A Reputational Theory of Firm Dynamics*

Simon Board† and Moritz Meyer-ter-Vehn‡

This Version: May 1, 2014
First Version: April 2010.

Abstract

We propose an industry lifecycle model in which each firm privately invests into its quality and thereby its reputation. Over time, both the firm and the market learn about the firm's evolving quality via infrequent breakthroughs. The firm can also exit if its value becomes negative, giving rise to selection within the industry. In a pure-strategy equilibrium, incentives are single-peaked: the firm shirks immediately following a breakthrough, works for intermediate levels of reputation and shirks again when it is about to exit. This investment behavior yields predictions for the distribution of firm productivity and the turnover rate. Finally, we compare the model to two variants: one in which the firm's investment is publicly observed, and a second where the firm has private information about its product quality.

1 Introduction

> "New technologies come and go. This is simply the nature of our business. The Samsung brand is the only asset that will live on beyond our products."
>
> Sue Shim, CMO at Samsung.

Models of firm dynamics seek to generate the large variability in productivity and profits seen within industries, with some firms investing in their assets and growing, while others disinvest and shrink (e.g. Syverson (2011)). One of a firm's most important assets is its reputation. Philip Kotler writes "In my field of marketing, brand reputation is everything", with Interbrand valuing Apple's brand at $98B (from a market cap of $475B) and Coca-Cola's brand at $79B (from a market cap of $170B). [1] Furthermore, reputation risk, in particular pertaining to product quality and the firm's public perception is a primary concern for boards of directors.[2].

*We have received helpful comments from Andy Atkeson, Heski Bar-Isaac, V. Bhaskar, Alessandro Bonatti, Christian Hellwig, Hugo Hopenhayn, Johannes Hörner, Yuliy Sannikov and seminar audiences at ES Winter Meetings, Gerzensee, IO Theory Conference at Duke, Mannheim Reputation Conference, Princeton, Toronto, SAET, SED, SWET. We gratefully acknowledge financial support from NSF grant 0922321. Keywords: Reputation, Self-esteem, Exit, Lifecycle, Brands, Firm dynamics, Career concerns. JEL: C73, L14

†Department of Economics, UCLA. http://www.econ.ucla.edu/sboard/

‡Department of Economics, UCLA. http://www.econ.ucla.edu/mtv

[1]Source: Interbrand's 2013 Global Brand Survey. The brand values come from measuring a product's demand after controlling for price and product features.

[2]Source: EisnerAmper's 2012 Board of Directors' Survey.

This paper proposes a model of lifecycle dynamics in which a firm's quality and it reputation are its most important assets. We suppose a firm privately chooses its level of investment, while the market and firm learn about the resulting quality. The firm's reputation, which is the market's belief of the firm's quality, therefore lags behind its actual quality, which lags behind its investments. Our analysis yields predictions concerning the distribution of firm revenue product, and the resulting selection within the industry. Consequently, this is a natural lens through which to study the evolution of reputation data, e.g. JD Power scores, Yelp reviews, brand perceptions.

Our model draws inspiration from two canonical models of firm dynamics. In Jovanovic (1982) firms learn about their capabilities over time; In Ericson and Pakes (1995) firms invest in the quality of their products and are subject to idiosyncratic shocks. By combining these forces, we provide a role for reputation to take center stage. Reputation reflects the market's belief about quality, and is therefore very different from traditional capital assets. First, reputation evolves according to Bayes' rule and therefore may be volatile even though the underlying product quality is fairly stable. For example, movie reviews can dramatically change box office numbers while the movie's quality is unchanged. This also means the particulars of the information structure will affect the nature of investment incentives. Second, reputation depends on the market's beliefs about the firm's investment, rather than the actual investment. This is intrinsically an equilibrium object which the firm controls only indirectly, implying that investment incentives are dampened by a moral hazard problem.

In the model, a long-lived firm sells a product of high or low quality to a continuum of identical short-lived consumers. Product quality is a stochastic function of the firm's past investments. Consumers observe neither the firm's investment nor the resulting quality. Rather, they learn about quality via public breakthroughs that can only be produced by a high-quality product; the market's belief that the quality is high, x_t, is called the *reputation* of the firm. The firm also learns about its quality via breakthroughs but, unlike the market, also recalls its past investments; the firm's belief that the quality is high, z_t, is called the *self-esteem* of the firm. In a pure strategy equilibrium, reputation and self-esteem coincide on path. At each point in time, the consumers willingness to pay and the firm's revenue equals its reputation. This reputation changes over time as a function of (a) the equilibrium beliefs of the firm's investments, and (b) market learning via product breakthroughs. The firm can exit the market at any time, and does so when its reputation falls below some threshold. We aim to characterize the firm's optimal investment and exit decisions over its lifecycle. Since the market only sees breakthroughs, we consider recursive equilibria where the state variables are the time since a breakthrough and the self-esteem of the firm.

We first study the firm's optimal strategy given any beliefs of the market. A little investment raises the firm's self-esteem but, since investment is not observed by the market, does not affect reputation. Hence investment incentives are determined by the marginal value of self-esteem. In turn, this can be characterized as the net present values of a stream of reputational dividends. Intuitively, a higher self-esteem raises the chance of a breakthrough that boosts in revenue; these dividends are then discounted by the interest rate, and the rate the increment in self-esteem depreciates.

Next, we turn to equilibrium, in which the market's beliefs are correct, first showing an equilibrium exits. As the firm nears the exit threshold, the firm will always shirk. Intuitively, investment only pays off if it affects quality and the firm realizes this via a breakthrough. When the firm is dt from the end, this event is of order dt^2, and therefore the firm prefers to shirk. The exit threshold itself is determined by an indifference condition where the option value of obtaining a breakthrough offsets the immediate losses in operating profit.

In a pure strategy equilibrium, investment incentives are single peaked in the time since a breakthrough. This means any nontrivial equilibrium is work-shirk or shirk-work-shirk. In the latter, the firm shirks immediately following a breakthrough, works for intermediate levels of reputation and shirks again if its reputation is close to the exit threshold. Intuitively, as the firm's reputation falls the benefit of a breakthrough grows, leading to increasing incentives; however, the firm gets closer to the end point, leading to decreasing incentives. We show that the sum of these two effects is single-peaked. Figure 1 illustrates a simulated pure strategy work-shirk-work equilibrium, showing how reputation quickly declines when the firm is believed to be shirking, and slowly declines when the firm is believed to be working. In a mixed strategy equilibrium, the firm will still shirk near the end point, but reputation may increase, and therefore incentives may not be single-peaked. We also discuss how the single-firm model can be placed within a competitive market.

Finally, we consider two variants of the baseline model (see Figure 1). First, we suppose the market observes the firm's investment, although both the firm and the customers still learn about the firm's quality over time. Here, investment incentives decrease monotonically as time passes without a breakthrough and the firm approaches the exit threshold. When compared to the baseline model, the elimination of moral hazard means that investment increases both a firm's self-esteem and its reputation. In equilibrium, this increases the amount of effort, slows the decline in reputation, delays the exit time and raises the firm's value. Second, we suppose the firm is privately informed about its own quality, on which it can condition its exit and investment choices. In this case, the high quality firm can signal its type by remaining in the market, implying that the low type randomizes over exiting in order to keep reputation constant, as in Bar-Isaac (2003). Investment incentives are then increasing monotonically over time, as the value of a breakthrough increases. Unlike the baseline case, investment incentives stay large close to the exit threshold because the firm can immediately see if investment pays off, and will then choose to remain in the market.

1.1 Literature

The paper embeds the reputation framework of Board and Meyer-ter-Vehn (2013), into a firm lifecycle model. The possibility of exit means that we must keep track of the firm's self-esteem; it also qualitatively changes the investment incentives of a firm. The "known quality" case also draws heavily on Bar-Isaac (2003), although that paper abstracts from any investment decision.

We also contribute to the growing literature on learning models with moral hazard; such models have the feature that private and public beliefs differ off-path. Kovrijnykh (2007) introduces exit

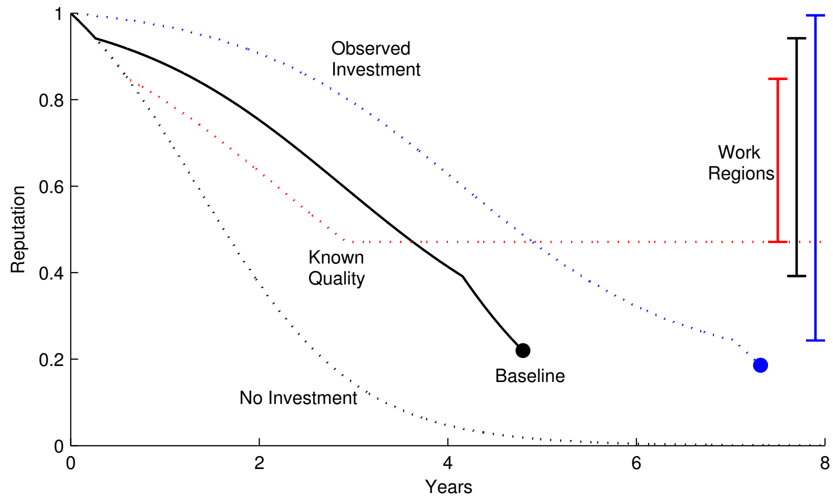

Figure 1: **Reputation Trajectories.** This figure shows the equilibrium trajectory of reputation under the baseline model and the two extensions: where the market observes the firm's investment, and where the firm knows its own quality. It also shows the trajectory of reputation if the firm never invests. The parameters are described in Section 4.

into a three-period career concerns model. Bonatti and Hörner (2011, 2013) consider effort incentives in a strategic experimentation game. Sannikov (2014) considers a contract design problem in which the agent's effort has long-run effects on her employer's performance. Cisternas (2014) analyzes a general model of two-sided learning with moral hazard; his incentive equation is analogous to our "marginal value of self-esteem".

In addition to Jovanovic (1982) and Ericson and Pakes (1995), there are a variety of other models of firm dynamics. Hopenhayn (1992) assumes firm capabilities change over time according to a Markov process, and looks at the resulting entry and exit patterns. Cabral (2014) and Abito, Besanko, and Diermeier (2012) consider reduced form models of reputational firm dynamics, whereby reputation is modeled as a state variable akin to capital stock, but is not derived from Bayes' rule. Gale and Rosenthal (1994) and Rob and Fishman (2005) consider the dynamics of repeated games equilibria where incentives arise from punishment strategies. On the empirical side, Foster, Haltiwanger, and Syverson (2013) model the slow demand growth of new entrants by assuming the level of current demand depends on the stock of past demand, which the authors interpret as the "growth of customer base or building a reputation". Bronnenberg, Dubé, and Gentzkow (2012) study the dynamics of brand shares when customers move between cities via a model of brand capital, where customers' preferences depend on their past purchases.

2 The Baseline Model

Players and actions: Time $t \in [0, \infty)$ is continuous. At every time t the firm chooses an *investment* level $A_t \in [0, \overline{a}]$ where $\overline{a} < 1$; it may also choose to exit the market, thereby ending the game.

Following Holmström (1999) and Mailath and Samuelson (2001), consumers are assumed to purchase the firm's single unit of output at a price equal to their willingness to pay.

At time t the firm's product *quality* is $\theta_t \in \{L, H\}$, where $L = 0$ and $H = 1$. Initial quality θ_0 is exogenous; subsequent quality depends on investment and technology shocks. Specifically, shocks are generated according to a Poisson process with arrival rate $\lambda > 0$. Quality θ_t is constant between shocks, and determined by the firm's investment at the most recent technology shock $s \leq t$; i.e., $\theta_t = \theta_s$ and $\Pr(\theta_s = H) = A_s$. This captures the idea that quality is a lagged function of past investments.

Information: Consumers observe neither quality nor investment, but learn about quality through public *breakthroughs*. Given quality θ, breakthroughs are generated according to a Poisson process with arrival rate $\mu\theta$. We write h^{t-} for histories of breakthrough arrival times before time t, h for infinite histories, and $\varnothing$ for histories with no breakthroughs.

The firm does not observe product quality either, but does recall its past actions. It chooses an investment plan $A = \{A_t\}_{t\geq 0}$ and an exit time $T \in [0, \infty)$ that is predictable with respect to the associated filtration; intuitively, $A_t = A_t(h^{t-})$. From the firm's perspective, investment A controls the distribution of quality $\{\theta_t\}_{t\geq 0}$ and thereby the histories of breakthroughs h; we write $\mathbb{E}^A$ for expectations under this measure and call $Z_t = \mathbb{E}^A\left[\theta_t | h^t\right]$ the firm's *self-esteem* at time $t < T$. This reflects the firm's belief of its own quality given its past actions and the history of breakthroughs.

We write pure market beliefs over investment and exit as $\tilde{A}$ and $\tilde{T}$ and mixed beliefs as distributions over beliefs $F = F(\tilde{A}, \tilde{T})$. If market beliefs are focused on a unique strategy $\tilde{A}, \tilde{T}$ its belief about quality, the firm's *reputation*, is given by $X_t := \mathbb{E}^{\tilde{A}}[\theta_t | h^t]$ as long as $t < \tilde{T}$. If the market holds mixed beliefs $F(\tilde{A}, \tilde{T})$ it faces an additional layer of uncertainty. Such F induces a joint distribution over $\{\theta_t\}_{t\geq 0}$, h, and exit times $\tilde{T}$; writing $\mathbb{E}^F$ for expectations under this measure, the firm's reputation is given by $X_t = \mathbb{E}^F[\theta_t | h^t, \tilde{T} > t]$ for all $t < T(F)$ where $T(F) := \min\{t : F(\tilde{T} \leq t) = 1\}$ is the first time at which the market expects the firm to exit with certainty. After the exit time, $t \geq T(F)$, the market revises its beliefs about the firm strategy to some arbitrary $F'(\tilde{A}, \tilde{T})$ with $t < T(F')$ and reputation equals $X_t = \mathbb{E}^{F'}[\theta_t | h^t, \tilde{T} > t]$.

