Notes on Product Bundling with Perfect Negative Dependence

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Preliminary and Incomplete
September 10, 2012

1. Introduction

Stigler (1963) presented a compelling example showing product bundling could be optimal for a monopolist in absence of product complementarity, economies of scope, or an exclusionary motive. This insight led to a sustained literature on the profitability of product bundling, culminating in Chen and Riordan (2012) demonstrating reasonably generally, using copulas to represent joint distributions, that product bundling is profitable if preferences for component products are negatively dependent, independent, or exhibit sufficiently limited positive dependence.

Additional remarkable features of Stigler (1963)'s example are that pure bundling is a profit-maximizing strategy and makes all consumers worse off. The example assumes two types of consumers. Both types value the goods enough that a monopolist selling a good separately finds it profitable to sell to both types, and pure bundling enables a multiproduct monopolist to charge more for the pair of goods without reducing demand.

My purpose is to extend Stigler (1963)'s insight that pure bundling of two goods can be a profit-maximizing strategy that uniformly reduces consumer welfare to a class of cases featuring continua of consumers, including cases for which not all consumers purchase both goods under separate selling. This class of cases has the properties that consumer preferences for the component goods are perfectly negatively dependent, while consumer preferences for a preferred component and for the bundle are perfectly positively dependent. In the special case of an exponential marginal distribution of values for components, profit-maximizing pure bundling improves social welfare even though consumers uniformly are worse off. The results are robust to generalizations that substantially relax perfect negative dependence.

2. Model

There are two goods, X and Y. A consumer's value for X is $u(x) = \mu + \sigma\varphi(x)$, and for Y symmetrically is $u(y)$, with $(x, y) \in [0,1]^2$. The function $\varphi(x)$ is assumed to be strictly increasing and twice differentiable on $[0,1]$ with $\int_0^1 \varphi(x)dx = 0$ and $\int_0^1 \varphi(x)^2 dx$. Therefore, μ is the mean and σ the standard deviation of consumer values for each product. The population of consumers, the size of which is normalized to 1, is described by a symmetric copula $C(x, y)$.

A copula is a function that couples marginal distributions to form a joint distribution. The marginal distribution of $u = u(x)$ is $F(u) \equiv \varphi^{-1}(\frac{u-\mu}{\sigma})$, and the marginal distribution of $v = u(y)$ is $F(v)$. The joint distribution of u and v is $C(F(u), F(v))$. The values of the two goods are perfectly negatively dependent if $C(x, y) = \max\{0, x + y - 1\}$. In this case, consumer type is effectively one dimensional, with $y = 1 - x$ and x uniformly distributed on [0,1].

Constant average costs are normalized to zero for both goods. This normalization should be interpreted carefully because costs do matter for the profitability of bundling. With this normalization, the values u and v should be interpreted as net of constant average costs, and prices should be interpreted as markups. Thus, a negative value of u or v corresponds to a consumer whose reservation value falls short of cost for one product or the other. If $u(0) \geq 0$, i.e. $\bar{\mu} \equiv \frac{\mu}{\sigma} \geq -\varphi(0)$, then all consumers value both products above cost.

3. Separate Selling

If the goods are sold separately at price $p = u(x)$, then profit is

$$\pi(x) = 2[\mu + \sigma\varphi(x)](1 - x).$$

The first-order condition for profit maximization is

$$2\sigma[(1 - x)\varphi'(x) - \varphi(x)] - 2\mu = 0$$

and the second-order condition for a unique maximum is

$$2\sigma[(1 - x)\varphi''(x) - 2\phi'(x)] < 0.$$

These conditions lead to the following characterizations of monopoly pricing when the two goods are sold separately.

Proposition 1: If

$$\varphi'(0) - \varphi(0) \geq \bar{\mu} \geq -\varphi(1)$$

and

$$(1 - x)\varphi''(x) < 2\varphi'(x)$$

for $x \in [0,1]$, then the monopoly price for separately sold goods is $p^s = u(x^s)$ with

$$(1 - x^s)\varphi'(x^s) - \varphi(x^s) = \bar{\mu}.$$

The two conditions of the proposition respectively assure the first-order and second-order conditions for an interior maximum. If $\bar{\mu} \geq \varphi'(0) - \varphi(0)$, then $x^s = 0$ and separate selling covers the entire market. If $\bar{\mu} \leq -\varphi(1)$, then $x^s = 1$ and markets fail.

