists some n_* such that $V_B^{n_*}(v) \in (0, \underline{v})$, so $V_B^{-(n_*-1)}(\underline{v}) \in (V_B(v), v)$. This implies (by Proposition 2.7) that $\bar{\psi}(V_B(v)) \leq \underline{\psi}(v') < \bar{\psi}(v') \leq \underline{\psi}(v)$. Therefore, the expected effort choice is strictly lower than ν .

In order to compare the equilibrium payoff correspondences of the two models, for each $v \in [0, \bar{v}]$, let $\underline{\psi}_1(v)$ and $\underline{\psi}_\nu(v)$ be defined as in (2.8) for the perfect-monitoring and original models, respectively. Our goal is to prove that $\underline{\psi}_1(v) = \underline{\psi}_\nu(v)$ for all $v \in [0, \bar{v}]$ (we can proceed similarly to prove the same for $\bar{\psi}(\cdot)$ defined in (2.9).) Note that $\underline{\psi}_1(v) \geq \underline{\psi}_\nu(v)$ for

all $v \in [0, \bar{v}]$, since as is shown previously in this proof, $\alpha(h^t) \leq \nu$ for all equilibria in our reduced model. Let's define for each $v \in [0, \bar{v}]$, $d_q(v) \equiv 1/\underline{\psi}_\nu(v) - 1/\underline{\psi}_1(v) \geq 0$, and let $d_q^{\max} \equiv \sup_v(d_q(v))$. Assume, by contradiction, that $d_q^{\max} > 0$.

The main difficulty in proving $\underline{\psi}_1(v) = \underline{\psi}_\nu(v)$ for all v is that there are histories in our original model where it may be the case that the L -seller has strict incentives to exert high effort. It is easy to show that also in the perfect-monitoring model the L -seller is indifferent about exerting high or low effort at some history h^t only if $V(h^t, \xi) = V_\xi(V(h^t))$ for all $\xi \in \{B, G\}$, as in the perfect-monitoring model. She is strictly willing to exert effort only if $V(h^t, G) > V_G(V(h^t))$ and $V(h^t, B) < V_B(V(h^t))$, in which case $\psi(h^t, \xi) = \psi(h^t)$ for all $\xi \in \{B, G\}$. Also, the seller is willing to exert low effort if $V(h^t, B) = V_B(V(h^t))$ and $V(h^t) \leq V_G^{-1}(\bar{v})$.

Note that for any h^t and $V(h^t)$ such that $\psi(h^t) = \underline{\psi}_1(V(h^t))$, either $\psi(h^t, \xi) = \underline{\psi}_1(V_\xi(V(h^t)))$ for $\xi \in \{B, G\}$ (if $\alpha(h^t) < \nu$, using the same argument as in Lemma 2.1) or $\underline{\psi}_1(V(h^t, G)) \leq \underline{\psi}_1(V(h^t))$ (if $\alpha(h^t) = \nu$.) Note that since $\underline{\psi}_1(V_G(v)) > \underline{\psi}_1(v) + m$ for some $m > 0$, and $\underline{\psi}(V(h^t, G)) \geq V_G(V(h^t))$, we have that in the second case $d_q(V(h^t, G)) > d_q(V(h^t, G))$. Therefore, if $\psi(h^t, \xi) \neq \underline{\psi}(V_\xi(V(h^t)))$, then $d_q(V(h^t, H)) > d_q(V(h^t)) + m'$ for some $m' > 0$.

Consider a strictly decreasing sequence $(\varepsilon_n)_n$ such that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Also, let's define

$$v^{\inf} \equiv \lim_{n \rightarrow 0} \underbrace{\inf \{ v > 0 \mid d_q(v) > d_q^{\max} - \varepsilon_n \}}_{\equiv v_n}.$$

Now, we can use an argument similar to the proof of Lemma A.1 to complete the proof. Note that it can be used since equation (A.4) still holds, given that otherwise there would exist $v > v^{\inf}$ with $d_q(v) > d_q(v) + m'$, which is a contradiction. Therefore, as in the proof of Lemma A.1, $d_q^{\max} = 0$, so $\underline{\psi}_1(\psi) = \underline{\psi}_\nu(\psi)$ for all $\psi \in [0, 1]$.

