On arbitrages arising with honest times

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Abstract

In the context of a general continuous financial market model, we study whether the additional information associated with an *honest time* τ gives rise to arbitrage. By relying on the theory of progressive enlargement of filtrations, we explicitly show that arbitrage profits can never be realized strictly before τ, while classical arbitrage opportunities can be realized exactly at τ and stronger arbitrages of the first kind always exist after τ. We carefully study the behavior of local martingale deflators and consider no-arbitrage-type conditions weaker than NFLVR.

Keywords: honest time, progressive enlargement of filtrations, arbitrage, free lunch with vanishing risk, local martingale deflator.

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1 Introduction and motivation

The study of *insider trading* behavior represents a classical issue in mathematical finance and financial economics. Loosely speaking, insider trading phenomena occur when agents having access to different information sets operate in the same financial market. In particular, the better informed agents may try to realize profits by relying on their deeper private knowledge and trading with the less informed agents. Typically, the presence of two distinct layers of information is mathematically represented by two filtrations $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ with $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \geq 0$. Intuitively, the filtration $\mathbb{G}$ represents the information in possession of the insider trader. Assuming that the less informed agents cannot realize arbitrage profits by trading in the market, the fundamental question can be formulated in the following terms: can the insider trader realize arbitrage profits by making use of

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the information contained in the larger filtration $\mathbb{G}$? And, if yes, what is the appropriate notion of "arbitrage profit" and what is the trading strategy which yields that arbitrage profit?

The main goal of the present paper is to give complete and precise answers to the above questions in the context of a general continuous financial market model where the information of the insider is associated to an *honest time* τ. Referring to Section 2 for a precise definition of the concept of honest time (which has been first introduced by Meyer et al. [25]), we would like to quote the following passage from Dellacherie et al. [9] (page 137) which intuitively explains the notion and seems particularly well suited to the present discussion:

> *Par exemple S_t peut représenter le cours d'une certaine action à l'instant t, et τ est le moment idéal pour vendre son paquet d'actions. Tous les spéculateurs cherchent à connaître τ sans jamais y parvenir, d'où son nom de variable aléatoire honnête.*

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and let $S = (S_t)_{t \geq 0}$ represent the discounted price process of some risky assets. The filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is constructed as the *progressive enlargement* of $\mathbb{F}$ with respect to an honest time τ, which is assumed to avoid all $\mathbb{F}$-stopping times. For a detailed account of the theory of progressive enlargement of filtrations, we refer the reader to Chapitres IV-V of Jeulin [19] (see also Section 5.9.4 of Jeanblanc et al. [18] and Section VI.2 of Protter [29] for more rapid accounts and the book Mansuy & Yor [24] for a presentation of the theory in the case where $\mathbb{F}$ is a Brownian filtration). In this context, the investors who have access only to the information contained in the filtration $\mathbb{F}$ represent the "spéculateurs" referred to in the passage quoted above. In particular, they are not allowed to construct portfolio strategies based on τ, simply because τ is not an $\mathbb{F}$-stopping time. In contrast, an insider trader having access to the full information of the filtration $\mathbb{G}$ can rely on his private information on τ when trading in the market and, hence, may have the possibility of realizing arbitrage profits.

In the present paper, we provide a complete analysis of the kinds of arbitrage that can be realized by an insider trader having access to the additional information generated by an honest time. We do not confine ourselves to the classical no-arbitrage theory based on the notions of *Arbitrage Opportunity* and *Free Lunch with Vanishing Risk*, as developed by Delbaen & Schachermayer [7], but we also consider several stronger notions of arbitrage, namely *Unbounded Increasing Profits*, *Arbitrages of the First Kind* and *Unbounded Profits with Bounded Risk*, which are of current interest in mathematical finance, as documented by the recent papers Hulley & Schweizer [13], Karatzas & Kardaras [21] and Kardaras [22]. In particular, this allows us to make precise the severity of the arbitrages induced by an honest time. Furthermore, and this is a major aspect of the present paper, we carefully distinguish what kinds of arbitrage can be realized *before*, *at* and *after* time τ. In that sense, the present paper is to the best of our knowledge the first systematic study of the relations existing between progressive enlargements of filtrations with respect to honest times and no-arbitrage-type conditions.

It has already been shown that an honest time τ induces arbitrage opportunities in the progressively enlarged filtration $\mathbb{G}$ immediately *after* time τ, see e.g. Imkeller [14] and Zwierz [32]. In comparison with these papers, our results provide two main innovations. On the one hand, we show that an insider trader can always realize an arbitrage opportunity not only *after* τ but also exactly *at* time τ. On the other hand, we can explicitly exhibit in a simple way the trading strategies which yield the arbitrage profits. This contrasts with the approach adopted in Imkeller [14] and Zwierz [32], where the existence of arbitrage opportunities is shown by relying on the abstract results of Delbaen & Schachermayer [8]. Moreover, our approach permits to recover the results obtained in Imkeller [14] and Zwierz [32] in a very simple way. A key tool in our approach is the multiplicative decomposition of the Azéma supermartingale $Z = (Z_t)_{t \geq 0}$ associated to the random time τ established in Nikeghbali & Yor [26] (see Section 2 for precise definitions).

We now illustrate some of the main results of the present paper in the simplest possible setting (detailed proofs and also sharper results in a general setting will be given in Sections 3-4). Let $W = (W_t)_{t \geq 0}$ be a one-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}^W, P)$, where $\mathbb{F}^W = (\mathcal{F}_t^W)_{t \geq 0}$ is the natural filtration of W (augmented by the P-nullsets of $\mathcal{F}_\infty^W$), with $\mathcal{F} := \mathcal{F}_\infty^W$. Let the process $S = (S_t)_{t \geq 0}$ represent the discounted price of a risky asset and be given as the solution of the following SDE, for some $\sigma > 0$:

$$\begin{cases} dS_t = S_t \, \sigma dW_t \\ S_0 = s \in (0, \infty) \end{cases} \tag{1.1}$$

Let us introduce the finite random time $\tau := \sup \{ t \geq 0 : S_t = \sup_{u \geq 0} S_u \}$ and the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$, assumed to be the progressive enlargement of $\mathbb{F}$ with respect to τ (see Section 2 for precise definitions). We call *informed agent* an agent who can invest in the risky asset S and has access to the information contained in the enlarged filtration $\mathbb{G}$. Then, the main results of the present paper can be essentially summarized as follows:

(a) for any $T \in (0, \infty)$, an informed agent can never realize an arbitrage profit on the time interval $[0, \tau \wedge T]$ (see Theorem 3.10). In particular, it is never possible to realize arbitrage profits *strictly before* time τ, i.e., on the time interval $[0, \varrho]$, where ϱ is any $\mathbb{G}$-stopping time with $\varrho < \tau$ P-a.s. (see Corollary 3.12);

(b) an informed agent can always realize a "classical" arbitrage opportunity *exactly at* time τ, i.e., on the time interval $[0, \tau]$ (see Proposition 3.1 and Theorem 3.7);

(c) an informed agent can always realize arbitrage profits which are better than "classical" arbitrage opportunities (i.e. arbitrages of the first kind, see Definition 2.3) *after* time τ (see Proposition 4.2 and Theorem 4.5).

Furthermore, we can explicitly construct the trading strategies which realize the arbitrage profits for the informed agent in (b) and (c): it will be enough to hold appropriate long and short positions, respectively, in the portfolio which replicates the non-negative $\mathbb{F}$-local martingale $N = (N_t)_{t \geq 0}$ appearing in the multiplicative decomposition of the Azéma supermartingale $Z = (Z_t)_{t \geq 0}$ of the random time τ (see Lemma 2.7).

The study of the impact of the additional information associated to a random time on the no-arbitrage-type properties of a financial market and on the behavior of market participants has already attracted attention in the mathematical finance literature. In particular, Imkeller [14] and Zwierz [32] are the closest precursors to our work (related results also appear in Ankirchner & Imkeller [1]). In the context of credit risk modelling, a study of the no-arbitrage-type properties of a market model with a filtration progressively enlarged with respect to a random time has also been recently undertaken in Coculescu et al. [6]. We also want to mention that, in the case of *initially* enlarged filtrations (see Jeulin [19], Chapitre III, or Protter [29], Section VI.2), the possibility of realizing arbitrage profits has been studied in Grorud & Pontier [12] and Imkeller et al. [15]. Finally, we refer the interested reader to Nikeghbali & Platen [27] and Nikeghbali & Platen [28] for a detailed analysis of the role of honest times in financial modelling.

The paper is structured as follows. Section 2 describes the general setting and recalls several no-arbitrage-type conditions as well as some key technical results from the theory of progressive enlargement of filtrations. Sections 3 and 4 contain the main results. More specifically, in Section 3 we prove the existence of arbitrage opportunities arising *at* an honest time, while Section 4 deals with the validity of no-arbitrage-type conditions *after* an honest time. Section 5 concludes by discussing the role played by the standing Assumptions I-III introduced in Section 2 and by pointing out some extensions of the results contained in the present paper.