Payoffs: The firm and consumers are risk-neutral and discount future payoffs at rate $r > 0$. At time t, the firm produces one unit with flow value θ_t. Given the public information h^{t-}, consumers' willingness to pay then equals the firm's reputation X_t. We assume that the price equals the willingness to pay, so consumers' expected utility is 0. Investment has a constant marginal flow cost of $c > 0$ and the firm's operating costs equal $k \in (0, 1)$. The firm's flow profits are thus given by $X_t - k - cA_t$.

Given the firm's strategy A, T and the market's belief about this strategy $F(\tilde{A}, \tilde{T})$, the firm's expected present value equals

$$\mathbb{E}^A\left[\int_{t=0}^{T} e^{-rt}(X_t - k - cA_t)dt\right]. \tag{2.1}$$

Market beliefs determine the firm's revenue $X_t = \mathbb{E}^F[\theta_t | h^t, \tilde{T} > t]$ for a given history h, while actual investment A determines the distribution over histories of breakthroughs h, and the exit time T determines the integration domain.

To make the analysis interesting we assume throughout the paper that

$$z^\dagger := \lambda/\mu < 1 \quad \text{and} \quad z^\dagger - k + \mu z^\dagger(1-k)/r < 0. \tag{2.2}$$

This assumption ensures that, in the absence of a breakthrough, the firm's reputation declines and the firm eventually exits.

2.1 Recursive Strategies

Both reputation and self-esteem are reset to $X = Z = 1$ at a breakthrough; in between breakthroughs the market observes no information about the firm's performance. For this reason, we consider recursive strategies which only depend on the time since the last breakthrough. Formally, we call strategy $\{A_t\}, T$ *recursive* if there exists $\{a_t\}$ with $a_t \in [0, \overline{a}]$ and $\tau \in [0, \infty]$ such that if the last breakthrough before t was at $s < t$, then $A_t = a_{t-s}$ and $T \leq t$ iff $\tau \leq t - s$. We write recursive strategies as $\{a_t\}, \tau$ and the resulting self-esteem as $\{z_t\}$, where $z_0 = 1$. Similarly we call beliefs F recursive if they assign probability one to recursive strategies $\{\tilde{a}_t\}, \tilde{\tau}$, and denote the induced reputation by $x_t = \mathbb{E}^F[\theta_t | h^t = \varnothing, \tilde{\tau} > t]$, where $x_0 = 1$.

Self-esteem evolves according to Bayes' rule. At a breakthrough, self-esteem jumps to one. Absent a breakthrough, self-esteem is governed by $\dot{z}_t = g(a_t, z_t)$ where the drift g is given by

$$g(a, z) = \lambda(a - z) - \mu z(1 - z). \tag{2.3}$$

The first term derives from the technology process: with probability λdt a technology shock hits in $[t, t + dt)$, previous quality becomes obsolete, and the current quality is determined by the firm's investment. This term is positive if investment a_t exceeds the firm's self-esteem z_t and negative otherwise. The second term derives from the absence of breakthroughs.

Given recursive reputation $\{x_t\}$, we can restrict the firm's problem to recursive strategies. From the firm's perspective breakthroughs arrive at rate μz_t, so truncating the integral in (2.1) at the first breakthrough the firm's continuation value at time t is given by

$$V(t, z_t) = \sup_{\{a_s\}_{s \geq t}, \tau} \int_{s=t}^{\tau} e^{-\int_t^s (r + \mu z_u) du} (x_s - k - c a_s + \mu z_s V(0, 1)) ds. \tag{2.4}$$

We write optimal recursive strategies as $\{a_t^*\}, \tau^*$ and the associated self-esteem as $\{z_t^*\}$.

Lemma 1 (Existence of Optimal Strategy) *For any reputation $\{x_t\}$, an optimal strategy $\{a_t^*\}, \tau^*$ maximizing (2.4) exists. Furthermore, there exists $\overline{\tau} < \infty$ such that for all $\{x_t\}$ and optimal strategies $\{a_t^*\}, \tau^*$, we have $\tau^* \leq \overline{\tau}$.*

Proof. We prove this lemma by defining a topology on the space of strategies $\{a_t\}, \tau$ such that the space is compact and the objective function in (2.4), is continuous in $\{a_t\}, \tau$ in this topology.

For the compactness argument, we first argue the second part of the Lemma, that there exists $\overline{\tau} < \infty$ at which the firm exits for any reputational trajectory $\{x_t\}$ induced by beliefs F. Given $\overline{a} < 1$ and assumption (2.2), reputational drift $g(a, z)$ is negative and bounded away from zero for $z \in [z^\dagger, 1]$ and any $a \in [0, \overline{a}]$. Write $z_t(\{a_t\})$ for the solution of $\dot{z}_t = g(a_t, z_t)$ for fixed investment strategy $\{a_t\}$ and define $\overline{\tau}$ such that $z_{\overline{\tau}}(\{\overline{a}\}) = z^\dagger$. Hence $z_t(\{a_t\}) \leq z^\dagger$ for all $t \geq \bar{\tau}$ and any $\{a_t\}$. By the law of iterated expectations, reputation $x_t = \mathbb{E}^F[z_t(\{a_t\})|t < \tilde{\tau}, h^t = \varnothing]$ is a conditional expectation over self-esteem and is thus bounded by $x_t \leq z_t(\bar{a})$ both on-path and off-path. Thus, $x_t \leq z^\dagger$ for all $t \geq \bar{\tau}$. Equation (2.1) implies $V(0, 1) < (1 - k)/r$, and so the integrand in firm value (2.4) is bounded above by $z^\dagger - k + \mu z^\dagger (1 - k)/r$, which is negative by assumption (2.2). Hence the firm exits by time $\bar{\tau}$, as required.

To prove the existence of an optimal strategy, let B be the space of measurable investment functions $\{a_t\}_{t\in[0,\overline{\tau}]}$. B is naturally embedded in the (rescaled) unit ball of $L^2([0, \overline{\tau}], \mathbb{R})$. In the weak topology this unit ball is compact by Alaoglu's theorem, and as a closed subset of this unit ball, B is also compact. In this topology a sequence $\{a_t^n\}$ converges to $\{a_t\}$ if $\int_0^{\overline{\tau}} (a_t^n - a_t)\xi_t dt \to 0$ for all test functions $\xi \in L^2([0, \overline{\tau}], \mathbb{R})$. While this topology is coarse enough to make B compact, it is fine enough for the trajectory $\{z_t\}_{t\in[0,\overline{\tau}]}$ to be continuous (in the sup-norm) in $\{a_t\}_{t\in[0,\overline{\tau}]}$ (see Davis (1993, Theorem 43.5)). Thus, firm value (2.4) is continuous in $\{a_t\}, \tau$ and is maximized by some $\{a_t^*\}, \tau^*$. $\square$

Remarks: We focus on learning based on breakthroughs that reveal high quality with certainty, termed 'perfect good news learning' in Board and Meyer-ter-Vehn (2013), for two reasons. First, it is tractable because it makes the model recursive in the time since the last breakthrough. Second, it precludes downward jumps in self-esteem and reputation, allowing us to study investment when exit is imminent (in comparison, under 'perfect bad news', exit would follow a breakdown). Our general approach also applies to other learning processes based on Brownian motion or imperfect Poisson signals. While such learning processes complicate our qualitative analysis significantly, our numerical strategy based on the firm's optimal investment condition (3.1) remains valid.[3]

3 Firm's Problem

In this section we analyze the firm's optimal strategy, $\{a_t^*\}, \tau^*$ for arbitrary (recursive) beliefs, inducing reputation $\{x_t\}$. First, we characterize the firm's investment incentives as the integral of a series of dividends that result from having higher self-esteem. Second, we explore the qualitative properties of the firm's strategy, showing that the firm shirks when it is close to bankruptcy and,

[3]As there is no "time since the last breakthrough t" for such learning processes, the natural Markovian state variables are reputation x_t and self-esteem z_t. These states capture all payoff-relevant information if the market holds point beliefs about investment, e.g. in a pure strategy equilibrium. Otherwise, if the market has mixed beliefs about investment, x_t is not a sufficient statistic for firm beliefs and one would have to keep track of the market's belief about z_t, that is, the market's second-order belief about quality.

if reputation is decreasing over time, investment incentives are single-peaked. In Section 4 we close the model in equilibrium by assuming that $\{x_t\}$ is derived from correct market beliefs $F(\tilde{a}, \tilde{\tau})$.

Investment does not directly affect reputation, but raises the firm's self-esteem and thereby raises the probability of breakthroughs that do boost reputation and revenue. Using (2.3), a small increase in investment raises self-esteem by λ. Assuming that the value function V is differentiable with respect to z, the marginal benefit of investing at time t is $\lambda V_z(t, z_t^*)$; thus optimal investment must satisfy

$$a_t^* = \begin{cases} 0 & \text{if} \quad \lambda V_z(t, z_t^*) < c, \\ \bar{a} & \text{if} \quad \lambda V_z(t, z_t^*) > c. \end{cases} \tag{3.1}$$

Next, observe that the firm's value $V(t, z)$ is convex in z. This follows because z is the firm's private belief about its quality, and the value of information is convex since firms with extreme values of z can attain the same average value as a firm with moderate z by mimicking its strategy.[4] This result has two implications. First, even where it is not differentiable, the value function admits directional derivatives $V_z(t, z-), V_z(t, z+)$. Second, investment today and investment in the future are dynamic complements. Intuitively, investment today raises the firm's self-esteem z_{t+dt} and life expectancy; this raises the marginal benefit of self-esteem and investment tomorrow since the firm has more chance of benefiting from the resulting breakthroughs. This strategic complementarity is in contrast to the strategic substitutability in Bonatti and Hörner (2011). There, a player who exerts more effort today is more pessimistic about the state of the project tomorrow when his effort fails to result in the desired breakthrough. Here, to the contrary, investment today makes the firm more optimistic about its prospects tomorrow.

We next express the marginal value of self-esteem in terms of future reputational dividends. This expression is the work-horse of our paper, helping us study optimal investment.

Lemma 2 (Marginal Value of Self-Esteem) *If $V_z(t, z_t^*)$ exists, it equals*

$$\Gamma(t) := \int_t^{\tau^*} e^{-\int_t^s r+\lambda+\mu(1-z_u^*)du} \mu(V(0,1) - V(s, z_s^*))ds. \tag{3.2}$$

More generally $V_z(t, z_t^*-) \leq \Gamma(t) \leq V_z(t, z_t^*+)$.

Proof. This follows by applying the envelope theorem to a variant of (2.4). See Appendix A.1. □

Equation (3.2) is an integral version of the adjoint equation for the firm's control problem. Economically, self-esteem raises the probability of a breakthrough and, since it is persistent, pays off dividends over time. That is, incremental self-esteem dz raises the probability of a breakthrough by $\mu dz dt$; the value of a breakthrough equals $V(0, 1) - V(t, z_t^*)$. We thus call the integrand $\mu(V(0, 1) - V(s, z_s^*))$ the *reputational dividend of self-esteem.* The dividend stream from the increment dz depreciates for three reasons. First, time discounting at rate r. Second, at rate μz_t^* a

[4] More formally, let $\{a_t^*\}, \tau^*$ be an optimal strategy for a firm with self-esteem z at time t. If two neighboring firms, high and low, with initial states $(x, z + \varepsilon)$ and $(x, z - \varepsilon)$ both mimic strategy $\{a_t^*\}, \tau^*$, then their average expected payoff equals $V(t, z)$. Since these firms can raise their value by reoptimizing, V is weakly convex.

breakthrough arrives, self-esteem jumps to one, and the increment disappears. Third, reputational drift (2.3) is not constant in z, but its derivative equals $g_z(a_t^*, z_t^*) = -(\lambda + \mu(1 - 2z_t^*))$. Summing these yields the discounting term in (3.2).

If $V(t, z_t^*)$ is differentiable in z, then it coincides with $\Gamma(t)$. If there are multiple optimal strategies $\{a_s^*\}_{s \geq t}, \tau^*$ for which $\Gamma(t)$ does not coincide, then V is not differentiable at (t, z_t^*). However, all the different expressions in (3.2) are well-defined and are bounded by the directional derivatives of V. Hence if $\lambda\Gamma(t) > c$ then $\lambda V_z(t, z_t^*+) > c$ and the firm finds it profitable to work, whereas if $\lambda\Gamma(t) < c$ then $\lambda V_z(t, z_t^*-) < c$ and the firm finds it profitable to shirk. This implies:

Lemma 3 (Optimal Investment) *Given $\{x_t\}$, any optimal strategy $\{a_t^*\}, \tau^*$ satisfies*

$$a_t^* = \begin{cases} 0 & \text{if } \lambda\Gamma(t) < c \\ \overline{a} & \text{if } \lambda\Gamma(t) > c \end{cases} \tag{3.3}$$

for almost all t.