Proof of Proposition 2.5 (page 9)

Proof. To prove the result, it is important to notice that, in equilibrium, if at some public history h^t we have $\Pr(\tilde{P}(h^t) = \bar{V}_H | \tilde{P}(h^t) \neq -\infty) = 1$ (i.e., conditional on a buyer arriving at h^t he offers $\bar{V}_H$ for sure), then the L -seller is indifferent about exerting effort if and only if $V(h^t, \xi) = V_\xi(V(h^t))$ (it follows from the indifference conditions).¹⁴ This implies that the seller is willing to generate returns equal to π_ξ if and only if $V(h^t, \xi) = V_\xi(V(h^t))$.

We solve each part separately:

1. Assume first $V_B(V(h^t)) \geq 0$ (so $V(h^t) \geq \underline{v}$). This implies, by Proposition 2.6 that if a buyer arrives at t he offers $\bar{V}_H$ for sure (recall that we do not use Proposition 2.5 to prove it.) By contradiction, assume $V(h^t, B) \neq V_B(V(h^t))$. If $V(h^t, B) < V_B(V(h^t))$, then the seller has strict incentives to generate high returns. Therefore, low returns

¹⁴As mentioned in the proof of Proposition 2.4, the proofs of Propositions 2.5-2.10 are done allowing the L -seller to choose G for sure (so she generates high returns for sure) at cost c_L .

would convince future buyers that the type of the seller is H , providing the seller with a continuation payoff $\bar{v}$. This clearly generates high returns. If $V(h^t, B) > V_B(V(h^t))$, then generating low returns reports a value of $\underline{v} + \delta(1 - \lambda)V(h^t, B)$, which, since $V(h^t) < \bar{v}$, is strictly higher than $V(h^t)$, so we also have a contradiction.

2. Assume now $V_G(V(h^t)) \leq \bar{v}$. If $V(h^t, G) > V_G(V(h^t))$ then the L -seller has strict incentives to generate high returns. Nevertheless, if low returns are generated, she convinces future buyers that her type is H , and this clearly dominates generating high returns. If, instead, $V(h^t, G) < V_G(V(h^t))$, then the L -seller strictly prefers not to generate high returns. If she instead generates high returns, she convinces the buyers that her type is H . Using that expression of $V_G(\cdot)$ and the fact that $V(h^t) < V_G^{-1}(\bar{v})$, we see that this is a profitable deviation. Therefore, $V(h^t, G) = V_G(V(h^t))$.

□

Proof of Proposition 2.6 (page 9)

Let's prove the proposition by parts.

1. Assume $\psi(h^t) < \psi^*$ for some $h^t \in \mathcal{H}$. By Proposition 2.2 if a buyer arrives at h^t , he makes an unacceptable offer. Assume $\alpha(h^t) \leq \nu$ (the other case is analogous). Since $\psi(h^t, B) \leq \psi(h^t)$ the L -seller is weakly willing to proceed to a history where, if a buyer arrives, he offers an unacceptable offer. This argument can be iteratively used, so we have that the L -seller is willing to choose a continuation play where an acceptable offer is not made at any point in the future. This proves our result.
2. The fact that $\hat{V}(\psi^*) \supset [0, \underline{v}]$ comes from the explicit construction of equilibria that we provide in Section 2.2.¹⁵ In order to prove that $\hat{V}(\psi^*) \subset [0, \underline{v}]$, note first that if there is some h^t such that $V(h^t) > \underline{v}$ and $\psi(h^t) = \underline{\psi}$ it needs to be the case that $\alpha(h^t) = \nu$. Otherwise, $\psi(h^t, \xi) < \underline{\psi}$ for some $\xi \in \{B, G\}$, so $V(h^t, \xi) = 0$. If the L -seller is willing to choose ξ , then $V(h^t) = V_\xi^{-1}(0) \leq 0$, which is a contradiction. So, assume that $\alpha(h^t) = \nu$ and $\psi(h^t) = \psi^*$. Let $v^{\max} \equiv \sup\{\hat{V}(\psi^*)\}$, and assume $v^{\max} > \underline{v}$. Consider $\varepsilon > 0$ small and an equilibrium and history such that $V(h^t) \geq v^{\max} - \varepsilon$. By the observation made at the beginning of the proof of Proposition 2.5, we have that $V(h^t, G) = V_G(V(h^t)) \geq V_G(v^{\max} - \varepsilon)$. Therefore, since $V_G(v^{\max}) > v^{\max}$, if ε is close enough to 0, we have $V(h^t, G) > v^{\max}$, which is a contradiction.
3. Proving $\hat{V}(\psi) \subset [\underline{v}, \bar{v}]$ if $\psi > \psi^*$ is trivial, since $\bar{v}$ is clearly the maximum payoff achievable by a L -seller, while at least she gets the probability that a buyer arrives multiplied by a high offer $\bar{V}_H$, that is, $\lambda \bar{V}_H$, that is exactly $\underline{v}$.