2 General setting and preliminary results

Let $(\Omega, \mathcal{F}, P)$ be a given probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions, where P denotes the physical probability measure and $\mathcal{F} := \mathcal{F}_\infty$. We consider a financial market comprising $d+1$ assets, with prices described by the $\mathbb{R}^{d+1}$-valued process $\bar{S} = \left(\bar{S}_t\right)_{t \geq 0}$. To allow for greater generality, we consider a financial market model on an infinite time horizon. Of course, financial markets on a finite time horizon $[0, T]$ can be imbedded by simply considering the stopped process $\bar{S}^T$. We assume that $\bar{S}^0$ represents a numéraire or reference asset and is P-a.s. strictly positive. Without loss of generality, we express the prices of all $d+1$ assets in terms of $\bar{S}^0$-discounted quantities, thus obtaining the $\mathbb{R}^d$-valued process $S = (S_t)_{t \geq 0}$, with $S^i := \bar{S}^i / \bar{S}^0$ for each $i = 1, \ldots, d$. We assume that the process S is a *continuous semimartingale* on $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

Let the random time $\tau : \Omega \to [0, \infty]$ be a P-a.s. finite *honest time* on $(\Omega, \mathcal{F}, \mathbb{F}, P)$. This means that τ is an $\mathcal{F}$-measurable random variable such that, for all $t > 0$, there exists an $\mathcal{F}_t$-measurable random variable ζ_t with $\tau = \zeta_t$ on $\{\tau < t\}$ (see e.g. Jeulin [19], Chapitre V). We define the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ as the *progressive enlargement* of $\mathbb{F}$ with respect to τ, i.e., $\mathcal{G}_t := \bigcap_{s > t} \bigl(\mathcal{F}_s \vee \sigma\left(\tau \wedge s\right)\bigr)$ for all $t \geq 0$, augmented by the P-nullsets of $\mathcal{G}_\infty = \mathcal{F}_\infty$. It is well-known that $\mathbb{G}$ is the smallest filtration satisfying the usual conditions which contains $\mathbb{F}$ and makes τ a $\mathbb{G}$-stopping time. Furthermore, the *(H')-hypothesis* holds between $\mathbb{F}$ and $\mathbb{G}$, meaning that any $\mathbb{F}$-semimartingale is also a $\mathbb{G}$-semimartingale (see Jeulin [19], Théorème 5.10). In particular, this implies that the discounted price process S is also a $\mathbb{G}$-semimartingale.

In order to model the activity of trading, we need to define the notion of admissible trading strategy, following Delbaen & Schachermayer [7]. Let $\mathbb{H}$ denote a generic filtration, i.e., in our setting $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$. We denote by $L^{\mathbb{H}}(S)$ the set of all $\mathbb{R}^d$-valued $\mathbb{H}$-predictable processes $\theta = (\theta_t)_{t \geq 0}$ which are S-integrable in $\mathbb{H}$ and we write $\theta \cdot S$ for the corresponding stochastic integral process.

Definition 2.1. *Let $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$. For $a \in \mathbb{R}_+$, an element $\theta \in L^{\mathbb{H}}(S)$ is said to be an* a-admissible $\mathbb{H}$-strategy *if $(\theta \cdot S)_\infty := \lim_{t \to \infty} (\theta \cdot S)_t$ exists and $(\theta \cdot S)_t \geq -a$ P-a.s. for all $t \geq 0$. We denote by $\mathcal{A}^{\mathbb{H}}_a$ the set of all a-admissible $\mathbb{H}$-strategies. We say that an element $\theta \in L^{\mathbb{H}}(S)$ is an* admissible $\mathbb{H}$-strategy *if $\theta \in \mathcal{A}^{\mathbb{H}} := \bigcup_{a \in \mathbb{R}_+} \mathcal{A}^{\mathbb{H}}_a$.*

We assume that there are no frictions or trading constraints and that trading is done in a self-financing way. This implies that the wealth process generated by trading according to an admissible $\mathbb{H}$-strategy θ starting from an initial endowment of $x \in \mathbb{R}$ is given by $V(x, \theta) := x + \theta \cdot S$, for $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$. We call the tuple $\mathcal{M}^{\mathbb{F}} := \bigl(\Omega, \mathcal{F}, \mathbb{F}, P; S, \mathcal{A}^{\mathbb{F}}\bigr)$ the *restricted* financial market, as opposed to the *enlarged* financial market $\mathcal{M}^{\mathbb{G}} := \bigl(\Omega, \mathcal{G}, \mathbb{G}, P; S, \mathcal{A}^{\mathbb{G}}\bigr)$, with $\mathcal{G} := \mathcal{G}_\infty = \mathcal{F}_\infty$. Intuitively, agents operating in the enlarged financial market are better informed than agents operating in the restricted financial market, due to the additional information associated to the random time τ.

Remark 2.2. Note that, since $\mathbb{F} \subseteq \mathbb{G}$ and all $\mathbb{F}$-semimartingales are also $\mathbb{G}$-semimartingales, we have $L^{\mathbb{F}}(S) \subseteq L^{\mathbb{G}}(S)$, as can be deduced from Proposition 2.1 of Jeulin [19]. In turn, this implies that $\mathcal{A}^{\mathbb{F}} \subseteq \mathcal{A}^{\mathbb{G}}$, thus accounting for the fact that agents in the enlarged financial market are allowed to use a richer information set to construct their portfolios.

As mentioned in the Introduction, the present paper aims at answering the following question: how does the additional information associated to the honest time τ give rise to arbitrage? To this end, let us first recall several notions of arbitrage which have appeared in the literature.

Definition 2.3. *Let* $(\mathcal{H}, \mathbb{H}) \in \{(\mathcal{F}, \mathbb{F}), (\mathcal{G}, \mathbb{G})\}$.

(i) An element $\theta \in \mathcal{A}_0^{\mathbb{H}}$ *yields an* Unbounded Increasing Profit *if*

$$P\big(V(0,\theta)_s \le V(0,\theta)_t\,,\ \textit{for all } 0 \le s \le t \le \infty\big) = 1 \qquad \textit{and} \qquad P\big(V(0,\theta)_\infty > 0\big) > 0$$

If there exists no such $\theta \in \mathcal{A}_0^{\mathbb{H}}$ *we say that the financial market* $\mathcal{M}^{\mathbb{H}}$ *satisfies the* No Unbounded Increasing Profit (NUIP) *condition.*

(ii) A non-negative $\mathcal{H}$*-measurable random variable* ξ *with* $P(\xi > 0) > 0$ *yields an* Arbitrage of the First Kind *if for all* $x > 0$ *there exists an element* $\theta^x \in \mathcal{A}_x^{\mathbb{H}}$ *such that* $V(x, \theta^x)_\infty \ge \xi$ *P-a.s. If there exists no such random variable we say that the financial market* $\mathcal{M}^{\mathbb{H}}$ *satisfies the* No Arbitrage of the First Kind (NA1) *condition.*

(iii) An element $\theta \in \mathcal{A}^{\mathbb{H}}$ *yields an* Arbitrage Opportunity *if* $V(0,\theta)_\infty \ge 0$ *P-a.s. and* $P\big(V(0,\theta)_\infty > 0\big) > 0$. *If there exists no such* $\theta \in \mathcal{A}^{\mathbb{H}}$ *we say that the financial market* $\mathcal{M}^{\mathbb{H}}$ *satisfies the* No Arbitrage (NA) *condition.*

(iv) A sequence $\{\theta^n\}_{n\in\mathbb{N}} \subset \mathcal{A}^{\mathbb{H}}$ *yields a* Free Lunch with Vanishing Risk *if there exist an* $\varepsilon > 0$ *and an increasing sequence* $\{\delta_n\}_{n\in\mathbb{N}}$ *with* $0 \le \delta_n \nearrow 1$ *such that* $P\big(V(0,\theta^n)_\infty > -1 + \delta_n\big) = 1$ *and* $P\big(V(0,\theta^n)_\infty > \varepsilon\big) \ge \varepsilon$. *If there exists no such sequence we say that the financial market* $\mathcal{M}^{\mathbb{H}}$ *satisfies the* No Free Lunch with Vanishing Risk (NFLVR) *condition.*

For a (possibly infinite-valued) $\mathbb{H}$*-stopping time* ϱ*, we say that NUIP/NA1/NA/NFLVR holds in the financial market* $\mathcal{M}^{\mathbb{H}}$ *on the time horizon* $[0,\varrho]$ *if the financial market* $\big(\Omega, \mathcal{H}, \mathbb{H}, P; S^{\tau}, \mathcal{A}^{\mathbb{H}}\big)$ *satisfies NUIP/NA1/NA/NFLVR, where* S^{ϱ} *denotes the stopped process* $(S_{t\wedge\varrho})_{t\ge 0}$.