Fixing a candidate strategy $\{a_t\}, \tau$, equation (3.3) gives a necessary condition for a best response. However, this is not sufficient since this approach is the continuous-time analogue to checking only "one-step deviations on path". Since actions are dynamic complements, the possibility of multi-stage deviations must be taken seriously.

With these preliminary results, we can characterize the qualitative properties of the equilibrium.

Theorem 1 (Shirk at End) *Given $\{x_t\}$ and any optimal strategy $\{a_t^*\}, \tau^*$, there exists $\varepsilon > 0$ such that $a_t^* = 0$ for almost all $t \in [\tau^* - \varepsilon, \tau^*]$.*

Proof. Lemma 2 implies that $\Gamma(t)$ is of order $O(\tau^* - t)$, and so $\lambda\Gamma(t) < c$ for $t \in [\tau^* - \varepsilon, \tau^*]$. The Theorem thus follows from Lemma 3. $\square$

Intuitively, investment incentives vanish at the exit time τ^* because there is no time left for the investment to pay off. More formally, the benefit of investment is of second order because both a technology shock and a breakthrough must arrive in the remaining time interval for the investment to avert exit.

When the firm is close to exit it will therefore cease to invest, accelerating the its demise. For example, in the beer industry, Goldfarb (2007) argues that Schlitz realized that the rise of Miller would have a large impact on its future profitability. The firm therefore disinvested in the brand, switching to lower quality accelerated batch fermentation, fired much of its marketing team, and changed the preservatives which lead to green flakes in the beer. More generally, this implies that the death of firms should be a quick process, with reputation quickly declining (this will contrast with the case of known quality in Section 5.2).

Next, we assume that reputation $\{x_t\}$ strictly decreases in time. This assumption means the longer the firm fails to prove itself by generating a breakthrough, the more pessimistic the market becomes about its product quality. This is satisfied if market beliefs $F(\tilde{a}, \tilde{\tau})$ are focused on a single strategy $\tilde{a}, \tilde{\tau}$ and the market draws no positive inference about $\{\tilde{a}_t\}_{t \leq \tilde{\tau}}$ in the off-path event that the firm fails to exit at time $\tilde{\tau}$.[5]

When $\{x_t\}$ strictly decreases, the firm's value $V(t, z)$ strictly decreases in t and strictly increases in z. Value is increasing in self-esteem because a firm with high self-esteem can mimic a firm with low self-esteem, yielding the same revenue prior to a breakthrough, but a higher probability of a breakthrough. Similarly, a low-t firm can mimic the strategy of a high-t firm, yielding the same probability of a breakthrough, but higher revenue prior to a breakthrough.[6]

The next Lemma computes the partial derivative $V_t(t, z_t^*)$. We then use this in Theorem 2 to show that investment incentives are single-peaked.

Lemma 4 (Evolution of Firm Value) *Assume that $\{x_t\}$ strictly decreases. Whenever the partial derivative $V_t(t, z_t^*)$ exists, it is equal to*

$$\Psi(t) := \int_t^{\tau^*} e^{-\int_t^s r + \mu z_u^* du} dx_s \tag{3.4}$$

Moreover, $\Psi(t) < 0$ for $t < \tau^*$.

Proof. Rewrite the firm's continuation value (2.4) by letting $\sigma = s - t$ be the time since t and $\{a_\sigma^*\}, \zeta^*$ for the optimal strategy starting at t,

$$V(t, z_t^*) = \int_{\sigma=0}^{\zeta^*} e^{-\int_0^\sigma (r + \mu z_{t+u}^*) du} (x_{t+\sigma} - k - c a_\sigma^* + \mu z_{t+\sigma}^* V(0, 1)) d\sigma$$

As $z_{t+\sigma}^*$ is determined by initial self-esteem z_t^* and $\{a_\sigma^*\}$, it is independent of t. The envelope theorem thus yields (3.4). As $\{x_t\}$ is assumed to strictly decrease, $\Psi(t)$ must be negative. □

Theorem 2 (Single-Peaked Incentives) *If $\{x_t\}$ strictly decreases, then investment incentives $\Gamma(t)$ are single-peaked with boundary values $\Gamma(0) > 0$, $\dot{\Gamma}(0) > 0$ and $\Gamma(\tau^*) = 0$.*

Proof. Taking the derivative of investment incentives (3.2) and setting $\rho(t) := r + \lambda + \mu(1 - z_t^*)$ yields the adjoint equation

$$\dot{\Gamma}(t) = \rho(t)\Gamma(t) - \mu(V(0, 1) - V(t, z_t^*)). \tag{3.5}$$

Now assume that $\rho(t)$ and $V(t, z_t^*)$ are differentiable; then $\dot{\rho}(t) = -\mu \dot{z}_t^*$ and $\frac{d}{dt} V(t, z_t^*) = \dot{z}_t^* \Gamma(t) + \Psi(t)$; in Appendix A.3 we show that these functions are indeed absolutely continuous and extend

[5] This assumption is not satisfied in a mixed strategy equilibrium, where reputation will rise when the low-investment firm exits (see Section 4.2).

[6] See Appendix A.2 for a formal argument.

our arguments to that case. The derivative of the adjoint equation equals

$$\begin{aligned}\ddot{\Gamma}(t) &= \rho(t)\dot{\Gamma}(t) + \dot{\rho}(t)\Gamma(t) - (-\mu\frac{d}{dt}V(t, z_t^*)) \\ &= \rho(t)\dot{\Gamma}(t) - \mu\dot{z}_t^*\Gamma(t) + \mu\dot{z}_t^*\Gamma(t) + \Psi(t) \\ &= \rho(t)\dot{\Gamma}(t) + \Psi(t)\end{aligned} \tag{3.6}$$

Since $\Psi(t) < 0$, $\dot{\Gamma}(t) = 0$ implies $\ddot{\Gamma}(t) < 0$, and $\Gamma(t)$ is single-peaked.

At $t = \tau^*$, equation (3.2) immediately implies that $\Gamma(\tau^*) = 0$. At $t = 0$, equation (3.2) implies $\Gamma(0) > 0$ because the integrand $\mu(V(0, 1) - V(s, z_s^*))$ are strictly positive for $s > 0$ and $\tau^* > 0$. Equation (3.6) then implies that $\dot{\Gamma}(0) = \rho(0)\Gamma(0) > 0$. □

Theorem 2 implies that the optimal strategy takes one of three forms:

1. Full-shirk. The firm shirks for almost all t.[7]
2. Work-shirk-work. The firm shirks immediately following a breakthrough, then works for a while, and shirks near the exit time.
3. Work-shirk. The firm works after a breakthrough, but shirks when they are close to exit.

Intuitively, the evolution of investment incentives is shaped by two countervailing forces. On the downside, as t increases the firm forgoes the reputational dividends over $[t, t + dt]$. This effect is captured by the second term in (3.5). This negative effect becomes more important over time as the reputational dividend increases; this is captured by the positive term $-\mu\frac{d}{dt}V(t, z_t^*)$ in (3.6). On the upside, an increase in t brings future and larger dividends closer, as captured by the first term in (3.5). Ignoring the time dependence of $\rho(t)$, this positive effect becomes less important over time once incentives start decreasing. Thus, once incentives decrease, the negative effect keeps growing while the positive effect decreases and so incentives decrease until the end.

Theorem 2 is a surprisingly robust result. First, much of the analysis would be identical for convex cost function $c(a)$; then single-peaked incentives would translate into single-peaked investment. Second, the result only requires that payoffs $\{x_t\}$ decrease over time so holds if, say, high-reputation firms make more sales, implying revenue is convex in reputation. Third, the result applies to the firm's optimal strategy, treating payoffs $\{x_t\}$ as an exogenous process, and therefore does not require that market beliefs be correct.

Turning to the firm's exit behavior, we next assume that reputation $\{x_t\}$ is continuous. In equilibrium, reputation $\{x_t\}$ must be continuous on-path. Off-path reputation is continuous if the market does not draw inferences about previous investment from a failure to exit.

[7] The firm can change its strategy at a measure zero set of times without affecting payoffs, so these statements about investment only hold almost everywhere. This qualification can be eliminated by restricting the firm to forward-continuous strategies (see Board and Meyer-ter-Vehn (2013)).

Theorem 3 (Exit Time) *If $\{x_t\}$ is continuous, the optimal exit time satisfies*

$$x_{\tau^*} - k + \mu z_{\tau^*} V(0,1) = 0. \tag{3.7}$$

Proof. Recall that the firm's value is given by (2.4). When the firm shirks, its flow payoff equals $x_t - k$ and its option value of staying in the market has a flow value of $\mu z_t^* V(0,1)$. Thus, if $x_t - k + \mu z_t^* V(0,1) > 0$ then the firm can secure itself strictly positive payoffs by shirking and staying in the market until (3.7) holds. Conversely, if $x_{\tau^*} - k + \mu z_{\tau^*}^* V(0,1) < 0$ then the continuity of $\{x_t\}$ implies this inequality also holds for t just before τ^*, and the firm would have been better off exiting a little earlier. □

At the end of its life the firm's flow profits $x_t - k$ are negative but it remains in the market for the option value of a last-minute breakthrough that boosts its reputation and self-esteem to one. Over time, losses grow and the option value diminishes, and the firm exits when they exactly offset each other.

The exit condition (3.7) implies that $V(t,z)$ is strictly convex in z on $\{(t,z) : V(t,z) > 0\}$. This follows because, in the argument in footnote 4, a high/low self-esteem firm mimicking a firm with intermediate self-esteem can strictly increase its profits by exiting later/earlier. Strict convexity and the investment condition (3.1) imply the existence of a threshold $z(t)$ with the property that the firm (almost always) invests when $z_t^* > z(t)$ and disinvests when $z_t^* < z(t)$. Strict convexity also implies that best responses are strictly ordered in the following sense: Let $\{a_t^+\}, \tau^+$ and $\{a_t^-\}, \tau^-$ be optimal strategies and $\{z_t^+\}, \{z_t^-\}$ the associated trajectories and assume that $z_t^+ > z_t^-$ for some $t < \min\{\tau^+, \tau^-\}$. Then $a_t^+ \geq a_t^-$ for almost all $s > t$, $z_s^+ > z_s^-$ for all $s > t$, and $\tau^+ > \tau^-$.

4 Equilibrium Analysis

So far we have studied the firm's optimal strategy for arbitrary beliefs $F = F(\tilde{a}, \tilde{\tau})$ and associated revenue trajectories $\{x_t\}$. In this section, we close the model by using the rationality of market beliefs. To do this, it will be useful to have a more explicit equation for reputation, x_t. Breakthroughs arrive with intensity $\mu z_s(\tilde{a})$, so the probability of no breakthrough before time t equals $w_t(\{a_t\}) := \exp(-\mu \int_0^t z_s(\tilde{a}) ds)$. Bayes' rule then implies

$$x_t = \frac{\mathbb{E}^F \left[z_t(\tilde{a}) w_t(\tilde{a}) \mathbb{I}_{\{t<\tilde{\tau}\}} \right]}{\mathbb{E}^F \left[w_t(\tilde{a}) \mathbb{I}_{\{t<\tilde{\tau}\}} \right]} \tag{4.1}$$

for $t < \tau(F)$.

Definition: An *equilibrium* consists of a distribution over (recursive) investment and exit strategies $F = F(\{a_t\}, \tau) \in \Delta(B \times [0, \bar{\tau}])$ and a (recursive) revenue trajectory $\{x_t\}_{t \in [0, \bar{\tau}]} \in B$ such that:
(a) Given $\{x_t\}$, any strategy $\{a_t\}, \tau$ in the support of F solves the firm's problem (2.4).
(b) Reputation $\{x_t\}$ is derived from F by Bayes' rule via (4.1) for $t < \tau(F)$.

One should note that this definition does not impose sequential optimality of strategy $\{a_t\}, \tau$ and thus corresponds to Nash equilibrium rather than sequential equilibrium. However, this is merely for notational convenience: as the firm's investment is unobservable, deviations do not affect beliefs and revenue. Thus, any equilibrium is outcome-equivalent to a sequential equilibrium. In fact, all of the analysis in the last section starting at on-path at states t, z_t^* extends immediately to optimal strategies starting at any state t, z.

Theorem 4 (Existence) *An equilibrium exists.*

Proof. See Appendix A.4. □

The proof of Theorem 4 applies the Kakutani-Fan-Glicksberg Theorem to the best-response correspondence, mapping revenue $\{x_t\}$ to optimal strategies $\{a_t^*\}, \tau^*$, and the Bayesian updating correspondence (4.1), mapping mixed strategies $F(\tilde{a}, \tilde{\tau})$ to revenue $\{x_t\}$.[8] The key step in the proof is to define the appropriate weak topology that renders the strategy space compact and the two correspondences continuous.[9]

4.1 Pure Strategy Equilibria

In a pure strategy equilibrium the market is certain of the firm's strategy $\{a_t^*\}, \tau^*$ and so $x_t = z_t^*$ for all $t < \tau^*$. Thus $\{x_t\}$ decreases and investment incentives $\Gamma(t)$ are single-peaked with $\Gamma(0) > 0$ and $\Gamma(\tau^*) = 0$ (Theorem 2).