¹⁵In this section we assume that an equilibrium exists and we then construct it explicitly.

Proof of Lemma 2.1 (page 10)

Proof. Note that the standard Bayes rule implies

$$\frac{1}{\psi(h^t)} = \frac{1-\rho}{\psi(h^t, G)} + \frac{\rho}{\psi(h^t, B)} . \quad (\text{A.1})$$

We will prove the result for $\underline{\psi}(\cdot)$, and the proof for $\bar{\psi}(\cdot)$ is analogous. Assume it is not true. Then, there exists an equilibrium and history with $V(h^t) > \underline{v}$ and $\psi(h^t) = \underline{\psi}(V(h^t))$ such that $\psi(h^t, B) > \underline{\psi}(V_B(V(h^t)))$ (the other case, that is, when $\psi(h^t, G) > \underline{\psi}(V_G(V(h^t)))$, is done analogously.) Then, let's define

$$\begin{aligned} \tilde{\psi} &\equiv \frac{1}{\frac{\nu}{\psi(h^t, G)} + \frac{1-\nu}{\underline{\psi}(V_B(V(h^t)))}} < \frac{1}{\frac{\nu}{\psi(h^t, G)} + \frac{1-\nu}{\psi(h^t, B)}} = \psi(h^t) \quad \text{and} \\ \tilde{\alpha} &\equiv \frac{\nu}{\nu + (1-\nu) \frac{\psi(h^t, G) (1-\underline{\psi}(V_B(V(h^t))))}{(1-\psi(h^t, G)) \underline{\psi}(V_B(V(h^t)))}} \in (0, 1) . \end{aligned}$$

Note that $\tilde{\alpha}$ is such that $\psi_B(\tilde{\psi}, \tilde{\alpha}) = \underline{\psi}(V_B(V(h^t)))$ and $\psi_G(\tilde{\psi}, \tilde{\alpha}) = \psi(h^t, G)$. Since, by assumption, there are equilibrium continuation paths at $\underline{\psi}(V_B(V(h^t)))$ and $\psi(h^t, G)$ providing, respectively, $V_B(V(h^t))$ and $V_G(V(h^t))$, when $\psi_0 = \tilde{\psi}$ there exists an equilibrium providing continuation value $V(h^t)$ to the L -seller. This implies that $\tilde{\psi} \geq \underline{\psi}(V(h^t))$, which is a clear contradiction. $\square$

Proof of Proposition 2.7 (page 11)

Let's define

$$\alpha_B(\psi, \psi') \equiv 1 - \frac{\psi(1-\psi')}{(1-\psi)\psi'}(1-\nu) \quad \text{and} \quad \alpha_G(\psi, \psi') \equiv \frac{\psi(1-\psi')}{(1-\psi)\psi'}\nu . \quad (\text{A.2})$$

Using the standard Bayes rule, it can be shown that for each history h^t , $\alpha_\xi(\psi, \psi')$ provides the (unique) equilibrium mixing $\alpha(h^t)$ compatible with $\psi = \psi(h^t)$ and $\psi(h^t, \xi) = \psi'$.

In order to prove the results, we first introduce a lemma that is very useful to characterize the functions $\underline{\psi}(\cdot)$ and $\bar{\psi}(\cdot)$.

Lemma A.1. $\underline{\psi}(\cdot)$ is the unique function $\underline{q} : (0, \bar{v}] \rightarrow [0, 1]$ that satisfies

1. $\underline{q}(v) = \psi^*$ for $v \in (0, \underline{v}]$,
2. $\alpha_B(\underline{q}(v), \underline{q}(V_B(v))) = \alpha_G(\underline{q}(v), \underline{q}(V_G(v)))$ for $v \in (\underline{v}, V_G^{-1}(\bar{v}))$, and
3. $\alpha_B(\underline{q}(v), \underline{q}(V_B(v))) = 0$ for $v \in [V_G^{-1}(\bar{v}), \bar{v})$.

