Tractable characterizations of nonnegativity on a closed set via Linear Matrix Inequalities

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Tractable characterizations of polynomials (and even semi-algebraic functions) which are nonnegative on a set, is a topic of independent interest in Mathematics but is also of primary importance in many important applications, and notably in global optimization.

We will review two kinds of *tractable* characterizations of polynomials which are nonnegative on a basic closed semi-algebraic set $\mathbf{K} \subset \mathbb{R}^n$. Remarkably, both characterizations are through *Linear Matrix Inequalities* and can be checked by solving a hierarchy of semidefinite programs or generalized eigenvalue problems.

The first type of characterization is when knowledge on $\mathbf{K}$ is through its defining polynomials, i.e., $\mathbf{K} = \{\mathbf{x} : g_j(\mathbf{x}) \geq 0,\ j = 1, \ldots, m\}$, in which case some powerful certificates of positivity can be stated in terms of some sums of squares (SOS)-weighted representation. For instance, in global optimization this allows to define a hierarchy fo semidefinite relaxations which yields a monotone sequence of *lower bounds* converging to the global optimum (and in fact, finite convergence is generic). Another (dual) way of looking at nonnegativity is when knowledge on $\mathbf{K}$ is through *moments* of a measure whose support is $\mathbf{K}$. In this case, checking whether a polynomial is nonnegative on $\mathbf{K}$ reduces to solving a sequence of *generalized eigenvalue* problems associated with a countable (nested) family of real symmetric matrices of increasing size. When applied in global optimization over $\mathbf{K}$, this results in a monotone sequence of *upper bounds* converging to the global minimum, which complements the previous sequence of upper bounds. These two (dual) characterizations provide convex *inner* (resp. *outer*) approximations (by spectrahedra) of the convex cone of polynomials nonnegative on $\mathbf{K}$.