Concentrated Ownership and Equilibrium Asset Prices *

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Abstract

I study the dynamics of asset prices in an economy in which investors choose whether to hold diversified or levered concentrated portfolios of risky assets. The latter are valuable, as they increase the productivity of the corresponding enterprises. I capture the tradeoff between risk sharing and productivity gains by introducing what I call "active capital": people who participate in such investments are restricted in their outside opportunities but receive extra compensation. In equilibrium, active and standard capital coexist. The willingness to provide active capital is mainly determined by risk considerations. Therefore, the quantity of active capital fluctuates jointly with risk premia, amplifying their variations. As a consequence, the price of volatility risk exposure can be large and return volatility is mainly induced by fluctuations in future expected returns. These results are particularly strong when fundamental volatility is low, because at such time, a large number of concentrated owners are likely to exit their positions and sell off their assets.

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1 Introduction

A number of economic activities can run more efficiently if some agents invest a significant fraction of their wealth in the enterprise. The benefits of such positions are one of the reasons advanced for stock-based compensation of executives and for why entrepreneurs keep a large equity stake in their businesses. The gains from concentrated, levered ownership can also come from investors outside the firm, typically financial institutions, exerting direct control or monitoring insiders. Venture capitalists and private equity funds exemplify this type of behavior, but one can also think of the activity of investment banks and hedge funds. A common feature of many of these financing activities is a pattern of cyclical behavior linked in particular to fluctuations in asset prices. For instance, broker-dealers (Adrian, Moench and Shin 2010), buyout funds (Haddad, Loualiche and Plosser 2011) and venture capitalists (Gompers, Kovner, Lerner and Scharfstein 2008) diminish their activities in periods of high risk premium, as can be seen on Figure 1. The financial crisis of 2007-2009 is such an episode: business creation dropped and many leveraged financial institutions largely reduced of or ceased completely their activities[1] as asset prices dropped across markets. These facts suggest incentives to take on concentrated investment vary with changes in asset prices. As large quantities of concentrated investment affect aggregate risk sharing, these fluctuations could feed back into asset prices. This paper investigates how the aggregate quantity of concentrated investment is determined jointly with asset prices. In particular, I study how various sources of fundamental fluctuations are transmitted to asset prices in the presence of such a form of investment.

I present a dynamic general equilibrium model with a role for concentrated investment. Agents are allowed to pick what I call "active capital" as an alternative form of asset ownership. Active investors constrain themselves to a concentrated risky position in a firm, which makes the firm more productive. I represent this activity by a constraint on the portfolio shares in risky assets for active agents. This constraint reproduces the high portfolio leverage typical of these investors and is close to the optimal contract as a solution of a moral hazard problem.[2] This framework allows me

[1] Between December 2007 and March 2009, the hedge fund industry equity went from $1975billion to $973billion according to the Barclay Hedge database. For broker-dealers, He, Khang and Krishnamurthy (2010) estimate a change of trading assets from $2601billion to $1810billion using balance sheets of three pure broker-dealers. Private equity activity was also largely impaired for an extended period of time: the CityUK report on global private equity reports a drop of funds raised from $480billion in 2007 to $140billion in 2009.

[2] See Holmstrom (1979) for the original derivation and Holmstrom and Tirole (1997) for a general

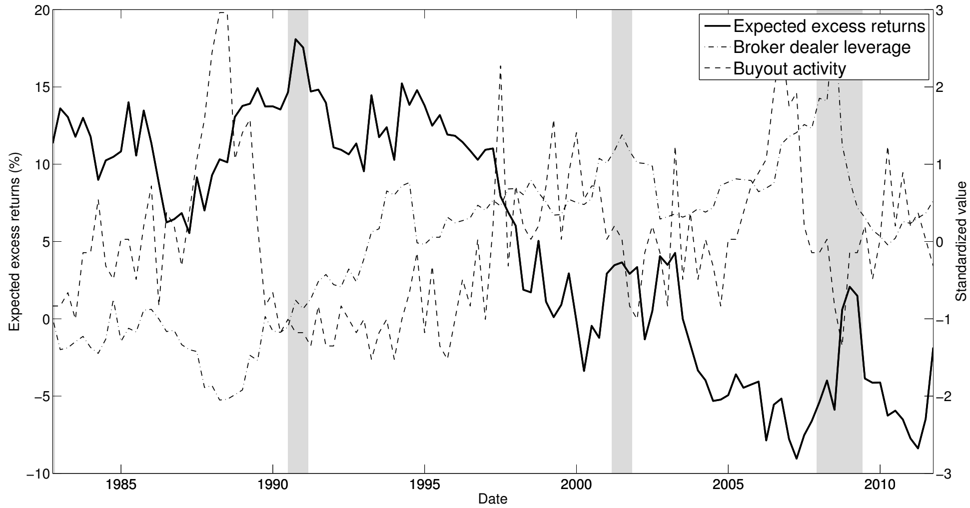

Figure 1: **Financial activity and expected returns**

Expected excess returns are for the CRSP value-weighted portfolio and are predicted for the two years following the current date using *d-p* and *cay* for the quarter immediately prior. Broker dealer leverage is the standardized logarithm of the leverage of broker-dealers using L.127 of the Flow of Funds (correlation of −.65 with expected returns). Buyout activity is the standardized number of buyout of public firms each quarter (correlation −.42 with expected returns). The shaded areas correspond to the NBER recessions.

to study the joint determination of the quantity of active capital and asset prices in a variety of stochastic structures. I show the effects of active capital on asset prices and the real economy crucially depend on the nature of fundamental risk in the economy.

Active capital affects asset prices through two channels: *distorted risk-sharing* and *deleveraging risk*. The static effect of active capital is a distortion of the risk-sharing arrangement in the economy. Active agents hold a disproportionate fraction of the risky assets. Therefore, the passive agents bear less risk in equilibrium. Consequently, they require a lower risk premium for the asset. This channel tells us risks that can be borne by active agents will tend to have a lower price than those for which there is no active ownership. Diverging from the standard perfect risk sharing is optimal in this framework as it improves the productivity of firms. I show, however, that a competitive market yields an excess amount of active capital. Taxing firms that use this source of capital raises the welfare of all agents by improving risk sharing.

The second channel, *deleveraging risk*, is driven by the dependence of the quantity of active capital on economic fundamentals. For instance, if fundamental risk increases, the quantity of active capital decreases. We observe a deleveraging episode: some

equilibrium application.

active investors switch back to passive investments. This switch requires selling assets to reduce their excess risky portfolio holdings. Existing passive agents have to absorb these assets arriving on the market, which tends to lower prices further than the direct impact of the increase in risk. In this sense, active capital amplifies the fundamental volatility risk. Ex ante, this effect will tend to increase the price of this risk. The general finding is that shocks that affect the supply or demand of active capital are amplified and become more costly.

The determination of the quantity of active capital is key to understanding characteristics of these two effects. Equilibrium in the active capital market equates the quantity of active capital firms demand with the number of investors willing to accept this particular portfolio. Firms demand active owners because they increases cash flow. They trade off these productivity gains with the extra cost of active capital. I assume the gains per fraction of active capital are independent of the state of the economy. Therefore, the demand curve for active capital is constant over time. On the other hand, the supply of active capital is endogenously determined. Because all agents are ex-ante identical, the extra returns paid to active capital must exactly compensate active agents for the extra risk they bear. The required compensation (cost of active capital) depends positively on risk aversion, the riskiness of the asset, and the size of the deviation from the optimal portfolio. This result points at two shocks that shift the amount of active capital: volatility and risk aversion shocks.

In general equilibrium, asset prices change with different levels of active capital. Market clearing implies that with more active agents, passive agents hold a smaller quantity of risky assets. For this condition to be consistent with optimization by passive agents, the asset must be more expensive. Therefore, the portfolio of active agents becomes more costly and they ask for more compensation for their activity. This feedback of activity on risk sharing makes the supply of active capital an increasing function of its price. Because deleveraging risk plays a role through variations in the quantity and not the price of active capital, the effects are more dramatic when the demand and supply are more elastic and when supply is more responsive to economic conditions.

My analysis provides a framework for understanding a number of asset-pricing facts. I show risk premia and the quantity of active capital are negatively related. This relation is consistent with the findings of Adrian et al. (2010): they show the aggregate risk premium covaries negatively with the balance sheet of financial intermediaries. Similarly, Haddad et al. (2011) find fluctuations in buyout activity are strongly negatively correlated with a dynamic measure of the equity risk premium. In my model, fluctuations in risk premium are a priced risk. Therefore, covariance with shocks to the quantity of active capital, as a measure of exposure to this risk, should help rationalize the cross-section of expected return. Adrian, Etula and Muir (2011) confirm this result: loadings on shocks to the leverage of broker-dealers explain the cross-section of equity expected returns. The model also provides insights regarding the sources of variation in prices. Since the "excess volatility puzzle" of Shiller (1981) and Campbell and Shiller (1988), understanding the link between fundamental fluctuations and price fluctuations has been problematic. I show changes in the quantity of active capital amplify the impact of some shocks (i.e., volatility) on prices. For the price of risks, the role of active capital can go in two directions: the prices of shocks that do not affect its quantity are lower relative to the standard endowment economy, whereas those that affect it can be larger. As cash-flow shocks fall in the first category, the model has the potential to explain the relative lack of success of approaches using measures of cash-flow risk to determine expected returns. Conversely, mild shocks to volatility can have a large impact on prices and command a high risk price, as they generate variation in the supply of active capital.

After discussing related work, section 2 presents a simple case of the model showing how equilibrium in the active capital market is determined. I detail the general model in section 3. Section 4 focuses on the pricing implications of the presence of active capital in an economy with changes in uncertainty and growth prospects. Finally, I discuss extensions of the model in section 5.

Related Literature

This paper fits in the literature studying the interaction of financing frictions and heterogeneous ownership of assets in general equilibrium. Following the Great Depression, a large body of work studied how financial contracts respond to economic fluctuations.[3] Fisher (1933) explains how deflation feeds back into more expensive nominal debt, and therefore tighter credit constraints, further depressing economic activity. Kiyotaki and Moore (1997) focus on the feedback of asset prices into collateral constraints.

I focus on the role of equity constraints on agents linked to particular firms for asset-pricing dynamics. Bernanke and Gertler (1989) derive such constraints as the solution of an agency problem due to costly state verification. They study how fluctuations in the net worth of entrepreneurs creates persistence in economic shocks. Carlstrom and Fuerst (1997) provides a quantitative exploration of this model. Bernanke, Gertler and

[3] Brunnermeier, Eisenbach and Sannikov (2010) survey extensively this literature.

Gilchrist (1999) obtain amplification through technological illiquidity. An alternative approach to the costly state verification as a motivation for these constraint is the standard moral hazard problem of Holmstrom (1979). Holmstrom and Tirole (1997) provide a static general equilibrium model featuring such a friction, and emphasize how changes in the supply of entrepreneurs or monitors can affect equilibrium investment and prices. He and Krishnamurthy (2008a) solves the contracting problem in a dynamic framework. Closest to my paper are Brunnermeier and Sannikov (2010), and He and Krishnamurthy (2008b). They both study the dynamics of asset prices in models with an equity constraint. Related is Danielsson, Shin and Zigrand (2009): they study volatility dynamics in the presence of a Value At Risk constraint.

Brunnermeier and Sannikov (2010) focus on the interaction of exogenous fluctuations in net worth with precautionary motives of entrepreneurs. They show this interaction generates substantial nonlinearities not captured by log-linear approximations used in the previous literature. In particular, they find that deleveraging following negative shocks creates instability in the economy. This instability is akin the deleveraging risk in my model. However, they generate this effect through precautionary motives rather than risk aversion. Therefore, because agents are risk-neutral, no risk premium is present in the model.

He and Krishnamurthy (2008b) features risk averse agents, and therefore can study the dynamics of the risk premium. Compared to my model, passive and active investors play a different role. Their active investors are intermediary, the only agents marginal in asset markets, and are constrained to hold a fraction of the total supply of risky assets. Following negative shocks, they have low net worth and the constraint becomes binding. Because they cannot sell their assets, the price must adjust and the risk premium must increase. This is symmetric to my model where in poor economic times, active investors sell off their assets and passive investors, who are marginal in asset markets, have to bear more risk. Therefore, although they obtain similar asset pricing implications as my model, they find an opposite relation between risk premium and leverage.

Another important difference of my approach relative to these other models is the focus on the entry and exit decision in active investment. Indeed, most papers in this literature take as given the sets of active and passive investors. This ex-ante segmentation makes the net worth of active investors an important state variables. Following losses, because their wealth loads disproportionately on aggregate risk, active agents represent a lower fraction of the economy. By assumption, new active agents cannot enter, and therefore a lack of active investment is present and affects the economy. I shut down this channel and focus on how variations in economic conditions affect the incentives to provide active capital.

Another model featuring endogenous segmentation is Rampini (2004). He generates variations in entrepreneurial activity through the interaction of decreasing absolute risk aversion and variations in productivity. My model focuses on variations in the uncertainty of the economy. Another important difference is that he focuses on a planner problem, and therefore, is unable to study asset prices.