Similarly, $\bar{\psi}(\cdot)$ is the unique function $\bar{q} : [0, \bar{v}] \rightarrow [0, 1]$ the previous conditions changing $\bar{q}(v) = \psi^*$ for $v \in [0, \underline{v})$ in the first part and $v \in [\underline{v}, V_G^{-1}(\bar{v})]$ in the second.

Proof. We do the proof for $\underline{\psi}(\cdot)$ (for $\bar{\psi}(\cdot)$ the proof is analogous.) Let's first prove existence. Using Bayes' rule (given in equation (A.1)) it is easy to show that $\underline{q}(\cdot)$ exists if and only if $\underline{\Gamma}(\cdot) \equiv \frac{1-\underline{q}(\cdot)}{\underline{q}(\cdot)}$ exists satisfying:

$$\underline{\Gamma}(v) = \begin{cases} \frac{1-\underline{\psi}^*}{\underline{\psi}^*} & \text{if } v \in (0, \underline{v}] \\ \nu \underline{\Gamma}(V_B(v)) + (1-\nu) \underline{\Gamma}(V_G(v)) & \text{if } v \in (\underline{v}, V_G^{-1}(\bar{v})) \\ \nu \underline{\Gamma}(V_B(v)) & \text{if } v \in [V_G^{-1}(\bar{v}), \bar{v}] \end{cases}$$

For each $v \in (0, \bar{v})$ let $\underline{\mathcal{H}}(v) \equiv \{h^t | V_{h^t}(v) \leq \underline{v} \text{ \& } V_{h^s}(v) \in (\underline{v}, \bar{v}) \ \forall s < t\}$ be the set of (continuation) histories where, provided that the initial continuation payoff is v , $\underline{\psi}(v)$ reaches $\underline{\psi}^*$ for first time.¹⁶ Then, it is easy to show that a solution for $\underline{\Gamma}(\cdot)$ is given by:

$$\underline{\Gamma}(v) \equiv \frac{1-\underline{\psi}^*}{\underline{\psi}^*} \sum_{h^t \in \underline{\mathcal{H}}(v)} \prod_{s=1}^t \nu^{\mathbb{I}(h_s^t=B)} (1-\nu)^{\mathbb{I}(h_s^t=G)} . \quad (\text{A.3})$$

Note that $\underline{\Gamma}(\cdot)$ is left-continuous.

Using Lemma 2.1 we know that $\underline{\psi}(\cdot)$ satisfies parts 1-3 of Lemma A.1. Consider two functions, $q(\cdot)$ and $\tilde{q}(\cdot)$, both satisfying the conditions of Lemma A.1. Define $d_q(v) \equiv \frac{1}{q(v)} - \frac{1}{\tilde{q}(v)}$ and assume that $d_q^{\max} \equiv \sup_{v \in (0, \bar{v})} d_q(v) > 0$. Consider a sequence $(\varepsilon_n)_n$, with $\varepsilon_n > 0$ for all n and such that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Define also

$$v^{\inf} \equiv \liminf_{n \rightarrow \infty} \underbrace{\inf \{ v > 0 \mid d_q(v) > d_q^{\max} - \varepsilon_n \}}_{\equiv v_n} .$$

Note that $(v_n)_n$ is a non-decreasing sequence (indeed, when n increases, the set over which the infimum is taken gets smaller). Also, note that $d_q(v_n) \rightarrow d_q^{\max}$. Let's first show that $v^{\inf} \leq V_G^{-1}(\bar{v})$. Assume otherwise, so for some sequence $v_n > V_G^{-1}(\bar{v})$ for all n we have that¹⁷

$$d_q(V_B(v_n)) = \frac{\frac{1}{q(v_n)} - \rho}{1-\rho} - \frac{\frac{1}{\tilde{q}(v_n)} - \rho}{1-\rho} = \frac{d_q(v_n)}{1-\rho} \rightarrow \frac{d_q^{\max}}{1-\rho} > d_q^{\max} .$$

