Equilibrium Technology Diffusion, Trade, and Growth

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ABSTRACT ——

How do reductions in barriers to international trade affect aggregate economic growth and welfare? We develop a novel dynamic model of growth and trade, driven by technology adoption, to better understand the interaction between technology diffusion, openness, and growth. In the model, heterogeneous firms choose to produce and trade or pay a cost and search within the economy to upgrade their technology. These upgrading and production choices determine the productivity distribution from which firms can acquire new technologies and, hence, the rate of technological diffusion and growth. In equilibrium, low productivity firms choose to upgrade their technology to remain competitive and profitable. Lower trade barriers enhance competitive forces that differentially affect firms of varying productivity levels. Lower barriers tend to reduce profits for all domestic firms by creating added competition from foreign firms, but improve profits for the highly productive firms by providing expanded opportunities through exporting. This shift in the relative value of firms provides increased incentives to upgrade technology, which are counterbalanced by an increasing cost of upgrading technology due to general equilibrium effects. In our baseline calibration, an increased growth rate generates a dynamic component of welfare that magnifies the traditional static component to increase the welfare gains from openness.

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1. Introduction

This paper studies how reductions in barriers to international trade affect aggregate economic growth and welfare. We develop a dynamic model of growth and trade and we use the model to study the relationship between technology diffusion, openness, and growth. The novel feature of our model is that technology diffusion occurs in equilibrium, as heterogeneous firms choose to acquire productivity-increasing technology from other firms producing in the economy. Reductions in barriers to trade increase the profitability of high productivity firms and decrease the profitability of low productivity firms, providing incentives to upgrade technology. General equilibrium effects counterbalance these incentives and act to lower growth rates by raising the cost of upgrading technology. In our baseline calibration, we find that the dynamic gains from trade roughly double the overall benefits of trade relative to the traditional static gains.

We model firms as monopolistic competitors who are heterogeneous in their productivity/technology. Our model of a firm's production decision is standard, with each firm having the opportunity to export after paying a fixed cost.[1] Our model of technology adoption and diffusion builds on Perla and Tonetti (2012), where firms choose to either upgrade their technology or continue to produce in order to maximize expected discounted profits for the infinite horizon. If a firm decides to upgrade its technology, it pays a fixed cost in return for a random productivity draw from the distribution of producing firms in equilibrium. Thus, the key aggregate state variable for a firm is the distribution of firms producing at any instant. Economic growth is a result, as firms are continually able upgrade their technology by learning from other, better firms in the economy. Thus, this is a model of growth driven by endogenous technology diffusion.[2]

We compute and analyze a balanced growth path equilibrium of this economy. There are essentially two steps to establishing the existence of a balanced growth path equilibrium. First, we characterize the evolution of the technology distribution over time, given the evolution of the firms' dynamic policy rule (i.e. upgrade or not). We show that the technology distribution evolves according to a repeated truncation of the time zero distribution. This result plus the assumption that the initial distribution is Pareto implies that every subsequent distribution of technology is Pareto itself. This allows us to completely characterize the path of the static-trade equilibrium as in Chaney (2008) or Eaton, Kortum, and Kramarz (2011) at every point in time.[3]

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[1]This setup follows the heterogeneous productivity monopolistic-competition frameworks of Melitz (2003), Chaney (2008), and Eaton, Kortum, and Kramarz (2011).

[2]This type of technology diffusion is closely related to the models of Lucas (2009), and Lucas and Moll (2012), who study knowledge/idea diffusion amongst individuals in a closed-economy. Kortum (1997) is an antecedent of these models where knowledge diffusion comes from an external source.

[3]These results are independent of the balanced growth requirement which will allow us to study off balanced growth path dynamics in future versions.

Second, we solve the firm's dynamic optimization problem to obtain the optimal policy rule of when to upgrade technology, given a perceived law of motion for the distribution.

The key equilibrium requirement (amongst others) is that the actual evolution of the technology distribution conforms with firms' perceived law of motion for the distribution of technology. On the balanced growth path we require that the distribution of technologies is stationary when appropriately scaled and that real GDP grows at a constant rate.

We calibrate the model and perform several comparative dynamics, showing how changes in parameters affect growth rates on the balanced growth path. The main comparison focuses on how changes in iceberg trade costs affect growth rates. Changes in iceberg trade costs are interesting because they control how much each country trades with other countries and hence the degree of openness. We find that decreases in the iceberg trade costs can optimally increase or decrease the growth rate of the economy.

When technology adoption costs must be paid in output, growth rates increase. As the economy becomes more open, the value of a low productivity firm changes relative to the value of a high productivity firm. Low productivity firms lose value in response to reduced trade barriers, as increased competition from foreign firms reduces their profits. High productivity firms are able to expand and export, increasing their profits and the value of the firm. Additionally, wages increase, especially decreasing the profits of domestic producers. The net effect of these forces is to push more low productivity firms to upgrade their technology sooner, as the costs of searching in terms of forgone production are smaller and the potential benefits (i.e., becoming an exporter) are now larger. Because the amount and frequency of firms upgrading their technology is intimately tied to aggregate growth, the growth rate increases as the economy becomes more open.

When technology adoption costs must be paid by hiring labor, a general equilibrium affect dominates and growth rates decrease in response to a reduction in barriers to trade. Since the wage increases due to increased demand for labor to produce for international sale, the cost of technology adoption also rises. The increase in the cost of upgrading technology dominates the increased convexity of the value function and growth declines as firms wait longer before upgrading their technology.

Compared to many static models of trade, the potential welfare gains from reduced trade barriers are large and depend crucially on the interest rate, the shape of the productivity distribution, and the cost of technological adoption. Welfare can even improve modestly if growth rates decline, as the initial increase in consumption from imports can offset lower growth. The model features strong externalities, since firms do not internalize how their search decisions influence the evolution of the productivity distribution, and thus the future opportunities of other firms. These strong externalities and general equilibrium effects can lead to reduced welfare in equi-

librium, as the large rise in the cost of technology adoption and the socially suboptimal rate of technology adoption lead to low growth rates in response to lower trade barriers.

A second comparative static changes the "thickness" of the right tail of the initial productivity distribution, which is governed by the shape parameter in the Pareto distribution. We show that as the right tail of the initial productivity distribution becomes thicker, the elasticity of growth with respect to the degree of openness increases; the cost of autarky and benefit of frictionless trade (in terms of growth rates) both become larger. This result is distinct from, but related to, the findings in other models of knowledge diffusion that growth increases with the thickness of the right tail of the productivity distribution (see, e.g., Alvarez, Buera, and Lucas (2008), Lucas (2009), Perla and Tonetti (2012), Lucas and Moll (2012)). Moreover, this result suggests ways to use cross-country evidence on the relationship between trade and growth to discipline this all important parameter.

The third comparative static focuses on scale effects. We keep the parameterization of the model the same, but double the number of countries to understand how the scale or size of the economy matters. We show that the relationship between growth and openness is unchanged when we increase the scale of the economy. Substantively, this result suggests that the key force behind the relationship between growth and openness in our model does not operate through scale effects per-se (i.e. firms upgrade faster because markets and profits are larger). Because scale seems to be absent, this result reinforces the idea that the driving force is how openness changes the relative value of firms across different productivity levels. Given the emphasis on scale effects in previous endogenous growth models (see, e.g., Jones (2005a) and the discussion in Ramondo, Rodriguez-Clare, and Saborio-Rodriguez (2012)) this result seems surprising. However, scale effects in newer models of knowledge diffusion are not well understood and we hope to explore them further in the future.

Our main contribution is to develop a new framework where opening up to trade affects the dynamic incentives of firms to adopt technology and, hence, aggregate productivity growth. Broadly speaking, the core mechanism—firm level technology adoption—is distinct from others emphasized in the literature that studies the effects of opening to trade. First, this is not a model of how technology evolves at the frontier and how openness affects the pace of innovation as in Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992) and the open economy studies of Rivera-Batiz and Romer (1991) and Baldwin and Robert-Nicoud (2008). Our model is one of firms at the bottom of the distribution who make small, incremental improvements in their productivity.

A second distinction is that our mechanism focuses on within-firm productivity gains and how they translate to aggregate productivity gains from trade. In contrast, Melitz (2003) studies how opening to trade reallocates production across firms as the least productive firms exit and

high productivity exporters expand their scale. Eaton and Kortum (2002) and Bernard, Eaton, Jensen, and Kortum (2003) are other examples that emphasize allocative productivity gains rather than within-firm productivity gains. Empirically, the distinction between within-firm effects and allocative effects is relevant, as there is much evidence that trade liberalization leads to significant within-firm productivity gains (see, e.g., Pavcnik (2002), Holmes and Schmitz (2010), and Syverson (2011)).

Alvarez, Buera, and Lucas (2012) is perhaps the most closely related paper to ours. They develop an open economy model to study the diffusion of ideas across countries. Moreover, in Alvarez, Buera, and Lucas (2012) idea arrivals are exogenous and hence not a choice by the firm in response to changes in the degree of openness.[4] We focus only on intra-country technology adoption and study how openness affects firms' dynamic incentives to adopt technology and, in turn, aggregate productivity growth.

2. Model

2.1. Countries, Time, Consumers

There are N countries with subscripts i denoting the identity of each country. Time is continuous and evolves for the infinite horizon. The representative consumer in country i is risk neutral with period utility function

$$U_i(t) = \int_t^\infty e^{-r(\tau-t)} C_i(\tau) d\tau. \tag{1}$$

The utility function $U_i(t)$ is the discounted value of future consumption for the infinite future, where r is the exogenously given discount rate. Consumers supply labor to firms for the production of varieties, the fixed costs of production, and possibly for technology acquisition. Labor is supplied inelastically and the total units of labor in a country are L_i. Consumers also own the firms (described below) operating within their country, thus, their income is the sum of total payments to labor and profits.

Consumption is defined over a final good that is an aggregate bundle of varieties aggregated by a constant elasticity of substitution (CES) function, where $P_i(t)$ is the CES aggregate price index. We abstract from borrowing or lending decisions, so consumers face the following budget constraint

$$w_i(t)L_i + P_i(t)\Pi_i(t) = P_i(t)C_i(t), \tag{2}$$

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[4]This distinction is the key advancement of Perla and Tonetti (2012) and Lucas and Moll (2012) in the idea diffusion literature. Modeling when agents *choose* to upgrade their productivity permits analysis of how the economic environment affects incentives and how policy can be implemented to change behaviors and improve welfare.

where $\Pi_i(t)$ is aggregate profits (net of investment costs) in consumption units. These relationships are elaborated in detail below.

2.2. Firms

In each country there is a final good producer that produces and supplies the aggregate consumption good competitively. This final good producer aggregates individual varieties v. In each country there is a unit mass of infinitely lived, monopolistically competitive firms. Each firm alone can supply variety v.

Final Good Producer. The final good producer in each country is the purchaser of these varieties and solves the problem:

$$\max_{q_{ij}(v,t)} \quad P_i(t)Q_i(t) - \sum_{j=1}^{N} \int_{\Omega_{ij}(t)} q_{ij}(v,t)p_{ij}(v,t)dv$$

$$\text{s.t.} \quad Q_i(t) = \left(\sum_{j=1}^{N} \int_{\Omega_{ij}(t)} q_{ij}(v,t)^{\frac{\sigma-1}{\sigma}} dv \right)^{\frac{\sigma}{\sigma-1}}.$$

The measure $\Omega_{ij}(t)$ defines the set of varieties consumed in country i from country j. The parameter σ controls the elasticity of substitution across varieties. The solution to this problem yields the demand function for a firms variety in each market:

$$q_{ij}(v,t) = Q_i(t) \left(\frac{p_{ij}(v,t)}{P_i(t)} \right)^{-\sigma}.$$

Individual Variety Producers. Firms producing individual varieties are heterogeneous over their productivity z and each firm alone can supply a unique variety v. We will drop the notation carrying around the variety identifier, as it is sufficient to identify each firm with its productivity level, z.

Firms producing individual varieties hire labor, ℓ, to produce quantity q with a linear production technology,

$$q = z\,\ell.$$

The cumulative distribution function $F_i(z,t)$ describes how productivity varies across firms, within a country.

Each instant, all firms can pay a fixed cost $x_i(t)$ to draw a new productivity. If the firm decides to pay this cost, they stop producing and receive a random draw from the distribution of only active producers in the economy, as in Perla and Tonetti (2012). Thus the random productivity

draw will be from a transformation of the equilibrium productivity distribution $F_i(z,t)$. This transformed distribution will be a function of the optimal policy of all firms, i.e. produce or draw a new productivity. Recursively, the optimal policy of firms will depend on the expected evolution of this distribution.

There are several interpretations of this technology choice. Mathematically, it is similar to the models of Lucas (2009) and Lucas and Moll (2012) where agents randomly meet and acquire each others technology. In this model, however, there is a sense in which "meetings" are directed. In equilibrium, there is a threshold productivity, $h_i(t)$, such that all firms below it will randomly meet a non-searching, producing firm above the threshold. Hence, this meeting structure represents limited directed search towards more productive firms. Empirically, this technology choice can be thought of as intangible investments that manifest themselves as improvements in productivity like improved production practices, work practices, advertising, supply-chain and inventory management, etc. See, for example, the discussion of changes in productivity within a plant or firm in Holmes and Schmitz (2010) and Syverson (2011).

Firms also have the ability to export at some cost. To export, a firm must pay a fixed flow cost in units of labor, $w_j \kappa_j$, to export to foreign market j. Exporting firms also face iceberg trade costs, $d_{ji} \geq 1$, to ship goods abroad from i to destination j.

Given this environment, firms must make choices regarding how much to produce, how to price their product, whether to export, and whether to change their technology. These choices can be separated into problems that are static and dynamic. Below we first describe the dynamic problem of a firm in country i, taking the profit functions and evolution of the productivity distribution as given. We then describe the static problem of the firm to derive the profit functions.