The next proposition distinguishes two cases under separate selling for interior solutions. In Case (a), some consumers purchase both goods and other consumers purchase one or the other. In Case (b), the market segments into consumers who purchase only X, those who purchase only Y, and those who purchase neither.

Corollary 1: Assume the conditions of Proposition 1 hold.

(a) If $\varphi'(0) - \varphi(0) \geq \bar{\mu} \geq \frac{1}{2}\varphi'\left(\frac{1}{2}\right) - \varphi\left(\frac{1}{2}\right)$, then $0 \leq x^s \leq \frac{1}{2}$, and consumer types $x \in [x^s, 1 - x^s]$ purchase both X and Y under separate selling at the monopoly price, while $x \in (1 - x^s, 1]$ purchase only X and $x \in [0, x^s)$ only Y.

(b) If $\frac{1}{2}\varphi'\left(\frac{1}{2}\right) - \varphi\left(\frac{1}{2}\right) > \bar{\mu} \geq -\varphi(1)$, then $\frac{1}{2} < x^s \leq 1$, and consumer types $x \in (x^s, 1 - x^s)$ purchase neither good under separate selling at the monopoly price, while $x \in [1 - x^s, 1]$ purchase only X and $x \in [0, x^s]$ only Y.

4. Bundling

A (symmetric) mixed bundling strategy sets a price q for the bundle composed of X and Y, and a price p for X or Y on a standalone basis. Pure bundling occurs when only the bundle is available, i.e. the standalone price is prohibitively high.

With perfect negative dependence the value of the bundle composed of X and Y is $w(x) \equiv u(x) + u(1 - x)$. Furthermore, $w'(x) = \sigma[\varphi'(x) - \varphi'(1 - x)]$ and $w''(x) = \sigma[\varphi''(x) + \varphi''(1 - x)]$. Therefore, $\varphi''(x) > 0$ is sufficient for $w''(x) > 0$, in which case consumers who value the bundle most are l at the extremes of the unit interval, and the consumer who values the bundle least is at the midpoint. In this case, $w(x)$ and $\max\{u(x), u(1 - x)\}$ are correlated perfectly. Furthermore, when some consumers purchase both goods under separate selling, the median consumer earns a strictly positive surplus at the monopoly price, i.e. $w\left(\frac{1}{2}\right) > 2u(x^s)$.

The corner solution case $\bar{\mu} \geq \varphi'(0) - \varphi(0)$ corresponds closely to Stigler (1963)'s example. In this case, all consumers purchase both goods under separate selling, and pay $u(0) \geq 0$ for each. Offering the only bundle at price $2u(0)$ changes nothing, but the monopolist can raise the price of the bundle to $w\left(\frac{1}{2}\right) \equiv 2u\left(\frac{1}{2}\right)$ and still cover the market. Thus pure bundling increases profit and uniformly reduces consumer welfare.

The next proposition establishes that pure bundling generally is optimal with perfect negative dependence for an interior solution if the value of bundle is convex in x and all consumers value both goods above cost.

Proposition 2: Assume X and Y are perfectly negative dependent, and the conditions of Proposition 1 hold. If $w''(x) > 0$ and $u(x) \geq 0$, then the profit-maximizing strategy is pure bundling.

Proof: Consider a profit-maximizing bundling strategy with standalone price p^* and bundle price q^*, such that $2p^* \geq q^* \geq p^*$. At these prices, consumers are willing to purchase X on a standalone basis if

$$u(x) \geq p^*$$

and

$$q^* - p^* \geq u(1-x).$$

Therefore, if consumer *x* purchase X on a standalone basis, then so do all consumers on the interval $[x, 1]$. It follows that a non-degenerate mixed bundling strategy defines a critical value $\hat{x} > \frac{1}{2}$ such that $x > \hat{x}$ purchase X on a standalone basis, and, by symmetry, $x < 1 - \hat{x}$ purchase Y on a standalone basis. Furthermore, it must be that

$$q^* - p^* = u(1-\hat{x});$$

otherwise, $p^* = u(\hat{x})$, $q^* - p^* > u(1-\hat{x})$, and $q^* > w(\hat{x})$, i.e. with $w''(x) > 0$, no consumer on the interval $[1-\hat{x}, \hat{x}]$ would purchase the bundle.