This is a clear contradiction. Assume then that $v_n \leq V_G^{-1}(\bar{v})$ for all n . Note that since $q(v) = \tilde{q}(v) = \underline{\psi}^*$ when $v \in (0, \underline{v}]$, we have that $v_n > \underline{v}$ for all n . Using equation (A.1) we have

$$d_q(v_n) = \nu d_q(V_G(v_n)) + (1-\nu) d_q(V_B(v_n)) . \quad (\text{A.4})$$

Note that the LHS is asymptotically (when $n \rightarrow \infty$) equal to $d_q^{\max}$. Each of the terms in the RHS is bounded above by $d_q^{\max}$. So, since the LHS is a linear combination of them, their limit must be equal to $d_q^{\max}$. In particular, $\lim_{n \rightarrow \infty} d_q(V_B(v_n)) = d_q^{\max}$. So, we have a contradiction, since $\lim_{n \rightarrow \infty} V_B(v_n) < v^{\inf}$, but $v^{\inf}$ is an infimum. $\square$

¹⁶ $V_{h^t}(v)$ is defined recursively as $V_{\emptyset}(v) = v$ and $V_{h^t}(v) = V_{h^t}(V_{h^{t-1}}(v))$.

¹⁷Note that when $v > V_G^{-1}(\bar{v})$ we have $\alpha = 0$, so the update of beliefs follows $\frac{1}{\psi_B(\psi, 0)} = \frac{1/\psi - \nu}{1-\nu}$.

(Continuation of the proof of Proposition 2.7) We first prove that $v \in \mathcal{V} \cup \{0\}$ if and only if $\underline{\psi}(v) < \bar{\psi}(v)$. Let's first prove the "only if" implication. Note first that $\underline{\psi}(\underline{v}) = \psi^\star < \bar{\psi}(\underline{v})$. Indeed, note that the solution for x of

$$\frac{1}{x} = \frac{\nu}{\psi^\star} + \frac{1 - \nu}{\underline{\psi}(V_G^{-1}(\underline{v}))}$$

belongs to $(\psi^\star, \underline{\psi}(V_G^{-1}(\underline{v})))$ and, if $\psi_0 = x$, we have continuation payoffs that support an equilibrium with initial continuation value equal to $\underline{v}$. Given that $\underline{\psi}(\underline{v}) < \bar{\psi}(\underline{v})$, it is easy to prove using induction (note that $\mathcal{V}$ can be constructed recursively applying V_B^{-1} and V_G^{-1}) that $\underline{\psi}(v) < \bar{\psi}(v)$ for all $v \in \mathcal{V} \cup \{0\}$.

To prove the "if" implication, we prove that if $v \notin \mathcal{V} \cup [0, \underline{v})$ then $\underline{\psi}(v) = \bar{\psi}(v)$. To do this, consider

$$v^{\inf} \equiv \liminf_{n \rightarrow 0} \inf \left\{ v \notin \mathcal{V} \mid \frac{1}{\underline{\psi}(v)} - \frac{1}{\bar{\psi}(v)} > d_q^{\max} - \varepsilon \right\},$$

where now $d_q^{\max} \equiv \sup_{v \notin \mathcal{V}} \left(\frac{1}{\underline{\psi}(v)} - \frac{1}{\bar{\psi}(v)} \right)$. Note that $V \notin \mathcal{V} \Rightarrow V_\xi(v) \notin \mathcal{V} \forall \xi \in \{B, G\}$. Therefore, applying an argument similar to that in the first part of the proof of Lemma A.1, the result holds. The argument works because when we restrict the domain of q in Lemma A.1 to $(\mathcal{V} \cup [0, \underline{v}))^c$, both $\underline{\psi}(\cdot)$ and $\bar{\psi}(\cdot)$ are both the unique functions satisfying the three conditions of the lemma.¹⁸ Nevertheless, since $\underline{v} \notin (\mathcal{V} \cup [0, \underline{v}))^c$, they satisfy exactly the same conditions, so they are equal. Note that this argument fails when $v \in \mathcal{V}$, since in this case we cannot rule out $\sup_{v \in \mathcal{V} \geq \underline{v}} \left(\frac{1}{\underline{\psi}(v)} - \frac{1}{\bar{\psi}(v)} \right) = \frac{1}{\underline{\psi}(\underline{v})} - \frac{1}{\bar{\psi}(\underline{v})} > 0$.