The tractability of the model allows me to study asset pricing frictions in the context of rich asset-pricing models. Indeed, most of the previous literature has focused on a single shock to economic conditions, affecting the level of output. I am able to study economies with a variety of shocks, in particular to the long-run growth rate and the volatility of the economy. The diversity of priced shocks is a recurrent theme of the finance literature, see Fama and French (1993) for instance. In particular there is a debate in the macro-finance literature on the sources of fluctuations in risk premium. Campbell and Cochrane (1999) argue that habit preferences can explain these fluctuations whereas Bansal and Yaron (2004) obtain them by assuming a combination of recursive preferences and changes in the long-run volatility of consumption. I study the effect of the financing friction in the framework of Bansal and Yaron (2004) and show that the friction amplifies fluctuations in expected returns due to volatility shocks through deleveraging. In this literature, the role of market incompleteness has been studied, for instance in Heaton and Lucas (1996). Most of these studies focus on an exogenous, constantly present source of incompleteness. I argue that an important source of variation in prices is due to dynamic changes in risk-sharing.

Other sources of heterogeneity in the behavior of agents have been pointed at as potential sources of fluctuations in risk premia. Dumas (1989) shows that even with i.i.d. dynamics, heterogeneity in risk aversion can generate fluctuations in expected returns. Most of the above papers also assume some degree of preference heterogeneity, either in risk aversion or time discount. Some other sources of heterogeneity have been shown to interact with financing constraints. Gennaioli, Shleifer and Vishny (2011) study the implications of neglected risks for deleveraging and asset prices. Geanakoplos (2009) focuses on how belief heterogeneity interacts with margins. These assumptions might not be innocuous for asset prices. I assume ex-ante identical agents, thereby focusing solely on the financing friction.

2 Basic Model

In this section, I present a case of my model with constant economic conditions to illustrate how the quantity of concentrated capital and asset prices are jointly determined in equilibrium. I study an infinite horizon, continuous-time economy. I first explain how I model the role of concentrated positions in increasing productivity. Then I move on to determining the equilibrium of the model and emphasize properties of prices in my economy that will drive the results in the general model with time-varying conditions.

I depart from the standard framework by relaxing the assumption that the production outcomes of firms are independent of their ownership structure. In a Walrasian equilibrium, ownership is determined only by concerns of consumption smoothing across time and states of the world; having agents that influence the production of the firm own it provides no benefit. Many (e.g., Berle and Means (1932)) have argued the development of larger firms and financial markets causing more diffuse ownership, has led this model to be an increasingly accurate representation of the world. However, this argument is at odds with the data. Holderness, Kroszner and Sheehan (1999) find the mean percentage of common stock held by a firm's officers and directors for exchange-listed firms actually increased from 13% in 1935 to 21% in 1995. Additionally, private firms still represent a large fraction of the economy and most of their equity is owned by their workers. I capture the particular role of concentrated ownership by introducing the notion of active investors: agents that concentrate their asset holdings in a given firm increase its productivity.

I do not explicitly model the labor and production decisions, but rather focus on the implications of an exogenously specified constraint for asset allocation. Specifically, each firm can choose to pay some agents to actively invest in it. Firms thereby trade off the cost of hiring these agents with the additional productivity they provide. The additional productivity is proportional to the fraction of capital active investors own, where the marginal return λ is exogenously specified. Agents, on the other hand, choose whether to allocate their wealth optimally without focusing on any precise firm or investing actively in a given firm. An agent investing actively must allocate a fraction $\bar{\theta} > 1$ exogenously specified of his wealth in claims to the output of the firm, financing this position by taking up risk-free debt. The motive to concentrate holdings is that the firm in which an agent invests actively will compensate him in addition to the regular asset returns.

Similar to this is the decision of inside ownership by firms. They can choose whether to provide their employees with fixed or stock-based compensation. Conversely, people can choose “safe” career paths that do not link their labor decision[4] to their wealth-allocation decisions, or to concentrate their wealth in one firm where its evolution depends on the enterprise's performance. However, note that many other forms of active investment exist. For instance, entrepreneurs usually keep a large stake in the firms they create. Active investment also does not need to come from agents working directly inside the firm. Holmstrom and Tirole (1997) emphasize that outside investors can affect a firm's outcomes through their monitoring activity. Typical of such activity are private equity funds and venture capitalists, whether they fund new projects or buy out firms, but one can also think of the investment activities of a number of hedge funds or investment banks.

2.1 The hiring decision of firms

I assume a continuum of identical firms indexed by $j \in [0, 1]$. Firms can go on the occupation market and hire the services of active investors in order to increase their productivity. Let m_t^j be the fraction of total firm value held by active agents at time t. The evolution of the firm cash flow D_t^j is given by:

$$\frac{dD_t^j}{D_t^j} = (\mu_D + \lambda m_t^j)dt + \sigma_D dZ_t. \tag{2.1}$$

The parameters μ_D and σ_D control the fundamental drift and volatility of cash-flow growth and $\{Z_t\}$ is a univariate brownian motion. Active investment increases cash-flow growth, with a marginal return λ. Such an effect is similar to the effect of investment in a standard *q-theory* framework. For instance, active investors can help the firm make better decisions or work harder, thereby increasing productivity while they are at the firm and permanently increasing the scale of production.

Firms have to pay active investors for their services. I assume the payment takes the form of a fee $f_t dt$ per unit of capital. This fee is determined by the competitive equilibrium of the occupation market, and the firm takes it as given. Denoting P_t^j as the market value of the firm, the total payment to active investors at time t is $f_t m_t^j P_t^j dt$. Equivalent to a direct payment, f_t can be thought of as a rate of share issuance: for each unit of capital they provide, active investors receive $f_t dt$ extra shares. As this payment is infinitesimal, whether investors receive it before or after the resolution of uncertainty is irrelevant.

[4] I only focus on the active capital friction. In particular, I assume the wealth of all other agents is perfectly liquid and tradable at all times.

The firm chooses how many investors, as a fraction of its capital, it hires in order to maximize its share value. The firm takes the process for the stochastic discount factor $\{S_\tau\}$ and the fee $\{f_\tau\}$ as given. As in the standard investment theory, the firm faces a static tradeoff between productivity increase and the fee payment. The marginal benefit of increasing m_t^j is a gain in scale generating a value $\lambda P_t^j dt$, whereas the marginal cost is the payment $f_t P_t^j dt$. As neither the marginal benefit nor the cost depend on m_t^j, we obtain a perfectly elastic demand for active capital from the firm:

$$\begin{cases} m_t^j = 1 & \text{if } \lambda > f_t, \\ m_t^j \in [0,1] & \text{if } \lambda = f_t, \\ m_t^j = 0 & \text{if } \lambda < f_t. \end{cases}$$

In the case of an interior equilibrium, $\lambda = f_t$, cost and benefit exactly cancel each other out. Firms are indifferent between any level of active capital. Their valuation does not depend on the level they choose; that is, the valuation is the same as that of a firm without active capital. In section 3, I provide a more complete derivation of this result and justify the time consistency of the policy function, even though the cost depends on the value the firm is optimizing.

2.2 The occupation decision of agents

I assume a continuum of ex-ante identical agents indexed by $i \in [0,1]$. They value risky consumption plans with the standard power utility function:

$$\mathcal{U}\left(\{C_\tau\}_{\tau=t}^{\infty}\right) = \mathbb{E}_t\left[\int_0^\infty e^{-\beta\tau}\frac{C_{t+\tau}^\gamma}{\gamma}d\tau\right],$$

where β is the rate of time discount and $\Gamma = 1-\gamma$ is the relative risk aversion. Agents are all endowed with an equal fraction of all firms at time 0. Let W_t^i be their wealth at time t. At each point in time, agents can choose to be either a passive or active investor. If they decide to be passive, they can also choose their portfolio. They make this decision in order to maximize their lifetime utility, taking asset returns and the fee for active capital as given.

Practically, most forms of active investment have some degree of illiquidity. Compensation contracts often involve some long-term relation, at least at the yearly frequency with the annual payment of bonuses. Similarly, entrepreneurs cannot always liquidate their firms on short notice. My model does not feature this long-run illiquidity, but captures the idea that at any point in time, some agents decide to take on or leave active investments. Indeed, we observe a lot of mobility in the workforce, and the landscape of firms is constantly changing.[5]

Passive investors are standard neoclassical agents. Let this (endogenous) subset of investors at time t be $\mathcal{P}_t^*$ They have unrestricted access to the asset markets: they can buy and sell claims to any payoff. I note $\theta_t^{i,j*}$ as the number of shares of firm j bought by agent i, and $\mu_{R,t}^j$ and $\sigma_{R,t}^j$ as the drift and volatility of these shares' returns. The wealth evolution for a passive agent is then

(2.2)

$$dW_t^i = \left(W_t^i \left(\int_0^1 \theta_t^{i,j*} \, (\mu_{R,t}^j - r_{f,t}) dj + r_{f,t} \right) - C_t \right) dt + W_t^i \left(\int_0^1 \theta_t^{i,j*} \sigma_{R,t}^j dj \right) dZ_t.$$

Active investors (set $\mathcal{A}_t^{j*}$ for firm j and $\mathcal{A}_t^*$ in aggregate) focus on a single firm j and help increase its productivity. I assume a fraction $\bar{\theta} > 1$ of shares of firm i financed by risk-free borrowing must comprise the investors' portfolios. The assumption that $\bar{\theta} > 1$ implies aggregate risk is concentrated in the hands of active investors.[6] As a compensation for focusing on firm j, they receive an extra return f_t per unit of investment. The wealth evolution for an active agent investing in firm j is therefore

$$dW_t^i = \left(W_t^i (\bar{\theta} \, (\mu_{R,t}^j - r_{f,t}) + r_{f,t} + \bar{\theta} f_t) - C_t \right) dt + W_t^i \, \bar{\theta} \, \sigma_{R,t}^j dZ_t. \tag{2.3}$$

This constraint departs from the standard optimal contract in the presence of moral hazard(Holmstrom 1979) in three ways: no benchmarking of aggregate risk, contract on market price rather than actual output, and constraint proportional to wealth.[7] The standard theory predicts the contract should only take into account a measure of the idiosyncratic part of cash flow, not an overall stock position. However, in practice, equity-based compensation is widely used and little evidence points to relative-performance evaluation.[8] The concentrated positions even seem to make agents bear an excessive amount of aggregate risk.[9] One could argue agents can go on markets and choose whether to hedge any excess exposure to aggregate risk. To my knowledge of the literature, little evidence supports that agents engage in such shorting of

[5]Puri and Zarutskie (2011) find that about 3 million firms are created in any 5-year period between 1981 and 2005.

[6]It is easy to check that an inequality constraint of $\theta \geq \bar{\theta}$ would always bind. If $\bar{\theta} \leq 1$ with an inequality constraint, the constraint would always be slack.

[7]The first two are common assumptions of the literature on the macroeconomic role of financial constraints and are present in Bernanke et al. (1999), He and Krishnamurthy (2008a) and Brunnermeier and Sannikov (2010).

[8]Janakiraman, Lambert and Larcker (1992) and Aggarwal and Samwick (1999) do not find significant evidence in favor of relative performance evaluation for firms' executives.

[9]Moreira (2009) finds small-business owners' income loads excessively on aggregate risk.

aggregate risk. This lack of hedging might be due to a limited ability to take short positions. My results still hold if agents cannot hedge as much as they would like to. In this case $\bar{\theta}$ becomes the loading on risk after all possible hedging is done. Whether compensation should be commensurate with changes in stock prices or proportional to the percentage change is ambiguous from the theoretical point of view. Empirically, percentage-percentage measures appear to give more sensible results and are more stable across firms.[10]

When choosing his occupation, an agent faces a tradeoff between optimizing his portfolio and receiving the fee f_t. As passive agents are unconstrained, without the fee, the utility of a passive investor would be smaller than that of an active investor. Concentrating a portfolio on a levered position in one firm would serve no purpose.

To solve for the optimal decision of an agent, we can make a few simplifying remarks. First note that because all firms are identical, they all have the same return and volatility, $\mu^j_{R,t}$ and $\sigma^j_{R,t}$, so I drop the superscript j and note θ^{i*}_t as the optimal risky position of agent i if he is passive. Also note the opportunity set of agents is linear in their wealth and independent of their past occupation, preferences are homogenous of degree γ, and the opportunity set of a firm is linear in its current size. The model is therefore stationary, and no endogenous state variable is present. In particular, the utility level of each agent as a function of wealth does not depend on i and t. It is given by

$$\mathcal{U}^i_t = \frac{(W^i_t)^\gamma}{\gamma} G,$$

for some endogenous constant G.

We can then focus on the Hamilton-Jacobi-Bellman equation, determining the utility of an agent starting with one unit of wealth:

$$\begin{cases} 0 = \max\{\text{HJB}_P, \text{HJB}_A\} \\ \text{HJB}_A = \sup_c c^\gamma - \beta G + \gamma G(\bar{\theta}(\mu_R - r_f) + r_f + \bar{\theta} f_t - c) + \frac{1}{2}\gamma(\gamma-1)G\bar{\theta}^2\sigma_R^2 \\ \text{HJB}_P = \sup_{c,\theta} c^\gamma - \beta G + \gamma G(\theta(\mu_R - r_f) + r_f - c) + \frac{1}{2}\gamma(\gamma-1)G\theta^2\sigma_R^2, \end{cases}$$

where the first maximization corresponds to the occupation choice and the next two correspond to the consumption and portfolio choices of an agent in each occupation. The first-order condition with respect to consumption is the same for both occupations.