Suppose $w(x) \geq q^*$ for some $x \in \left[\frac{1}{2}, \hat{x}\right)$. Then all consumers on the interval $[x, \hat{x}]$ are willing to purchase the bundle. Thus there might also exist a second critical value $\frac{1}{2} \leq \hat{\hat{x}} < \hat{x}$, such that consumers on the intervals $\left(1-\hat{\hat{x}}, \hat{\hat{x}}\right)$ purchase the outside good, while consumers on the intervals $\left[1-\hat{x}, 1-\hat{\hat{x}}\right]$ and $\left[\hat{\hat{x}}, \hat{x}\right]$ purchase the bundle. Furthermore, it must be that

$$w\left(\hat{\hat{x}}\right) = q^*$$

at a profit-maximizing strategy; otherwise, the firm could raise prices without changing purchasing patterns.

If follows from the above that

$$p^* = w\left(\hat{\hat{x}}\right) - u(1-\hat{x}),$$

and, using the symmetry of the two halves of the unit interval, profit from the mixed bundling strategy can be written as

$$2\left\{(1-\hat{x})\left[w\left(\hat{\hat{x}}\right) - u(1-\hat{x})\right] + \left(\hat{x} - \hat{\hat{x}}\right)w\left(\hat{\hat{x}}\right)\right\}$$

or, equivalently,

$$2\left\{\left(1-\hat{\hat{x}}\right)w\left(\hat{\hat{x}}\right)-(1-\hat{x})u(1-\hat{x})\right\}.$$

Note that $u(1-x)>0$ for $x<1$ implies profit is increasing in $\hat{x}$. Therefore, it is always profitable to raise the standalone price until consumers stop purchasing the standalone good, i.e. set $\hat{x}=1$, and the optimal bundling strategy must be a pure bundling. Q.E.D.

Since $\hat{x}=1$, profits under an optimal pure bundling strategy maximize

$$\Pi(x)\equiv(1-x)w(x)$$

on $[\frac{1}{2},1]$ under the conditions of Proposition 2. The next proposition places a regularity condition on this function to show that pure bundling leaves consumers uniformly worse off compared to separate selling in Case (a) or Corollary 1.

<u>Proposition 3</u>: Assume the conditions of Propositions 1 and 2 hold. If $\Pi(x)$ is quasi-concave on $\left[\frac{1}{2},1\right]$, and

$$\varphi'(0)-\varphi(0)>\bar{\mu}\geq\tfrac{1}{2}\varphi'\left(\tfrac{1}{2}\right)-\varphi\left(\tfrac{1}{2}\right),$$

then all consumers are worse off under pure bundling compared to separate selling.

<u>Proof</u>: In Case (a) of Corollary 1, by setting $q=2u(x^s)$ and raising the standalone price to a prohibitive level, the firm increases profit by selling the bundle to all consumers who purchase under separate selling. Consumers who purchased only one good under separate selling are worse off by revealed preference, while consumers who purchased both goods are indifferent. Since $w'\left(\frac{1}{2}\right)=0$, and therefore $\Pi'\left(\frac{1}{2}\right)=-w\left(\frac{1}{2}\right)<0$, the corner solution $\hat{\hat{x}}=\frac{1}{2}$ maximizes $\Pi(x)$ on $[\frac{1}{2},1]$ under the quasi-concavity assumption. Therefore, the profit-maximizing price of the bundle is $q^*=w\left(\frac{1}{2}\right)>2u(x^s)$, and all consumers are strictly worse off compared to separate selling. Q.E.D.

<u>Remark</u>: Since $\Pi\left(\frac{1}{2}\right)<0$, quasi-concavity of $\Pi(x)$ on $[\frac{1}{2},1]$ is equivalent to $\Pi'(x)\leq 0$ on $[\frac{1}{2},1]$.