Equilibrium investment behavior depends on the level of the investment cost, c. If the cost c is high, the firm always wishes to shirk. If the cost c is intermediate, initial incentives $\lambda\Gamma(0)$ is insufficient to motivate effort, and any equilibrium is shirk-work-shirk. After a breakthrough, such a firm rests on its laurels because it has little to gain from an additional breakthrough; as its reputation and self-esteem drop, it starts investing and works hard for its survival, but eventually gives up and shirks before exiting the market. Finally, if the cost c is small, then any equilibrium is work-shirk. This has the flavor of a probationary equilibrium where the market assumes a firm invests for a fixed period of time after each breakthrough, but then grows suspicious.[10]

The incentives at high reputations depend critically on the level of $\overline{a}$. As $\overline{a} \to 1$, investment at $t \approx 0$ is impossible to sustain in equilibrium because with such market beliefs, reputation would remain close to 1 and dividends would remain small forever, undermining investment incentives. This same force is seen in Mailath and Samuelson (2001).

[8]The latter correspondence is multi-valued because x_t can take any value in $[z_t(0), z_t(\overline{a})]$ for $t \geq \tau(F)$.

[9]Unfortunately, we cannot prove the existence of a pure strategy equilibrium. Given Theorem 2, we know that if costs are small any equilibrium is work-shirk. Hence the natural approach is to pick a belief cutoff t and map this into a optimal cutoff for the firm $t^*(t)$ and find a fixed point. Since actions are dynamic complements, we cannot rule out the possibility that this relationship is discontinuous.

[10]Formally, if $\overline{a}$ is large and a pure strategy equilibrium exists for all parameter values of $c > 0$, then there exist parameter values $c < c' < c''$ such that equilibrium investment $\{a_t^*\}$ must be work-shirk for low costs c, shirk-work-shirk for intermediate costs c', and shirk for high costs c''.

Figures 2-4 illustrate a pure strategy equilibrium. This simulation considers a restaurant that has revenues of $\$x$ million a year, capital cost of $k = \$500,000$, investment cost of $c = \$125,000$ and an interest rate of $r = 20\%$ (incorporating a risk premium). Good news arrives when the restaurant is written up in the local paper; we set $\mu = 1$, so a good restaurant is reviewed positively once a year on average. Finally, we set $\lambda = 0.2$, so a technology shock arrives every 5 years on average. In these figures we replace the firm's state variable t with its time-t reputation x_t to aid comparison with the models in Section 5.[11]

The pure strategy equilibrium is shirk-work-shirk, exhibiting work on $x \in [0.39, 0.94]$ with a exit threshold of $x^e = 0.22$. In Figure 2, the left panel shows the work region and the resulting equilibrium value function. The value function exhibits kinks at the edges of the work region, but smooth pasting at x^e. The right panel shows the distribution of surviving firms' reputation after 10 years, if all firms start at $x = z = 1$. This shows how firms tend to bunch in the work region, where drift is relatively slow. In Figure 3, the left panel shows the investment incentives $V_z(x, z)$ on the entire state-space (x, z). One can see that V_z is increasing in z, illustrating the convexity of the value function; it is also single-peaked in x, illustrating how incentives are low when reputation is high and when near the exit point. The right panel shows the incentives along the equilibrium path where $x = z$, which coincides with the 45° line in the left panel. These are clearly single-peaked, as shown in Theorem 2. Finally, Figure 4 shows three typical 10-year lifecycles for firms starting at $x = z = 1$. The left and center firms survive the 10-year period, experiencing 11 and 10 breakthroughs respectively; the right firm exits after 6 years after only a single breakthrough.

For empirical implementation, one may be concerned about the realism of having a large number of firms close to the upper end of the distribution, as in the right panel of Figure 2. With restaurants, one can view this as representing the degree of occupancy, say, on a weekend evening. If one is interested in explaining the size distribution of firms, there are a number of natural model extensions. First, as the mass of firms at the top results from our stylized breakthrough learning process, one could consider other learning processes as discussed at the end of Section 2. Second, one could relax the linear relationship between revenue and reputation. In competitive industries this relationship is convex, and so revenue has a thinner right tail than reputation. Third, one could append the model with regular capital in addition to reputational capital, so a restaurant with great reviews would need time to expand the franchise.

4.2 Mixed Strategy Equilibria

The model may potentially exhibit mixed strategy equilibria, as illustrated in Figure 5. In this picture, the work region is the area above some function $z(t)$. There are then two optimal trajectories of self-esteem for the firm: the lower path is full-shirk, while the upper path is shirk-work-shirk. The dynamic complementarity of the actions means that the firm that works when the paths divide then

[11] We numerically solve for equilibrium by fixing a candidate strategy of the firm a_t, calculating the resulting payoffs for the firm and verifying that the action induced by the first-order condition (3.1) coincides with the candidate strategy. This approach is valid using a standard verification argument (e.g. Davis (1993, Theorem 45.16)). We calculate one equilibrium but, since we have not shown uniqueness, there may be others.

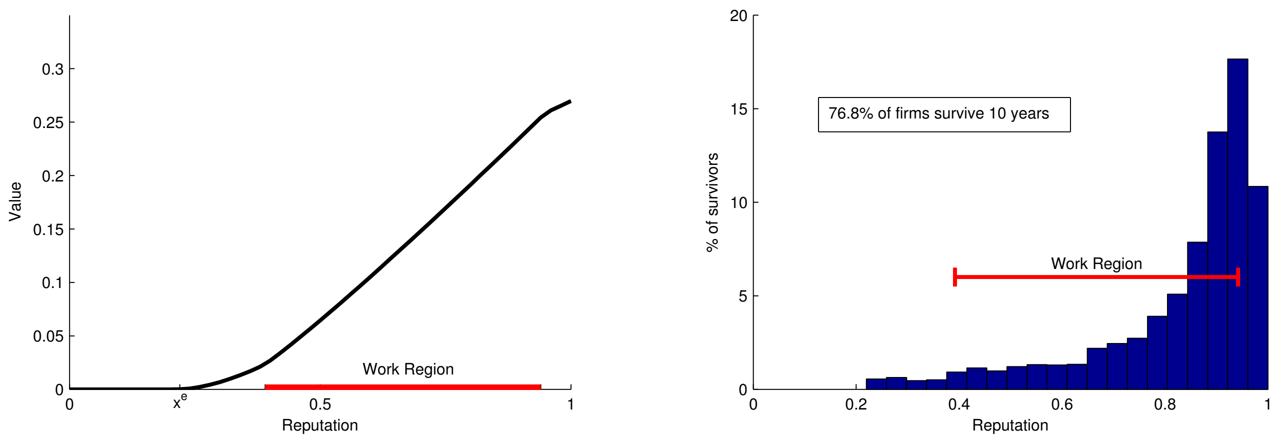

Figure 2: **Value Function and Distribution of Firms.** The **left** panel shows the firm's value as a function of the firm's reputation. The **right** panel shows the resulting distribution of firms after 10 years, assuming all firms start at $x = z = 1$. This figure assumes capital cost $k = 0.5$, interest rate $r = 0.2$, maximum effort $\overline{a} = 0.9$, investment cost $c = 0.125$, technology shocks $\lambda = 0.2$ and breakthrough rate $\mu = 1$.

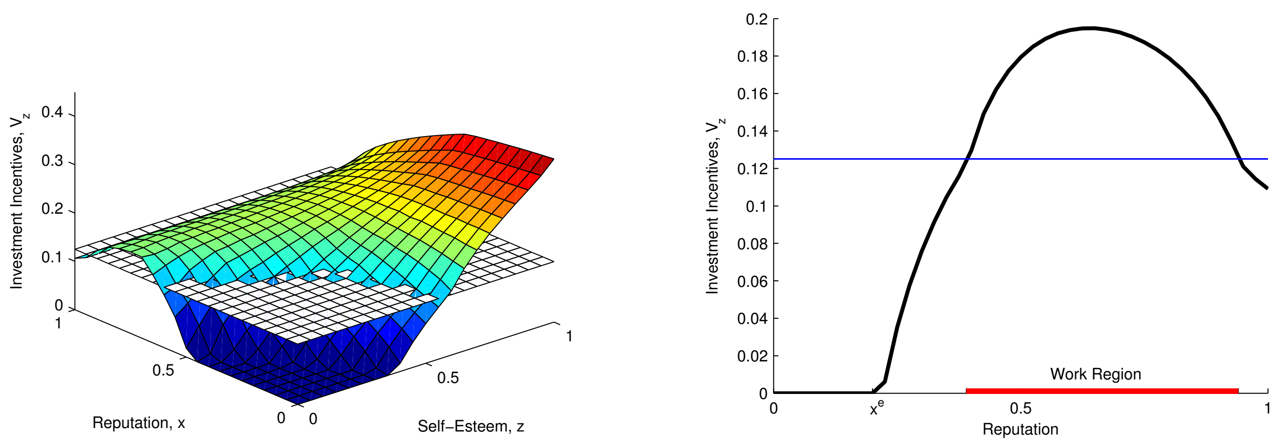

Figure 3: **Investment Incentives.** The **left** panel shows the firm's investment incentives $V_z(x_t, z_t)$ as a function of reputation x_t and self-esteem z_t. The **right** panel shows the investment incentives along the equilibrium path, where $x_t = z_t$. The parameters are the same as Figure 2.

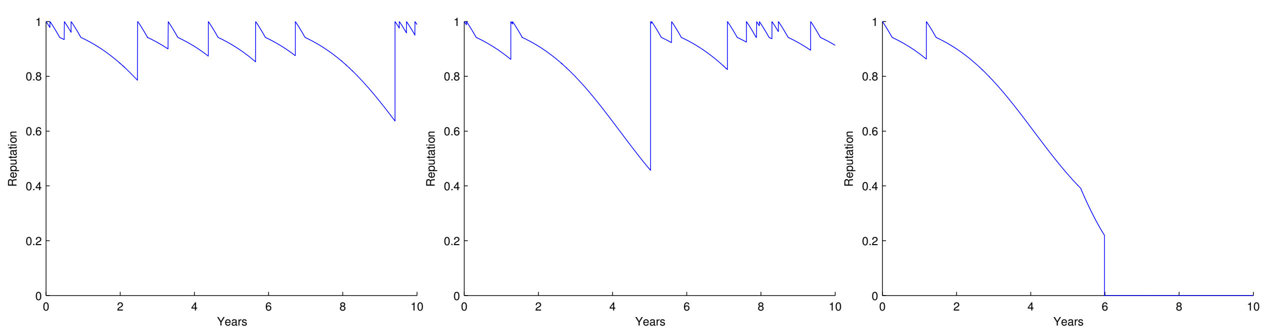

Figure 4: **Firm Lifecycles** This figure shows three sample paths starting at $x = z = 1$. The **left** and **center** firms survive 10 years, the **right** firm exits after 6 years. The parameters are the same as Figure 2.

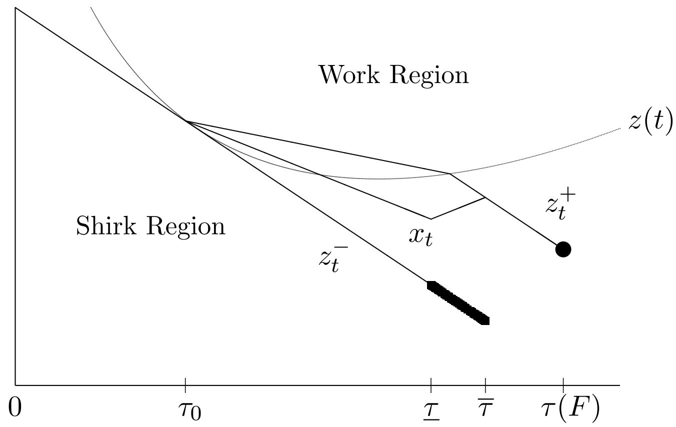

Figure 5: **Mixed Strategy Equilibrium.** This picture illustrates the qualitative features of a mixed strategy equilibrium. At time τ_0 the firm mixes between working and shirking resulting in self-esteem paths $z_t^{\pm}$ and reputation x_t. The z_t^- firms exit over $[\underline{\tau}, \overline{\tau}]$, while the z_t^+ firms exit at $\tau(F)$.

strictly prefers to continue working, whereas the firm that shirks at the dividing line then strictly prefers to continue shirking.

The firm's reputation, which equals the market's belief about the firm's self-esteem, is sandwiched between these two self-esteem paths. When both firms remain in the market, reputation and self-esteem decline as normal. After sufficient time without a breakthrough, the 'low' firm will wish to exit. If the firm exits with probability one at a single time, reputation would jump up, so equilibrium requires that the 'low' firm randomizes between exiting and remaining in the market over some region $\tau \in [\underline{\tau}, \overline{\tau}]$. During this exit period, self-esteem declines, and so reputation has to rise so as to satisfy the firm's indifference condition (3.7). Once the 'low' firm has exited with probability one, reputation coincides with the self-esteem of the 'high' firm. Such a firm will exit instantly after their self-esteem has dropped sufficiently low. Since reputation increases at times, we can no longer conclude that investment incentives are single-peaked. However, Theorem 1 holds, and the firm shirks near the exit time.[12]

More generally, a mixed strategy equilibrium could have more than two optimal investment plans $\{a_t^*\}$. The qualitative features of the above example, such as gradual exit during which reputation recovers, carry over from the above discussion.