2.3. Firms Dynamic Problem

Given the static profit functions and a perceived law of motion for the productivity distribution which are described below, each firm has the choice to acquire a new technology, z, and also whether to export to market j or not. If a firm chooses to search and upgrade its technology, it will not produce any output in that instant, it will pay a search cost, and it will meet another firm that has chosen to produce and copy their productivity level. In other words, a firm is able to replicate (at a cost) the technology of another producer that is currently operating. Thus, the new productivity level is a random variable that's distribution is the equilibrium distribution of technology, conditional on the productivity being above the search threshold. The essential trade-off that a firm faces is between the benefits of operating its existing technology versus the expected net benefit of operating with a new technology. The firm's objective is to maximize the present discounted expected value of real profits, since it is owned by the consumers. With

all profits $\pi_{ji}(z,\tau)$ and costs $x_i(\tau)$ in units of the final consumption good, the firm problem is

$$V_i(z,t) = \max_{\substack{T_{ji} \geq t \\ T_i \geq t}} \left\{ \int_t^{T_i} e^{-r(\tau-t)}\pi_{ii}(z,\tau)d\tau + \sum_{j\neq i} \int_t^{T_{ji}} e^{-r(\tau-t)}\pi_{ji}(z,\tau)d\tau + e^{-r(T_i-t)}\left[W_i(T_i) - x_i(T_i)\right] \right\}$$

(3)

where

$$W_i(t) := \int V_i(\tilde{z},t)dF_i(\tilde{z},t|\tilde{z} > h_i(t))$$

(4)

A firm chooses an absolute time, T_i, at which it will search for a new technology. For the waiting time before searching, $T_i - t$, the firm produces and earns profits from operating domestically. The firm also chooses an absolute time at which it will stop exporting to country j, T_{ji}. While $T_{ji} - t > 0$ the firm is an exporter to destination j and receives profits from this activity. Given the fixed cost of exporting, every exporter will also operate domestically, i.e., $T_{ji} \leq T$ and search occurs at time T_i when a firm is only operating domestically. When a firm chooses to search, it gets a new productivity draw with expected benefit $W_i(T_i)$ and it pays the fixed cost of searching $x_i(T_i)$.[5] By standard arguments, the solution to this problem can be shown to be reservation productivity functions, $h_i(t)$ and $\phi_{ji}(t)$. All firms with productivity less than or equal to $h_i(t)$ will search and all other firms will produce. All firms with productivity greater than or equal to $\phi_{ji}(t)$ will export to destination j and all firms with lower productivity will not export to j. Define the search and exporter thresholds as these indifference points:

$$h_i(t) := \max\{ z \mid T_i(z,t) = t \}$$

(5)

$$\phi_{ji}(t) := \max\{ z \mid T_{ji}(z,t) = t \}$$

(6)

The function $h_i(t)$ maps time into the largest productivity level such that the firm with that productivity level is upgrading its technology. Given this definition, the function $h_i^{-1}(z)$ defines the *time* at which a firm with productivity level z will draw a new technology. Then, since a draw comes from the equilibrium distribution of *producers*, the expected value of the new technology level, $W_i(T_i)$, is defined in (4). Notice that the value of the new technology is integrated with respect to the conditional productivity distribution $F_i(z,t|z > h_i(t))$ and hence is a function of the choices of the individual firms. We detail the evolution of this distribution—in equilibrium—in more detail below. This problem takes the profit functions as given, but they are the result of a

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[5]The effects of particular specifications of the search cost are detailed in Section 3.5.A. In particular, we examine the importance of the degree to which costs require hiring labor versus spending goods.

static optimization problem.

2.4. Firms Static Problem

Below we describe a firm's static problem and suppress any explicit dependence upon time to ease notation. Given a firm's location, productivity level, aggregate prices, and final good producers' demand, the firm's static decision is to chose the amount of labor to hire, the price to set, and exporting decisions to each destination to maximize profits each instant. Formally, the optimization problem is

$$P_i \pi_{ii}(z) = \max_{p_{ii}, \ell_{ii}} \ p_{ii} z \ell_{ii} - w_i \ell_{ii}.$$

where $\pi_{ii}(z)$ is defined in units of the final consumption good. Using the demand function from the final goods producer, the profit function satisfying this problem is

$$P_i \pi_{ii}(z) = \left(\frac{1}{\sigma} \right) \left(\frac{m \ w_i}{z} \right)^{1-\sigma} \frac{Y_i}{P_i^{1-\sigma}}, \quad \text{where} \quad m := \frac{\sigma}{\sigma - 1}, \tag{7}$$

where m is the standard markup over marginal cost, w_i is the wage rate in country i, and $Y_i = P_i Q_i$ is total expenditures on final goods in country i.

The decision to export to market j is similar, but differs in that the firm faces variable iceberg trade costs and a fixed cost to sell in the foreign market, or

$$P_i \pi_{ji}(z) = \max_{p_{ji}, \ell_{ji}} \left\{ p_{ji} d_{ji}^{-1} z \ell_{ji} - w_i \ell_{ji} - w_j \kappa_j \ , 0 \right\}.$$

Conditional on exporting, the profits from exporting to market j are

$$P_i \pi_{ji}(z) = \frac{1}{\sigma} \left(\frac{m \ d_{ji} \ w_i}{z} \right)^{1-\sigma} \frac{Y_j}{P_j^{1-\sigma}} - w_j \kappa_j \tag{8}$$

The productivity level ϕ_{ji}, which determines the cutoff productivity level above which firms from market i will export to market j, is

$$\phi_{ji} = k_1 \frac{w_i}{P_j} \left(\frac{w_j \kappa_j}{Y_j} \right)^{\frac{1}{\sigma-1}}, \quad \text{where} \quad k_1 := m \ d_{ji} \sigma^{\frac{1}{\sigma-1}}. \tag{9}$$

All firms in market i with productivity level greater than or equal to ϕ_{ji} will export to market j, earning positive profits. Note that the exporter threshold, ϕ_{ji}, is directly related to time T_{ji}, as defined in (6).

3. Equilibrium

An equilibrium of the model economy consists of a set of initial productivity distributions and sequences of productivity distributions, firms' search and exporter thresholds, prices, and allocations, that solve firms' static and dynamic problems and satisfy market clearing and rational expectation conditions. Below, we describe key equilibrium relationships, which can be separated into dynamic and static equilibrium relationships. We then formally define a balanced growth path equilibrium and state Proposition 2 which says that one exists and is proved by construction.

3.1. Dynamic Equilibrium Relationships

Describing and deriving the dynamic equilibrium relationships is done two steps. First, we derive the law of motion of the productivity distribution given a time path for the threshold $h_i(t)$ defined in (5). Second, we derive a system of equations thats solution is the optimal dynamic firm policy, i.e., the search threshold $h_i(t)$, given a perceived law of motion of the productivity distribution.

Deriving the Law of Motion of the Productivity Distribution. Here we derive the law of motion for the distribution. This law of motion is a function of individual firms' optimal times to draw a new productivity and, hence, the threshold $h_i(t)$ below which firms upgrade their productivity. This first step in describing the equilibrium takes the threshold as given and then derives how the productivity distribution evolves.

Some formalities: As a tie-breaking rule, it is assumed that agents at the threshold search, and hence the function is right-continuous. The description in this section holds for regions of continuity in $h_i(t)$. There are instances when $h_i(t)$ may not be continuous, particularly at "special times" that reset the economy like time 0 or potentially when a closed economy unexpectedly opens to foreign trade. Technical details surrounding discontinuities in $h_i(t)$ and more detailed derivations are provided in the appendix.

In the economic environment described in Section 2.2, we specified that firms who decide to draw a new productivity only draw from the set of firms that are producing. Therefore, firms drawing at time t only receive a draw from the productivity distribution strictly above $h_i(t)$. This implies that $h_i(t)$ is an absorbing barrier sweeping through the distribution from below and, thus, the infimum of support of the productivity density is

$$\inf \text{support}\{F_i(\cdot, t)\} = h_i(t). \tag{10}$$

Given this observation, the distribution from which firms upgrading their technology receive a

draw is then the existing productivity distribution

$$f_i(z,t|z > h_i(t)) = f_i(z,t). \tag{11}$$

Law of Motion: Kolmogorov Forward Equation. A key determinant of the growth rate of the economy and of the evolution of the productivity distribution is the flow of searchers upgrading their technology, $S_i(t)$. There exists a flow of searchers during each infinitesimal time period, where the flow of searchers is the net flow of the probability current through the search threshold, $h_i(t)$. As is derived in the appendix,

$$S_i(t) = h_i'(t)f_i(h_i(t),t). \tag{12}$$

While $h(t)$ is an absorbing barrier removing mass from the system, the flow of searchers are a source that are redistributed back into the system.[6] These agents who search have an equal probability to draw any z in $f(z,t)$, as stated in equation 11. Hence, since the only time a firm's productivity changes is when it searches, the Kolmogorov forward equation (KFE) for $z > h_i(t)$ is simply the flow of searchers (source) times the density they draw from (redistribution density):

$$\frac{\partial f_i(z,t)}{\partial t} = S_i(t)f_i(z,t) \tag{13}$$

Using equation 12

$$\frac{\partial f_i(z,t)}{\partial t} = f_i(z,t)f_i(h_i(t),t)h_i'(t). \tag{14}$$

In words, this says that the search threshold is sweeping across the density at rate $h_i'(t)$ and as the search boundary sweeps across the density from below it collects $f_i(h_i(t),t)$ amount firms. Then $f_i(h_i(t),t)h_i'(t)$ is the flow of searchers to be returned back into the distribution. Since the economic environment is such that searchers only meet existing producers above $h_i(t)$, but $h_i(t)$ is the infimum of support of $f_i(z,t)$, then the searchers are redistributed across the entire support of $f_i(z,t)$. Since agents draw directly from the productivity density, they are redistributed throughout the distribution in proportion to the density and thus, the flow of searchers $f_i(h_i(t),t)h_i'(t)$ multiplies the density $f_i(z,t)$.

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[6]This system is related to the "return process" featured in Luttmer (2007). Luttmer (2007) focuses on how entry and exit driven by exogenous stochastics shape the productivity distribution, while in this paper existing firms' productivities improve as they choose to upgrade their technology because they internalize the value of increased future profits.

Solving the KFE.

Proposition 1. *$f_i(z,t)$ evolves according to repeated left truncations at $h_i(t)$ for **any** $h_i(t)$ and $F_i(0)$.*

A solution to the Kolmogorov forward equation 14 is

$$f_i(z,t) = \frac{f_i(z,0)}{1 - F_i(h_i(t),0)}. \tag{15}$$

That is, the distribution at date t is a truncation of the initial distribution at the minimum of support at time t, $h_i(t)$.

Solving the Firm Dynamic Problem. Solving (3) consists of jointly finding the optimal search policy function, $h_i(t)$, and the expected value of search, $W_i(t)$, given profit functions, a productivity distribution, $F_i(z,t)$, and it's law of motion. Below, we describe the general steps to finding this solution.

Recall that the equilibrium search threshold $h_i(t)$ is the minimum of the productivity distribution. Given parameter constraints (particularly a positive fixed cost of exporting) the exporter productivity threshold is greater than the technology adoption search threshold. Thus, only non-exporters optimally choose to search, and the first order condition that determines the optimal search time is the derivative of the value function with respect to the search timing decision, where the discounted stream of export profits earned before searching is 0 with certainty:

$$V_i(z,t)_{|(\pi_{ji}=0)} = \max_{T_i \geq t} \left\{ \int_t^{T_i} e^{-r(\tau-t)} \pi_{ii}(z,\tau) d\tau + e^{-r(T_i-t)} \left[W_i(T_i) - x_i(T_i) \right] \right\},$$

Taking the derivative of the value function of a non-exporting firm with respect to T_i yields

$$\frac{\partial V_i(z,t)_{|(\pi_{ji}=0)}}{\partial T_i} = \left[\frac{\partial \int_t^{T_i} e^{-r(\tau-t)} \pi_{ii}(z,\tau) d\tau}{\partial T_i} + \frac{e^{-r(T_i-t)} \partial W_i(T_i)}{\partial T_i} - \frac{\partial e^{-r(T_i-t)} x_i(T_i)}{\partial T_i} \right] \tag{16}$$

$$= e^{-r(T_i-t)} \left[\pi_{ii}(z,T_i) - rW_i(T_i) + W_i'(T_i) + rx_i(T_i) - x_i'(T_i) \right] \tag{17}$$

Setting $T_i = t$, i.e., where the firm is just indifferent between switching technologies and producing, and recognizing that the productivity level of the indifferent firm is $z = h_i(t)$ by defini-

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tion, we have the first order condition

$$0 = \pi_{ii}(h_i(t), t) - rW_i(t) + W_i'(t) + rx_i(t) - x_i'(t)$$

$$r(W_i(t) - x_i(t)) = \pi_{ii}(h_i(t), t) + W_i'(t) - x_i'(t) \tag{18}$$

To provide intuition, this FOC is analogous to the standard bellman equation in asset pricing, $rV(t) = \pi(t) + \frac{dV(t)}{dt}$, where the flow (net) value of an asset must equal its dividend plus capital gains. Since in our problem, this is the equity value of a firm, there is a natural arbitrage free pricing interpretation. If the LHS was larger than the RHS, then the current value of the firm would be larger than its dividend and resale value warrants, and an agent could make money by shorting the firm this instant and buying it an instant later.

Equation 18 is one equation in $h_i(t)$ and $W_i(t)$. We now want to find another equation in $h_i(t)$ and $W_i(t)$, providing two equations in two unknowns.

The second equation we focus on is the expected value of acquiring a new technology.

Since $h_i(t)$ is the minimum of support of $F_i(z, t)$ as stated in equation 10, we can rewrite equation 4 as

$$\begin{aligned}
W_i(t) &= \int V_i(z, t) dF_i(z, t) \\
&= \int_{h_i(t)}^{\infty} \left\{ \int_t^{h_i^{-1}(z)} e^{-r(\tau - t))} \pi_{ii}(z, \tau) d\tau + \sum_{j \neq i} \int_t^{\phi_{ji}^{-1}(z)} e^{-r(\tau - t)} \pi_{ji}(z, \tau) d\tau \right. \\
&\quad + \left. e^{-r(h_i^{-1}(z) - t)} \left[W_i(h_i^{-1}(z)) - x_i(h_i^{-1}(z)) \right] \right\} dF_i(z, t)
\end{aligned} \tag{19}$$

The first integral in the inside bracket is the discounted value of domestic profits until the next change of technology, where the search time T_i has been replaced with the function $h_i^{-1}(z)$. The second integral in the inside bracket is the discounted value of profits from exporting. Similarly, the final exporting time, T_{ji}, has been replaced with the function $\phi_{ji}^{-1}(z)$, which is defined in (6). The function $\phi_{ji}(z)$ is the largest z such that a firm *stops* exporting to market j. Thus the inverse of this function defines the time when the firm stops exporting to market j. The final term in the inside bracket is the discounted value of the new technology net of search costs evaluated at the date $h_i^{-1}(z)$.