Finally, we prove part $\underline{\psi}(v) = \sup\{\bar{\psi}(v') \mid v' < v \text{ \& } v' \in \mathcal{V}\}$ (the other case is analogous). First note if $[v, v'] \cap \mathcal{V} = \emptyset$ for some $v < v'$, then $\underline{\psi}(v) = \underline{\psi}(v')$ (and $\bar{\psi}(v) = \bar{\psi}(v')$.) Indeed, we can see from equation (A.3) that $\underline{\psi}(v) \neq \underline{\psi}(v')$ only if $\mathcal{H}(v) \neq \mathcal{H}(v')$. It is easy to see that this implies that there exists some $v'' \in (v, v']$ and history h^t such that $V_{h^t}(v') = \underline{v}$. Nevertheless, this implies that $v'' \in \mathcal{V}$, which is a contradiction.

Note that since $\underline{\psi}(\cdot)$ is increasing and left-continuous (since $\underline{\Gamma}(\cdot)$ defined in Lemma A.1 is increasing and left-continuous), we have that

$$\underline{\psi}(v) = \sup\{\underline{\psi}(v') \mid v' < v\} \leq \sup\{\bar{\psi}(v') \mid v' < v\} = \sup\{\bar{\psi}(v') \mid v' < v \text{ \& } v' \in \mathcal{V}'\},$$

where the last inequality comes from the fact that, as we showed, $\bar{\psi}(v)$ is constant in the intervals outside $\mathcal{V}$. We apply the same technique as in the proof of Lemma A.1, using

$$v^{\inf} \equiv \liminf_{n \rightarrow 0} \inf \left\{ v \mid \lim_{w \searrow v} (1/\underline{\psi}(v) - 1/\bar{\psi}(w)) > d_q^{\max} - \varepsilon \right\},$$

where now $d_q^{\max} \equiv \sup_v \left(\lim_{w \nearrow v} \left(\frac{1}{\underline{\psi}(v)} - \frac{1}{\bar{\psi}(w)} \right) \right)$. Note that the argument works because it is trivially true for $v = \underline{v}$ and $\underline{\psi}(v) = \psi^\star$ for $v \in (\underline{v}, V_L^{-1}(\underline{v}))$, so $v^{\inf} > \underline{v}$.

¹⁸Indeed, a corollary to Lemma A.1 is given by stating the same result but restricting $q : \mathcal{V}^c \cup \rightarrow [0, 1]$. In this case, since $0, \underline{v} \notin \mathcal{V}^c$, $\underline{\psi}(\cdot)$ and $\bar{\psi}(\cdot)$ have exactly the same conditions, so they are equal.

Proof of Proposition 2.8 (page 12)

Let's first prove that if $V_B^{-1}(\underline{v}) > V_G^{-1}(\bar{v})$ then $\mathcal{V}$ is not dense. Indeed, note that $v \in \mathcal{V}$ only if $V_\xi(v) \in \mathcal{V}$ for some $\xi \in \{B, G\}$. Nevertheless, for any $v \in (V_G^{-1}(\bar{v}), V_B^{-1}(\underline{v}))$ we have $V_G(v) > \bar{v}$ and $V_B(v) < \underline{v}$.

If $\underline{v} \geq V_G^{-1}(\bar{v})$ then note that $\alpha(h^t) = 0$ for all equilibria and histories. So, $\mathcal{V} = \{V_B^{-(n-1)}(\underline{v}) | \forall n \in \mathbb{N}\}$, so the only accumulation point of $\mathcal{V}$ is $\bar{v}$ (note that $\lim_{n \rightarrow \infty} V_B^{-n}(\underline{v}) = \bar{v}$).

If, instead, $V_G^{-1}(\bar{v}) \in (\underline{v}, V_B^{-1}(\underline{v}))$ then note that $V_G^{-1}(\bar{v})$ is an accumulation point of $\mathcal{V}$. Indeed, $\bar{v}$ is an accumulation point of $\mathcal{V}$ by the same argument as above. Since $V_G(\underline{v}) < \bar{v}$, we have that $V_G^{-1}(\{V_B^{-(n-1)}(\underline{v}) | \forall n \in \mathbb{N}\} \cap (V_G(\underline{v}), \bar{v})) \subset \mathcal{V}$, so $V_G^{-1}(\bar{v})$ is an accumulation point of $\mathcal{V}$. Many other accumulation points can be found by using a similar procedure.