[10]The seminal paper of Jensen and Murphy (1990) finds an apparently small dollar-dollar sensitivity of 0.3% for CEOs. Edmans, Gabaix and Landier (2009) propose a model predicting percentage-percentage pay. They find empirically this measure is 9 on average and is stable across different sizes of firms.

It tells us that if both occupations occur in equilibrium, the consumption-wealth ratio of all agents will be the same, equal to

$$c = G^{\frac{1}{\gamma-1}}.$$

The other first-order condition is the portfolio choice of a passive investor. Because a passive investor is just a regular investor with power utility, we obtain the standard Merton formula:

$$\theta^* = \frac{\mu_R - r_f}{(1-\gamma)\sigma_R^2}.$$

The two HJB are the same linear-quadratic function of the portfolio share, with the exception of the fee $\bar{\theta}f$. Depending if the fee exceeds the quadratic cost of deviating from the optimal portfolio, agents will choose one or the other activity. Agents are indifferent between occupations if:

$$\bar{\theta}f = \frac{1}{2}(1-\gamma)(\bar{\theta}-\theta^*)^2\sigma_R^2. \tag{2.4}$$

If the left-hand side is larger than the right-hand side, all agents are active; if the right-hand side is larger, all agents are passive. The supply of active capital, taking asset prices as given, is therefore perfectly elastic.

The cost of deviating from the optimal portfolio is proportional to relative risk aversion $(1-\gamma)$, return volatility σ_R^2, and the distance between the active portfolio and the optimal one. Note this cost is not a measure of the absolute excessive risk taken by an active investor, but rather of how far is his portfolio is from that of the optimal one in terms of risk.

2.3 Equilibrium

We can now turn to the determination of the equilibrium. To do so, I add market-clearing conditions to the problems of agents of firms. I generalize the standard Walrasian equilibrium by adding a market for active investment where firms and agents are price takers. This market structure represents the idea that firms compete for hiring active investors, and investors compete for the active positions.

Definition 2.1. *Given $\bar{\theta}_A$ and λ, an equilibrium constitutes $\theta_t^{i,j}$, c_t^i, a partition $\mathcal{A}_t^{j*}$ and $\mathcal{P}_t^*$, m_t^j, S_t, and f_t for $i, j \in [0,1]$ and $t \in [0,\infty)$ such that*

(i) The portfolio choices of active investors satisfy the equity constraint:

$$\forall t \in [0,\infty), \forall j \in [0,1], \forall i \in \mathcal{A}_t^{j*}, \theta_t^{i,j} = \bar{\theta} \text{ and } \forall j' \neq j, \theta_t^{i,j'} = 0.$$

(ii) Occupation $\mathcal{O}_t$(A_j if $i \in \mathcal{A}_t^{j*}$, P if $i \in \mathcal{P}_t^*$) portfolio, and consumption choices are feasible and maximize utility given aggregate prices, the active fee f, and the wealth evolutions (2.2) and (2.3).

$$\max_{\mathcal{O}_t, C_t, \{\theta^{i,j}\}_t} \mathbb{E}_t \left[\int_0^\infty e^{-\beta\tau} \frac{C_{t+\tau}^\gamma}{\gamma} d\tau \right] \textit{ such that}$$

$$dW_t^i = \left(W_t^i \left(\int_0^1 \theta_t^{i,j*} \, (\mu_{R,t}^j - r_{f,t}) dj + r_{f,t} \right) - C_t \right) dt$$

$$+ W_t^i \left(\int_0^1 \theta_t^{i,j*} \sigma_{R,t}^j dj \right) dZ_t \qquad \textit{if } \mathcal{O}_t = P$$

$$dW_t^i = \left(W_t^i (\bar{\theta} \, (\mu_{R,t}^j - r_{f,t}) + r_{f,t} + \bar{\theta} f_t) - C_t \right) dt$$

$$+ W_t^i \, \bar{\theta} \, \sigma_{R,t}^j dZ_t \qquad \textit{if } \mathcal{O}_t = A_j$$

$$W_t, C_t \geq 0, \ \ \forall t$$

(iii) Levels of active investment maximize firm value:

$$\forall t \in [0, \infty), \forall j \in [0, 1],$$

$$\{m_\tau^j\} \in \arg\max_{\{m_\tau\}} P_t = \arg\max_{\{m_\tau\}} \mathbb{E}_t \left[\int \frac{S_{t+\tau}}{S_t} D_{t+\tau} - f_{t+\tau} m_{t+\tau} P_{t+\tau} d\tau \right]$$

given the cash-flow evolution (2.1).

(iv) The occupation market clears:

$$\forall j \in [0, 1], m_t^j P_t^j = \int_{\mathcal{A}_t^{j*}} \theta_t^{i,j} W_t^i di.$$

(v) The market for assets clears:

$$\forall j \in [0, 1], P_t^j = \int_0^1 \theta_t^{i,j} W_t^i di.$$

(vi) The market for goods clears:

$$\int_0^1 C_t^i di = \int_0^1 D_t^j dj.$$

Some of the integrals in the definition above are a slight abuse of notation in order to keep clarity. Indeed, individual passive investors' stock positions in a given firm will typically be negligible compared to those of active investors. These problems disappear once we aggregate across firms. To do so, let $P_t = \int_0^1 P_t^j dj$ be the price of the aggregate endowment. It is equal to the aggregate wealth $W_t = \int_0^1 W_t^i di$. We can note the aggregate fraction of active capital as $M_t = \int_0^1 m_t^j P_t^j / P_t dj$. Finally, let θ_t^* be the portfolio of a passive agent, as we saw they all make the same choice. Combining the market-clearing conditions for the occupation and the asset markets, we obtain the following market-clearing condition:

$$\frac{M_t}{\bar{\theta}} + \frac{1 - M_t}{\theta_t^*} = 1. \tag{2.5}$$

This condition is summarized in Figure 2. All wealth is owned either by active investors, who have in total a fraction M_t of risky assets by each taking a position $\bar{\theta}$ or passive investors, who have in total a fraction $1 - M_t$ of risky assets by taking a position θ_t^*.

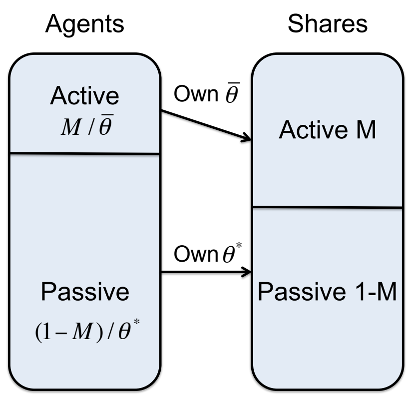

Figure 2: **Market clearing condition**

This market-clearing condition, combined with the individual supply of active capital (2.4), determines the aggregate supply of active capital. This function, linking the quantity M to the price f, is increasing. Indeed, as M increases, the fraction of risky asset owned by active investors increases. The market-clearing condition (2.5) shows that as passive agents have to sell risky assets to the new active investors, their portfolio share θ^* decreases. This change in positions increases the distance between the active and passive portfolios and, as shown by the individual supply (2.4), increases the fee required by agents to invest actively.

To determine the equilibrium of the active capital market, it suffices to equate the aggregate supply to the perfectly elastic demand of firms at the fee level λ. Figure 3

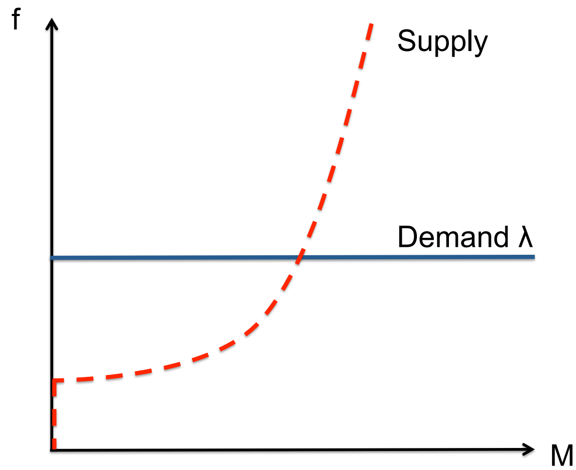

Figure 3: **Equilibrium of the active capital market**

illustrates this equilibrium, which corresponds to the quantity M such that

$$\lambda = \frac{\frac{1}{2}(1-\gamma)(\bar{\theta}-\theta^*)^2\sigma_R^2}{\bar{\theta}}.$$

This equilibrium pins down the portfolio share θ^* of passive agents and therefore the level of active capital. We obtain directly the following comparative statics:

Proposition 2.2. *The portfolio share of passive agents θ^* and the fraction of passive capital $1-M$ are*

(i) increasing in return volatility σ_D and,

(ii) increasing in relative risk aversion $1-\gamma$.

2.4 Asset prices

Let us turn to the behavior of asset prices. Because of the absence of state variables, we can start by noticing the price–cash-flow ratio is constant. This result implies the volatility of return will exactly equal the volatility of cash-flow σ_D. The active capital market does not affect price volatility in this setting. The price–cash-flow ratio satisfies the standard Gordon growth formula:

$$V = (r_f + \sigma_D rp - \mu_D)^{-1},$$

where rp is the risk price of the innovation dZ of cash flow and r_f the risk-free rate. To determine these quantities, note that passive investors are standard investors à la

Merton. Therefore, the stochastic discount factor corresponds to the marginal utility of these agents. In particular, using the first-order condition of the portfolio decision, we obtain the risk price:

$$rp = (1-\gamma)\sigma_D \theta^*.$$

This formula helps us detail the two main effects of the active capital market: distorted risk sharing and deleveraging risk.

Distorted risk sharing. Because $\theta^* < 1$, we can conclude the risk price is lower than in an economy without active capital (which corresponds to $\theta^* = 1$). Active investors hold a disproportionate share of the aggregate risk of the economy; therefore, in equilibrium, passive investors have to bear less risk. The risk active investors take does not affect the risk price. Indeed, though investors take asset prices into account in their occupation choice, active investors are not marginal in the asset market: their portfolio is constrained to take the value $\bar{\theta}$. Of course, they are compensated for taking this risk by the fee f[11]. The importance of the distortion in risk sharing is clearly linked to the fraction M of active investors. As shown by proposition 2.2, risk prices will be relatively lower in economies with low fundamental volatility and populated by agents with low risk aversion.

Deleveraging risk. This dependency with respect to risk conditions yields the second main effect of active capital: deleveraging risk. As the quantity of active capital is sensitive to risk, fluctuations in risks will generate large fluctuations in prices and risk premium. I illustrate this effect here as a comparative static. I detail it in a completely dynamic model in section 4. Figure 4 represents the equilibrium changes after an increase in fundamental volatility σ_D. First, on the right panel, we see the demand for risky assets decreases: passive agents ask a higher risk price. This effect is standard in an economy without active capital. To hold the same quantity of a more risky asset, agents demand a larger compensation. Therefore, we move vertically from the initial demand curve to the new one. But this change is not the only one: the relative cost of providing active capital increases. We can see this increase on the first panel: the supply curve for active capital shifts left. As the demand is perfectly elastic, this shift results in a lower quantity M of active capital. As active agents sell off their assets, passive agents have to hold more risky assets. The middle panel shows

[11] The fee f cannot exactly be interpreted as a different expected return incentivizing active agents to take on a portfolio $\bar{\theta}$. One can prove active agents, even in presence of the fee, would always choose a lower share of risky assets if relieved of the constraint.

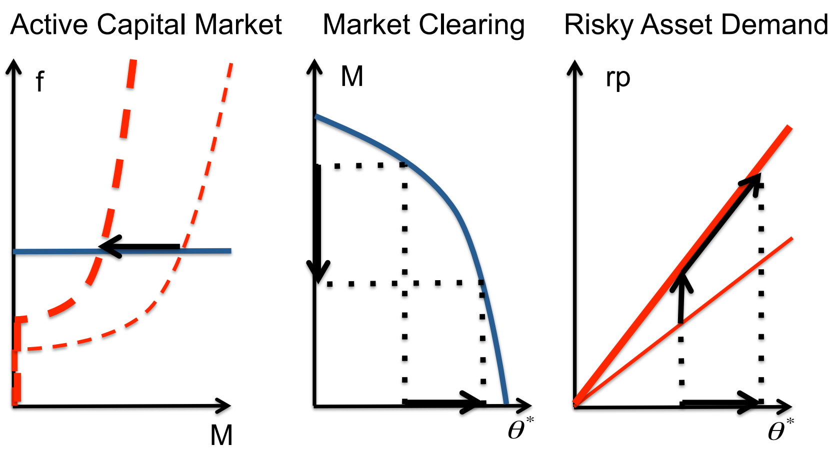

Figure 4: **Deleveraging following an increase in volatility**

this move along the market-clearing condition. Finally, returning to the right panel, we move along the demand curve of passive agents having to hold more risky assets and therefore, asking for a larger risk price.