Outside the brackets, we then integrate over productivity levels with the existing (equilibrium) productivity distribution of producers, $F_i(z, t)$, since that is the distribution from which firms draw. This defines the expected value of acquiring a new technology.

Equations (18) and (19) give us two equations from which we can solve for the policy function,

$h_i(t)$, and the expected value of a new productivity draw, $W_i(t)$, for a given a law of motion for the productivity distribution, $F_i(z,t)$.

3.2. The Pareto Distribution

The shape of the productivity distribution plays an important role, affecting both the dynamic technology acquisition decision of the firm and the firm's static production and export decisions. The parametric form of the initial productivity distributions across countries is an essential initial condition specified by the researcher. The Pareto distribution has a history in the growth (Kortum (1997); Jones (2005b); Perla and Tonetti (2012)), trade (Melitz (2003); Chaney (2008)), and industrial organization (Gabaix (2009)) literature as being both empirically motivated and particularly tractable. To maintain analytical tractability in the static firm problem and to allow for a balanced growth path, we will solve for the equilibrium of our baseline model under the assumption that the initial distributions are all Pareto with the same tail index.

Assumption 1. *The initial distributions of productivity are Pareto, $F_i(z,0) = 1 - \left(\dfrac{h_i(0)}{z} \right)^\theta$ $\forall i$, with densities, $f_i(z,0) = \theta h_i(0)^\theta z^{-1-\theta}$.*

Lemma 1. *Assumption 1 together with Proposition 1 implies*

$$f_i(z,t) = \theta h_i(t)^\theta z^{-1-\theta} \tag{20}$$

That is, if $F_i(z,0)$ is Pareto with tail index θ and minimum of support $h_i(0)$, then $F_i(z,t)$ remains Pareto with the same tail index θ and new minimum of support $h_i(t)$. This greatly simplifies the derivation of static equilibrium relationships and solving for the model along a balanced growth path.

3.3. Static Equilibrium Relationships

At every date t, there are essentially three aggregate equilibrium objects that determine the static allocation problem of production and trade across countries. These are the aggregate price index, trade shares, and aggregate sales. Below we describe each of these objects as an explicit function of time, with detailed derivations provided in the appendix.

Price Index. From standard CES arguments, the price index in market i is

$$P_i(t)^{1-\sigma} = \int_{h_i(t)}^\infty p_{ii}(z,t)^{1-\sigma} f_i(z,t) dz + \sum_{j \neq i} \int_{\phi_{ij}(t)}^\infty p_{ij}(z,t)^{1-\sigma} f_j(z,t) dz.$$

Given the assumption that initial productivity distributions are Pareto, the CES price index is

$$P_i(t)^{1-\sigma} = k_2(mw_i(t))^{1-\sigma}h_i(t)^{\sigma-1} + \sum_{j\neq i} k_3(md_{ji}w_j(t))^{-\theta}h_j(t)^\theta P_i(t)^{\theta+1-\sigma}\left(\frac{w_i(t)\kappa_i}{Y_i(t)}\right)^{\frac{\sigma-1-\theta}{\sigma-1}} \quad (21)$$

where $k_2 := \frac{\theta}{\theta+1-\sigma}$ and $k_3 := k_2\sigma^{\frac{\sigma-1-\theta}{\sigma-1}}$.

Note that this expression is similar—but not the same—as standard results for the CES price index in monopolistic competition models (e.g. Chaney (2008) or Eaton, Kortum, and Kramarz (2011)). The key differences regard the term for the home country effect on the left-hand side of (21). This term is not multiplied by the price index $P_i^{\theta+1-\sigma}$ as the foreign country terms are. The power term is $1-\sigma$ not θ as in the foreign country terms. Finally, the constant multiplying each of these terms are different as well (k_2 vs. k_3). The reason for this difference is that there is not a fixed cost of operating domestically as models such as Chaney (2008) or Eaton, Kortum, and Kramarz (2011) have.

Trade shares. Trade shares, λ_{ji}, equal the expenditure country j spends on goods from country i relative to total expenditure in country j. Mathematically, the trade share is given by

$$\lambda_{ji}(t) = \int_{\phi_{ji}(t)}^\infty \frac{p_{ji}(z,t)q_{ji}(z,t)}{Y_j(t)}f_i(z,t)dz$$

Given the distributional assumptions, the optimal price and quantity rules for firms of productivity level z, and the price index in equation 21, the trade share is

$$\lambda_{ji}(t) = \frac{k_3(md_{ij}w_i(t))^{-\theta}h_i(t)^\theta P_j(t)^{\theta+1-\sigma}\left(\frac{w_j(t)\kappa_j}{Y_j(t)}\right)^{\frac{\sigma-1-\theta}{\sigma-1}}}{k_2(mw_j(t))^{1-\sigma}h_j(t)^{\sigma-1} + \sum_{n\neq j} k_3(md_{nj}w_n(t))^{-\theta}h_n(t)^\theta P_j(t)^{\theta+1-\sigma}\left(\frac{w_n(t)\kappa_n}{Y_n(t)}\right)^{\frac{\sigma-1-\theta}{\sigma-1}}}. \quad (22)$$

Note again, that this expression is similar—but not the same—as standard results for trade shares in monopolistic competition models. The key differences are the same issues arising in the price index discussed above.

From equation 22 we can derive a simple expression for the home trade share, $\lambda_{ii}(t)$, in terms of the real wage and technology parameters

$$\lambda_{ii}(t) = k_2 m^{1-\sigma}\left(\frac{w_i(t)}{P_i(t)}\right)^{1-\sigma}h_i(t)^{\sigma-1} \quad (23)$$

By inverting this expression, one can relate the wage to the home trade share in a way that is similar to the expression for the welfare gains from trade as discussed in Arkolakis, Costinot,

and Rodriguez-Clare (2011), with a difference. The key difference is that the elasticity of the real wage with respect to the home trade share is not dictated by the shape parameter in the productivity distribution, θ, but by the preference parameter, σ.

Given the trade share formula, we want to express the profit functions in equations 7 and 8 in a more convenient format. Noting that domestic profits are a function of the real wage and the real wage's relationship to trade shares in (23), we have

$$P_i(t)\pi_{ii}(z,t) = k_4\lambda_{ii}(t)\left(\frac{z}{h_i(t)}\right)^{\sigma-1}Y_i(t), \quad \text{where} \quad k_4 = \frac{1}{k_2\sigma} \tag{24}$$

A similar formula for profits from a firm in market i exporting to market j is

$$P_i(t)\pi_{ji}(z,t) = k_4\lambda_{jj}(t)d_{ij}^{1-\sigma}\left(\frac{z}{h_j(t)}\right)^{\sigma-1}\left(\frac{w_i(t)}{w_j(t)}\right)^{1-\sigma}Y_j(t) - w_j(t)\kappa_j \tag{25}$$

Aggregate Sales. Total sales to country i, $Y_i(t)$, can be expressed as

$$Y_i(t) = w_i(t)L_i + \int_{h_i(t)}^{\infty}P_i(t)\pi_{ii}(z,t)f_i(z,t)dz + \sum_{j\neq i}\int_{\phi_{ji}(t)}^{\infty}P_i(t)\pi_{ji}(z,t)f_i(z,t)dz. \tag{26}$$

This simply says that total sales must equal all income earned from labor plus profits earned by firms and rebated to consumers. Substituting 23 into the profit functions and then integrating over productivity we have

$$Y_i(t) = w_i(t)L_i + k_3k_4\lambda_{ii}(t)Y_i(t) \tag{27}$$

$$+ \sum_{j\neq i}\left\{k_3k_4\lambda_{jj}(t)d_{ij}^{1-\sigma}\left(\frac{w_i(t)}{w_j(t)}\right)^{1-\sigma}Y_j(t)\left(\frac{\phi_{ji}(t)}{h_i(t)}\right)^{\sigma-1-\theta} - w_j(t)\kappa_j\left(\frac{\phi_{ji}(t)}{h_i(t)}\right)^{-\theta}\right\} \tag{28}$$

3.4. Market Clearing

Before constructing the market clearing conditions, we must specify a functional form for the search cost of upgrading technology, $x_i(t)$. The cost to draw a new productivity level is a convex combination of hiring domestic labor and spending final goods, given by $x(t) = \zeta\left[(1-\eta)\frac{w(t)}{P(t)} + \eta\mathbb{E}_t[z_i]\right]$. $\eta \in [0,1]$ controls the degree to which the cost of search requires labor as opposed to goods, while ζ affects the overall cost of upgrading technology. $\frac{w(t)}{P(t)}$ is the real cost of hiring a unit of labor, while $\mathbb{E}_t[z_i]$ is the amount of goods required to search. As is standard, the search cost in goods must grow with the economy or become irrelevant over

time.[7]

Goods Market Clearing. Final goods are spent on either consumption or paying technology adoption costs. Since $\zeta\eta\mathbb{E}_t[z_i]$ final goods are spent per search and there is a flow of $S_i(t)$ searchers each instant, the goods market clearing condition is

$$\frac{Y_i(t)}{P_i(t)} = Q_i(t) = C_i(t) + \zeta\eta\mathbb{E}_t[z_i]S_i(t) \tag{29}$$

Labor Market Clearing. Wages, $w_i(t)$, are determined by the labor market clearing conditions. Aggregating the labor in market i used for domestic production, $L_{i,d}$, and for export production, $L_{i,ex}$, yields

$$L_{i,d} = \int_{h_i(t)}^{\infty} \frac{p_{ii}(z,t)^{-\sigma}Y_i(t)}{zP_i(t)^{1-\sigma}}f_i(z,t)dz \quad \text{and} \quad L_{i,ex} = \sum_{j\neq i}\int_{\phi_{ji}(t)}^{\infty} \frac{d_{ji}p_{ji}(z,t)^{-\sigma}Y_j(t)}{zP_j(t)^{1-\sigma}}f_i(z,t)dz. \tag{30}$$

Since the fixed cost of exporting from j to i requires units of i labor, the total amount of labor from i used in the production of fixed costs equals

$$L_{i,\kappa} = \sum_{j\neq i}\int_{\phi_{ij}(t)}^{\infty} \kappa_i f_j(z,t)dz = \kappa_i \sum_{j\neq i}\left(\frac{h_j(t)}{\phi_{ij}(t)}\right)^{\theta}. \tag{31}$$

Since the technology upgrade search cost is partially paid to hire labor, the search component of labor demand is

$$L_{i,x} = \zeta(1-\eta)S_i(t) \tag{32}$$

Equating aggregate labor supply, L_i, with aggregate labor demand yields the labor market clearing condition $L_i = L_{i,d} + L_{i,ex} + L_{i,\kappa} + L_{i,x}$,

$$L_i = \int_{h_i(t)}^{\infty} \frac{p_{ii}(z,t)^{-\sigma}Y_i(t)}{zP_i(t)^{1-\sigma}}f_i(z,t)dz + \sum_{j\neq i}\int_{\phi_{ji}(t)}^{\infty} \frac{d_{ji}p_{ji}(z,t)^{-\sigma}Y_j(t)}{zP_j(t)^{1-\sigma}}f_i(z,t)dz$$

$$+ \kappa_i \sum_{j\neq i}\left(\frac{h_j(t)}{\phi_{ij}(t)}\right)^{\theta} + \zeta(1-\eta)S_i(t). \tag{33}$$

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[7]This could alternatively be achieved by indexing the cost to the minimum productivity in the economy instead of the average or by making the cost a fixed fraction of output. Average productivity was chosen as the goods cost since it more closely corresponds with the benefit of searching, the expected value of a new productivity.

3.5. A Balanced Growth Path Equilibrium

Definition 1. *A balanced growth path (BGP) equilibrium is a set of initial distributions $F_i(0)$ with support $[zmin_i, \infty)$, search and exporter thresholds $\{h_i(t), \phi_{ji}(t)\}_{t=0}^{\infty}$, firm price and labor policies $\{p_{ji}(z,t), \ell_{ji}(z,t)\}_{t=0}^{\infty}$, wages $\{w_i(t)\}_{t=0}^{\infty}$, aggregate trade shares and price indexes $\{\lambda_i(t), P_i(t)\}_{t=0}^{\infty}$, and a growth rate $g > 0$ such that for all countries i:*

- *Given aggregate prices and distributions*

    - *$h_i(t)$ is the optimal search threshold,*

    - *$\phi_{ji}(t)$ is the optimal export threshold to market j,*

    - *$p_{ji}(z,t)$ and $\ell_{ji}(z,t)$ solve the static optimization problem,*

- *Markets clear at each date t.*

- *Sales grow at a constant rate $Y(t) = Y_0 e^{gt}$,*

- *The distribution of productivities is stationary when re-scaled:*

$$f_i(z,t) = e^{-gt} f_i(z e^{-gt}, 0) \ \forall \ t, \ z \geq \ zmin_i e^{gt}$$

The initial distribution must have infinite right tailed support or the economy would not be able to grow indefinitely. Requiring sales to grow at a constant rate, the productivity distributions to be constant after rescaling, and trade shares and prices to be constant ensures that the BGP equilibrium features balanced growth. Restricting $g > 0$ ensures that the BGP equilibrium has growth.

3.5.A. Solving for a BGP

We will now prove existence of a BGP equilibrium by construction in a particular environment.

Assumption 2. *There are N symmetric countries.*

Assumption 3. *Preferences are such that $\sigma = 2$.*

Assumption 4. $L = 1$

We will guess that along the BGP, the search threshold, wages, and the value of search also grow at constant rate g. These guesses will be verified as part of the solution methodology.

Guess 1. The optimal search threshold grows at the same rate g as total sales: $h(t) = h_0 e^{gt}$.