The notion of *Unbounded Increasing Profit* has been introduced under that name in Karatzas & Kardaras [21] and represents the strongest possible notion of arbitrage among those listed above. Indeed, it can be checked directly from Definition 2.3 that the existence of an unbounded increasing profit implies that none of the NA1, NA and NFLVR conditions can hold. The notion of *Arbitrage of the First Kind* is due to Kardaras [22] and can be shown to be equivalent to the boundedness in probability of the set $\{V(1,\theta)_\infty : \theta \in \mathcal{A}_1^{\mathbb{H}}\}$, see Proposition 1 of Kardaras [22]. The latter condition has appeared under the name *No Unbounded Profit with Bounded Risk (NUPBR)* in Karatzas & Kardaras [21] but its importance was first recognized by Delbaen & Schachermayer [7] and Kabanov [20]. Both the NA1 and the NFLVR conditions can be characterized in purely probabilistic terms. As a preliminary, let us recall the following Definition.

Definition 2.4. *Let* $(\mathcal{H}, \mathbb{H}) \in \{(\mathcal{F}, \mathbb{F}), (\mathcal{G}, \mathbb{G})\}$ *and* ϱ *a (possibly infinite-valued)* $\mathbb{H}$*-stopping time.*

(i) A strictly positive $\mathbb{H}$*-local martingale* $L = (L_t)_{t\ge 0}$ *with* $L_0 = 1$ *and* $L_\infty > 0$ *P-a.s. is said to be a* local martingale deflator in $\mathbb{H}$ on the time horizon $[0,\varrho]$ *if the process* LS^{ϱ} *is an* $\mathbb{H}$*-local martingale;*

(ii) a probability measure $Q \sim P$ *on* $(\Omega, \mathcal{H})$ *is said to be an* Equivalent Local Martingale Measure in $\mathbb{H}$ ($\mathrm{ELMM}_{\mathbb{H}}$) on the time horizon $[0,\varrho]$ *if the process* S^{ϱ} *is an* $\mathbb{H}$*-local martingale under* Q.

Note that the notion of local martingale deflator corresponds to the notion of *strict martingale density* first introduced by Schweizer [30]. We then have the following fundamental Theorem. The first assertion is a partial statement of Theorem 4 of Kardaras [22] (noting that the proof carries over to the infinite time horizon case), while the last two assertions are due to Delbaen & Schachermayer [7].

Theorem 2.5. *Let* $(\mathcal{H},\mathbb{H}) \in \{(\mathcal{F},\mathbb{F}),(\mathcal{G},\mathbb{G})\}$ *and* ϱ *a (possibly infinite-valued)* $\mathbb{H}$*-stopping time. Then, on the time horizon* $[0,\varrho]$*, the following hold:*

(i) NA1 (or, equivalently, NUPBR) holds in the financial market $\mathcal{M}^{\mathbb{H}}$ *if and only if there exists a local martingale deflator in* $\mathbb{H}$*;*

(ii) NFLVR holds in the financial market $\mathcal{M}^{\mathbb{H}}$ *if and only if there exists an Equivalent Local Martingale Measure in* $\mathbb{H}$*.*

(iii) NFLVR holds in the financial market $\mathcal{M}^{\mathbb{H}}$ *if and only if both NA1 (or, equivalently, NUPBR) and NA hold in the financial market* $\mathcal{M}^{\mathbb{H}}$*.*

We shall always work under the following standing assumption, which ensures that the restricted financial market $\mathcal{M}^{\mathbb{F}}$ does not allow for any kind of arbitrage.

Assumption I. The restricted financial market $\mathcal{M}^{\mathbb{F}}$ satisfies NFLVR.

We aim at studying the no-arbitrage-type properties (or the lack thereof) of the enlarged financial market $\mathcal{M}^{\mathbb{G}}$. In the remaining part of the paper, we shall give a clear answer to this issue under the two following standing assumptions, where we denote by $M = (M_t)_{t\geq 0}$ the $\mathbb{F}$-local martingale part in the canonical decomposition of S in the filtration $\mathbb{F}$.

Assumption II. The random time τ avoids $\mathbb{F}$-stopping times, i.e., for any $\mathbb{F}$-stopping time T we have $P(\tau = T) = 0$.

Assumption III. The continuous $\mathbb{F}$-local martingale $M = (M_t)_{t\geq 0}$ has the $\mathbb{F}$-predictable representation property in the filtration $\mathbb{F}$.

Assumption II is classical when dealing with progressive enlargements of filtrations. Assumption III means that any $\mathbb{F}$-local martingale $U = (U_t)_{t\geq 0}$ with $U_0 = 0$ can be represented as $U = \varphi \cdot M$, where $\varphi = (\varphi_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued $\mathbb{F}$-predictable process such that $\int_0^t \varphi_s{}' d\langle M, M\rangle_s \varphi_s < \infty$ P-a.s. for all $t \geq 0$, see e.g. Chapter III of Jacod & Shiryaev [16]. In particular, Assumption III implies that all $\mathbb{F}$-martingales are continuous. We postpone to Section 5 a discussion of the importance of Assumptions I-III and of possible extensions thereof.

Remark 2.6 (*On the completeness of the restricted financial market*)**.** Under the additional assumption that the initial σ-field $\mathcal{F}_0$ is trivial, Assumptions I and III together imply that there exists an unique ELMM$_{\mathbb{F}}$ Q for the restricted financial market $\mathcal{M}^{\mathbb{F}}$, see e.g. Theorem 9.5.3.1 of Jeanblanc et al. [18]. In turn, this implies that for any non-negative $\mathcal{F}$-measurable random variable $\xi \in L^1(Q)$ there exists a strategy $\theta^\xi \in \mathcal{A}^{\mathbb{F}}$ such that $\xi = x + (\theta^\xi \cdot S)_\infty$, for some $x \in \mathbb{R}$ and where $\theta^\xi \cdot S$ is a uniformly integrable $(Q,\mathbb{F})$-martingale (see e.g. Ansel & Stricker [2], Théorème 3.2, or Delbaen & Schachermayer [7], Theorem 5.2).

We close this Section by recalling two technical results obtained by Nikeghbali & Yor [26] under the hypothesis that all $\mathbb{F}$-local martingales are continuous and Assumption II holds. Recall also that a P-a.s. finite random time τ is an honest time if and only if it is the end of an $\mathbb{F}$-optional set (see Jeulin [19], Proposition 5.1) and note that, due to Assumption III together with the continuity of S, the $\mathbb{F}$-optional sigma field coincides with the $\mathbb{F}$-predictable sigma field. In the following, we denote by $Z = (Z_t)_{t\geq 0}$ the *Azéma supermartingale* of the random time τ, i.e., $Z_t = P(\tau > t|\mathcal{F}_t)$ for all $t \geq 0$.

Lemma 2.7 (Nikeghbali & Yor [26], Theorem 4.1)**.** *There exists a continuous non-negative $\mathbb{F}$-local martingale $N = (N_t)_{t\geq 0}$ with $N_0 = 1$ and $\lim_{t\to\infty} N_t = 0$ such that Z admits the following multiplicative decomposition, for all $t \geq 0$:*

$$Z_t = P(\tau > t|\mathcal{F}_t) = \frac{N_t}{N_t^*}$$

where $N_t^* := \sup_{s\leq t} N_s$. Furthermore, we have that:

$$\tau = \sup\{t \geq 0 : N_t = N_t^*\} = \sup\{t \geq 0 : N_t = N_\infty^*\}$$

Lemma 2.8 (Nikeghbali & Yor [26], Proposition 2.5)**.** *Let $X = (X_t)_{t\geq 0}$ be an $\mathbb{F}$-local martingale. Then X has the following canonical decomposition as a semimartingale in $\mathbb{G}$:*

$$X_t = \widetilde{X}_t + \int_0^{t\wedge\tau} \frac{d\langle X, N\rangle_s}{N_s} - \int_\tau^{t\vee\tau} \frac{d\langle X, N\rangle_s}{N_\infty^* - N_s}$$

where $\widetilde{X} = \big(\widetilde{X}_t\big)_{t\geq 0}$ is a $\mathbb{G}$-local martingale and $N = (N_t)_{t\geq 0}$ is as in Lemma 2.7.

3 Arbitrages up to a honest time

The goal of this Section is to determine whether the information associated to an honest time τ does give rise to arbitrage in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0, \tau]$. We always suppose that Assumptions I, II and III are satisfied.

Without any loss of generality, we may and do assume that P is already an ELMM for the restricted financial market $\mathcal{M}^{\mathbb{F}}$. Indeed, Assumption I together with part *(ii)* of Theorem 2.5 ensures the existence of an ELMM$_{\mathbb{F}}$ Q. Since $Q \sim P$, it is easy to check that all the properties of the general setting described in Section 2 still hold under Q. More precisely, the random time τ is still an honest time which avoids $\mathbb{F}$-stopping times under any ELMM$_{\mathbb{F}}$ Q and the $(Q, \mathbb{F})$-local martingale $S = (S_t)_{t\geq 0}$ has the predictable representation property under the measure Q (see Jacod & Shiryaev [16], part *a)* of Theorem III.5.24). Hence, Assumptions I, II and III hold under any ELMM$_{\mathbb{F}}$ Q. Finally, observe that the notion of admissible strategy given in Definition 2.1 is stable under an equivalent change of measure (see e.g. Protter [29], Theorem IV.25). As a consequence, all the NUIP, NA1, NA and NFLVR no-arbitrage-type conditions introduced in Definition 2.3 hold for the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ under the measure Q if and only if they hold under the measure P.