4.3 Competitive Equilibrium

So far, we have analyzed a model with a single firm. Following Atkeson, Hellwig, and Ordonez (2012), we can embed our model into a competitive equilibrium context analogous to Jovanovic (1982) or Hopenhayn (1992). Suppose there is a continuum of firms $i \in [0, 1]$ and firm i produces $x_{t,i}$ units of the experience good at time t. Let the total output of the experience good be $X_t = \int_i x_{t,i} di$, and

[12]A mixed strategy equilibrium has the interesting technical property that smooth-pasting fails at the exit time. In particular, for the 'low' firm, the investment incentives depend on the choice of the exit time. For example, $\Gamma(\underline{\tau}) = 0$ if $\tau^* = \underline{\tau}$ but $\Gamma(\underline{\tau}) > 0$ if $\tau^* > \underline{\tau}$. As $V_z(\underline{\tau}, z_{\underline{\tau}}-) \leq \Gamma(\underline{\tau}) \leq V_z(\underline{\tau}, z_{\underline{\tau}}+)$ for any exit time, firm value $V(t, z)$ cannot be differentiable at $(t, z) = (\underline{\tau}, z_{\underline{\tau}})$. This failure of smooth-pasting is analogous to that in Keller and Rady (2014).

assume consumers have quasilinear utility $U(X_t) - Y$, where Y is money. Competitive equilibrium then yields a price for the experience good that equals the marginal utility, $P_t = U'(x_t)$, so the revenue of firm i equals $x_{t,i}P_t$. Appealing to the law of large numbers, one can then consider a stationary equilibrium in which the price $P_t = P$ is constant over time, inducing a value for firm i given by $V_i(t, z_t; P)$.

To close the model, one needs to model entry. The usual approach is to assume a firm pays an entry cost ξ at which point it has high quality with probability $\check{x}$. In a pure-strategy equilibrium this corresponds to some time $\check{t}$ since a breakthrough, i.e. $z_{\check{t}} = \check{x}$. Free entry then determines the price level, $V_i(\check{t}, \check{x}; P) = \xi$.

5 Model Variations

In this section, we consider three natural variations of our baseline model. In Section 5.1 we study the model in which the market observes the firm's investment. Here, both the firm and the market learn about the firm's quality, but there is no moral hazard. In Section 5.2 we analyze the model in which the firm knows its own quality. Here there is still moral hazard, but only the market learns about the firm's quality. Finally, Section 5.3 considers the model in between the baseline model and the version with known quality, assuming the firm does not know its quality, but that it observes additional private signals in addition to the public signals.

5.1 Observable Investment

In the baseline model, the firm's investment was unobserved by the market; in this section we eliminate the moral hazard, and assume that the market can directly observe the firm's investment. Theorem 5 shows that investment incentives are decreasing over time. Theorem 6 shows that moral hazard decreases investment.

Suppose that the market observes the history of signals h^t and the firm's past investment $\{a_s\}_{s \le t}$. Since the market has the same information as the firm, reputation and self-esteem coincide $x_t = z_t$. We can thus write firm value as a function of self-esteem alone. Analogous to (2.4), we truncate the integral at a breakthrough, yielding

$$\hat{V}(z_t) = \sup_{\{a_s\} s \ge t, \tau} \int_t^\tau e^{-\int_t^s r + \mu z_u du} \left[z_s - a_s c - k + \mu z_s \hat{V}(1) \right] ds \tag{5.1}$$

Denote $\{\hat{a}_t\}, \hat{\tau}$ as the optimal strategy and $\{\hat{z}_t\}$ the associated self-esteem. Since the firm controls both self-esteem and reputation, the analysis reduces to a decision problem (rather than finding an equilibrium). Existence of an optimal solution is analogous to Lemma 1.

The value function $\hat{V}(z)$ is strictly convex, as in footnote 4. Using equation (2.3), investment raises self-esteem (and reputation) at rate λ. Analogous to equation (3.1), optimal investment is thus characterized by

$$\hat{a}_t = \begin{cases} 0 & \text{if } \lambda \hat{V}'(\hat{z}_t) < c \\ 1 & \text{if } \lambda \hat{V}'(\hat{z}_t) > c. \end{cases}$$

In addition, the optimal exit time $\hat{\tau}$ satisfies

$$\hat{z}_{\hat{\tau}} - k + \mu \hat{z}_{\hat{\tau}} \hat{V}(1) = 0.$$

Adapting the proof of Lemma 2 to this model we show in Appendix A.5 that if $\hat{V}'(z)$ exists it is given by

$$\hat{\Gamma}(t) := \int_t^{\hat{\tau}} e^{-\int_t^s r+\lambda+\mu(1-\hat{z}_u)du} \left[1 + \mu(\hat{V}(1) - \hat{V}(\hat{z}_s))\right] ds \tag{5.2}$$

More generally, $\hat{\Gamma}(t)$ is sandwiched between the upper and lower derivatives of $\hat{V}(z)$, which exist by convexity. When compared to the investment incentives with moral hazard (3.2), investment now affects reputation and revenue directly. This accounts for the "1" term in the integrand. In any optimal strategy, investment therefore satisfies

$$\hat{a}_t = \begin{cases} 0 & \text{if } \lambda \hat{\Gamma}(t) < c \\ 1 & \text{if } \lambda \hat{\Gamma}(t) > c. \end{cases}$$

Theorem 5 (Observable Investment Characterization) *Assume that the market observes the firm's investment. Then investment incentives $\hat{\Gamma}(t)$ decrease in t with $\hat{\Gamma}(\hat{\tau}) = 0$.*

Proof. Given assumption (2.2), the drift $g(a_t, z_t)$ is bounded below zero on $[z^\dagger, 1]$ and the firm exits before hitting $z^\dagger$. Since the drift is negative z_t decreases and, by the convexity of the value function, $\hat{V}'(z_t)$ strictly decreases in t. Hence there exists $\hat{t}$ such that $a_t = 1$ for almost all $t < \hat{t}$ and $a_t = 0$ for almost all $t > \hat{t}$. □

Theorem 5 means that investment incentives are either full-shirk or work-shirk, depending on the cost of investment. As time progresses without a breakthrough, the firm gets closer to its exit point and any investment has a shorter lifespan, reducing incentives. With moral hazard, the firm benefits from its investment solely via the reputational dividends when the market learns about the quality at a breakthrough. Right after at breakthrough, these dividends are zero and so investment incentives increase. This initial increase is absent here.

When comparing the investment incentives with moral hazard (3.2) and without (5.2), the additional "1" term indicates that investment incentives are indeed higher when investment is observable; however, many other terms in the integral are endogenous. To state our theorem, for any baseline equilibrium we write $t^* := \sup\{t : \Pr(\lambda\Gamma(t) \geq c) > 0\}$ for the last time at which investment is optimal.[13]

[13] Theorem 6 holds for both pure and mixed equilibria. In a mixed strategy equilibrium, firm investment $\{a_t\}$ and thus investment incentives $\Gamma(t)$ are a random variable. In a pure equilibrium one can also state the result in terms of reputation space. That is, if the firm with moral hazard works at any x, then the firm with observable investment also works at this x.

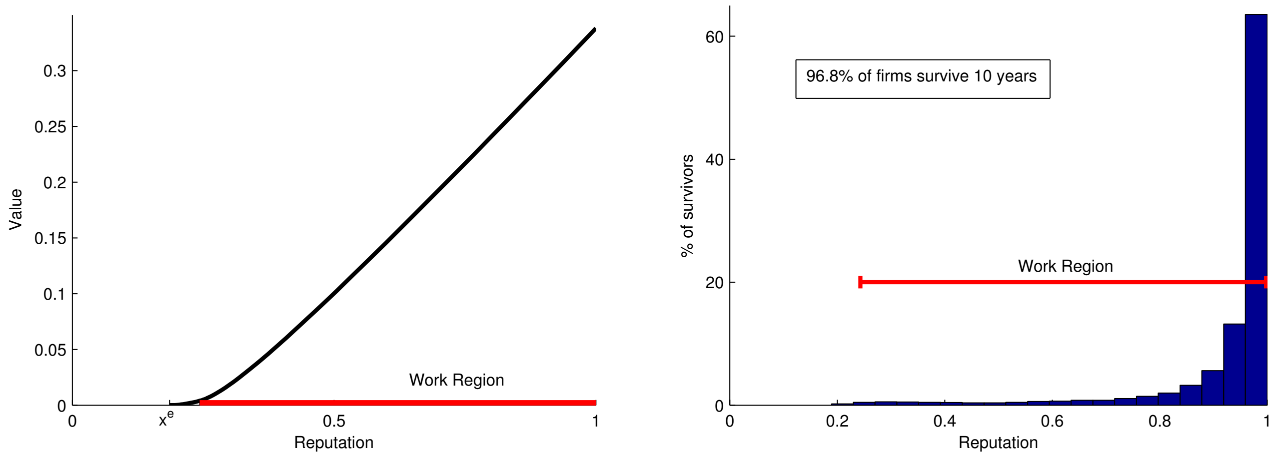

Figure 6: **Market Observes Firm's Investment.** The **left panel** shows the firm's value as a function of the firm's reputation. The **right panel** shows the resulting distribution of firms after 10 years, assuming all firms start at $x = z = 1$. The parameters are the same as Figure 2.

Theorem 6 (Impact of Moral Hazard) *With observable investment, the firm works strictly longer than in any baseline equilibrium, $t^* < \hat{t}$.*

Proof. See Appendix A.6. □

Intuitively, in the baseline case, when a firm invests, it raises its self-esteem. With observable investment, the firm raises its self-esteem and its reputation. Since a higher reputation is good for the firm, it has a higher marginal benefit of investment and, in equilibrium, invests more.

Figure 6 simulates the equilibrium for the same parameters as Figures 1-4. In this example, the firm works for $z \in [0.24, 1]$ with an exit cutoff $\hat{z}^e = 0.19$. The left panel shows that the extra work at high reputations is very valuable for the firm, raising the value of a firm with perfect reputation from 0.27 to 0.34. Intuitively, the investment at the top means that reputation initially falls slowly, and a breakthrough is likely before the reputation has fallen significantly (see Figure 1). As a result, the right panel shows that most firms remain close to $z = 1$ after 10 years, with 96.8% surviving.

5.2 Privately Known Quality

In the baseline model, we assume that the firm learns about its quality through the same public breakthroughs as the market and acquires private information only if the market's beliefs about investment are incorrect. In this section we suppose the firm knows its own quality, although the market still does not. This model is reasonable if the firm has much better information than the market; for example, a restaurant owner may receive direct, non-public feedback from his patrons. This opens up the possibility for the firm to signal its quality by remaining in business.

As before, we focus on strategies and beliefs that are recursive in the time since the last breakthrough t. Additionally, we assume that the firm conditions its strategy on current quality but not quality in the past, which is payoff-irrelevant. Thus, a recursive strategy consists of an investment plan $\{a_t\}$ and an exit time τ for the firm with either quality level.

To analyze the value of a low quality firm at time t, we truncate its cash flow expansion at the first technology shock

$$V(t,0) = \sup_{\{a_s\},\tau} \int_t^\tau e^{-(r+\lambda)(s-t)} \left[x_s - ca_s - k + \lambda(a_s V(s,1) + (1-a_s)V(s,0))\right] ds \tag{5.3}$$

Compared to the low-quality firm, the high-quality firm additionally enjoys breakthroughs with present value $V(0,1) - V(s,1)$ and arrival rate μ, so that

$$V(t,1) = \sup_{\{a_s\},\tau} \int_t^\tau e^{-(r+\lambda)(s-t)} \left[x_s - ca_s - k + \lambda(a_s V(s,1) + (1-a_s)V(s,0)) + \mu(V(0,1) - V(s,1))\right] ds \tag{5.4}$$

Investment raises the quality of the firm at rate λ. Writing $\Delta(s) = V(s,1) - V(s,0)$ for the *value of quality*, optimal investment is thus characterized by the bang-bang condition

$$a_s = \begin{cases} 0 & \text{if } \lambda\Delta(s) < c \\ 1 & \text{if } \lambda\Delta(s) > c \end{cases} \tag{5.5}$$

for almost all s. Importantly, optimal investment is independent of the firm's quality, allowing us to write it as $\{a_s^*\}$. Intuitively, investment only pays off if there is a technology shock, in which case the firm's current quality is irrelevant. For the firm's exit decision, write τ^θ for the optimal exit time(s) of a firm with current quality θ.

An equilibrium in this model variation with privately known quality consists of a distribution over strategies $F^\theta(\{a_t\},\tau)$ for $\theta \in \{L,H\}$ and a reputation trajectory $\{x_t\}$ such that: (1) all equilibrium strategies are optimal, and (2) reputation x_t is derived from the distributions via Bayes' rule whenever possible. We restrict attention to equilibria where reputation $\{x_t\}$ is continuous. This assumption holds on the equilibrium path but, off the equilibrium path, implies the firm cannot be punished for failing to exit by beliefs that jump down. We ignore such equilibria because it is implausible for the market to interpret the failure to exit as a signal of low quality. We also restrict attention to equilibria with weakly decreasing reputation; this is satisfied in any markovian equilibrium.

The following result is from Bar-Isaac (2003).

Lemma 5 (Exit) *In an equilibrium with continuous, weakly decreasing reputation $\{x_t\}$, there exists a time $\underline{\tau} < \infty$ such that the exit time of the low-quality firm τ^L has support $[\underline{\tau},\infty)$, revenue and firm value are constant for $t \in [\underline{\tau},\infty)$ and satisfy*

$$x_t - k + \overline{a}\max\{\lambda V(t,1) - c, 0\} = 0. \tag{5.6}$$

The high-quality firm never exits, i.e. $\tau^H = \infty$.