Another way to look at this phenomenon is to consider the elasticity of the risk price with respect to volatility or relative risk aversion; the results are the same. In the standard model without active capital, the risk price is proportional to volatility and therefore the elasticity is 1. With active capital, as the amount of risk as well as the quantity of risky assets held by passive agents increase, this elasticity is larger than 1:

$$\begin{aligned}\frac{\partial \log(rp)}{\partial \log(\sigma_D)} &= 1 + \frac{\partial \log(\theta^*)}{\partial \log(\sigma_D)} > 1 \\ &= 1 + \frac{\bar{\theta} - \theta^*}{\theta^*}.\end{aligned}$$

This elasticity indicates the determinant of the magnitude of the deleveraging effect. The deleveraging effect is proportional to the leverage of the active investors' risky position relative to that of passive investors. For instance, if $\bar{\theta} > 2$ this effect is always more important that the standard quantity of risk effect. When the quantity active capital in the economy is large, the price of risk is more sensitive to volatility, as θ^* gets smaller.

On the other hand, no such deleveraging occurs following shocks to the level of cash flow, as shown by the constant equilibrium fraction of active capital M. As a comparative static, changes in future expected cash flow, created by a change in fundamental growth rate μ_D, also do not yield any change in the fraction of active capital. These results point to the key idea that shocks to uncertainty or risk aversion interact strongly with the fraction of active capital, whereas shocks to the level of present or future cash-flow do not. This difference confers a particular importance of changes in volatility and expected returns to explain asset price volatility and expected returns once we turn to the fully dynamic model.

3 General model

I now turn to the general case of the model. I characterize Markov equilibria in the general case of Duffie-Epstein-Zin preferences and an arbitrary Markov diffusion for cash-flow growth. I then explain how to obtain all quantities of the model from the solution of a single partial differential equation. The dimension of the equation corresponds to the number of state variables determining fundamental dynamics. Finally, I discuss some properties of the wealth distribution in equilibrium.

3.1 Firms

The aggregation results regarding firms derived in the previous section still hold as I maintain linear dynamics in the size of firms for cash flow. Therefore, I focus on a representative firm. The evolution of its cash flow is given by

$$\frac{dD_t}{D_t} = (\mu_D(s_t) + \lambda m_t)dt + \sigma_D(s_t)dZ_t,$$

where $\{Z_t\}$ is now a multivariate brownian motion of dimension K. Aggregate conditions are characterized by $\{s_t\}$, a set of S state variables following a Markov diffusion:

$$ds_t = \mu_s(s_t) + \sigma_s(s_t)dZ_t.$$

As before, firms choose their fraction of active capital m_t dynamically, taking the process for the active fee $\{f_t\}$ and the stochastic discount factor $\{S_t\}$ as given. The firms maximize the net present value P_t of their payoffs after payment of the active fee. This decision corresponds to the following problem:

$$\begin{aligned} P_t = & \sup_{\{m_{t+\tau}\}_{0\le\tau<\infty}} \mathbb{E}_t\left[\int_0^\infty \frac{S_{t+\tau}}{S_t}(D_{t+\tau} - f_{t+\tau}m_{t+\tau}P_{t+\tau})d\tau\right] \\ \text{s.t. } & \frac{dD_{t+\tau}}{D_{t+\tau}} = (\mu_D(s_{t+\tau}) + \lambda m_{t+\tau})\,dt + \sigma_D(s_{t+\tau})dZ_{t+\tau}. \end{aligned}$$

The linearity of both the objective function and the dynamics in the current level of cash flow implies the value function is linear in the level of cash flow and the optimal policy does not depend on this level. In other words, P_t/D_t and m_t^* are deterministic functions of s_t. By abuse of notation, $P/D(s_t) = V(s_t)$ and $m(s_t)$, respectively, in the remainder of the paper.

Though the payments of the fee depend of current and future values of the price P_t, this problem is time consistent. To see this result, we can rewrite $\forall t < T$:

$$\frac{P_t}{D_t} = \sup_{\{m_{t+\tau}\}_{0\leq\tau<T-t}} \mathbb{E}_t \left[\int_0^{T-t} \frac{S_{t+\tau}}{S_t} \left(\frac{D_{t+\tau}}{D_t} - f_{t+\tau} m_{t+\tau} \frac{P_{t+\tau}}{D_t} \right) d\tau + \frac{S_T}{S_t} \frac{D_T}{D_t} \tilde{V}_T \right]$$

$$s.t. \frac{dD_{t+\tau}}{D_{t+\tau}} = \left(\mu_D(s_{t+\tau}) + \lambda m_{t+\tau}\right) dt + \sigma_D(s_{t+\tau}) dZ_{t+\tau},\ 0 \leq \tau \leq T-t$$

$$\tilde{V}_T = \sup_{\{m_{t+\tau}\}_{0\leq\tau<\infty}} \mathbb{E}_T \left[\int_0^{\infty} \frac{S_{T+\tau}}{S_T} \left(\frac{D_{T+\tau}}{D_T} - f_{T+\tau} m_{T+\tau} \frac{\tilde{V}_{T+\tau}}{\tilde{V}_T} \right) d\tau \right]$$

$$s.t. \frac{dD_{T+\tau}}{D_{T+\tau}} = \left(\mu_D(s_{T+\tau}) + \lambda m_{T+\tau}\right) dt + \sigma_D(s_{T+\tau}) dZ_{T+\tau},\ 0 \leq \tau < \infty,$$

where $\tilde{V}_{T+\tau}$ for $\tau > 0$ is defined similarly to $\tilde{V}_T$. Examining the problems for $\{\tilde{V}_{T+\tau}\}_{0\leq\tau<\infty}$ shows it exactly corresponds to the problem for $\{P_{T+\tau}/D_{T+\tau}\}_{0\leq\tau<\infty}$, which confirms time consistency.

The recursive structure of the problem allows us to write it in the form of a Hamilton-Jacobi-Bellman equation, as can be seen in appendix A.1. As is standard for this type of investment model, the choice of optimal active capital turns out to be static. The first-order condition to have an interior optimum is, as it was for the stationary case,

$$\lambda = f_t.$$

This condition pins down the fee paid to active capital but not individual policies. The indeterminacy can generate ex-post heterogeneity in the size of firms. However, because the dynamics of cash-flow are linear in the current level, and the way m^j aggregates to M, one can see aggregate dynamics are invariant to the distribution of individual firms' policies.

Another implication of this first-order condition, which is also derived in appendix, is that the price of the firm is the same as that of an identical firm without active capital:

$$P_t = \mathbb{E}_t\left[\int_0^\infty \frac{S_{t+\tau}}{S_t} D_{t+\tau} d\tau\right]$$

$$\text{s.t. } \frac{dD_{t+\tau}}{D_{t+\tau}} = \mu_D(s_{t+\tau})dt + \sigma_D(s_{t+\tau})dZ_{t+\tau}.$$

In particular, any difference in the price of the firm compared to an economy without active capital must come from different processes for the stochastic discount factor. In this sense, my model emphasizes that fluctuations in the quantity of active capital, even if they do not affect the payoffs to passive investors, affect asset prices through the changes in the valuation of this cash flow.

3.2 Asset markets

Because there are always some passive agents that are marginal in complete asset markets, we know there are no arbitrage opportunities. Therefore, a stochastic discount factor exists. It follows a diffusion given by:

$$\frac{dS_t}{S_t} = -r_{f,t}dt - rp_t dZ_t,$$

where r_f is the risk-free rate and rp is the vector of risk prices of the K shocks.

Without loss of generality, I assume a set of K assets are available to investors. The first one is a share of any of the firms. As noted previously, they all have the same price and cash-flow evolution, so their shares have the same returns. The other $K-1$ asset returns are in zero net supply, have unit variance, and complete the market. Their expected excess returns can directly be inferred from the stochastic discount factor $\{S_t\}$, as they correspond to the risk prices. I note $\mu_{R,t}$ as the vector of expected returns and $\sigma_{R,t}$ as the vector of volatility of these assets. The risk-free asset is in zero net supply. In equilibrium, all these quantities are deterministic functions of the state s_t of the economy.

3.3 Agents

Agents now rank consumption streams according to the stochastic differential utility of Duffie and Epstein (1992). It is a continuous-time version of the recursive preferences of Epstein and Zin (1989). Let $\mathcal{U}_t$ be the utility of the agent at time t and $f(C,\mathcal{U})$ the aggregator. The utility value $\mathcal{U}_t$ is defined recursively by:

$$\mathcal{U}_t = \mathbb{E}_t\left[\int_t^\infty g(C_s,\mathcal{U}_s)ds\right].$$

For the aggregator g, I use the standard function:

$$g(C,\mathcal{U}) = \beta\frac{\gamma}{\rho}\mathcal{U}\left[\frac{C^\rho}{\gamma^{\frac{\rho}{\gamma}}\mathcal{U}^{\frac{\rho}{\gamma}}} - 1\right].$$

β is the rate of time preference. $\Gamma = 1 - \gamma$ is the relative risk aversion (RRA) of the agent. $\psi = \frac{1}{1-\rho}$ is the intertemporal elasticity of substitution (IES). When $\Gamma = \frac{1}{\psi}$, or, equivalently, $\gamma = \rho$, the utility function reduces to the standard power utility specification of section 2. This utility function is homogenous of degree γ and therefore preferences are homothetic. An important motivation for using these preferences, in addition to the fact that they have proven useful in obtaining good quantitative results for asset-pricing models, is that they generate a volatility risk premium.

Agents can choose whether they are active or passive investors. Let $\mathcal{A}_t^*$ and $\mathcal{P}_t^*$ these sets of investors at each time. Agents can move freely between occupations. They can choose their consumption C_t without any constraint.

Passive investors choose freely their portfolio $\boldsymbol{\theta}_t^*$ across all assets. Their wealth evolution is then

$$dW_t^i = \left(W_t^i(\boldsymbol{\theta}^{*\prime}(\mu_{R,t} - r_{f,t}) + r_{f,t}) - C_t\right)dt + W_t^i\,\boldsymbol{\theta}^{*\prime}\sigma_R dZ_t.$$

Active investors are constrained to choose a portfolio $\bar{\boldsymbol{\theta}} = [\bar{\theta}, 0, \ldots, 0]$ that only consists of a position in shares of the firm. Again, I assume $\bar{\theta} > 1$. This assumption constrains them to hold more of the asset than anybody would hold in a world without active investors. Additionally, they cannot hedge the risk coming from changes in the state variables. As a compensation for accepting this constraint, they receive the fee $f_t dt$ for each unit of capital of the firm in which they invest. The evolution of their wealth is driven by

$$dW_t^i = \left(W_t^i(\bar{\boldsymbol{\theta}}'(\mu_{R,t} - r_{f,t}) + r_{f,t} + \bar{\theta}_A f_t) - C_t\right)dt + W_t^i\,\bar{\boldsymbol{\theta}}'\sigma_R dZ_t.$$

Because all these dynamics are linear in wealth and the homogeneity of utility functions, one can see all agents will face the same tradeoff, irrespective of their current wealth, when choosing their occupations. Additionally, agents in each occupation will all have the same consumption-wealth ratio, written as $c_t^i = C_t^i/W_t^i$. To have an interior equilibrium for the level of active capital, agents, given their wealth, have to be indifferent between activities at each point in time. Let G_t/γ be the utility of an agent with wealth one at date t. The utility of an agent with wealth W_t^i is then

$$\mathcal{U}_t = \frac{(W_t^i)^\gamma}{\gamma}G_t,$$

where G_t is a deterministic function of the state variables I note as $G(s_t)$. In particular, $G(.)$ must be the value function of an agent being active or passive for some interval of time. The following proposition formalizes this idea.

Proposition 3.1. *The fee f_t is such that the value function per unit of wealth $G(s_t)$ solves the Hamilton-Jacobi-Bellman problems:*

(i) ***Passive investor:***

$$\begin{aligned} 0 &= \max_{c\geq 0, \boldsymbol{\theta}\in\mathbb{R}^K} g(\gamma^{1/\gamma}c, G) + \frac{\mathbb{E}\left[d\left(W^\gamma G\right)\right]}{W^\gamma dt} \\ s.t.\ dW_t &= \left(W_t(\boldsymbol{\theta}'\left(\mu_{R,t} - r_{f,t}\right) + r_{f,t}) - C_t\right) dt + W_t\, \boldsymbol{\theta}'\, \sigma_R dZ_t. \end{aligned}$$

(ii) ***Active investor:***

$$\begin{aligned} 0 &= \max_{c\geq 0} g(\gamma^{1/\gamma}c, G) + \frac{\mathbb{E}\left[d\left(W^\gamma G\right)\right]}{W^\gamma dt} \\ s.t.\ dW_t &= \left(W_t(\bar{\boldsymbol{\theta}}'\left(\mu_{R,t} - r_{f,t}\right) + r_{f,t} + \bar{\theta} f_t) - C_t\right) dt + W_t\, \bar{\boldsymbol{\theta}}'\, \sigma_R dZ_t. \end{aligned}$$

I derive these problems in appendix A.2.1. This proposition helps us sidestep an issue with the continuous-time model: a few of the agents switch often. The aggregate level of active capital is a function of the state variables. As they follow a diffusion, the fraction of active agents will also follow a diffusion. For instance, one can prove it is impossible all agents follow stopping-time strategies to change their activities. This problem is not present in the discrete-time version of the model. Proposition 3.1 holds in the limit of the discrete-time case, as it characterizes agents indifference, not their actual occupation trajectory.