Guess 2. Wages grow at the same constant rate g as total sales: $w(t) = w_0 e^{gt}$.

Guess 3. The optimal value of search grows at the same rate g as total sales: $W(t) = W_0 e^{gt}$.

Here we solve for the special case of N symmetric countries with CES substitution parameter $\sigma = 2$. Imposing the balanced growth path guess that h evolves according to $h(t) = e^{gt} h_0$, w evolves according to $w(t) = e^{gt} w_0$, the BGP restriction that total sales evolves with $Y(t) = e^{gt} Y_0$, and dropping the notation identifying the country, simplifies the profit functions from equations 24 and 25 to

$$P(t)\pi_d(z,t) = k_4 \lambda(t) \left(\frac{z}{h_0}\right) Y_0 \tag{34}$$

$$P(t)\pi_{ex}(z,t) = \pi_d(z,t)d^{-1} - e^{gt} w_0 \kappa. \tag{35}$$

with the explicit dependence on time now noted. Here the country identifier notation is changed such that $\pi_d(z,t)$ denotes the domestic profits of a firm and $\pi_{ex}(z,t)$ denotes the exporting profits to a single country.

Before proceeding we should note that the trade share, $\lambda(t)$, and price index, $P(t)$, potentially vary with time. We proceed to verify that they do not.

$\lambda(t)$ and $P(t)$ are constant if $h(t)$ and $w(t)$ grow at the same rate as sales, as stated in guess 1 and guess 2. Imposing symmetry, the trade share and associated price index are

$$P(t)^{1-\sigma} = k_2 (mw(t))^{1-\sigma} h(t)^{\sigma-1} + (N-1)k_3 (mdw(t))^{-\theta} h(t)^{\theta} P(t)^{\theta+1-\sigma} \left(\frac{w(t)\kappa}{Y(t)}\right)^{\frac{\sigma-1-\theta}{\sigma-1}} \tag{36}$$

$$\lambda(t) = \frac{k_3 (mdw(t))^{-\theta} h(t)^{\theta} P(t)^{\theta+1-\sigma} \left(\frac{w(t)\kappa}{Y(t)}\right)^{\frac{\sigma-1-\theta}{\sigma-1}}}{k_2 (mw(t))^{1-\sigma} h(t)^{\sigma-1} + (N-1)k_3 (mdw(t))^{-\theta} h(t)^{\theta} P(t)^{\theta+1-\sigma} \left(\frac{w(t)\kappa}{Y(t)}\right)^{\frac{\sigma-1-\theta}{\sigma-1}}}. \tag{37}$$

$$\tag{38}$$

Careful examination of equation (36) shows that only the ratio of wages, $w(t)$, to the search threshold, $h(t)$, affects the evolution of $P(t)$. Thus, if $h(t)$ and $w(t)$ both grow at constant rate g as stated in guess 1 and guess 2, the aggregate price index must be constant.

Next, careful examination of equation (37) shows that only the ratio of wages, $w(t)$, to either the minimum of support of the distribution, $h(t)$, or sales, $Y(t)$, affects the evolution of $\lambda(t)$. Thus, the trade share must be constant on a balanced growth path.

From equation 35, the cutoff value for exporting is

$$\phi(t) = \chi h(t) \quad \text{where} \quad \chi := \frac{d\kappa w_0}{k_4 Y_0 \lambda} \tag{39}$$

Sales are then

$$Y(t) = w(t)k_5 \quad \text{where} \quad k_5 = \frac{1 - \kappa\chi^{-\theta}}{1 - [k_3 k_4 \lambda (1 + (N-1)d^{-1}\chi^{1-\theta})]}. \tag{40}$$

Thus, guess 2, that wages are growing at the same rate as sales, is verified.

Summarizing, we have shown that given assumptions and guesses,

$$\lambda(t) = \lambda(0) \; \forall \, t, \quad P(t) = P(0) \; \forall \, t, \quad \text{and} \quad Y(t) = k_5 w_0 e^{gt}. \tag{41}$$

We have characterized completely how the profit functions are growing over time. The next step is to use this information in the firms dynamic problem and verify that the balanced growth path guesses solve the firm problem and satisfy the BGP equilibrium requirements.

The broad outline for the proof is to verify the economy is on a balanced growth path. A key aspect of this is to verify guess 3 that $W(t)$, the expected value of search, is growing at a constant rate, i.e., $W(t) = e^{gt}W_0$. To accomplish this, we will plug in our guesses above into the formula for the value of search (equation 19) and verification comes if we can solve for a growth rate g and initial value of search W_0 that are independent of time.

Proposition 2. *Given assumptions 1-4, there exists a balanced growth path.*

Note, since P is constant, we can normalize the numeraire such that $P(t) = 1 \; \forall \, t$.

Proof

Given the balanced growth path guess, $h^{-1}(z)$ equals

$$h^{-1}(z) = \frac{\log\left(\dfrac{z}{h_0}\right)}{g}. \tag{42}$$

Using equation 39, $\phi^{-1}(z)$ equals

$$\phi_{ji}^{-1}(z) = \frac{\log\left(\dfrac{z}{\chi h_0}\right)}{g} \tag{43}$$

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Recall from Lemma 1 that

$$f_i(z,t) = \theta h_i(t)^\theta z^{-1-\theta}. \tag{44}$$

Starting from equation 19, substituting the profit functions from equations 34 and 35 and using the guess $W(t) = W_0 e^{gt}$ we have

$$W_0 e^{gt} = \int_{h(t)}^{\infty} \int_t^{h^{-1}(z)} e^{-r(\tau-t)} k_4 \lambda \left(\frac{z}{h_0}\right) Y_0 \, d\tau \, d\mathrm{F}(z,t) \tag{45}$$

$$+ (N-1) \int_{\phi_{ji}(t)}^{\infty} \int_t^{\phi_{ji}^{-1}(z)} e^{-r(\tau-t)} k_4 d^{-1} \lambda \left(\frac{z}{h_0}\right) Y_0 d\tau \, d\mathrm{F}(z,t)$$

$$- (N-1) \int_{\phi_{ji}(t)}^{\infty} \int_t^{\phi_{ji}^{-1}(z)} e^{-r(\tau-t)} \kappa w_0 e^{g\tau} \, d\tau \, d\mathrm{F}(z,t)$$

$$+ \int_{h(t)}^{\infty} e^{-r(h^{-1}(z)-t)} \left[e^{gh^{-1}(z)} \left(W_0 - \zeta \left((1-\eta)w_0 + \eta \frac{\theta}{\theta-1} h_0 \right) \right) \right] d F(z,t).$$

Computing the integrals and dividing by e^{gt} gives the initial value of search,

$$W_0 = \frac{(N-1)w_0 \kappa \chi^{-\theta} + \theta \left(g \left(W_0 - \zeta \left[(1-\eta)w_0 - \eta \frac{\theta}{\theta-1} h_0 \right] \right)(\theta-1) + k_4 Y_0 \lambda \right)}{(r + g(\theta-1))(\theta-1)}. \tag{46}$$

Note that W_0 is not explicitly a function of time, a key step in verifying the guess that g is constant and $W(t) = W_0 e^{gt}$.

Given that the initial wage, w_0, and initial sales, Y_0, are solved from the sales and labor market clearing equations, equation 46 is one equation in two unknowns, g and W_0. We now turn to the FOC of the dynamic firm problem to derive a second equation in g and W_0.

Given the assumptions, the value function is

$$V(z,t) = \max_{T \geq t} \left\{ \int_t^T e^{-r(\tau-t)} \pi_d(z,\tau) d\tau + (N-1) \int_t^T e^{-r(\tau-t)} \pi_{ex}(z,\tau) d\tau \right. \tag{47}$$

$$\left. + e^{-r(T-t)} \left[W(T) - \zeta \left((1-\eta)w(T) + \eta \frac{\theta}{\theta-1} h(T) \right) \right] \right\}, \tag{48}$$

Following the solution to the firm problem in Section 3.1, the first order condition is

$$\frac{\partial V(z,t)_{|(\pi_{ex}=0)}}{\partial T}\bigg|_{(T=t,z=h(t))} = (g-r)\left[W_0 - \zeta\left((1-\eta)w_0 + \eta\frac{\theta}{\theta-1}h_0\right)\right] + k_4 Y_0 \lambda = 0. \quad (49)$$

Now we have two equations (46) and (49) in W_0 and g for which we can solve for the value of search and the growth rate on the BGP. As t has dropped out of equations 46 and 49, W_0 and g are not functions of time, confirming the guess that the value of search grows geometrically along the balanced growth path at constant rate g ($W(t) = W_0 e^{gt}$). Finally,

$$g = \frac{(N-1)w_0\kappa\chi^{-\theta} + k_4 k_5 w_0 \lambda}{(\theta-1)^2\zeta\left((1-\eta)w_0 + \eta\frac{\theta}{\theta-1}h_0\right)} - \frac{r}{\theta-1} \quad (50)$$

where w_0 satisfies the labor market clearing condition.∎

In the following section, we demonstrate the mechanisms at work in the model by using a calibrated model to explore the link between openness to trade, growth, and welfare.

4. Comparative Statics

In this section, we calibrate the model and solve for the balanced growth path. We then perform several comparative statics to illustrate the workings of the model.

Table 1: Parameterization

Parameter	Source or Target
$\sigma = 2$	Assumption 3 (consistent with Broad and Weinstein (2006))
$\theta = 4$	Simonovska and Waugh (2012)
$r = 0.10$	—
$N = 10$	—
$\eta = 1$	search cost all in goods
Search cost, ζ	match 2 percent growth rate
Fixed export cost, κ	match 10 percent of firms exporting
Iceberg trade cost, d	match 80 percent home trade share

Table 1 outlines the parameterization of the model. One set of parameters we calibrate based on previous work or introspection. These are described in the top panel. The curvature parameter σ is pinned down by Assumption 3. However, we should note that this assumption is not inconsistent with the best available evidence. Estimates of this CES parameter from Broda and Weinstein (2006) find a median estimate of σ near two. Inferences from high-frequency changes in trade flows and relative prices support a value of two as well, see, e.g., the discussion in Ruhl (2008).

The θ parameter is set equal to four as a baseline. There are various ways to get at this parameter, i.e. by looking at the distribution of sales or sizes across firms or from how trade flows respond to various shocks. The specific value of four is from Simonovska and Waugh (2012), who use price and trade flow data to estimate the heterogeneity parameter in the Eaton and Kortum (2002) trade model. In our comparative statics, we illustrate how θ affects the response of growth to changes in trade flows.

We picked the interest rate to equal ten percent and set the number of countries equal to ten. There is nothing deep about these choices, though we do explore scale effects and how the number of countries in the economy affects the growth rate.

The bottom panel of Table 1 outlines the remaining parameters are the cost to search for a new technology, ζ, the fixed cost to export, κ, and the iceberg trade cost, d. We jointly pick these parameters to match a two percent growth rate, ten percent of all firms exporting, and an 80 percent home trade share. These targets are roughly consistent with properties of the U.S. economy.

To illustrate how growth depends on openness, we started from the baseline calibration and varied the iceberg trade costs to trace out how growth responds on the balanced growth path. Figure 1 plots the results. The vertical axis reports the growth rate in percent. The horizontal axis reports the import share for a country, i.e., $1 - \lambda_{ii}$, which grows as trade costs decrease. As an orientation device, note that when the import share equals 20 percent, the growth rate is 2 percent as calibrated.

Figure 1 shows that the growth rate of the economy increases as countries trade more and become open. For example, when trade costs are lowered such that the trade share increases from 20 percent to 40 percent, the growth rate increases from 2 percent to 3 percent. At the other extreme, when the economy is closed and countries do not trade with each other, the growth rate is about 1.6 percent.

What drives this result is that reductions in trade costs change the relative value of being a firm with productivity level z. This in turn changes the incentives for a firm to draw a new productivity, which in turn changes the growth rate of the economy. While these forces are complex, there are essentially two forces at work changing the relative value of a firm. First, all
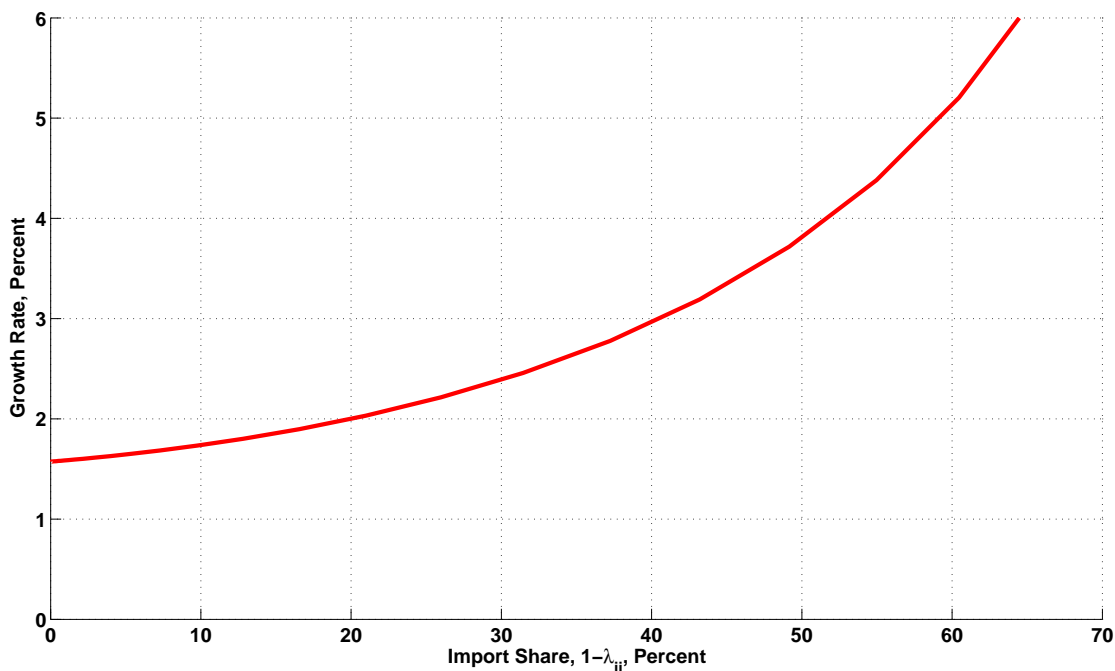
Figure 1: Openness Increases Growth: Growth Rate vs. Imports

domestic firms face more competition from foreign firms which reduces the market share for domestic firms and reduces their profits. Second, high z firms are able to expand and export increasing profits for high z firms. The net effect of these two forces is to change the relative value of a high z firm versus a low z firm. This provides an incentive for a low z firm to draw a new technology term soon than later. Since growth is generated by search, as can be seen in the BGP relationship $S(t) = \theta g$, greater incentives to search generates higher growth.