Proof. Since reputation $\{x_t\}$ continuously decreases, firm value $V(t,z)$ for $z \in \{0,1\}$ continuously decreases in t. For an optimal investment strategy $\{a_t^*\}$ the flow-payoff of the low- and high-quality firm, the integrands in (5.3) and (5.4), continuously decrease in t as well. Thus, exiting is optimal exactly when flow-payoffs are zero; since firm value is zero at an exit-time, flow-payoffs of the low quality firm are given by (5.6). As flow payoffs of the high-quality firm exceed those of the low-quality firm by the last, positive term in (5.4), the latest exit time of the low-quality firm must strictly precede the earliest exit time of the high-quality firm.

To see that the low-quality firm starts exiting at some finite $\underline{\tau}$, note first that (2.2) implies the low-quality firm exits with certainty before its reputation falls to $x^\dagger := \lambda/\mu$; for then the negative flow profits $x^\dagger - k$ from staying in the market exceed the option value of staying in the market $\lambda V(t, 1)$, which is bounded above by $(1 - k)/r$. Moreover, reputational drift $g(\tilde{a}, x)$ is strictly negative on $[x^\dagger, 1]$ and takes reputation below $x^\dagger$ in finite time, unless the market expects the low-quality firm to start exiting and draws a positive inference from its failure to exit. Thus, in equilibrium the low-quality firm must eventually exit and we define $\underline{\tau}$ as the earliest time when it does so.

Reputation must be constant after $\underline{\tau}$. For otherwise, if it started to decrease at some time t, the flow-payoffs of the low-quality firm turn strictly negative and the low-quality firm exits with certainty; thus, reputation would jump to one, undermining incentives to exit. Thus, the firm's problem becomes stationary after $\underline{\tau}$, all exit times $\tau^L \in [\underline{\tau}, \infty)$ are optimal and the high-quality firm never exits. Finally, in order to keep reputation at $x_{\underline{\tau}}$, the low-quality firm must exit at constant rate $-g(\overline{a}, x_{\underline{\tau}})/x_{\underline{\tau}}(1 - x_{\underline{\tau}})$. □

Given the low-quality firm's indifference at the exit threshold, we can assume that it always remains in the market, and therefore shares the same strategy as the high-quality firm. Subtracting (5.3) from (5.4), we obtain the following expression for the value of quality

$$\Delta(t) = \int_t^\infty e^{-(r+\lambda)(s-t)} \mu(V(0, 1) - V(s, 1))ds. \tag{5.7}$$

The integrand in (5.7) represents the *reputational dividend of quality*: High quality does not affect the firm's reputation and revenue immediately but gives rise to future breakthroughs that arrive at rate μ and boost the firm's reputation to one by resetting its clock to zero. These dividends depreciate at both the time-discount rate r and the quality obsolescence rate λ. This is analogous to the investment incentives (3.2) in the baseline model.

Theorem 7 (Known Quality Characterization) *In an equilibrium with continuous, weakly decreasing reputation $\{x_t\}$, investment incentives $\Delta(t)$ increase over time.*

Proof. As s rises, the firm's value $V(s, 1)$ falls and reputational dividends $V(0, 1) - V(s, 1)$ grow. Hence an increase in t leads to an increase in the value of quality (5.7) and in investment via the optimality equation (5.5). □

Intuitively, breakthroughs are most valuable to a firm with low reputation since a breakthrough takes the firm from its current reputation to $x = 1$. Thus, the optimal investment strategy is either full shirk, shirk-work or full work, depending on the investment cost c.

In contrast to the baseline model, all states $(t, z) \in [0, \infty) \times \{0, 1\}$ are on-path in this model variation, and so conditions (5.5) and (5.6) are sufficient as well as necessary for an equilibrium strategy. Thus, disinvestment at times $t \in [0, \underline{t})$, investment thereafter and exit of low-quality firms after time $\underline{\tau}$ constitutes an equilibrium if (5.5) and (5.6) are satisfied. Equilibrium existence can thus be established by Brouwer's fixed-point theorem applied to $(\underline{t}, \underline{\tau}) \in [0, \underline{\tau}] \times [0, \overline{\tau}]$, as in Board and Meyer-ter-Vehn (2013, Theorem 2).

The increasing investment incentives in Theorem 7 are in sharp contrast to the single-peaked and eventually vanishing investment incentives in Theorem 2. In that case, the firm gives up near the exit threshold and coasts into liquidation; with privately known quality, the firm fights until the bitter end. Intuitively, in the baseline model, the firm's investment at times $t \in [\tau^* - \varepsilon, \tau^*]$ pays off only if a technology shock arrives and a breakthrough arrives that averts exit. The probability of this joint event is of order ε^2 which eventually falls short of the investment costs. With known quality, only a technology shock is required for investment to pay off because a boost in quality averts exit. Thus, investment incentives are of order ε at all times, and are actually maximized when the firm is about to exit as discussed above.

With observed investments, incentives decrease as the firm gets closer to exit (Theorem 5). With known quality, incentives increase as the breakthroughs become more valuable to the firm (Theorem 7). The single-peaked incentives in the baseline model (Theorem 2) can be viewed as a combination of these two effect.

Figures 7-8 illustrate an equilibrium for the same parameters as Figures 1-4. In Figure 7, the left panel shows the value function of high- and low-quality firms with different reputations. One can see the kink at the start of the work region. In this example, the value of a firm with perfect reputation is 0.22, compared with 0.27 under unknown quality. The right panel plots the distribution of firms after 10 years, starting at $x = 1$. One can see a bulge of firms in the work region as firm's reputation declines more slowly. Relative to the baseline distribution, this distribution appears censored. The exit threshold is much higher, with a large number of firms massing at this threshold, where low quality firms randomize between exiting and remaining in the market. Overall 71.8% firms survive 10 years, compared to 76.8% in the baseline case. Figure 8 shows typical life-cyles for three firms. The first never goes near the exit threshold; the second is temporarily indifferent between exiting and not, and ultimately survives; the third exits after 9 years.

5.3 Imperfect Private Information

The model variation with privately known quality predicts the opposite investment behavior than the baseline model when the firm is about to exit. To further illuminate this contrast, we now bridge the two extremes by nesting them in a class of models with imperfect private information.

We model the firm's new information via private breakthroughs that arrive at rate ν and reveal high quality with certainty. Thus, at a public breakthrough reputation and self-esteem jump to one; at a private breakthrough reputation is continuous while self-esteem jumps to one; absent either

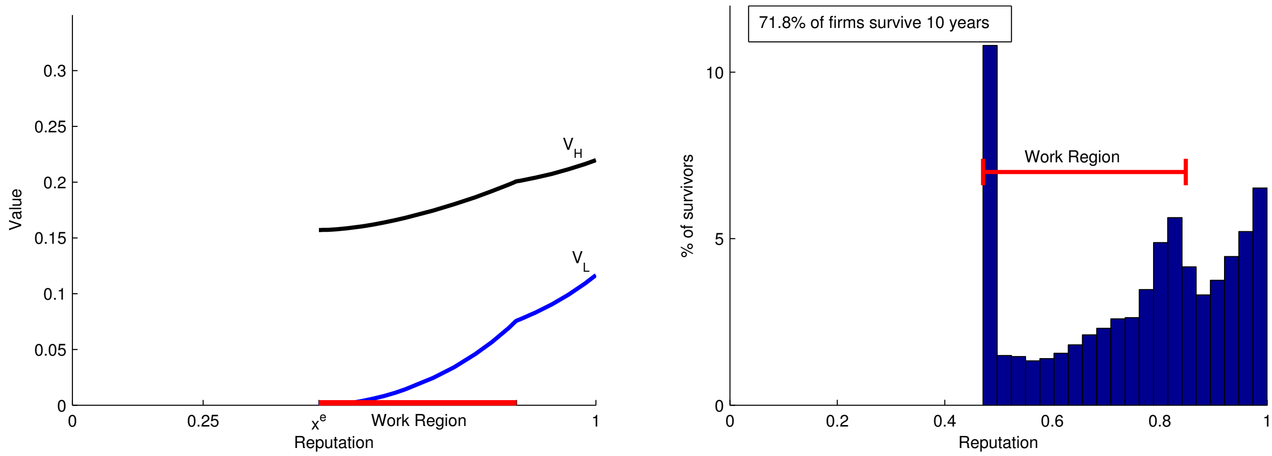

Figure 7: **Privately Known Quality.** The **left** panel shows the firm's value as a function of the firm's reputation. The **right** panel shows the resulting distribution of firms after 10 years, assuming all firms start at $x = z = 1$. The parameters are the same as Figure 2.

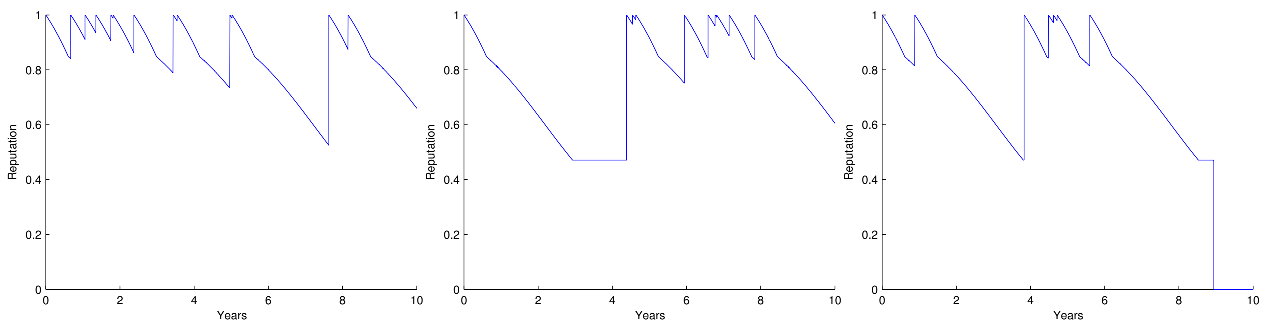

Figure 8: **Firm Lifecycles with Privately Known Quality** This figure shows three sample paths starting at $x = z = 1$. The **left** and **center** firms survive 10 years, although the center firm hits the exit point and survives the randomization. The **right** firm exits after 9 years. The parameters are the same as Figure 2.

breakthrough, self-esteem is governed by $\dot{z} = g(a, z)$ with

$$g(a, z) = \lambda(a - z) - (\mu + \nu)z(1 - z).$$

When $\nu = 0$ we recover the unknown quality case. As $\nu \to \infty$, this model approximates the known quality case in the sense that self-esteem z_t converges to 0 or 1 in distribution for any time t and any investment strategy $\{a_t\}$.[14]

This model is recursive in the time since the last public breakthrough, t. A recursive strategy for the firm then specifies investment a_t and exit time τ as a function of the history of private breakthroughs. Writing optimal strategies as $\{a_t^*\}_{t \geq 0}, \tau^*$ and the resulting process of self-esteem as

[14] To see this, note that for any $\varepsilon > 0$ there exists $\delta > 0$ and $\nu^* > 0$ such that for all $\nu > \nu^*$ we have either $z_{t+\delta} < \varepsilon$ if no breakthrough arrived in $[t, t + \delta]$, or $\Pr(z_{t+\delta} > 1 - \varepsilon) > 1 - \varepsilon$ if a breakthrough arrived at $t' \in [t, t + \delta]$ and thus $\Pr(\theta_{t+\delta} = H | \theta_{t'} = H) \geq e^{-\lambda\delta}$.

$\{z_t^*\}$, we truncate the firm's cash flow expansion at either kind of breakthrough to obtain

$$V(t, z_t^*) = \int_t^{\tau^*} e^{-\int_t^s (r+(\mu+\nu)z_u^*)du} \left[x_s - ca_s^* - k + \mu z_s^* V(0,1) + \nu z_s^* V(s,1)\right] ds.$$

The additional term $\nu z_s^* V(s,1)$ captures the firm's continuation value after a private breakthrough. As in Lemmas 2-3, investment incentives are given by

$$\Gamma(t) = \int_t^{\tau^*} e^{-\int_t^s (r+\lambda+(\mu+\nu)(1-z_u^*))du} \left[\mu(V(0,1) - V(s,z_s^*)) + \nu(V(s,1) - V(s,z_s^*))\right] ds. \tag{5.8}$$

These disappear at the exit time τ^*, so the firm will shirk close to the exit threshold, as in Theorem 1. Thus, even as $\nu \to \infty$ and the model approaches the known-quality case, the firm shirks before exiting, in contrast to Theorem 7. However, this does not imply a discontinuity: The integrand in (5.8) increases in ν, so while the firm shirks for some time before exit in equilibrium, the length of this shirking time may converge to zero as ν grows large.

6 Conclusion

This paper models the lifecycle of a firm whose primary assets are its quality and its reputation. In the baseline model, the firm privately invests in its quality, while both the market and firm learn about the success of past investments. We characterize investment incentives and show they are are single-peaked in the time since a breakthrough. This yields predictions about the distribution of firm revenue product and the level of industry turnover. Finally, we investigate two variants of our model: one where there is no moral hazard, and a second where the firm privately knows its quality.

We believe this model has a wide variety of applications that lend themselves to empirical investigation For example, a film studio invests in its personnel and its products; the studio and Hollywood then learn about the resulting quality via its hit movies. In labor markets, academics and other professionals invest in their skills; the agent and the market then learn about the success through publications. At the international level, countries makes policy choices concerning government spending and privatizations; both the country and their sovereign debt holders then learn about the country's solvency via public statistics.