Recall that Lemma 2.7 gives the existence of a continuous non-negative $\mathbb{F}$-local martingale $N = (N_t)_{t\geq 0}$ with $N_0 = 1$ and $\lim_{t\to\infty} N_t = 0$ such that $\tau = \sup\{t \geq 0 : N_t = N_\infty^*\}$. It is clear that $N_\tau \geq 1$ P-a.s. and $P(N_\tau > 1) > 0$. Furthermore, due to Assumption III, there exists an $\mathbb{R}^d$-valued $\mathbb{F}$-predictable process $\varphi = (\varphi_t)_{t\geq 0} \in L^{\mathbb{F}}(S)$ such that $N = 1 + \varphi \cdot S$. By relying on the above reasoning, we can easily construct an admissible $\mathbb{G}$-strategy which yields an arbitrage opportunity (in the sense of part *(iii)* of Definition 2.3) in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$.

Proposition 3.1. *The process $\bar{\varphi} := \mathbf{1}_{[\![0,\tau]\!]}\varphi$ yields an arbitrage opportunity in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$. As a consequence, NA fails to hold in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0, \tau]$.*

Proof. Due to Remark 2.2 and since τ is a $\mathbb{G}$-stopping time, it is clear that $\bar{\varphi} \in L^{\mathbb{G}}(S)$. Since $V(0, \bar{\varphi})_t = \big(\mathbf{1}_{[\![0,\tau]\!]}\varphi \cdot S\big)_t = N_{t\wedge\tau} - 1 \geq -1$ P-a.s. for all $t \geq 0$, we also have $\bar{\varphi} \in \mathcal{A}_1^{\mathbb{G}}$. Note that $V(0, \bar{\varphi})_\infty = V(0, \bar{\varphi})_\tau = N_\tau - 1$, thus implying $V(0, \bar{\varphi})_\tau \geq 0$ P-a.s. and $P\big(V(0, \bar{\varphi})_\tau > 0\big) > 0$. This shows that NA fails in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0, \tau]$. □

Despite its simplicity, the result of Proposition 3.1 is quite interesting. Indeed, it shows that, as soon as Assumptions I, II and III hold, we can explicitly construct an admissible $\mathbb{G}$-strategy which realizes an arbitrage opportunity *at* the honest time τ. To the best of our knowledge, this result is new: as mentioned in the Introduction, all previous works in the literature have only shown the existence of arbitrage opportunities *immediately after* τ (see e.g. Imkeller [14] and Zwierz [32]).

Remark 3.2. We want to point out that an arbitrage opportunity similar to that constructed in the proof of Proposition 3.1 can be obtained, somewhat more generally, for any random time τ which avoids $\mathbb{F}$-stopping times and such that its Azéma supermartingale satisfies $Z_\tau = 1$ P-a.s. This can be shown by relying on the multiplicative decomposition (see e.g. Jacod & Shiryaev [16], Theorem II.8.21) of the supermartingale $Z = (Z_t)_{t\geq 0}$ into a positive $\mathbb{F}$-local martingale $N = (N_t)_{t\geq 0}$ and a positive $\mathbb{F}$-predictable decreasing process $D = (D_t)_{t\geq 0}$ with $N_0 = D_0 = 1$. Since the present paper is focused on the important class of random times represented by honest times, we shall not consider this generalization in the following.

Our next goal consists in studying the validity of NA1 (or, equivalently, NUPBR) and NUIP in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0,\tau]$. In view of part *(i)* of Theorem 2.5, the NA1 condition holds in the financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0,\tau]$ if and only if there exists a local martingale deflator in $\mathbb{G}$ for S^τ. Due to Lemma 2.8, the stopped process S^τ admits the following canonical decomposition in the filtration $\mathbb{G}$:

$$S_t^\tau = \widetilde{S}_t^\tau + \int_0^{t\wedge\tau} \frac{d\langle S, N\rangle_s}{N_s} = \widetilde{S}_t^\tau + \int_0^{t\wedge\tau} d\langle S, S\rangle_s \frac{\varphi_s}{N_s} = \widetilde{S}_t^\tau + \int_0^t d\langle \widetilde{S}^\tau, \widetilde{S}^\tau\rangle_s \frac{\varphi_s}{N_s} \tag{3.1}$$

where $\widetilde{S} = \big(\widetilde{S}_t\big)_{t\geq 0}$ is a continuous $\mathbb{G}$-local martingale. We say that a $\mathbb{G}$-local martingale $(M_t)_{t\geq 0}$ is a $\mathbb{G}$-martingale on $[0,\tau]$ if $E[M_\varrho] = E[M_0]$ for every $\mathbb{G}$-stopping time ϱ with $\varrho \leq \tau$ P-a.s. If this is not the case, we say that $(M_t)_{t\geq 0}$ is a *strict* $\mathbb{G}$-local martingale, following the terminology of Elworthy et al. [10].

Proposition 3.3. *The process $1/N^\tau = (1/N_{t\wedge\tau})_{t\geq 0}$ is a local martingale deflator in $\mathbb{G}$ on the time horizon $[0,\tau]$. Furthermore, the process $1/N^\tau$ is a strict $\mathbb{G}$-local martingale on $[0,\tau]$.*

Proof. Let us first define the $\mathbb{F}$-stopping time $\nu := \inf\{t \geq 0 : N_t = 0\} = \inf\{t \geq 0 : Z_t = 0\}$, where $Z = (Z_t)_{t\geq 0}$ is the Azéma supermartingale associated to τ. Since $P(\tau > \nu|\mathcal{F}_\nu) = Z_\nu = 0$, it follows that $\tau \leq \nu$ P-a.s. Since τ avoids $\mathbb{F}$-stopping times (Assumption II), we furthermore have $\tau < \nu$ P-a.s. Thus, the process $1/N^\tau$ is well-defined. By Itô's formula together with equation (3.1):

$$\frac{1}{N^\tau} = 1 - \frac{1}{(N^\tau)^2}\cdot N^\tau + \frac{1}{(N^\tau)^3}\cdot\langle N\rangle^\tau = 1 - \frac{\varphi}{(N^\tau)^2}\cdot S^\tau + \frac{1}{(N^\tau)^3}\,\varphi\cdot\langle S^\tau, N\rangle = 1 - \frac{\varphi}{(N^\tau)^2}\cdot\widetilde{S}^\tau \tag{3.2}$$

This shows that $1/N^\tau$ is a strictly positive continuous $\mathbb{G}$-local martingale with $1/N_0^\tau = 1$ and $1/N_\infty^\tau = 1/N_\tau > 0$ P-a.s. Furthermore, using the product rule together with equations (3.1)-(3.2):

$$\frac{S^\tau}{N^\tau} = S_0 + \frac{1}{N^\tau}\cdot S^\tau + S^\tau\cdot\frac{1}{N^\tau} + \left\langle S^\tau, \frac{1}{N^\tau}\right\rangle = S_0 + \frac{1}{N^\tau}\cdot\widetilde{S}^\tau + S^\tau\cdot\frac{1}{N^\tau}$$

This shows that $1/N^\tau$ is a local martingale deflator in $\mathbb{G}$ on the time horizon $[0,\tau]$. Being a positive $\mathbb{G}$-local martingale, the process $1/N^\tau$ is also a $\mathbb{G}$-supermartingale. It is a true $\mathbb{G}$-martingale on $[0,\tau]$ if and only if $E[1/N_\tau] = E[1/N_0] = 1$. However, $N_\tau \geq 1$ P-a.s. and $P(N_\tau > 1) > 0$ imply that $E[1/N_\tau] < 1$. $\square$

Proposition 3.3 shows that there always exists at least one local martingale deflator in $\mathbb{G}$ on the time horizon $[0,\tau]$, given by the reciprocal of the $\mathbb{F}$-local martingale N appearing in the multiplicative decomposition of the Azéma supermartingale Z of the random time τ (see Lemma 2.7).

Remark 3.4. The arbitrage strategy $\bar{\varphi}$ constructed in Proposition 3.1 admits a special interpretation. Indeed, the corresponding value process $V(1,\bar{\varphi}) = N^\tau$ is the reciprocal of the local martingale deflator $1/N^\tau$. According to Theorem 7 of Hulley & Schweizer [13] (see also Karatzas & Kardaras [21], Section 4.4), this implies that $V(1,\bar{\varphi})$ represents the value process of the *growth-optimal portfolio*, which also coincides with the *numeraire portfolio*, for the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0,\tau]$.

The following Lemma shows the general structure of all local martingale deflators in $\mathbb{G}$ on the time horizon $[0,\tau]$.