Figure 2 illustrates this by plotting the value function of a firm (normalized by the average value) versus the log of its productivity level under different levels of openness. The blue line plots the value functions when the economy is closed. The red and black line plot the value function when the economy is open. Notice that as the economy opens up, the value functions as a function of z begin to rotate counterclockwise, becoming increasingly convex. The value of having a low z firm is becoming worse relative to a closed economy. As trade barriers decrease, foreign competition increases. Additionally, increased labor demand by high productivity firms to increase exports causes domestic wages to rise. Ultimately, lower trade costs increase the value of having a high z relative to having a low z.

This change in the relative profitability of firms is best illustrated in Figure 3, which plots the static profits of a firm on the vertical axis and the log of firm productivity on the horizontal axis, for different values of the iceberg trade costs. The profits from exporting are more than offsetting any loss in domestic profits from foreign competition and increased labor costs. This
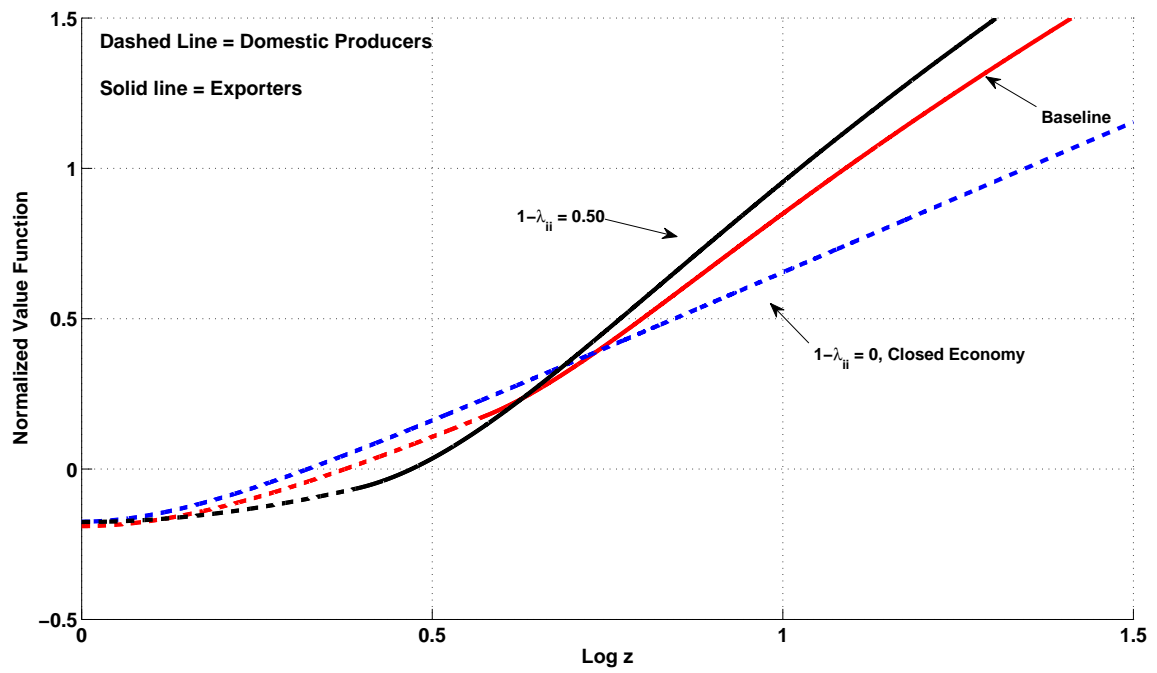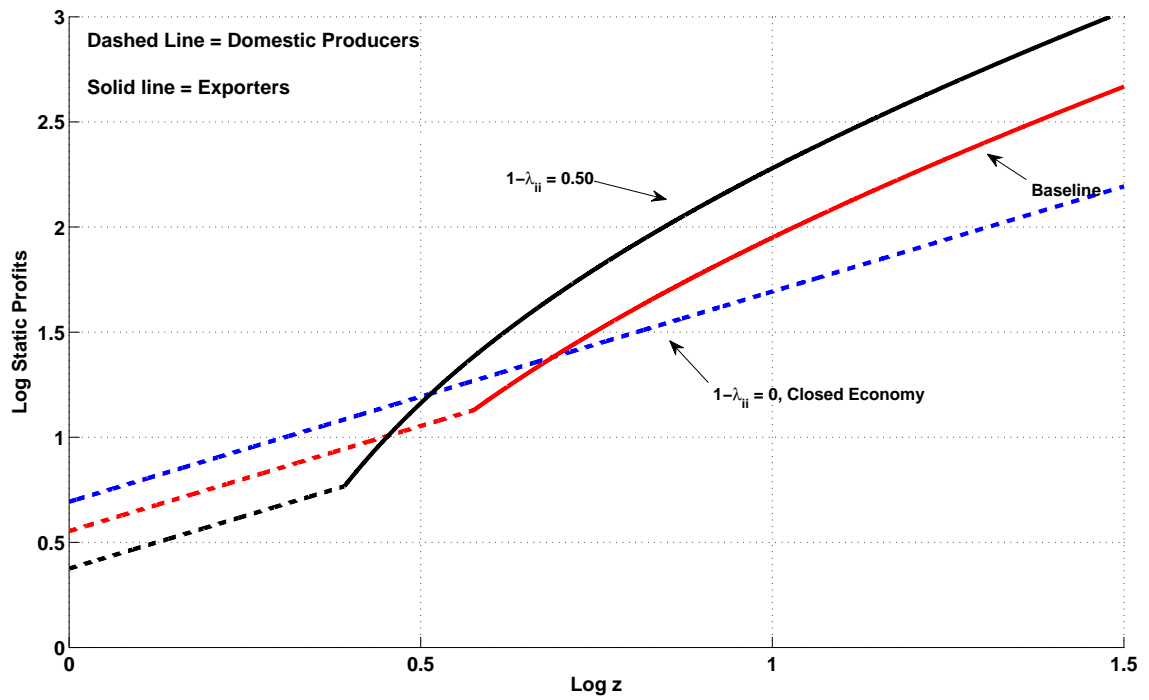
Figure 2: Value Functions and Openness

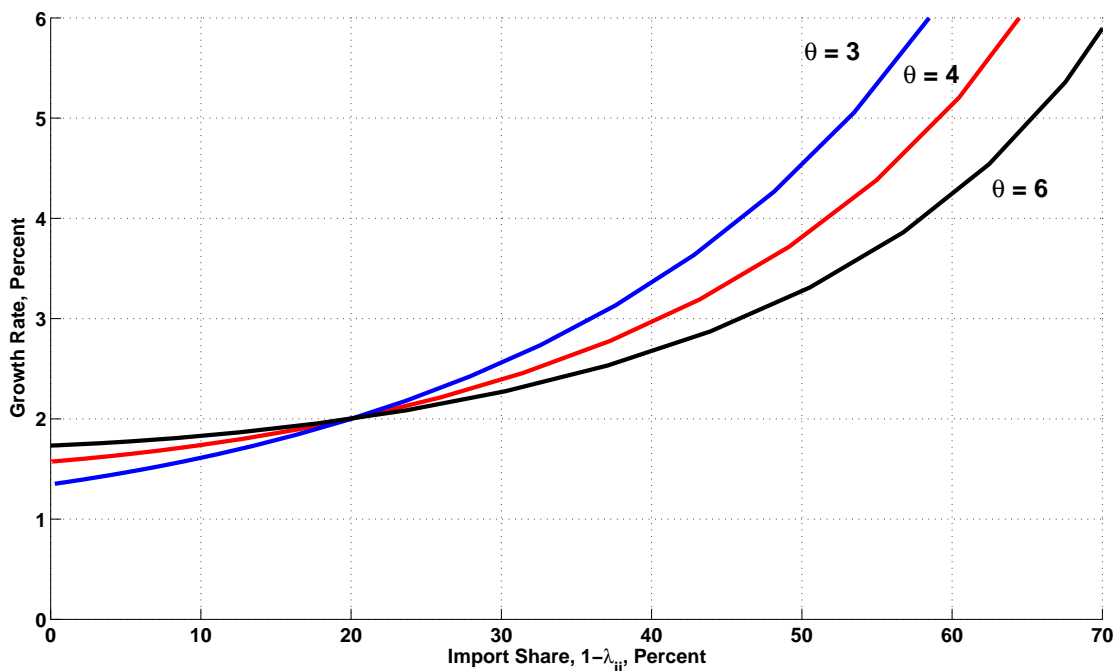**Figure 3: Static Profits and Openness**

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Figure 4: Higher θ, More Elastic Growth

then provides an incentive for the low z firm to draw a new technology level sooner.

The positive relationship between growth and openness—in the absence of cross-country idea diffusion—is a unique feature of the model. Alvarez, Buera, and Lucas (2012) generate an increase in output from openness in a model with equilibrium technology diffusion because opening to trade changes the number of ideas/productivity that (exogenously) an agent has an opportunity to meet in a given instant. In our model, the number of ideas sampled per instant is fixed. Independent of whether the country is open or closed, a firm has the opportunity to draw one new productivity from the distribution of domestic firms, once the cost is paid. The critical force is the dynamic, forward looking nature of firms in our model. Firms *choose* to draw a new productivity more frequently in response to opening to trade as higher productivity levels associated with exporting have become relatively more valuable.

A critical parameter in this model is θ. Figure 4 plots the relationship between growth and openness under several parameterizations of θ. In all the parameterizations of θ, we recalibrate the other parameters to match the same targets discussed above. As Figure 4 illustrates, the relationship between growth and openness becomes steeper. For example, when θ is three, a move to a 40 percent trade share increases growth to about 3.5 percent (compared to 3 when θ is two). Similarly, a move to autarky decreases growth to about 1.25 percent versus 1.6 percent when θ equals four.

This observation is related to the standard role θ plays in idea flow models (i.e. Alvarez, Buera,
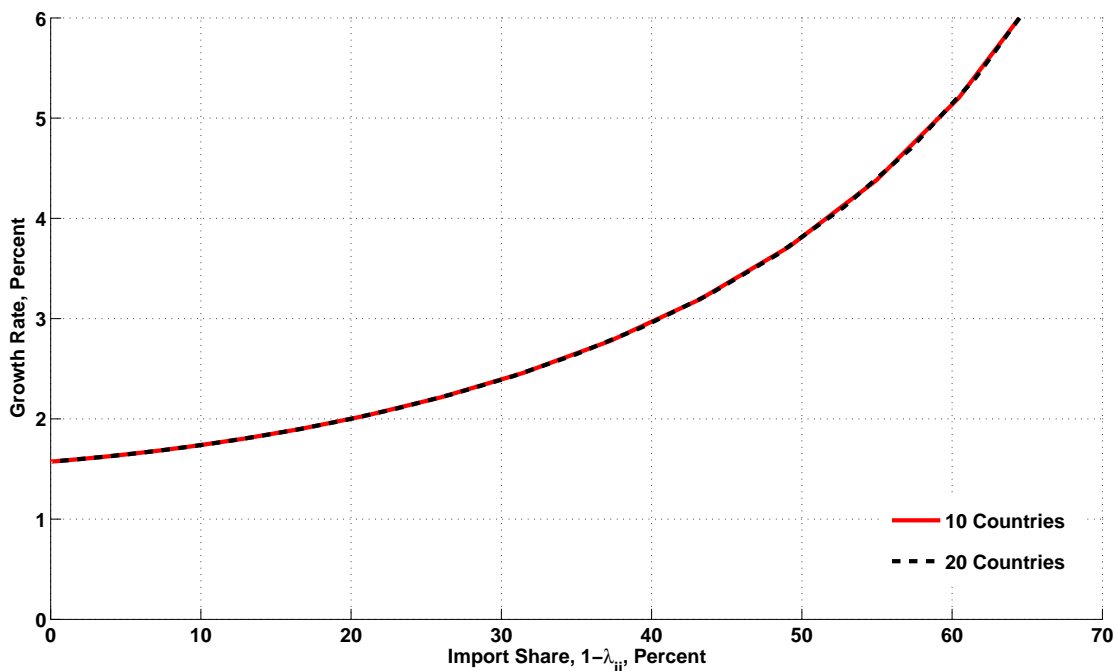
Figure 5: More Countries, No Change in Growth Rates

and Lucas (2008), Lucas (2009), Perla and Tonetti (2012), Lucas and Moll (2012), Alvarez, Buera, and Lucas (2012). A lower θ is associated with a thicker right tail of the idea distribution, meaning draws from the idea distribution lead to larger jumps in productivity and, in equilibrium, faster growth rates. The same force is present here. Holding everything else constant, a smaller θ will lead to a faster growth rate. However, Figure 4 is saying something more. The response of growth to a change in trade costs is more sensitive the thicker the tail of the underlying idea distribution.

Figure 5 plots how the results depend upon the number of countries. Here we did not recalibrate or change the parameters. We kept all parameters from the baseline parameterization the same and doubled the number of countries from ten to 20. As Figure 5 shows, doubling the number of countries does not change the relationship between the scale of the economy and the growth rate.