Examining the problems of proposition 3.1, we can derive the consumptions and portfolios of agents as well as the activity fee:

Proposition 3.2. *At equilibrium:*

(i) All agents (active and passive) have the same consumption-wealth ratio, determined by:

$$c = \beta^{\frac{-1}{(\rho-1)}} G^{\frac{\rho}{(\rho-1)\gamma}}.$$

(ii) The portfolio θ^* of passive agents is:

$$\boldsymbol{\theta}^* = \frac{1}{1-\gamma}\left(\sigma_R\sigma_R'\right)^{-1}\left(\mu_R - r_f\right) + \frac{1}{1-\gamma}(\sigma_R\sigma_R')^{-1}\sigma_R\sigma_s' G_s.$$

(iii) The activity fee f is given by:

$$\bar{\theta} f = \frac{1}{2}(1-\gamma)\left(\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\right)' \sigma_R \sigma_R' \left(\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\right).$$

The proof is in appendix A.2.2. Points (i) and (ii) are the standard portfolio results for a passive agent with recursive preferences. Note that active agents do not choose a consumption policy distinct from that of passive agents. The reason for this result is that the only determinant of consumption is the marginal utility of wealth. It has to be the same, active or passive, as agents are indifferent between occupations at all levels of wealth.

The formula for the fee f_t is similar to the stationary case, proportional to relative risk aversion and the volatility of the portfolio $\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}^*$. One can note that it is purely a compensation for risk taken as a departure from the optimal portfolio. In particular, it does not depend on the covariance of returns with the state variables, because, in a diffusion framework, hedging demands are linear in the amount of risk. Therefore the already larger returns from the levered position exactly offset the additional loading on the state variables the agent has to take.

3.4 Equilibrium

As in the stationary model, the demand of active capital from firms pins down the fee to be constant, equal to the marginal productivity: $f(s_t) = \lambda$. Note that because hedging assets are in zero net supply and all passive investors take the same position, this position has to be zero. The optimal portfolio takes the form $\boldsymbol{\theta}^* = [\theta^*, 0, \ldots, 0]$ in equilibrium. We can therefore simplify the supply of active capital to obtain

$$\lambda = \frac{1}{2\bar{\theta}}(1-\gamma)(\bar{\theta} - \theta^*)^2 \sigma_{R,1}^2. \tag{3.1}$$

In the stationary model, this condition was sufficient to pin down the equilibrium portfolio θ^*, because the volatility of returns corresponded to the fundamental volatility σ_D. Now, because time-varying conditions are present, this quantity is endogenous and depends on the quantity of active capital. To see this result, first apply Ito's lemma to the asset returns:

$$\frac{dR}{R} = \frac{D_t}{P_t}dt + \frac{dD_t}{D_t} + \frac{dV_t}{V_t} + \frac{< dD_t, dV_t >}{P_t} \tag{3.2}$$

The volatility comes from fluctuations in cash flow dD_t as well as fluctuations in the price–cash-flow ratio dV_t. Any changes in the properties of the stochastic discount factor or in expected future cash flow affect this valuation ratio. Fluctuation in the fraction of active investors amplifies these changes. To see this effect, consider, for instance, a change tomorrow in the volatility of cash flow. If we happen to be in a high-volatility state, the price will be lower than in a low-volatility state for three reasons. First, future cash flows are riskier and as such are discounted more. Second, because the environment is more risky, the risk price for cash-flow shocks is larger. Finally, because the environment is more risky, less active capital will be present, therefore the risk price will be even larger as passive investors bear more of the risk. This third channel, present only with an active sector, creates additional price volatility. Going back to equation (3.1), we can see, however, this endogenous channel creates an important self-limiting force on the development of the active investment sector. If the active sector is developed and is at risk of fluctuations, returns are more volatile. Therefore, agents are less willing to provide this large quantity of active capital in the first place.

Finally, the market-clearing conditions for the occupation and asset markets can be combined to pin down the fraction of active capital:

$$\frac{M}{\bar{\theta}} + \frac{1-M}{\theta^*} = 1$$

3.5 Solving for the equilibrium

In the determination of the equilibrium, we must simultaneously solve three dynamic problems: the valuation of the firm, the utility maximization of passive agents, and the utility maximization of active agents. In this section, I show how to combine them to obtain only one problem, represented by a single partial differential equation. To obtain this result, I focus on the determination of the price–cash-flow ratio $V(s_t)$ of the firm.

First, remember all agents have the same consumption-wealth ratio. Total wealth corresponds to the market value of the firm and total consumption must equal the cash flow of the firm. Therefore, the consumption-wealth ratio c is equal to the inverse of the price–cash-flow ratio V. The valuation ratio also determines the normalized value function G of agents from the first-order condition for consumption:

$$\frac{1}{V} = c = \beta^{\frac{-1}{(\rho-1)}} G^{\frac{\rho}{(\rho-1)\gamma}}$$

Then, using formula (3.2), we can see the the only endogenous parts of the volatility of the share returns $\sigma_{R,1}$ are the elasticities of V with respect to the state variables.

Using this result, we can directly obtain the optimal portfolio share θ^* from the equilibrium of the active capital market (3.1). Inverting this equation in closed form is trivial.

Turning to the optimization problem of active agents, first note expected excess returns are the product of the known volatilities of returns and the unknown risk prices. As we derived $\boldsymbol{\theta}^*$ and the value function G_s, the first-order condition for the portfolio of agents in proposition 3.2 becomes a linear system of equations pinning down the risk prices. The last part of the stochastic discount factor, the risk-free rate r_f, can be obtained by plugging in all the quantities just determined in the HJB equation of the passive agent.

We have now seen all endogenous quantities of the model are closed-form functions of V. Now V can be directly determined by the pricing HJB:

$$0 = \frac{1}{V} + \frac{\mathbb{E}[d(SDV)]}{SDV dt},$$

which is a second-order partial differential equation in V, as the dynamics of the cash flow D are exogenous and those of the stochastic discount factor S are known functions of V. In the stationary case of section 2, there are no state variables: this equation determines the number V. With only one state variable, this equation is an ordinary differential equation of degree 2.

3.6 Individual policies and the wealth distribution

Before studying the asset-pricing implications of the model with changing fundamental conditions, let us consider possible wealth distribution evolutions in the population. Let us first recapitulate properties of wealth trajectories and then examine a couple of possible implementations of the equilibrium.

At each point in time, active investors receive on average a higher wealth increase than passive investors. This difference comes through two channels: they take on more leverage and therefore receive extra asset returns, and they receive the fee. Active investors are also more exposed to shocks than passive investors. Therefore, when returns are good, active investors gain relatively more wealth; when returns are bad, active investors lose relatively more wealth. These differences in wealth evolution will create dispersion in wealth across agents. However, because agents can choose their occupation at each point in time, and preferences are homothetic, this wealth heterogeneity does not create a related heterogeneity in behavior. The individual policies are not pinned down by the equilibrium conditions, and many occupation-choice trajectories are consistent with the evolution of aggregate quantities. To get a better idea of how much switching is necessary at equilibrium, consider two simple implementations.

Remember we have a continuum of agents indexed by i on $[0,1]$, and note $w_t^i = W_t^i/W_t$ their fraction of total wealth at time t. A fraction $M(s_t)/\bar{\theta}$ must be in the active sector. A first implementation is that the agents with the lowest indices are active. To determine which agents are active, define implicitly the threshold I_t by

$$\int_0^{I_t} w_t^i di = \frac{M(s_t)}{\bar{\theta}}.$$

A unique solution to this equation always exists, as all individual wealth fractions w_t^i clearly stay strictly positive and integrate to 1, which is strictly larger than the right-hand side. Also, because all individual wealth fractions w_i^t and $M(s_t)$ follow a diffusion, I_t does as well. Using properties of diffusion processes, we can derive local properties of career dynamics in this implementation:

Proposition 3.3. *At each point in time t:*

(i) For almost every agent i, $\varepsilon > 0$ almost surely exists such that i does not change occupation in $[t, t+\varepsilon)$,

(ii) Agent I_t, for any interval $[t, t+\varepsilon)$, almost surely changes occupation an uncountable set of times, without isolated points.

This proposition is a direct consequence of the properties of the zeros of the Brownian motion found, for instance, in Morters and Peres (2010). In other words, the proposition tells us most agents do not change jobs on any finite interval. Only the agents at the border will go back and forth an infinite number of times. Because these agent only represent an infinitesimal fraction of aggregate wealth, their back-and-forth do not affect aggregate dynamics. This implementation of the equilibrium has the undesirable effect of not allowing a stationary distribution of wealth. To see this result, consider the wealth of agent 0 relative to any other agents. Because he is always active, his wealth has a larger drift than that of any other agent; therefore, the ratio of his wealth relative to any other agent tends to increase, and in the limit diverges to infinity almost surely.

An alternative implementation that insures the wealth distribution does not diverge is to make the group of active agents change over time. For instance, given $\zeta > 0$, we can choose the subset $[\zeta t, \zeta t + I'_t] \mod 1$, where I'_t is now defined by:

$$\int_{\zeta t}^{\zeta t + I'_t} w_t^{(i \mod 1)} di = \frac{M(s_t)}{\tilde{\theta}_A}.$$

As the group of active agents cycles through the population, no individual wealth can drift permanently over that of the rest of the population. Relative to proposition 3.3, we now have two agents switching occupation at each point in time. Agent $\zeta t \mod 1$ becomes passive, and agent $(\zeta t + I'_t) \mod 1$ switches an infinite number of times.

Finally, note that in both implementations considered here, neither the wealth distributions nor the threshold I_t are deterministic functions of the state s_t. Indeed, past shocks affect the relative wealth evolutions of the two groups and modify the fraction of agents to include in order to obtain the equilibrium fraction of active capital.

4 Asset-pricing implications

I now turn to the asset-pricing implications of the model. In particular, I focus on the consequences of fluctuations in the quantity of active capital for the volatility of returns and the price of various sources of risk.

4.1 Setup

I study a continuous time version of the long-run risk model of Bansal and Yaron (2004), as in Hansen, Heaton, Lee and Roussanov (2007). There are two state variables: $s_t = (X_t, \sigma_t^2)$. The dynamics of cash-flow growth are given by

$$\begin{aligned} \frac{dD_t}{D_t} &= (\mu_D + X_t + \lambda m_t)dt + \sigma_t dZ_t^D \\ dX_t &= -\kappa_X X_t dt + \phi \sigma_t dZ_t^X \\ d\sigma_t^2 &= -\kappa(\sigma_t^2 - \sigma_0^2)dt + \nu \sigma_t dZ_t^\sigma, \end{aligned}$$

where Z^D, Z^X, and Z^σ are independent brownian motions. The variable X_t controls the persistent component of cash-flow growth and σ_t the volatility of shocks. For σ_t^2 to stay positive, I impose the parameter restriction $2\kappa\sigma_0^2 > \nu^2$.

To illustrate the theoretical results, I follow the calibration of Bansal and Yaron (2004) I report in table 1. All parameters are at the monthly frequency. An important feature of this calibration is that the intertemporal elasticity of substitution is larger than 1, making the price increasing in expected cash flow. For the active investment parameters, I use $\bar{\theta} = 1.1$ and and $\lambda = \mu_D$.

Preferences			Consumption		State Variables			
β	RRA	IES	μ_D	σ_0	κ_X	κ_σ	ϕ_X	ν
0.0013	10	1.5	0.13%	0.79%	0.0212	0.0131	5.64	0.0003

Table 1: **Preferences and consumption dynamics**

I compare the solution of my model to the case of no active capital, which is equivalent to set $\theta^* = 1$ and $m = 0$ in all previous calculations. I call the latter model the baseline model.

4.2 Price–cash-flow ratio and quantity of active capital

First, note the price is always larger with active capital than without. Indeed, the first-order condition of passive agents for consumption tells us the utility level is increasing in the price. Agents in the economy can choose at any moment to use their shares of risky assets to finance the consumption plan they would have in an economy without active capital. Indeed, we proved the price of the risky asset is the same as that of a firm employing no active investors, which is the output consumed in the baseline case. The fact that agents choose not to do this trade tells us they are better off in this equilibrium.

I now look separately at the asset price and the quantity of active capital along changes in the two state variables.

4.2.1 Role of the fundamental growth rate

As shown in the stationary case, the growth rate of cash flow does not affect active capital, because the growth rate of cash flow does not affect the volatility of returns. This property still holds approximately in the long-run risk model. Bansal and Yaron (2004) obtain this property exactly in their log-linear approximation, and the result is fairly robust in this parameter region. The reason changes in growth rate do not change the volatility is that the corresponding changes in the timing of risk are small.

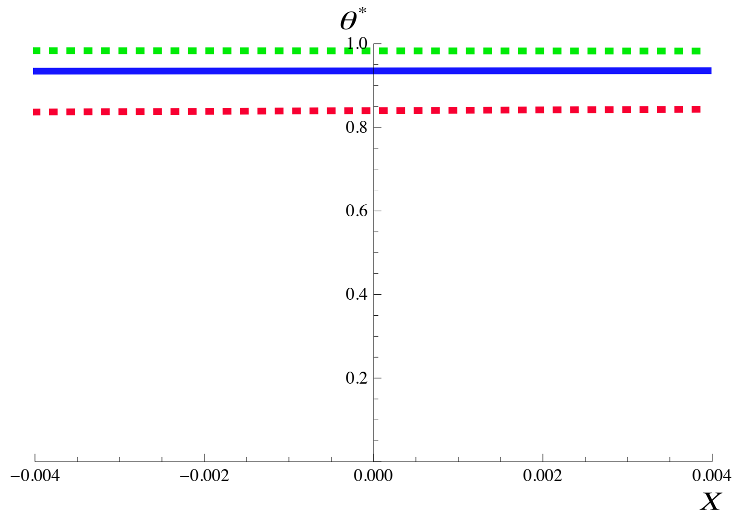

Figure 5: **Optimal portfolio of passive agents** θ^* **as a function of** X *(blue:* σ_0^2*, green:* $1.5\sigma_0^2$*, red:* $0.5\sigma_0^2$*)*

The absence of an impact of the growth rate on return volatility translates into an absence of dependence on X_t of the portfolio of passive agents. Figure 5 confirms this result. This figure represents, for various levels of the volatility state, the portfolio of passive agents as a function of the growth rate X_t.