Lemma 3.5. *Let $L = (L_t)_{t\geq 0}$ be a local martingale deflator in $\mathbb{G}$ on the time horizon $[0,\tau]$. Then L admits the following representation:*

$$L = \frac{\mathcal{E}(R)}{N^\tau}$$

where $R = (R_t)_{t\geq 0}$ is a $\mathbb{G}$-local martingale with $R_0 = 0$, purely discontinuous on $[0,\tau]$ and with $\{\Delta R \neq 0\} \subseteq [\![\tau]\!]$ and $\Delta R_\tau > -1$ P-a.s. Furthermore, all local martingale deflators in $\mathbb{G}$ on the time horizon $[0,\tau]$ are strict $\mathbb{G}$-local martingales on $[0,\tau]$.

Proof. We already know from Proposition 3.3 that the set of all local martingale deflators in $\mathbb{G}$ on the time horizon $[0,\tau]$ is non-empty. Let $L = (L_t)_{t\geq 0}$ be an element of that set. Theorem 1 of Schweizer [31] (or also Choulli & Stricker [4], Théorème 2.2) together with equation (3.1) shows that L can be written as follows:

$$L = \mathcal{E}\left(-\frac{\varphi}{N}\cdot \widetilde{S}^\tau\right)\mathcal{E}(R)$$

where $R = (R_t)_{t\geq 0}$ is a $\mathbb{G}$-local martingale with $R_0 = 0$ and $\Delta R > -1$ P-a.s. such that $\langle R, \widetilde{S}^i\rangle^\tau = 0$ for all $i = 1,\ldots,d$. The uniqueness of the Doléans-Dade exponential together with equation (3.2) implies that $1/N^\tau = \mathcal{E}\big(-\frac{\varphi}{N}\cdot \widetilde{S}^\tau\big)$. Let $R = R^c + R^d$ be the decomposition of the $\mathbb{G}$-local martingale R into its continuous and purely discontinuous $\mathbb{G}$-local martingale parts. Since $\langle R^c, \widetilde{S}_i\rangle^\tau = \big\langle (R^c)^\tau, \widetilde{S}^i\big\rangle = 0$ for all $i = 1,\ldots,d$ by orthogonality, Proposition 5.4 of Barlow [3] implies that $(R^c)^\tau = 0$, thus showing that R is purely discontinuous on $[0,\tau]$. Furthermore, Théorème 5.12 of Jeulin [19] implies that $\{\Delta R \neq 0\} \subseteq [\![\tau]\!]$, since all $\mathbb{F}$-local martingales are continuous. It remains to show that L is a strict $\mathbb{G}$-local martingale on $[0,\tau]$. For that, it suffices to observe that:

$$E[L_\tau] = E\left[\frac{\mathcal{E}(R)_\tau}{N_\tau}\right] < E\left[\mathcal{E}(R)_\tau\right] \leq 1$$

where the first inequality follows since $N_\tau \geq 1$ P-a.s. and $P(N_\tau > 1) > 0$ and the last inequality is due to the supermartingale property of the positive $\mathbb{G}$-local martingale $\mathcal{E}(R)$. □

Remarks 3.6. 1) We want to point out that the structure of the $\mathbb{G}$-local martingale R appearing in Lemma 3.5 can be described a bit more explicitly by relying on the general martingale representation results recently established in Jeanblanc & Song [17] for progressively enlarged filtrations. Indeed, noting that the dual $\mathbb{F}$-predictable projection of the process $\left(\mathbf{1}_{\{\tau\leq t\}}\right)_{t\geq 0}$ is given by $\left(\log(N_t^*)\right)_{t\geq 0}$ (see Nikeghbali & Yor [26], Corollary 2.4, or also Mansuy & Yor [24], Exercise 1.8), Theorem 6.2 of Jeanblanc & Song [17] shows that the following representation holds true:

$$L^\tau = \frac{1}{N^\tau}\exp\left(-\frac{k}{N^*}\cdot N^*\right)\left(1 + k_\tau \mathbf{1}_{[\![\tau,\infty)\!)} + \zeta \mathbf{1}_{[\![\tau,\infty)\!)}\right) \tag{3.3}$$

where $k = (k_t)_{t\geq 0}$ is an $\mathbb{F}$-predictable process such that $1 + k_\tau > 0$ P-a.s. and ζ is a $\mathcal{G}_\tau$-measurable random variable such that $E[\zeta|\mathcal{G}_{\tau-}] = 0$.

2) According to the terminology of Hulley & Schweizer [13], the process $1/N^\tau = \mathcal{E}\big(-\frac{\varphi}{N} \cdot \widetilde{S}^\tau\big)$ represents the *minimal martingale density* for the stopped process S^τ in the progressively enlarged filtration $\mathbb{G}$, i.e., the candidate density process of the *mimimal martingale measure* (when the latter exists).

3) It is interesting to note that Lemma 3.5 gives a recipe for constructing a whole class of possibly discontinuous *strict* $\mathbb{G}$-local martingales. To the best of our knowledge, apart from the particular case considered in Chybiryakov [5], there exist very few non-trivial examples of strict local martingales which are not necessarily continuous.

Propositions 3.1 and 3.3 directly yield the following Theorem, which gives a definite answer to the question of whether an agent in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ can profit from the additional information and realize arbitrage profits on the time horizon $[0,\tau]$.

Theorem 3.7. *NA1 (or, equivalently, NUPBR) and NUIP hold in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0,\tau]$. However, NA and NFLVR fail to hold in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0,\tau]$.*

Proof. The first assertion follows from Proposition 3.3 together with part *(i)* of Theorem 2.5, recalling also that NUIP is implied by NA1. The second assertion follows from Proposition 3.1 together with part *(iii)* of Theorem 2.5. □

Remark 3.8 (*A probabilistic proof of the failure of NFLVR*)**.** It is worth noting that the failure of NFLVR in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0,\tau]$ can also be proved in a purely probabilistic way, without relying on Proposition 3.1. Indeed, suppose on the contrary that NFLVR holds on $[0,\tau]$. In view of part *(ii)* of Theorem 2.5, this gives the existence of a probability measure $Q \sim P$ such that S^τ is a $(Q,\mathbb{G})$-local martingale, with density process $L_t := \frac{dQ|_{\mathcal{G}_t}}{dP|_{\mathcal{G}_t}}$, $t \geq 0$. Obviously, the process $L = (L_t)_{t\geq 0}$ is a local martingale deflator in $\mathbb{G}$ on the time horizon $[0,\tau]$ and also a uniformly integrable $\mathbb{G}$-martingale. This contradicts the last statement of Lemma 3.5 and, hence, NFLVR cannot hold in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0,\tau]$.

At this point, one may also wonder whether it is possible to construct arbitrage opportunities in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ *before* the honest time τ. As a preliminary, we shall need the following Lemma. Recall that $\nu = \inf\{t \geq 0 : Z_t = 0\}$, as in the proof of Proposition 3.3. The proof of the following Lemma is technical and, hence, postponed to the Appendix.

Lemma 3.9. *Let $L = (L_t)_{t\geq 0}$ be a local martingale deflator in $\mathbb{G}$ on the time horizon $[0,\tau]$ and let σ be an $\mathbb{F}$-stopping time. Then the following holds:*

$$E\big[L_{\sigma\wedge\tau}\big] = E\left[1 - \exp\left(-\int_0^\tau \frac{1+k_s}{N_s^*}\, dN_s^*\right) \mathbf{1}_{\{\nu\leq\sigma\}}\right] \tag{3.4}$$

where $k = (k_t)_{t\geq 0}$ is the $\mathbb{F}$-predictable process appearing in the representation (3.3)*. Furthermore, we have $\int_0^\tau \frac{1+k_s}{N_s^*}\, dN_s^* > 0$ P-a.s.*

We can then prove the following Theorem, which in particular holds true for all deterministic times $T \in (0,\infty)$.

Theorem 3.10. *Let σ be an $\mathbb{F}$-stopping time. Then NFLVR holds in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0,\sigma\wedge\tau]$ if and only if $P(\sigma \geq \nu) = 0$.*

Proof. If $P(\sigma \geq \nu) = 0$, equation (3.3) (with $k = \zeta = 0$) together with Lemma 3.9 (compare also Proposition 3.3) implies that $1/N^{\sigma\wedge\tau}$ is a uniformly integrable $\mathbb{G}$-martingale. Together with Proposition 3.3, this shows that $1/N^{\sigma\wedge\tau}$ can be taken as the density process of an ELMM$_{\mathbb{G}}$ for $S^{\sigma\wedge\tau}$. Due to part *(ii)* of Theorem 2.5, it follows that NFLVR holds in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0,\sigma\wedge\tau]$. Conversely, if $P(\sigma \geq \nu) > 0$ then Lemma 3.9 implies that $E[L_{\sigma\wedge\tau}] < 1$ for any local martingale deflator $L = (L_t)_{t\geq 0}$ in $\mathbb{G}$ on the time horizon $[0,\tau]$. This implies that L cannot be a uniformly integrable martingale and, hence, no ELMM can exist for the enlarged financial market $\mathcal{M}^{\mathbb{G}}$. □

Remark 3.11. Note that, for any $\mathbb{F}$-stopping time σ with $P(\sigma \geq \nu) = 0$, we always have $P(\sigma < \tau) = E[Z_\sigma] = E\left[Z_\sigma \mathbf{1}_{\{\sigma<\nu\}}\right] > 0$. Hence, there is no contradiction between Theorem 3.10 and the second assertion of Theorem 3.7.