This result is suggestive about the workings of the model. First, this result suggests that the key mechanism behind the relationship between growth and openness in our model is not coming through a scale effect per-se. What we mean by the previous sentence is the intuition that firms simply upgrade faster because markets and profits are larger. If this were true, then we would expect to see a relationship between the scale of the economy and growth. Because scale seems to be absent, this result reinforces the idea that the driving force is how openness changes the relative value of firms across different productivity levels.
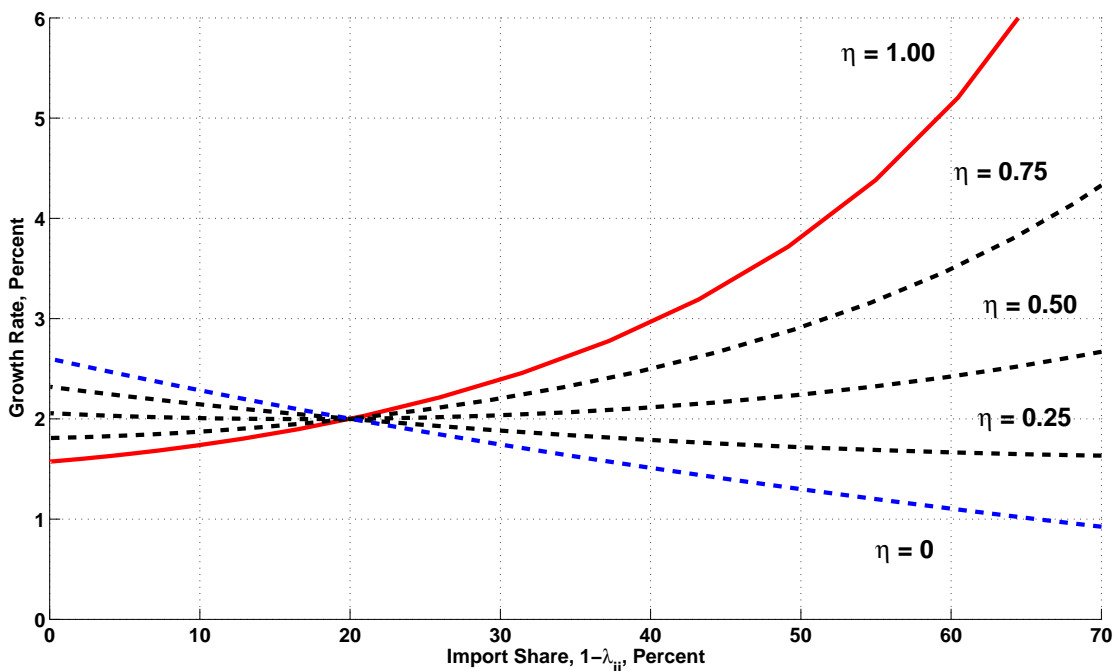
26

Figure 6: Lower η, Slower Growth

This result differs substantially from Alvarez, Buera, and Lucas (2012). Scale effects play a prominent role with the number of ideas a country has access to depending upon the number of countries. The growth rate on a balanced growth path with symmetric countries is linear in the number of countries in the economy. The critical difference is that our economy does not have cross-country idea diffusion as the Alvarez, Buera, and Lucas (2012) economy does.

Even without cross-country idea diffusion, the absence of growth scale effects does seem surprising. In fact, endogenous growth models have previously emphasized scale effects and openness as a way to increase scale and hence growth (see, e.g., the discussion of scale effects in Jones (2005a) and Ramondo, Rodriguez-Clare, and Saborio-Rodriguez (2012)). We should note that scale effects in newer models of knowledge diffusion are not well understood and we hope to explore more in the future.

One of the most important determinants of the effect of openness on growth is whether the process of technology adoption requires more labor or more goods. Figure 6 repeats the exercise of reducing iceberg trade costs as featured in Figure 1, but now varies the composition of labor and goods in the search cost.

As η decreases, labor becomes a larger component of the cost of technology adoption. For small η, the growth rate actually decreases as iceberg trade costs fall. Thus, opening to trade does not always increase growth rates. This is the result of strong general equilibrium effects on the wage rate. Decreased trade costs lead to increased demand for labor, as exporting firms want

to produce more to sell abroad and more firms become exporters. Wages increase in response to the increased labor demand, and thus the larger the labor component in the search cost the larger the increase in the cost of upgrading technology. The economy continues to grow as trade costs fall, but the economy grows more slowly for low η, as the cost of search increases more rapidly. Given that the empirical literature on the relationship between growth and trade has found mixed evidence, this theory suggests future research into the costs of technology adoption and technological progress across countries may prove insightful.

Table 2: Welfare Cost of Autarky, $\eta = 1$ (cost of search in goods)

	Open	Autarky	Ratio
Welfare	13.45	11.86	1.13
Dynamic Component	12.50	11.86	1.05
Static Component	1.07	1.00	1.07
Imports/GDP	0.20	0	—
Real GDP	1.09	1.00	1.09
Growth rate	2.00	1.57	1.27

Table 3: Welfare Cost of Autarky, $\eta = 0$ (cost of search in labor)

	Open	Autarky	Ratio
Welfare	14.05	13.51	1.04
Dynamic Component	12.50	13.51	0.93
Static Component	1.12	1.00	1.12
Imports/GDP	0.20	0	—
Real GDP	1.12	1.00	1.12
Growth rate	2.00	2.60	0.77

Using the representative consumer's utility function, we can analyze the welfare implications of these growth patterns. Although growth rates can increase or decrease in response to reduced trade costs, there exist welfare gains from trade even if growth rates decline. Table 2 presents the welfare costs of autarky for an economy where the cost of search is in goods ($\eta = 1$). In the open economy, welfare is 1.13 times higher than under autarky. Moreover, the change in welfare can be decomposed into two components, static and dynamic. The static component captures the time zero change in the level of consumption while the dynamic component accounts for the change in the growth rate. The static gain of 1.07 results from the more standard increase in varieties exported and increase in quantities produced for a given variety as in Chaney (2008). Real GDP is 9 percent higher with international trade, with most of the increased production going to increase consumption, and some going to lay for higher technology adoption costs as firms upgrade more frequently. The dynamic gains of 1.05 come from the

increased growth rate under openness and multiplies the static gains of 1.07 to generate total welfare improvements of 1.13.

Table 3 documents the welfare gains from openness when the cost of search is in labor ($\eta = 0$). Even though growth rates increase in autarky from 2 percent to 2.6 percent, there are still welfare gains of 1.04 percent from openness. Initially, real GDP is 12 percent larger in the open economy due to increased production and exports, which more than offsets the drag small growth rates to increase overall welfare.

5. Conclusion

This paper contributes a novel dynamic model of growth and international trade, driven by technology diffusion based on Perla and Tonetti (2012). Firms choose to upgrade their productivity through technology adoption to remain competitive and profitable, with the incentives to upgrade dependent on the shape of the endogenously determined productivity distribution. Highly productive firms benefit from a decline in trade costs, as they are the exporters who can take advantage of increased sales abroad. Low productivity firms only sell domestically and are hurt by the increased competition from foreign firms and by increased wages. Under most calibrations, in equilibrium this leads lower productivity firms to upgrade their technology more frequently, which increases aggregate growth. The increased pace of technology adoption has aggregate benefits beyond those to the individual firm, since in the future upgrading firms may adopt its improved technology. However, aggregate growth rates do not always increase in response to reduced trade costs, since the growth response to increased openness depends on the cost of technological improvement and the strength of general equilibrium wage effects. Nonetheless, while the gains and losses from reduced trade barriers are not distributed evenly across firms, the representative consumer who owns all firms benefits from openness.

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A. Appendix

1.1. Dynamic Problem

Let the aggregate price index, P_i, be the numeraire. Then the firm solves the following dynamic programming problem, that's solution is a sequence of search thresholds, $h_i(t)$, and export thresholds for each destination, $\phi_{ji}(t) \; \forall j \neq i$.

$$V_i(z,t) = \max_{\substack{T_{ji} \geq t \\ T_i \geq t}} \left\{ \int_t^{T_i} e^{-r(\tau-t)} \pi_{ii}(z,\tau) d\tau + \sum_{j \neq i} \int_t^{T_{ji}} e^{-r(\tau-t)} \pi_{ji}(z,\tau) d\tau + e^{-r(T_i-t)} \left[W_i(T_i) - x_i(T_i) \right] \right\}$$

(51)

where

$$W_i(t) := \int V_i(z,t) dF_i(z,t | z > h_i(t))$$

(52)

Define the search and exporter thresholds as these indifference points:

$$h_i(t) := \max\{ \, z \mid T_i(z,t) = t \, \}$$

(53)

$$\phi_{ji}(t) := \max\{ \, z \mid T_{ji}(z,t) = t \, \}$$

(54)

The function $h_i(t)$ maps time into the largest productivity level such that the firm with that productivity level is upgrading its technology. Given this definition, the function $h_i^{-1}(z)$ defines the *time* at which a firm with productivity level z will draw a new technology. Then, since a draw comes from the equilibrium distribution of *producers*, the expected value of the new technology level, $W_i(T)$, is defined in (4). Notice that the value of the new technology is integrated with respect to the conditional productivity distribution $F_i(z,t | z > h_i(t))$ and hence is a function of the choices of the individual firms.

1.1.A. Derivation of Law of Motion and Searchers

Now we will derive the law of motion for the distribution, which is a function of the mass of searchers and the evolution of the firms dynamic control $h_i(t)$. One key issue, that may be particularly relevant at the beginning of time, is whether $h_i(t)$ is continuous. The law of motion for $F_i(z,t)$ is written for continuous and discontinuous regions. Recall that $h_i(t)$ is defined as the reservation value below which agents search. As a tie-breaking rule, it is assumed that agents at the threshold search, and hence the function is right-continuous. Define $\mathbb{S}_i(t)$ as the *mass* of searchers at time t. At points of continuity in $h_i(t)$ it should be equal to 0. Define

$S_i(t)$ as the *flow* of searchers at time t, which is not defined at points of discontinuity of $h_i(t)$. It is important to recognize that $h_i(t)$ may not be continuous, particularly at "special times," that reset the economy like time 0 or potentially when a closed economy unexpectedly opens to foreign trade. A discontinuity can be introduced by unexpectedly discretely changing the value of search or cost of search, i.e., a sudden change in iceberg trade costs.

The search technology of the environment is that agents only match other agents in the producing region, as in Perla and Tonetti (2012). Therefore, agents searching at time t only meet agents strictly above $h_i(t)$, when thinking of the evolution of the distribution $h_i(t)$ is an absorbing barrier, and the minimum of support of the productivity density is

$$\lim_{\Delta \to 0} \inf \text{support}\{F_i(\cdot, t + \Delta)\} = h_i(t) \tag{55}$$

$$\inf \text{support}\{F_i(\cdot, t)\} = h_i(t), \quad \text{at points of continuity} \tag{56}$$

Thus $f_i(z, t + \Delta)$ equals $f_i(z, t)$ at points of continuity in $h_i(t)$.

1.1.B. Mass of Searchers

The mass of agents searching at time t are agents below the $h_i(t)$ threshold

$$\mathbb{S}_i(t) := F_i(h_i(t), t) \tag{57}$$

Note from equation 56, at points of continuity,

$$\mathbb{S}_i(t) = F_i(\inf \text{support}(\cdot, t), t) = 0$$

1.1.C. Flow of Searchers

A key determinant of the growth rate of the economy and of the evolution of the productivity distribution is the flow of searchers upgrading their technology. At points of continuity of $h_i(t)$, there exists a flow of searchers during each infinitesimal time period. The flow of searchers is net flow of the probability current through the search threshold, $h_i(t)$.[8] To derive this, at time t fix the reference frame of the probability distribution of z with respect to the search threshold $h_i(t)$. That is, consider the change of variables $\tilde{z} := z - h_i(t)$. Let the transformed probability distribution function be $\tilde{f}_i(\tilde{z}, t)$. Note that the only time a firms productivity changes is when they search, i.e., z does not change in the continuation region, and hence the process is $dZ = 0 \cdot dt$ while the firm is producing. Using a special case of Ito's lemma with no diffusion (or more basic methods with a Taylor series), the process for $\tilde{Z}$ is $d\tilde{Z} = -h_i'(t)dt$. Equation 5.1.3 in Gardiner

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[8]Equivalently, this is the flux of a vector field through the barrier (of 1 dimension, z in this case).

(2009) gives the more general case of the probability current with diffusion at $\tilde{z}, t$:

$$J_i(\tilde{z}, t) = -h_i'(t)\tilde{f}_i(\tilde{z}, t)$$

We are interested in the flow of searchers at the search threshold $h_i(t)$, and thus the flow at $\tilde{z} = 0$. This flow of searchers is given in equation 5.1.13 of Gardiner (2009). Since this is a simple 1 dimensional problem, the surface integral is trivial and the normal is $n = -1$. Note that since this is just a translation of z, the probability current at $\tilde{z} = 0$ and $z = h_i(t)$ are identical

$$S_i(t) = -1 \times J_i(0, t) \tag{58}$$
$$= h_i'(t)\tilde{f}_i(0, t) \tag{59}$$
$$= h_i'(t)f_i(h_i(t), t) \tag{60}$$

1.1.D. Law of Motion at Points of Discontinuity

At points of discontinuity in $h_i(t)$, a mass $\mathbb{S}_i(t)$ "exit" and draw from $\lim_{\Delta \to 0} F_i(\cdot, t + \Delta)$. The law of motion at $t+ := \lim_{\Delta \to 0} t + \Delta$ is therefore

$$F_i(z, t+) = \underbrace{F_i(z, t)}_{\text{Was below } z} - \underbrace{\mathbb{S}_i(t)}_{\text{Searched}} + \underbrace{\mathbb{S}_i(t)F_i(z, t+)}_{\text{Searched and drew} \leq z} , \quad \text{for } z \geq h_i(t+) \tag{61}$$
$$F_i(z, t+) - F_i(z, t) = -(1 - F_i(z, t+))F_i(h_i(t), t) \tag{62}$$

1.1.E. Law of Motion at Points of Continuity

The flow of "exiting" firms is defined by equation 60. These agents have an equal probability to draw any z in $f_i(z, t + \Delta)$, which equals $f_i(z, t)$ at points of continuity in $h_i(t)$. Hence, since $dZ = 0 \cdot dt$, the Kolmogorov forward equation for $z > h_i(t)$ is simply the flow of searchers who draw z:

$$\frac{\partial f_i(z, t)}{\partial t} = S_i(t)f_i(z, t) \tag{63}$$

Using equation 60

$$\frac{\partial f_i(z, t)}{\partial t} = f_i(z, t)f_i(h_i(t), t)h_i'(t) \tag{64}$$

In words, this says that the absorbing barrier (i.e., the search threshold) is sweeping across the density at rate $h_i'(t)$ and as the barrier sweeps across the density from below, it collects $f_i(h_i(t), t)$ amount firms. Then $f_i(h_i(t), t)h_i'(t)$ is the flow of searchers to be distributed back into the distribution. Since the economic environment is such that searchers only meet existing producers

above $h_i(t)$, but $h_i(t)$ is the minimum of support of $f_i(z,t)$, then the searchers are redistributed across the entire support of $f_i(z,t)$. Since agents draw directly from the productivity density, they are redistributed throughout the distribution in proportion to the density and thus, the flow of searchers $f_i(h_i(t),t)h_i'(t)$ multiplies the density $f_i(z,t)$.