Finally, let's prove that if $V_B^{-1}(\underline{v}) \leq V_G^{-1}(\bar{v})$ then $\mathcal{V}$ is dense. Consider otherwise, that is, $\mathcal{V}$ is not dense in $(\underline{v}, \bar{v})$, and let $A \subset (\underline{v}, \bar{v})$ be an interval with maximal length satisfying $A \cap \mathcal{V} = \emptyset$. Note that if $v \notin \mathcal{V}$ then $V_\xi(v) \notin \mathcal{V}$ for all $\xi \in \{B, G\}$. If $\sup(A) < V_G^{-1}(\bar{v})$ then $V_G(A) \cap \mathcal{V} = \emptyset$, but then $V_G(A) \subset (\underline{v}, \bar{v})$, $V_G(A) \cap \mathcal{V} = \emptyset$ and $V_G(A)$ has more length than A ,¹⁹ which is a contradiction. If $\inf(A) < V_G^{-1}(\bar{v}) \leq \sup(A)$ then $(V_G(\inf(A)), \bar{v}]$ is not in $\mathcal{V}$. Nevertheless, since $V_B^{-n}(v) \in \mathcal{V}$ for all $v \in \mathcal{V}$ and $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} V_B^{-n}(v) \rightarrow \bar{v}$, this implies that $\mathcal{V}$ is empty, which is a contradiction. If $V_G^{-1}(\bar{v}) \leq \inf(A)$ then $V_B(A) \subset (\underline{v}, \bar{v})$, $V_B(A) \cap \mathcal{V} = \emptyset$ and $V_B(A)$ has more length than A , which again is a contradiction.

Proof of Proposition 2.9 (page 13)

Note that the equilibrium is not unique only if there is some $v \in \mathcal{V}$ such that $V_\xi(v) \in \mathcal{V}$ for all $\xi \in \{B, G\}$. Indeed, otherwise, if $\psi_0 \in \hat{V}^{-1}(v)$ and, for example, $V_G(v) \notin \mathcal{V}$, $\alpha(\emptyset)$ is uniquely given by $\alpha_G(\psi_0, \underline{\psi}(V_G(v)))$, where $\alpha_G(\cdot, \cdot)$ is defined in (A.2).

So, there is multiplicity in equilibrium only if $V_{h^t}(v) = \underline{v} = V_{\tilde{h}^s}(v)$ for two different histories $h^t \neq \tilde{h}^s$ (where V_{h^t} is defined as in footnote 16). It is easy to verify that this does not hold generically. So, since the public outcome distribution is only a function of $\alpha(h^t)$, the result holds.

Proof of Lemma 2.2 (page 14)

The proof of this lemma is given by the fact that $\mathcal{V}$ is dense, given the properties of $\underline{\psi}(\cdot)$ and $\bar{\psi}(\cdot)$ stated in Proposition 2.7 (which imply that that $\hat{V}^{-1}(v) = [\underline{\psi}(v), \bar{\psi}(v)]$ for all v and $\underline{\psi}(\cdot)$ and $\bar{\psi}(\cdot)$ are increasing).

¹⁹Indeed, it is easy to verify that $\sup(V_G(A)) - \inf(V_G(A)) = \frac{\sup(A) - \inf(A)}{\delta}$.

Proof of Lemma 2.3 (page 14)

This is trivial given that $\hat{V}(\cdot)$ is a devil's staircase in the domain $(\psi^*, 1]$. Indeed, since $\underline{\psi}(v) < \bar{\psi}(v)$ for all $v \in \mathcal{V}$ and $\mathcal{V}$ is dense, the result holds.

Proof of Proposition 2.10 (page 16)

Proof. Note that, by Proposition 2.2, $\mu(h^t) = 0$ if $\psi(h^t) < \psi^*$ and $\mu(h^t) = 1$ if $\psi(h^t) > \psi^*$. Given Propositions 2.2 and 2.3, it is easy to see that $V(\cdot)$ satisfies equations (2.3) and (2.4) if and only if it satisfies the following equation:

$$V(h^t) = \max_{\alpha \in [0, \nu]} \left(\lambda \mu(h^t) \bar{V}_H + \alpha (1 - \lambda \mu(h^t)) (\pi - c_L) \right) \quad (\text{A.5})$$

$$+ (1 - \lambda \mu(h^t)) \delta \left(\alpha V(h^t, G) + (1 - \alpha) V(h^t, B) \right) . \quad (\text{A.6})$$

Note that if $\mu(h^t) = 1$, the previous equation coincides with (2.10).