The next section illustrates the above results for exponential marginal distributions, while also exploring in more detail the welfare properties of pure bundling.

5. Exponential Case

If

$$\varphi(x) = -\ln(1-x) - 1,$$

then

$$F(u) = 1 - e^{-\left(\frac{u-\mu}{\sigma}\right)-1},$$

i.e. u is distributed exponentially with mean μ and variance σ^2. Case (a) of Corollary 1 corresponds to $2 \geq \bar{\mu} \geq 2 + \ln\left(\frac{1}{2}\right) \approx 1.307$,

Proposition 4: If preferences for X and Y are perfectly negatively dependent, the marginal distribution of values is exponential, and $2 > \bar{\mu} \geq 2 + \ln\left(\frac{1}{2}\right)$, then pure bundling is profit maximizing, consumers uniformly are worse off compared to separate selling, and social surplus is higher under pure bundling than separate selling.

Proof: Since $\varphi'(x) = 1/(1-x)$ and $\varphi''(x) = 1/(1-x)^2$, the conditions of Proposition 1 are satisfied if $\bar{\mu} \leq 2$. The interior solution is

$$x^s = 1 - e^{\bar{\mu}-2}.$$

Note that $w''(x) \equiv \sigma\left[\frac{1}{(1-x)^2} + \frac{1}{x^2}\right] > 0$, and $u(0) \geq 0$ if $\bar{\mu} \geq -\varphi(0) = 1$. Therefore, pure bundling is profit maximizing by Proposition 2.

Case (a) of Corollary 1 corresponds to $2 \geq \bar{\mu} \geq 2 + \ln\left(\frac{1}{2}\right) \approx 1.307$. As observed in Remark 1, $\Pi(x)$ is quasi-concave if

$$\Pi'(x) \equiv 4 - 2\bar{\mu} + \Psi(x) < 0$$

where

$$\Psi(x) \equiv \ln(1-x) + \ln(x) - \frac{1}{x}$$

The function $\Psi(x)$ achieves a maximum on $[\frac{1}{2}, 1]$ at $x = \frac{1}{\sqrt{2}} \approx 0.707$. Therefore,

$$\Pi'(x) < 0 \Leftrightarrow \bar{\mu} > 2 + \frac{1}{2}\Psi\left(\frac{1}{\sqrt{2}}\right) \approx 0.506$$

Therefore, the conditions of Proposition 3 are satisfied, and consumers uniformly are worse off compared to separate selling.

Social surplus under separate selling is

$$2\int_{x^s}^{1} u(x)dx = 2\sigma\left\{(1-x^s)\bar{\mu} - \int_{x^s}^{1}[\ln(1-x)+1]dx\right\}$$
$$= 2\sigma\{(1-x^s)\bar{\mu} - (1-x^s)\ln(1-x^s)\} = 2\sigma(1-x^s)[\bar{\mu} - \ln(1-x^s)]$$
$$= 2\sigma(1-x^s)[\bar{\mu} - \ln(e^{\bar{\mu}-2})] = 4\sigma(1-x^s) = 4\sigma e^{\bar{\mu}-2}$$

Social surplus when the entire market is covered by a pure bundling strategy is

$$2\int_{0}^{\frac{1}{2}} w(x)dx = 2\sigma\left\{\bar{\mu} - 1 - \int_{0}^{\frac{1}{2}}[\ln(x) + \ln(1-x)]dx\right\} = 2\sigma\bar{\mu}$$

Social surplus is higher under pure bundling if $\Psi(\bar{\mu}) \equiv \bar{\mu} - 2e^{\bar{\mu}-2} > 0$. Since $\Psi(\bar{\mu})$ is concave, $\Psi\left(2 + \ln\left(\frac{1}{2}\right)\right) = 1 + \ln\left(\frac{1}{2}\right) > 0$, and $\Psi(2) = 0$, it follows that $\Psi(\bar{\mu}) > 0$ for $\bar{\mu} \in \left(2 + \ln\left(\frac{1}{2}\right), 2\right)$. Therefore, social surplus therefore is higher under pure bundling compared to separate selling in this case. Q.E.D.

6. Robustness

[Incomplete]

References

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