1.1.F. Solving the KFE

A solution to equation (64) is a truncation for **any** $h_i(t)$ and $F_i(0)$

$$f_i(z,t) = \frac{f_i(z,0)}{1 - F_i(h_i(t),0)}. \tag{65}$$

That is the distribution at date t is a truncation of the initial distribution.

Note by properties of the Pareto distribution, given that the distribution evolves by repeated truncations, if $F_i(z,0)$ is Pareto with tail index θ and minimum of support $h_i(0)$, then $F_i(z,t)$ remains Pareto with tail index θ and minimum of support $h_i(t)$.

1.1.G. Solving the Firm Dynamic Problem

The steps to solving (3) consists of jointly finding an optimal waiting policy function, $h_i(t)$, and the expected value of search, $W_i(t)$, given a productivity distribution $F_i(z,t)$ and it's law of motion. Below, we describe the general steps to finding this solution.

Recall that the equilibrium search threshold $h_i(t)$ is the minimum of the productivity distribution. Given parameter constraints (particularly the fixed cost of exporting) such that not all firms export, the exporter threshold is greater than the search threshold. Thus, only non-exporters would want to search, and the FOC that determines the optimal search time is the derivative of the value function with respect to the search timing decision, where the discounted stream of export profits earned before searching is 0 with certainty:

$$V_i(z,t)_{|(\pi_{ji}=0)} = \max_{T_i \geq t} \left\{ \int_t^{T_i} e^{-r(\tau-t)} \pi_{ii}(z,\tau) d\tau + e^{-r(T_i-t)} \left[W_i(T_i) - x_i(T_i) \right] \right\},$$

Taking the derivative of the value function of a non-exporting firm with respect to T yields

$$\frac{\partial V_i(z,t)_{|(\pi_{ji}=0)}}{\partial T_i} = \frac{\partial \int_t^{T_i} e^{-r(\tau-t)} \pi_{ii}(z,\tau) d\tau}{\partial T_i} + \frac{e^{-r(T_i-t)} \partial W_i(T_i)}{\partial T_i} - \frac{\partial e^{-r(T_i-t)} x_i(T_i)}{\partial T_i} \tag{66}$$

$$= e^{-r(T_i-t)} \left[\pi_{ii}(z,T_i) - rW_i(T_i) + W_i'(T_i) + rx_i(T_i) - x_i'(T_i) \right] \tag{67}$$

Setting $T_i = t$, i.e., where the firm is just indifferent between switching technologies and producing, and recognizing that the productivity level of the indifferent firm is $z = h_i(t)$ by definition, we have the first order condition

$$0 = \pi_{ii}(h_i(t), t) - rW_i(t) + W_i'(t) + rx_i(t) - x_i'(t) \tag{68}$$

Now equation (68) gives us one equation in $h_i(t)$ and $W_i(t)$. We now want to find another equation in $h_i(t)$ and $W_i(t)$ essentially giving us two equations in two unknowns.

The second equation we focus on is the expected value of acquiring a new technology:

Since $h_i(t)$ is the minimum of support of $F_i(z, t)$ as stated in equation 56, we can rewrite equation 52 as

$$
\begin{aligned}
W_i(t) &= \int V_i(z, t) dF_i(z, t) \\
&= \int_{h_i(t)}^{\infty} \left\{ \int_t^{h_i^{-1}(z)} e^{-r(\tau - t))} \pi_{ii}(z, \tau) d\tau + \sum_{j \neq i} \int_t^{\phi_{ji}^{-1}(z)} e^{-r(\tau - t)} \pi_{ji}(z, \tau) d\tau \right. \\
&\quad \left. + e^{-r(h_i^{-1}(z)-t)} \left[W_i(h_i^{-1}(z)) - x_i(h_i^{-1}(z)) \right] \right\} dF_i(z, t)
\end{aligned} \tag{69}
$$

The first integral in the inside bracket is the discounted value of domestic profits until the next change of technology. Here we substituted for the time T_i with the function $h_i^{-1}(z)$. The second integral in the inside bracket is the discounted value of profits from exporting. Again, we substituted for the final exporting time, T_{ji}, for the function $\phi_{ji}(z)$, which is defined in (54). The function $\phi_{ji}(z)$ is the largest z such that a firm stops exporting to market j. Thus the inverse of this function defines the time when the firm stops exporting to market j. The final term in the inside bracket is the discounted value of the new technology net of search costs evaluated at the date $h_i^{-1}(z)$.

Outside the brackets, we then integrate over productivity levels with the existing (equilibrium) productivity distribution $F_i(z, t)$. This defines the expected value of acquiring a new technology.

Equations (68) and (69) give us two equations from which we can solve for the policy function $h_i(t)$ and the value of search $W_i(t)$ for a given a law of motion for the productivity distribution $F_i(z, t)$.

1.2. Static Problem

This derivation is for the specification where the fixed cost of exporting is $w_j \kappa_j$.

We suppress dependence on time for clarity of notation.

1.2.A. Prices, Quantities, and Profits

$$p_{ii}(z) = \frac{m w_i}{z} \tag{70}$$

$$p_{ij}(z) = \frac{m d_{ji} w_i}{z} \tag{71}$$

$$q_{ii}(z) = \frac{p_{ii}(z)^{-\sigma} Y_i}{P_i^{1-\sigma}} = z \ell_{ii}(z) \tag{72}$$

$$q_{ji}(z) = \frac{p_{ji}(z)^{-\sigma} Y_j}{P_j^{1-\sigma}} = \frac{z}{d_{ji}} \ell_{ji}(z) \tag{73}$$

$$P_i \pi_{ii}(z) = \frac{1}{\sigma} \left(\frac{m\, w_i}{z} \right)^{1-\sigma} \frac{Y_i}{P_i^{1-\sigma}} \tag{74}$$

$$P_i \pi_{ji}(z) = \max\{ \frac{1}{\sigma} \left(\frac{m\, d_{ji}\, w_i}{z} \right)^{1-\sigma} \frac{Y_j}{P_j^{1-\sigma}} - w_j \kappa_j, 0 \} \tag{75}$$

1.2.B. Exporter Threshold

ϕ_{ji} solves $\pi_{ji}(z) = 0$.

$$\left(\frac{m\, d_{ji}\, w_i}{\phi_{ji}} \right)^{1-\sigma} = \sigma w_j \kappa_j \frac{P_j^{1-\sigma}}{Y_j}$$

$$\frac{m\, d_{ji}\, w_i}{\phi_{ji}} = \left(\frac{\sigma w_j \kappa_j}{Y_j} \right)^{\frac{1}{1-\sigma}} P_j$$

$$\phi_{ji} = \frac{m\, d_{ji}\, w_i}{P_j} \left(\frac{\sigma w_j \kappa_j}{Y_j} \right)^{\frac{1}{\sigma-1}}$$

Define $k_1 := m\, d_{ji} \sigma^{\frac{1}{\sigma-1}}$

$$\phi_{ji} = k_1 \frac{w_i}{P_j} \left(\frac{w_j \kappa_j}{Y_j} \right)^{\frac{1}{\sigma-1}} \tag{76}$$

1.2.C. Aggregate Price Index

$$P_i^{1-\sigma} := \int_{h_i}^{\infty} p_{ii}(z)^{1-\sigma} f_i(z) \, dz + \sum_{j \neq i} \int_{\phi_{ij}}^{\infty} p_{ij}(z)^{1-\sigma} f_j(z) \, dz \tag{77}$$

Let $f_i(z) = \theta h_i^{\theta} z^{-1-\theta}$. Recall, $p_{ii}(z) = \frac{m w_i}{z}$ and $p_{ij}(z) = \frac{m d_{ji} w_i}{z}$.

$$P_i^{1-\sigma} = (m w_i)^{1-\sigma} \theta h_i^{\theta} \int_{h_i}^{\infty} \frac{z^{-1-\theta}}{z^{1-\sigma}} \, dz + \sum_{j \neq i} (m d_{ji} w_j)^{1-\sigma} \theta h_j^{\theta} \int_{\phi_{ij}}^{\infty} \frac{z^{-1-\theta}}{z^{1-\sigma}} \, dz$$

Define $k_2 := \frac{\theta}{\theta + 1 - \sigma}$.

$$P_i^{1-\sigma} = k_2 (m w_i)^{1-\sigma} h_i^{\theta} h_i^{\sigma-1-\theta} + \sum_{j \neq i} k_2 (m d_{ji} w_j)^{1-\sigma} h_j^{\theta} \phi_{ij}^{\sigma-1-\theta}$$

$$P_i^{1-\sigma} = k_2 (m w_i)^{1-\sigma} h_i^{\sigma-1} + \sum_{j \neq i} k_2 (m d_{ji})^{1-\sigma} w_j^{1-\sigma} h_j^{\theta} (m \, d_{ji} \sigma^{\frac{1}{\sigma-1}})^{\sigma-1-\theta} \left(\frac{w_j}{P_i} \right)^{\sigma-1-\theta} \left(\frac{w_i \kappa_i}{Y_i} \right)^{\frac{\sigma-1-\theta}{\sigma-1}}$$

Define $k_3 := k_2 \sigma^{\frac{\sigma-1-\theta}{\sigma-1}}$.

$$P_i^{1-\sigma} = k_2 (m w_i)^{1-\sigma} h_i^{\sigma-1} + \sum_{j \neq i} k_3 (m d_{ji} w_j)^{-\theta} h_j^{\theta} P_i^{\theta+1-\sigma} \left(\frac{w_i \kappa_i}{Y_i} \right)^{\frac{\sigma-1-\theta}{\sigma-1}} \tag{78}$$

1.2.D. Aggregate Trade Shares

$$\lambda_{ji} := \int_{\phi_{ji}}^{\infty} \frac{p_{ji}(z) q_{ji}(z)}{Y_j} f_i(z) dz \tag{79}$$

Recall $q_{ji}(z) = \frac{p_{ji}(z)^{-\sigma} Y_j}{P_j^{1-\sigma}}$.

$$\lambda_{ji} = \int_{\phi_{ji}}^{\infty} \frac{p_{ji}(z)^{1-\sigma}}{P_j^{1-\sigma}} f_i(z) dz$$

$$\lambda_{ji} = \frac{1}{P_j^{1-\sigma}} \int_{\phi_{ji}}^{\infty} p_{ji}(z)^{1-\sigma} f_i(z) dz$$

Recall from the derivation of the Aggregate Price Index,

$$\int_{\phi_{ij}}^{\infty} p_{ij}(z)^{1-\sigma} f_j(z) dz = k_3 (md_{ji} w_j)^{-\theta} h_j^{\theta} P_i^{\theta+1-\sigma} \left(\frac{w_i \kappa_i}{Y_i} \right)^{\frac{\sigma-1-\theta}{\sigma-1}} \tag{80}$$

so,

$$\lambda_{ji} = \frac{1}{P_j^{1-\sigma}} \left[k_3 (md_{ij} w_i)^{-\theta} h_i^{\theta} P_j^{\theta+1-\sigma} \left(\frac{w_j \kappa_j}{Y_j} \right)^{\frac{\sigma-1-\theta}{\sigma-1}} \right] \tag{81}$$

substituting the aggregate price index from equation 78,

$$\lambda_{ji} = \frac{k_3 (md_{ij} w_i)^{-\theta} h_i^{\theta} P_j^{\theta+1-\sigma} \left(\frac{w_j \kappa_j}{Y_j} \right)^{\frac{\sigma-1-\theta}{\sigma-1}}}{k_2 (mw_j)^{1-\sigma} h_j^{\sigma-1} + \sum_{n \neq j} k_3 (md_{nj} w_n)^{-\theta} h_n^{\theta} P_j^{\theta+1-\sigma} \left(\frac{w_n \kappa_n}{Y_n} \right)^{\frac{\sigma-1-\theta}{\sigma-1}}} \tag{82}$$

Deriving λ_{ii}.

Note $\sum_i \lambda_{ji} = 1$. That is, $\sum_{i \neq j} \lambda_{ji} + \lambda_{ii} = 1$.

$$\sum_{i \neq j} \lambda_{ji} = \frac{\sum_{i \neq j} k_3 (md_{ij} w_i)^{-\theta} h_i^{\theta} P_j^{\theta+1-\sigma} \left(\frac{w_j \kappa_j}{Y_j} \right)^{\frac{\sigma-1-\theta}{\sigma-1}}}{k_2 (mw_j)^{1-\sigma} h_j^{\sigma-1} + \sum_{n \neq j} k_3 (md_{nj} w_n)^{-\theta} h_n^{\theta} P_j^{\theta+1-\sigma} \left(\frac{w_n \kappa_n}{Y_n} \right)^{\frac{\sigma-1-\theta}{\sigma-1}}}$$

Define $\Gamma_{ji} := \sum_{i \neq j} k_3 (md_{ij} w_i)^{-\theta} h_i^{\theta} P_j^{\theta+1-\sigma} \left(\frac{w_i \kappa_i}{Y_i} \right)^{\frac{\sigma-1-\theta}{\sigma-1}}$. Then,

$$\sum_{i \neq j} \lambda_{ji} = \frac{\Gamma_{ji}}{k_2 (mw_j)^{1-\sigma} h_j^{\sigma-1} + \Gamma_{ji}}$$

Since $\sum_{i \neq j} \lambda_{ji} + \lambda_{ii} = 1$

$$\lambda_{ii} = \frac{k_2 (mw_j)^{1-\sigma} h_j^{\sigma-1}}{k_2 (mw_j)^{1-\sigma} h_j^{\sigma-1} + \Gamma_{ji}}$$

By definition of P_j and Γ_{ji},

$$\lambda_{ii} = \frac{k_2 (mw_j)^{1-\sigma} h_j^{\sigma-1}}{P_j^{1-\sigma}} \tag{83}$$

Deriving π_{ii} and π_{ji} in terms of trade shares.