4.2.2 Role of the fundamental volatility

Fundamental volatility has a much more important impact on the active capital market. On figure 6, we see important variations in the portfolio of passive agents with changes in volatility. This result corresponds to the idea that fundamental volatility is reflected in the volatility of returns and subsequently in the supply of active capital. When volatility is low, agents are more willing to supply active capital relative to passive investment. These active agents buy a larger fraction of the assets, and the remaining agents are left holding less risky assets.

Interestingly, we observe that θ^* is a concave function of σ_t. The portfolio of passive agents changes more in response to a volatility change at lower levels of fundamental volatility than at higher levels. This result corresponds to the idea that the economy is more susceptible to large waves of deleveraging when large amounts of concentrated investments are present. Naturally, this deleveraging affects prices. Figure 7 shows the price–cash-flow ratio V as a function of fundamental volatility σ_t for the model and the

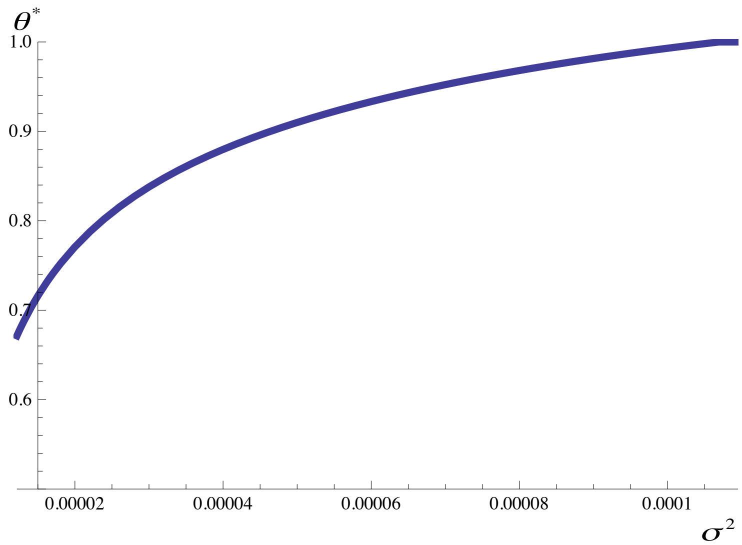

Figure 6: **Optimal portfolio of passive agents θ^* as a function of σ^2** *($X = 0$)*

baseline case. As volatility increases and active capital disappears, the price converges to the baseline case. Though risky allocations are the same in both models when θ^* reaches 1, the two prices are not equalized. Indeed, risky allocations will depart from each other when mean-reversion and shocks brings σ_t down. Corresponding to the concavity of θ^*, the price converges in a concave way to that in the baseline model. However, when reaching very low volatility states, because passive agents do not bear little risk, the price becomes less sensitive to changes in volatility.

4.3 Return volatility and fundamental volatility

As fluctuations in active capital impact prices, they create additional volatility for the asset. The risky-asset returns dynamics are:

$$\frac{dR_t}{R_t} = \mu_R(s_t)dt + \sigma_t dZ^D + \phi\sigma_t \frac{\partial \log V}{\partial X}(s_t)dZ_t^X + \nu\sigma_t \frac{\partial \log V}{\partial \sigma^2}(s_t)dZ^\sigma.$$

The volatility of returns comes from the three shocks affecting the economy: the instantaneous cash-flow shock dZ^D, the shock to expected cash-flow growth dZ^X, and the shock to uncertainty dZ^σ. First note that direct shocks to cash flow, as in the baseline model, are directly transmitted to returns. They do not affect the active capital market; therefore, their impact on prices is left unchanged. Similarly, because the sensitivity of the valuation ratio to the growth rate is unchanged, the volatility from changes in future expectations of cash flow is the same as in the baseline model.

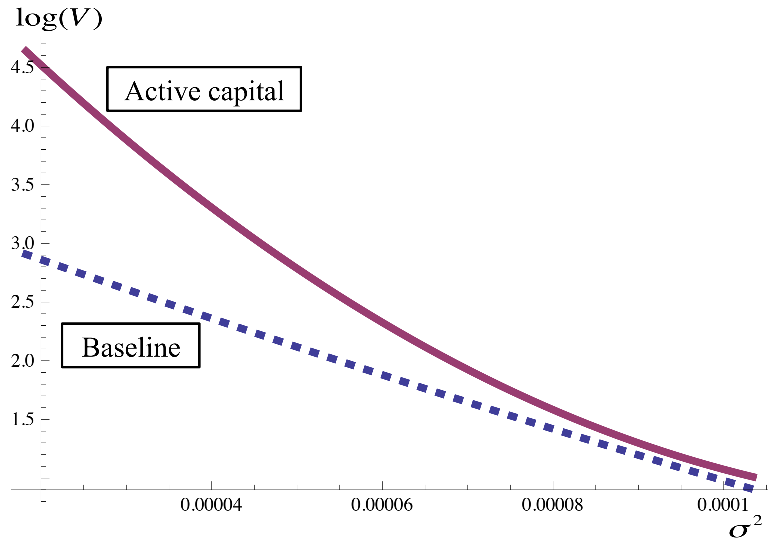

Figure 7: **Price–cash-flow ratio as a function of volatility** σ^2 *(*$X = 0$*)*

Finally, the volatility shock now has a larger effect on prices: the sensitivity $\partial \log V/\partial\sigma^2$ is larger than in the baseline case. This larger volatility comes through the deleveraging effect. When volatility increases, the asset becomes less attractive to passive agents, and they must absorb the assets sold off by active agents who change occupations. From these results, we see active capital not only increases the volatility of returns but also changes the composition of its sources. In particular, it puts relatively less weight on cash-flow shocks than to volatility shocks. Figure 8 illustrates this effect. We can see the volatility shocks explain a larger part of returns variance. Additionally, as we noticed for the sensitivity of prices, this distortion of the composition of risk is larger in low-volatility states.

The source of return volatility has been a puzzle for the asset-pricing literature since Campbell and Shiller (1988). They point out the volatility of returns appears to be too large relative to the volatility of cash flow. The long-run risk model of Bansal and Yaron (2004) explains this puzzle by the presence of small persistent shocks affecting consumption growth that have a large impact on the utility of agents with recursive preferences. However, Beeler and Campbell (2009) point out that the model still generates a counter-factual level of cash-flow predictability. In other words, it creates too tight a link between prices and expected cash-flow. Volatility shocks do not affect the level of future cash flows but create variation in prices through variation in discount rates and therefore have the potential to explain return volatility. Bansal, Kiku and

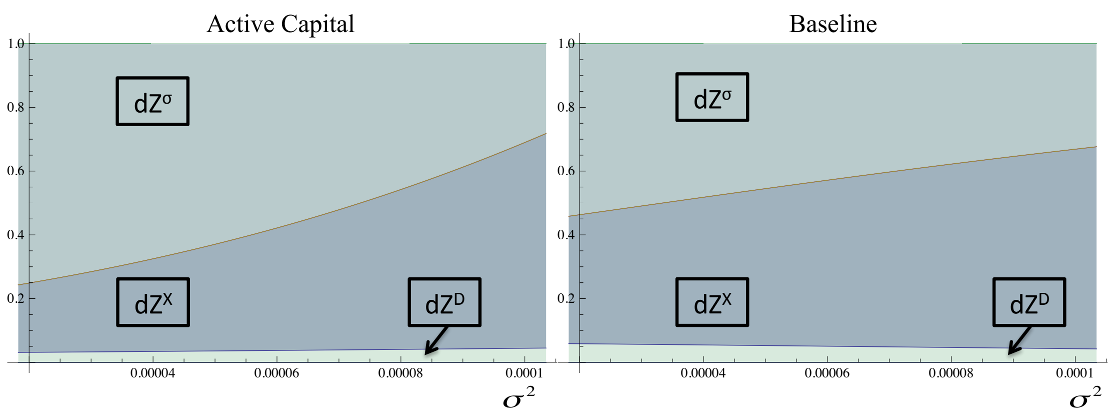

Figure 8: **Decomposition of the volatility of returns as a function of** σ_t *($X = 0$; left panel: active capital; right panel: baseline case)*

Yaron (2007) present a calibration of the long-run risk model in which volatility shocks are extremely persistent, and avoid the excess cash-flow predictability. My model offers an endogenous channel that gives more importance to volatility shocks. This effect comes through the endogenous variation in risk-sharing between active and passive investors. In particular, we should observe that variations in prices, due to changes in risk premia, coincide with variation in the quantity of active capital. In a similar vein, Adrian et al. (2010) find that intermediary leverage is negatively related to the macroeconomic risk premium. Also, Haddad et al. (2011) find the quantity of leveraged buyouts, a transaction concentrating ownership of risky assets, is strongly negatively correlated with a measure of the equity risk premium.

4.4 Price of risks

As we saw when examining the firm problem, the price of the risky asset is equal to that of a similar firm employing no active capital. Because active capital has no effect on payoff of the risky asset in equilibrium, all changes in valuation relative to the baseline model must come through changes in the stochastic discount factor, particularly risk prices. For these, the two effects described in section 2 play a role.

The risk prices for the three shocks are

$$\begin{cases} rp_D = (1-\gamma)\theta^*\sigma_t \\ rp_X = \phi\sigma_t \frac{\partial \log V}{\partial X}[(1-\gamma)\theta^* - \frac{\gamma\rho}{1-\rho}] \\ rp_\sigma = \nu\sigma_t \frac{\partial \log V}{\partial \sigma^2}[(1-\gamma)\theta^* - \frac{\gamma\rho}{1-\rho}]. \end{cases}$$

The first effect is risk dampening. In equilibrium, passive agents hold fewer risky assets than in the baseline model: $\theta^* < 1$. Because agents hold less risk, they require a lower compensation for marginal risk. The dampening is direct for the instantaneous shock price rp_D. For the other two shocks, the dampening is less important, because $\gamma\rho/(1-\rho) < 0$. The reason for this property is that with recursive preferences, the long-run impact of shocks on utility affects risk prices. The first term in brackets for rp_X and rp_σ comes from the instantaneous loading on risk. Passive agents instantaneously bear less risk than in the baseline case, causing dampening. However, at the next instant, all agents are identical again. Therefore, they all have the same value functions and all have to bear equally the long run consequences of shocks. The long-run component of risk prices is not dampened.

The second effect is deleveraging risk. Shocks to fundamental volatility are more risky, because they increase not only the riskiness of the asset, but the fraction of risky assets passive agents hold. This extra risk corresponds to the larger sensitivity $\partial \log V/\partial\sigma^2$ relative to the baseline case. As for the volatility of returns, deleveraging only occurs with volatility shocks and as such this amplification only affects the risk price of volatility rp_σ. A consequence of this phenomenon is a change in the relative values of risk prices for the various sources of risk. Active capital affects not only the price of shocks, but the nature of priced shocks as well. The volatility risk price becomes relatively more important than the other two risk prices.

These last results have implications for understanding the cross-section of expected returns. Indeed, though I assume firms are identical, in reality, they have different loadings on the various sources of risk. If the volatility risk price is relatively large, exposure to volatility risk is likely to be an important determinant of the cross-section. Ang, Hodrick, Xing and Zhang (2006) find exposure to shocks to aggregate volatility lines up with expected returns. Additionally, because conditional expected returns move in tandem with volatility in this model, measuring exposure to shocks to expected returns should provide a good proxy for exposure to this long-run volatility shock. Kozak and Santosh (2012) confirm this result using a model-free measure of expected returns: stocks that covary positively with aggregate expected returns earn lower average returns.

The specific prediction of this model is that the shocks to volatility correspond to changes in the quantity of levered investors. Larger exposure to measures of leveraged investment should correspond to larger expected returns. Adrian et al. (2011) measure covariance of returns with the leverage of broker-dealers and find this exposure is able to predict returns.

5 Extensions

In this section, I present a few extensions of the model. First I discuss the efficiency of the equilibrium. I show that taxing firms for the use of active capital increases welfare in a static framework. Then I discuss the effect of changes in the productivity of active investment. To do so, I relax two assumptions: constant returns to scale and no dependence on the state variables. Finally, I show how to incorporate idiosyncratic risk and investment in physical capital.

5.1 Tax on active capital

One can wonder whether the equilibrium allocation of this model is efficient. Indeed, the presence of the portfolio constraint for active investors is a source of market incompleteness. In such frameworks, pecuniary externalities are generically sources of externality. Changing asset prices affects different agents differently and generate redistribution. Gromb and Vayanos (2002) study such welfare implications in an exogenously segmented market framework.