These applications suggest a number of interesting extensions to our analysis. One could change the learning process or add more quality levels. One could introduce common learning, so news about one firm impacts its competitors. One could also append the model with regular capital in addition to reputational capital.

A Appendix

A.1 Proof of Lemma 2

Fix time t, self-esteem z_t, firm strategy $\{a_s\}_{s\geq 0}$, τ (not necessarily optimal), write $\{z_s\}_{s\geq t}$ for future self-esteem, and let

$$\Pi(t,z_t)=\int_{s=t}^{\tau}e^{-\int_t^s r+\mu z_u du}(x_s-ca_s-k+\mu z_s\Pi(0,1))ds \tag{A.1}$$

be the firm's continuation value, where we truncated the integral of the cash-flows at the first breakthrough as in (2.4). We will show that $\Pi(t,z)$ is differentiable in z with derivative

$$\Pi_z(t,z_t)=\int_{s=t}^{\tau}e^{-\int_t^s r+\lambda+\mu(1-z_u)du}\mu(\Pi(0,1)-\Pi(s,z_s))ds \tag{A.2}$$

Equation (3.2) then follows by the envelope theorem, Milgrom and Segal (2002).

To show (A.2) we first state two claims, both of which follow immediately from Board and Meyer-ter-Vehn (2013).

Claim 1: For any bounded, measurable functions $\phi,\rho:[0,\tau]\to\mathbb{R}$, the function

$$\psi(t)=\int_t^{\tau}e^{-\int_t^s \rho(u)du}\phi(s)ds \tag{A.3}$$

is the unique solution to the integral equation

$$f(t)=\int_{s=t}^{\tau}(\phi(s)-\rho(s)f(s))ds. \tag{A.4}$$

This is proved for $\tau=\infty$ and constant ρ in Board and Meyer-ter-Vehn (2013, Lemma 5). The proof generalizes immediately to finite τ and measurable functions $\rho(t)$.

Claim 2: For any times $s>t$ and fixed investment $\{a_u\}_{u\in[s,t]}$, time-s self-esteem z_s is differentiable in time-t self-esteem z_t with derivative

$$\frac{dz_s}{dz_t}=\exp\left(-\int_{u=t}^{s}(\lambda+\mu(1-2z_u))du\right).$$

This follows by the same arguments as in Board and Meyer-ter-Vehn (2013, Lemma 8B).

Setting $\psi(s)=e^{-r(s-t)}\Pi(s,z_s)$, $\rho(s)=\mu z_s$ and $\phi(s)=e^{-r(s-t)}(x_s-ca_s-k+\mu z_s\Pi(0,1))$, yields equation (A.3). Applying Claim 1, equation (A.4) becomes

$$\Pi(t,z_t)=\int_{s=t}^{\tau}e^{-r(s-t)}(x_s-ca_s-k+\mu z_s(\Pi(0,1)-\Pi(s,z_s)))ds.$$

Taking the derivative with respect to z at $z = z_t$, and applying Claim 2 we get

$$\begin{aligned}\Pi_z(t, z_t) &= \int_{s=t}^{\tau} e^{-r(s-t)} \frac{dz_s}{dz_t} (\mu(\Pi(0,1) - \Pi(s, z_s)) - \mu z_s \Pi_z(s, z_s)) ds \\ &= \int_{s=t}^{\tau} e^{-\int_t^s r+\lambda+\mu(1-2z_u)du} (\mu(\Pi(0,1) - \Pi(s, z_s)) - \mu z_s \Pi_z(s, z_s)) ds.\end{aligned}$$

Setting $\rho(s) = \mu z_s$, $\phi(s) = e^{-\int_t^s r+\lambda+\mu(1-2z_u)du}\mu(\Pi(0,1)-\Pi(t, z_s))$ and $f(s) = e^{-\int_t^s r+\lambda+\mu(1-2z_u)du}\Pi_z(s, z_s)$ yields equation (A.4). Applying Claim 1, equation (A.3) becomes

$$\Pi_z(t, z_t) = \int_t^{\tau} e^{-\int_t^s \mu z_u du} e^{-\int_t^s r+\lambda+\mu(1-2z_u)du} \mu(\Pi(0,1) - \Pi(t, z_s)) ds$$

implying (A.2).

A.2 Monotonicity of Value Function in Section 3

Lemma 6 *If $\{x_t\}$ strictly decreases, then $V(t, z)$ strictly decreases in t and strictly increases in z on $\{(t, z) : V(t, z) > 0\}$.*

Proof. Fix $t \geq t'$ and $z \leq z'$ and consider a 'low' firm with initial state (t, z) and a 'high' firm with initial state (t', z'). We can represent the firms' uncertainty as an increasing sequence of 'potential breakthrough' times $\{t_i\}_{i\in\mathbb{N}}$ that follow a Poisson distribution with parameter λ, and a sequence of uniform $[0, 1]$ random variables $\{\zeta_i\}_{i\in\mathbb{N}}$, with the interpretation that the firm experiences an actual breakthrough after time σ (that is at time $t + \sigma$ for the low firm and at time $t' + \sigma$ for the high firm) if $\sigma = t_i$ for some i and $\zeta_i \leq Z_{t-}$. Fix any realization of uncertainty $\{t_i, \zeta_i\}_{i\in\mathbb{N}}$, let $\{A^*_\sigma\}, T^*$ be the low firm's optimal strategy given this realization, and assume that the high firm mimics this strategy. Note that this strategy is in generally not recursive for the high firm. Given $\{t_i, \zeta_i\}$ and $\{A^*_\sigma\}, T^*$, we can compute revenue and self-esteem of low and high firm (X_σ, Z_σ) and (X'_σ, Z'_σ) for any $\sigma \geq 0$. We now argue inductively that

$$X_\sigma \leq X'_\sigma \text{ and } Z_\sigma \leq Z'_\sigma \tag{A.5}$$

for any $\sigma < t_i$ and any $i \in \mathbb{N}$. For $i = 1$, that is for $\sigma \in [0, t_1)$, we have $X_\sigma = x_{t+\sigma} < x_{t'+\sigma} = X'_\sigma$ because $\{x_t\}$ decreases, and the self-esteem trajectories Z_σ, Z'_σ are governed by the ODE $\dot{z} = g(a, z)$, implying (A.5) for $\sigma \in [0, t_1)$. At $\sigma = t_1$, the low (resp. high) firm experiences a breakthrough if $\zeta_1 \leq Z_{\sigma-}$ (resp. $\zeta_1 \leq Z'_{\sigma-}$). As $Z_{\sigma-} \leq Z'_{\sigma-}$, we get (A.5) for $\sigma = t_1$. Inductive application of these steps yields (A.5) for all σ. Thus, by mimicking the low firm's optimal strategy $\{A^*_\sigma\}, T^*$ for any realization $\{t_i, \zeta_i\}$, the high firm can guarantee itself weakly higher cash-flows $X'_\sigma - cA^*_\sigma - k$ at all times σ, implying $V(t', z') \geq V(t, z)$. As long as firm value is strictly positive and the firms don't exit immediately, the inequality $X_\sigma \leq X'_\sigma$ is strict for a positive measure of times with positive probability, implying $V(t', z') > V(t, z)$. □

A.3 Proof of Theorem 2

The discount rate $\rho(t) = r + \lambda + \mu(1 - z_t^*)$ is Lipschitz continuous with derivative $\mu \dot{z}_t^*$ where $\dot{z}_t^* = g(a_t^*, z_t^*) = \lambda(a_t^* - z_t^*) - \mu z_t^*(1 - z_t^*)$ for almost all t. Firm value as a function of time $t \mapsto V(t, z_t^*)$ is also Lipschitz continuous with derivative $\frac{d}{dt}V(t, z_t^*) = \Psi(t) + \dot{z}_t^* \Gamma(t)$ for almost all t.

Now assume that $\dot{\Gamma}(t) \leq 0$. Then

$$\begin{aligned}
\dot{\Gamma}(t+\varepsilon) - \dot{\Gamma}(t) &= \int_t^{t+\varepsilon} \frac{d}{ds}\left[\rho(s)\Gamma(s) - \mu(V(0,1) - V(s, z_s^*))\right] ds \\
&= \int_t^{t+\varepsilon} \left[\rho(s)\dot{\Gamma}(s) + \dot{\rho}(s)\Gamma(s) + \mu \frac{d}{ds} V(s, z_s^*)\right] ds \\
&= \int_t^{t+\varepsilon} \left[\rho(s)\dot{\Gamma}(s) - \mu \dot{z}_s^* \Gamma(s) + \mu(\Psi(s) + \dot{z}_s^* \Gamma(s))\right] ds \\
&= \int_t^{t+\varepsilon} \left[\rho(s)\dot{\Gamma}(s) + \mu \Psi(s)\right] ds.
\end{aligned}$$

As $\dot{\Gamma}(t) \leq 0$ and $\Psi(t) < 0$ and both functions are continuous, the integrand is strictly negative for small ε, so $\dot{\Gamma}$ strictly decreases on some small interval $[t, t+\varepsilon)$. If $\dot{\Gamma}$ did not strictly decrease on $[t, \tau^*]$ there would exist $t' > t$ with $\dot{\Gamma}(t') < 0$ and $\dot{\Gamma}(t'+\varepsilon) \geq \dot{\Gamma}(t')$ for arbitrarily small ε, which is impossible by the above argument.

A.4 Proof of Theorem 4

Proof strategy: The firm's payoff from strategy $\{a_t\}, \tau$ is given by

$$\Pi(\{a_t\}, \tau; \{x_t\}) = \frac{\int_0^\tau e^{-\int_{s=0}^t r + \mu z_s ds}\,(x_t - c a_t - k)\,dt}{1 - \int_0^\tau e^{-\int_{s=0}^t r + \mu z_s ds} \mu z_t dt}$$

The proof idea is to show that the firm's best response correspondence

$$BR(\{x_t\}) = \arg \max_{\{a_t^*\}, \tau^*} \Pi(\{a_t^*\}, \tau^*; \{x_t\})$$

and the Bayesian updating formula $\mathcal{B}$ defined by (4.1) admit a fixed point.

To establish existence of a fixed point we define topologies on the space of mixed strategies F and reputation trajectories $\{x_t\}$ with the property that both spaces are compact, locally convex, and Hausdorff, and both correspondences are upper-hemicontinuous. Then the existence of the fixed point follows by the Kakutani-Fan-Glicksberg theorem.

Defining the topologies: In the proof of Lemma 1 we interpreted investment strategies $\{a_t\}_{t \in [0, \overline{\tau}]}$ as elements of a space B with a weak topology under which B is compact. We now also interpret revenue trajectories $\{x_t\}$ as elements of B with this topology. As for the firm's mixed strategies, we equip $\Delta(B \times [0, \overline{\tau}])$ with the topology of convergence in distribution. Standard arguments (Aliprantis and Border (1999, Theorem 14.11)) show that this space is compact. By definition it is locally convex.

Upper hemi-continuity of Bayes' rule: We now prove that the correspondence $\mathcal{B} : \Delta(B \times [0, \overline{\tau}]) \to B$ mapping beliefs F to the set of measurable trajectories $\{x_t\}$ that satisfy (4.1) for $t < \tau(F)$ is upper hemi-continuous. Consider a sequence of beliefs F^n (with expectation $\mathbb{E}^n$) that converges to F in distribution. $\mathcal{B}(F^n)$ consists of all measurable trajectories $\{x_t^n\}$ that satisfy (4.1) (when replacing $\mathbb{E}^F$ by $\mathbb{E}^n$) for $t < \tau(F^n)$. As F assigns probability less than one to the event $\{\tilde{\tau} < t\}$ for any $t < \tau(F)$ so does F^n for sufficiently large n; thus, $\lim_{n\to\infty} \tau(F^n) \geq \tau(F)$.