Using $\pi_{ii}(z)$ derived in equation 74,

$$\pi_{ii}(z) = k_4\lambda_{ii}\left(\frac{z}{h_i}\right)^{\sigma-1}Y_i, \quad \text{where} \quad k_4 = \frac{1}{k_2\sigma} \tag{84}$$

Similarly, using $\pi_{ji}(z)$ derived in equation 75,

$$\pi_{ji}(z) = k_4\lambda_{jj}d_{ij}^{1-\sigma}\left(\frac{z}{h_j}\right)^{\sigma-1}\left(\frac{w_i}{w_j}\right)^{1-\sigma}Y_j - w_j\kappa_j \tag{85}$$

1.2.E. Aggregate Sales

The cost to draw a new productivity level is a convex combination of hiring domestic labor and spending final goods, given by

$$x(t) = \zeta\left[(1-\eta)\frac{w(t)}{P(t)} + \eta\mathbb{E}_t[z_i]\right]. \tag{86}$$

$$Y_i = w_iL_i + \int_{h_i}^{\infty}P_i\pi_{ii}(z)f_i(z)dz + \sum_{j\neq i}\int_{\phi_{ji}}^{\infty}P_i\pi_{ji}(z)f_i(z)dz \tag{87}$$

$$Y_i = w_iL_i + k_4\lambda_{ii}h_i^{1-\sigma}Y_i\int_{h_i}^{\infty}z^{\sigma-1}f_i(z)dz$$

$$+ \sum_{j\neq i}\left\{k_4\lambda_{jj}d_{ij}^{1-\sigma}h_j^{1-\sigma}\left(\frac{w_i}{w_j}\right)^{1-\sigma}Y_j\int_{\phi_{ji}}^{\infty}z^{\sigma-1}f_i(z)dz - w_j\kappa_j\int_{\phi_{ji}}^{\infty}f_i(z)dz\right\}$$

$$Y_i = w_iL_i + k_3k_4\lambda_{ii}Y_i + \sum_{j\neq i}\left\{k_3k_4\lambda_{jj}d_{ij}^{1-\sigma}\left(\frac{w_i}{w_j}\right)^{1-\sigma}Y_j\left(\frac{\phi_{ji}}{h_i}\right)^{\sigma-1-\theta} - w_j\kappa_j\left(\frac{\phi_{ji}}{h_i}\right)^{-\theta}\right\} \tag{88}$$

1.2.F. Goods Market Clearing

$$\frac{Y_i}{P_i} = Q_i = C_i + \zeta\eta\mathbb{E}_t[z_i] \tag{89}$$

1.2.G. Wages: Labor Market Clearing

For each producer, the quality of labor demanded to produce a good in country i for destination i is

$$\ell_{ii}(z) = \frac{p_{ii}(z)^{-\sigma}Y_i}{zP_i^{1-\sigma}} \tag{90}$$

Aggregating the labor in market i used for domestic production

$$L_{i,d} = \int_{h_i}^{\infty} \frac{p_{ii}(z)^{-\sigma}Y_i}{zP_i^{1-\sigma}} f_i(z)dz \tag{91}$$

For each producer, the quality of labor demanded to produce a good in country i for destination j is

$$\ell_{ji}(z) = \frac{d_{ji}p_{ji}(z)^{-\sigma}Y_j}{zP_j^{1-\sigma}} \tag{92}$$

Aggregating the labor in market i used for export production

$$L_{i,ex} = \sum_{j\neq i} \int_{\phi_{ji}}^{\infty} \frac{d_{ji}p_{ji}(z)^{-\sigma}Y_j}{zP_j^{1-\sigma}} f_i(z)dz \tag{93}$$

Since the fixed cost of exporting from j to i requires units of i labor, the total amount of labor from i used in the production of fixed costs equals

$$L_{i,\kappa} = \sum_{j\neq i} \int_{\phi_{ij}}^{\infty} \kappa_i f_j(z)dz$$

$$L_{i,\kappa} = \kappa_i \sum_{j\neq i} \left(\frac{h_j}{\phi_{ij}}\right)^{\theta} \tag{94}$$

Since the technology upgrade search decision is partially paid to hire labor, the search component of labor demand is

$$L_{i,x} = \zeta(1-\eta)S_i(t) \tag{95}$$

Equating aggregate labor supply, L_i, with aggregate labor demand yields the labor market clearing condition:

$$L_i = L_{i,d} + L_{i,ex} + L_{i,\kappa} + L_{i,x}$$

$$L_i = \int_{h_i}^{\infty} \frac{p_{ii}(z)^{-\sigma} Y_i}{z P_i^{1-\sigma}} f_i(z) dz + \sum_{j \neq i} \int_{\phi_{ji}}^{\infty} \frac{d_{ji} p_{ji}(z)^{-\sigma} Y_j}{z P_j^{1-\sigma}} f_i(z) dz + \kappa_i \sum_{j \neq i} \left(\frac{h_j}{\phi_{ij}} \right)^{\theta} + \zeta(1 - \eta) S_i(t)$$

$$(96)$$

1.3. BGP with Symmetry

Solving for a BGP given Assumptions 2, 3, 4 and Guesses 1, 2, and 3.

$$P(t)\pi_d(z,t) = k_4 \lambda(t) \left(\frac{z}{h(t)} \right) Y(t)$$

$$= k_4 \lambda(t) \left(\frac{z}{h_0} \right) Y_0 \qquad (97)$$

$$P(t)\pi_{ex}(z,t) = k_4 \lambda(t) d^{-1} \left(\frac{z}{h(t)} \right) Y(t) - w(t)\kappa$$

$$= \pi_d(z,t) d^{-1} - e^{gt} w_0 \kappa. \qquad (98)$$

Imposing symmetry, the price index and trade share are

$$P(t)^{1-\sigma} = k_2 (mw(t))^{1-\sigma} h(t)^{\sigma-1} + (N-1) k_3 (mdw(t))^{-\theta} h(t)^{\theta} P(t)^{\theta+1-\sigma} \left(\frac{w(t)\kappa}{Y(t)} \right)^{\frac{\sigma-1-\theta}{\sigma-1}} \qquad (99)$$

$$\lambda(t) = \frac{k_3 (mdw(t))^{-\theta} h(t)^{\theta} P(t)^{\theta+1-\sigma} \left(\frac{w(t)\kappa}{Y(t)} \right)^{\frac{\sigma-1-\theta}{\sigma-1}}}{k_2 (mw(t))^{1-\sigma} h(t)^{\sigma-1} + (N-1) k_3 (mdw(t))^{-\theta} h(t)^{\theta} P(t)^{\theta+1-\sigma} \left(\frac{w(t)\kappa}{Y(t)} \right)^{\frac{\sigma-1-\theta}{\sigma-1}}}. \qquad (100)$$

$$(101)$$

Given Guesses 1 and 2, the price index and trade share must be constant on a balanced growth path.

BGP total sales is

$$Y(t) = w(t) + k_3 k_4 \lambda Y(t) + (N-1) \left[k_3 k_4 \lambda d^{-1} Y(t) \left(\frac{\phi(t)}{h(t)} \right)^{1-\theta} - \kappa w(t) \left(\frac{\phi(t)}{h(t)} \right)^{-\theta} \right] \qquad (102)$$

From equation 98, the cutoff value for exporting is

$$\phi(t) = \chi h(t) \quad \text{where} \quad \chi := \frac{d\kappa w_0}{k_4 Y_0 \lambda} \tag{103}$$

Thus,

$$Y(t) = w(t) + k_3 k_4 \lambda Y(t) + (N-1) \left[k_3 k_4 \lambda d^{-1} Y(t) \chi^{1-\theta} - \kappa w(t) \chi^{-\theta} \right] \tag{104}$$

Then rearranging and collecting terms we have

$$Y(t) = w(t) k_5, \quad \text{where} \quad k_5 = \frac{1 - \kappa \chi^{-\theta}}{1 - \left[k_3 k_4 \lambda \left(1 + (N-1) d^{-1} \chi^{1-\theta} \right) \right]}. \tag{105}$$

Thus, Guess 2 is verified that wages are growing at the same rate as sales.

Summarizing,

$$\lambda(t) = \lambda(0) \ \forall \ t, \quad P(t) = P(0) \ \forall \ t, \quad \text{and} \quad Y(t) = k_5 w_0 e^{gt}. \tag{106}$$

1.3.A. Proof of Proposition 2

Normalize the numeraire such that $P(t) = 1 \ \forall \ t$.

Proof

First, note that given the balanced growth path guess, $h^{-1}(z)$ equals

$$h^{-1}(z) = \frac{\log\left(\frac{z}{h_0}\right)}{g}. \tag{107}$$

Second, using equation 103, $\phi^{-1}(z)$ equals

$$\phi_{ji}^{-1}(z) = \frac{\log\left(\frac{z}{\chi h_0}\right)}{g} \tag{108}$$

Finally, recall Assumption 1 that the initial distribution is Pareto and hence, by the solution to the Kolmogorov forward equation from equation 15, the pdf at time t is

$$f_i(z,t) = \theta h_i(t)^\theta z^{-1-\theta}. \tag{109}$$

Starting from equation 69, substituting the profit functions from equations 97 and 98, using the

search cost in equation 86, and using the guess $W(t) = W_0 e^{gt}$ we have

$$W_0 e^{gt} = \int_{h(t)}^{\infty} \int_{t}^{h^{-1}(z)} e^{-r(\tau-t)} k_4 \lambda \left(\frac{z}{h_0} \right) Y_0 \, d\tau \, d\mathrm{F}(z, t) \tag{110}$$

$$+ \ (N-1) \int_{\phi_{ji}(t)}^{\infty} \int_{t}^{\phi_{ji}^{-1}(z)} e^{-r(\tau-t)} k_4 d^{-1} \lambda \left(\frac{z}{h_0} \right) Y_0 d\tau \, d\mathrm{F}(z, t)$$

$$- \ (N-1) \int_{\phi_{ji}(t)}^{\infty} \int_{t}^{\phi_{ji}^{-1}(z)} e^{-r(\tau-t)} \kappa w_0 e^{g\tau} \, d\tau \, d\mathrm{F}(z, t)$$

$$+ \int_{h(t)}^{\infty} e^{-r(h^{-1}(z)-t)} \left[W_0 e^{gh^{-1}(z)} - \zeta((1-\eta)w_0 + \eta \frac{\theta}{\theta-1} h_0) e^{gh^{-1}(z)} \right] d\mathrm{F}(z, t).$$

Computing integrals yields

$$W_0 = \frac{k_4 Y_0 \theta \lambda}{(r + g(\theta-1))(\theta-1)} + \frac{(N-1)w_0 \theta \kappa \left(\frac{\kappa d w_0}{k_4 \lambda Y_0} \right)^{-\theta}}{(r + g(\theta-1))(\theta-1)} \tag{111}$$

$$- \frac{(N-1)w_0 \kappa \left(\frac{\kappa d w_0}{k_4 \lambda Y_0} \right)^{-\theta}}{r + g(\theta-1)} + \frac{g W_0 \theta}{r + g(\theta-1)} - \frac{g \zeta((1-\eta)w_0 + \eta \frac{\theta}{\theta-1} h_0)\theta}{r + g(\theta-1)}.$$

Combining terms and dividing by e^{gt} finally gives the initial value of search,

$$W_0 = \frac{(N-1)w_0 \kappa \chi^{-\theta} + \theta(g(W_0 - \zeta((1-\eta)w_0 - \eta \frac{\theta}{\theta-1} h_0))(\theta-1) + k_4 Y_0 \lambda)}{(r + g(\theta-1))(\theta-1)}. \tag{112}$$

Note that W_0 is not explicitly a function of time, a key step in verifying the guess that g is constant and $W(t) = W_0 e^{gt}$.

Given that the initial wage, w_0, and initial sales, Y_0, are solved from the sales and labor market clearing equations we now have one equation in two unknowns, g and W_0.

The relevant value function for a marginal searcher is

$$V(z,t)_{|(\pi_{ex}=0)} = \max_{T \geq t} \left\{ \int_t^T e^{-r(\tau-t)} \pi_d(z,\tau) d\tau + e^{-r(T-t)} \left[W(T) - \zeta \left[(1-\eta)w(T) + \eta \frac{\theta}{\theta-1} h_T \right] \right] \right\},$$
(113)

Taking the derivative of the value function of a non-exporting firm with respect to T yields

$$\frac{\partial V(z,t)_{|(\pi_{ex}=0)}}{\partial T} = e^{gT-r(T-t)}(g-r)W_0 - e^{gT-r(T-t)}(g-r)\zeta \left[(1-\eta)w_0 + \eta \frac{\theta}{\theta-1} h_0 \right] + \frac{e^{-r(T-t)} k_4 Y_0 z \lambda}{h_0}$$

Setting $T = t$, i.e., where the firm is just indifferent between switching technologies and producing, we have

$$0 = e^{gt}(g-r)(W_0 - \zeta \left[(1-\eta)w_0 + \eta \frac{\theta}{\theta-1} h_0 \right]) + \frac{k_4 Y_0 z \lambda}{h_0}$$

Finally recall that the productivity level of the indifferent firm, where $T = t$, is $z = h(t)$ by definition. Substituting $z = h(t) = h_0 e^{gt}$ gives

$$0 = (g-r)(W_0 - \zeta \left[(1-\eta)w_0 + \eta \frac{\theta}{\theta-1} h_0 \right]) + k_4 Y_0 \lambda.$$
(114)

Now we have two equations (46) and (49) in W_0 and g for which we can solve for the value of search and the growth rate on the BGP. As t has dropped out of equations 46 and 49, W_0 and g are not functions of time, confirming the guess that the value of search grows geometrically along the balanced growth path at constant rate g ($W(t) = W_0 e^{gt}$). ∎