Note that Theorem 3.10 implies in particular that, for any $T \in (0,\infty)$ with $P(\nu \leq T) = 0$, the NFLVR condition holds in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0,\tau\wedge T]$. We also have the following Corollary, which shows that one can never construct arbitrage opportunities in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ *strictly before* the honest time τ.

Corollary 3.12. *Let ϱ be a $\mathbb{G}$-stopping time with $\varrho < \tau$ P-a.s. Then NFLVR holds in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon* $[0,\varrho]$.

Proof. If ϱ is a $\mathbb{G}$-stopping time with $\varrho < \tau$, the Lemma on page 370 of Chapter VI of Protter [29] implies that there exists an $\mathbb{F}$-stopping time σ with $\sigma = \varrho$ P-a.s. Noting that $\tau < \nu$ P-a.s. (see the beginning of the proof of Proposition 3.3), the claim then follows from Theorem 3.10. □

Remarks 3.13. 1) As considered in the Introduction, let $d = 1$ and suppose that the real-valued process $S = (S_t)_{t\geq 0}$ is given as the solution of the SDE (1.1) on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}^W, P)$, where $\mathbb{F}^W$ is the (P-augmented) natural filtration of W. Since S is a $(P,\mathbb{F})$-martingale, Assumption I is trivially satisfied and, clearly, Assumption III holds as well. Furthermore, since $\lim_{t\to\infty} S_t = 0$ P-a.s. (due to the law of large numbers for Brownian motion), Corollary 2.4 of Nikeghbali & Yor [26] implies that $\tau = \sup\{t \geq 0 : S_t = \sup_{u\geq 0} S_u\}$ is an honest time which avoids all $\mathbb{F}$-stopping times. Note also that, in the context of this example, we have $S = N$, as can be deduced from Proposition 2.2 of Nikeghbali & Yor [26], and $\nu = \infty$ P-a.s. Then, Theorem 3.10 together with Corollary 3.12 and Proposition 3.1 together with Theorem 3.7 directly imply claims (a)-(b), respectively, in the Introduction. Observe that, in the context of this simple example, the arbitrage opportunity constructed in Proposition 3.1 reduces simply to a buy-and-hold position on S until time τ.

2) Theorem 3.10 implies that if σ is an $\mathbb{F}$-stopping time such that $P(\sigma \geq \nu) > 0$, then there exist arbitrage opportunities in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0,\sigma\wedge\tau]$. Let us illustrate this fact by means of a simple example, in the same setting of the previous remark. Suppose that $S_0 = 1$ and define the $\mathbb{F}$-stopping time $\tau^* := \inf\{t \geq 0 : S_t = 1/2\}$, which is P-a.s. finite, and the honest time $\tau := \sup\{t \in [0,\tau^*] : S_t = \sup_{u\in[0,\tau^*]} S_u\}$. Let us also introduce the $\mathbb{F}$-stopping time $\sigma := \inf\{t \geq 0 : S_t = 3/2\}$. As can be easily seen, we have $\nu = \tau^*$ and $P(\sigma > \nu) = P(\sigma > \tau^*) > 0$. Hence, due to Theorem 3.10, NFLVR fails to hold in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on $[0,\sigma\wedge\tau]$. Indeed, the buy-and-hold strategy $\mathbf{1}_{[\![0,\sigma\wedge\tau]\!]}$ provides an arbitrage opportunity on the time interval $[0,\sigma\wedge\tau]$, since $S_{\sigma\wedge\tau} - S_0 \geq 0$ and $P(S_{\sigma\wedge\tau} > S_0) > 0$.

4 Arbitrages on the global time horizon

This Section deals with the characterization of arbitrage in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the *global* time horizon $[0,\infty]$, taking into account especially what can happen *after* the honest time τ. As in Section 3, we shall always suppose that Assumptions I, II and III are satisfied and that P is already an ELMM$_{\mathbb{F}}$ for S, without loss of generality. As a preliminary, note that Lemma 2.8 gives the following canonical decomposition of $S=(S_t)_{t\geq 0}$ in the enlarged filtration $\mathbb{G}$:

$$S_t=\widetilde{S}_t+\int_0^{t\wedge\tau}\frac{d\langle S,N\rangle_s}{N_s}-\int_\tau^{t\vee\tau}\frac{d\langle S,N\rangle_s}{N^*_\infty-N_s}=\widetilde{S}_t+\int_0^t d\langle\widetilde{S},\widetilde{S}\rangle_s\,\tilde{\alpha}_s=:\widetilde{S}_t+\widetilde{A}_t \tag{4.1}$$

for a $\mathbb{G}$-local martingale $\widetilde{S}=\bigl(\widetilde{S}_t\bigr)_{t\geq 0}$, with $\tilde{\alpha}_t:=\mathbf{1}_{\{\tau\geq t\}}\frac{\varphi_t}{N_t}-\mathbf{1}_{\{\tau<t\}}\frac{\varphi_t}{N^*_\infty-N_t}$ and where the process $\varphi=(\varphi_t)_{t\geq 0}\in L^{\mathbb{F}}(S)$ is the integrand in the stochastic integral representation $N=1+\varphi\cdot S$.

We first have the following simple Lemma, which shows that Unbounded Increasing Profits can never occur in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$.

Lemma 4.1. *NUIP holds in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$.*

Proof. Suppose that $\theta\in\mathcal{A}_0^{\mathbb{G}}$ generates an unbounded increasing profit. Then, according to part *(i)* of Definition 2.3, the process $V(0,\theta)=\theta\cdot S$ is increasing, hence of finite variation. This implies that the $\mathbb{G}$-local martingale $\theta\cdot\widetilde{S}=\theta\cdot S-\theta\cdot\widetilde{A}$ is null, being $\mathbb{G}$-predictable and of finite variation. The Kunita-Watanabe inequality implies then that $\bigl|\bigl\langle\theta\cdot\widetilde{S},\widetilde{S}^i\bigr\rangle\bigr|=0$ P-a.s. for all $i=1,\ldots,d$. It then follows, for all $t\geq 0$:

$$V(0,\theta)_t=(\theta\cdot S)_t=(\theta\cdot\widetilde{A})_t=\int_0^t\theta'_s\,d\bigl\langle\widetilde{S},\widetilde{S}\bigr\rangle_s\,\tilde{\alpha}_s=\int_0^t d\bigl\langle\theta\cdot\widetilde{S},\widetilde{S}\bigr\rangle_s\,\tilde{\alpha}_s=0$$

thus contradicting the assumption that $P\bigl(V(0,\theta)_\infty>0\bigr)>0$. ☐

Since Lemma 4.1 only excludes the existence of almost pathological arbitrages, it is of interest to study the validity of stronger no-arbitrage-type conditions in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$. We already know from Theorem 3.7 that NA and NFLVR fail to hold in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0,\tau]$ and, hence, also on the global time horizon $[0,\infty]$. One can then naturally ask whether NA1 (or, equivalently, NUPBR) holds in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the global time horizon $[0,\infty]$, i.e., even after the honest time τ. The following Proposition shows that this is not the case, since we are able to explicitly construct an arbitrage of the first kind, in the sense of part *(ii)* of Definition 2.3.

Proposition 4.2. *The random variable $\xi:=N_\tau-1$ yields an arbitrage of the first kind. As a consequence, NA1 (or, equivalently, NUPBR) fails to hold in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the global time horizon $[0,\infty]$.*

Proof. Clearly, $\xi:=N_\tau-1$ is a $\mathcal{G}$-measurable non-negative random variable with $P(\xi>0)>0$. Let $\hat{\varphi}:=-\mathbf{1}_{(\!(\tau,\infty]\!]}\varphi$. Due to Remark 2.2 and since τ is a $\mathbb{G}$-stopping time, it is clear that $\hat{\varphi}\in L^{\mathbb{G}}(S)$. Since $V(0,\hat{\varphi})=-\mathbf{1}_{(\!(\tau,\infty]\!]}\varphi\cdot S=N^\tau-N$, we have $V(0,\hat{\varphi})_t=0$ on $\{t\leq\tau\}$ and $V(0,\hat{\varphi})_t>0$ on $\{t>\tau\}$, because $\tau=\sup\{t\geq 0:N_t=N^*_\infty\}$ (see Lemma 2.7), implying that $\hat{\varphi}\in\mathcal{A}_0^{\mathbb{G}}$. For all $x>0$, we then have $V(x,\hat{\varphi})_\infty=x+N_\tau-N_\infty=x+1+\xi>\xi$, thus showing that ξ yields an arbitrage of the first kind. ☐

Remarks 4.3. **1)** As shown in the proof of Proposition 4.2, the arbitrage strategy $\hat{\varphi} \in \mathcal{A}_0^{\mathbb{G}}$ satisfies $\hat{\varphi} = \hat{\varphi}\mathbf{1}_{(\!(\tau,\infty]\!]}$ and $(\hat{\varphi} \cdot S)_t > 0$ for all $t > \tau$. According to Definition 3.2 of Delbaen & Schachermayer [8], the strategy $\hat{\varphi}$ generates an *immediate arbitrage opportunity* at the $\mathbb{G}$-stopping time τ. This intuitively means that one can realize an arbitrage profit *immediately after* the $\mathbb{G}$-stopping time τ has occurred, i.e., on the time interval $[\tau, \tau + \varepsilon]$, for all $\varepsilon > 0$. This possibility has been also pointed out in Zwierz [32].