We now show that $x_t^n \to x_t$ for all $t < \tau(F)$ at which the marginal distribution $F(\tilde{\tau})$ is continuous.[15] Consider the numerator of (4.1) (the argument for the denominator is identical). The integrand $\chi_t^-(\tilde{a}, \tilde{\tau}) := z_t(\tilde{a}) w_t(\tilde{a}) \mathbb{I}_{\{\tilde{\tau} > t\}}$ is continuous in $\tilde{a}$ (see, e.g., Davis (1993, Theorem 43.5)) and lower semi-continuous in $\tilde{\tau}$; similarly, $\chi_t^+(\tilde{a}, \tilde{\tau}) := z_t(\tilde{a}) w_t(\tilde{a}) \mathbb{I}_{\{\tilde{\tau} \geq t\}}$ is continuous in $\tilde{a}$ and upper semi-continuous in $\tilde{\tau}$. The portmanteau theorem thus implies $\liminf \mathbb{E}^n[\chi_t^-(\tilde{a}, \tilde{\tau})] \geq \mathbb{E}^F[\chi_t^-(\tilde{a}, \tilde{\tau})]$ and $\mathbb{E}^F[\chi_t^+(\tilde{a}, \tilde{\tau})] \geq \limsup \mathbb{E}^n[\chi_t^+(\tilde{a}, \tilde{\tau})]$. As χ_t^- and χ_t^+ are bounded and disagree only for $\tilde{\tau} = t$, which happens with probability zero under F and thus with vanishing probability under F^n, we have $\mathbb{E}^F[\chi_t^-(\tilde{a}, \tilde{\tau})] = \mathbb{E}^F[\chi_t^+(\tilde{a}, \tilde{\tau})]$ and $\limsup \mathbb{E}^n[\chi_t^+(\tilde{a}, \tilde{\tau})] = \limsup \mathbb{E}^n[\chi_t^-(\tilde{a}, \tilde{\tau})]$. Thus,

$$\liminf \mathbb{E}^n[\chi_t^-(\tilde{a}, \tilde{\tau})] \geq \mathbb{E}^F[\chi_t^-(\tilde{a}, \tilde{\tau})] = \mathbb{E}^F[\chi_t^+(\tilde{a}, \tilde{\tau})] \geq \limsup \mathbb{E}^n[\chi_t^+(\tilde{a}, \tilde{\tau})] = \limsup \mathbb{E}^n[\chi_t^-(\tilde{a}, \tilde{\tau})]$$

and so $\lim \mathbb{E}^n[\chi_t^-(\tilde{a}, \tilde{\tau})]$ exists and equals $\mathbb{E}^F[\chi_t^-(\tilde{a}, \tilde{\tau})]$ as desired. Thus, $\{x_t^n\}$ converges to $\{x_t\}$ pointwise for almost all $t \in [0, \tau(F)]$, and thus also in the $L^2([0, \tau(F)], [0, 1])$-norm and a fortiori in the weak topology. As the set $\mathcal{B}(F)$ allows for any measurable trajectories after $\tau(F)$, all trajectories $\{x_t^n\}_{t \in [0, \overline{\tau}]} \in \mathcal{B}(F^n)$ are uniformly close to $\mathcal{B}(F)$; that is, $\mathcal{B}$ is upper hemi-continuous.

Upper hemi-continuity of the firm's best responses: Self-esteem z_t is continuous in $\{a_t\} \in B$, so firm payoff $\Pi(\{a_t\}, \tau; \{x_t\})$ is continuous in $\{a_t\}, \tau$ and $\{x_t\}$, and thus also continuous in $F = F(\{a_t\}, \tau)$ (Aliprantis and Border (1999, Theorem 14.5)); thus Berge's maximum theorem implies that the best response mapping $BR : B \to \Delta(B \times [0, \overline{\tau}])$ is upper-hemicontinuous.

Summary: We have shown that $(\{x_t\}, F) \mapsto (\mathcal{B}(F), BR(\{x_t\}))$ is an upper hemi-continuous, convex-valued mapping of the compact, locally convex, Hausdorff space $B \times \Delta(B \times [0, \overline{\tau}])$ to itself. The Kakutani-Fan-Glicksberg theorem implies that this mapping has a fixed point. This fixed point constitutes an equilibrium.

A.5 Proof of Equation (5.2)

This proof is analogous to the Proof of Lemma 2 in Appendix A.1. With observable investment, the firm's payoff from strategy $\{a_s\}_{s \geq t}, \tau$ is given by

$$\hat{\Pi}(z_t) = \int_{s=t}^{\tau} e^{-\int_t^s r + \mu z_u du} (z_s - c a_s - k + \mu z_s \hat{\Pi}(1)) ds.$$

[15] It can have at most countably many discontinuities.

Setting $\psi(s) = e^{-r(s-t)}\hat{\Pi}(z_s)$, $\rho(s) = \mu z_s$ and $\phi(s) = e^{-r(s-t)}(z_s - ca_s - k + \mu z_s \hat{\Pi}(1))$ yields equation (A.3). Applying Claim 1, equation (A.4) becomes

$$\hat{\Pi}(z_t) = \int_{s=t}^{\tau} e^{-r(s-t)}(z_s - ca_s - k + \mu z_s(\hat{\Pi}(1) - \hat{\Pi}(z_s))ds.$$

Taking the derivative and applying Claim 2 we get

$$\begin{aligned}\hat{\Pi}'(z_t) &= \int_{s=t}^{\tau} e^{-r(s-t)}\frac{dz_s}{dz_t}(1 + \mu(\hat{\Pi}(1) - \hat{\Pi}(z_s)) - \mu z_s \hat{\Pi}'(z_s))ds \\ &= \int_{s=t}^{\tau} e^{-\int_t^s r+\lambda+\mu(1-2z_u)du}(1 + \mu(\hat{\Pi}(1) - \hat{\Pi}(z_s)) - \mu z_s \hat{\Pi}'(z_s))ds.\end{aligned}$$

Setting $\rho(s) = \mu z_s$ and $\phi(s) = e^{-\int_t^s r+\lambda+\mu(1-2z_u)du}\mu(\hat{\Pi}(1) - \hat{\Pi}(z_s))$, $f(s) = e^{-\int_t^s r+\lambda+\mu(1-2z_u)du}\hat{\Pi}'(z_s)$ satisfies (A.4). Applying Claim 1, equation (A.3) becomes

$$\hat{\Pi}'(z_t) = \int_{s=t}^{\tau} e^{-\int_t^s r+\lambda+\mu(1-z_u)du}\mu(1 + \hat{\Pi}(1) - \hat{\Pi}(z_s))ds.$$

The envelope theorem then implies equation (5.2).

A.6 Proof of Theorem 6

Suppose otherwise, that $t^* \geq \hat{t}$. Then $\hat{z}_{\hat{t}} = z_{\hat{t}}(\overline{a}) \geq z^*_{\hat{t}}$ because $\hat{a}_t = \overline{a}$ on $[0, \hat{t}]$; as reputational drift $g(a, z)$ is negative for $z \geq \hat{z}_{\hat{t}}$ and any investment $\{a_t\}$, this implies $\hat{z}_{\hat{t}} \geq z^*_{t^*}$ and there exists $\tilde{t} \geq \hat{t}$ such that $\hat{z}_{\tilde{t}} = z^*_{t^*} =: z$. By definition both firms shirk after reaching self-esteem z and so the trajectories coincide, $\hat{z}_{\tilde{t}+t} = z^*_{t^*+t}$. For convenience we write this joint trajectory as z_t; that is, we restart the clock at $t = 0$ when self-esteem reaches z. Writing $x^*_t = x_{t^*+t}$ for the revenue of the unobservable firm, we have $x^*_t \leq z_t$ because the equilibrium strategy that invests the longest, until t^*, leads to the highest self-esteem z_t while x^*_t is a weighted average of z_t and the self-esteem resulting from equilibrium strategies with lower investment.

We will now show that the observable firm with self-esteem z has strictly higher investment incentives than the unobservable firm $\hat{\Gamma}(\tilde{t}) > \Gamma(t^*) = c/\lambda$. This contradicts our assumption that the observable firm weakly prefers to at time $\tilde{t} \geq \hat{t}$. The investment incentives of the two firms are given by

$$\hat{\Gamma}(\tilde{t}) = \int_0^{\hat{\tau}-\tilde{t}} e^{-\int_0^t r+\lambda+\mu(1-z_s)ds}\left[1 + \mu(\hat{V}(1) - \hat{V}(z_t))\right]dt \tag{A.6}$$

$$\Gamma(t^*) = \int_0^{\tau^*-t^*} e^{-\int_0^t r+\lambda+\mu(1-z_s)ds}\mu(V(0,1) - V(t^*+t, z_t))dt \tag{A.7}$$

First, we argue that the observable firm exits later than the unobservable firm, i.e. $\hat{\tau} - \tilde{t} > \tau^* - t^*$. To see this, note that the former enjoys higher flow payoffs than the latter

$$z_t - k + \mu z_t \hat{V}(1) > x^*_t - k + \mu z_t V(0,1)$$

because $z_t \geq x_t^*$ and the observable firm's value at a breakthrough is higher than the unobservable firm's, $\hat{V}(1) \geq V(0,1)$; the latter follows because the observable firm can ensure itself weakly higher flow payoffs by mimicking the equilibrium strategy.

Next, consider the integrands in (A.6) and (A.7). We can write the value functions as,

$$\begin{aligned}
\hat{V}(z_t) &= \int_{s=t}^{\hat{\tau}-\tilde{t}} e^{-\int_t^s r+\mu z_u du} \left[z_s - k + \mu z_s \hat{V}(1)\right] ds \\
V(t^*+t, z_t) &= \int_{s=t}^{\tau^*-t^*} e^{-\int_t^s r+\mu z_u du} \left[x_s^* - k + \mu z_s V(0,1)\right] ds
\end{aligned}$$

Obviously the second value decreases if we force the firm to exit at the suboptimal time $\hat{\tau} - \tilde{t}$ and also omit the positive revenue x_s^*, i.e.

$$V(t^*+t, z_t) > \int_{s=t}^{\hat{\tau}-\tilde{t}} e^{-\int_t^s r+\mu z_u du} \left[-k + \mu z_s V(0,1)\right] ds.$$

Taking differences

$$\hat{V}(z_t) - \hat{V}(t^*+t, z_t) < \int_{s=t}^{\hat{\tau}-\tilde{t}} e^{-\int_t^s r+\mu z_u du} z_s \left[1 + \mu(\hat{V}(1) - V(0,1))\right] ds < \frac{1}{\mu} + \hat{V}(1) - V(0,1),$$

where the last inequality follows from

$$\int_{s=t}^{\hat{\tau}-\tilde{t}} e^{-\int_t^s r+\mu z_u du} z_s ds < \int_{s=t}^{\infty} -\frac{1}{\mu}\frac{d}{ds}\left(e^{-\int_t^s \mu z_u du}\right) ds = \frac{1}{\mu}\left[1 - \exp\left(\int_t^{\infty} \mu z_u du\right)\right] \leq \frac{1}{\mu}.$$

Rearranging, we get

$$1 + \mu(\hat{V}(1) - \hat{V}(z_t)) > \mu(V(0,1) - \hat{V}(t^*+t, z_t)).$$

Thus both the integrand and the integration domain are larger in (A.6) than in (A.7), implying $\hat{\Gamma}(\tilde{t}) > \Gamma(t^*)$ and completing the proof.

References

Abito, J.-M., D. Besanko, and D. Diermeier (2012): "Corporate Reputational Dynamics, Private Regulation, and Activist Pressure," Working paper, Northwestern.

Aliprantis, C. D., and K. C. Border (1999): *Infinite Dimensional Analysis*, vol. 32006. Springer.

Atkeson, A., C. Hellwig, and G. Ordonez (2012): "Optimal Regulation in the Presence of Reputation Concerns," Working paper, Penn.

Bar-Isaac, H. (2003): "Reputation and Survival: learning in a dynamic signalling model," *Review of Economic Studies*, 70(2), 231–251.

Board, S., and M. Meyer-ter-Vehn (2013): "Reputation for Quality," *Econometrica*, 81(6), 2381–2462.

Bonatti, A., and J. Hörner (2011): "Collaborating," *American Economic Review*, 101, 632–663.

——— (2013): "Career Concerns and Market Structure," Working paper, MIT.

Bronnenberg, B. J., J.-P. H. Dubé, and M. Gentzkow (2012): "The Evolution of Brand Preferences: Evidence from Consumer Migration," *American Economic Review*, 102(6), 2472–2508.

Cabral, L. (2014): "Living Up To Expectations: Corporate Reputation and Sustainable Competitive Advantage," Working paper, NYU.

Cisternas, G. (2014): "Two-Sided Learning and Moral Hazard," Working paper, MIT.

Davis, M. (1993): *Markov Models and Optimization*. Chapman & Hall.

Ericson, R., and A. Pakes (1995): "Markov-Perfect Industry Dynamics: A Framework for Empirical Work," *Review of Economic Studies*, 62(1), 53–82.

Foster, L., J. Haltiwanger, and C. Syverson (2013): "The Slow Growth of New Plants: Learning about Demand," Working paper, Chicago.

Gale, D., and R. Rosenthal (1994): "Price and Quality Cycles for Experience Goods," *RAND Journal of Economics*, 25(4), 590–607.

Goldfarb, A. (2007): "Schlitz: Why Schlitz Hit the Fan," in *Industry and Firm Studies, 4th edition*, ed. by V. J. Tremblay, and C. H. Tremblay, pp. 321–341. M. E. Sharpe Inc.

Holmström, B. (1999): "Managerial Incentive Problems: A Dynamic Perspective," *Review of Economic Studies*, 66(1), 169–182.

Hopenhayn, H. A. (1992): "Entry, Exit, and firm Dynamics in Long Run Equilibrium," *Econometrica*, 60(5), 1127–1150.

Jovanovic, B. (1982): "Selection and the Evolution of Industry," *Econometrica*, 50(3), 649–670.

Keller, G., and S. Rady (2014): "Breakdowns," *Theoretical Economics*, forthcoming.

Kovrijnykh, A. (2007): "Career Uncertainty and Dynamic Incentives," Working paper, University of Chicago.

Mailath, G., and L. Samuelson (2001): "Who Wants a Good Reputation," *Review of Economic Studies*, 68(2), 415–441.

Milgrom, P., and I. Segal (2002): "Envelope theorems for arbitrary choice sets," *Econometrica*, 70(2), 583–601.

Rob, R., and A. Fishman (2005): "Is Bigger Better? Customer Base Expansion through Word-of-Mouth Reputation," *Journal of Political Economy*, 113(5), 1146–1175.

Sannikov, Y. (2014): "Moral Hazard and Long-Run Incentives," Working paper, Princeton University.

Syverson, C. (2011): "What Determines Productivity?," *Journal of Economic Literature*, 49(2), 326–365.