Let's use tildes to denote a solution to (2.10). We first prove that $\tilde{V}(h^t) = 0$ whenever $\psi(h^t) < \psi^*$. Assume otherwise; so we have that $v_L^* \equiv \sup \{ \tilde{V}(h^t) \mid h^t \text{ such that } \psi(h^t) < \psi^* \} > 0$. Let h^t be such that $\psi(h^t) < \psi^*$ and $\tilde{V}(h^t) > v_L^* - \varepsilon$, for some $\varepsilon > 0$. First notice that if $\alpha(h^t) > 0$, then

$$\tilde{V}(h^t, G) \geq \frac{\tilde{V}(h^t) + c_L - \pi}{\delta} . \quad (\text{A.7})$$

If $\varepsilon > 0$ is small enough, we have $\tilde{V}(h^t, G) > v_L^*$. Therefore, it must be the case that $\alpha(h^t) < \nu$. This implies that $\tilde{V}(h^t, B) \geq \frac{\tilde{V}(h^t)}{\delta} > v_L^*$, but this is a contradiction, since $\psi(h^t, B) < \psi(h^t) < \psi^*$. So, $v_L^* = 0$.

Now, let's prove that $\tilde{V}(h^t) = \lambda \bar{\mu}(h^t) \bar{V}_H$ when $\psi(h^t) = \psi^*$. If $\alpha(h^t) < \nu$, then the result trivially holds. In order to finally prove that it also holds for $\alpha(h^t) = \nu$, let's define, similar to before, $v_L^* \equiv \sup \{ \tilde{V}(h^t) \mid h^t \text{ such that } \psi(h^t) = \psi^* \}$, and let's first prove that $v_L^* \leq \lambda \bar{V}_H$. Assume otherwise; i.e., $v_L^* > \lambda \bar{V}_H$, and consider an equilibrium and history where $\psi(h^t) = \psi^*$ and $\tilde{V}(h^t) = \lambda \bar{V}_H - \varepsilon$ for some $\varepsilon > 0$. Note that necessarily $\alpha(h^t) = \nu$. Then, it is easy to show that

$$\tilde{V}(h^t, G) - \tilde{V}(h^t) > \frac{c_L - \pi - \lambda \bar{V}_H}{\delta} .$$

Therefore, if $\varepsilon > 0$ is small enough, $\tilde{V}(h^t, G) > v_L^*$, but since $\alpha(h^t) = \nu$ we have $\psi(h^t, G) = \psi^*$, which is a contradiction. Therefore, $v_L^* = \lambda \bar{V}_H$. Since $\tilde{V}(h^t, B) \geq 0$, equation (A.7) implies that $\tilde{V}(h^t, B) > \lambda \bar{V}_H$, so $\alpha(h^t) < \nu$.

As we mentioned, it is trivial to show that equations (2.10) and (A.5) are equivalent when $\psi(h^t) > \psi^*$. Therefore, the statement holds. $\square$

Proof of Proposition 2.11 (page 17)

Proof. We will make this proof only for the case $\psi_0 = \psi^*$, by showing that there exists an equilibrium where $\Pr(\psi_t = \psi^*) = 1$ for all t . Extending it to the case $\psi_0 > \psi^*$ only requires “pasting” this continuation play every time ψ^* is reached. Note that if $c_L - \pi \leq \delta \lambda \bar{V}_H$ and $\psi_0 = \psi^*$, then there is an equilibrium where $\psi(h^t) = \psi^*$ for all h^t . Indeed, consider an equilibrium where

$$\Pr(\tilde{P}(h^t) = \bar{V}_H) = \mu \mathbb{I}_{h_t^t=G} ,$$

for some $\mu \in [0, 1]$ to be determined. In this equilibrium, we want the continuation payoff after a history h^t to depend only on the last signal. To verify that this equilibrium exists, let $V(\xi)$ denote the continuation payoff after signal ξ . Then

$$V(\xi) = \mu \mathbb{I}_{\xi=H} \lambda \bar{V}_H + (1 - \mu \mathbb{I}_{\xi=H} \lambda) (\nu (\pi - c_L) + \delta (\nu V(H) + (1 - \nu) V(L))) .$$

The condition $c_L - \pi \leq \delta \lambda \bar{V}_H$ is necessary to ensure $V(\xi) \geq 0$ for all $\xi \in \{B, G\}$. We can then solve for μ by making the L -seller indifferent about exerting effort, in which case we find

$$\mu = \frac{c_L - \pi}{\lambda \bar{V}_H + \lambda \nu (c_L - \pi)} \in (0, 1) .$$

□