2) Proposition 4.2 can be seen as a counterpart of Proposition 3.1. Indeed, Proposition 3.1 shows that one can realize an arbitrage opportunity *at* time τ by taking a *long* position (up to τ) in the strategy φ which replicates N, while Proposition 4.2 shows that one can realize an immediate arbitrage opportunity *after* time τ by taking a *short* position in the strategy φ. It is interesting to observe that in both cases the arbitrage strategy is directly related to the $\mathbb{F}$-local martingale N appearing in the multiplicative decomposition of the Azéma supermartingale Z of τ. Note that admissibility constraints prevent the arbitrage of the first kind $\xi = N_\tau - 1$ to be realized at time τ.

3) It has already been shown in Imkeller [14] and Zwierz [32] that NFLVR fails to hold *after* τ in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$. However, the proofs given in those papers are somewhat long and technical. In contrast, the proof of Proposition 4.2 is extremely simple and explicitly shows the trading strategy which realizes the arbitrage. Furthermore, we have shown that not only NA and NFLVR but also the weaker NA1 and NUPBR no-arbitrage-type conditions fail to hold in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the global time horizon $[0,\infty]$.

It is worth pointing out that the failure of NA1 in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ after τ can also be proved in a purely probabilistic way, by relying on the characterization of NA1 given in part *(i)* of Theorem 2.5. More precisely, we have the following Proposition.

Proposition 4.4. *The enlarged financial market $\mathcal{M}^{\mathbb{G}}$ does not admit any local martingale deflator in $\mathbb{G}$ on the global time horizon $[0,\infty]$.*

Proof. Suppose that $L = (L_t)_{t\geq 0}$ is a local martingale deflator in $\mathbb{G}$ for S on the global time horizon $[0,\infty]$. Similarly as in the proof of Lemma 3.5, Theorem 1 of Schweizer [31] together with equation (4.1) implies that L admits the following representation:

$$L = \mathcal{E}\big(-\tilde{\alpha} \cdot \widetilde{S}\big)\,\mathcal{E}\,(R)$$

where $R = (R_t)_{t\geq 0}$ is a purely discontinuous $\mathbb{G}$-local martingale with $R_0 = 0$, $\{\Delta R \neq 0\} \subseteq [\![\tau]\!]$ and $\Delta R_\tau > -1$ P-a.s. By using the definition of $\tilde{\alpha}$ together with Lemma 3.5, we can write as follows:

$$L = \mathcal{E}\left(-\mathbf{1}_{(\!(0,\tau]\!]}\frac{\varphi}{N} \cdot \widetilde{S}\right)\mathcal{E}\left(\mathbf{1}_{(\!(\tau,\infty]\!]}\frac{\varphi}{N^*_\infty - N} \cdot \widetilde{S}\right)\mathcal{E}\,(R) = \frac{\hat{L}}{N^\tau}\mathcal{E}\,(R)$$

with $\hat{L} := \mathcal{E}\big(\mathbf{1}_{(\!(\tau,\infty]\!]}\frac{\varphi}{N^*_\infty - N} \cdot \widetilde{S}\big)$. Lemma 2.7 implies that $Z_t < 1$ P-a.s. for all $t > \tau$ (see also Barlow [3], Lemma 2.4). Hence, using Itô's formula together with Lemma 2.7 and Lemma 2.8, we can write as follows, for all for $\tau < s \leq t$:

$$\begin{aligned}
\frac{1}{1-Z_t} - \frac{1}{1-Z_s} &= \frac{N^*_\tau}{N^*_\tau - N_t} - \frac{N^*_\tau}{N^*_\tau - N_s} = \int_s^t \frac{N^*_\tau}{(N^*_\tau - N_u)^2}dN_u + \int_s^t \frac{N^*_\tau}{(N^*_\tau - N_u)^3}d\langle N,N\rangle_u \\
&= \int_s^t \frac{N^*_\tau}{(N^*_\tau - N_u)^2}\varphi_u dS_u + \int_s^t \frac{N^*_\tau}{(N^*_\tau - N_u)^3}\varphi'_u d\langle S,N\rangle_u \\
&= \int_s^t \frac{N^*_\tau}{(N^*_\tau - N_u)^2}\varphi_u d\widetilde{S}_u = \int_s^t \frac{1}{1-Z_u}\frac{\varphi_u}{N^*_\tau - N_u}d\widetilde{S}_u
\end{aligned}$$

The uniqueness of the Doléans-Dade exponential implies that $\hat{L}_t - \hat{L}_s = \frac{1}{1-Z_t} - \frac{1}{1-Z_s}$ for all $\tau < s \leq t$. So, we can write:

$$\lim_{s \downarrow \tau} \frac{1}{1-Z_s} = \frac{1}{1-Z_t} - \hat{L}_t + \hat{L}_\tau < \infty \qquad P\text{-a.s.}$$

Since $Z = (Z_t)_{t \geq 0}$ is continuous and $Z_\tau = 1$ P-a.s., this yields a contradiction, thus showing that $L = (L_t)_{t \geq 0}$ cannot be a local martingale deflator in $\mathbb{G}$ for S on the global time horizon $[0, \tau]$. $\square$

We summarize the results of this Section on the existence of arbitrages in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ after the random time τ and, hence, on the global time horizon $[0, \infty]$, in the following Theorem, which is a direct consequence of Lemma 4.1, Proposition 4.2 and Theorem 2.5.

Theorem 4.5. *NUIP holds in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the global time horizon $[0, \infty]$. However, NA1 (or, equivalently, NUPBR), NA and NFLVR all fail in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ after τ and, hence, also on the global time horizon $[0, \infty]$.*

Remark 4.6. As considered in the Introduction, let $d = 1$ and suppose that the real-valued process $S = (S_t)_{t \geq 0}$ is given as the solution of the SDE (1.1) on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}^W, P)$, where $\mathbb{F}^W$ is the (P-augmented) natural filtration of W. As for Remark 3.13, the random time τ is an honest time which avoids all $\mathbb{F}$-stopping times. Hence, Proposition 4.2 and Theorem 4.5 directly imply claim (c) in the Introduction. Observe that, in this simple example, the arbitrage strategy $\hat{\varphi}$ constructed in the proof of Proposition 4.2 reduces simply to a short position on S immediately after time τ.

5 Conclusions and extensions

In the present paper, we have dealt with the question of whether the additional information associated to an honest time does give rise to arbitrage. Under Assumptions I-III, we have given a complete and precise answer in the context of a general continuous financial market model. In particular, we have studied the validity of no-arbitrage-type conditions which go beyond the classical NFLVR criterion, such as the NUIP, NA1 and NUPBR conditions. We have shown in a simple and direct way that an informed agent can realize arbitrage opportunities *at* an honest time as well as arbitrages of the first kind *after* an honest time, while it is impossible to make arbitrage profits strictly *before* an honest time. The present paper significantly extends previous results in the literature, providing at the same time simpler and more transparent proofs.

We conclude the paper by commenting on the role of Assumptions I-III and discussing some possible extensions and generalizations. The present paper aims at understanding the impact of an honest time on the validity of suitable no-arbitrage-type conditions in the *enlarged* financial market $\mathcal{M}^{\mathbb{G}}$ and, hence, we assumed from the beginning that the *restricted* financial market $\mathcal{M}^{\mathbb{F}}$ is free from any kind of arbitrage, in the classical sense of NFLVR (Assumption I). However, we want to point out that analogous results can be obtained if the restricted financial market $\mathcal{M}^{\mathbb{F}}$ satisfies NA1 (or, equivalently, NUPBR) but the stronger NFLVR condition fails to hold. In that case, Theorems 3.7 and 4.5 continue to hold, provided that Assumptions II-III are still satisfied. Indeed, due to Remark 2.2, if NFLVR fails to hold in the restricted financial market $\mathcal{M}^{\mathbb{F}}$, then it fails in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ as well (and can be also shown to fail on the time horizon $[0, \tau]$). Moreover, by relying on part *(i)* of Theorem 2.5 together with Lemma 2.8 and Assumptions II-III, one can show that NA1 (or, equivalently, NUPBR) holds in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0, \tau]$ but fails on the global time horizon $[0,\infty]$. For the sake of brevity, we omit the details and refer the interested reader to Section 4.4.3 of Fontana [11].