I focus on the following intervention: a tax on firms per unit of active capital, given back to them as a lump sum payment. This is equivalent to reducing the fee paid by firms to active investors. An interesting aspect of this policy is that it does not affect the agents side of the markets. In particular, agents in the two categories still have to be indifferent. This way, the policy will be unambiguously Pareto ranked with the market equilibrium. I only look at the first-order effect of an increase in tax from the market equilibrium. Additionally I focus on a small interval dt of the stationary model.

The tax affects the conditions of the agents' problem in three ways: a direct effect of reduction of the fee f and indirect effects on the relative valuation of risky and risk-free payoffs and on the cash flow of the firm through a different level M of active capital. This last effect has no first-order impact as in the market equilibrium, the fee f and the marginal productivity λ cancel out. The effect of the reduced fee is to increase the utility of passive investors and reduce that of active investors. To equate the two utilities again, the relative price of risky assets has to decrease. Indeed passive agents, as sellers of risky assets, lose from this change and active agents, as buyers of risky assets, gain. If markets were complete, as marginal rates of substitutions are equalized across agents, the total first-order effect of all these changes would be exactly zero. This can alternatively be seen as a direct consequence of the first welfare theorem.

In our incomplete market framework, as active agents are constrained to own more risky assets than they would desire, they value risky assets relatively less than passive agents. Thus, the effect of a change in the fee relative to a change in the relative price of risky assets is larger for passive agents than active agents. The net effect is an increase of the utility level of both agents. The first unit of tax creates welfare; therefore, too much active capital is present in the market equilibrium.

This conclusion is specific to the stationary model where there is no intertemporal link in the decision to provide active investment. In the general case, future changes in the level of active capital affects the present decision to enter active investment, potentially creating other inefficiencies. Another concern before taking this result as a policy implication is the exogeneity of the contract. Practically, policy interventions could affect the contracts offered on markets.

5.2 Time-varying productivity of active capital

Similarly to this concern, the productivity of active capital could change over time. I study two sources for this variation: decreasing returns to scale and exogenously time-varying productivity.

5.2.1 Decreasing returns to active capital

Up until now, I have assumed active capital exhibits constant returns to scale. However, as the number of active investors increases, their quality could decrease, and the opportunities to create value are lower. A way to model this is to assume decreasing returns to scale at the firm level. This approach complicates the resolution of the model as it creates surplus at the firm level: the residual payoff increases from the use of active capital.

To avoid this issue, we can use decreasing returns to scale at the aggregate level, but constant returns at the individual level. This corresponds to a decrease in the quality of active capital when more firms use it. At the firm level, there is no difference in returns as a function of the individual quantity it uses.[12] This model corresponds to the following dynamics of cash-flow

$$\frac{dD_t}{D_t} = (\mu_D(s_t) + \lambda(M_t)m_t)dt + \sigma_D(s_t)dZ_t,$$

where $\lambda(.)$ is a decreasing function. This specification leaves the individual demand of firms perfectly elastic, but makes the aggregate demand a decreasing function, given

[12]Empirically, the evidence is mixed. Some have argued the impact of active capital on firm value is hump shaped.

by the first-order condition of the individual firm problem:

$$\lambda(M_t) = f.$$

The decreasing aggregate demand mitigates deleveraging. When the supply of active capital is lowered, both a decrease in the quantity and an increase in the fee for active investment absorb this change. Fluctuations in the fee have no impact on risk sharing and therefore do not affect risk premia. Panels (a) and (b) of Figure 9 show the effect of an increase in volatility in the standard model and in the decreasing returns case, respectively The more inelastic the aggregate demand for active capital is, the smaller the fluctuations in the quantity of active capital are. The limiting effect of productivity increase on the deleveraging decision can be seen for instance in the Q1:2009 letter to investors of Pershing Square Capital, a large activist hedge fund. They explain that they exited some of their positions due to their "inability to accurately forecast with confidence the duration and depth of the current recessionary environment". However, they note: "the fact that a number of our competitors have closed their doors or withdrawn from an activist approach obviously makes for a less competitive environment for the surviving participants.[...] Judged by these standards, the ingredients for profitable shareholder activism are more present than ever before, and we continue to be highly capable of implementation."

5.2.2 Dependence on state-variables

Alternatively, the productivity of active capital varies in and of its own or with economic conditions. It corresponds to assuming marginal productivity is a function $\lambda(s_t)$ of the state variables. This assumption does not affect the resolution of the model. If $\lambda(s_t)$ changes independently from other conditions, it corresponds to shifts in the demand for active capital. The fee is always equal to the marginal productivity and quantity changes along the supply curve clear the market. When the productivity increases, the fee and the fraction of active capital increase. When active capital becomes more productive, its quantity increases, more agents lever up and risk prices decrease. Such fluctuations would be priced in equilibrium.

If $\lambda(s_t)$ is correlated with economic conditions, the corresponding demand shocks can amplify or dampen the endogenous fluctuations in supply. If $\lambda(s_t)$ is negatively related to fundamental volatility, deleveraging cycles are amplified. In times of high volatility, neither agents are willing to provide active capital nor firms demand it either. This case corresponds to panel (c) of Figure 9. However, if the relation is positive and

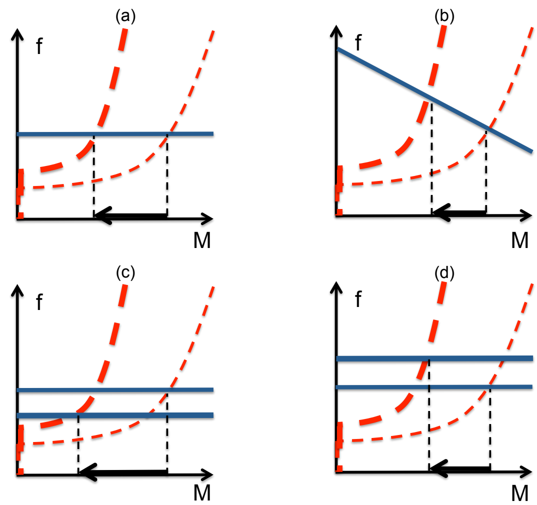

Figure 9: **Changes in active capital market equilibrium following an increase in volatility** *(a: standard model; b: decreasing returns; c: negative relation of λ and σ; d: positive relation)*

active capital is more effective in volatile periods, deleveraging is dampened. If the productivity gains are large enough, they could suppress deleveraging altogether or even create more active capital in volatile times (panel (d) of Figure 9). Empirically, the sign of this correlation is an open question.

5.3 Idiosyncratic risk

Another important feature left out is idiosyncratic risk. Indeed, firms' output does not depend only on aggregate conditions, but on individual level shocks. A simple way to modify the model is to introduce firm-specific shocks to cash-flow growth. This assumption corresponds to changing the dynamics of the static model to

$$\frac{dD_t^j}{D_t^j} = (\mu_D + \lambda m_t)dt + \sigma_D dZ^D + \sigma_j dZ^j,$$

where Z^j is a brownian motion specific to firm j, independent from all other shocks of the economy. Idiosyncratic shocks do not affect the price–cash-flow ratio; they are directly reflected into returns. The supply of active capital is then

$$\bar{\theta} f = \frac{1}{2}(1-\gamma)(\bar{\theta} - \theta^*)^2(\sigma_D^2 + \sigma_j^2).$$

We see idiosyncratic risk causes agents to require a larger compensation than in the basic model. In equilibrium, the lower supply generates a decrease in the quantity of active capital. Because there is less active capital, and active capital is less sensitive to aggregate conditions at low levels, the excess sensibility to the level of volatility will be milder. However, empirically, idiosyncratic volatility correlates positively with aggregate volatility, as showed by Campbell, Lettau, Malkiel and Xu (2001). In this case, fluctuations in the two types of volatility concur and increase the magnitude of fluctuations in the level of active capital.

5.4 Physical investment

A last extension is to include physical investment in the model. We can introduce physical investment without loosing tractability of the mode. To do so, I introduce a CES aggregator of active investment and physical investment in the the cash-flow dynamic:

$$\frac{dD_t}{D_t} = (\mu_D + \lambda(m_t, i_t))\, dt + \sigma_D dZ_t^D$$

$$\lambda(m, i) = \lambda_0 (a m^r + (1-a) i^r)^{\frac{1}{r}},$$

where i_t is the level of physical investment. The investment i_t is directly taken out of the cash flow of the firm. The first-order conditions of the firm problem become:

$$\begin{cases} \frac{\partial \lambda}{\partial m}(m, i) = f \\ \frac{\partial \lambda}{\partial i}(m, i) = 1/V. \end{cases}$$

Because λ is homogenous of degree 1, only the ratio of active capital to physical investment is pinned down by these conditions. Also, as in the standard model, this homogeneity implies all gains are used to pay for the two resources: active capital and investment goods. The price of the firm does not depend on the scale of its investment and all effects of active capital come through changes in the stochastic discount factor.

This model naturally generates comovement between active capital and physical investment. The elasticity of substitution between m and i is $1/(1-r)$. If they are complements, active investment and physical investment can both decrease in periods of high volatility. The decrease in physical investment now has two rationales. First, the standard q-theory one: in times of high uncertainty, valuations are low, investment is not valuable. This effect can be seen in the first condition of the firm. Second, in times of high uncertainty, little active investment is used and therefore physical investment is less valuable.

6 Conclusion

In this paper, I introduced a tractable model of asset pricing in the presence of concentrated ownership. In particular, I showed the price of risky assets and the provision of active capital are strongly tied. Variations in risk premia affect the supply of active capital, thereby generating the documented negative relation between expected returns and the amount of active capital. The quantity of active capital feeds back into aggregate risk sharing. As active investors deleverage to get out of their positions in times of high volatility, passive investors must bear the risk exactly when they do not want to. This feedback is a source of amplification for shocks to risk premium. On the other hand, cash flow shocks, because they are borne by active agents, are less costly for passive investors and command a lower risk premium. This mechanism can help us understand why cash flow shocks are not necessarily the key determinant of expected returns. On the other hand, fluctuations in risk premium might command a higher risk price than the standard model, due to their amplification because they cause variations in the quantity of active capital. Similarly, a large fraction of the volatility of asset prices might be due to fluctuations in fundamental volatility.

These qualitative predictions call for a more quantitative exploration. The main obstacle in this task is to measure the quantity of active capital. Because active capital takes many forms, inside and outside the firm, building aggregate measures of it in order to study its dynamics is an important challenge. For this task, a better understanding of the heterogeneity in sources of active capital is necessary. These sources present various levels of leverage or of productivity gains. An interesting direction along those lines is to take a more detailed look at how the structure of financing interacts with the investment policy. I have sketched a way to introduce physical investment into the model, and some work at the microeconomic level addresses how ownership and investment interact. Aggregating these results at the macroeconomic level is a challenge.

Another interesting avenue of study would be the development of the active technology at a lower frequency. For instance, the last few decades have seen an important expansion of the financial industry at the same time as a lowering of fundamental volatility. Some have argued a financial volatility has replace fundamental volatility. Again, to understand how changes in the financing of firms and in technology interact, a better understanding of the role of physical investment in the model would be necessary.

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A Proofs for the general model

A.1 Firm problem

The sequence problem for the valuation of the firm is:

$$P_t = \sup_{\{m_{t+\tau}\}_{0\le\tau<\infty}} \mathbb{E}_t\left[\int_0^\infty \frac{S_{t+\tau}}{S_t}(D_{t+\tau} - f_{t+\tau}m_{t+\tau}P_{t+\tau})d\tau\right]$$
$$\text{s.t. } \frac{dD_{t+\tau}}{D_{t+\tau}} = (\mu_D(s_{t+\tau}) + \lambda m_{t+\tau})\,dt + \sigma_D(s_{t+\tau})dZ_{t+\tau}.$$

Because of the homogeneity of the problem, the price–cash-flow ratio $V_t = P_t/D_t$ is just a function $V(s_t)$ of the state variables s_t. We can transform the sequence problem in a HJB equation. After dividing by $S_tD_tV_t$, we get:

$$0 = \max_m \frac{1}{V} - fm + \frac{\mathbb{E}[d(SDV)]}{SDVdt}.$$

Expanding, we obtain:

$$\begin{aligned} 0 = \max_m & \frac{1}{V} - fm + \frac{\mathbb{E}[dS]}{Sdt} + \frac{\mathbb{E}[dD]}{Ddt} + \frac{\mathbb{E}[dV]}{Vdt} \\ & + \frac{< dS, dV >}{[SVdt]} + \frac{< dS, dD >}{[SDdt]} + \frac{< dD, dV >}{[DVdt]}. \end{aligned}$$

The activity level only appears in the flow term $1/V - fm$ and the drift of the size of the firm $\mathbb{E}[dD]/Ddt = \mu_D(s) + \lambda m$. The optimization in m is therefore linear. This linearity gives us two key implications in the case of an interior optimum. First the slope of the equation in m has to be 0; this relation links the equilibrium fee for active capital to its productivity:

$$f(s) = \lambda. \tag{A.1}$$

Second, has the slope is zero, all terms in m cancel out in the HJB. The equation determining V corresponds to:

$$\begin{aligned} 0 = \max_m & \frac{1}{V} + \frac{\mathbb{E}[dS]}{Sdt} + \frac{\mathbb{E}[d\tilde{D}]}{\tilde{D}dt} + \frac{\mathbb{E}[dV]}{Vdt} \\ & + \frac{< dS, dV >}{[SVdt]} + \frac{< dS, d\tilde{D} >}{[S\tilde{D}dt]} + \frac{< d\tilde{D}, dV >}{[\tilde{D}Vdt]}, \end{aligned}$$

where

$$\frac{d\tilde{D}_{t+\tau}}{\tilde{D}_{t+\tau}} = \mu_D(s_{t+\tau})dt + \sigma_D(s_{t+\tau})dZ_{t+\tau}.$$

This equation corresponds to the price of a firm that never uses any amount of active capital.