The assumption that the honest time τ avoids all $\mathbb{F}$-stopping times (Assumption II) seems to be crucial. Indeed, if NFLVR holds in the restricted financial market $\mathcal{M}^{\mathbb{F}}$ (Assumption I) but Assumption II does not hold, then an honest time τ does not necessarily give rise to arbitrage opportunities in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0,\tau]$. As an example, let $\tilde{\tau}$ be an honest time which avoids all $\mathbb{F}$-stopping times and let σ be any $\mathbb{F}$-stopping time such that $\tilde{Z}_\sigma > 0$ P-a.s., where $\tilde{Z} = (\tilde{Z}_t)_{t\geq 0}$ is the Azéma supermartingale of $\tilde{\tau}$. Then $\tau := \tilde{\tau}\wedge\sigma$ is easily seen to be an honest time which does not avoid $\mathbb{F}$-stopping times and, as can be deduced from Theorem 3.10, NFLVR still holds in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on $[0,\tau]$.

Observe that our results have been obtained under Assumption III, which implies that, under any $\mathrm{ELMM}_{\mathbb{F}}$ Q, the $(Q,\mathbb{F})$-local martingale N appearing in the multiplicative decomposition of the Azéma Q-supermartingale of τ (see Lemma 2.7) can be written as $N = 1+\varphi\cdot S$. As can be easily checked (see in particular the proofs of Propositions 3.1 and 4.2), only the latter condition is necessary and, hence, the assumption that *all* $(Q,\mathbb{F})$-local martingales can be represented as stochastic integrals of S can be significantly relaxed.

Finally, we want to emphasize that the present paper gives a complete picture of the relations between honest times and arbitrage in the context of general financial market models based on *continuous* semimartingales. However, at least under suitable additional assumptions, our results can also be extended to the case where the discounted price process S has possibly discontinuous paths. For instance, all the results of the present paper still hold if S is assumed to be (under some $\mathrm{ELMM}_{\mathbb{F}}$ Q) a Lévy process and τ an honest time such that its Azéma supermartingale Z admits a multiplicative decomposition as in Lemma 2.7, with $N = 1+\varphi\cdot S^c$, where S^c denotes the continuous $(Q,\mathbb{F})$-local martingale part of S. Note also that a result analogous to Lemma 2.7, which plays a key role in the present paper, has been recently established in the discontinuous case by Kardaras [23] (see also Nikeghbali & Yor [26], Proposition 4.6). For reasons of space, we omit the details and postpone a complete study of the discontinuous case to a forthcoming work.

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A Appendix

Proof of Lemma 3.9.

Note first that, since all $\mathbb{F}$-local martingales are continuous (due to Assumption III), Corollary 2.4 of Nikeghbali & Yor [26] (see also Mansuy & Yor [24], Exercise 1.8) implies that the dual $\mathbb{F}$-predictable projection of the process $\left(\mathbf{1}_{\{\tau\leq t\}}\right)_{t\geq 0}$ is given by $\left(\log\left(N_t^*\right)\right)_{t\geq 0}$. Then, due to (3.3), we can write the following:

$$\begin{aligned}E\left[L_{\sigma\wedge\tau}\right] &= E\left[L_\sigma\mathbf{1}_{\{\sigma<\tau\}}\right]+E\left[L_\tau\mathbf{1}_{\{\tau\leq\sigma\}}\right]\\ &= E\left[\frac{1}{N_\sigma}\exp\left(-\int_0^\sigma\frac{k_s}{N_s^*}\,dN_s^*\right)\mathbf{1}_{\{\sigma<\tau\}}\right]+E\left[\frac{1}{N_\tau}\exp\left(-\int_0^\tau\frac{k_s}{N_s^*}\,dN_s^*\right)(1+k_\tau+\zeta)\,\mathbf{1}_{\{\tau\leq\sigma\}}\right]\end{aligned} \tag{A.1}$$

Let us first focus on the first term on the right-hand side of (A.1). Recall that $\tau < \nu$ P-a.s. (see the proof of Proposition 3.3) and that $Z = (Z_t)_{t \geq 0}$ is the $\mathbb{F}$-optional projection of $\left(\mathbf{1}_{\{\tau > t\}}\right)_{t \geq 0}$ and $Z_\sigma / N_\sigma = 1/N^*_\sigma$ on the set $\{\sigma < \nu\}$ (see Lemma 2.7). Then, we can write as follows:

$$
\begin{aligned}
&E\left[\frac{1}{N_\sigma}\exp\left(-\int_0^\sigma \frac{k_s}{N^*_s}\,dN^*_s\right)\mathbf{1}_{\{\sigma<\tau\}}\right] = E\left[\frac{1}{N_\sigma}\exp\left(-\int_0^\sigma \frac{k_s}{N^*_s}\,dN^*_s\right)\mathbf{1}_{\{\sigma<\tau\}}\mathbf{1}_{\{\sigma<\nu\}}\right] \\
&= E\left[\frac{1}{N_\sigma}\exp\left(-\int_0^\sigma \frac{k_s}{N^*_s}\,dN^*_s\right)Z_\sigma\mathbf{1}_{\{\sigma<\nu\}}\right] = E\left[\frac{1}{N^*_\sigma}\exp\left(-\int_0^\sigma \frac{k_s}{N^*_s}\,dN^*_s\right)\mathbf{1}_{\{\sigma<\nu\}}\right] \\
&= E\left[\exp\left(-\int_0^\sigma \frac{1+k_s}{N^*_s}\,dN^*_s\right)\mathbf{1}_{\{\sigma<\nu\}}\right]
\end{aligned}
\tag{A.2}
$$

Now, let us compute more explicitly the second term on the right-hand side of (A.1). Recall that $E\left[\zeta | \mathcal{G}_{\tau-}\right] = 0$ (see Jeanblanc & Song [17], Theorem 6.2). Recall also that, since all $\mathbb{F}$-local martingales are continuous (due to Assumption III), the dual $\mathbb{F}$-predictable projection of the process $\left(\mathbf{1}_{\{\tau \leq t\}}\right)_{t \geq 0}$ is given by $\left(\log\left(N^*_t\right)\right)_{t\geq 0}$ (see Nikeghbali & Yor [26], Corollary 2.4, or also Mansuy & Yor [24], Exercise 1.8) and that the measure dN^*_s is supported by the set $\{s \geq 0 : Z_s = 1\}$. Then, we can write as follows, where the first equality follows by first taking the $\mathcal{G}_{\tau-}$-conditional expectation:

$$
\begin{aligned}
&E\left[\frac{1}{N_\tau}\exp\left(-\int_0^\tau \frac{k_s}{N^*_s}\,dN^*_s\right)(1+k_\tau+\zeta)\,\mathbf{1}_{\{\tau\leq\sigma\}}\right] = E\left[\frac{1}{N_\tau}\exp\left(-\int_0^\tau \frac{k_s}{N^*_s}\,dN^*_s\right)(1+k_\tau)\,\mathbf{1}_{\{\tau\leq\sigma\}}\right] \\
&= E\left[\int_0^\sigma \frac{1}{N_s}\exp\left(-\int_0^s \frac{k_u}{N^*_u}\,dN^*_u\right)(1+k_s)\,\frac{1}{N^*_s}\,dN^*_s\right] \\
&= E\left[\int_0^\sigma \frac{1}{N^*_s}\exp\left(-\int_0^s \frac{k_u}{N^*_u}\,dN^*_u\right)(1+k_s)\,\frac{1}{N^*_s}\,dN^*_s\right] \\
&= E\left[\int_0^\sigma \exp\left(-\int_0^s \frac{1+k_u}{N^*_u}\,dN^*_u\right)(1+k_s)\,\frac{1}{N^*_s}\,dN^*_s\right] = E\left[1-\exp\left(-\int_0^\sigma \frac{1+k_s}{N^*_s}\,dN^*_s\right)\right]
\end{aligned}
\tag{A.3}
$$

Equation (3.4) then follows by combining (A.2) and (A.3), using the fact that, since $\tau < \nu$ P-a.s., we have $\sigma > \tau$ on the set $\{\sigma \geq \nu\}$ and noting that the process N^* is constant after τ. It remains to show that $\int_0^\tau \frac{1+k_s}{N^*_s}\,dN^*_s > 0$ P-a.s. For that, it suffices to notice the following, where we use the fact that $1 + k_\tau > 0$ P-a.s. (see the first Remark after Lemma 3.5):

$$
E\left[\frac{1+k_\tau}{N^*_\tau}\mathbf{1}_{\left\{\int_0^\tau \frac{1+k_s}{N^*_s}dN^*_s=0\right\}}\right] = E\left[\int_0^\infty \frac{1+k_s}{N^*_s}\mathbf{1}_{\left\{\int_0^s \frac{1+k_u}{N^*_u}dN^*_u=0\right\}}\frac{1}{N^*_s}\,dN^*_s\right] = 0
$$

thus implying that $\int_0^\tau \frac{1+k_u}{N^*_u}dN^*_u > 0$ P-a.s.

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