A.2 Agent problem

A.2.1 Rescaling the HJB

The HJB equation determining the value function of an agent in any equation is the standard formulation for recursive preferences:

$$0 = \max_{C\ge 0,\theta\in\Theta} g(C,\mathcal{U}) + \mathbb{E}(d\mathcal{U})/dt.$$

First, using the homogeneity of the utility function and the linearity of wealth dynamics, we can express $\mathcal{U}$ as a separated function of wealth and the state variables:

$$\mathcal{U} = \frac{(W_t^i)^\gamma}{\gamma} G(s_t).$$

Then, using the homogeneity of the aggregator g, we simplify:

$$0 = \max_{c \geq 0, \theta \in \mathbb{R}^K} f(\gamma^{1/\gamma} c, F) \frac{W^\gamma}{\gamma} + \mathbb{E}\left[d\left(\frac{W^\gamma}{\gamma} F \right) \right] / dt.$$

Dividing, by W^γ gives directly the equations of Proposition 3.1.

A.2.2 Solving for consumption and the fee

First compute the various derivatives of the value function $\mathcal{U}$:

$$\mathcal{U} = \frac{W^\gamma}{\gamma} G,\ \mathcal{U}_W = \gamma \frac{W^{\gamma-1}}{\gamma} G,\ \mathcal{U}_{WW} = \gamma(\gamma-1) \frac{W^{\gamma-2}}{\gamma} G,\ \text{and } \mathcal{U}_{Ws} = \gamma \frac{W^{\gamma-1}}{\gamma} G_s.$$

Let us focus first on the HJB of the passive agent:

$$0 = \max_{C, \boldsymbol{\theta}} g(C, \mathcal{U}) + \frac{\mathbb{E}[d\mathcal{U}]}{dt}.$$

Expanding gives:

$$\begin{aligned} 0 = \max_{C,\boldsymbol{\theta}} g(C,\mathcal{U}) &+ \mathcal{U}_W \left[W(\boldsymbol{\theta}'(\mu_R - r_f) + r_f) - C \right] + \mathcal{U}_s' \mu_s \\ &+ \frac{1}{2} \mathcal{U}_{WW} W^2 \boldsymbol{\theta}' \sigma_R \sigma_R' \theta + W \boldsymbol{\theta}' \sigma_R \sigma_s' \mathcal{U}_{Ws} + \frac{1}{2} (\sigma_s' \sigma_s) * \mathcal{U}_{ss}, \end{aligned}$$

where $*$ is the elementwise multiplication. The first-order condition for consumption is:

$$g_C(C, \mathcal{U}) = \mathcal{U}_W.$$

The first-order condition for the portfolio share is:

$$\boldsymbol{\theta}^* = -\frac{\mathcal{U}_W W}{\mathcal{U}_{WW} W^2} (\sigma_R \sigma_R')^{-1} (\mu_R - r_f) - (\sigma_R \sigma_R')^{-1} \sigma_R \sigma_S \frac{\mathcal{U}_{Ws} W}{\mathcal{U}_{WW} W^2}.$$

Plugging in using the formulas for the derivatives, we get:

$$\begin{aligned} c = C/W &= \beta^{\frac{-1}{(\rho-1)}} G^{\frac{\rho}{(\rho-1)\gamma}} \\ \boldsymbol{\theta}^* &= \frac{1}{1-\gamma} (\sigma_R \sigma_R')^{-1} (\mu_R - r_f) + \frac{1}{1-\gamma} (\sigma_R \sigma_R')^{-1} \sigma_R \sigma_s' G_s. \end{aligned}$$

These formulas are the first two points of Proposition 3.2. Now we can turn to the problem of the passive agent. It is:

$$\begin{aligned} 0 = \max_{C} f(C, \mathcal{U}) &+ \mathcal{U}_W \left[W(\bar{\boldsymbol{\theta}}'(\mu_R - r_f) + r_f + \bar{\theta} f) - C \right] + \mathcal{U}_s' \mu_s \\ &+ \frac{1}{2} \mathcal{U}_{WW} W^2 \bar{\boldsymbol{\theta}}' \sigma_R \sigma_R' \bar{\boldsymbol{\theta}} + W \bar{\boldsymbol{\theta}}' \sigma_R \sigma_s' \mathcal{U}_{Ws} + \frac{1}{2} (\sigma_s' \sigma_s) * \mathcal{U}_{ss}. \end{aligned}$$

The first order condition for consumption is clearly the same as for the active agent which proves that they pick the same consumption wealth ratio. To solve for the fee, just substract this HJB from that of the passive investor. It gives:

$$\mathcal{U}_W \bar{\theta} f = \left[\mathcal{U}_W W\ \boldsymbol{\theta}^{*\,\prime}\,(\mu_R - r_f) + \frac{1}{2}\mathcal{U}_{WW} W^2\ \boldsymbol{\theta}^{*\,\prime}\,\sigma_R \sigma_R' \boldsymbol{\theta}^* + W\ \boldsymbol{\theta}^{*\,\prime}\,\sigma_R \sigma_s' \mathcal{U}_{Ws}\right]$$
$$- \left[\mathcal{U}_W W \bar{\boldsymbol{\theta}}'(\mu_R - r_f) + \frac{1}{2}\mathcal{U}_{WW} W^2 \bar{\boldsymbol{\theta}}'\sigma_R \sigma_R' \bar{\boldsymbol{\theta}} + W\bar{\boldsymbol{\theta}}'\sigma_R \sigma_s' \mathcal{U}_{Ws}\right].$$

The right-hand side is the difference of two values of an affine-quadratic form where one of the evaluation points is the optimum. It is therefore exactly quadratic, and equal to, after dividing by $\mathcal{U}_W$:

$$\bar{\theta} f = \frac{1}{2}(1-\gamma)(\boldsymbol{\theta}^* - \bar{\boldsymbol{\theta}})'\sigma_R \sigma_R'(\boldsymbol{\theta}^* - \bar{\boldsymbol{\theta}})$$

which concludes the proof.

B Solving the model of section 4

Stochastic discount factor

The stochastic discount factor evolution is given by

$$\frac{dS_t}{S_t} = -r_f(s_t) - rp_D(s_t) dZ_t^D - rp_X(s_t) dZ_t^X - rp_\sigma(s_t) dZ_t^\sigma.$$

Firm problem

The HJB detemining the price–cash-flow ratio $V(s)$ is

$$0 = \frac{1}{V} + \frac{\mathbb{E}[d(SDV)]}{SDVdt}.$$

Expanding, we obtain:

$$\begin{aligned}
-\frac{1}{V} =& \frac{\mathbb{E}[dS]}{Sdt} + \frac{\mathbb{E}[dD]}{Ddt} + \frac{\mathbb{E}[dV]}{Vdt} + \frac{< dS, dD >}{SDdt} + \frac{< dS, dV >}{SVdt} \\
=& - r_f + \mu_D - \kappa_X X_t \frac{V_X}{V} - \kappa_\sigma(\sigma_t^2 - \sigma_0^2)\frac{V_\sigma}{V} \\
&+ \frac{1}{2}\nu^2 \sigma_t^2 \frac{V_{\sigma\sigma}}{V} + \frac{1}{2}\phi^2 \sigma_t^2 \frac{V_{XX}}{V} \\
&- rp_D \sigma_t - rp_X \phi \sigma_t \frac{V_X}{V} - rp_\sigma \nu \sigma_t \frac{V_\sigma}{V}.
\end{aligned}$$

To obtain a single partial differential equation in V, we need to express the risk-free rate and all three risk prices as functions of V and the state variables. To do so, we first use the optimal portfolio of the passive agent

Stock return

The stock return for the firm's share is given by:

$$\begin{aligned}\frac{dR}{R} &= \left(\frac{1}{V} + \mu_D - \kappa_X \frac{V_X}{V} X_t - \kappa_\sigma \frac{V_\sigma}{V}(\sigma_t^2 - \sigma_0^2) + \frac{1}{2}\frac{V_{XX}}{V}\phi^2\sigma_t^2 + \frac{1}{2}\frac{V_{\sigma\sigma}}{V}\nu^2\sigma_t^2\right) dt \\ &+ \sigma_t \left(dZ^D + \phi \frac{V_X}{V} dZ^X + \nu \frac{V_\sigma}{V} dZ^\sigma\right).\end{aligned}$$

The expected return is:

$$\begin{aligned}\mathbb{E}_t\left[\frac{dR}{Rdt}\right] &= \frac{1}{V} + \mu_D - \kappa_X \frac{V_X}{V} X_t - \kappa_\sigma \frac{V_\sigma}{V}(\sigma_t^2 - \sigma_0^2) + \frac{1}{2}\frac{V_{XX}}{V}\phi^2\sigma_t^2 + \frac{1}{2}\frac{V_{\sigma\sigma}}{V}\nu^2\sigma_t^2 \\ &= r_f + rp_D\sigma_t + rp_X\phi\sigma_t\frac{V_\sigma}{V} + rp_\sigma\nu\sigma_t\frac{V_X}{V}.\end{aligned}$$

Return dynamics for state variable insurance claims

The other two assets of the economy are zero-cost assets that pay off $rp_X dt + dZ^X$ and $rp_\sigma dt + dZ^\sigma$ We need to check that their price is indeed zero. For the volatility shock insurance this corresponds to

$$\begin{aligned}\mathbb{E}\left[\frac{S_{t+dt}}{S_t}X_{t+dt}\right] &= \mathbb{E}\left[X_{t+dt}\right] + \mathbb{E}\left[\frac{dS_t}{S_t}X_{t+dt}\right] \\ &= rp_\sigma dt - rp_\sigma dt = 0.\end{aligned}$$

Portfolio problem of a passive agent

We can directly replace by the results of the general model.

Consumption-wealth ratio c:

Market-clearing imposes the consumption-wealth ratio to equal the cash-flow–price ratio:

$$\frac{1}{V} = c = \beta^{\frac{-1}{(\rho-1)}} G^{\frac{\rho}{\gamma(\rho-1)}}.$$

Volatility insurance position θ_σ:

In equilibrium, agents do not take any position in the insurance claims.

$$\begin{aligned}0 &= \gamma rp_\sigma + \gamma(\gamma-1)\theta^*\nu\frac{V_\sigma}{V}\sigma_t + \gamma\frac{G_\sigma}{G}\nu\sigma_t \\ rp_\sigma &= \nu\sigma_t\left[(1-\gamma)\theta^*\frac{V_\sigma}{V} - \frac{G_\sigma}{G}\right] \\ &= \nu\sigma_t\frac{V_\sigma}{V}\left[(1-\gamma)\theta^* - \frac{\gamma\rho}{1-\rho}\right],\end{aligned}$$

where the last equality is obtained by using the market-clearing condition for consumption linking V and G.

Growth-rate insurance position θ_X:

Similarly, we obtain:

$$rp_X = \phi\sigma_t\frac{V_X}{V}\left[(1-\gamma)\theta^* - \frac{\gamma\rho}{1-\rho}\right].$$

Stock position θ^*

We invert the formula for the optimal position θ^* to find the risk price of the shock dZ^D.

$$0 = (\mu_R - r_f) + (\gamma - 1)\left[\theta^* \sigma_t^2 + \theta^* \nu^2 \left(\frac{V_\sigma}{V}\right) + \theta^* \phi^2 \left(\frac{V_X}{V}\right)^2 \sigma_t^2\right] + \phi^2 \frac{G_X}{G}\frac{V_X}{V}\sigma_t^2 + \nu^2 \frac{G_\sigma}{G}\frac{V_\sigma}{V}\sigma_t^2.$$

Plugging in for $\mu_R - rf$ as a function of risk premia and recognizing the formulas for the other risk prices, we obtain:

$$rp_D = (1 - \gamma)\theta^* \sigma_t.$$

Risk-free rate

The risk-free rate r_f is obtained by plugging in all the quantities just derived in the HJB of the passive agent.

At this stage, we have expressed all the risk prices and the risk-free rate as functions of $V(s_t)$ and the optimal portfolio $\theta^*(s_t)$. The equilibrium of the market for active capital provides us this last quantity.

Market clearing in the active capital market

From the firms' FOC, we know the fee f must equal the marginal productivity of active capital λ. Equating this with the indifference condition between agents of different occupations gives:

$$\bar{\theta}\lambda = \frac{1}{2}(1 - \gamma)(\bar{\theta} - \theta^*)^2 \sigma_t^2 \left[1 + \nu^2 \left(\frac{V_\sigma}{V}\right)^2 + \phi^2 \left(\frac{V_X}{V}\right)^2\right]$$

The only unknown quantity is therefore $V(s_t)$. It is determined by the valuation equation of the firm.